# Distributed Robust Partial State Consensus Control for Chain Interconnected Delay Systems

Zidong Liu, Dongya Zhao, *Member, IEEE*, Jiehua Feng, Ye Zhang, and Sarah K. Spurgeon, *Fellow, IEEE*

***Abstract*— Partial state consensus (PSC) is investigated for chain interconnected systems with time-varying delays and parameter uncertainties. A novel design philosophy of PSC control is proposed and a sequential calculation method is presented to guarantee the robustness of the controller. A sufficient condition based on linear matrix inequalities (LMIs) is derived and the stability is proven by the Lyapunov method. The proposed approach can ensure that the states which are subject to a consensus constraint achieve consensus, while those without a consensus constraint track their own set points. Finally, numerical simulations and a solution proportioning experiment are developed to validate the effectiveness of the proposed method.**

***Index Terms*— Chain interconnected delay systems, distributed robust control, partial state consensus (PSC).**

## I. Introduction

IN MODERN industrial plants, a process network is composed of many units arranged in a possibly complex and particular structure. There have been many control publications which seek to classify the network types according to their structure, such as chain interconnected systems (sometimes named cascade interconnected systems) [1], [2], [3] and parallel interconnected systems [4], [5]. Chain interconnected systems are common and many process networks can be modeled in this form, such as a fossil fuel power unit [6] and a continuous annealing line [7]. A multistage flash distillation process for desalination, which is also a typical example, is shown in Fig. 1.

There are many common characteristics in chain interconnected systems. According to Fig. 1, it is straightforward to conclude that the product (freshwater) can be obtained by evaporation and liquefaction of material (seawater) under successively decreasing pressure. Each grade can be regarded as a subsystem and it is only affected by the adjacent front and rear grades. Due to material transfer and modeling error, delays and uncertainties make control design difficult for chain interconnected systems. There exist many results on identical interconnected systems in [8] and [9], but these control design methods are not suitable due to the heterogeneity and uncertainties of chain interconnected systems. Although some methods have been developed for heterogeneous systems in [10] and [11], there are many restrictions on the interconnection terms or requirements on the availability of full information. A more effective method is investigated in [2], but the case of delays is not addressed.

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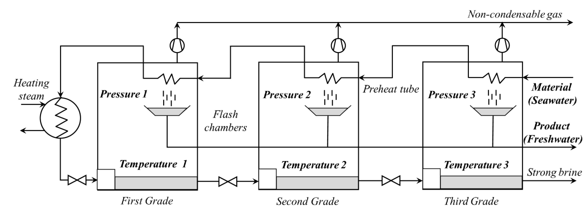


Fig. 1. Multistage flash distillation method for desalination with three grades.

Also in chain interconnected systems, there is a common dynamic behavior named partial state consensus (PSC), which means that only some of the states are subject to consensus constraints. For example in Fig. 1, the pressure of each grade has a fixed relationship with others and its control can be considered as a consensus problem. There is no consensus constraint for the other states, for example the temperatures, which are required to maintain their own set-points. However, most of the existing results on large-scale systems [8], [9], [10], [11], [12], [13] can only achieve consensus, that is, PSC has not been addressed. Further, the existing work on PSC in [14] and [15] cannot guarantee the stability of the states which do not have consensus constraints. Note that the methods in [16], [14], and [15] cannot process the interconnections. The effects of delays exist in the consideration of [17], but only PSC for integrator systems is investigated. A static optimization problem with a partial consensus constraint is investigated in [18], but it is not suitable for solving the PSC problem. In this work, a new distributed robust PSC control (DR-PSC) has been proposed to address these outstanding issues. This approach is more applicable to chain interconnected uncertain systems than existing methods.

The main contributions are as follows.

1) A novel design approach is proposed to achieve PSC in a distributed manner. By establishing augmented systems, the original systems can be converted into auxiliary systems, making PSC control design convenient and

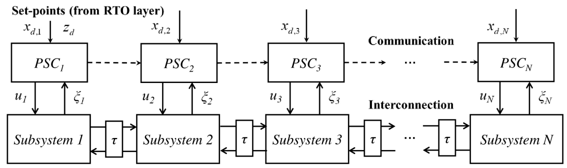


Fig. 2. Network of chain interconnected systems.

intuitive. For the case of tracking control, the proposed distributed PSC control can make the states with and without consensus constraints reach common and local set points respectively.

2) A sufficient condition for DR-PSC control is derived for chain interconnected systems with time-varying delay in the interactions and uncertainty in the parameters. With DR-PSC, the dynamic performance of the systems can be effectively guaranteed.

The brief is structured as follows. In Section II, the dynamics of chain interconnected systems are formulated and the definition of PSC is given. The main result is presented in Section III, including the PSC control design and a sufficient condition for DR-PSC. Section IV shows the results of simulations and an experiment. Section V summarizes the results and provides conclusions.

## II. Preliminaries and Problem Formulation

Consider a chained network composed of $N$ continuous-time linear uncertain subsystems as follows:

$$\dot{\xi}_i(t) = \tilde{A}_{ii}\xi_i(t) + \tilde{B}_i u_i(t) + \sum_{j\in\mathcal{P}^i} \tilde{A}_{ij}\xi_j(t-\tau(t)) \qquad (1)$$

where $i \in \mathcal{V}$, $\xi_i(t) \in R^{p_i}, p_i \geq 2$ is the state vector, $u_i(t) \in R^{q_i}$ is the input vector. The state consists of two parts, the "Nonconsensus" state $x_i \in R^{m_i}$ and the "Consensus" state $z_i \in R^n$. It can be formulated as $\xi_i = \text{col}(x_i, z_i)$, where $p_i = m_i + n$. $\tau(t)$ is the time-varying delay and the parameter matrices for $i \in \mathcal{V}$, $j \in \mathcal{P}_i$ are

$$\tilde{A}_{ij} = A_{ij} + \Delta A_{ij} \in R^{p_i\times p_j}, \quad \tilde{B}_i = B_i + \Delta B_i \in R^{p_i\times q_i}.$$

In Fig. 2, it is straightforward to see that there are time-varying delays in the interconnections among the subsystems and the neighbors of each subsystem are the adjacent front and rear elements. Then, the neighbor set $\mathcal{P}^i$ in (1) can be written as $\mathcal{P}^i = \{i-1, i+1\}\cap\mathcal{V}$. There are two special cases where the neighbor sets are $\mathcal{P}^1 = \{2\}$ and $\mathcal{P}^N = \{N-1\}$, as $i = 1$ and $i = N$.

There are several assumptions given as follows.

*Assumption 1:* The pairs $(A_{ii}, B_{ii}), i \in \mathcal{V}$ are controllable and the following equality holds:

$$\text{rank}\left(\begin{bmatrix} A_{ii} & B_i \\ I_{p_i} & \mathbf{0}_{p_i\times q_i}\end{bmatrix}\right) = 2p_i.$$

*Assumption 2 [19]:* The time-varying delay $\tau(t)$ in interconnected terms satisfies $0 \leq \tau(t) \leq \bar{\tau}$ and $|\dot{\tau}(t)| \leq \mu \leq 1$, where $\bar{\tau}, \mu > 0$ are both known boundary values. The initial states $\xi_i(0)$ are known and $\xi_i(t) = \xi_i(0), t \in [-\bar{\tau}, 0], i \in \mathcal{V}$.

*Assumption 3:* The uncertain terms in (1) satisfy

$$\begin{bmatrix} \Delta A_{ii} & \Delta A_{ij} & \Delta B_i \end{bmatrix} = D_i F_i(t) \begin{bmatrix} E_{1,ii} & E_{1,ij} & E_{2,i}\end{bmatrix}$$

where $i \in \mathcal{V}$. The matrices $D_i \in R^{p_i\times r_i}$, $E_{1,ii} \in R^{s_i\times p_i}$, $E_{1,ij} \in R^{s_i\times p_j}$, $E_{2,i} \in R^{s_i\times q_i}$ are known and the unknown time-varying one $F_i(t) \in R^{r_i\times s_i}$ satisfies $F_i^\top(t)F_i(t) \leq I$.

The definition of PSC can now be presented.

*Definition 1:* For any initial condition $\xi_i(0)$ and each subsystem $i \in \mathcal{V}$, the PSC for (1) is achieved if and only if all "Nonconsensus" state converge to their individual set points $x_{d,i}$, $\lim_{t\to\infty} x_i(t) = x_{d,i}$, and all "Consensus" state converge to the common set point (the consensus value) $z_d$, $\lim_{t\to\infty} z_i(t) = z_d$, $\lim_{t\to\infty} z_i(t) = z_j(t)$, $j \in \mathcal{V}$.

*Remark 1:* In the multistage flash distillation method for desalination, the pressure and the temperature of each grade can be modeled as "Nonconsensus" states and "Consensus" states, respectively. Then, PSC controllers can be designed using the method to be proposed in Section III.

According to the framework in Fig. 2, local set points $x_{d,i}, i \in \mathcal{V}$ are assigned to each subsystem and the common one $z_d$ is only assigned to subsystem 1, which can be considered as a "leader." To achieve PSC for (1) based on a distributed method, subsystems have to exchange information between neighbors through the one-way topology, whose Laplacian matrix $L$ can be formulated as

$$L_{ij} = \begin{cases} 1, & i = j \\ -1, & j = i-1 \\ 0, & \text{otherwise.}\end{cases}$$

For convenience, "Nonconsensus" and "Consensus" tracking errors are defined as

$$\begin{cases} e_{z,1} = z_1(t) - z_d \\ e_{z,i} = z_i(t) - z_{i-1}(t), \quad i \in \mathcal{V}\backslash\{1\}\end{cases}$$
$$e_{x,i} = x_i(t) - x_{d,i}, \quad i \in \mathcal{V}. \qquad (2)$$

*Remark 2:* Different from the fully connected topology in traditional process control, the topology shown in Fig. 2 is one-way and fixed. It can not only reduce the communication cost, but also facilitate scalability. When a new subsystem is added to the network, there is only the link between it and the last subsystem to be established; the others are not affected.

There are two goals:

1) To propose a PSC control design method for (1) which will satisfy

$$\lim_{t\to\infty}\left\|e_{z,1}(t)\right\| = \lim_{t\to\infty}\|z_1(t) - z_d\| = 0$$
$$\lim_{t\to\infty}\left\|e_{z,i}(t)\right\| = \lim_{t\to\infty}\|z_i(t) - z_{i-1}(t)\| = 0, \quad i \in \mathcal{V}\backslash\{1\}$$
$$\lim_{t\to\infty}\left\|e_{x,i}(t)\right\| = \lim_{t\to\infty}\left\|x_i(t) - x_{d,i}\right\| = 0, \quad i \in \mathcal{V}.$$

2) To give a sufficient condition for DR-PSC control in the presence of uncertainties and delays.

## III. Partial State Consensus Control Design

In this section, a novel method for designing a DR-PSC control is proposed for (1) and a constructive stability condition based on linear matrix inequalities (LMIs) is presented. Several useful lemmas are first presented.

*Lemma 1 [19]:* Suppose that $U(t) \in R^p$ is a vector-valued function with continuous first-order derivatives. For any matrices $\mathcal{M}_1, \mathcal{M}_2 \in R^{p\times p}$, and $\mathcal{S} = \mathcal{S}^\top > 0$, the following inequality holds:

$$-\int_{t-\tau(t)}^{t} \dot{U}^\top(s)S\dot{U}(s)ds \leq \bar{U}^\top(t)\begin{bmatrix} {M_1}^\top + M_1 & * \\ -M_1 + {M_2}^\top & -{M_2}^\top - M_2 \end{bmatrix}\bar{U}(t) + \tau(t)\bar{U}^\top(t)\begin{bmatrix} {M_1}^\top \\ {M_2}^\top \end{bmatrix} S^{-1}\begin{bmatrix} M_1 & M_2 \end{bmatrix}\bar{U}(t)$$

where $\bar{U}(t) = \mathrm{col}(U(t), U(t-\tau(t)))$.

*Lemma 2 (Cholesky Decomposition [20]):* If $V \in R^{n\times n}$ is a symmetric positive definite matrix, then there exists a unique lower triangular matrix $W$ with positive diagonal entries such that $V = WW'$.

*Lemma 3 [21]:* There is a symmetric block matrix

$$V = \begin{bmatrix} V_{11} & V_{21}^\top & & \mathbf{0} \\ V_{21} & V_{22} & \ddots & \\ & \ddots & \ddots & V_{N,N-1}^\top \\ \mathbf{0} & & V_{N,N-1} & V_{NN} \end{bmatrix}$$

where $V_{i,j}$ is a block matrix with appropriate dimension. The inequality $V > 0$ holds if and only if

$$M_i > 0, \quad i = 1, 2, \ldots, N$$
$$M_i = \begin{cases} V_{i,i}, & i = 1 \\ V_{i,i} - V_{i,i-1}M_{i-1}^{-1}V_{i,i-1}^\top, & i = 2, \ldots, N. \end{cases} \tag{3}$$

*Lemma 4 [22]:* (Schur complement) The LMI

$$\begin{bmatrix} Q(x) & S(x) \\ S^\top(x) & R(x) \end{bmatrix} < 0 \tag{4}$$

where $Q(x), R(x)$ are symmetric matrices and $S(x)$ depends affinely on $x$, is equivalent to

$$R(x) < 0, \quad Q(x)R^{-1}(x)S^\top(x) < 0. \tag{5}$$

*Lemma 5 [23]:* Given matrices $D, E$ with appropriately dimensions, for any $\varepsilon > 0$ and $F^\top(t)F(t) \leq I$, there is

$$DF(t)E + E^\top F^\top(t)D^\top \leq \varepsilon DD^\top + \frac{1}{\varepsilon}E^\top E. \tag{6}$$

Inspired by the method in [24], the following novel augmented systems are established as:

$i = 1$:
$$\begin{cases} \dot{\xi}_1(t) = \tilde{A}_{11}\xi_1(t) + \tilde{B}_1u_1(t) + \tilde{A}_{12}\xi_2(t-\tau(t)) \\ \dot{r}_{x,1}(t) = e_{x,1} = \theta_{x_1}\xi_1(t) - x_{d,1} \\ \dot{r}_{z,1}(t) = e_{z,1} = \theta_{z_1}\xi_1(t) - z_d \end{cases} \tag{7}$$

$i = 2, \ldots, N$:
$$\begin{cases} \dot{\xi}_i(t) = \tilde{A}_{ii}\xi_i(t) + \tilde{B}_iu_i(t) + \sum_{j\in\mathcal{P}^i}\tilde{A}_{ij}\xi_j(t-\tau(t)) \\ \dot{r}_{x,i}(t) = e_{x,i} = \theta_{x_i}\xi_i(t) - x_{d,i} \\ \dot{r}_{z,i}(t) = e_{z,i} = \theta_{z_i}\xi_i(t) - \theta_{z_{i-1}}\xi_{i-1}(t) \end{cases} \tag{8}$$

where $\theta_{x_i} = \mathrm{row}(I_{m_i}, \mathbf{0}_{m_i\times n})$, $\theta_{z_i} = \mathrm{row}(\mathbf{0}_{n\times m_i}, I_n)$, $r_{x,i} \in R^{m_i}$ is the variable associated with the error in the "Nonconsensus" state and $r_{z,i} \in R^n$ is the variable associated with the error in the "Consensus" state.

For convenience, define the augmented variable as $\zeta_i(t) := \mathrm{col}(\xi_i(t), r_{x,i}(t), r_{z,i}(t))$. The systems (7), (8) can then be expressed as follows:

$$\dot{\zeta}_1(t) = \tilde{G}_c^{11}\zeta_1(t) + \tilde{H}_1u_1(t) + \sum_{j\in\mathcal{P}^1}\tilde{G}_d^{1j}\zeta_j(t-\tau(t)) - \eta_1$$
$$\dot{\zeta}_i(t) = \tilde{G}_c^{ii}\zeta_i(t) + \tilde{H}_iu_i(t) + \tilde{G}_c^{i,i-1}\zeta_{i-1}(t) + \sum_{j\in\mathcal{P}^i}\tilde{G}_d^{ij}\zeta_j(t-\tau(t)) - \eta_i, \quad i = 2, \ldots, N \tag{9}$$

where

$$\begin{aligned} \tilde{G}_c^{ii} &= c_i\tilde{A}_{ii}c_i^\top + \theta_{1,i} + \theta_{2,i} \\ \tilde{G}_d^{ij} &= c_i\tilde{A}_{ij}c_i^\top, \quad \tilde{G}_c^{i,i-1} = -\theta_{2,i-1} \\ \eta_1 &= \mathrm{col}(\mathbf{0}_{p_1}, x_{d,1}, z_d), \quad \eta_i = \mathrm{col}(\mathbf{0}_{p_i}, x_{d,i}, \mathbf{0}_{n_i}) \\ \tilde{H}_i &= c_i\tilde{B}_i, c_i = \begin{bmatrix} I_{p_i} & \mathbf{0}_{p_i\times p_i} \end{bmatrix}^\top \\ K_i &= \mathrm{row}\big(K_{1,i}, K_{2,i}, K_{3,i}\big) \\ \theta_{1,i} &= \begin{bmatrix} \mathbf{0}_{p_i\times p_i} & \mathbf{0}_{p_i\times p_i}\begin{bmatrix} \theta_{x,i} \\ \mathbf{0}_{n\times p_i} \end{bmatrix} & \mathbf{0}_{p_i\times p_i} \end{bmatrix} \\ \theta_{2,i} &= \begin{bmatrix} \mathbf{0}_{p_i\times p_i} & \mathbf{0}_{p_i\times p_i} \\ \begin{bmatrix} \mathbf{0}_{m_i\times p_i} \\ \theta_{z_i} \end{bmatrix} & \mathbf{0}_{p_i\times p_i} \end{bmatrix}. \end{aligned}$$

A distributed PSC control for (9) is proposed as

$$u_i(t) = K_i\zeta_i = K_{i,1}\xi_i + K_{i,2}r_{x,i} + K_{i,3}r_{z,i} \tag{10}$$

where $i \in \mathcal{V}$, $K_{i,1} \in R^{q_i\times p_i}$, $K_{i,2} \in R^{q_i\times m_i}$ and $K_{i,3} \in R^{q_i\times n}$ are gain matrices for the subsystem $i$.

An infinite-horizon quadratic PSC performance index is defined as follows:

$$J_i = \int_0^\infty \big(x_i(t) - x_{d,i}\big)^\top Q_{x,i}\big(x_i(t) - x_{d,i}\big)dt + \int_0^\infty \sum_{j\in\mathcal{P}_i}\big(z_i(t) - z_j(t)\big)^\top Q_{z,i}\big(z_i(t) - z_j(t)\big)dt + \int_0^\infty u_i(t)^\top R_iu_i(t)dt \tag{11}$$

where $i \in \mathcal{V}$, $Q_{x,i} \in R^{m_i\times m_i}$, $Q_{z,i} \in R^{n_i\times n_i}$ and $R_i \in R^{q_i\times q_i}$ are symmetric positive definite matrices.

The equilibrium point of (9) can be represented by $(\zeta_{i,0}, u_{i,0})$, $i \in \mathcal{V}$. Define the following new auxiliary variables:

$$\begin{aligned} w_i &= \zeta_i - \zeta_{i,0} \\ v_i &= u_i - u_{i,0} = K_iw_i. \end{aligned} \tag{12}$$

Then the following auxiliary systems can be expressed by:

$$\dot{w}_1(t) = \tilde{G}_c^{11}w_1(t) + \tilde{H}_1v_1(t) + \sum_{j\in\mathcal{P}^1}\tilde{G}_d^{1j}w_j(t-\tau)$$
$$\dot{w}_i(t) = \tilde{G}_c^{ii}w_i(t) + \tilde{H}_iv_i(t) + \tilde{G}_c^{i,i-1}w_{i-1}(t) + \sum_{j\in\mathcal{P}^i}\tilde{G}_d^{ij}w_j(t-\tau), \quad i = 2, \ldots, N. \tag{13}$$

The performance index (11) is then replaced by

$$J_i = \int_0^\infty w_i^\top(t)Q_iw_i(t) + v_i^\top(t)R_iv_i(t)dt$$

$$= \int_0^{\infty} w_i^\top(t)\big(Q_i + K'_i R_i K_i\big) w_i(t) dt \tag{14}$$

where $Q_i \in R^{2p_i \times 2p_i}$ is a symmetric positive definite matrix.

Now a key Lemma is given and proved.

*Lemma 6:* The following three propositions are equivalent.

1) System (1) can achieve PSC asymptotically.
2) The equilibrium points of (9) are asymptotically stable.
3) The auxiliary systems (13) are asymptotically stable.

*Proof:* Necessity. If PSC for (1) is achieved, there exists $\xi_i(t)$ such that $\lim_{t\to\infty}|\theta_x \xi_i(t) - x_{d,i}| = 0$, and $\lim_{t\to\infty}|\theta_z \xi_i(t) - \theta_z \xi_j(t)| = 0$, $\lim_{t\to\infty}|\theta_z \xi_i(t) - z_d| = 0$, $i \in \mathcal{V}$, $j \in \mathcal{P}_i$, i.e., the equilibrium points $\zeta_{i,0}$ of (9), or the auxiliary systems (13), are asymptotically stable. Sufficiency. If systems (13) are asymptotically stable, then the states $\dot{\zeta}_i, i \in \mathcal{V}$ converge to $\dot{\zeta}_{i,0} = 0$, and $e_{x,i}, e_{z,i}$ are both equal to 0. ∎

*Remark 3:* Guaranteed cost control is a typical robust control that is usually used to solve stabilization problems. The states and inputs converge to 0 as $t$ approaches $\infty$ and the quadratic performance index is bounded. Since $r_{x,i}$, $r_{z,i}$ and $u_i, i \in \mathcal{V}$ converge to nonzero vectors, the value of (11) is unbounded. Therefore, the auxiliary systems (13) are established to formulate the dynamics of reaching the equilibrium points for (9). To guarantee performance, (14) is defined and a corresponding robust control can be designed.

Now the main result is ready to be presented.

*Theorem 1:* Let Assumptions 1–3 hold. For positive scalars $\varepsilon_{1,i}, \varepsilon_{2,i}, \varepsilon_{3,i}$ and known constants $\bar{\tau}$ and $\mu$, there exist symmetric positive definite matrices $X_{1,i}, X_{2,i}, X_{3,i}$ and $W_i$ that satisfy the following local LMIs for subsystem $i$:

$$\begin{aligned} &\text{Find } \varepsilon_{1,i}, \varepsilon_{2,i}, \varepsilon_{3,i}, X_{1,i}, X_{2,i}, X_{3,i}, W_i \\ &\text{s.t. } X_{1,i}, X_{2,i}, X_{3,i} > 0,\ \tilde{\Xi}_i < 0 \end{aligned} \tag{15}$$

where

$$\tilde{\Xi}_i = \begin{cases} \tilde{\Xi}_{i,i}, & i = 1 \\ \tilde{\Xi}_{i,i} - \tilde{\Xi}_{i,i-1}\tilde{\Xi}_{i-1}^{-1}\tilde{\Xi}_{i,i-1}^\top, & i \in \mathcal{V}\backslash\{i\} \end{cases}$$

$\tilde{\Xi}_{i,i}$ is

$$\begin{bmatrix} \hat{\Xi}_{1,i} & * & * & * & * & * & * & * & * & * \\ \hat{\Xi}_{2,i} & \hat{\Xi}_{6,i} & * & * & * & * & * & * & * & * \\ \hat{\Xi}_{3,i} & 0 & \hat{\Xi}_{7,i} & * & * & * & * & * & * & * \\ 0 & X_{3,i} & 0 & -\bar{\tau}^{-1}X_{3,i} & * & * & * & * & * & * \\ X_{1,i} & 0 & 0 & 0 & -X_{2,i} & * & * & * & * & * \\ X_{1,i} & 0 & 0 & 0 & 0 & -Q_i^{-1} & * & * & * & * \\ W_i & 0 & 0 & 0 & 0 & 0 & -R_i^{-1} & * & * & * \\ \hat{\Xi}_{4,i} & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_{1,i}I & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_{2,i}I & * \\ \hat{\Xi}_{5,i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_{3,i}I \end{bmatrix}$$

where

$$\begin{aligned} \hat{\Xi}_{1,i} &= \big(\big(c_i A_{ii} c_i^\top + \theta_{1,i} + \theta_{2,i}\big) X_{1,i} + c_i B_i W_i\big) \\ &\quad + \big(\big(c_i A_{ii} c_i^\top + \theta_{1,i} + \theta_{2,i}\big) X_{1,i} + c_i B_i W_i\big)^\top \\ &\quad + (\mu - 1) X_{2,i} + \big(\varepsilon_{1,i} + \varepsilon_{2,i}\big) c_i D_i D_i^\top c_i^\top \\ \hat{\Xi}_{2,i} &= X_{1,i} + (\mu - 2) X_{2,i} \\ \hat{\Xi}_{3,i} &= \bar{\tau}\big(\big(c_i A_{ii} c_i^\top + \theta_{1,i} + \theta_{2,i}\big) X_{1,i} + c_i B_i W_i\big) \\ \hat{\Xi}_{4,i} &= E_{1,ii} c_i^\top X_{1,i} + E_{2,ii} W_1, \quad \hat{\Xi}_{5,i} = \hat{\Xi}_{4,i} \\ \hat{\Xi}_{6,i} &= (\mu - 3) X_{2,i}, \quad \hat{\Xi}_{7,i} = -\bar{\tau} X_{3,i} + \varepsilon_{3,i}\bar{\tau}^2 c_i D_i D_i^\top c_i^\top \end{aligned}$$

and $\tilde{\Xi}_{i,i-1}$ is

$$\begin{bmatrix} \breve{\Xi}_{1,i} & \breve{\Xi}_{4,i} & \breve{\Xi}_{6,i} & 0 & \cdots & 0 & \breve{\Xi}_{9,i} & 0 & \breve{\Xi}_{9,i} \\ \breve{\Xi}_{2,i} & 0 & \breve{\Xi}_{7,i} & 0 & \cdots & \cdots & 0 & \breve{\Xi}_{9,i} & \breve{\Xi}_{9,i} \\ \breve{\Xi}_{3,i} & \breve{\Xi}_{5,i} & 0 & & \cdots & & \cdots & & 0 \\ 0 & 0 & & & & & & & \\ \vdots & \vdots & \vdots & & \ddots & & & & \vdots \\ 0 & \vdots & & & & 0 & & & \\ \breve{\Xi}_{8,i} & 0 & \vdots & & & & \ddots & & \vdots \\ 0 & \breve{\Xi}_{8,i} & & & & & & & \\ \breve{\Xi}_{8,i} & \breve{\Xi}_{8,i} & 0 & & \cdots & & \cdots & & 0 \end{bmatrix}$$

where

$$\begin{aligned} \breve{\Xi}_{1,i} &= X_{2,i}\big(c_{i-1} A_{i-1,i} c_{i-1}^\top\big)^\top + c_i A_{i,i-1} c_i^\top X_{2,i-1} \\ &\quad - \theta_{2,i-1} X_{1,i-1} \\ \breve{\Xi}_{2,i} &= X_{2,i}\big(c_{i-1} A_{i-1,i} c_{i-1}^\top\big)^\top \\ \breve{\Xi}_{3,i} &= \bar{\tau}\big(c_i A_{i,i-1} c_i^\top X_{2,i-1} - \theta_{2,i-1} X_{1,i-1}\big) \\ \breve{\Xi}_{4,i} &= c_i A_{i,i-1} c_i^\top X_{2,i-1},\ \breve{\Xi}_{5,i} = \bar{\tau} c_i A_{i,i-1} c_i^\top X_{2,i-1} \\ \breve{\Xi}_{6,i} &= \breve{\Xi}_{7,i} = \bar{\tau} X_{2,i}\big(c_{i-1} A_{i-1,i} c_{i-1}^\top\big)^\top \\ \breve{\Xi}_{8,i} &= E_{1,i,i-1} c_i^\top X_{2,i-1},\ \breve{\Xi}_{9,i} = X_{2,i} c_{i-1} E_{1,i-1,i}^\top. \end{aligned}$$

Then the DR-PSC control for (1) with $K_i = W_i X_{1,i}^{-1}$ can achieve PSC and guarantee the following performance index is satisfied:

$$J < \sum_{i \in \mathcal{V}} \big(\zeta_i(0) - \zeta_{0,i}\big)^\top \big(P_{1,i} + \bar{\tau} P_{2,i}\big)\big(\zeta_i(0) - \zeta_{0,i}\big). \tag{16}$$

*Proof:* Denote $w(t) := \mathrm{col}(w_1(t), w_2(t), \ldots, w_N(t))$. The dynamics in (13) can be written as

$$\dot{w}(t) = \big(\tilde{G}_c + \tilde{H}K\big) w(t) + \tilde{G}_d w(t - \tau(t)) \tag{17}$$

where $\tilde{G}_c = c\tilde{A}_c c^\top + \Theta$, $\tilde{H} = c\tilde{B}$, $\tilde{G}_d = c\tilde{A}_d c^\top$, $\tilde{A}_c = A_c + DFE_{1,c}$, $\tilde{B} = B + DF(t)E_d$, $\tilde{A}_d = A_d + DF(t)E_{2,c}$, $\Theta = \theta_x + L \circ (1_N \otimes \theta_z)$, $\theta_z = \mathrm{row}(\theta_{z_1}, \theta_{z_2}, \ldots, \theta_{z_N})$. Besides, $\theta_x$, $c$, $A_c$, $B$, $D$, $F(t)$, $E_{1,c}$, and $E_{2,c}$ are block diagonal matrices composed of heterogeneous matrices $\theta_{x,i}$, $c_i$, $A_c$, $B_i$, $D_i$, $F_i(t)$, $E_{1,ii}$, and $E_{2,i}$ of each subsystem. $A_d$ and $E_{2,c}$ can be written as

$$A_d = \begin{bmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ A_{21} & 0 & A_{23} & \ddots & \vdots \\ 0 & A_{32} & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & A_{N-1,N} \\ 0 & \cdots & 0 & A_{N,N-1} & 0 \end{bmatrix}$$

$$E_{2,c} = \begin{bmatrix} 0 & E_{1,12} & 0 & \cdots & 0 \\ E_{1,21} & 0 & E_{1,23} & \ddots & \vdots \\ 0 & E_{1,32} & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & E_{1,N-1,N} \\ 0 & \cdots & 0 & E_{1,N,N-1} & 0 \end{bmatrix}.$$

Consider a Lyapunov–Krasovskii function of (13) as

$$V(t) = V_1(t) + V_2(t) + V_3(t) \tag{18}$$

where

$$\begin{aligned} V_1(t) &= w^\top(t) P_1 w(t) \\ V_2(t) &= \int_{t-\tau}^{t} w^\top(s) P_2 w(s) ds \\ V_3(t) &= \int_{-\bar{\tau}}^{t} \int_{t+\theta}^{t} \dot{w}^\top(s) P_3 \dot{w}(s) ds d\theta \end{aligned}$$

and $P_1, P_2, P_3 \in R^{2p\times 2p}$, $p = \sum_{i=1}^{N} p_i$ are symmetric positive definite block diagonal matrices composed of $N$ dimensional heterogeneous matrices. There must be constants $a, b, c > 0$ to ensure the following inequality holds:

$$\begin{aligned} a\|w(t)\|^2 &\le V(t) \\ &\le b \sup_{s\in[-h,0]} \|w(t+s)\|^2 + c \sup_{s\in[-h,0]} \|\dot{w}(t+s)\|^2. \end{aligned}$$

Then according to Lemma 1, the derivative of $V(t)$ is

$$\dot{V}(t) \le \begin{bmatrix} w(t) \\ w(t-\tau) \end{bmatrix}^\top (\Pi_1 + \Pi_2 + \Pi_3) \begin{bmatrix} w(t) \\ w(t-\tau) \end{bmatrix} \tag{19}$$

where

$$\begin{aligned} \Pi_1 &= \begin{bmatrix} P_1(\tilde{G}_c + \tilde{H}K) + (\tilde{G}_c + \tilde{H}K)^\top P_1 & * \\ \tilde{G}_d^\top P_1 & \mathbf{0} \end{bmatrix} \\ \Pi_2 &= \begin{bmatrix} P_2 & \mathbf{0} \\ \mathbf{0} & (\mu - 1)P_2 \end{bmatrix} \\ \Pi_3 &= \bar{\tau} \begin{bmatrix} (\tilde{G}_c + \tilde{H}K)^\top \\ \tilde{G}_d^\top \end{bmatrix} P_3 \begin{bmatrix} \tilde{G}_c + \tilde{H}K & \tilde{G}_d \end{bmatrix} \\ &+ \begin{bmatrix} M_1{}^\top + M_1 & * \\ -M_1 + M_2{}^\top & -M_2{}^\top - M_2 \end{bmatrix} \\ &+ \bar{\tau} \begin{bmatrix} M_1{}^\top \\ M_2{}^\top \end{bmatrix} P_3^{-1} \begin{bmatrix} M_1 & M_2 \end{bmatrix}. \end{aligned}$$

To obtain the distributed robust control, the following inequality must hold:

$$\Pi_1 + \Pi_2 + \Pi_3 + \begin{bmatrix} Q + K^\top RK & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} < 0. \tag{20}$$

Then, let $X_1 = P_1^{-1}$, $X_2 = P_2^{-1}$, $X_3 = P_3^{-1}$, $W = KX_1$, choose $M_1 = -P_1$, $M_2 = P_2$, and define a nonsingular matrix

$$T = \begin{bmatrix} P_1 & \mathbf{0} \\ -P_1 & P_2 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} P_1^{-1} & \mathbf{0} \\ P_2^{-1} & P_2^{-1} \end{bmatrix}.$$

According to Lemmas 4, Lemma 5 and by multiplying both sides of (20) by $\mathrm{diag}(T^{-1}, I, X_3, I, I, I)$, the following LMI can be obtained:

$$\Xi = \begin{bmatrix} \Xi_{1,1} & * & * & * & * & * & * & * & * & * \\ \Xi_{2,1} & \Xi_{2,2} & * & * & * & * & * & * & * & * \\ \Xi_{3,1} & \Xi_{3,2} & \Xi_{3,3} & * & * & * & * & * & * & * \\ 0 & X_3 & 0 & -\bar{\tau}^{-1}X_3 & * & * & * & * & * & * \\ X_1 & 0 & 0 & 0 & -X_2 & * & * & * & * & * \\ X_1 & 0 & 0 & 0 & 0 & -Q^{-1} & * & * & * & * \\ W & 0 & 0 & 0 & 0 & 0 & -R^{-1} & * & * & * \\ \Xi_{8,1} & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_1 I & * & * \\ 0 & \Xi_{9,2} & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_2 I & * \\ \Xi_{10,1} & \Xi_{10,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_3 I \end{bmatrix} < 0 \tag{21}$$

where

$$\begin{aligned} \Xi_{1,1} &= \big((cA_c c^\top + \Theta)X_1 + cA_d c^\top X_2 + cBW\big) \\ &\quad + \big((cA_c c^\top + \Theta)X_1 + cA_d c^\top X_2 + cBW\big)^\top \\ &\quad + (\mu - 1)X_2 + (\varepsilon_1 + \varepsilon_2)cDD^\top c^\top \\ \Xi_{2,1} &= X_2 c^\top A_d^\top c + X_1 + (\mu - 2)X_2, \quad \Xi_{2,2} = (\mu - 3)X_2 \\ \Xi_{3,1} &= \bar{\tau}\big((cA_c c^\top + \Theta)X_1 + cBW\big) + \bar{\tau} cA_d c^\top X_2 \\ \Xi_{3,2} &= \bar{\tau} cA_d c^\top X_2, \quad \Xi_{3,3} = -\bar{\tau} X_3 + \varepsilon_3 \bar{\tau}^2 cDD^\top c^\top \\ \Xi_{8,1} &= E_{1,c} c^\top X_1 + E_d c^\top X_2 + E_{2,c} W \\ \Xi_{9,2} &= E_d c^\top X_2, \quad \Xi_{10,1} = \Xi_{8,1}, \quad \Xi_{10,2} = \Xi_{9,2}. \end{aligned}$$

Define a permutation matrix $\Upsilon$ as follows:

$$\Upsilon = \mathrm{col}(\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_N) \tag{22}$$

where $\Upsilon_i = \mathrm{col}(\gamma_{1,i}, \gamma_{2,i}, \ldots, \gamma_{10,i})$ and let $I_\Xi$ be an identity matrix with the same dimension as $\Xi$. Then $\gamma_{k,l} = I_\Xi(2p_l(k-1) + l + 1 : 2p_l k + l, :)$. Multiplying the right and left sides of (21) by $\Upsilon$ and $\Upsilon^\top$ respectively,

$$\begin{aligned} \tilde{\Xi} &= \Upsilon \Xi \Upsilon^\top \\ &= \begin{bmatrix} \tilde{\Xi}_{1,1} & \tilde{\Xi}_{2,1}^\top & 0 & \cdots & 0 \\ \tilde{\Xi}_{2,1} & \tilde{\Xi}_{2,2} & \tilde{\Xi}_{3,2}^\top & \ddots & \vdots \\ 0 & \tilde{\Xi}_{3,2} & \tilde{\Xi}_{3,3} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \tilde{\Xi}_{N,N-1}^\top \\ 0 & \cdots & 0 & \tilde{\Xi}_{N,N-1} & \tilde{\Xi}_{NN} \end{bmatrix} < 0 \end{aligned} \tag{23}$$

where $\tilde{\Xi}_{i,i}$, $\tilde{\Xi}_{i,i-1}$ in (15) can be obtained. Then, according to Lemmas 2 and 3, the feasible LMI problem in (15) can be obtained. From (20), the following inequality holds:

$$\dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) < -w^\top(t)\big(Q + K^\top RK\big)w(t). \quad (24)$$

Because of the positive definiteness of $Q$ and $R$, the derivative of $V(t)$ is less than 0. According to the Lyapunov Theorem and Lemma 6, (13) is asymptotically stable and PSC for (1) under (10) can be achieved. In addition, according to the initial states in Assumption 1 and integrating both sides of (24), bounded performance in (16) can be obtained. Theorem 1 is proven. ■

Based on Theorem 1, Algorithm 1 is shown to describe how to calculate the DR-PSC control for (1).

**Algorithm 1** DR-PSC for Chain Interconnected Systems

**Input:** $N, \Delta T, T, \xi_i(0), x_{d,i}, z_d$
**Output:** $K_i$ and the responses of $\xi_i$
Initialization.Let $i = 2$.
Obtain $K_1$ by solving (15) for subsystem 1.
**while** $i \leq N$ **do**
  Receive $\tilde{\Xi}_{i-1}, \tilde{A}_{i-1,i}, E_{i-1,i}$ from subsystem $i-1$.
  Obtain $K_i$ by solving (15) and set $i \mapsto i+1$.
**end**
The calculation of gain matrices is ended. Let $t = 0$.
**while** $t \leq T$ **do**
  State information communication.
  Locally calculate $r_{x,i}$, $r_{z,i}$, $i \in \mathcal{V}$ by integrators.
  Calculate and implement $u_i(t)$ in (10).
  Set $t \mapsto t + \Delta T$.
**end**

*Remark 4:* In Algorithm 1, the LMI feasibility problems can be solved locally and sequentially to obtain the control gain matrices $K_i$ and PSC control $u_i, i \in \mathcal{V}$. This method is fully distributed because global information on the network is not needed during the calculations. In contrast to the centralized form, high-dimensional matrix calculations and the concept of the central node are avoided to reduce the computational burden and improve the operability of the algorithm in practice.

*Remark 5:* Due to the heterogeneity of subsystems and the special structure of networks, it is infeasible to adopt general decoupling methods to design controllers for chain interconnected systems without restricting the interconnection terms. Here, the method based on Cholesky decomposition in Lemma 2 includes no requirements on the interconnected parameters $A_{ij}$. During operation, limited neighbor information is required for local calculations and the method does not introduce too great a burden on the communication network.

## IV. Simulation and Experimental Validation

### A. Numerical Simulation

Consider a chained network composed of three subsystems, whose parameters are shown as

$$A_{11} = [1, 0.5; 1, 1], \quad B_1 = [1, 0.5; 0, 3]$$
$$A_{22} = [1.5, 1; 0.5, 0.5], \quad B_2 = [1.5, 0.5; 0, 2]$$

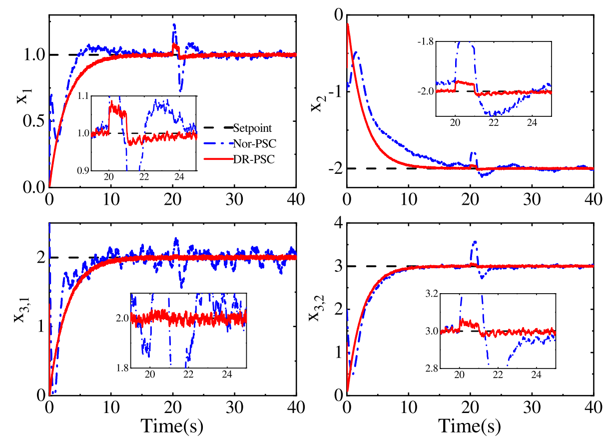


Fig. 3. Responses of "Nonconsensus" states.

$$A_{33} = [1.7, 1, 0.5; 1, 0.3, 0; 0.6, 0, 0.5]$$
$$B_3 = [1.7, 0.5, 0; 0, 0, 1.7; 0, 1.5, 0]$$
$$A_{12} = [0, 0; 0.4, 0], \quad A_{23} = [0, 0, 0.5; 0.6, 0, 0]$$
$$A_{21} = [0.7, 0; 0, 0], \quad A_{32} = [1.3, 0; 0, 0; 0, -0.3].$$

The parameter uncertainties are assumed to satisfy $D_1 = D_2 = D_3 = 0.1 * I_2$. $E^1_{ii}, E^1_{ij}, E^2_i$ are matrices with random elements and the unknown parameter $F_i(t)$ is a time-varying random scalar whose range is $[-10, 10]$. The delay is set as $\tau(t) = 0.2\sin(t)$, and $\bar{\tau} = 0.2, \mu = 0.2$. The matrices in (14) are chosen as $Q_i = 10 * \mathrm{diag}(I_{p_i}, 0.05 * I_{p_i})$, $R_i = 0.5 * I_{q_i}$. For "Nonconsensus" states $x_i(t)$, the set-points are set as $x_{d,1} = -3, x_{d,2} = -2, x_{d,3} = [5; 3]$. The common set point of $z_i(t)$ is set as $z_d = 1$.

To demonstrate the superiority, effectiveness, and robustness of the proposed PSC algorithm, the following simulation results are compared.

1) *Simple PSC Control (Sim-PSC):* There is no consideration of the delays. The solution is only based on the Lyapunov inequality $P(\tilde{G}_c + \tilde{G}_d + \tilde{H}K) + (\tilde{G}_c + \tilde{G}_d + \tilde{H}K)^\top P < 0$.
2) *Centralized Robust PSC Control (CR-PSC):* The algorithm is the centralized form of the one proposed in Algorithm 1.
3) *DR-PSC:* The algorithm is proposed in Algorithm 1.

The performance with CR-PSC is not significantly different from that with DR-PSC. Thus, there are only responses of the states with Sim-PSC and DR-PSC shown in Figs. 3 and 4. It is straightforward to see that all subsystems can achieve PSC and converge to the desired setpoints. The dynamic performance of DR-PSC is much better than that of Sim-PSC. Further, disturbances (amplitude: 2, duration: 20-21 s) are added to verify the robustness of the algorithm. In the detailed view of Figs. 3 and 4, it is not difficult to see that the systems with DR-PSC are less affected by disturbances. The DR-PSC has better robustness than Sim-PSC.

In Table I, the simulation results of the three algorithms are shown. DR-PSC has slightly worse performance than CR-PSC, but it requires much lower computational effort. Although sim-PSC also has a very small amount of computation, its

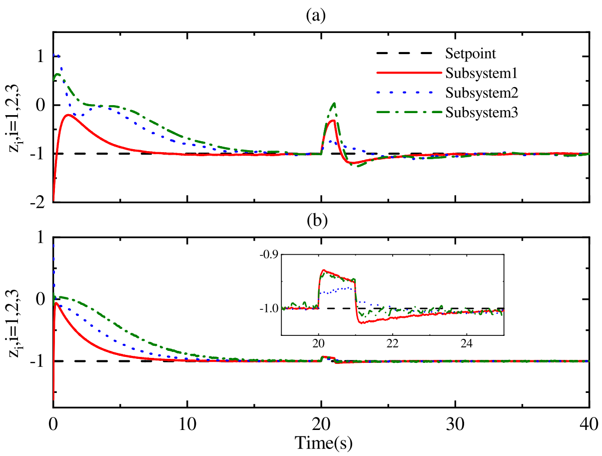


Fig. 4. Responses of "Consensus" states. (a) Nor-PSC. (b) DR-PSC.

TABLE I

CONVERGENCE TIME AND PERFORMANCE INDEX OF THREE ALGORITHMS

| | Sim-PSC | CR-PSC | DR-PSC |
|---|---|---|---|
| Convergence time | 0.0477s | 1.9111s | 0.8711s |
| Performance Index | 1132.64 | 441.12 | 449.09 |

Fig. 5. Distributed cooperative optimization control experimental platform.

performance is very poor, and even the algorithm is invalid. The effectiveness, robustness, and superiority of the proposed method are demonstrated.

## B. Experiment

In this section, a solution proportioning experiment is carried out to validate the effectiveness of the proposed PSC control. There is a distributed cooperative optimization control experimental platform developed by the research team at the China University of Petroleum (East China), which is shown in Fig. 5. There are two tanks R101 and R102 which are used as reaction vessels. V111 and V112 are material tanks containing water and NaOH, respectively. The tank V113 contains hot water to store the heat needed for the experiment. The materials and hot water are transported to the main tanks and jackets by pumps.

In this experiment, the two-tank process is configured as shown in Fig. 6. The tanks are chain interconnected with R101 and R102 being connected by a pipeline. The partial product of the former is sent to the latter as material. This flow and the water flow of R101 are uncontrollable and they remain $75 \pm 10$ L/h and $50 \pm 10$ L/h, which are regarded as the uncertainties in R101 and R102. Since the medium needs to pass through a

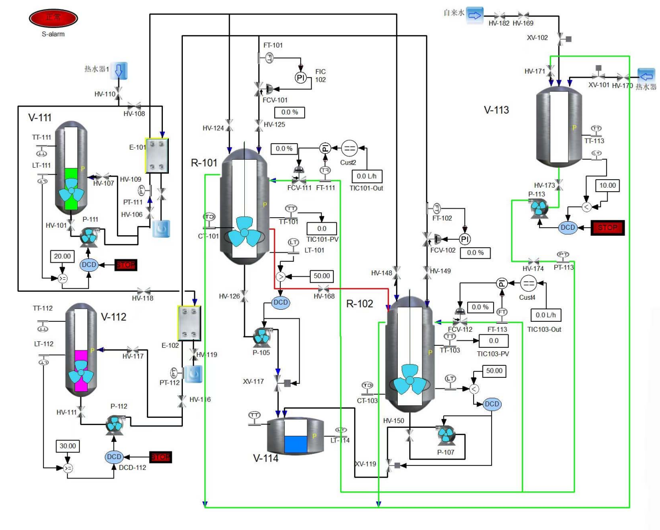


Fig. 6. Scheme of the two-tank process.

pipe from R101 to R102, there is an uncertain lag phenomenon between them. Moreover, there is a coupling between the two tanks in the supply of material. The coupling mentioned above can be modeled as interconnections with delay in the state equation.

The solution proportioning experiment is designed as follows. There is ample hot water at 55 °C in V113. Water in V111 is at room temperature (about 20 °C) and the 10.0 kmol/m$^3$ NaOH solution in V112 is at about 30 °C. To ensure the progress of the experiment, R101 and R102 are preheated to about 30 °C. Then, the jackets are heated by the hot water from V113 and the NaOH solution is injected into R101 and R102. At the same time, PSC control is implemented. PSC1 and PSC2 control the NaOH feed valves FCV101, FCV102 and the jacket feed valves FCV111, FCV112 to achieve the set-point tracking control of concentration $c_1, c_2$, and temperature $t_1, t_2$.

The experimental objective is to configure 4.0 kmol/m$^3$ NaOH solution at 38 °C, 43 °C in R101 and R102, respectively. In Fig. 6, the local set-points (38 °C, 43 °C) are assigned to PSC1, PSC2, and the common set-point (4.0 kmol/m$^3$, converted to conductivity) is only assigned to PSC1. $c_1, c_2$ are modeled as "Nonconsensus" states and $t_1, t_2$ as "Consensus" states. Then, PSC control can be calculated by the distributed method in Theorem 1. The model parameters are shown as

$$
\begin{aligned}
A_{11} &= [0.56825, 0.00541; -0.00001, 0.63425] \\
A_{12} &= [0.00022, -0.00922; -0.00003, 0.00022] \\
A_{21} &= [-0.00045, 0.00081; -0.00002, -0.00018] \\
A_{22} &= [0.60400, -0.00309; -0.00001, 0.59800] \\
B_1 &= [0.43165, 0.00131; -0.00001, 0.36512] \\
B_2 &= [0.39561, -0.00026; -0.00002, 0.40123] \\
E_{11} &= [0.00010, 0.00001; 0, -0.00021] \\
E_{22} &= [0.00009, 0.00011; 0, 0.00012] \\
E_{12} &= [0, 0.00001; 0, 0] \\
E_{21} &= [-0.00001, 0.00001; 0, 0]
\end{aligned}
$$

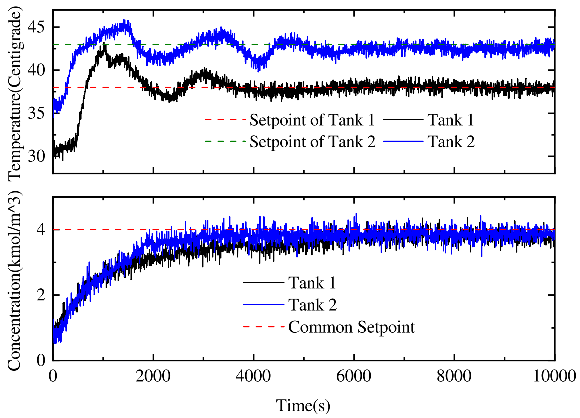


Fig. 7. Temperature and concentration of two tanks.

$$E_1 = [0.00021, 0.00001; 0, 0]$$
$$E_2 = [-0.00020, 0; 0, 0.00008]$$

and $D_1 = D_2 = I_2$, $F_i(t)$ are time-varying random scalars. The sampling period is selected as $T_s = 1$ second and the sampling length as $T = 10\,000$. The time-varying delay is modeled as $\tau(t) = \bar{\tau}\sin(\mu t)$ where $\bar{\tau} = 1$, $\mu = 0.01$. Set $Q_1 = Q_2 = 10 * I_2$ and $R_1 = R_2 = I_2$. The gain matrices obtained by the proposed method are shown as

$$K_1 = \begin{bmatrix} -45.010 & 0.073 & -21.572 & 0.033 \\ -0.075 & -49.914 & -0.044 & -22.618 \end{bmatrix}$$
$$K_2 = \begin{bmatrix} -55.173 & 0.017 & -24.284 & 0.005 \\ 0.051 & -56.726 & 0.025 & -28.735 \end{bmatrix}.$$

The experimental results are shown below.

Temperature and concentration trends are given in Fig. 7. With the PSC control, $t_1, t_2$ approach 38 °C and 43 °C, respectively, and $c_1, c_2$ achieve consensus. Since the process of temperature variation is slow, stabilization of $t_1, t_2$ takes a long time. The effectiveness and robustness of the proposed algorithm are validated.

## V. Conclusion

A PSC problem for chain interconnected delay systems with parameter uncertainty has been investigated. Augmented systems and auxiliary systems have been established to design the distributed PSC control. This control is fully distributed and can take into account the optimization and robustness of the systems. A sufficient condition is given in LMI form and the gain matrices can be obtained by solving feasible problems locally. The results of numerical simulations and a solution proportioning experiment are shown to validate the effectiveness of the proposed method. Future work will extend this to nonlinear systems.

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