Published for SISSA by Springer

Received: *June 21, 2022*
Accepted: *July 20, 2022*
Published: *August 17, 2022*

Non-perturbative aspects of the self-dual double copy

Kymani Armstrong-Williams, Chris D. White and Sam Wikeley

*Centre for Theoretical Physics, Department of Physics and Astronomy,
Queen Mary University of London,
327 Mile End Road, London E1 4NS, U.K.*

E-mail: k.t.k.armstrong-williams@qmul.ac.uk, christopher.white@qmul.ac.uk, s.wikeley@qmul.ac.uk

Abstract: The double copy is by now a firmly-established correspondence between amplitudes and classical solutions in biadjoint scalar, gauge and gravity theories. To date, no strongly coupled examples of the double copy in four dimensions have been found, and previous attempts based on exact non-linear solutions of biadjoint theory in Lorentzian signature have failed. In this paper, we instead look for biadjoint solutions in Euclidean signature, which may be relatable to Yang-Mills or gravitational instantons. We show that spherically symmetric power-like Euclidean solutions do not exist in precisely four spacetime dimensions. The explanation for why this is the case turns out to involve the Eguchi-Hanson instanton, whose single copy structure is found to be more complicated (and interesting) than previously thought. We provide a more general prescription for double-copying instantons, and explain how our results provide a higher-dimensional complement to a recently presented non-perturbative double copy of exact solutions in two spacetime dimensions. In doing so, we demonstrate how the replacement of colour by kinematic Lie algebras operates at the level of exact classical solutions.

Keywords: Scattering Amplitudes, Classical Theories of Gravity, Solitons Monopoles and Instantons

ArXiv ePrint: 2205.02136

https://doi.org/10.1007/JHEP08(2022)160

Contents

1 Introduction

In recent years, a correspondence known as the *double copy* has received a great deal of attention. Originating in the study of field theory scattering amplitudes [1, 2], and inspired by earlier work in string theory [3], it states that quantities in non-abelian gauge theories can be straightforwardly mapped to counterparts in gravity theories. Similarly, one can take objects in gauge theory, and translate them to a so-called *biadjoint scalar theory*, in which a scalar field carries two types of colour charge. This is known as the *zeroth copy*, and it is common to depict the ladder of correspondences between these different theories as in figure 1, although this is itself a subset of a much wider web of theories (see e.g. [4] for a review).

Much of the work in recent years has focused on how generally we are to interpret the scheme of figure 1. We unfortunately lack a complete understanding of the double copy at the level of e.g. Lagrangians or equations of motion (although see refs. [2, 5–12] for interesting developments), so that it is not known whether figure 1 applies to the complete theories. If it does, it suggests a profound and previously hidden commonality between our theories of nature, that traditional ways of thinking have obscured. Furthermore, there is a practical way to test how generally figure 1 applies. Namely, to take different objects in each of the theories, and to find rules for matching these up that can be seen to generalise the original double copy for scattering amplitudes. The first such work in this regard was ref. [13], which extended the double and zeroth copies to a special family of exact classical solutions of the relevant theories: those associated with *Kerr-Schild metrics* in General Relativity (see also refs. [14, 15] for related work in a different context). Follow-up work has considered the implications for specific solutions, including in different numbers of spacetime dimension [16–23]. A second exact double copy procedure is the *Weyl double*

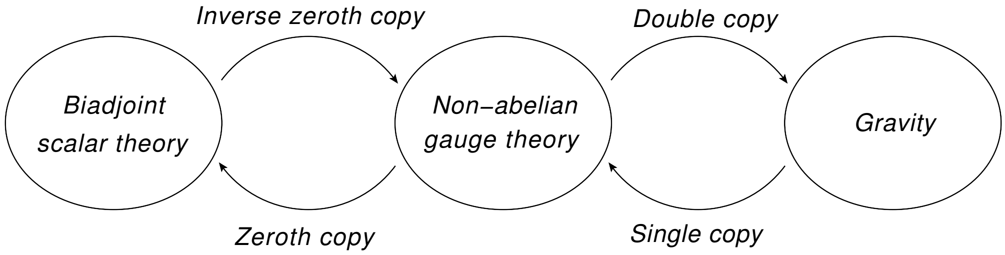

Figure 1. Correspondences between various types of field theory.

copy introduced in ref. [24], and studied further in refs. [25–34]. This is complementary to the Kerr-Schild double copy in that it relies on the spinorial formulation of field theory rather than the tensorial approach. Nevertheless, it agrees where overlap exists. In addition to exact solutions, one may double-copy classical solutions order-by-order in perturbation theory. For a variety of classical double copy approaches, see e.g. refs. [9–11, 35–53].

In all of the above double copies, solutions of *linearised* biadjoint scalar theory play a crucial role. They emerge as the denominators in scattering amplitudes, or as intermediate quantities in classical double copies. To date, there has been no concrete realisation of figure 1 for genuinely non-perturbative / strongly coupled solutions in four dimensions, involving fully non-linear solutions of each theory, including the biadjoint scalar case. A number of ideas have been proposed for exploring non-perturbative aspects of the double copy, including the study of how symmetries in different theories can be mapped [5, 11, 54–56], or non-trivial geometric / topological properties [19, 57, 58]. Recently, a proposal for a non-perturbative double copy in two spacetime dimensions has been made [52], and we return to this below. First, however, we continue a programme of work initiated in ref. [59], which found exact non-linear solutions of biadjoint scalar theory. Further solutions were found in refs. [60, 61], and the hope is that by assembling a catalogue of such solutions, it might be possible to identify their counterparts in gauge or gravity theory, thus providing a non-perturbative realisation of the double copy and related correspondences.[1]

The simplest non-linear solutions of biadjoint theory consist of spherically-symmetric monopole-like objects, involving a singularity at the origin. It is then natural to propose that these might be associated with singular monopoles in Yang-Mills theory, namely the *Wu-Yang monopoles* of ref. [64]. Indeed, this conjecture was tentatively made in ref. [59], but subsequent work has shown that it cannot be true [57, 65]. The Wu-Yang monopole turns out to be related (by a singular gauge transformation) to a non-abelian version of the well-known Dirac magnetic monopole, whose double and zeroth copies are already known: in gravity, it corresponds to a so-called *NUT charge* [66, 67], as first shown in ref. [16]. Thus, there is no room left for the "biadjoint monopole" to correspond to an obvious gauge theory solution, and it remains unclear how to proceed.

Given the above uncertainty regarding monopole solutions — which are in standard Lorentzian signature — we take here a different approach. Besides monopoles, some of the

[1] Interesting non-linear solutions that may be double-copiable can also be found in other theories, in various numbers of dimensions [62, 63].

most well-known non-perturbative solutions of gauge and gravity theories are *instantons*, which are solutions of the field equations in *Euclidean* signature. There is thus a clear motivation for trying to find non-linear solutions of Euclidean biadjoint scalar theory, in the hope that they may be relatable to known instantons. That this should be possible is further motivated by the fact that instanton solutions, as commonly referred to in both Yang-Mills (YM) theory and gravity, are (anti)-self-dual. It is known, at least in principle, that the self-dual sectors of YM and gravity can be written as manifest double copies of each other [5], where the zeroth copy to biadjoint theory also takes a simple form. Generalisations of this idea to more exotic theories are also known [68], although it is not clear what the precise implications are for classical solutions rather than scattering amplitudes.

In this paper, we will start by trying to find simple power-like spherically symmetric solutions of Euclidean biadjoint scalar field theory, analogous to the Lorentzian solutions found in ref. [59]. Curiously, we will see that whilst non-linear solutions do indeed exist in $d \neq 4$ spacetime dimensions, they are entirely absent in $d = 4$. This may at first seem surprising, but is in fact easily explainable: the power-like solutions one obtains for general dimensions solve the *linearised* biadjoint equation in $d = 4$, and thus cannot be solutions of the non-linear equation. Based on previous insights [19, 24], we are able to identify these linear solutions with the zeroth copy of the Eguchi-Hanson (gravitational) instanton [69–71]. However, in doing so, we find that the single copy of the Eguchi-Hanson instanton is more intricate than previously thought: one may provide a fully non-abelian single copy of this solution, in addition to its previously understood abelian counterpart. This mirrors the situation found for monopoles in refs. [57, 65], where either an abelian or non-abelian monopole are found to double copy to the same gravity solution (a NUT charge). We will also be able to write a more general ansatz for double-copying instantons than has previously been used. However, ultimately we find that it applies only to those gauge or gravity solutions which linearise the equations of motion.

As mentioned above, ref. [52] recently proposed a non-perturbative double copy procedure for exact solutions in a variety of theories in two spacetime dimensions. The role of the gravity theory is played by *Special Galileon (SG) theory*, and that of the gauge theory by so-called *Zakharov-Mikhailov (ZM) theory*. The equations of motion for these theories, which we write explicitly in section 5, bear a strong resemblance to those of self-dual Yang-Mills theory and gravity in four spacetime dimensions. This will allow us to interpret the results of ref. [52] as a close counterpart of the known four-dimensional self-dual double copy. Conversely, the ideas of ref. [52] will also clarify aspects of our four-dimensional results. In particular, the original double copy for scattering amplitudes relies on a phenomenon known as *BCJ duality* [72], which states that there is a *kinematic algebra* underlying gauge theory amplitudes, mirroring the Lie algebra describing the colour degrees of freedom. The gravity theory has no colour algebra, but instead has two copies of the kinematic algebra. The nature of this algebra has remained mysterious in general, but it can be made explicit at the level of equations of motion in the (anti-)self-dual sector, where it is known to correspond to a certain infinite-dimensional Lie algebra of area-preserving diffeomorphisms [5]. Until now, it has remained unclear how this algebra relates to properties of (exact) classical solutions, but ref. [52] provides an answer to this question in two spacetime dimensions. In turn, this allows us to interpret how the kinematic algebra is made manifest in the case of four-dimensional (anti-)self-dual solutions, thus resolving a long-standing conceptual issue in the double copy literature.

The structure of our paper is as follows. In section 2, we detail our attempts to find power-like solutions of Euclidean biadjoint scalar field theory in various numbers of spacetime dimension, showing that non-trivial examples are absent in $d = 4$. In section 3, we interpret this result as being due to known properties of the Eguchi-Hanson instanton. In section 4, we present a more general ansatz for double-copying instanton solutions. In section 5, we explain how our results relate to the recent two-dimensional non-perturbative double copy proposal of ref. [52], and clarify certain conceptual issues in the four-dimensional non-perturbative double copy. Finally, we discuss our results and conclude in section 6.

2 Solutions of Euclidean biadjoint scalar theory

In Lorentzian $(-,+,+,+)$ signature, the equation of motion for biadjoint scalar theory is as follows:

$$\partial^2\Phi^{aa'} + y f^{abc}\tilde{f}^{a'b'c'}\Phi^{bb'}\Phi^{cc'} = 0. \tag{2.1}$$

Here $\Phi^{aa'}$ is a scalar field with two colour indices in the adjoint representation of two global gauge groups, whose structure constants are f^{abc} and $\tilde{f}^{a'b'c'}$ respectively. Furthermore, y is a coupling constant. The quadratic interaction term in eq. (2.1) arises from a cubic Lagrangian, such that the energy of biadjoint scalar theory is unbounded from below. This in turn makes it a not particularly physical theory, given that its solutions will be dynamically unstable. However, this does not prevent us from looking for such solutions, as in previous instances of figure 1 for both scattering amplitudes and classical solutions, quantities in biadjoint theory are indeed related to physically well-behaved objects in gauge and gravity theory. Thus, it is worthwhile and meaningful to classify solutions in biadjoint theory, regardless of whether or not they make sense if considered in isolation.

We may proceed to Euclidean signature in eq. (2.1) by analytically continuing the time coordinate $t \to \tau = it$, such that one has

$$\partial^2 \to \Delta, \tag{2.2}$$

where Δ is the Laplacian operator, and we work in d spacetime dimensions in general. We then have

$$\Delta\Phi^{aa'} + y f^{abc}\tilde{f}^{a'b'c'}\Phi^{bb'}\Phi^{cc'} = 0, \tag{2.3}$$

for which we may attempt to find solutions as in the Lorentzian case of ref. [59]. First, we take the gauge groups to coincide, such that $f^{abc} = \tilde{f}^{abc}$. We may also write

$$f^{abc}f^{a'bc} = T_A\delta^{aa'}, \tag{2.4}$$

where the constant T_A depends on the common gauge group G, and the normalisation of the generators. Next, we restrict to spherically symmetric solutions via the ansatz

$$\Phi^{aa'} = \frac{\delta^{aa'}}{yT_A} f(r), \quad r^2 = x_\mu x^\mu. \tag{2.5}$$

Substituting this into eq. (2.3), one obtains

$$\frac{1}{r^{d-1}} \frac{d}{dr} \left(r^{d-1} \frac{df(r)}{dr} \right) + f^2(r) = 0, \tag{2.6}$$

where we have used the known form of the Laplacian in d-dimensional spherical polar coordinates:

$$\Delta f = \frac{1}{r^{d-1}} \frac{\partial}{\partial r} \left(r^{d-1} \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \Delta_{S^{d-1}} f. \tag{2.7}$$

The second term contains the *Laplace-Beltrami operator* $\Delta_{S^{d-1}}$, which depends only upon angular coordinates. It thus does not contribute to eq. (2.6) given the dependence upon only the radial coordinate.

Let us now look for pure power-like solutions by stipulating

$$f(r) = Ar^\alpha, \tag{2.8}$$

for some constants A and α. Substituting this into eq. (2.6) yields

$$A\alpha(d + \alpha - 2)r^{\alpha-2} + A^2 r^{2\alpha} = 0. \tag{2.9}$$

If this is to be true for all $r > 0$, then we must have

$$\alpha = -2 \quad \Rightarrow \quad A[A - 2(d - 4)] = 0. \tag{2.10}$$

We thus find $A = 0$ or $A = 2(d - 4)$, such that translating back to eq. (2.5) implies that there are two power-like solutions in general. The first is not interesting — it is the trivial (vacuum) solution $\Phi^{aa'} = 0$. The second is a non-trivial power-like solution

$$\Phi^{aa'} = \frac{2\delta^{aa'}}{yT_A} \frac{d-4}{r^2}. \tag{2.11}$$

Interestingly, the power of r^{-2} is common to all spacetime dimensions, which can be confirmed from dimensional analysis: the dimensions of the field $\Phi^{aa'}$ and the coupling constant y both vary with the number of dimensions, in just such a way as to fix the power of radial distance for solutions involving an inverse power of the coupling. This is in contrast to solutions of the *linearised* field equation, whose power of distance must vary in order to maintain the correct dimensions of the field. A consistency check of eq. (2.11) is that it reproduces the Lorentzian monopole solutions of ref. [59] for $d = 3$. These are static solutions, and thus the field equation of eq. (2.1) reduces to that of eq. (2.3), with a three-dimensional Laplacian involving the spatial coordinates.

Arguably the most curious feature of eq. (2.11) is the presence of $(d-4)$ in the numerator, which tell us that the non-trivial power-like solution of the full Euclidean biadjoint scalar field equation is absent in four spacetime dimensions. To gain more insight into what is going on, it is instructive to examine more general spherically symmetric solutions, as was done for the Lorentzian case in ref. [60]. That paper looked for solutions in which the divergence of the biadjoint field at the origin was (partially) screened.[2] This motivates the form

$$f(r) = \frac{K(r) - 1}{r^2}, \tag{2.12}$$

which we are always entitled to write, and for which $K(r) \to 1$ (everywhere) constitutes the trivial solution. By further introducing the variable ξ via

$$r = e^{-\xi}, \quad -\infty < \xi < \infty, \tag{2.13}$$

one may show that eq. (2.6) amounts to

$$\frac{\partial^2 K}{\partial \xi^2} - (d-6)\frac{\partial K}{\partial \xi} + (K-1)(K-2d+7) = 0. \tag{2.14}$$

This is a non-linear second-order differential equation, which cannot be solved analytically in general.[3] However, we may visualise solutions as follows. Defining $\psi \equiv \partial K/\partial\xi$, we may write eq. (2.14) as two coupled first-order equations:

$$\left(\frac{\partial K}{\partial \xi}, \frac{\partial \psi}{\partial \xi}\right) = \Big(\psi, (d-6)\psi - (K-1)(K-2d+7)\Big). \tag{2.15}$$

This defines a vector field in the (K,ψ) plane, whose integral curves correspond to solutions of eq. (2.14). We show such curves for the cases of $d = 2$, 3, 4 and 5 in figure 2. For general d, there are fixed points for

$$K \in \{1, 2d-7\}, \tag{2.16}$$

as is evident from eq. (2.14). Indeed, these correspond to the trivial solution and the non-trivial power-like solution of eq. (2.11) respectively. As the number of dimensions increases from $d < 4$, the non-trivial solution moves to the right in the (K,ψ) plane. For precisely $d = 4$, the two fixed points coincide, so that there is only the trivial solution, as found above.

If we want to find solutions that partially screen the divergence at the origin, we must look for bounded curves in the (K,ψ) plane i.e. those that correspond to finite numerators in eq. (2.12). For $d \neq 4$, there is always precisely one such bounded curve, connecting the points $(1,0)$ and $(2d-7,0)$. Thus, there is a single extended spherically symmetric solution that corresponds to a screened charge. This broadens the implications of the absence of a second fixed point in the $d = 4$ case: not only is there no non-trivial power-like solution, but there are no non-trivial extended solutions of the type in eq. (2.12) either (with bounded numerators).

[2] Finite energy static solutions of biadjoint scalar theory are impossible, as a consequence of *Derrick's theorem* [73].

[3] By a further transformation, one may recast eq. (2.14) as an *Abel equation of the second kind*, albeit not one that has a tractable solution in terms of known functions.

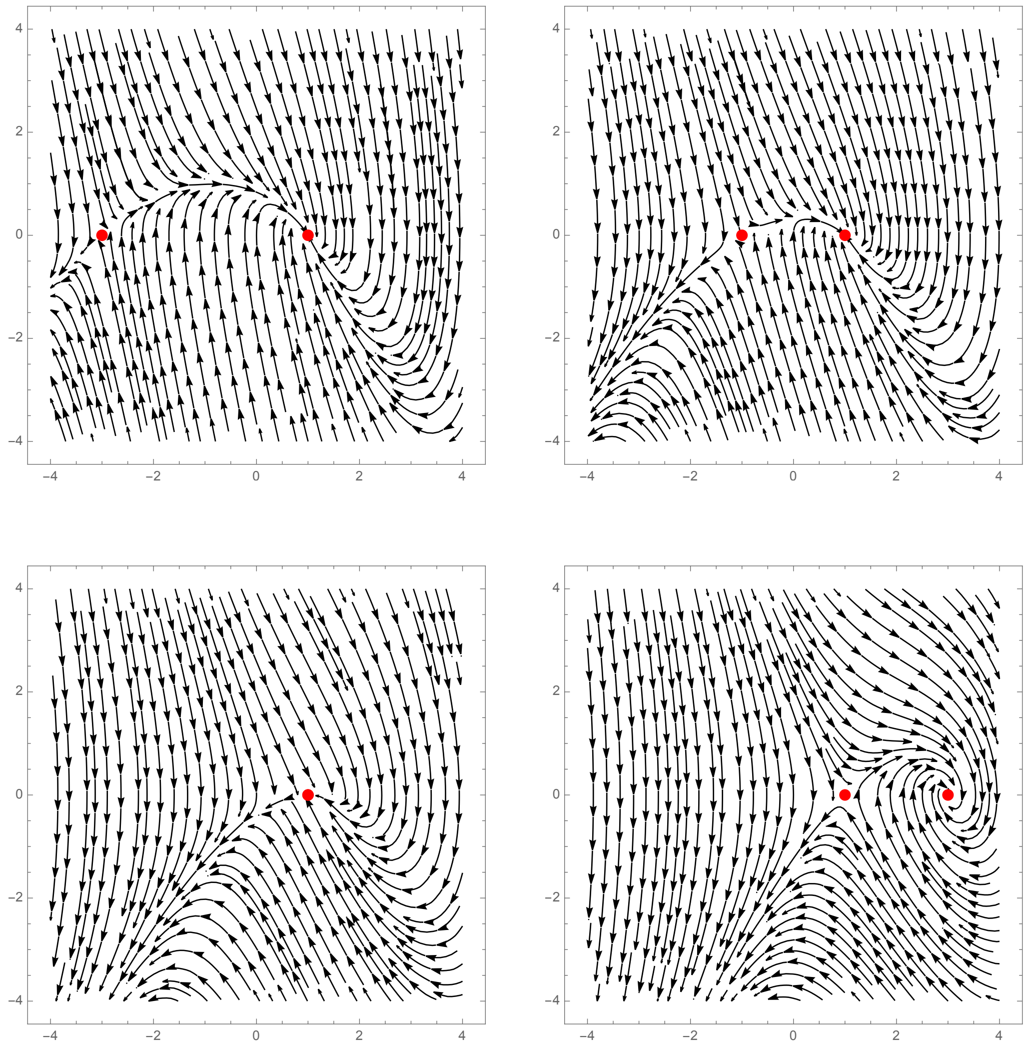

Figure 2. Integral curves of the vector field of eq. (2.15) in the (K, ψ) plane, corresponding to solutions of eq. (2.14). Shown are the cases $d = 2$, 3, 4, and 5 respectively. The red dots correspond to the fixed point solutions $K = 1$ and $K = 2d - 7$.

In this section, we have undertaken a first investigation of the spectrum of solutions of Euclidean biadjoint scalar field theory. We find non-trivial power-like solutions in all spacetime dimensions $d \neq 4$, with concomitant extended solutions. These solutions deserve further study, but for the remainder of this paper we will explain fully why there are no non-trivial power-like solutions for $d = 4$, and examine related implications.

3 The Eguchi-Hanson instanton revisited

In the previous section, we saw that there are no non-trivial power-like solutions (or spherically symmetric solutions) of Euclidean biadjoint theory in $d = 4$. There is in fact a very simple reason why this is the case. First, we may recall the observation made above that non-linear power-like solutions always have a radial dependence $\sim r^{-2}$, where this power can be fixed by dimensional analysis for solutions that involve an inverse power of the coupling constant y. However, in $d=4$, r^{-2} is a harmonic function, whose Laplacian vanishes for $r\neq 0$. To see this, note that in d dimensions, eq. (2.7) implies

$$\Delta r^{-n} = n(n+2-d)r^{-n-2}, \tag{3.1}$$

which indeed vanishes if $r>0$ for $n=2$ and $d=4$ (at $r=0$, there is a singularity, leading to an appropriately normalised delta function on the right-hand side of eq. (3.1)). If r^{-2} is harmonic in $d=4$, this means that it solves the *linearised* biadjoint equation of eq. (2.3). There is thus no room left for it to solve the non-linear equation, which is why there is no non-trivial power-like solution of Euclidean biadjoint theory in four spacetime dimensions. Indeed, we could simply have started with this observation, and not bothered with the analysis of the previous section at all. We maintain, however, that the results of the previous section remain useful: there are indeed non-trivial power-like solutions in other numbers of dimensions. Furthermore, the observation that there are no bounded extended solutions of Euclidean biadjoint theory, as well as power-like forms, is itself interesting.

As is well-known [13], one can turn harmonic functions into solutions of the full biadjoint scalar theory by dressing them with constant colour vectors $\{c^a,\tilde{c}^{a'}\}$, which in the case of our power-like solution in $d=4$ becomes

$$\Phi^{aa'} = \frac{\alpha c^a \tilde{c}^{a'}}{r^2}, \tag{3.2}$$

where we have included an arbitrary constant α that remains unfixed by the requirement that the kinematic dependence is harmonic. It is straightforward to check that, upon substitution into eq. (2.3), the non-linear term vanishes, leaving only the linear term as required. Given that previous examples of the classical double copy have focused on solutions that linearise biadjoint theory, we can then ask if it is possible to identify the gauge and gravity solutions for which eq. (3.2) constitutes the zeroth copy. Indeed, the solution turns out to be already known: it is related to the *Eguchi-Hanson (EH) solution* in gravity. First derived and discussed in refs. [69–71], this is a solution whose finite energy, self-dual nature and asymptotically Euclidean character lead to its interpretation as a *gravitational instanton*. As pointed out in e.g. refs. [19, 24, 74], it is particularly convenient to express the EH solution in (2,2) signature, using the coordinate system

$$u=\frac{\tau-iz}{\sqrt{2}},\quad v=\frac{\tau+iz}{\sqrt{2}},\quad X=\frac{ix-y}{\sqrt{2}},\quad Y=\frac{ix+y}{\sqrt{2}}, \tag{3.3}$$

in terms of Euclidean Cartesian coordinates[4]

$$x_\mu = (x,y,z,\tau). \tag{3.4}$$

Then the EH metric may be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{3.5}$$

[4]As is common in the instanton literature, we let Greek indices take the values $\mu,\nu,\ldots = 1,2,3,4$, with $x_4=\tau$.

where $h_{\mu\nu}$ is the graviton field

$$h_{\mu\nu} = \phi k_\mu k_\nu, \quad \phi = \frac{\lambda}{(uv - XY)}, \quad k_\mu = \frac{1}{(uv - XY)}(v, 0, 0, -X). \tag{3.6}$$

The metric of eqs. (3.5), (3.6) is in so-called *Kerr-Schild form*, where the vector k_μ satisfies the null and geodesic conditions

$$k^2 = 0 \quad k \cdot \partial k_\mu = 0. \tag{3.7}$$

This in turn means that its single and zeroth copies may be straightforwardly taken, using the general procedure defined in ref. [13]. One simply writes

$$A^a_\mu = c^a \phi k_\mu, \quad \phi^{aa'} = c^a \tilde{c}^{a'} \phi, \tag{3.8}$$

where c^a and $\tilde{c}^a$ are constant colour vectors. Then, the gauge and biadjoint fields thus constructed are guaranteed to solve the Yang-Mills and biadjoint equations, which happen to linearise in both cases. Upon translating to (Euclidean) Cartesian coordinates in eq. (3.6), one finds

$$\phi = \frac{2\lambda}{r^2}, \tag{3.9}$$

so that the zeroth copy of the Eguchi-Hanson solution in eq. (3.8) precisely matches the power-like solution of eq. (3.2) as claimed, provided we identify[5] $\alpha = 2\lambda$.

There is another way to interpret the above results. First writing the gauge field of eq. (3.8) in terms of an abelian gauge field A_μ:

$$A^a_\mu = c^a A_\mu, \quad A_\mu = \frac{\lambda}{(uv - XY)^2}(v, 0, 0, -X), \tag{3.10}$$

we may recognise the latter as

$$A_\mu = \hat{k}_\mu \phi, \quad \hat{k}_\mu = -(\partial_u, 0, 0, \partial_Y), \tag{3.11}$$

such that the single copy is given by the action of a differential operator on ϕ. It is easily checked that the Eguchi-Hanson graviton is given in terms of the same operator:

$$h_{\mu\nu} = \hat{k}_\mu \hat{k}_\nu \phi, \tag{3.12}$$

such that the double copy is formulated as a product in momentum, rather than position, space. A similar idea has occurred in the literature before. Reference [13] pointed out that setting up the double copy in terms of differential operators reproduces known descriptions of (anti-)self-dual Yang-Mills theory and gravity. That is, for a $\hat{k}_\mu$ satisfying the two conditions

$$\hat{k}^2 = 0, \quad \partial \cdot \hat{k} = 0, \tag{3.13}$$

[5] Alternatively, one can absorb the arbitrary constant α into the additional colour vector $\tilde{c}^{a'}$ that appears in the biadjoint field.

substituting the ansatz of eq. (3.12) into the Einstein equations reduces the latter to the *Plebanski equation* of self-dual gravity. For a suitable choice of $\hat{k}_\mu$, this can be written in the lightcone coordinate system as

$$\partial^2\phi + \kappa\,\{\partial_Y\phi, \partial_u\phi\} = 0, \tag{3.14}$$

where the Poisson bracket is defined by

$$\{A, B\} = (\partial_Y A)(\partial_u B) - (\partial_u A)(\partial_Y B). \tag{3.15}$$

Alternatively, one may write the Plebanski equation directly in terms of a general operator $\hat{k}_\mu$ satisfying eq. (3.13), as [13][6]

$$\partial^2\phi - \frac{1}{2}(\hat{k}_\mu\hat{k}_\nu\phi)(\partial_\mu\partial_\nu\phi) = 0. \tag{3.16}$$

Similarly, substituting the ansatz

$$A^a_\mu = \hat{k}_\mu\Phi^a, \tag{3.17}$$

into the Yang-Mills equations, for $\hat{k}_\mu$ satisfying eq. (3.13), leads to a known formulation of self-dual Yang-Mills theory [75][7]

$$\partial^2\Phi^a - \frac{1}{2}\epsilon^{abc}(\hat{k}_\mu\Phi^b)(\partial_\mu\Phi^c) = 0, \tag{3.18}$$

where the anti-self-dual sectors of both gauge and gravity theory can be obtained similarly (i.e. by a different choice of $\hat{k}_\mu$). The (anti-)self dual sectors constitute explicit cases in which the double copy can be made manifest at the level of equations of motion [5], and generalisations of this construction to exotic deformed theories are also known [68]. The Eguchi-Hanson instanton is a special case in which the fields are of Kerr-Schild form. In the gauge theory, this specialises the ansatz of eq. (3.17) to

$$\Phi^a = c^a\phi, \tag{3.19}$$

which linearises the Yang-Mills equations, such that one may consider the abelian field A_μ of eq. (3.10). As noted in ref. [19], the observation that (anti-)self-dual gravity solutions can be defined in terms of differential operators, and associated with (null) electromagnetic fields, was made long ago in ref. [74]. The double copy reinterprets this observation, and provides a framework for potential generalisations.

Returning to the present study, it is not obvious that the double copy between eqs. (3.10), (3.11) and eq. (3.12) is a special case of the known self-dual double copy of ref. [24], as the $\hat{k}_\mu$ operator of eq. (3.11) does not satisfy both conditions in eq. (3.13).

[6] Following convention, we do not raise or lower indices given that we are in Euclidean signature.

[7] Our eq. (3.18) can be obtained from eq. (28) of ref. [13]. Our conventions differ in that we are in Euclidean signature. We have also set coupling constants to unity in both eqs. (3.16) and (3.18), and normalised the vector $\hat{k}_\mu$ such that numerical constants are the same in all theories, for reasons that will become clear later on.

For our purposes, it will be useful to transform the operator to Cartesian coordinates, in which it takes the form[8]

$$\hat{k}_\mu = -\frac{1}{2}\left(\partial_x + i\partial_y, \partial_y - i\partial_x, \partial_z - i\partial_\tau, \partial_\tau + i\partial_z\right), \tag{3.20}$$

which may be written more compactly as

$$\hat{k}_\mu = -\frac{1}{2}\left(\delta_{\mu\nu} + i\bar{\eta}^3_{\mu\nu}\right)\partial_\nu. \tag{3.21}$$

Here $\bar{\eta}^3_{\mu\nu}$ is a special case of the *'t Hooft symbols* $\{\bar{\eta}^a_{\mu\nu}\}$, which arise in the study of instantons. For convenience, we review the properties of these symbols, as well as useful identities, in appendix A. More briefly, the 't Hooft symbols $\{\bar{\eta}^a_{\mu\nu}\}$ form a representation of an SU(2) subalgebra of SO(4), where the latter group is equivalent to the Lorentz group in Euclidean signature. Thus, the presence of the 't Hooft symbol in eq. (3.21) means that the operator $\hat{k}_\mu$ involves a particular "rotation" of the derivative operator ∂_ν. Using eq. (A.9), it is then straightforward to verify that

$$\hat{k}^2 = 0, \quad \partial \cdot \hat{k} = -\frac{1}{2}\Delta, \tag{3.22}$$

and thus that the second condition in eq. (3.13) is not satisfied. However, one may instead consider the alternative differential operator

$$\hat{k}'_\mu = (0, \partial_Y, \partial_u, 0), \tag{3.23}$$

In Cartesian coordinates, this translates as

$$\hat{k}'_\mu = \frac{1}{2}\left(\bar{\eta}^2_{\mu\nu} - i\bar{\eta}^1_{\mu\nu}\right)\partial_\nu = -\bar{\eta}^2_{\mu\nu}\hat{k}_\nu, \tag{3.24}$$

which does indeed satisfy both of the properties in eq. (3.13). Furthermore, we see that $\hat{k}'_\mu$ is a coordinate transformation of k_μ to a new frame whose coordinates are

$$x'_\mu = -\bar{\eta}^2_{\mu\nu}x_\nu \quad \Rightarrow \quad \begin{pmatrix} x' \\ y' \\ z' \\ \tau' \end{pmatrix} = \begin{pmatrix} z \\ \tau \\ -x \\ -y \end{pmatrix}. \tag{3.25}$$

We then have

$$(r')^2 = x'_\mu\, x'_\mu = x_\mu\, x_\mu = r^2, \tag{3.26}$$

such that one may write the Eguchi-Hanson solution and its single copy in the primed coordinate system as

$$A'_\mu = c^a \hat{k}'_\mu \phi(r'), \quad h'_{\mu\nu} = \hat{k}'_\mu \hat{k}'_\nu \phi(r'), \tag{3.27}$$

which is indeed a special case of the general self-dual construction of eqs. (3.12)–(3.17).

[8] In an abuse of notation, we will refer to the operator as $\hat{k}_\mu$ in both the (x, y, z, τ) and (u, v, X, Y) coordinate systems, given that the explicit coordinates that appear in any given equation imply no ambiguity.

In this section, we have explained the absence of non-linear power-like solutions in Euclidean biadjoint scalar theory, by pointing out that simple power-like solutions in four dimensions are in fact already "claimed" by the zeroth copy of the Eguchi-Hanson instanton, for which the biadjoint field equations linearise. A natural language for describing the single and zeroth copies of the EH solution is in terms of differential operators satisfying eq. (3.13), such that one obtains a special case of the self-dual double copy construction proposed in ref. [13]. Particularly compelling in our present paper is the fact that differential operators satisfying eqs. (3.13) can be written in terms of 't Hooft symbols, as in eq. (3.24). This suggests a general ansatz for double-copying certain instanton solutions, that we explore in the following section. We will also find that one may easily construct *non-abelian* single copies of the Eguchi-Hanson instanton, thus making its single copy structure more intricate than previously thought.

4 A general ansatz for double-copying instantons

In the previous section, we discussed the single copy of the Eguchi-Hanson instanton, which can be taken to be an abelian-like gauge field. In general, however, there are many instanton solutions of non-abelian gauge theories, namely (anti-)self-dual classical solutions, of finite energy (see e.g. refs. [76, 77] for pedagogical reviews). To be concrete, let us consider the case of SU(2) gauge theory. Finite energy demands that the field become pure gauge at infinity, and a given solution then constitutes a map from the boundary of spacetime (S^3) to the gauge group manifold, which is also S^3 for SU(2). Instantons can then be classified by their *winding*, or *instanton number*, which has a simple interpretation as the number of times the first S^3 space wraps around the second, in mapping the two manifolds.[9] The winding number is given by the volume integral

$$k = \frac{1}{16\pi^2} \int d^4x \mathrm{Tr}\left[\tilde{\mathbf{F}}_{\mu\nu}\, \mathbf{F}_{\mu\nu}\right], \tag{4.1}$$

where the field strength $\mathbf{F}_{\mu\nu} \equiv F^a_{\mu\nu}\mathbf{T}^a$ and its dual $\tilde{\mathbf{F}}_{\mu\nu} \equiv \tilde{F}^a_{\mu\nu}\mathbf{T}^a$ are given respectively by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu, \qquad \tilde{F}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^a_{\rho\sigma}. \tag{4.2}$$

There will be a non-trivial parameter space of solutions for definite winding number k in general, with the various parameters representing the width or size of the solution, its position in space, rotation angles in spacetime or in the gauge space etc. Some of these parameters are redundant under gauge transformations or other redundancies, but the set of independent parameters that label instantons of given k are called *moduli*, and they form a *moduli space*. The metric in this space is not completely smooth, but can be singular at certain points e.g. for instantons that have zero size, or where the centres of multiple

[9] Mathematically, one talks about the *third homotopy group* of a manifold $\mathcal{M}$, which classifies non-trivial maps from S^3 to a given manifold $\mathcal{M}$. Choosing $\mathcal{M}$ to be the gauge group manifold for SU(2), one has $\pi_3(S^3) = \mathbb{Z}$, meaning that there are topologically distinct maps labelled by different integers. This is precisely the winding number mentioned above, where positive (negative) values correspond to the (anti-)self dual sectors.

JHEP08(2022)160

instantons coincide. Remarkably, it is known how to classify *all* possible instanton solutions in pure YM theory [78].

A large family of SU(2) instanton solutions is given by dressing a vector field V_μ according to the *'t Hooft ansatz* [79]

$$A^a_\mu = -\bar{\eta}^a_{\mu\nu} V_\nu, \quad A^a_\mu = -\eta^a_{\mu\nu} V_\nu, \tag{4.3}$$

for the case of self-dual and anti-self dual fields respectively, and where $\{\eta^a_{\mu\nu}, \bar{\eta}^a_{\mu\nu}\}$ are the 't Hooft symbols encountered above, and reviewed briefly here in appendix A.[10] In our previous use of a 't Hooft symbol in eq. (3.21), this was acting merely as a representation of a particular spacetime rotation, where the upper index labelled which particular infinitesimal rotation we were talking about. In eq. (4.3), however, the upper indices on the 't Hooft symbols are to be interpreted as adjoint indices associated with the SU(2) gauge group. That this is possible is due to the fact that the $\{\eta^a_{\mu\nu}\}$ and $\{\bar{\eta}^a_{\mu\nu}\}$ separately form SU(2) algebras of SO(4) rotations. They can thus be mapped to the SU(2) gauge algebra. Focusing on the self-dual case, substitution of eq. (4.3) into the Yang-Mills equations yields the condition

$$\partial_\mu V_\mu + V_\mu V_\mu = 0, \tag{4.4}$$

as well as

$$\tilde{f}_{\mu\nu} = f_{\mu\nu}, \quad f_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu. \tag{4.5}$$

Here we can recognise $f_{\mu\nu}$ as being analogous to an abelian-like field strength tensor. However, V_μ cannot necessarily be interpreted as an abelian gauge field, given that $f_{\mu\nu}$ is not guaranteed to satisfy the Maxwell equation

$$\partial_\mu f_{\mu\nu} = 0 \tag{4.6}$$

in general. Equation (4.4) is usually satisfied by taking V_μ to be the gradient of the logarithm of a harmonic function:

$$V_\mu = \partial_\mu \log V, \quad \Delta V = 0. \tag{4.7}$$

However, motivated by the discussion in the previous section, there is another ansatz we can make. Let us construct a vector field according to the prescription:

$$A_\mu = \hat{k}_\mu \phi, \quad \hat{k}_\mu = \left(A\delta_{\mu\nu} + B_i\, \bar{\eta}^i_{\mu\nu}\right) \partial_\nu, \quad i \in \{1, 2, 3\}, \tag{4.8}$$

where the scalar A and three-vector B_i are possibly complex constants. This generalises the definitions of $\hat{k}_\mu$ and $\hat{k}'_\mu$ given by eqs. (3.21), (3.24). Note that, as in eq. (3.21), the upper index on each 't Hooft symbol is not a gauge (adjoint) index, but merely labels which infinitesimal rotations we are talking about. With this field we can compute

$$f_{\mu\nu} = B_i \left(\bar{\eta}^i_{\nu\rho}\partial_\mu - \bar{\eta}^i_{\mu\rho}\partial_\nu\right) \partial_\rho \phi, \tag{4.9}$$

[10] The minus signs in eq. (4.3) are conventional. Also, note that use of the anti-self-dual matrix $\bar{\eta}^a_{\mu\nu}$ actually results in a self-dual field. Whether or not $\eta^a_{\mu\nu}$ or $\bar{\eta}^a_{\mu\nu}$ appears in the gauge field can depend upon the gauge.

JHEP08(2022)160

from which one finds

$$\tilde{f}_{\mu\nu} = f_{\mu\nu} + B_i \bar{\eta}^i_{\mu\nu} \Delta\phi. \tag{4.10}$$

Thus, we see that the requirement for A_μ to satisfy the self-dual equations in eq. (4.5) is such as to force the field ϕ to be a harmonic function ($\Delta\phi = 0$). If we further insist that the double copy of eq. (4.8), defined according to eq. (3.12), be a solution of self-dual gravity, then the differential operator $\hat{k}_\mu$ must satisfy the dual constraints of eq. (3.13). For the first, we find

$$\hat{k}^2 = (A^2 + B^2)\Delta = 0 \quad \Rightarrow \quad A^2 = -B^2, \tag{4.11}$$

where $B^2 = B_i B^i$ and we have used the identities in appendix A. The second condition gives

$$\partial_\mu \hat{k}_\mu = A\Delta = 0. \tag{4.12}$$

Thus, the requirement that the abelian self-dual gauge field of eq. (4.8) double-copies to a self-dual gravity solution imposes

$$A = 0, \quad B^2 = 0. \tag{4.13}$$

The components $\{B_i\}$ are then required to be complex in general. Indeed, the abelian single copy of Eguchi-Hanson, with a differential operator defined as in eq. (3.24), emerges as a special case. Interestingly, though, eq. (4.13) implies that eq. (4.4) is automatically satisfied: one finds

$$\partial_\mu A_\mu + A_\mu A_\mu = A\Delta\phi + (A^2 + B^2)(\partial_\mu\phi)(\partial_\mu\phi) = 0. \tag{4.14}$$

This immediately implies that we may dress the solution of eq. (4.8) to form a non-abelian SU(2) instanton, as in eq. (4.3):

$$A^a_\mu = -\bar{\eta}^a_{\mu\nu} \hat{k}_\nu \phi. \tag{4.15}$$

A suggestive way to write this is as

$$A^a_\mu = -\hat{k}^a_\mu \phi, \quad \hat{k}^a_\mu \equiv \bar{\eta}^a_{\mu\nu} \hat{k}_\nu, \tag{4.16}$$

i.e. in terms of a "non-abelian" differential operator. Intriguingly, the non-abelian instanton can be double-copied directly, upon tracing over the colour indices: using eq. (A.9), one finds

$$\hat{k}^a_\mu \hat{k}^a_\nu = -\hat{k}_\mu \hat{k}_\nu. \tag{4.17}$$

Hence, if we construct the graviton field

$$h_{\mu\nu} = -\hat{k}^a_\mu \hat{k}^a_\nu \phi, \tag{4.18}$$

this yields precisely the same gravity solution as double copying the field of eq. (4.8) according to eq. (3.12). As noted above, eq. (4.8) is not guaranteed to be an abelian gauge field in general, as it may not necessarily satisfy the Maxwell equation of eq. (4.6). In fact

JHEP08(2022)160

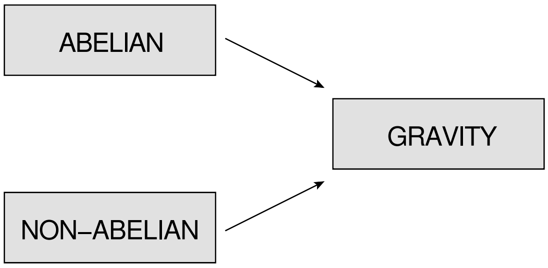

Figure 3. Schematic depiction of the multiple single copies of the gravitational instanton solutions considered in this paper. The same scheme has previously been obtained for magnetic monopoles [57, 65], and for infrared singularities of scattering amplitudes [80].

it does, however, due to the consequence derived above that ϕ be harmonic. To see this, note that eq. (4.9) implies

$$\partial_\mu f_{\mu\nu} = B_i \bar{\eta}^i_{\nu\rho} \partial_\rho (\Delta\phi) = 0, \tag{4.19}$$

where we have used the antisymmetry property $\bar{\eta}^i_{\mu\nu} = -\bar{\eta}^i_{\nu\mu}$. We therefore find that one may construct either an *abelian* or *non-abelian* single copy of certain (anti-)self-dual gravity solutions, for which the Eguchi-Hanson solution discussed in the previous section is a special case. Pleasingly, this mirrors the situation that has been found for magnetic monopole solutions in refs. [57, 65]. That is, one may regard the single copy of the pure NUT solution in gravity as an abelian-like (Dirac) magnetic monopole, dressed by a constant colour vector, or as a genuinely non-abelian Wu-Yang monopole, where there is a singular gauge transformation that relates the two forms. This scheme is shown in figure 3, and is interesting to note that a similar idea has occurred before in the original double copy for scattering amplitudes. For example, the infrared singularities of both abelian and non-abelian gauge theory map to the *same* infrared singularities in gravity, to all orders in perturbation theory [80]. There is presumably a gauge transformation (that we have not been able to find) that relates the two forms of the single copy instantons considered here.

In this section, we have given a general ansatz for single-copying a class of gravitational instantons, that allows us to construct non-abelian as well as abelian single copies. However, all of the solutions thus obtained turn out to be special, in that they linearise the equations of motion in both gravity and gauge theory (and hence, by association, biadjoint theory). The question then arises of whether one can single-copy more general instantons, or more formally: what portion of the moduli space of gauge theory instantons is captured by the ansatz of eq. (4.15)? The answer to the latter question appears to be rather limited. For example, we may attempt to calculate the winding number integral of eq. (4.1), for the single copy of the Eguchi-Hanson solution. This turns out to yield

$$\mathrm{Tr}[\tilde{\mathbf{F}}_{\mu\nu}\mathbf{F}_{\mu\nu}] \propto \frac{\lambda^2 B^2}{r^8}, \tag{4.20}$$

which vanishes due to being proportional to $B^2 = 0$. Thus, the non-abelian single copy of the Eguchi-Hanson instanton is topologically trivial.[11] Also, as a singular solution whose

[11]Similar conclusions were reached for the abelian single copy in ref. [19], albeit using a different single copy field. See ref. [24] for a discussion of the relation between the single copy of ref. [19], and that used in this paper.

JHEP08(2022)160

singularity appears non-removable by a gauge transformation, it should not properly be regarded as part of the moduli space of gauge theory instantons. Similar conclusions will be reached for more general functions ϕ: the winding number cannot depend on which particular basis of the rotation generators entering $\hat{k}_\mu$ we choose i.e. the 't Hooft symbols $\{\bar{\eta}^i_{\mu\nu}\}$. This in turn implies invariance under rotations of the vector B_i, such that the result can only depend on B^2, which is trivial. Interestingly, though, the space of harmonic functions ϕ in gauge theory still allows for some potentially interesting gravity solutions. For example, one may choose

$$\phi = \sum_{i=1}^{N} \frac{c_i}{(x-a_i)^2}, \tag{4.21}$$

which has N point-like disturbances at 4-positions $\{a_i\}$. The double copy of this would appear to be a multi-centre generalisation of the Eguchi-Hanson solution. Given that the single-centre Eguchi-Hanson case is related to a two-centre Gibbons-Hawking metric [81, 82], the graviton obtained from eqs. (3.12), (4.21) may also be of multi-centre Gibbons-Hawking type. We have not been able to find an explicit coordinate transformation that realises this, which by no means rules out that such a transformation is possible.

Of course, we have not considered the most general ansatz for gauge theory solutions in this paper. Equation (4.15) relies on a single function ϕ which is dressed by additional factors, rather than the full adjoint-valued field ϕ^a of eq. (3.17). However, it is not clear how to generate the additional information required by eq. (3.17) upon taking the single copy, i.e. how to turn the single gravitational function ϕ into the multiple functions $\{\phi^a\}$.[12] A prescription for achieving something similar has recently been proposed in various two-dimensional theories [52]. Indeed, comparing our analysis in this paper with the results of ref. [52] yields a number of useful insights, which we explore in the following section.

5 Relation to the two-dimensional non-perturbative double copy

In the previous section, we provided an ansatz for single-copying exact solutions of (anti-)self-dual gravity. Recently, another non-perturbative single copy procedure has appeared [52], and the aim of this section is to compare their approach with our analysis in this paper. As we will see, this comparison reveals a number of useful insights, that clarify long-standing questions in the double copy literature, but also generalise the results of ref. [52] itself. The latter reference considered various field theories in two spacetime dimensions, including the biadjoint scalar field theory we have already encountered in this paper. In the conventions of ref. [52], and allowing for arbitrary gauge groups, this has equation of motion

$$\partial^2\phi^{aa'} - \frac{1}{2} f^{abc}\tilde{f}^{a'b'c'}\phi^{bb'}\phi^{cc'} = 0, \tag{5.1}$$

[12] Reference [53] recently considered the more general ansatz of eq. (3.17) in exploring how asymptotic symmetries in self-dual Yang-Mills and gravity are related by the double copy. However, replacements between theories were at the level of commutators, or equivalently structure constants.

JHEP08(2022)160

in Lorentzian signature. Solutions in this theory were argued to obey a similar scheme to figure 1, but where the gauge theory is replaced by *Zakharov-Mikhailov (ZM)* theory [83]:

$$\partial^2\phi^a - \frac{1}{2} f^{abc} \partial_\mu \phi^a \tilde{\partial}^\mu \phi^b = 0. \tag{5.2}$$

Here the field ϕ^a carries a single adjoint index, and we have introduced the dual derivative operator

$$\tilde{\partial}^\mu = \epsilon^{\mu\nu} \partial_\nu, \tag{5.3}$$

where $\epsilon^{\mu\nu}$ is the two-dimensional Levi-Civita symbol. Finally, the gravity theory is replaced by *Special Galileon (SG) theory* [84–86], whose equation of motion is

$$\partial^2\phi - \frac{1}{2}(\partial_\mu\partial_\nu\phi)(\tilde{\partial}^\mu\tilde{\partial}^\nu\phi) = 0. \tag{5.4}$$

There is a clear pattern of replacements in proceeding from eq. (5.1) through eqs. (5.2), (5.4): adjoint indices are progressively removed, and the number of spacetime (and dual) derivatives increases. More formally, given two adjoint-valued fields V^a and W^a, one may define the formal replacement rules

$$V^a \to V, \quad f^{abc} V^a W^b \to (\partial_\mu V)(\tilde{\partial}^\mu W), \tag{5.5}$$

which can indeed be used to transform between the various theories mentioned above. The second replacement can be interpreted as replacing the structure constants of the colour algebra with those of an infinitely dimensional kinematic algebra. The kinematic structure constants are more easily viewed in momentum space, and to find them we may Fourier transform the right-hand side of the second replacement in eq. (5.5) to get

$$\begin{aligned} \int d^2x e^{ip_1\cdot x} \partial_\mu V \tilde{\partial}^\mu W &= \int d^2x \int \frac{d^2p_2}{(2\pi)^2} \int \frac{d^2p_3}{(2\pi)^2} e^{i(p_1-p_2-p_3)\cdot x} [-\epsilon^{\mu\nu} p_{2\mu} p_{3\nu}] \tilde{V}(p_2) \tilde{W}(p_3) \\ &= \int \frac{d^2p_2}{(2\pi)^2} \int \frac{d^2p_3}{(2\pi)^2} [-\epsilon^{\mu\nu} p_{2\mu} p_{3\nu} \delta^2(p_2+p_3-p_1)] \tilde{V}(p_2) \tilde{W}(p_3), \end{aligned} \tag{5.6}$$

where we have introduced momentum modes of the spacetime fields, denoted with tildes. We thus see that eq. (5.5) replaces adjoint-valued fields with scalars, which are then combined according to the momentum-space kinematic structure constant

$$f_{p_2 p_3}{}^{p_1} = X(p_2,p_3)\delta^2(p_2+p_3-p_1), \quad X(p_2,p_3) = -\epsilon^{\mu\nu} p_{2\mu} p_{3\nu}. \tag{5.7}$$

As to the nature of this kinematic algebra, it describes area-preserving diffeomorphisms of the two-dimensional spacetime. To see this, we may write the general form of an such an infinitesimal diffeomorphism in two-dimensional spacetime:

$$\mathbf{V} = -(\tilde{\partial}^\mu V)\partial_\mu. \tag{5.8}$$

According to standard differential geometry results, an infinitesimal diffeomorphism $f^\mu \partial_\mu$ is area-preserving (in more than two dimensions, volume-preserving) provided that $\partial_\mu f^\mu = 0$. Equation (5.3) implies

$$\partial_\mu \tilde{\partial}^\mu V = 0, \tag{5.9}$$

so that the diffeomorphism of eq. (5.8) is area-preserving as claimed. As explained in ref. [52], evaluating the Lie algebra of the generators in eq. (5.8) yields

$$[\mathbf{V}, \mathbf{W}] = \mathbf{Z}, \quad \mathbf{Z} = -\tilde{\partial}^{\mu} \left(\partial_{\nu} V \tilde{\partial}^{\nu} W \right) \partial_{\mu}. \tag{5.10}$$

To recognise the above structure constants, we can take generators corresponding to individual momentum modes:

$$\mathbf{V}_{p_i} = -\tilde{\partial}^{\mu}(V_{p_i})\partial_{\mu}, \quad V_{p_i} = e^{ip_i \cdot x}, \tag{5.11}$$

from which one finds

$$[\mathbf{V}_{p_2}, \mathbf{V}_{p_3}] = X(p_2, p_3)\mathbf{V}_{p_2+p_3} = f_{p_2 p_3}{}^{p_1}\mathbf{V}_{p_1}, \tag{5.12}$$

where we have recognised the form of the structure constant of eq. (5.7).

Above, we have seen that one may transform between the theories of eqs. (5.1), (5.2), (5.4) by successively replacing colour algebras with a Lie algebra of area-preserving diffeomorphisms. This does not explain *why* one should make such replacements, however, and ref. [52] provided the following motivation. It is a known fact that the group U(N) becomes isomorphic, at large N, to the group of diffeomorphisms of a torus [87]. To make this precise, one may use the fact that there is a particular basis $\{\mathbf{T}_p\}$ of the generators of U(N) (for odd N), where p is a 2-vector whose components are integer modulo N, and such that the structure constants are [52, 87]

$$f_{p_2 p_3}{}^{p_1} = -\frac{N}{2\pi} \sin\left(\frac{2\pi}{N} \epsilon^{\mu\nu} p_{2\mu} p_{3\nu}\right). \tag{5.13}$$

Upon taking the large N limit, one may reinterpret the vectors $\{p_i\}$ as specifying momentum modes on a torus, and the structure constants of eq. (5.13) reproduce precisely those of eq. (5.7). Thus, in the large N limit, the three theories of eqs. (5.1), (5.2), (5.4) (for gauge group U(N), repeated in the case of biadjoint theory) become mutually isomorphic. Reference [52] then uses this to argue that the colour-kinematic replacements of eq. (5.5) should be made also for finite N, and also proposes a scheme for turning non-perturbative solutions of SG theory into counterparts in ZM and biadjoint theory, which is accurate up to subleading corrections in N.

Regardless of this motivation, a very similar scheme — albeit perhaps not noticeably so — has appeared in the literature before, namely in the study of (anti-)self dual Yang-Mills and gravity [5]. We have quoted the relevant field equations in eqs. (3.18), (3.16), where different choices of the differential operator $\hat{k}_{\mu}$ correspond to the (anti-)self-dual cases respectively. Comparing these with eqs. (5.2), (5.4), we see that four-dimensional (anti-)self-dual Yang-Mills theory and gravity have *precisely* the same forms as two-dimensional ZM and SG theory respectively, but where the dual derivative operator $\tilde{\partial}^{\mu}$ is replaced by the differential operator $\hat{k}^{\mu}$.[13] Indeed, as is the role of $\tilde{\partial}^{\mu}$ in two dimensions, we can associate

[13]It should now hopefully be clear why we have chosen different conventions in our eqs. (3.16), (3.18) relative to those in existing literature [13]: it is to make the similarity between our four-dimensional theories and the two-dimensional theories of ref. [52] more striking.

$\hat{k}_\mu$ with area-preserving diffeomorphisms, where the most general such transformation will be given by

$$\mathbf{V} = -(\hat{k}^\mu V)\partial_\mu. \tag{5.14}$$

That this is area-preserving follows from our imposition that $\partial \cdot \hat{k} = 0$, and the cases of self-dual and anti-self-dual YM theory or gravity arise, as explained above, from the general ansätze

$$\hat{k}_\mu\Big|_{\rm SD} = \frac{1}{2}B_i\bar{\eta}^i_{\mu\nu}\partial^\nu, \quad \hat{k}_\mu\Big|_{\rm ASD} = \frac{1}{2}B_i\eta^i_{\mu\nu}\partial^\nu, \tag{5.15}$$

respectively. The role of the vector B_i is to pick out the two-dimensional planes in which the area-preserving diffeomorphisms of eq. (5.14) act. Specifically, we may write

$$B_i\bar{\eta}^i_{\mu\nu} = b^{(1)}_{[\mu}b^{(2)}_{\nu]}, \tag{5.16}$$

where the explicit forms of the vectors on the right-hand side, as may be verified using eq. (A.8), are

$$b^{(1)}_\mu = (B_1, B_2, B_3, 0) \quad b^{(2)}_\mu = \left(0, \frac{B_3}{B_1}, \frac{-B_2}{B_1}, -1\right), \tag{5.17}$$

where we have assumed $B_1 \neq 0$ and $B^2 = 0$ as before. Then

$$(\hat{k}^\mu\phi)\partial_\mu = (b^{(1)[\mu}b^{(2)\nu]}\partial_\nu\phi)\partial_\mu \tag{5.18}$$

will generate diffeomorphisms in the plane defined by the *bivector* $b^{(1)}_{[\mu}b^{(2)}_{\nu]}$. As an example, we may consider the self-dual operator of eq. (3.24), which has

$$B_1 = -i, \quad B_2 = 1, \quad B_3 = 0. \tag{5.19}$$

This gives (in Cartesian coordinates)

$$b^{(1)}_\mu = (-i, 1, 0, 0), \quad b^{(2)}_\mu = (0, 0, -i, -1). \tag{5.20}$$

Translating to the lightcone coordinate system, one finds diffeomorphisms in the (u, Y) plane as expected. Note that the vectors $\{b^{(i)}\}$ satisfy the conditions

$$b^{(i)} \cdot b^{(j)} = 0, \quad \forall i \in \{1, 2\}. \tag{5.21}$$

This makes the plane defined by the above bivector an example of a *null plane*. Null planes defined by self-dual and anti-self-dual bivectors are called α-planes and β-planes respectively, and they play a key role in the study of instantons (see e.g. ref. [88] for a pedagogical review). To summarise, we have found that (anti-)self-dual YM theory and gravity provide a four-dimensional generalisation of the non-perturbative double copy construction of ref. [52], but where the area-preserving diffeomorphisms take place in α- or β-planes. To visualise this over the entire space, we may foliate four-dimensional Euclidean space by a family of parallel α- or β-planes, whose orientation is determined by the vector B_i. The vector field A_μ will then generate area-preserving diffeomorphisms in each one, as shown in figure 4.

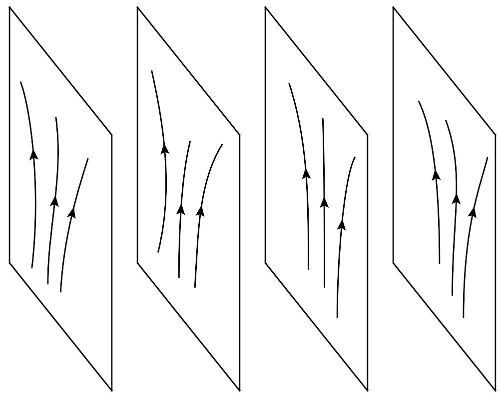

Figure 4. Foliation of four-dimensional Euclidean space by a family of α- or β-planes. The abelian single copy gauge field A_μ generates area-preserving diffeomorphisms in each plane, represented here by field lines (integral curves).

It was already known that the kinematic algebra of self-dual YM and gravity consisted of area-preserving diffeomorphisms [5]. New to this paper, however, are the geometric construction of area-preserving diffeomorphisms in arbitrary null planes, and the recognition that this provides a four-dimensional analogue of the two-dimensional non-perturbative double copy proposed in ref. [52]. Regarding the latter, it would be possible, for example, to argue for the double copy replacements

$$\phi^a \to \phi, \quad f^{abc}\phi_1^b\phi_2^c \to \partial_\mu\phi_1\,\hat{k}^\mu\phi_2 \tag{5.22}$$

on similar grounds to eq. (5.5). That is, one could take the large N limit of self-dual U(N) Yang-Mills theory (or U(N)×U(N) biadjoint theory), consider periodic solutions in the planes associated with $\hat{k}^\mu$, and then argue that the theories become mutually isomorphic.

Although the presence of an area-preserving diffeomorphism algebra in (anti-)self-dual gauge and gravity theory has been known for some time, it has not been known how these transformations are visible or relevant when considering exact classical solutions. For scattering amplitudes, the situation is much clearer, as first explained in ref. [5]: amplitudes in all theories defined by eqs. (3.16), (3.18), (5.1) are given by an expansion in cubic diagrams, each of whose vertices involves a product of two structure constants appropriate to the theory of interest. Thus, amplitudes in one theory can simply be obtained from amplitudes in another by replacing the appropriate structure constants, which gives a direct operational meaning to phrases such as "replacing the colour algebra with a kinematic algebra". For classical solutions, no structure constants manifestly appear, and thus it is not clear how moving from one theory to another involves a replacement of algebras. This conceptual problem is especially pronounced given that BCJ duality for amplitudes is an intrinsically non-linear phenomenon: it crucially relies on higher orders in perturbation theory, such that products of structure constants appear. The above ideas, however, indeed allow us to interpret what is happening, even though our exact classical solutions *linearise* the equations of motion. For a given gravity solution of the form of eq. (3.12), let us choose its abelian single copy

$$A^a_\mu = c^a A_\mu, \quad A_\mu = \hat{k}_\mu \phi. \tag{5.23}$$

Using standard results from differential geometry (see e.g. [89]), we may regard the vector field A_μ as generating an infinitesimal diffeomorphism

$$A^\mu \partial_\mu, \tag{5.24}$$

whose physical interpretation is that it performs a simultaneous translation along the integral curves of the field (figure 5). Vector fields that are the single copies of gravitational solutions will then generate diffeomorphisms of form

$$(\hat{k}^\mu \phi)\partial_\mu, \tag{5.25}$$

which, as remarked above, generate area-preserving diffeomorphisms in each of the null planes associated with the operator $\hat{k}_\mu$. Thus, even for the case of linearised solutions that do not involve higher-order contractions of structure constants, there is still a well-defined way in which their properties are governed by the area-preserving diffeomorphism algebra. Furthermore, translating between biadjoint, gauge and gravity theory entails replacing the diffeomorphism generators with colour generators, or vice versa. This is straightforward to see in the case of solutions with abelian-like single copies, which give rise to the following fields in different theories (contracted with appropriate generators):

$$\Phi = (c^a \mathbf{T}^a) \otimes (\tilde{c}^{a'} \tilde{\mathbf{T}}^{a'})\phi, \quad \mathbf{A}^\mu \partial_\mu = (c^a \mathbf{T}^a)(\hat{k}^\mu \phi)\partial_\mu, \quad h^{\mu\nu}\partial_\mu\partial_\nu = (\hat{k}^\mu \hat{k}^\nu \phi)\partial_\mu\partial_\nu. \tag{5.26}$$

Upon proceeding from left to right, we can see directly that colour generators are replaced by generators of area-preserving diffeomorphisms. There is nothing particularly profound going on here: the replacements of generators simply constitute the statement that the field in each theory must be Lie-algebra valued in two Lie algebras. These will be colourful or kinematical, as dictated by which theory we are in. Notably, this correspondence works for any value of N, not just the large N limit. But, as explained above, it applies only to those solutions that can be chosen to linearise their respective field equations. Interestingly, the non-perturbative double copy discussed in ref. [52] suffers from a similar specialism, in that SG theory is ultimately related to a free field theory.

Given the ideas of this section, it is not clear how to fully interpret the non-abelian single copy of eq. (4.15), which does not obviously translate to replacing an area-preserving diffeomorphism generator with a colour counterpart. It would be interesting to find non-trivial examples of double copies of the form of eq. (3.17), which in turn relates to the question of whether all gauge theory instantons can be double-copied. It may well turn out that, whilst all solutions of the Plebanksi equation can be single-copied to make non-abelian instantons, the converse is not possible. Indeed, similar comments were made in ref. [52] regarding how all solutions of SG theory can be used to make solutions of ZM theory, but that the opposite is not true. Complementary statements were made, some time ago and completely independently of the double copy, in ref. [90]. The authors showed that every gravitational instanton solution can be mapped to a SU(2) non-abelian instanton, albeit

JHEP08(2022)160

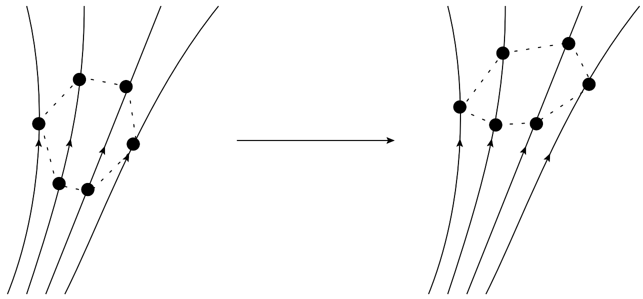

Figure 5. A vector field defines a diffeomorphism consisting of simultaneous translations along the integral curves of the field. For an area-preserving diffeomorphism, the areas of the two shapes shown on the left and right will be the same.

one that lives in the *same* Ricci-flat background as that defined by the gravitational solution i.e. the SU(2) solutions are "self-gravitating". In double copy lingo, this corresponds to a so-called *type B* curved-space double copy [18], in which a classical solution in gravity can be identified with a gauge field living on a non-dynamical curved background. Although this is a different situation to that considered in this paper, it nevertheless has the property that not all gauge theory solutions can be mapped to those in gravity. The reason in this case, though, is unique to the particular set-up considered in ref. [90]: one cannot copy a SU(2) solution on a particular curved background to obtain a gravity solution corresponding to a *different* curved space.

6 Discussion

In this paper, we have performed a first investigation of the spectrum of non-linear solutions of biadjoint scalar field theory in four Euclidean dimensions. Our motivation stems from the known double copy relationships between various field theories, summarised here in figure 1. It is not known how general this scheme is, and finding a genuinely non-perturbative incarnation would be a big step forward in this regard. We found that the spectrum of Euclidean solutions is rather rich in dimensions other than four, mirroring similar results that have been obtained previously in Lorentzian signature [59–61]. In precisely four spacetime dimensions, however, there are no simple power-like solutions, with a consequent absence of dressed solutions that screen a power-like divergence at the origin. This can be traced to the fact that the power-like form that is required is a harmonic function in $d = 4$, and thus solves the linearised biadjoint field equation. It can be identified with the zeroth copy of the Eguchi-Hanson solution, which has previously been considered from a double copy point of view in refs. [19, 24]. The single and zeroth copies can be formulated in terms of certain differential operators, obeying conditions that are similar (but not the same) as the Kerr-Schild conditions underlying the exact classical double copy of ref. [13].

We have here reinterpreted these differential operators as a special case of a more general ansatz, that involves the well-known 't Hooft symbols that arise in the study of

SU(2) gauge theory instantons. We provide a geometric interpretation of this ansatz, showing that our general (anti-)self dual operators are associated with area-preserving diffeomorphisms in given families of null planes. Further use of the 't Hooft symbols allows us to construct exact non-abelian single copies of (anti-)self-dual gravity solutions, living in an SU(2) gauge theory. However, the gauge theory requirements of (anti-)self-duality restrict the class of solutions to those that are ultimately linearisable. Nevertheless, the presence of both abelian and non-abelian single copies realises the same scheme (figure 3) that has previously arisen in the study of magnetic monopoles [57, 65].

Our results make contact with a recently proposed non-perturbative double copy in two spacetime dimensions. In particular, the replacement of colour algebras by area-preserving diffeomorphism algebras in four dimensions is a direct analogue of similar replacements that relate biadjoint scalar, Zakharov-Mikhailov and Special Galileon theory in two dimensions. Whilst the presence of such algebras in the (anti-)self dual sectors of YM theory and gravity have been known for some time [5], we show for the first time how to interpret the replacement of colour by kinematic algebras for exact classical solutions, rather than amplitudes.

Given that the full set of instanton solutions in non-abelian gauge theories is known [78], the question arises of how general our methods for explicitly double-copying them are. In more formal language, it remains unclear how the moduli space of gauge theory instantons maps onto that of gravitational instantons. Intriguing in this regard is that the explicit examples of solutions that we have given here seem to be excluded from the moduli space of SU(2) instanton solutions. Based on similar conclusions in other contexts [52, 90], it seems likely that not all gauge theory instantons can be double-copied to make gravitational solutions. However, it is correct to say that a full classification of *which* solutions can be copied is both useful, and missing.

There are many potential avenues for further work. One might look for non-spherically symmetric solutions of Euclidean biadjoint theory, and also interpret the spectrum of existing solutions we have found in various numbers of dimension, including their relationship with the Lorentzian solutions of refs. [59–61]. Examining how general our methods are — in terms of mapping out the known moduli space of gauge theory instantons — would be useful, and a first step in this regard would perhaps be to try to make sense of the more general ansatz of eq. (3.17), as noted explicitly in ref. [53]. Finally, we note that twistor methods are ubiquitous in the study of instantons, and have recently arisen in the context of the exact classical double copy [28, 30–32, 36, 91, 92]. Some sort of twistorial description of the (anti-)self-dual double copy is surely possible. We look forward to reporting on these various topics in the future.

A 't Hooft symbols and their properties

In this appendix, we introduce and briefly review the properties of *'t Hooft symbols*, which are used in the study of instanton solutions in non-abelian gauge theories. They are used in two different ways throughout this paper. First, they arise as infinitesimal generators of Euclidean rotations (the Euclidean signature equivalents of Lorentz transformations), which constitute the group SO(4). This group is six-dimensional, comprising three rotations $\{J_i\}$ in the (x_i, x_j) plane, and three rotations $\{K_i\}$ in the (x_i, x_4) plane, where $i \in \{1, 2, 3\}$, and we have defined coordinates as in eq. (3.4). Note that the latter are the analogue of boosts in Lorentzian signature. One may then form the combinations

$$M_i = \frac{1}{2}(J_i + K_i), \quad N_i = \frac{1}{2}(J_i - K_i), \tag{A.1}$$

which furnish two independent SU(2) subalgebras:

$$[M_i, M_j] = -\epsilon_{ijk} M_k \quad [N_i, N_j] = -\epsilon_{ijk} N_k, \quad [M_i, N_j] = 0. \tag{A.2}$$

To interpret these, we may note that a representation of $\{M_i, N_i\}$ acting on four-dimensional vectors can be given in terms of the *'t Hooft symbols*

$$\eta^a_{\mu\nu} = \epsilon^a{}_{\mu\nu 4} + \delta^a_\mu \delta_{\nu 4} - \delta^a_\nu \delta_{\mu 4}, \tag{A.3}$$

$$\bar{\eta}^a_{\mu\nu} = \epsilon^a{}_{\mu\nu 4} - \delta^a_\mu \delta_{\nu 4} + \delta^a_\nu \delta_{\mu 4}, \tag{A.4}$$

satisfying (in matrix notation)

$$[\eta^a, \eta^b] = -2\epsilon^{abc}\eta^c, \quad [\bar{\eta}^a, \bar{\eta}^b] = -2\epsilon^{abc}\bar{\eta}^c, \quad [\eta^a, \bar{\eta}^b] = 0, \tag{A.5}$$

such that one may set

$$M_i \to \frac{1}{2}\bar{\eta}^i_{\mu\nu}, \quad N_i \to \frac{1}{2}\eta^i_{\mu\nu}. \tag{A.6}$$

From eq. (A.4), one may verify that the 't Hooft symbols satisfy

$$\eta^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\eta^a_{\rho\sigma}, \qquad \bar{\eta}^a_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\bar{\eta}^a_{\rho\sigma}, \tag{A.7}$$

and thus are self-dual and anti-self-dual respectively. Their explicit matrix representation, from eq. (A.4) is

$$\eta^1_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \eta^2_{\mu\nu} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \eta^3_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix};$$
$$\bar{\eta}^1_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \bar{\eta}^2_{\mu\nu} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \bar{\eta}^3_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{A.8}$$

Further useful properties of these symbols, that we use throughout the paper, are

$$\begin{aligned} \bar{\eta}^a_{\mu\nu}\,\bar{\eta}^a_{\rho\sigma} &= \delta_{\mu\rho}\,\delta_{\nu\sigma} - \delta_{\mu\sigma}\,\delta_{\nu\rho} - \epsilon_{\mu\nu\rho\sigma}; \\ \bar{\eta}^a_{\mu\rho}\,\bar{\eta}^b_{\mu\sigma} &= \delta^{ab}\,\delta_{\rho\sigma} + \epsilon^{abc}\bar{\eta}^c_{\rho\sigma}; \\ \epsilon^{abc}\,\bar{\eta}^b_{\mu\nu}\,\bar{\eta}^c_{\rho\sigma} &= \delta_{\mu\rho}\bar{\eta}^a_{\nu\sigma} + \delta_{\nu\sigma}\bar{\eta}^a_{\mu\rho} - \delta_{\mu\sigma}\bar{\eta}^a_{\nu\rho} - \delta_{\nu\rho}\bar{\eta}^a_{\mu\sigma} \\ \bar{\eta}^a_{\mu\nu}\bar{\eta}^b_{\mu\nu} &= 4\delta^{ab}. \end{aligned} \tag{A.9}$$

These and additional identities may be found in e.g. appendix B of ref. [93].

JHEP08(2022)160

Thus far, the 't Hooft symbols have been described as representing particular rotations in spacetime. However, they have a second use in the study of non-abelian instantons, namely in embedding vector fields into non-abelian gauge groups. Considering the case of pure SU(2) Yang-Mills theory, this will have a matrix-valued gauge field $\mathbf{A}_\mu = A^a_\mu \mathbf{T}^a$, where $\{\mathbf{T}^a\}$ are the generators of the gauge group. Given a vector field V_μ, a general procedure for turning this into a non-abelian field with components A^a_μ is the *'t Hooft ansatz* of eq. (4.3). In this equation, the upper index on the 't Hooft symbol is to be interpreted as an adjoint index associated with the gauge group, rather than labelling a given rotation generator. The 't Hooft ansatz thus represents a map from a selected SU(2) subalgebra of SO(4), to the SU(2) colour algebra. Embedding vector fields into higher gauge groups is also possible, given that the latter will contain SU(2) subgroups.

Acknowledgments

We thank Silvia Nagy for many useful discussions, and collaboration on related topics. We are also grateful to the participants of the Amplitudes meeting at the Higgs Centre for Theoretical Physics for useful feedback and advice. This work has been supported by the U.K. Science and Technology Facilities Council (STFC) Consolidated Grant ST/P000754/1 "String theory, gauge theory and duality", and by the European Union Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 764850 "SAGEX". KAW is supported by a studentship from the U.K. Engineering and Physical Sciences Research Council (EPSRC). No new data were generated or analysed during this study.

References

[1] Z. Bern, J.J.M. Carrasco and H. Johansson, *Perturbative Quantum Gravity as a Double Copy of Gauge Theory*, *Phys. Rev. Lett.* **105** (2010) 061602 [arXiv:1004.0476] [INSPIRE].

[2] Z. Bern, T. Dennen, Y.-t. Huang and M. Kiermaier, *Gravity as the Square of Gauge Theory*, *Phys. Rev. D* **82** (2010) 065003 [arXiv:1004.0693] [INSPIRE].

[3] H. Kawai, D.C. Lewellen and S.H.H. Tye, *A Relation Between Tree Amplitudes of Closed and Open Strings*, *Nucl. Phys. B* **269** (1986) 1 [INSPIRE].

[4] Z. Bern, J.J. Carrasco, M. Chiodaroli, H. Johansson and R. Roiban, *The Duality Between Color and Kinematics and its Applications*, arXiv:1909.01358 [INSPIRE].

[5] R. Monteiro and D. O'Connell, *The Kinematic Algebra From the Self-Dual Sector*, *JHEP* **07** (2011) 007 [arXiv:1105.2565] [INSPIRE].

[6] C. Cheung and G.N. Remmen, *Twofold Symmetries of the Pure Gravity Action*, *JHEP* **01** (2017) 104 [arXiv:1612.03927] [INSPIRE].

[7] C. Cheung and G.N. Remmen, *Hidden Simplicity of the Gravity Action*, *JHEP* **09** (2017) 002 [arXiv:1705.00626] [INSPIRE].

[8] M. Tolotti and S. Weinzierl, *Construction of an effective Yang-Mills Lagrangian with manifest BCJ duality*, *JHEP* **07** (2013) 111 [arXiv:1306.2975] [INSPIRE].

[9] L. Borsten and S. Nagy, *The pure BRST Einstein-Hilbert Lagrangian from the double-copy to cubic order*, *JHEP* **07** (2020) 093 [arXiv:2004.14945] [INSPIRE].

[10] L. Borsten, B. Jurčo, H. Kim, T. Macrelli, C. Sämann and M. Wolf, *Becchi-Rouet-Stora-Tyutin-Lagrangian Double Copy of Yang-Mills Theory*, *Phys. Rev. Lett.* **126** (2021) 191601 [arXiv:2007.13803] [INSPIRE].

[11] L. Borsten, H. Kim, B. Jurčo, T. Macrelli, C. Sämann and M. Wolf, *Double Copy from Homotopy Algebras*, *Fortsch. Phys.* **69** (2021) 2100075 [arXiv:2102.11390] [INSPIRE].

[12] M. Beneke, P. Hager and A.F. Sanfilippo, *Double copy for Lagrangians at trilinear order*, *JHEP* **02** (2022) 083 [arXiv:2106.09054] [INSPIRE].

[13] R. Monteiro, D. O'Connell and C.D. White, *Black holes and the double copy*, *JHEP* **12** (2014) 056 [arXiv:1410.0239] [INSPIRE].

[14] V.E. Didenko, A.S. Matveev and M.A. Vasiliev, *Unfolded Description of AdS_4 Kerr Black Hole*, *Phys. Lett. B* **665** (2008) 284 [arXiv:0801.2213] [INSPIRE].

[15] V.E. Didenko and M.A. Vasiliev, *Static BPS black hole in 4d higher-spin gauge theory*, *Phys. Lett. B* **682** (2009) 305 [*Erratum ibid.* **722** (2013) 389] [arXiv:0906.3898] [INSPIRE].

[16] A. Luna, R. Monteiro, D. O'Connell and C.D. White, *The classical double copy for Taub–NUT spacetime*, *Phys. Lett. B* **750** (2015) 272 [arXiv:1507.01869] [INSPIRE].

[17] A.K. Ridgway and M.B. Wise, *Static Spherically Symmetric Kerr-Schild Metrics and Implications for the Classical Double Copy*, *Phys. Rev. D* **94** (2016) 044023 [arXiv:1512.02243] [INSPIRE].

[18] N. Bahjat-Abbas, A. Luna and C.D. White, *The Kerr-Schild double copy in curved spacetime*, *JHEP* **12** (2017) 004 [arXiv:1710.01953] [INSPIRE].

[19] D.S. Berman, E. Chacón, A. Luna and C.D. White, *The self-dual classical double copy, and the Eguchi-Hanson instanton*, *JHEP* **01** (2019) 107 [arXiv:1809.04063] [INSPIRE].

[20] M. Carrillo-González, R. Penco and M. Trodden, *The classical double copy in maximally symmetric spacetimes*, *JHEP* **04** (2018) 028 [arXiv:1711.01296] [INSPIRE].

[21] M. Carrillo González, B. Melcher, K. Ratliff, S. Watson and C.D. White, *The classical double copy in three spacetime dimensions*, *JHEP* **07** (2019) 167 [arXiv:1904.11001] [INSPIRE].

[22] I. Bah, R. Dempsey and P. Weck, *Kerr-Schild Double Copy and Complex Worldlines*, *JHEP* **02** (2020) 180 [arXiv:1910.04197] [INSPIRE].

[23] G. Alkac, M.K. Gumus and M.A. Olpak, *Kerr-Schild double copy of the Coulomb solution in three dimensions*, *Phys. Rev. D* **104** (2021) 044034 [arXiv:2105.11550] [INSPIRE].

[24] A. Luna, R. Monteiro, I. Nicholson and D. O'Connell, *Type D Spacetimes and the Weyl Double Copy*, *Class. Quant. Grav.* **36** (2019) 065003 [arXiv:1810.08183] [INSPIRE].

[25] S. Sabharwal and J.W. Dalhuisen, *Anti-Self-Dual Spacetimes, Gravitational Instantons and Knotted Zeros of the Weyl Tensor*, *JHEP* **07** (2019) 004 [arXiv:1904.06030] [INSPIRE].

[26] R. Alawadhi, D.S. Berman and B. Spence, *Weyl doubling*, *JHEP* **09** (2020) 127 [arXiv:2007.03264] [INSPIRE].

[27] H. Godazgar, M. Godazgar, R. Monteiro, D. Peinador Veiga and C.N. Pope, *Weyl Double Copy for Gravitational Waves*, *Phys. Rev. Lett.* **126** (2021) 101103 [arXiv:2010.02925] [INSPIRE].

[28] C.D. White, *Twistorial Foundation for the Classical Double Copy*, *Phys. Rev. Lett.* **126** (2021) 061602 [arXiv:2012.02479] [INSPIRE].

[29] R. Monteiro, D. O'Connell, D. Peinador Veiga and M. Sergola, *Classical solutions and their double copy in split signature*, *JHEP* **05** (2021) 268 [arXiv:2012.11190] [INSPIRE].

[30] E. Chacón, S. Nagy and C.D. White, *The Weyl double copy from twistor space*, *JHEP* **05** (2021) 2239 [arXiv:2103.16441] [INSPIRE].

[31] E. Chacón, A. Luna and C.D. White, *The double copy of the multipole expansion*, arXiv:2108.07702 [INSPIRE].

[32] E. Chacón, S. Nagy and C.D. White, *Alternative formulations of the twistor double copy*, *JHEP* **03** (2022) 180 [arXiv:2112.06764] [INSPIRE].

[33] H. Godazgar, M. Godazgar, R. Monteiro, D. Peinador Veiga and C.N. Pope, *Asymptotic Weyl double copy*, *JHEP* **11** (2021) 126 [arXiv:2109.07866] [INSPIRE].

[34] R. Monteiro, S. Nagy, D. O'Connell, D. Peinador Veiga and M. Sergola, *NS-NS spacetimes from amplitudes*, *JHEP* **06** (2022) 021 [arXiv:2112.08336] [INSPIRE].

[35] G. Elor, K. Farnsworth, M.L. Graesser and G. Herczeg, *The Newman-Penrose Map and the Classical Double Copy*, *JHEP* **12** (2020) 121 [arXiv:2006.08630] [INSPIRE].

[36] K. Farnsworth, M.L. Graesser and G. Herczeg, *Twistor Space Origins of the Newman-Penrose Map*, arXiv:2104.09525 [INSPIRE].

[37] A. Anastasiou, L. Borsten, M.J. Duff, L.J. Hughes and S. Nagy, *Yang-Mills origin of gravitational symmetries*, *Phys. Rev. Lett.* **113** (2014) 231606 [arXiv:1408.4434] [INSPIRE].

[38] G. Lopes Cardoso, G. Inverso, S. Nagy and S. Nampuri, *Comments on the double copy construction for gravitational theories*, *PoS* **CORFU2017** (2018) 177 [arXiv:1803.07670] [INSPIRE].

[39] A. Anastasiou, L. Borsten, M.J. Duff, S. Nagy and M. Zoccali, *Gravity as Gauge Theory Squared: A Ghost Story*, *Phys. Rev. Lett.* **121** (2018) 211601 [arXiv:1807.02486] [INSPIRE].

[40] A. Luna, S. Nagy and C. White, *The convolutional double copy: a case study with a point*, *JHEP* **09** (2020) 062 [arXiv:2004.11254] [INSPIRE].

[41] W.D. Goldberger, S.G. Prabhu and J.O. Thompson, *Classical gluon and graviton radiation from the bi-adjoint scalar double copy*, *Phys. Rev. D* **96** (2017) 065009 [arXiv:1705.09263] [INSPIRE].

[42] W.D. Goldberger and A.K. Ridgway, *Bound states and the classical double copy*, *Phys. Rev. D* **97** (2018) 085019 [arXiv:1711.09493] [INSPIRE].

[43] W.D. Goldberger, J. Li and S.G. Prabhu, *Spinning particles, axion radiation, and the classical double copy*, *Phys. Rev. D* **97** (2018) 105018 [arXiv:1712.09250] [INSPIRE].

[44] W.D. Goldberger and J. Li, *Strings, extended objects, and the classical double copy*, *JHEP* **02** (2020) 092 [arXiv:1912.01650] [INSPIRE].

[45] W.D. Goldberger and A.K. Ridgway, *Radiation and the classical double copy for color charges*, *Phys. Rev. D* **95** (2017) 125010 [arXiv:1611.03493] [INSPIRE].

[46] S.G. Prabhu, *The classical double copy in curved spacetimes: Perturbative Yang-Mills from the bi-adjoint scalar*, arXiv:2011.06588 [INSPIRE].

[47] A. Luna, R. Monteiro, I. Nicholson, A. Ochirov, D. O'Connell, N. Westerberg et al., *Perturbative spacetimes from Yang-Mills theory*, *JHEP* **04** (2017) 069 [arXiv:1611.07508] [INSPIRE].

[48] A. Luna, I. Nicholson, D. O'Connell and C.D. White, *Inelastic Black Hole Scattering from Charged Scalar Amplitudes*, *JHEP* **03** (2018) 044 [arXiv:1711.03901] [INSPIRE].

[49] C. Cheung and C.-H. Shen, *Symmetry for Flavor-Kinematics Duality from an Action*, *Phys. Rev. Lett.* **118** (2017) 121601 [arXiv:1612.00868] [INSPIRE].

[50] C. Cheung and J. Mangan, *Covariant color-kinematics duality*, *JHEP* **11** (2021) 069 [arXiv:2108.02276] [INSPIRE].

[51] C. Cheung, A. Helset and J. Parra-Martinez, *Geometry-Kinematics Duality*, arXiv:2202.06972 [INSPIRE].

[52] C. Cheung, J. Mangan, J. Parra-Martinez and N. Shah, *Non-perturbative Double Copy in Flatland*, arXiv:2204.07130 [INSPIRE].

[53] M. Campiglia and S. Nagy, *A double copy for asymptotic symmetries in the self-dual sector*, *JHEP* **03** (2021) 262 [arXiv:2102.01680] [INSPIRE].

[54] R. Alawadhi, D.S. Berman, B. Spence and D. Peinador Veiga, *S-duality and the double copy*, *JHEP* **03** (2020) 059 [arXiv:1911.06797] [INSPIRE].

[55] A. Banerjee, E.O. Colgáin, J.A. Rosabal and H. Yavartanoo, *Ehlers as EM duality in the double copy*, *Phys. Rev. D* **102** (2020) 126017 [arXiv:1912.02597] [INSPIRE].

[56] Y.-T. Huang, U. Kol and D. O'Connell, *Double copy of electric-magnetic duality*, *Phys. Rev. D* **102** (2020) 046005 [arXiv:1911.06318] [INSPIRE].

[57] L. Alfonsi, C.D. White and S. Wikeley, *Topology and Wilson lines: global aspects of the double copy*, *JHEP* **07** (2020) 091 [arXiv:2004.07181] [INSPIRE].

[58] R. Alawadhi, D.S. Berman, C.D. White and S. Wikeley, *The single copy of the gravitational holonomy*, *JHEP* **10** (2021) 229 [arXiv:2107.01114] [INSPIRE].

[59] C.D. White, *Exact solutions for the biadjoint scalar field*, *Phys. Lett. B* **763** (2016) 365 [arXiv:1606.04724] [INSPIRE].

[60] P.-J. De Smet and C.D. White, *Extended solutions for the biadjoint scalar field*, *Phys. Lett. B* **775** (2017) 163 [arXiv:1708.01103] [INSPIRE].

[61] N. Bahjat-Abbas, R. Stark-Muchão and C.D. White, *Biadjoint wires*, *Phys. Lett. B* **788** (2019) 274 [arXiv:1810.08118] [INSPIRE].

[62] A. Cherman, D. Dorigoni, G.V. Dunne and M. Ünsal, *Resurgence in Quantum Field Theory: Nonperturbative Effects in the Principal Chiral Model*, *Phys. Rev. Lett.* **112** (2014) 021601 [arXiv:1308.0127] [INSPIRE].

[63] C. Cheung and J. Mangan, *Scattering Amplitudes and the Navier-Stokes Equation*, arXiv:2010.15970 [INSPIRE].

JHEP08(2022)160

[64] T. . Wu and C.-N. Yang, *Some solutions of the classical isotopic gauge field equations*, *Properties of Matter Under Unusual Conditions* (1967).

[65] N. Bahjat-Abbas, R. Stark-Muchão and C.D. White, *Monopoles, shockwaves and the classical double copy*, *JHEP* **04** (2020) 102 [arXiv:2001.09918] [INSPIRE].

[66] A.H. Taub, *Empty space-times admitting a three parameter group of motions*, *Annals of Mathematics* **53** (1951) 472.

[67] E. Newman, L. Tamburino and T. Unti, *Empty-space generalization of the schwarzschild metric*, *J. Math. Phys.* **4** (1963) 915.

[68] E. Chacón, H. García-Compeán, A. Luna, R. Monteiro and C.D. White, *New heavenly double copies*, *JHEP* **03** (2021) 247 [arXiv:2008.09603] [INSPIRE].

[69] T. Eguchi and A.J. Hanson, *Asymptotically Flat Selfdual Solutions to Euclidean Gravity*, *Phys. Lett. B* **74** (1978) 249 [INSPIRE].

[70] T. Eguchi and A.J. Hanson, *Selfdual Solutions to Euclidean Gravity*, *Annals Phys.* **120** (1979) 82 [INSPIRE].

[71] T. Eguchi and A.J. Hanson, *Gravitational instantons*, *Gen. Rel. Grav.* **11** (1979) 315 [INSPIRE].

[72] Z. Bern, J.J.M. Carrasco and H. Johansson, *New Relations for Gauge-Theory Amplitudes*, *Phys. Rev. D* **78** (2008) 085011 [arXiv:0805.3993] [INSPIRE].

[73] G.H. Derrick, *Comments on nonlinear wave equations as models for elementary particles*, *J. Math. Phys.* **5** (1964) 1252 [INSPIRE].

[74] K.P. Tod, *Self-dual Kerr–Schild metrics and null Maxwell fields*, *J. Math. Phys.* **23** (1982) 1147 [INSPIRE].

[75] A. Parkes, *A Cubic action for selfdual Yang-Mills*, *Phys. Lett. B* **286** (1992) 265 [hep-th/9203074] [INSPIRE].

[76] E.J. Weinberg, *Classical solutions in quantum field theory*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, U.K. (2012).

[77] N.S. Manton and P. Sutcliffe, *Topological solitons*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, U.K. (2004).

[78] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Y.I. Manin, *Construction of Instantons*, *Phys. Lett. A* **65** (1978) 185 [INSPIRE].

[79] G. 't Hooft, *Computation of the Quantum Effects Due to a Four-Dimensional Pseudoparticle*, *Phys. Rev. D* **14** (1976) 3432 [*Erratum ibid.* **18** (1978) 2199] [INSPIRE].

[80] S. Oxburgh and C.D. White, *BCJ duality and the double copy in the soft limit*, *JHEP* **02** (2013) 127 [arXiv:1210.1110] [INSPIRE].

[81] G.W. Gibbons and S.W. Hawking, *Classification of Gravitational Instanton Symmetries*, *Commun. Math. Phys.* **66** (1979) 291 [INSPIRE].

[82] M.K. Prasad, *Equivalence of Eguchi-Hanson metric to two-center Gibbons-Hawking metric*, *Phys. Lett. B* **83** (1979) 310 [INSPIRE].

[83] V.E. Zakharov and A.V. Mikhailov, *Relativistically Invariant Two-Dimensional Models in Field Theory Integrable by the Inverse Problem Technique* (in Russian), *Sov. Phys. JETP* **47** (1978) 1017 [INSPIRE].

JHEP08(2022)160

[84] C. Cheung, K. Kampf, J. Novotny and J. Trnka, *Effective Field Theories from Soft Limits of Scattering Amplitudes*, *Phys. Rev. Lett.* **114** (2015) 221602 [arXiv:1412.4095] [INSPIRE].

[85] C. Cheung, K. Kampf, J. Novotny, C.-H. Shen and J. Trnka, *On-Shell Recursion Relations for Effective Field Theories*, *Phys. Rev. Lett.* **116** (2016) 041601 [arXiv:1509.03309] [INSPIRE].

[86] K. Hinterbichler and A. Joyce, *Hidden symmetry of the Galileon*, *Phys. Rev. D* **92** (2015) 023503 [arXiv:1501.07600] [INSPIRE].

[87] J. Hoppe, *Diffeomorphism Groups, Quantization and SU(infinity)*, *Int. J. Mod. Phys. A* **4** (1989) 5235 [INSPIRE].

[88] C. Nash and S. Sen, *Topology and Geometry for Physicists*, Academic Press (1988).

[89] B.F. Schutz, *Geometrical Methods of Mathematical Physics*, Cambridge University Press, Cambridge, U.K. (1980).

[90] J.J. Oh, C. Park and H.S. Yang, *Yang-Mills Instantons from Gravitational Instantons*, *JHEP* **04** (2011) 087 [arXiv:1101.1357] [INSPIRE].

[91] A. Guevara, *Reconstructing Classical Spacetimes from the S-matrix in Twistor Space*, arXiv:2112.05111 [INSPIRE].

[92] T. Adamo and U. Kol, *Classical double copy at null infinity*, *Class. Quant. Grav.* **39** (2022) 105007 [arXiv:2109.07832] [INSPIRE].

[93] S. Vandoren and P. van Nieuwenhuizen, *Lectures on instantons*, arXiv:0802.1862 [INSPIRE].

JHEP08(2022)160