On Gromov–Witten invariants of blowups and the classification of T-polygons

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Thesis submitted for the degree of
Doctor of Philosophy

Supervised by
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&
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Statement of Originality

I, Wendelin Lutz, confirm that this thesis is my own work unless otherwise stated.

Part I is solo work and based on the preprint [Lut21].

Part II is joint work with Tom Coates and Qaasim Shafi [CLS21], and has been accepted for publication in *Forum of Mathematics, Sigma* under the title *The Abelian/non-Abelian Correspondence and Gromov–Witten invariants of Blow-ups.*

Parts of Chapter 4 of Part I are submitted for publication in *Geometry & Topology* as an Appendix to an article of Paul Hacking and Ailsa Keating entitled *Homological mirror symmetry for log Calabi-Yau surfaces.*

Abstract

I prove a Torelli theorem for certain Laurent polynomials. This provides strong evidence for the idea that, under mirror symmetry, a Fano manifold corresponds to a single geometric object called a cluster variety. As things stand, mirror symmetry provides a one-to-many correspondence between a single Fano manifold and a collection of Laurent polynomials (or Landau–Ginzburg models); my result gives a geometric proof that, for smooth Fanos in dimension two, these Laurent polynomials assemble to give a single cluster variety.

My other theorem is joint work with Tom Coates and Qaasim Shafi, and determines, under mild hypotheses, how the genus-zero Gromov–Witten invariants of a space X change under blow-ups of X. This is a significant result in enumerative geometry; it also expands the range of Fano manifolds for which we can establish mirror symmetry.

Impact Statement

I expect the methods used to prove my results to generalize in various directions and I have ongoing projects and collaborations on these questions. For example, I expect many of the results in Part I of the thesis to hold for more general Laurent polynomials in two variables and Laurent polynomials in several variables. This will play an important role in the Fano classification problem, by giving a good criterion to single out those varieties that are mirror to a Fano manifold, Part II of the thesis expands the class of varieties we can verify mirror symmetry for, and thereby allows us to gather more evidence for our conjectures

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Contents

Chapter 1

Introduction

Mirror symmetry is a relatively new area of research that originated in the 1980s when the physicists Candelas et al [CdlOGP92] predicted the number of lines on the quintic threefold using arguments from string theory. What followed was a period of immense activity to prove the physicists' conjecture and understand the mathematics behind it, creating the field of mathematics now known as mirror symmetry. The conjecture of [CdlOGP92] was reformulated in precise mathematical terms by Kontsevich and subsequently proven by Givental [Giv96], [Giv98], and Lian, Liu, Yau [LLY02]. Since then mirror symmetry has evolved to combine mathematics ranging from Number Theory, Algebraic Geometry, Symplectic Geometry, Homological Algebra, and Analysis.

Originally, mirror symmetry was formulated by mathematicians as a duality between Calabi-Yau threefolds. Each such Calabi-Yau X has a so-called mirror Calabi-Yau $X^\vee$ whose Hodge diamond is obtained from the Hodge diamond of X by a reflection along a 45° line (hence the name "mirror symmetry"). There are now several introductory textbooks on mirror symmetry for Calabi-Yau manifolds, see for example [CK99].

More recently, mirror symmetry has been generalized to more general varieties. These include varieties for which $-K_X$ is merely nef: Givental [Giv95], [Giv98], produced a mirror to a nef complete intersection Y in a toric variety X. The mirror to Y is in general an open variety $Y^\vee$ with a regular function $W\colon Y^\vee \to \mathbb{C}$, called the superpotential. In case Y is Calabi-Yau, Givental proved that $Y^\vee$ admits a natural compactification to a Calabi-Yau variety and that W is constant and thereby recovers the duality between Calabi-Yau varieties mentioned above.

Givental phrases mirror symmetry as an equality of certain cohomology-valued generating functions, the J-function and the I-function. The J-function is a generating function of certain genus 0 Gromov–Witten invariants, see §6.3, whereas the I-function is built out of explicit hypergeometric factors depending on the cohomology classes of the toric divisors on X and the

Chern classes of the line bundles cutting out the complete intersection X. This approach, called 'Givental's symplectic formalism' has proved to be very fruitful, and has led to mirror theorems for a large class of varieties related to toric varieties: the Abelian/non-Abelian correspondence [CFKS08] expresses the J-function of a GIT quotient $V/\!\!/G$ of a vector space V by a reductive Lie group G in terms of the J-function of the corresponding quotient $V/\!\!/T$ by the maximal torus $T \subset G$ which has an explicit expression in terms of hypergeometric functions thanks to Givental's mirror theorem [Giv98]. Varieties of the form $V/\!\!/G$ with G non-Abelian include Grassmannians and flag manifolds. One of the main results of this thesis is Theorem C, which is a generalization of the Abelian/non-Abelian correspondence to certain zero loci in Grassmann bundles, which allows, for example, to compute the J-function of certain blowups $\tilde{X}$ in terms of the J-function of X.

The other part of my thesis concerns the Fano classification problem. The classification of Fano manifolds is a long-standing open problem. A recent breakthrough here was the realisation that there is a close link between Fano classification and mirror symmetry. This new approach has recovered the classification of 3 dimensional Fano manifolds [CCGK16], and has led to the discovery of more than 500 new 4-dimensional Fano manifolds [Kal19], [Pri20], [CKP15], [CKP19].

This correspondence takes the following form: according to [CCG$^+$13], a n-dimensional Fano manifold X should correspond under mirror symmetry to a Laurent polynomial $f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ in n variables, and under this correspondence, the *regularised quantum period* $\widehat{G}_X(t)$ of X, a certain generating function for genus 0 descendent Gromov–Witten invariants with one insertion, should match up with the classical period of f, defined by

$$\pi_f(t) = \left(\frac{1}{2\pi i}\right)^n \int_{S^1 \times \cdots \times S^1} \frac{\Omega}{1 - tf} = \sum_{d \geq 0} c_0(f^d) t^d$$

where $\Omega = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$ and $c_0(f^d)$ denotes the constant term of f^d. Here, the second equality follows by Taylor expanding $\frac{1}{1-tf}$ near $t = 0$ and applying the Cauchy Residue Theorem n times. Laurent polynomials mirror to a Fano variety have very special coefficients, they are expected to be *rigid maximally mutable Laurent polynomials*, see [CKPT21]. On the other hand, many different Laurent polynomials are usually mirror to the same Fano variety: an easy calculation with the change of variable formula for integrals shows that if $\varphi\colon (\mathbb{C}^\times)^2 \dashrightarrow (\mathbb{C}^\times)^2$ is any volume preserving map (i.e $\varphi^*\Omega = \Omega$) and $\varphi^* f = g$, then $\pi_f(t) = \pi_g(t)$. By a result of Blanc [Bla13] any such φ factors as a composition of certain easier birational maps $(\mathbb{C}^\times)^2 \dashrightarrow (\mathbb{C}^\times)^2$ called algebraic mutations. In all known examples where $\pi_f(t) = \pi_g(t)$, the Laurent polynomials f and g are actually mutation equivalent, meaning that there is a composition of such algebraic mutations relating f and g (see Definition 2.2.3), so it is natural to conjecture the following:

Conjecture 1.0.1. *Suppose that f and g are Laurent polynomials such that $\pi_f(t) = \pi_g(t)$. Then f and g are mutation equivalent.*

Very little is known about this and it seems to be a very deep question and out of reach in this generality. If true, it would imply that the analytic information provided by the power series $\pi_f(t)$ is somehow sufficient to recover f up to some algebraic change of variables.
We can reinterpret all this as follows: the phenomenon that a Fano variety X has several mirror Laurent polynomials can be explained by declaring that the true mirror object to a Fano variety X should not be a Laurent polynomial, but rather a cluster variety $X^\vee$ with holomorphic volume form ω, together with a regular function $W\colon X^\vee \to \mathbb{C}$, called the *superpotential*. $X^\vee$ admits a compactification to a log Calabi-Yau pair (Y, D). Given any Laurent polynomial f mirror to X, we can construct $X^\vee$ by viewing f as an affine fibration $f\colon (\mathbb{C}^\times)^n \to \mathbb{C}$: this fibration has a canonical compactification to a smooth projective variety Y_f with a proper morphism $Y_f \xrightarrow{\pi} \mathbb{P}^1$. Let $D_f = \pi^{-1}(\infty)$, then the pair $(X^\vee, W)$ is obtained by taking $X^\vee = Y_f \setminus D_f$, and $W = \pi|_{Y_f \setminus D_f}$.
To recover Laurent polynomials from $(X^\vee, \omega, W)$, we consider torus charts on $X^\vee$, i.e open sets $U \subset X^\vee$ isomorphic to $(\mathbb{C}^\times)^n$. $W|_U$ is then a regular function on $(\mathbb{C}^\times)^n$, i.e a Laurent polynomial in n variables, and $\omega|_U = \Omega$. The transition functions relating two torus charts U and V is a birational map $\phi : (\mathbb{C}^\times)^n \dashrightarrow (\mathbb{C}^\times)^n$ satisfying $\phi^*\Omega = \Omega$, so that all Laurent polynomials arising from restricting W to a torus chart have the same classical period. We can then unambiguously speak of the classical period $\pi_W(t)$ of the pair $(X^\vee, W)$.
This strongly suggests that $\pi_W(t)$, or equivalently $\widehat{G}_X(t)$ should be an object intrinsic to $X^\vee$, and we can reformulate Conjecture 1.0.1 as a Torelli theorem:

Conjecture 1.0.2. *Let X_1 and X_2 be Fano varieties with mirror Landau-Ginzburg models $(X_1^\vee, W_1)$ and $(X_2^\vee, W_2)$. If $\pi_{W_1}(t) = \pi_{W_2}(t)$ (or equivalently if $\widehat{G}_{X_1}(t) = \widehat{G}_{X_2}(t)$), then $X_1^\vee$ and $X_2^\vee$ are isomorphic.*

A more combinatorial, and coarser notion of mirror symmetry is obtained by replacing a Laurent polynomial f by its Newton polytope $\mathrm{Newt}\,(f)$. An algebraic mutation of f induces a mutation of $\mathrm{Newt}\,(f)$ (see Lemma 2.3.2) and this has led to a formulation of mirror symmetry for orbifold del Pezzo surfaces.

Conjecture 1.0.3. *[ACC+16] There is a one-to-one correspondence between*

- *$\mathbb{Q}$-Gorenstein (qG) deformation families of locally qG-rigid del Pezzo surfaces with cyclic quotient singularities of class TG (i.e admits a qG degeneration with reduced fibres to a normal toric del Pezzo surface)*

- *mutation equivalence classes of Fano polygons*

Here, a Fano polygon is a lattice polygon P such that the vertices of P are primitive lattice points and 0 is in the strict interior of P (see Definition 2.2.1). This correspondence is obtained by sending a Fano polygon P to any qG deformation of the toric variety defined by the fan spanned by the vertices of P. One expects to be able to read off the singularities of the general member of the qG deformation family from the combinatorics of the corresponding Fano polygon: given a Fano polygon P, [AK14] have defined the singularity content of P; this is a subdivison of the cones over edges of P into T-cones and R-cones. On the mirror del Pezzo surface, T-cones give rise to singularities that admit a qG-smoothing, and R-cones give rise to qG-rigid singularities. In particular, the generic member of a qG family mirror to P has singularities prescribed by the R-cones of P. If the Fano polygon P does not admit any R cones, P is called a T-polygon, and the general member of the qG family mirror to P is a smooth del Pezzo surface (see §2.2 for details). Fano polygons related by mutations give rise to the same qG-deformation family, and should correspond to the different toric degenerations of the general member of such a family.
So far, this conjecture has been verified in two cases: it is classically known that there are 10 deformation families of smooth del Pezzo surfaces, and [CH17] have shown that there are 29 deformation families of del Pezzo surfaces with $\frac{1}{3}(1,1)$-singularities, 26 of which are of class TG. On the Fano polygon side, Kasprzyk, Nill, and Prince [KNP17] have combinatorially classified certain Fano polygons: they have shown that there exist 10 mutation equivalence classes of T-polygons, and 26 mutation equivalence classes of Fano polygons with singularity content $\frac{1}{3}(1,1)$. The other main theorem from my thesis is a new geometric, and arguably more intuitive proof of the classification of T-polygons, see Theorem A.

1.1 Statement of results

Part I of this thesis is concerned with giving an entirely geometric proof of the classification of T-polygons.

Theorem A (see Theorem 4.2.2)**.** *There are* 10 *mutation equivalence classes of T-polygons.*

As mentioned in the introduction, the proof of the classification of T-polygons given in [KNP17] is combinatorial and proceeds by introducing a notion of *minimality* of T-polygons. The classification is then obtained by reducing every T-polygon to a minimal T-polygon. However, a geometric, and more direct proof is clearly desirable, not the least if we want to generalize these results to other Fano polygons or to Fano polyhedra.

Recall that a Looijenga pair (Y, D) is a smooth projective surface Y with a nodal anticanonical divisor D. Our geometric proof of Theorem A has three key ingredients: the first is the Torelli theorem for Looijenga pairs of Gross, Hacking, and Keel [GHK14]. The second is Friedman's classification [Fri16, Proposition 9.15] of deformation families of Looijenga pairs (Y, D) with D strictly negative semi-definite, of which we give a simplified proof, see Proposition 3.3.8 and Theorem 3.3.9.

The third key ingredient is a result on factorizations of birational maps:

Theorem B (see Theorem 4.1.5)**.** *Let (Y, D) be a Looijenga pair with two toric models*

$$\begin{tikzcd} & (Y,D) \arrow[dl, "p"'] \arrow[dr, "p'"] & \\ (\bar{Y},\bar{D}) \arrow[rr, dashed, "\varphi"] & & (\bar{Y}',\bar{D}') \end{tikzcd}$$

Then φ has a factorization

$$(\bar{Y},\bar{D}) = (\bar{Y}_0,\bar{D}_0) \overset{\varphi_1}{\dashrightarrow} (\bar{Y}_1,\bar{D}_1) \overset{\varphi_2}{\dashrightarrow} \dots \overset{\varphi_n}{\dashrightarrow} (\bar{Y}_n,\bar{D}_n) = (\bar{Y}',\bar{D}')$$

where each of the maps φ_k is a toric blowup, toric blowdown, or a mutation.

Moreover, let $p_k = \varphi_k \circ \dots \varphi_1 \circ p$. Then $p_k \colon (Y, D) \dashrightarrow (\bar{Y}_k, \bar{D}_k)$ extends to a regular map

$$\tilde{p}_k \colon (\widetilde{Y}, \widetilde{D}) \to (\bar{Y}_k, \bar{D}_k)$$

on some corner blowup $(\widetilde{Y}, \tilde{D})$ of (Y, D).

Theorem B implies a closely related theorem of Blanc [Bla13] and may be of independent interest. Using Theorem B, we also prove Conjecture 1.0.1 for normalized maximally mutable Laurent polynomial f with Newt (f) a T-polygon, see Theorem 4.2.3.

Part II of this thesis is joint work with Tom Coates and Qaasim Shafi and concerns a generalization of the Abelian/non-Abelian correspondence of [CFKS08]. The classical form of the Abelian/non-Abelian correspondence [CFKS08] expresses the J-function of a GIT quotient $V/\!\!/G$ of a vector space V by a reductive Lie group G in terms of the J-function of the corresponding quotient $V/\!\!/T$ by the maximal torus $T \subset G$. Since $V/\!\!/T$ is a toric variety, Givental's Mirror Theorem [Giv98] gives an explicit expression for $J_{V/\!\!/T}$ in terms of hypergeometric functions.

Varieties of the form $V/\!\!/G$ with G non-Abelian include Grassmannians and flag manifolds.
The Abelian/non-Abelian correspondence has been generalised to families by Brown [Bro14] and Oh [Oh21]. In more detail, let $E = L_0 \oplus \cdots \oplus L_n \to X$ be a direct sum of line bundles on a smooth projective variety X. The GIT quotient $E/\!\!/\operatorname{GL}_r(\mathbb{C})$ is the Grassmann bundle $\operatorname{Gr}(r, E) \to X$, and Brown and Oh have given an expression for $J_{\operatorname{Gr}(r,E)}$ in terms of $J_{E/\!\!/T}$, or equivalently in terms of J_X and explicit hypergeometric functions involving the Chern classes of E.
A representation $\rho\colon G \to \operatorname{GL}(V)$ induces a vector bundle $\mathcal{V}_\rho = E \times V/\!\!/G$ on $E/\!\!/G$, where G acts on $E \times V$ via

$$g \cdot (e, v) = (eg, \rho(g^{-1})v)$$

and we say that $\mathcal{V}_\rho$ is representation-theoretic. We extend the above results to certain zero loci of representation theoretic bundles in Grassmann bundles.

Theorem C (see Corollary 7.2.4 and Corollary 7.2.5)**.** *Let $E = L_0 \oplus \cdots \oplus L_n \to X$ be a direct sum of line bundles on a smooth projective variety X, let $\operatorname{Gr}(r, E)$ be the Grassmann bundle of E, and let $\mathcal{V}_\rho$ be a nef, representation-theoretic bundle on $\operatorname{Gr}(r, E)$. Let Z be the zero locus of a generic section of $\mathcal{V}_\rho$ and suppose that Z is Fano. Then there exist an expression for J_Z in terms of J_X and explicit hypergeometric functions involving the Chern classes of E and the Chern classes of $\mathcal{V}_\rho$.*

As mentioned before, the main application of this result is to compute the J-function of a blowup $\tilde{X}$ in terms of the J-function of X. This is achieved by a new geometric construction which expresses the blowup $\tilde{X}$ as the zero locus of a regular section of a vector bundle on a Grassmann bundle.

Theorem D (see Theorem 7.3.2)**.** *Let X be a smooth projective variety, let $E = L_0 \oplus \cdots \oplus L_n \to X$ be a direct sum of line bundles, and let $Z \subset X$ be the zero locus of a regular section s of E. Let $\pi\colon \operatorname{Gr}(n, E^\vee) \to X$ be the Grassmann bundle of subspaces and let $S \to \operatorname{Gr}(n, E^\vee)$ be the tautological subbundle. Then the composition*

$$S \hookrightarrow \pi^* E^\vee \xrightarrow{\pi^* s^\vee} \mathcal{O}$$

defines a regular section of $S^\vee$, and the zero locus of this section is the blow-up $\tilde{X} = \operatorname{Bl}_Z X$.

Applying C to the Grassmann bundle $\operatorname{Gr}(n, E^\vee)$ and $\mathcal{V}_\rho = S^\vee$, shows that under certain positivity assumptions, there exist an explicit expression for $J_{\tilde{X}}$ in terms of J_X, the Chern classes of E and the Chern classes of $S^\vee$.

Outline

This thesis is structured into two parts. The first part is an expanded version of the preprint [Lut21]. §2 gives the necessary background material on mutations of polygons and Laurent polynomials. For simplicity, we restrict ourselves to two dimensions, although many of the definitions generalize. We also introduce maximally mutable Laurent polynomials, those Laurent polynomials that are conjecturally mirror to Fano varieties. §3 gives background material on the Torelli theorem for Looijenga pairs of Gross, Hacking, and Keel [GHK14]. We then explain how to canonically associate a Looijenga pair to a Laurent polynomial and give a simplified proof of the classificaiton of negative semi-definite Looijenga pairs (Proposition 3.3.8 and Theorem 3.3.9). §4 is concerned with the proof of Theorem 4.1.5, from which we then deduce Theorem A. The second part is a joint paper with Qaasim Shafi and Tom Coates [CLS21]. This paper has been accepted for publication in Forum of Mathematics, Sigma. §5 explains our reformulation of the Abelian/non-Abelian correspondence of [CFKS08] in the language of Lagrangian cones, see [Giv95]. §6 gives the necessary background material: §6.1 explains the relationship between GIT quotients of a space A by a complex reductive Lie group G, and the corresponding quotient by the maximal torus $T \subset G$. §6.2 then specializes these results to the case of Grassmann and flag bundles. After that, we give the necessary background material on Givental's symplectic formalism, I-functions, and J-functions in §6.4. We then introduce the Givental-Martin cone in §6.5, which plays an important role in our formulation of the Abelian/non-Abelian correspondence. §7 is devoted to proving Theorem C and D. We review the theorems of Brown [Bro14] and [Oh21] in §7.1. After that, we prove our main Theorem C in §7.2. We continue by giving our new construction of certain blowups as zero loci of a regular section of a vector bundle on a Grassmann bundle in §7.3, and deduce Theorem D. Finally, we apply Theorem D to compute quantum periods of certain Fano manifolds which arise as blowups.

Part I

A Geometric Proof of the Classification of T-Polygons

Chapter 2

Mutations

We begin this section by collecting a few facts from toric geometry that will be used throughout this thesis. A good reference for the material is [CLS11]. In §2.2, we define mutations of polygons, and §2.3 is devoted to mutations of Laurent polynomials. We introduce Laurent polynomials of Tveiten class and maximally mutable Laurent polynomials and prove results about their geometry. §2.4 then constructs the rational elliptic surface Y_f associated to a Laurent polynomial.

2.1 Toric Geometry

We assume the reader is familiar with the basics of toric geometry, and only give a very brief recap. A toric variety is a variety Y containing an algebraic torus $\mathbb{T}$ as a dense subset, such that the action of $\mathbb{T}$ on itself by multiplication extends to Y. Toric varieties can be constructed from cones and fans: We fix throughout a rank n lattice M with dual lattice N. A strongly convex rational polyhedral cone $\sigma \subset N$ defines an affine toric variety

$$U_\sigma = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$$

where the dense torus $\mathbb{T}$ is given by $T_N := N \otimes_{\mathbb{Z}} \mathbb{C}^\times$. More generally, a *fan* $\Sigma \subset N$ defines a toric variety as follows: For every two full-dimensional cones σ and σ' that share a face τ we have the inclusions of affine toric varieties

$$U_\sigma \supset U_\tau \subset U_{\sigma'}$$

and the toric variety Y_Σ is obtained by glueing all affine open sets U_σ along these inclusions.
We will often start with a lattice polytope $P \subset M$. For any vertex v of P, consider the cone

$$C_v = \mathrm{Cone}(P \cap M - v) \subset M_{\mathbb{R}}$$

The collection of dual cones $C_v^\vee \subset N_{\mathbb{R}}$ assemble to a fan Σ_P, called the normal fan of P, and we define $Y_P = Y_{\Sigma_P}$. We will see below that the toric variety Y_P comes with a distinguished ample divisor D_P.
The polytope P is also known as the moment polygon of the toric variety Y_P and records the orbits of the action of T_N on Y_P: k-dimensional strata of P correspond to k-dimensional orbits of the T_N action. In particular, we have

- vertices of $P \leftrightarrow$ fixed points of the T_N-action $\leftrightarrow$ maximal cones in Σ_P
- facets of $P \leftrightarrow T_N$-invariant divisors $\leftrightarrow$ rays in Σ_P

We denote the T_N-invariant divisor associated to a ray ρ by D_ρ. If we are working with a polytope P, we denote the toric divisor associated to the facet E by D_E. We write $\Sigma(1)$ for the collection of rays in the fan Σ. There is a standard short exact sequence

$$0 \to M \to \bigoplus_\rho \mathbb{Z}D_\rho \to \mathrm{Cl}(Y_\Sigma) \to 0$$

where $m \in M$ maps to $\sum_\rho \langle m, u_\rho \rangle D_\rho$ and the surjection maps a divisor to its class in $\mathrm{Cl}(Y_\Sigma)$.
Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^\times)$ yields a short exact sequence

$$1 \to G \to (\mathbb{C}^\times)^{\Sigma(1)} \to T_N \to 1$$

where $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(Y_\Sigma), \mathbb{C}^\times)$. It is shown in [CLS11, Theorem 5.1.11] that under suitable conditions, we have an isomorphism

$$Y_\Sigma \cong \mathbb{C}^{\Sigma(1)} /\!\!/ G$$

with respect to a suitable stability condition depending on the fan Σ. Given a lattice polytope P, the toric variety Y_P has a distinguished ample Cartier divisor

$$D_P = \mathcal{O}(\sum_{E \subset P} h_E D_E)$$

where h_E is the unique integer such that the facet E is contained in the hyperplane $u_E \cdot m = -h_E$, where u_E is the minimal generator of the ray ρ corresponding to E (the inward normal to E). and we have an isomorphism

$$\Gamma(Y_P, \mathcal{O}(D_P)) \cong \bigoplus_{m \in P \cap M} \mathbb{C}\chi^m \tag{2.1}$$

We will often choose local coordinates on the toric variety Y_P around a nonsingular torus fixed point p: The fixed point p corresponds to a smooth cone σ in Σ_P, let $e_1, \dots e_n$ be the minimal generators of σ and define $x_i = \chi^{e_i^*}$. This defines a local coordinate system around p under which p corresponds to the origin and the n toric divisors containing p are given by the hyperplanes $x_i = 0$.

Suppose now that $n = 2$. If $\tilde{Y}_P \to Y_P$ denotes the toric minimal resolution, then D_P pulls back to a Cartier divisor on Y_P with isomorphic ring of sections ([CLS11, Proposition 6.2.7]). We will need the following result:

Proposition 2.1.1. *Let N be a 2-dimensional lattice, and let Y_Σ be the toric variety defined by a fan $\Sigma \subset N$. Let D be a component of the toric boundary of Y_Σ. A choice of orientation of N (i.e. a choice of generator of $H_1(D, \mathbb{Z}) \cong \mathbb{Z}$) gives a canonical identifiaction $D^{int} \cong \mathbb{C}^\times$.*

Proof. D^{int} is canonically the torus $T_{N/\mathbb{Z}v}$, where v is the minimal generator of the ray of Σ corresponding to D. We have

$$T_{N/\mathbb{Z}v} = \operatorname{Spec} \mathbb{C}[v^\perp \cap M]$$

The 1-dimensional lattice $N/\mathbb{Z}v$ has two generators, and the choice of orientation of N picks out a preferred generator e of $N/\mathbb{Z}v$. Writing $x = \chi^{e^*}$, this induces an isomorphism $\mathbb{C}[v^\perp \cap M] \cong \mathbb{C}[x, x^{-1}]$, yielding the required identification $D^{int} \cong \operatorname{Spec} \mathbb{C}[x, x^{-1}] \cong \mathbb{C}^\times$. □

2.2 Mutations of polygons

We now specialize to toric surfaces and polygons. Although many of the definitions carry over to higher dimensions, this adds an additional layer of complication for which we have no use in this thesis. So fix throughout a 2-dimensional lattice M with dual lattice N. We follow definitions of [ACC+16] and [AK14].

Definition 2.2.1. A Fano polygon is a full-dimensional lattice polygon $P \subset M_\mathbb{R}$ such that

- 0 is in the strict interior of P
- all vertices of P are primitive lattice points

Fix an edge E of P with primitive inward normal vector u. The lattice length ℓ_E is the number of lattice points on E minus one, and the lattice height h_E is the positive integer $-\langle u, E\rangle$. For every edge E, let m_E be the unique positive integer such that

$$\ell_E = m_E h_E + r_E, \quad 0 \leq r_E < h_E$$

A Fano polygon defines two different toric surfaces: we have the polarized toric surface (Y_P, D_P) defined by the normal fan of P as defined in Section 2.1. The edges of P correspond to the toric divisors of Y_P, and denoting the divisor corresponding to an edge E by D_E, the distinguished ample divisor D_P is defined as

$$D_P = \sum_{E \subset P} h_E D_E.$$

(see Section 2.1).

However P also defines a different toric surface $Y_{Sp(P)}$ via the *spanning fan*. This is the fan with maximal cones equal to the cones over the edges of P, so in particular, $Y_{Sp(P)}$ is only defined when 0 is contained in the interior of P. It is not hard to see that if P is a reflexive polygon (i.e the origin is the unique interior lattice point of P) and $Q = P^\star$ is the polar (or dual) polygon, then the normal fan of P is interchanged with the spanning fan of Q and vice-versa. The mirror symmetry conjecture of [ACC$^+$16] predicts a one-to-one correspondence between Fano polygons and deformation families of orbifold del Pezzo sufaces, up to an appropriate equivalence relation We analyze the structure of the toric surface $Y_{Sp(P)}$. Any edge E of P defines a maximal cone σ in $Y_{Sp(P)}$ which corresponds to a torus fixed point under the orbit-cone correspondence. Let $\{u, v\}$ be the rays spanning σ. We may find a basis e_1, e_2 for N such that σ is the cone spanned by e_1 and $de_1 - ke_2$ for coprime integers d and k such that $d > 0, 0 \leq k < d$. [CLS11, Proposition 10.1.2] shows that the affine toric surface Y_σ defined by σ is the cyclic quotient singularity $\frac{1}{d}(1, k)$, i.e the quotient $\mathbb{C}^2/\mu_d$ where μ_d acts with weights $(1, k)$.

We introduce some terminology: Fix an edge E and let σ the corresponding cone of $Y_{Sp(P)}$. If $r_E = 0$, we call σ a T-cone, and the singularity of Y_σ is called a T-singularity. If in addition $m_E = 1$ (or in other words $\ell(E) = h(E)$), σ is called a primitive T-cone and Y_σ is called a primitive T-singularity. If $m_E = 0$ and $r_E > 0$, then σ is called a R-cone, and the singularity of Y_σ is a R-singularity. We may (non-uniquely) subdivide the cone σ into m_E primitive T-cones and one R-cone σ_R (if $r_E = 0$ there is no R-cone), see Figure 2.1 for an example. Geometrically, this corresponds to a partial crepant resolution

$$\tilde{Y}_\sigma \to Y_\sigma$$

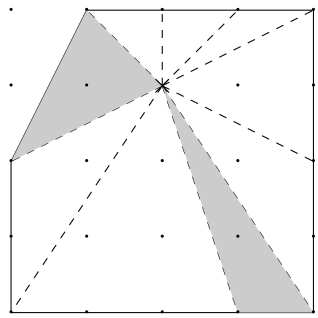


Figure 2.1: A Fano polygon with seven T-cones and two R-cones (shaded in grey). The singularity content of P is $(7, \{2 \times \frac{1}{3}(1,1)\})$

where $\tilde{Y}_\sigma$ now has m_E primitive T-singularities and one R-singularity (or no R-singularity if $r_E = 0$). If Y_{σ_R} is the cyclic quotient singularity $\frac{1}{d}(1,k)$, we define the *residue* $\mathrm{res}(\sigma)$ of σ to be

$$\mathrm{res}(\sigma) = \begin{cases} \frac{1}{d}(1,k) & \text{if } r_E \neq 0 \\ \emptyset & \text{if } r_E = 0 \end{cases}$$

This definition stems from the fact that T-singularities are $\mathbb{Q}$-Gorenstein-(abbreviated to qG) smoothable, whereas R-singularities are qG-rigid (see [ACC$^+$16, p.1-2], for definitions of these terms). Since there are no local to global obstructions for qG smoothings of toric del Pezzo surfaces ([ACC$^+$16, Lemma 6]), this shows that a general qG deformation Y of $\tilde{Y}_\sigma$ has exactly one singularity of type $\mathrm{res}(\sigma)$. [AK14] have defined the singularity content of a Fano polygon P. This is the pair $(k, \mathcal{B})$, where k is the number of T-cones of P and $\mathcal{B}$ is a cyclically ordered list of the residues of the R-cones of P. Choose now a decomposition of P into T-cones and R-cones. For each R-cone, choose moreover the right vertex with respect to the anticlockwise orientation.

Definition 2.2.2. We say that $p \in P \cap M$ is a residual point if $p = 0$ or if p is contained in the strict interior of an R-cone of P. We say that p is quasi-residual if p is residual or if p is the right vertex (with respect to the chosen orientation) of an R-cone.

Figure 2.2 shows residual and quasi-residual points for a Fano polygon P. The set of residual and quasi-residual points depends on choices, but this will be irrelevant in what follows.

As mentioned in the introduction, the classification of Fano polygons up to an appropriate equivalence relation is conjecturally mirror to the classification of orbifold del Pezzo surfaces admitting a toric degeneration. This equivalence relation is called mutation: while it is a bit technical to define, the idea behind it is rather simple, see Figure 2.3.

Definition 2.2.3. Let $P \subset M$ be a Fano polygon and let $v \in N$ be a primitive vector. Choose a

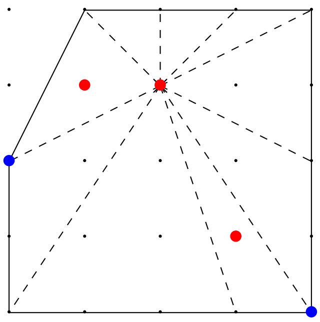


Figure 2.2: A Fano polygon with a choice of decomposition into T-cones and R-cones. Residual points in red, additional quasi-residual points in blue

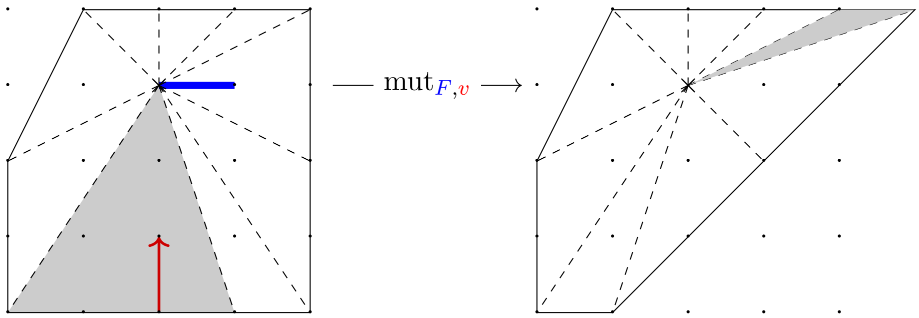


Figure 2.3: Mutation of the polygon P with respect to mutation data $v = (0, 1)$, $F = \text{Newt}\,(1+x)$. The mutation contracts a grey T-cone on the left and adds a grey T-cone on the right.

line segment $F \subset v^\perp \subset M$ and write P_d for the slice of P at height d with respect to v. Suppose that for all $d < 0$ we can decompose $P_d = R_d + (-d)F$ as a Minkowski sum for some line segment R_d (where we allow $R_d = \emptyset$). Then we say that P is *mutable* with respect to (v, F), and define

$$P' = mut_{v,F}P = \text{conv}\left(\bigcup_{d<0} R_d \cup \bigcup_{d\geq 0}(P_d + dF)\right)$$

We call F the factor of the mutation, and we say that two polygons P, P' are *mutation equivalent* if there is a sequence of mutations of polygons starting with P and ending with P'.

In the same notation as Definition 2.2.3, suppose $F = kF'$ for some primitive line segment F' and positive integer k and suppose that P is mutable with respect to (v, F). Then P' is obtained from P by contracting k T-cones on one edge of P and adding k T-cones on the opposite edge of P. In particular, we see that the condition of P to be mutable with respect to (v, F) just means that there is an edge of P perpendicular to v, long enough to contract k copies of F. We also note that this shows that the singularity content of P is invariant under mutation (see [AK14, Proposition 3.6]) for a proof). We now specialize to a special class of Fano polygons

Definition 2.2.4. A Fano polygon P is a T-polygon if for every edge E of P the lattice length

ℓ_E is divisible by the lattice height h_E.

T-polygons have singularity basket $(k, \emptyset)$ for some $k > 0$. Since the singularity basket of a Fano polygon is invariant under mutation, mutation descends to an equivalence relation on the set of T-polygons.

2.3 Mutations of Laurent polynomials

Let now $v \in N$ and $f \in \mathbb{C}[v^\perp] \subset \mathbb{C}[M]$. Following [GHK15] and [ACC$^+$16], we define the automorphism of the function field $\mathbb{C}(M)$

$$x^m \mapsto x^m f^{\langle m,v\rangle}$$

which induces a birational map

$$\varphi_f \colon T_N \dashrightarrow T_N$$

and we call φ_f an algebraic mutation, and f the factor of the mutation.

Definition 2.3.1. Given a Laurent polynomial $g \in \mathbb{C}[M]$, we say that g is *mutable* with respect to φ_f if $\varphi_f^*(g) \in \mathbb{C}[M]$, i.e $\varphi_f^*(g)$ is again a Laurent polynomial. We also say that φ_f is an algebraic mutation of g.
Given $g, g' \in \mathbb{C}[M]$, we say that g and g' are *mutation equivalent* if there exist algebraic mutations φ_i for $1 \leq i \leq n$ and Laurent polynomials $g_i \in \mathbb{C}[M]$ for $0 \leq i \leq n$ such that $g_0 = g, g_n = g'$ and $\varphi_i^* g_{i-1} = g_i$ for all i.

Let us interpret mutability more concretely. v gives a $\mathbb{Z}$-grading of M by height, let us write $g = \sum_{d=-h}^{m} g_d$ where g_d is the sum of the monomials of g at height d. Extend v to a basis $e_1 = v, e_2$ for N. Write $x = x^{e_2^*}$ and $y = x^{e_1^*}$. The factor f is a Laurent polynomial in x and $\varphi_f^*(g_d) = g_d f^d$. It follows that $g \in \mathbb{C}[M]$ is mutable with respect to φ_f if and only if g_{-d} is divisible by f^d for $d > 0$.
Note that if f is a monomial, then every Laurent polynomial is mutable with respect to φ_f, so we regard such mutations as trivial. If a factor is of the form $(\lambda + x^u)$ for some $\lambda \in \mathbb{C}^\times$ and *primitive* $u \in M$, we call the mutation *standard*. It is clear that any factor is a product of standard and trivial factors. We have the following easy lemma:

Lemma 2.3.2. *Suppose that $g \in \mathbb{C}[M]$ is mutable with respect to φ_f and that $P = \operatorname{Newt}(g)$ is a Fano polygon. Then P is mutable with respect to $(v, \operatorname{Newt}(f))$.*

Proof. Since P is Fano, the origin is an interior lattice point so we have that $g = \sum_{d=-h}^{m} g_d$ for some $h, m > 1$. Since g is mutable with respect to f, g_{-d} is divisible by f^d for all $d > 0$ and we

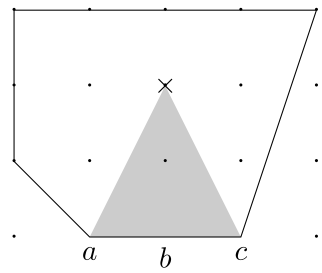


Figure 2.4

may write $g_{-d} = f^d r_{-d}$ for some r_{-d}. This implies that

$$\mathrm{Newt}\,(g_{-d}) = R_{-d} + d\,\mathrm{Newt}\,(f)$$

for all $d > 0$, where $R_{-d} = \mathrm{Newt}\,(r_{-d})$. This is exactly the condition for P to be mutable with respect to $(v, \mathrm{Newt}\,(f))$. □

We say that the algebraic mutation φ_f induces the mutation of $\mathrm{Newt}\,(g)$. The converse to 2.3.2 fails; it is not true that every mutation of $\mathrm{Newt}\,(g)$ is induced by a mutation of g:

Example 2.3.3. Consider the Fano polygon P in Figure 2.4. P is mutable with respect to $v = (0, 1)$ and $F = \mathrm{Newt}\,(1 + x)$. We have that

$$g_{-2} = y^{-2}(a + bx + cx^2)x^{-1}$$

For g to be mutable with respect to $f = \lambda + x$ for some $\lambda \in \mathbb{C}$, we must have that f^2 divides g_{-2} which happens if and only if $b^2 = 4ac$

We now define special classes of Laurent polynomials which are 'as mutable as possible' in a precise sense.

Definition 2.3.4. Let P be a Fano polygon. A Laurent polynomial $g \in \mathbb{C}[M]$ with $\mathrm{Newt}\,(g) = P$ is of Tveiten class if every mutation of P is induced by an algebraic mutation φ_f of g.

Let us investigate the consequences of this, keeping the same notation as before. Fix an edge E of P with inner normal v and write $\ell_E = mh + r$ as before. Then P is mutable with respect to (v, kF) for all $1 \leq k \leq m$ where $F \subset v^\perp$ be a primitive line segment. These mutations can only be induced by an algebraic mutation of g if there exists a polynomial $f \in \mathbb{C}[x]$ with $\mathrm{Newt}\,(f) = mF$ such that for all $0 \leq d \leq h$, g_{-d} is divisible by f^d in $\mathbb{C}[M]$. This is quite restrictive: up to a unit in $\mathbb{C}[M]$ we may write $f = \prod_{i=1}^{m}(\lambda_i + x)$ with $\lambda_i \neq 0$, so that we have

(again up to a unit)

$$g_{-d} = \prod_{i=1}^{m} (\lambda_i + x)^d \cdot r_{-d}$$

where $r_{-d} \in \mathbb{C}[x]$.

We see from this that a Laurent polynomial g of Tveiten class is mutable with respect to m (not necessarily distinct) standard factors $(\lambda_i + x)$ along the edge E, one for each primitive T-cone on E. Since $\deg(r_{-h}) = r < h$, it follows that any Laurent polynomial g can be mutable with respect to a maximum of m standard factors along E, this motivates the term *maximally mutable* used for Laurent polynomials of Tveiten class in [Tve15]. However, we reserve this notion for those Laurent polynomials where all of the factors have $\lambda_i \equiv 1$, see Definition 2.3.23.

Let $e_1 = v, e_2$ be a basis for one of the two smooth cones σ in Σ_P containing the ray $\mathbb{R}_{\geq 0} v$ and write $x = \chi^{e_2^*}, y = \chi^{e_1^*}$. Then D_E is given by the equation $y = 0$, and x is a local coordinate on D_E around the torus fixed point corresponding to the cone σ.

Definition 2.3.5. Suppose that g is mutable with factor $(\lambda + x)$, and let p be the point on D_E where $\lambda + x = 0$, i.e the point with coordinates $(-\lambda, 0)$. Then we say that g is mutable with respect to p.

Similarly, if g is mutable with factor $(\lambda + x)^m$ for some positive integer m, we say that g is mutable at mp. The set of all points with respect to which g is mutable defines a zero cycle Z supported on the interior of D_E, called the mutable cycle of g.

We see that g with $\mathrm{Newt}\,(g) = P$ is of Tveiten class if and only if the mutable cycle has degree m_E along the edge E, the maximal possible degree.

We next characterize 0-cycles Z that can appear as mutable cycles of a Laurent polynomial (see Proposition 2.3.7. Clearly, a necessary condition is that the intersection $Z \cap D_E$ must be of degree $\leq m_E$.

Definition 2.3.6. Let P be a Fano polygon, and let Z be a zero cycle supported on the interior of the toric boundary ∂X_P. We say that Z is admissible if

$$\deg(Z \cap D_E) \leq m_E$$

for every edge E. We say that Z is maximal admissible if

$$\deg(Z \cap D_E) = m_E$$

for every edge E.

We now give a classification of Laurent polynomials of Tveiten class. This is a slight generalization of [CKPT21, Proposition 3.7], and we closely follow their setup and notation. Let P be a Fano polygon and let Y_P be the associated toric variety. For every edge E of P, the choice of orientation of N gives identifications $D_E^{int} \cong \mathbb{C}^\times$ by Proposition 2.1.1. If Z is a maximal admissible zero-cycle, the restriction $Z \cap D_E$ is then a set of (not necessarily distinct) points $\lambda_1, \ldots, \lambda_{m_E} \in \mathbb{C}^\times$. We have the following result:

Proposition 2.3.7. *Let P be a Fano polygon and let Z be a maximal admissible zero cycle supported on the toric boundary ∂X_P.*

- *If P is a T-polygon, there exists a Laurent polynomial g of Tveiten class with Newton polygon P and mutable cycle Z if and only if*

$$\prod_{E \subset P} \prod_i \lambda_i = 1$$

  The coefficients of a generic such g are linear functions in the coefficient of the origin and the coefficient of one vertex.

- *If P is not a T-polygon, there always exists a Laurent polynomial g of Tveiten class with Newton polygon P and mutable cycle Z. The coefficients of a generic such g are linear functions of the coefficients of a choice of quasi-residual points*

Proof. The setup is almost identical to [CKPT21, Proposition 3.7], and we keep their notation. We fix a decomposition of P into T-cones and R-cones in such a way that the R-cones are always on the right of E with respect to the anticlockwise orientation. This fixes a set of residual points on P, and we take the quasi-residual points to be the right vertices of the R-cones, again with respect to the anticlockwise orientation. If P is not a T-polygon, we define $\mathcal{R}$ to be the set of quasi-residual points. If P is a T-polygon, we define $\mathcal{R} = \{0, v\}$, where v is any vertex of P. We define the P-height of a lattice point $p \in P$ to be the non-negative rational number s such that p lies on the boundary of sP. It is clear that s must lie in the interval $[0, 1]$ and we let $\mathcal{R}_{\geq s}$ be the set of points in $\mathcal{R}$ of P-height at least s. Fix a generic Laurent polynomial $g = \sum_{v \in P \cap M} a_v x^v$. Using downward induction on s, we will prove:

The coefficients a_v such that v has P-height s are linear functions of the coefficients a_w for $w \in \mathcal{R}_{\geq s}$. Moreover, these coefficients a_w of g are independent.

We need to prove (2.3) for all $0 \leq s \leq 1$. The base case is $s = 1$. The points of P-height 1 are exactly the points on the boundary $\partial P \cap M$. Fix an edge E. In suitable coordinates, the part of g along E are (up to overall multiplication by a constant monomial)

$$a_0 + a_1 x + \cdots + a_\ell x^\ell$$

such that a_0 is the coeffcient of the left vertex of E in our chosen orientation. Here $\ell = \ell_E$ is the length of E and we write $\ell = hm+r$ as before, where $h = h_E$ and $r = r_E$. The restriction of Z to D_E is a zero-cycle of degree $m = m_E$, let $-\lambda_1, \ldots, -\lambda_m$ be the coordinates of the corresponding (not necessarily distinct) points on D_E. The condition that g is mutable with respect to Z implies that

$$a_0 + a_1 x + \cdots + a_\ell x^\ell = \prod_{i=1}^{m} (\lambda_i + x)^h (\sum_{i=0}^{r} c_i x^i) \tag{2.2}$$

for some $c_i \in \mathbb{C}$. We first show that we can solve (2.2) uniqely for c_i in terms of linear functions of $a_0, a_{mh+1}, \ldots a_{mh+r}$. We argue by downward induction. Clearly we have $c_r = a_{mh+r}$, this is the base case. Suppose that c_i has been determined in such a way for $i > k$. Comparing the two sides of (2.2) we obtain

$$a_{mh+k} = \sum_{i=0}^{r-k} p_i c_{k+i}$$

for homogeneous polynomials $p_i(\lambda_1, \ldots \lambda_k)$ of degree i. Note that $p_0 = 1$ so that we can solve

$$c_k = a_{mh+k} - \sum_{i=1}^{r-k} p_i c_{k+i}$$

By the induction hypothesis, the right hand side is a linear function of $a_{mh+k}, \ldots a_{mh+r}$. This shows that $c_1, \ldots, c_r$ are uniquely determined as linear functions of $a_{mh+1}, \ldots a_{mh+r}$. We also see that $c_0 = \prod_i \lambda_i^h a_0$, so this proves that all coefficients of $a_0 + a_1 x + \cdots + a_{mh+r} x^r$ are linear functions in $a_0, a_{mh+1}, \ldots a_{mh+r}$.

We can repeat this procedure for every edge, however we need to ensure compatibility at the vertices: indeed, if E does not support an R-cone, then $r = 0$ and the coefficients a_0 and $a_\ell = a_{mh}$ are not independent since $a_0 = \prod_i \lambda_i^h a_\ell$. After eliminating as many vertex coefficients as possible, we obtain the following:

- If P is a T-polygon, then the induction hypothesis holds for $s = 1$ if and only if

$$\prod_{E \subset P} \prod_i \lambda_i = 1 \tag{2.3}$$

  In this case, the coefficients of P at P-height 1 are linear functions in the coefficient a_v where $\{v\} = \mathcal{R}_{\geq 1}$.

- If P is not a T-polygon, then the induction hypothesis always holds for $s = 1$, and the coefficients of P at P-height 1 are linear functions in the coefficients a_w for $a_w \in \mathcal{R}_{\geq 1}$.

The rest of the proof now goes through as in [CKPT21, Proposition 3.7]. We proceed by induction on the P-height s, where $0 < s < 1$ and we suppose that the coefficients of P along lattice points in qP for $q > s$ have been determined. Suppose that there is a lattice point $p \in P \cap M$ at P-height s. Since P is Fano, p must lie in the cone C_E over an unique edge E. As before, in suitable coordinates, the part of g supported on the line L through p parallel to E is given by

$$a_0 + a_1 x + \cdots + a_k x^k$$

up to overall multiplication by a constant monomial, such that a_0 is the coefficient of the left vertex of $L \cap P \cap M$. Write $h = h_E$ and $m = m_E$ as before. The line L is at height sh with respect to E, and since a T-cone supports exactly d lattice points at height d for $0 < d < h$, $L \cap P \cap M$ contains at least shm lattice points. The restriction of Z to D_E is a zero-cycle of degree $m = m_E$, let $-\lambda_1, \ldots, -\lambda_m$ be the coordinates of the corresponding (not necessarily distinct) points on D_E. The condition that g is mutable with respect to Z gives

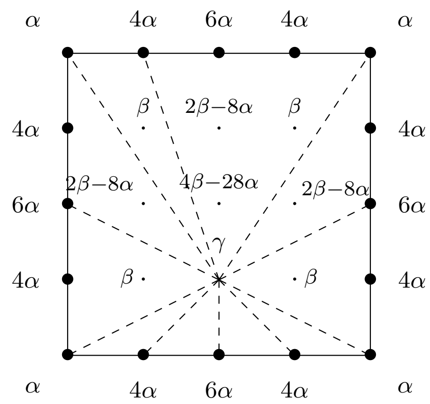
$$a_0 + a_1 x + \cdots + a_k x^k = \prod_{i=1}^{m} (\lambda_i + x)^{sh} \sum_{i=0}^{s} c_i x^i \tag{2.4}$$

for some $c_i \in \mathbb{C}$. Lattice points along L fall into three categories: residual points in C_E, nonresidual points in C_E and points outside of C_E. By convexity, the points outside of C_E are at P-height greater than s and hence their coefficients are determined by our induction hypothesis. so if L does not contain any residual points and no points outside C_E then $k = shm - 1$ and therefore $a_0 = \cdots = a_k = 0$, so (2.4) trivially holds. Otherwise, (2.4) holds with $s + 1$ equal to the number of points on $L \cap E$ that are either residual or outside C_E. The coefficients of these points are $a_0, \ldots a_i$ and $a_j, \ldots a_k$ for some $i < j$. As before we can inductively solve (2.4) for the c_i in terms of $a_0, \ldots a_k$ and $a_{k'}, \ldots a_m$, proving the theorem. □

Corollary 2.3.8. *Let P be a T-polygon. The space of Laurent polynomial of Tveiten class for P is a dense open subset of a vector space of dimension equal to the number of quasiresidual points if P is not a T-polygon, and equal to two if P is a T-polygon.*

Proof. We showed in Proposition 2.3.7 that the space of Laurent polynomials mutable with respect to Z is a vector space of the above stated dimension. Such a Laurent polynomial only fails to be of Tveiten class if $\mathrm{Newt}(g) \subsetneq P$, which only happens along the union of the loci $a_w = 0$ for w quasi-residual. The result follows. □

Example 2.3.9. (Adapted from [CKPT21, Example 3.8]). Let P the lattice polygon with vertices $(-2,-1)$, $(2,-1)$, $(2,3)$, $(-2,3)$ and take Z to be the maximal admissible cycle supported at the point $[-1:1]$ on each edge. P has singularity content $(9, \{\frac{1}{3}(1,1)\})$. The space of Laurent polynomials g with $\mathrm{Newt}\,(g) = P$ of Tveiten class with mutable cycle Z is the dense open subset $\alpha \neq 0$ of the three-dimensional subspace of $\mathbb{C}^{25}$ illustrated below (where $\alpha, \beta, \gamma \in \mathbb{C}$).



We will need the following lemma.

Lemma 2.3.10. *Let P be a Fano polygon, and let E be an edge of height h supporting an R-cone C. Then C has an interior lattice point at height $h-1$.*

Proof. This is essentially an easy exercise in trigonometry. Since P is a T-polygon, the vertices of P are primitive lattice points, and it follows that the vertices of every primitive T-cone on E are also primitive. We may therefore assume that there is a unique primitive T-cone supported on E. We can also assume that E is horizontal, as in Figure 2.5. If the R-cone has length ≥ 2 then the intersection of C with the horizontal line L at height $h-1$ has length > 1 and hence the lemma holds. It remains to check the case where the R-cone has length 1.

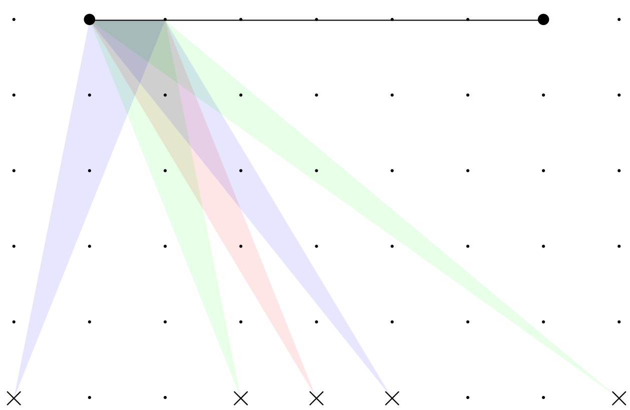


Figure 2.5: The R-cone over the edge E for different possible locations of the origin, in the case where $h = 5$.

Refer to Figure 2.6 for notation. We will only treat the case where the origin is to the right of v, the other case is entirely analogous.

The x-coordinates of the two blue points are $-n+\frac{n}{h}$ and $-n+1+\frac{n-1}{h}$, so it suffices to show that there is an integer in the interval $(-n+\frac{n}{h}, -n+1+\frac{n-1}{h})$. We may write $n = mh+\ell$ for $0 \leq \ell < h$ and $k \in \mathbb{N}_{\geq 0}$. Note that if $\ell = 0$ then v is not primitive, and if $\ell = 1$, then w is not primitive, so we may assume that $\ell \geq 2$. This shows that

$$\begin{aligned} -n+\tfrac{n}{h} &= -mh-\ell+m+\tfrac{\ell}{h} < -mh-\ell+m+1 \\ -n+1+\tfrac{n-1}{h} &= -mh-\ell+1+m+\tfrac{\ell-1}{h} > -mh-\ell+1+m \end{aligned}$$

It follows that $(-mh-\ell+1+m, h-1) \in M$ is in the interior of C as required.

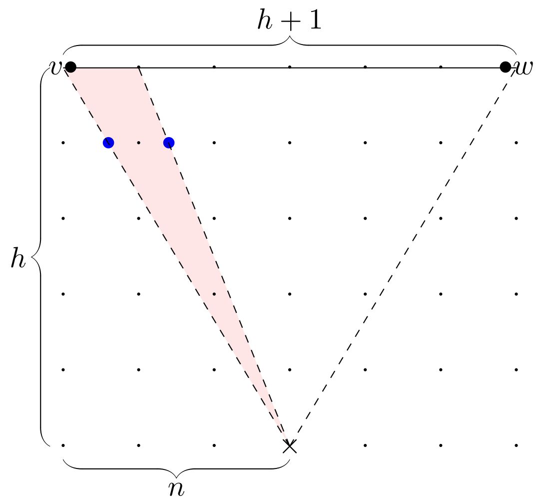


Figure 2.6

□

We now investigate the geometry of the curve $g = 0$, or rather a suitable compactification thereof in the toric variety Y_P. Fix a Fano polygon P, let $g \in \mathbb{C}[M]$ with $\mathrm{Newt}\,(g) = P$. g defines a section $\tilde{g}$ of $\mathcal{O}(D_P)$ by (2.1), and the vanishing locus $Z(\tilde{g})$ is an algebraic curve that restricts to the affine curve $g = 0$ on the dense torus $(\mathbb{C}^\times)^2 \subset Y_P$. For simplicity we also denote Z by g. Since $\mathrm{Newt}\,(g) = P$, the curve $g = 0$ does not contain any of the toric divisors. Moreover, we claim that g also does not pass through any torus fixed point. Indeed, the restriction of g to $D_E \cong \mathbb{P}^1$ is given by the sum of monomials supported on E, which is given by the vanishing of a homogeneous polynomial $p = a_0 y^\ell + a_1 x y^{\ell-1} + \dots a_\ell x^\ell$. The condition that $\mathrm{Newt}\,(g) = P$ implies that a_0 and a_ℓ are nonzero, so that p does not vanish at $x = 0$ or $y = 0$, which are the coordinates of the torus fixed points on D_E. It follows that the strict transform of g under the minimal resolution $\tilde{Y}_P \to Y_P$ is isomorphic to g and we may assume that Y_P is smooth for the purposes of studying the curve g.

We recall the following definition

Definition 2.3.11. Let C be a curve. We say that $p \in C$ is a point of multiplicity h if in suitable local coordinates centered at p, C has a Taylor expansion of the form $f = \sum_{i,j \geq h} c_{ij} x^i y^j = 0$. We say that p is an ordinary point of multiplicity h if additionally the discriminant of the initial polynomial $f_{in} = \sum_{i+j=h} c_{ij} x^i y^j$ is nonzero.

The condition that the discriminant of $f_{in} = \sum_{i+j=h} c_{ij} y^i x^j =: \sum_i c_i y^i x^{h-i}$ is nonzero is equivalent to the statement that we may factor

$$f_{in} = \prod_{i=1}^{h} (\alpha_i y + \beta_i x)$$

for $\alpha_i, \beta_i \in \mathbb{C}^\times$ such that $[\alpha_i : \beta_i] \in \mathbb{P}^1$ are distinct. This implies that locally at p, C has h distinct smooth branches with distinct tangents. We also note that if $c_h \neq 0$ (which will almost always be the case for us), then the line $x = 0$ is not tangent to any branch of C at p, and we can rewrite

$$f_{in} = \prod_{i=1}^{h} (y + \gamma_i x)$$

for $\gamma_i \in \mathbb{C}$. If in addition $c_0 \neq 0$, then the $\gamma_i \in \mathbb{C}^\times$. Since the discriminant of f_{in} is nonzero, the γ_i are all distinct, and the singularity $p \in C$ can be resolved by a single point blowup. The strict transform $\tilde{C}$ satisfies $\tilde{C} = \pi^* C - hE$, where E denotes the exceptional divisor.

Definition 2.3.12. Let C and D be curves on a smooth surface Y, and let $p \in C \cap D$ be a smooth point on C and D. We say that C and D are tangent to order k at p if the local intersection number $(C \cdot D)_p = k$. Equivalently, k successive point-blowups are necessary to separate the strict transforms $\tilde{C}$ and $\tilde{D}$.

In our definition, C is transverse to D at p iff C and D are tangent to order 1.

Definition 2.3.13. We say that $p \in C$ is an ordinary point of multiplicity h with k-fold tangency along a divisor D if we may choose local coordinates based at p such that D is given by $y = 0$ and C has a Taylor expansion

$$f = \sum_i c_i y^i x^{k(h-i)} + (\text{terms of degree at least } kh + 1)$$

with respect to the grading $\deg(x) = 1$ and $\deg(y) = k$ and moreover the discriminant of the polynomial $\sum_i c_i y^i z^{h-i}$ is nonzero, where $z = x^k$. If $k > 1$, we also require that $c_h \neq 0$.

A familiar example is the tacnode $(y - x^2)(y + x^2) = 0$, which in our definition is a double point with (exact) twofold tangency.
As before, the condition that the discriminant is nonzero is equivalent to the statement that we may factor

$$\sum_i c_i y^i x^{k(h-i)} = \prod_{i=1}^{h} (y + \gamma_i x^k) \tag{2.5}$$

for distinct $\gamma_i \in \mathbb{C}$. In our definition, an ordinary point of multiplicity h with 1-fold tangency along a divisor D is just an ordinary point p of multiplicity h which happens to lie on D. We will show that an ordinary point of multiplicity h with k-fold tangency along a divisor D indeed deserves its name: it is locally the union of distinct branches, each of which are tangent to order at least k to D at p (see Corollary 2.3.15 for a more precise statement). We have the following Lemma

Lemma 2.3.14. *Let $k > 1$, and let $p \in C$ be an ordinary point of multiplicity h with k-fold tangency along a smooth divisor D, and let $q \colon \tilde{Y} \to Y$ be the blowup of Y along p with exceptional divisor E. Then E intersects the strict transform $\tilde{C}$ at a unique point p', and $p' \in \tilde{C}$ is an ordinary point of multiplicity h with $(k-1)$-fold tangency along $\tilde{D}$.*

Proof. By definition, C locally has an equation of the form

$$\prod_{i=1}^{h} (y + \gamma_i x^k) + r(x, y)$$

for $\gamma_i \in \mathbb{C}$ and some polynomial $r(x, y)$ which consists of monomials of degree at least $kh + 1$ with respect to the grading $\deg(x) = 1, \deg(y) = k$. Let $q \colon \tilde{Y} \to Y$ be the blowup of p. Locally, we are blowing up $\mathbb{C}^2$ in the point $(0, 0)$, so that $\tilde{Y}$ is given by

$$\{z_0 y - z_1 x = 0\} \subset \mathbb{P}^1_{[z_0:z_1]} \times \mathbb{C}^2_{xy}$$

Let U' be the chart on $\tilde{Y}$ with coordinates $u = x, v = \frac{z_1}{z_0}$. On U', q is given by $(u, v) \mapsto (u, uv)$, so that the pullback of C is given by

$$u^h \prod_{i=1}^{h} (v + \gamma_i u^{k-1}) + r(u, uv) = u^h \left(\prod_{i=1}^{h} (v + \gamma_i u^{k-1}) + us(u, v) \right)$$

where $s(u, v) = r(u, uv)/u^{h+1}$. Note that s is a polynomial since each term in r has total degree at least $h + 1$ (with respect to the grading $\deg(x) = 1, \deg(y) = 1$). Since $u = 0$ is the equation of the exceptional divisor E, it follows that the strict transform $\tilde{C}$ is given by

$$\prod_{i=1}^{h}(v+\gamma_i u^{k-1})+us(u,v)$$

Since $k>1$, the point $(0,0)$ is the only point in $E\cap\tilde{C}$. The condition that every monomial in $r(x,y)$ has at least degree $kh+1$ with respect to the grading $\deg(x)=1, \deg(y)=k$, then implies that every monomial in $us(u,v)$ has at least degree $(k-1)h+1$ with respect to the grading $\deg(u)=1, \deg(v)=k-1$. The condition on the discriminant is still satisfied, and $v=0$ is the equation of the strict transform $\tilde{D}$, so it follows that the point $(0,0)$ is a point of multiplicity h with tangency $k-1$ along $\tilde{D}$. □

Corollary 2.3.15. *Let $p\in C$ be an ordinary point of multiplicity h with k-fold tangency along a smooth divisor D, then every branch of C is tangent to D at p to exact order k to D at p, except possibly for one branch, which might be tangent to higher order. If the coefficient $c_0\neq 0$ in* (2.5)*, then every branch is tangent to exact order k.*

Proof. Applying Lemma 2.3.14 several times shows that there is a unique composition of successive point blowups

$$Y_{k-1}\xrightarrow{q_{k-1}}\dots\xrightarrow{q_1}Y_0$$

such that the strict transform of C under $q_1\circ\cdots\circ q_i$ is an ordinary point of multiplicity h with $(k-i)$-fold tangency along the strict transform of D. In particular, the strict transform $\tilde{C}$ under $q_1\circ\cdots\circ q_{k-1}$ is an ordinary point of multiplicity h lying on the strict transform of D, i.e. $\tilde{C}$ is locally the union of h distinct curves C_i with distinct tangents meeting D at a point. It follows that $(C_i\cdot D)\geq 1$, and if $y=0$ is not tangent to any branch (or equivalently $c_0\neq 0$), then $(C_i\cdot D)=1$ for all i. Now C is locally the union of the curves $(q_1\circ\cdots\circ q_{k-1})(C_i)$, and since every blowup decreases the order of tangency by 1, the result follows. □

It follows from the proof of Corollary 2.3.15 that the singularity $(p\in C)$ can be resolved by a composition q of k successive point blowups, and the strict transform $\tilde{C}$ satisfies $\tilde{C}=q^*C-hE_1-\cdots-hE_k$, where the E_i are the exceptional divisors.
Inspired by Corollary 2.3.15 we make the following definition.

Definition 2.3.16. We say that $p\in C$ is an ordinary point of multiplicity h with exact k-fold tangency along D if p is an ordinary point of multiplicity h with k-fold tangency along D, and $c_0\neq 0$ in (2.5) (or equivalently, each branch of C is tangent to exact order k along D at p).

In terms of (2.5), exact k-fold tangency is equivalent to the statement that the $\gamma_i\in\mathbb{C}^\times$.

Lemma 2.3.17. *Let $f = \sum_{i=0}^{h} c_i y^i x^{h-i}$ and suppose that one of the following holds*

- *c_h is generic and $c_0 \neq 0$*
- *c_1 is generic and $c_h \neq 0$*

Then the discriminant Δ_f of f is nonzero.

Proof. The discriminant of f is defined as the resultant of f and its derivative f', i.e as the determinant

$$\Delta_f = \frac{1}{c_h}\begin{vmatrix} c_0 & c_1 & \dots & c_{h-2} & c_{h-1} & c_h & 0 & \dots & 0 \\ 0 & c_0 & \dots & c_{h-3} & c_{h-2} & c_{h-1} & c_h & \dots & 0 \\ \vdots & & \ddots & & & & & \ddots & \\ 0 & \dots & 0 & c_0 & c_1 & c_2 & c_3 & \dots & c_h \\ c_1 & 2c_2 & \dots & \dots & hc_h & 0 & \dots & & \\ 0 & c_1 & 2c_2 & \dots & \dots & hc_h & 0 & \dots & \\ 0 & 0 & c_1 & 2c_2 & \dots & 0 & hc_h & 0 & \dots \\ \vdots & & & \ddots & \ddots & & & \ddots & \\ 0 & 0 & \dots & 0 & c_1 & 2c_2 & \dots & & hc_h \end{vmatrix}$$

The terms of Δ_f only involving c_0 and c_h are obtained by setting all the other c_i equal to zero. We immediately see that the only contribution comes from the diagaonal, yielding a summand of the form $h^h(c_0c_h)^{h-1}$. We may write the discriminant as a polynomial p in c_h

$$p = h^h(c_0c_h)^{h-1} + (\text{terms of degree} < h-1 \text{ in } c_h)$$

Since c_h is generic, this can only be zero if all coefficients of p are zero. However, by assumption $c_0 \neq 0$, so the leading coeffcient is nonzero, and hence $\Delta_f \neq 0$ for generic c_h.
Similarly, the terms of Δ_f only involving c_1 and c_h are obtained by setting all other c_i equal to zero. Expanding along the first column gives

$$\frac{c_1}{c_h}\begin{vmatrix} c_1 & 0 & \dots & 0 & c_h & 0 & \dots & 0 \\ 0 & c_1 & 0 & \dots & 0 & c_h & \dots & 0 \\ \vdots & & \ddots & & & & \ddots & \vdots \\ 0 & \dots & 0 & c_1 & 0 & \dots & 0 & c_h \\ c_1 & 0 & \dots & \dots & hc_h & 0 & \dots & \\ 0 & c_1 & 0 & \dots & 0 & hc_h & 0 & \dots \\ \vdots & & \ddots & \ddots & & & \ddots & \\ 0 & \dots & 0 & c_1 & 0 & \dots & 0 & hc_h \end{vmatrix} = Nc_1^h c_h^{h-2}$$

for some integer N (which we will not bother to work out). Since $h > 1$, the rows of the matrix are linearly independent we must have that $N \neq 0$. We may expand the discriminant as a polynomial p in c_1:

$$p = Nc_1^h c_h^{h-2} + (\text{terms of degree} < h \text{ in } c_1)$$

The same argument as before now shows that if $c_h \neq 0$ and c_1 is generic, then $\Delta_f \neq 0$. $\square$

Proposition 2.3.18. *Let P be a Fano polygon, and let g with* $\mathrm{Newt}\,(g) = P$ *be a Laurent polynomial that is generic subject to the condition that g be mutable at $p \in D_E$. Then g has an ordinary point of multiplicity h at p, and D_E is not tangent to any branch of g at p.*

Proof. Choose local coordinates x and y around a torus fixed point v corresponding to a maximal cone σ as before, let E' be the other edge containing p, and write $h = h_E, h' = h_{E'}$. Suppose that $p = (-\lambda, 0)$ in these coordinates, then g is mutable with respect to the factor $\lambda + x$. This means that g is of the form

$$g = \sum_{i=-n}^{0} y^i(\lambda + x)^{-i} r_i(x) + \sum_{i=1}^{m} y^i r_i(x) \tag{2.6}$$

for generic Laurent polynomials $r_i(x), -n \leq i \leq m$. The condition that $\mathrm{Newt}\,(g) = P$ implies that $n = h$ and

$$(\lambda + x)^{-i} r_i(x) = cx^{-h'} + (\text{terms of degree} > -h')$$

for some $c \in \mathbb{C}^\times$. We want to find a local equation for the zero locus of the section $g \in \mathcal{O}(D_P)$. Note that

$$\mathcal{O}(D_P)_{|U_\sigma} = y^{-h} x^{-h'} \mathbb{C}[x, y]$$

it follows that we obtain a local equation for g by multiplying (2.6) through by $y^h x^{h'}$, giving

$$\sum_{i=0}^{h} y^i(\lambda + x)^{h-i} \tilde{r}_{i-h}(x) + \sum_{i=1}^{m} y^{i+h} \tilde{r}_i(x) = 0 \tag{2.7}$$

where $\tilde{r}_{i-h}(x) = r_{i-h}(x)x^{h'}$ is now a polynomial. Making the coordinate change $x \mapsto \lambda + x$, we see that

$$\begin{aligned} g &= \sum_{i=0}^{h} y^i x^{h-i} \tilde{r}_{i-h}(x - \lambda) + (\text{terms of total degree at least } h + 1) \\ &= \sum_{i=0}^{h} c_i y^i x^{h-i} + (\text{terms of total degree at least } h + 1) \end{aligned}$$

where $c_i = \tilde{r}_{i-h}(-\lambda)$. This shows that the point $p = (-\lambda, 0)$ (before the coordinate change) is a point of multiplicity at least h.

To prove the remaining statements, we consider the discriminant Δ of the polynomial $\sum_{i=0}^{h} c_i y^i x^{h-i}$. p is an ordinary h-uple point if and only if $\Delta \neq 0$. Note that c_h contains a summand of the form $\mathrm{const}(g) \cdot \lambda^{h'}$. Since $\mathrm{const}(g)$ is generic by assumption and $\lambda \neq 0$, it follows that c_h is generic. Since g is generic, $\tilde{r}_{i-h}$ is a generic polynomial of degree $\ell_E - h$, and therefore $c_0 = \tilde{r}_{i-h}(-\lambda) \neq 0$. By Lemma 2.3.17, this implies that $\Delta \neq 0$. We also see that y is not a factor of Δ, so that D_E is not tangent to g at p, as required. □

Proposition 2.3.19. *Let P be a Fano polygon, let E be an edge of P at height h, and let g be a be a Laurent polynomial with* Newt $(g) = P$ *that is generic subject to the condition that g be mutable with respect to $kp \in D_E$. Then p is an ordinary point of multiplicity h and exact k-fold tangency along D_E.*

Proof. Choose local coordinates as before around a torus fixed point of P contained in E as before, then g must be of the form

$$g = \sum_{i=-n}^{0} y^i (\lambda + x)^{-ki} r_i(x) + \sum_{i=1}^{m} y^i r_i(x)$$

and the same argument as in Proposition 2.3.20 shows that

$$g = \sum_{i=0}^{h} c_i y^i x^{k(h-i)} + (\text{terms of degree at least } kh + 1)$$

where $c_i = \tilde{r}_{i-h}(-\lambda)$. As before we must have that c_h is generic, and since g is generic, $\tilde{r}_{i-h}$ is a generic polynomial of degree $\ell_E - hk$, and therefore $c_0 = \tilde{r}_{i-h}(-\lambda) \neq 0$. By Lemma 2.3.17, the discriminant of the polynomial $\sum_{i=0}^{h} c_i y^i z^{(h-i)}$ is nonzero, and therefore p is an ordinary point of multiplicity h and exact k-fold tangency. □

Proposition 2.3.20. *Let P be a Fano polygon, let E be an edge of P at height h, and let Z be a maximal admissible zero cycle on P. Let g with* Newt $(g) = P$ *be mutable with respect to Z. Suppose that $p \in Z \cap D_E$ appears in Z with multiplicity k.*

If g has generic coefficients along residual points of P, then p is an ordinary point of multiplicity h with k-fold tangency along D_E.

If moreover P does not support an R-cone, then p is an ordinary point of multiplicity h with exact k-fold tangency along D_E.

Proof. The restriction of Z to D_E is a zero-cycle of degree $m = m_E$, let $p = -\lambda_0, -\lambda_1, \ldots, -\lambda_m$ be the coordinates of the corresponding (not necessarily distinct) points on D_E. The condition that g is mutable with respect to Z implies that

$$g_{-h} = \prod_{i=1}^{m} (\lambda_i + x)^h \tilde{r}_{-h} \tag{2.8}$$

where $\tilde{r}_{-h}$ is a nonzero polynomial of degree r_E. The same argument as in Corollary 2.3.19 shows that g has a point of multiplicity h at $x = -\lambda_i$ and locally has a Taylor expansion of the form

$$g = \sum_{i=0}^{h} c_i y^i x^{k(h-i)} + (\text{terms of degree at least } h+1)$$

where $c_i = \tilde{r}_{i-h}(-\lambda_0)$.

It remains to show that the discriminant Δ of the polynomial $\sum_{i=0}^{h} c_i y^i z^{(h-i)}$ is nonzero. As before, c_h is generic because the origin is a residual point. If E does not support any R-cone, then $\tilde{r}_{-h}$ is constant and we must have $c_0 \neq 0$ (otherwise we couldn't have $\operatorname{Newt}(g) = P$), and Lemma 2.3.17 implies that $\Delta \neq 0$. We also see that y is not a factor of Δ (i.e $c_0 \neq 0$), so that then each branch of g has exact tangency k to D_E at p. If E supports an R-cone, it might happen that $c_0 = 0$ (see Example 2.3.22). However, by Lemma 2.3.10, the R-cone contains a lattice point at height $h-1$, so that $\tilde{r}_{-h+1}$ has generic leading term and c_1 is generic. Lemma 2.3.17 now implies that $\Delta \neq 0$ in this case as well. □

We obtain the following Corollary for T-polygons

Corollary 2.3.21. *Let P be a T-polygon, let Z be a maximal admissible zero-cycle and let g be any (not necessarily generic) Laurent polynomial of Tveiten class. Suppose that p appears in the mutable cycle with multiplicity k, and let $\lambda \in \mathbb{C}$ be generic. Then $g + \lambda$ satisfies the conclusion of Proposition 2.3.20.*

Proof. Since P is a T-polygon, the origin is the only residual point on P, and hence $g + \lambda$ has generic coefficients at residual points. So Proposition 2.3.20 applies. □

Example 2.3.22. To illustrate Proposition 2.3.20, consider the polygon P from Example 2.3.9. Figure 2.7 is a sketch of the curve $g = 0$ close to the toric boundary, where g is of Tveiten class with $\operatorname{Newt}(g) = P$, has generic coefficients along residual points and has

- generic mutable cycle (on the left)
- mutable cycle supported entirely on the points $[-1 : 1] \in D_E \cong \mathbb{P}^1$ (on the right)

The bottom edge E of P has $m_E = 1$ and $h_E = 3$, so Proposition 2.3.20 implies that the curve $g = 0$ has an ordinary triple point along D_E. However, in the right picture, the condition that

g be mutable implies that the restriction of g to D_E is of the form

$$(1+x)^3(\alpha+\beta x)$$

for some $\alpha, \beta \in \mathbb{C}^\times$. Enforcing mutability along each edge shows that the coefficient of g at the bottom left vertex must be equal to the coefficient at the bottom right vertex, so that $\alpha = \beta$. It follows that $g = 0$ meets D_E only at the point $[-1:1] \in D_E \cong \mathbb{P}^1$. In the notation of the proof of Proposition 2.3.20) we have that $c_0 = 0$ for the bottom edge E. The curve $g = 0$ still has an ordinary triple point along the corresponding divisor D_E, but one branch of the curve now has second order tangency along D_E.

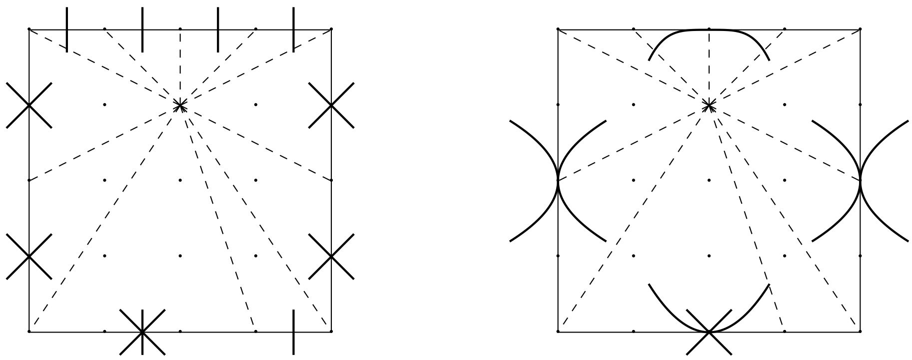


Figure 2.7

We now define maximally mutable Laurent polynomials (MMLP).

Definition 2.3.23. Let P be a Fano polygon. A Laurent polynomial g with $\mathrm{Newt}(g) = P$ is *maximally mutable* if every mutation of P is induced by an algebraic mutation φ_f of g and moreover, the factor f of φ_f can be taken to be $f = (1+x^u)^k$ for a primitive generator $u \in \mathbb{C}[v^\perp]$ and some $k \in \mathbb{Z}_{>0}$.

Suppose that g with $\mathrm{Newt}(g) = P$ is maximally mutable. Then we see that for $0 \leq d \leq h$, we must have up to a unit that

$$g_{-d} = \prod_{i=1}^{m}(1+x)^d \cdot r_{-d} = (1+x)^{dm} \cdot r_{-d}$$

where $r_{-d} \in \mathbb{C}[x]$. In particular, if P is a T-polygon, then $g_{-h} = c(1+x)^{mh}$ for some $c \in \mathbb{C}^\times$. In terms of the mutable cycle, we see that g is maximally mutable if and only if the mutable cycle Z of g is maximal and $Z \cap D_E$ is supported on the point $[-1:1] \in D_E \cong \mathbb{P}^1$.
We summarize the above in the following theorem, which is originally due to [Tve15] and [AK14].

Theorem 2.3.24. *Let P be a T-polygon. The MMLPs g with* $\mathrm{Newt}(g) = P$ *is two dimensional.*

If g is generic, then g has an ordinary h_E-uple point with m_E-fold tangency along D_E at $[-1,1] \in D_E \cong \mathbb{P}^1$.

Proof. The mutable cycle Z with $Z \cap D_E = m_E \cdot [-1:1]$ is maximal admissible, and satisfies condition (2.3). Proposition 2.3.7 implies that the space of MMLPs is two dimensional. The second part then follows from Proposition 2.3.20. □

Definition 2.3.25. A MMLP is *normalized* if the coefficient of g at each quasi-residual vertex of $\mathrm{Newt}\,(g)$ is 1, and the coefficient of the constant monomial is 0.

Since the mutable cycle of a MMLP is supported on the points $[-1:1] \in D_E$, it follows that the coefficients of a normalized MMLP along *all* vertices are equal to 1. Suppose that g with $\mathrm{Newt}\,(g) = P$ is normalized maximally mutable. In the same notation as before, we must have that up to a unit

$$g_{-h} = \prod_{i=1}^{m} (1+x)^h \cdot r_{-h} = (1+x)^{mh} r_{-h}$$

where $r_{-h} = 1 + a_1 x + \cdots + a_{r_E} x^{r_E - 1} + x^{r_E}$. In particular, if P is a T-polygon, then $g_{-h} = (1+x)^{mh}$. More generally, we immediately deduce from Corollary 2.3.24:

Corollary 2.3.26. *Let P be a T-polygon. There is a unique normalized MMLP g with* $\mathrm{Newt}\,(g) = P$.

Proof. The affine space of normalized MMLPs is obtained from the two-dimensional space of MMLPs by setting the two coordinates equal to 0 and 1. □

We make the following definition

Definition 2.3.27. Let D be the toric boundary of the toric variety $\bar{Y}_P$, let $p \in D^{int}$ and $m \geq 1$. The blowup of $\bar{Y}_P$ along $m \cdot p$ is defined as an iterated blowup

$$Y_m \to Y_{m-1} \to \cdots \to Y_1 \to Y_0 = \bar{Y}_P$$

where $Y_1 \to \bar{Y}_P$ is the blowup of $\bar{Y}_P$ at p and $Y_{i+1} \to Y_i$ is the blowup of Y_i along the intersection of the strict transform of D under $Y_i \to \bar{Y}_P$ and the exceptional divisor of $Y_i \to Y_{i-1}$.
Given a zero-cycle $Z = \sum_i m_i p_i$ supported on D^{int}, we similarly define the blowup of $\bar{Y}_P$ along Z as m_i iterated blowups at each p_i.

Definition 2.3.28. Given a Fano polygon P, and a generic Laurent polynomial f of Tveiten class with $\mathrm{Newt}\,(f) = P$, we define the surface $q \colon \tilde{Y}_Z \to \bar{Y}_P$ to be the blowup along the mutable cycle Z of f.

Let C be the curve in $\bar{Y}_P$ defined by f. Proposition 2.3.20 combined with the discussion after Corollary 2.3.15 shows that the strict transform $\tilde{C}$ of C under q is a desingularization of C. The genus of this curve has been computed by Tveiten [Tve15, Theorem 3.5]:

Theorem 2.3.29. *The genus of the curve $\tilde{C}$ is equal to the number of residual points of P.*

Proof. Let $q\colon \tilde{Y}_Z \to \bar{Y}_P$ be the blow-up of the mutable cycle of f. We have

$$\tilde{C} = \pi^* C - \sum_{E \subset P} \sum_{i=1}^{m_E} h_E E_i$$

on $\tilde{Y}_Z$, where the E_i are the pullbacks of the exceptional divisors. We compute

$$\tilde{C}^2 = (\pi^* C - \sum_{E \subset P} \sum_{i=1}^{m_E} h_E E_i)^2 = C^2 - \sum_{E \subset P} m_E h_E^2$$

$$K_{\tilde{Y}_Z} \cdot \tilde{C} = (\pi^* K_{\bar{Y}_P} + \sum_{E \subset P} \sum_{i=1}^{m_E} E_i) \cdot \tilde{C} = K_{\bar{Y}_P} \cdot C + \sum_{E \subset P} h_E m_E$$

Using the adjunction formula, we obtain

$$g(\tilde{C}) = \tfrac{1}{2}\tilde{C}(\tilde{C} + K_{\tilde{Y}_f}) + 1 = \tfrac{1}{2}C(C + K_{\bar{Y}_P}) + 1 - \tfrac{1}{2}\sum_{E \subset P} m_E h_E (h_E - 1)$$

The number $\frac{1}{2}C(C+K_{\bar{Y}_P})+1$ equals the genus of a generic Laurent polynomial g with $\mathrm{Newt}\,(g) = P$, which is equal to $|Int(P) \cap M|$ by [CLS11, Proposition 10.5.4]. Since a primitive T-cone of height h contains exactly k lattice points at height k for $1 \leq k < h$, the quantity

$$\tfrac{1}{2}\sum_{E \subset P} m_E h_E (h_E - 1)$$

equals the number of lattice points in P interior to T-cones, so that $g(\tilde{C})$ is equal to the number of residual points on P, as required. □

We can use similar ideas to give an alternative proof of Proposition 2.3.7 (almost, see the remark after the proof). The crucial point is the following: Fix a Fano polygon P and a maximal admissible zero-cycle. Then a Laurent polynomials g with $\mathrm{Newt}\,(g) \subset P$ is mutable with respect to Z if and only if g extends to a section of

$$\tilde{D}_P = \pi^* D_P - \sum_{E \subset P} \sum_{i=1}^{m_E} h_E E_i$$

where $\pi\colon \tilde{Y}_Z \to \bar{Y}_P$ is the blowup of the mutable cycle. It follows by Corollary 2.3.8 that the space of Laurent polynomial of Tveiten class for Z is a dense subset of $h^0(\tilde{Y}_Z, \mathcal{O}(\tilde{D}_P))$, so we can compute its dimension using the Riemann-Roch formula.

Proposition 2.3.30. *Let P be a Fano polygon, and suppose there exists a Laurent polynomial g of Tveiten class with* $\mathrm{Newt}\,(g) = P$. *If P is not a T-polygon, then $h^0(\tilde{Y}_Z, \mathcal{O}(\tilde{D}_P))$ is equal to the number of quasi-residual points on P. If P is a T-polygon, then $h^0(\tilde{Y}_Z, \mathcal{O}(\tilde{D}_P)) = 2$.*

Proof. Riemann-Roch for surfaces gives

$$\chi(\mathcal{O}(\tilde{D}_P)) = \chi(\mathcal{O}) + \tfrac{1}{2}\tilde{D}_P \cdot (\tilde{D}_P - K_{\tilde{Y}_Z})$$

and $\chi(\mathcal{O}) = 1$ since $\tilde{Y}_Z$ is rational. We will use [CLS11, Proposition 10.5.6] which says that

$$\begin{aligned} D_P^2 &= 2\mathrm{Area}(P) \\ -K_{\bar{Y}_P} \cdot D_P &= |\partial P \cap M| \end{aligned}$$

We compute

$$\begin{aligned} \tilde{D}_P^2 &= (\pi^* D_P - \sum_{E \subset P} \sum_{i=1}^{m_E} h_E E_i)^2 = 2\mathrm{Area}(P) - \sum_{E \subset P} m_E h_E^2 \\ -K_{\tilde{Y}_Z} \cdot \tilde{D}_P &= (\pi^*(-K_{\bar{Y}_P}) - \sum_{E \subset P} \sum_{i=1}^{m_E} E_i) \cdot \tilde{D}_P = -K_{\bar{Y}_P} \cdot D_P - (\sum_{E \subset P} \sum_{i=1}^{m_E} E_i) \cdot D \\ &= |\partial P \cap M| - \sum_{E \subset P} h_E m_E \end{aligned}$$

Putting everything together gives

$$\chi(\mathcal{O}(\tilde{D}_P)) = 1 + \mathrm{Area}(P) - \tfrac{1}{2} \sum_{E \subset P} m_E h_E (h_E + 1) + \tfrac{1}{2} |\partial P \cap M| \tag{2.9}$$

And using Pick's formula:

$$\mathrm{Area}(P) = |Int(P) \cap M| + \tfrac{1}{2} |\partial P \cap M| - 1$$

we can simplify (2.9) to

$$\chi(\mathcal{O}(\tilde{D}_P)) = |P \cap M| - \tfrac{1}{2} \sum_{E \subset P} m_E h_E (h_E + 1)$$

A primitive T-cone of height h contains in its closure exactly k lattice points at height k for $1 \leq k < h$ and $h+1$ lattice points at height h. This implies that $\frac{1}{2}h(h+1)+1$ is the number of lattice points in the closure of such a T-cone, and it follows that $\chi(\mathcal{O}(\tilde{D}_P)))$ is exactly equal to the number of quasi-residual points of P if P is not a T-polygon, and 1 otherwise.

We now need to compute the dimensions of the individual cohomology groups, for which we follow [Fri16, Lemma 4.13]. Note first that $\tilde{D}_P - K_{\tilde{Y}_Z}$ is effective since $\tilde{D}_P$ is linearly equivalent to the strict transform of D_P and $-K_{\tilde{Y}_Z}$ is linearly equivalent to the strict transform of the toric boundary. Using Serre duality, it follows that $h^2(\mathcal{O}(\tilde{D}_P)) = h^0(\mathcal{O}(-\tilde{D}_P) + K_{\tilde{Y}_Z}) = 0$. Consider now the short exact sequence

$$0 \to \mathcal{O}_{\tilde{Y}_Z} \to \mathcal{O}(\tilde{D}_P) \to \mathcal{O}(\tilde{D}_P)|_{\tilde{D}_P} \to 0$$

which gives rise to an isomorphism $h^1(\mathcal{O}(\tilde{D}_P)) = h^1(\mathcal{O}(\tilde{D}_P)|_{\tilde{D}_P})$ since $\tilde{Y}_Z$ is rational and has vanishing higher cohomology. The adjunction formula for $\mathcal{O}(\tilde{D}_P)$ now gives $K_{\tilde{D}_P} = K_{\tilde{Y}_Z}|_{\tilde{D}_P} \otimes \mathcal{O}(\tilde{D}_P)|_{\tilde{D}_P}$. Using Serre duality on $\tilde{D}_P$, we obtain $h^1(\mathcal{O}(\tilde{D}_P)|_{\tilde{D}_P}) = h^0(K_{\tilde{Y}_Z}|_{\tilde{D}_P})$.

Let C be the curve defined by $g = 0$. Since the base locus of the pencil generated by C and D_P consists of isolated points on the interior of the toric boundary, the strict transform $\tilde{C}$ of a general member of the pencil is smooth away from these basepoints, using Bertini's theorem. If P is not a T-polygon, then the curve C meets the toric boundary at least at one point away from the support of the mutable cycle and it follows that $\tilde{C} \in |\tilde{D}_P|$ meets the strict transform of the toric boundary $\tilde{D} \in |-K_{\tilde{Y}_Z}|$. We conclude that $\tilde{D} \cdot \tilde{C} > 0$ and therefore $h^0(K_{\tilde{Y}_Z}|_{\tilde{D}_P}) = 0$. It follows that $h^0(\tilde{D}_P) = \chi(\mathcal{O}(\tilde{D}_P))$ is equal to the number of quasi-residual points on P.

If P is a T-polygon, then $\tilde{C} \cdot \tilde{D} = 0$, so that we have $h^0(K_{\tilde{Y}_Z}|_{\tilde{D}_P}) = h^0(\mathcal{O}_{\tilde{D}_P}) = 1$, giving

$$\chi(\tilde{D}_P) = |P \cap M| - \tfrac{1}{2} \sum_{E \subset P} m_E h_E(h_E + 1) + 1 = 2$$

as required. □

Remark 2.3.31. While this proof is much more concise than Proposition 2.3.7 it requires the previous knowledge that there exists a Laurent polynomial of Tveiten class to compute the dimension of the space. Moreover, we cannot conclude that the coeffients of a general g can be expressed in terms of linear functions in the quasi-residual points on P, nor can we say anything about the existence of normalized Laurent polynomials.

2.4 The surface of a Laurent polynomial

Let f be a Laurent polynomial with $\mathrm{Newt}\,(f) = P$ a T-polygon and mutable cycle Z. Define Γ_f to be the pencil of sections of $\mathcal{O}(D_P)$ on Y_P generated by f and the constant Laurent polynomial

1. Since the singularities of Y_P are torus fixed points, Γ_f pulls back to a pencil without fixed component on the minimal resolution $\bar{Y}_P$, defining a rational map $\bar{Y}_P \dashrightarrow \mathbb{P}^1$ given by $[f : 1]$. We will now see that the surface $\tilde{Y}_Z$ (defined in the previous section) comes with a proper map to $\mathbb{P}^1$, obtained by resolving this rational map. Even though the surface $\tilde{Y}_Z$ only depends on the base locus of Γ_f, the map $\tilde{Y}_Z \to \mathbb{P}^1$ depends on a specific member $f \in \Gamma_f$, and for this reason, we will from now on denote $\tilde{Y}_Z$ by $\tilde{Y}_f$. We first recall a few definitions.

Given a linear system Γ without fixed components on a smooth projective surface Y and a birational map $q\colon X \dashrightarrow Y$, we define the strict transform $q_*^{-1}\Gamma$ to be the linear system on X whose general member is the birational transform of the general member of Γ. Equivalently, we may decompose $q^*\Gamma = F + M$ into fixed and mobile components, and define $q_*^{-1}\Gamma = M$. The linear system $q_*^{-1}\Gamma$ is generated by the strict transforms of general members of Γ, and both linear systems define the same map to projective space in the sense that the diagram

$$\begin{tikzcd} X \arrow[d, "q"'] \arrow[dr, dashed, "q_*^{-1}\Gamma"] & \\ Y \arrow[r, dashed, "\Gamma"] & \mathbb{P}^N \end{tikzcd}$$

commutes. We have the following result

Proposition 2.4.1. *Suppose that P is a T-polygon and that f with* $\mathrm{Newt}(f) = P$ *is any (not necessarily generic) Laurent polynomial of Tveiten class. Let $q\colon \tilde{Y}_f \to \bar{Y}_P$ be the blowup of the mutable cycle of f. Then the pencil $q_*^{-1}(\Gamma_f)$ is base point free, and the strict transform of D_P is a member of $q_*^{-1}(\Gamma_f)$.*

Proof. The zero scheme $Z(1)$ is equal to D_P, so that the basepoints of Γ_f are exactly the points where f intersects the toric boundary. Suppose that $p = (-\lambda, 0)$ appears in the mutable cycle with multiplicity k. This means that f is mutable with respect to the factor $(\lambda + x)^k$, but not with respect to the factor $(\lambda + x)^{k+1}$. Since P is a T-polygon, Corollary 2.3.21 then shows that on the generic member of Γ_f, p is a point of multiplicity h_E with exact k-fold tangency to D_E so that the general member of Γ_f has an equation of the form

$$\prod_{i=1}^{h}(y + \gamma_i x^k) + r(x, y) = 0$$

for distinct $\gamma_i \in \mathbb{C}^\times$, and some polynomial $r(x, y)$ which consists of monomials of degree at least $kh + 1$ with respect to the grading $\deg(x) = 1, \deg(y) = k$. Note that r must be divisible by y:

indeed, we have that

$$r = s_{-h} + s_{-h+1}y + \cdots + s_0 y^h + (\text{terms of degree at least } h+1 \text{ in } y)$$

where s_{-i} is the polynomial obtained from $\tilde{r}_{-i}$ by deleting the constant term (see the proof of Proposition 2.3.18 for the definition of $\tilde{r}_{-i}$). Since P is a T-polygon, the polynomial $\tilde{r}_{-h}$ is constant so that $s_{-h} = 0$, and we let $r' = r/y$. A similar calculation to Lemma 2.3.14 shows that if $q\colon \tilde{Y} \to \bar{Y}_P$ is the blowup of p, then $q_*^{-1}(\Gamma_f) = q^*(\Gamma_f) - hE$ and the general member of $q_*^{-1}(\Gamma_f)$ is given by

$$\prod_{i=1}^{h}(v + c_i u^{k-1}) + uvs(u,v)$$

where $s(u,v) = r'(u,uv)/u^h$. (Note as before that s is a polynomial since each term in r' has total degree at least h (with respect to the grading $\deg(x) = 1, \deg(y) = 1$). As before, we can also show that the general member of $q_*^{-1}(\Gamma_f)$, $(0,0)$ is a point of multiplicity h_E with $(k-1)$-fold tangency along D_E.

The local equation for D_P is $y^h(x-\lambda)^{h'}$, so that $\tilde{D}_P = \pi^* D_P - hE$ is a member of $q_*^{-1}(\Gamma_f)$, with local equation $v^h(u-\lambda)^{h'}$. This shows that the basepoints of $q_*^{-1}(\Gamma_f)$ are exactly those points where the general member of $q_*^{-1}(\Gamma_f)$ meets the strict transform of the toric boundary. Moreover, if $k > 1$, the only basepoint of $q_*^{-1}(\Gamma_f)$ lying on the exceptional divisor $u = 0$ is the point $(0,0)$, and if $k = 1$, then $q_*^{-1}(\Gamma_f)$ has no basepoint lying on $u = 0$. Inductively, this shows that blowing up the cycle $k \cdot p$ removes the basepoint p. To see that f does not have any basepoints away from the support of the mutable cycle, note that the restriction of f to D_E is given by $f_h = \prod_i (1 + \lambda_i x)^h$, so that the only basepoints on D_E are given by the points $x = -\lambda_i$, which all appear in the mutable cycle. □

Remark 2.4.2. We note that the mutable cycle is in general distinct from the base scheme of Γ_f, although they are supported on the same points. Blowing up the base scheme instead of the mutable cycle in general leads to a non-normal surface and destroys the log Calabi-Yau property that we need later on.

Let us explore the consequences of Proposition 2.4.1. Fix a T-polygon P, let f be of Tveiten class, and let $q\colon \tilde{Y}_f \to \bar{Y}_P$ be the blowup of the mutable cycle of f. Then the rational map $q\colon \bar{Y}_P \dashrightarrow \mathbb{P}^1$ defined by Γ_f extends to a regular map $\pi\colon \tilde{Y}_f \to \mathbb{P}^1$, defined by the base-point free pencil $q_*^{-1}\Gamma_f$. The surface $\tilde{Y}_f$ fits in a diagram

$$\begin{array}{ccc} (\mathbb{C}^\times)^2 & \longrightarrow & \tilde{Y}_f \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \pi} \\ \mathbb{C} & \longrightarrow & \mathbb{P}^1 \end{array}$$

such that the fibre of π over $[s:t] \in \mathbb{P}^1$ is a compactification of the curve $f = \frac{s}{t}$. Moreover, Proposition 2.4.1 also shows that $\pi^*(\infty)$ is the strict transform of D_P. Since P does not have any R-cones, it follows from Theorem 2.3.29 that the genus of the general fibre of π is one, so that $\tilde{Y}_f$ is a fibration by genus 1 curves over $\mathbb{P}^1$. This fibration might not be relatively minimal, but we can contract all (-1)-curves contained in fibres to obtain a surface Y_f. We note the following.

Lemma 2.4.3. *Any (-1)-curve contained in a fibre of $\tilde{Y}_f$ is a component of $\pi^*(\infty)$.*

Proof. Let $\tilde{D}$ denote the strict transform of the toric boundary of $\bar{Y}_P$ under $\tilde{Y}_f \to \bar{Y}_P$. If C is a (-1)-curve contained in a fibre different from $\pi^*(\infty)$, we must have $\tilde{D} \cdot C = 0$, since $\tilde{D}$ is the underlying curve of $\pi^*(\infty)$. However, in order to obtain $\tilde{Y}_f$, we only blew up points in the interior of the toric boundary, or in the interior of a strict transform of the toric boundary, and therefore $\tilde{D} \in |-K_{\tilde{Y}_f}|$. Adjunction then gives $\tilde{D} \cdot C = 1$, a contradiction. □

Given a relatively minimal genus 1 fibration $\pi\colon Y \to C$ over a curve C, Kodaira [Kod66] has classified all possible singular fibres of π (see for example [BHPVdV04, V.7]). It follows from the classification that if a singular fibre is not simply connected, it must be of the form $F = mD$ where D is either an irreducible nodal rational curve or a cycle of n reduced rational (-2)-curves. This means that the intersection matrix of D must be strictly negative semidefinite. If F is reduced, the singular fibre F is denoted I_n, and the corresponding multiple fibre mF is denoted ${}_m\mathrm{I}_n$.

Proposition 2.4.4. *Let f be a Laurent polynomial of Tveiten class with $P = \mathrm{Newt}\,(f)$ a T-polygon, and let $Y_f \to \mathbb{P}^1$ the associated genus 1 fibration. Then the fibre $\pi^*(\infty)$ is of Kodaira type ${}_m\mathrm{I}_n$ for some m and n.*

Proof. Consider first the fibre $(\tilde{Y}_f)_\infty$, which is equal to the strict transform of D_P by Proposition 2.4.1. By Lemma 2.4.3, performing a relative minimal model amounts to blowing down (-1)-curves in $(\tilde{Y}_f)_\infty$ to obtain $(Y_f)_\infty$, which must be one of the singular fibres in Kodaira's list. Since $(Y_f)_\infty$ is not simply connected, the only possibility is that $(Y_f)_\infty$ is of type ${}_m\mathrm{I}_n$. □

Remark 2.4.5. We will show in Proposition 3.2.6 that we must have $m = 1$ and $1 \leq n \leq 9$ in Proposition 2.4.4.

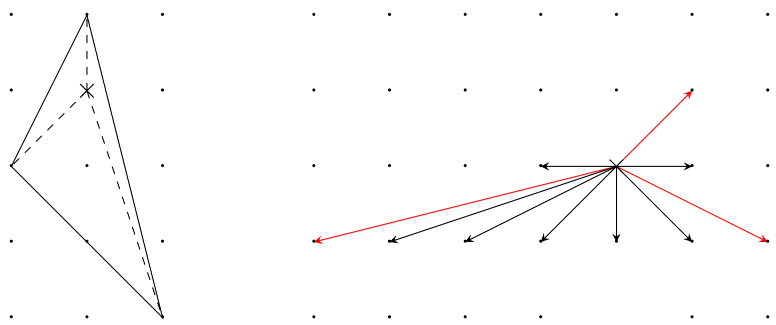


Figure 2.8: On the left a T-polygon P, divided into primitive T-cones. On the right the fan of the minimal resolution $\bar{Y}_P$, with the rays of the fan of Y_P in red.

We summarise all this as follows

Definition 2.4.6. Let f be a Laurent polynomial of Tveiten class with $P = \mathrm{Newt}\,(f)$ a T-polygon, and let Y_P be the toric variety defined by P. Then the associated surface Y_f is constructed by

- Performing a toric resolution of singularities $\bar{Y}_P \to Y_P$
- Blowing up the mutable locus of Γ_f, yielding a genus 1 fibration $\pi\colon \tilde{Y}_f \to \mathbb{P}^1$
- Contracting (-1)-curves contained in fibres of π, yielding a relatively minimal genus 1 fibration $\pi\colon Y_f \to \mathbb{P}^1$

Example 2.4.7. Consider the T-polygon P shown in the left of Figure 2.8. The unique normalized maximally mutable Laurent polynomial with Newton polygon P is $f = y + \frac{1}{xy} + \frac{2}{y^2} + \frac{x}{y^3}$. The fan of the minimal resolution $\bar{Y}_P$ of the toric variety Y_P is shown on the right of Figure 2.8. The generic member of the pencil Γ_f has one basepoint of multiplicity 2 along the edge of P of length 2, and one basepoint of multiplicity 1 along the two other edges. Blowing up these three basepoints, we arrive at the toric surface $\tilde{Y}_f$, as shown in Figure 2.9 (adapted from [Duc21]). The strict transform of the toric boundary $\bar{D}_P$ has 10 components, whose self-intersection numbers are shown in 2.9. Note that the component of self-intersection (-1) appears with multiplicity 2 in $\pi^*(\infty)$. To obtain Y_f, we contract the (-1) curve in $\pi^*(\infty)$. The fibre over ∞ is now a cycle of 9 (-2)-curves, and the elliptic surface $Y_f \to \mathbb{P}^1$ is well-known as the modular elliptic surface associated to the congruence subgroup $\Gamma_1(3)$.

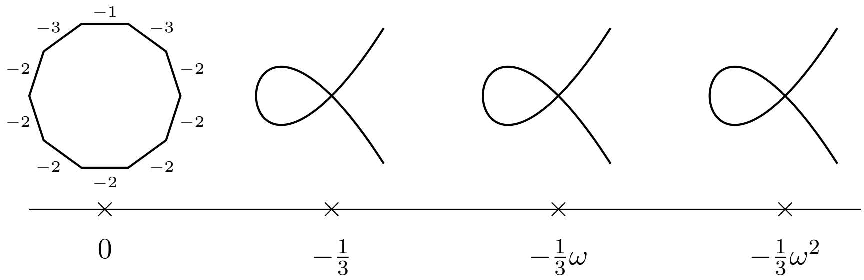


Figure 2.9: The singular fibres of $\tilde{Y}_f \to \mathbb{P}^1$.

Chapter 3

Looijenga pairs and Laurent polynomials

We will now rephrase the constructions of §2 in the language of Looijenga pairs. We start by briefly reviewing the relevant notions, and state the Torelli Theorem for Looijenga pairs of Gross, Hacking, and Keel [GHK14] in §3.1. After that, in §3.2 we associate to a Laurent polynomial f of Tveiten class a Looijenga pair (Y_f, D) and prove it has period point 1. In §3.3 we give a simplified proof of Friedman's classification of negative semidefinite Looijenga pairs.

Let Y be a smooth projective surface and $D \in |-K_Y|$ a singular anticanonical divisor with at worst nodal singularities. The pair (Y, D) is called a *Looijenga pair*. D is either a nodal curve, or a cycle of n rational curves. An orientation of (Y, D) is a choice of generator for $H_1(D, \mathbb{Z}) \cong \mathbb{Z}$, and a labeling of (Y, D) is an indexing $D = D_1 + \cdots + D_n$ of the components of D. An isomorphism $(Y, D) \cong (Y', D')$ is an isomorphism $f\colon Y \to Y'$ such that $f(D_i) = D_i'$ for all $i = 1, \dots, n$ and f is orientation-preserving. We fix a labeling and orientation of (Y, D) throughout. If Y is a toric surface with $D = Y \setminus (\mathbb{C}^\times)^2$ its toric boundary, then (Y, D) is called a *toric pair*. Given a Looijenga pair (Y, D), there are two elementary operations to produce another Looijenga pair:

- Let $p\colon Y' \to Y$ be the blowup of Y at a smooth point of D. Denoting by D' the strict transform of D, the pair (Y', D') is again a Looijenga pair. The map p is called an *interior blowup.*
- Let $p\colon Y' \to Y$ the blowup of a node of D. Denoting by D' the reduced inverse image of D, the pair (Y', D') is again a Looijenga pair. The map p is called a *corner blowup.*

By the birational classification of surfaces, Y must be rational, so that we have $\mathrm{Pic}(Y) =$

$H^2(Y, \mathbb{Z})$.

Remark 3.0.1. In the literature, corner blowups are often called toric blowups. To avoid confusion, we reserve this notion for the blowup of a toric surface along a torus fix point (which is a special case of a corner blowup).

Given a Looijenga pair (Y, D) it is shown in [GHK14, Proposition 1.3] that we may always find a diagram

$$(Y, D) \xleftarrow{p} (\tilde{Y}, \tilde{D}) \xrightarrow{p'} (\bar{Y}, \bar{D})$$

where p is a composition of corner blowups, p' is a composition of interior blowups, and the pair $(\bar{Y}, \bar{D})$ is a toric pair. If p is the identity map, we call p' a *toric model* for (Y, D).
Let $\pi\colon \mathcal{Y} \to S$ be a family of smooth projective surfaces and suppose that $\mathcal{D}$ is a relative anticanonical divisor with normal crossings on $\mathcal{Y}$ (i.e $\mathcal{D}$ restrict to a nodal anticanonical divisor on each fibre of π). We say that $(\mathcal{Y}, \mathcal{D})$ is a *family of Looijenga pairs* if the family $\pi_{|\mathcal{D}}$ is locally trivial on S. In particular, this implies that each anticanonical divisor D_s has the same number of components.
The adjunction formula implies that an exceptional curve E on Y (i.e $E \cong \mathbb{P}^1$ and $E^2 = -1$) is either a component of D, or meets D transversely in one point $p \in D^{int}$. In the latter case, we say that E is an *interior exceptional curve.* A (-2)-curve on Y disjoint from D will be called an *internal* (-2)-*curve.*
Given a toric model $(Y', D') \to (Y, D)$, each connected component of the exceptional locus is a chain $E_1 + \cdots + E_r$ of smooth rational curves (where r is the number of times we blow up the corresponding point), and we have $E_r^2 = -1$ and $E_i^2 = -2$ for $i \neq r$. The curves

$$C_1 = E_r, \quad C_2 = E_r + E_{r-1}, \dots, C_r = E_1 + \cdots + E_r$$

then all have self-intersection (-1) and are called the exceptional curves for this toric model.
We say that a Looijenga pair (Y, D) is negative definite if the matrix with entries $(D_i \cdot D_j)_{1 \leq i,j \leq n}$ is negative definite. Similarly, we call (Y, D) is negative semidefinite if the matrix with entries $(D_i \cdot D_j)_{1 \leq i,j \leq n}$ is negative semidefinite. We say that (Y, D) is positive if it is not negative semidefinite. Looijenga pairs have been classified to a certain extent by Mandel [Man19]. We will be most interested in strictly negative semidefinite (Y, D). Such a pair must satisfy $D^2 = 0$ and D is either an irreducible nodal 0-curve, or a cycle of (-2)-curves.

3.1 Torelli for Looijenga pairs

In this section, we review the Torelli Theorem for Looijenga pairs of Gross, Hacking, and Keel [GHK14].

Following [GHK14], Definition 1.7 we define the cone C^+ to be the connected component of $\{x \in \mathrm{Pic}(Y)_{\mathbb{R}} \mid x^2 > 0\}$ containing all the ample classes. For a given ample H define an *effective numerical exceptional curve* as a class E with $E^2 = K_Y \cdot E = -1$, and $E \cdot H > 0$. This notion is independent of H, see [GHK14, Lemma 2.13]. Let $C^{++} \subset C^+$ be the subcone defined by the inequalities $x \cdot E \geq 0$ for all effective numerical exceptional curves E, and let C_D^{++} be the subcone of C^{++} defined by $x \cdot D_i \geq 0$ for all i. The cone C_D^{++} should be thought of as an enlargement of $\mathrm{Nef}(Y)$ invariant under deformation (see [GHK14, Lemma 2.13]). Indeed, $\mathrm{Nef}(Y)$ is the subcone of C_D^{++} defined by the inequalities $x \cdot \alpha \geq 0$ for $\alpha \in \Delta_Y$, where Δ_Y is the set of internal -2-curves on Y. Since such a -2-curve might move to a difference of two (-1)-curves under parallel transport in a family of Looijenga pairs, $\mathrm{Nef}(Y)$ is in general not preserved under deformation. Note also that a generic Looijenga pair (Y, D) does not support any internal -2-curves, so that $\mathrm{Nef}(Y) = C_D^{++}$ for generic (Y, D). This explains the name 'generic ample cone' for C_D^{++} used in [Fri16]. The cone C^{++} determines the pair (Y, D) up to deformation:

Proposition 3.1.1. *[Fri16, Theorem 5.13], Two Looijenga pairs (Y, D) and (Y', D') are deformation equivalent if and only if there exists a labeling of D and D' and an integral isometry $\mu\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(Y')$ satisfying $\mu([D_i]) = [D_i']$ and $\mu(C^{++}) = C^{++}$.*

If we fix a toric model, then we can obtain deformation equivalent Looijenga pairs by simply varying the location of the blowups on the toric boundary. Therefore, Proposition 3.3.8 is most useful in the absense of a canonical choice of toric model. To distinguish Looijenga pairs in the same deformation family, we introduce the period point of a Looijenga pair:

Definition 3.1.2. Let (Y, D) be a Looijenga pair. Define the lattice

$$\Lambda = \{L \in \mathrm{Pic}(Y) \mid L \cdot D_i = 0 \text{ for all } i\}$$

By Lemma 2.1.1, the orientation of D induces a canonical identification $\mathrm{Pic}^0(D) = \mathbb{C}^\times$. The map

$$\phi_Y \colon \Lambda \to \mathrm{Pic}^0(D) = \mathbb{C}^\times, \quad L \mapsto L_{|D}$$

is called the *period point* $\phi_Y \in \mathrm{Hom}(\Lambda, \mathbb{C}^\times)$ of (Y, D).

The main result of [GHK14] is that the period point determines a Looijenga pair in a deformation family up to isomorphism:

Theorem 3.1.3. *[GHK14, Theorem 1.8](Weak Torelli Theorem) Let (Y, D) and (Y', D') be Looijenga pairs and let*

$$\mu\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(Y')$$

be an isomorphism of lattices. There exist an isomorphism of pairs $f : (Y, D) \to (Y', D')$ iff all of the following hold:

1. *$\mu([D_i]) = [D_i]$ for all i.*

2. *$\mu(C^{++}) = C^{++}$.*

3. *$\phi_{Y'} \circ \mu = \phi_Y$.*

Remark 3.1.4. In order to conclude $\mu = f^*$ in the statement above, one needs to add the condition that $\mu(\Delta_Y) = \Delta_{Y'}$. For our purposes, the weaker conclusion of Theorem 3.1.3 will be sufficient.

3.2 The Looijenga pair associated to a Laurent polynomial

We will now rephrase the construction of Y_f in the language of Looijenga pairs.
Consider the smooth toric surface $\bar{Y}_P$ as a Looijenga pair $(\bar{Y}_P, \bar{D})$, where $\bar{D}$ is the toric boundary of $\bar{Y}_P$. Recall from 2.4.1 that $q\colon \tilde{Y}_f \to \bar{Y}_P$ is constructed by blowing up the mutable cycle on $\bar{Y}_P$, which is a zero-cycle on $\bar{Y}_P$ supported in the interior of $\bar{D}$. q is a composition of interior blowups, so $\tilde{D}$, the strict transform of $\bar{D}$ under q, satisfies $\tilde{D} \in |-K_{\tilde{Y}_f}|$. It follows that $q\colon (\tilde{Y}_f, \tilde{D}) \to (\bar{Y}_P, \bar{D})$ is a toric model. Since $\tilde{D}$ is the underlying reduced curve of $D_P = \pi^*(\infty)$, Lemma 2.4.3 implies that the morphism $\tilde{Y}_f \to Y_f$ contracting all (-1)-curves contained in fibres of π is a corner blowdown $(\tilde{Y}_f, \tilde{D}) \to (Y_f, D)$. We summarise all this as follows.

Definition 3.2.1. Let f be a Laurent polynomial of Tveiten class with $P = \operatorname{Newt}(f)$ a T-polygon, let Y_P be the toric variety defined by P, and let $\bar{Y}_P$ be the toric minimal resolution. Then the associated Looijenga pair (Y_f, D) is constructed by

- Blowing up the mutable cycle on $\bar{Y}_P$, yielding a toric model $(\tilde{Y}_f, \tilde{D}) \to (\bar{Y}_P, \bar{D})$, with a genus 1 fibration $\pi\colon \tilde{Y}_f \to \mathbb{P}^1$

- Blowing down (-1)-curves contained in fibres of π, yielding a corner blowdown $(\tilde{Y}_f, \tilde{D}) \to (Y_f, D)$ with a relatively minimal genus 1 fibration $\pi\colon Y_f \to \mathbb{P}^1$, such that $mD = \pi^*(\infty)$ for some $m > 0$.

Since D is a fibre of Kodaira type ${}_m\mathrm{I}_n$ for some $n > 0$, the pair (Y_f, D) is strictly negative semidefinite.

We have the following result:

Proposition 3.2.2. *Let P be a T-polygon. The Looijenga pairs (Y_f, D) with f of Tveiten class and* $\mathrm{Newt}\,(f) = P$ *are all deformation equivalent.*

Proof. Let Y_P be the toric variety associated to P as before and label the components of the toric boundary as $D_0, \ldots, D_r$. As before, write $\ell_{D_E} = m_E h_E + r_E$, where $0 \leq r_E < h_E$. Since f is of Tveiten class, the mutable cycle of f is a cycle of degree $m_E h_E$ on D_E^{int}. We first show the following: Given a smooth surface Y and a twice punctured smooth rational curve $D^\circ \cong \mathbb{C}^\times$ in Y, there is a smooth family $\mathcal{Y} \to \mathbb{C}^\times$ such that the fibre over $\lambda \in \mathbb{C}^\times$ is the blowup of $\lambda \in D^\circ \cong \mathbb{C}^\times$ in Y.
Choose local coordinates x and y on an open set $U \subset Y$ around one of the punctures such that D° is given by $y = 0$. Then the required family is given by

$$\begin{array}{c} \mathcal{Y} = (x - \lambda)t + ys = 0 \subset U \times \mathbb{P}^1_{s:t} \times \mathbb{C}^\times_\lambda \\ \big\downarrow \\ \mathbb{C}^\times_\lambda \end{array}$$

Note that $\mathcal{D} = \{t = 0\} \subset \mathcal{Y}$ is a divisor on $\mathcal{Y}$ which restricts to the strict tranform of D° on each fibre, so that we have actually constructed a family of pairs $(\mathcal{Y}, \mathcal{D}) \to \mathbb{C}^\times$.
Resolve singularities of Y_P to obtain a smooth toric pair $(\bar{Y}_P, \bar{D})$. Iterating the construction above, we obtain a family of pairs

$$(\tilde{\mathcal{Y}}, \tilde{\mathcal{D}}) \to (D_0^{int})^{a_1} \times \cdots \times (D_r^{int})^{a_r} \cong (\mathbb{C}^\times)^N$$

where $N = \sum_E m_E h_E$ and the fibre over $(p_{11}, \ldots, p_{ra_r})$ is the pair (Y, D) obtained by blowing up the zero-cycle $Z = \sum_{i,j} p_{ij}$ on $(\bar{Y}_P, \bar{D})$, The mutable cycle of f is a zero cycle of that form, so $(\tilde{Y}_f, D)$ is isomorphic to a fibre of the family $(\tilde{\mathcal{Y}}, \tilde{\mathcal{D}})$. By construction, the family $\tilde{\mathcal{D}} \to \mathbb{C}^N$ is trivial and restricts to an anticanonical divisor on each fibre, so that $(\tilde{\mathcal{Y}}, \tilde{\mathcal{D}})$ is a family of Looijenga pairs.
Finally, if $\tilde{D} \to D$ is a toric blowdown, then we may contract the corresponding component in every fibre (this is well-defined since the family $\tilde{\mathcal{D}} \to \mathbb{C}^N$ is trivial, and the fibres have the same self-intersection sequences). This gives rise to a divisorial contraction $(\tilde{\mathcal{Y}}, \tilde{\mathcal{D}}) \to (\mathcal{Y}, \mathcal{D})$ over $(\mathbb{C}^\times)^N$, such that (Y_f, D_f) is isomorphic to a fibre of this family, as required. □

We now state the main result of this section, that if f is maximally mutable, then the period point of the surface (Y_f, D) is equal to 1. This follows easily from the results of [GHK14], but we first need a further definition.

Definition 3.2.3. Given a Looijenga pair (Y, D), a *marking* of D is a choice of point $p_i \in D_i$ for all i. Given a marking of D, we can define $\phi \in \operatorname{Hom}(\operatorname{Pic}(Y), \operatorname{Pic}^0(D))$ by

$$\phi(L) = (L_{|D}) \otimes \bigotimes_{i=1}^{n} \mathcal{O}_D(-(L \cdot D_i)p_i)$$

the *marked period point* of (Y, D, p_i). Note that $\phi_{|\Lambda} = \phi_Y$ is the period point of Y as defined before.

Proposition 3.2.4. *Let P be a lattice polygon and f be a maximally mutable Laurent polynomial with* Newt $f = P$. *Then the Looijenga pair (Y_f, D) associated to f has period point $\phi_{Y_f} = 1$, i.e $\phi_{Y_f} \colon \Lambda \to \mathbb{C}^\times$ is the constant function* 1.

Proof. Recall the diagram in Proposition 3.2.1 summarising the construction of (Y_f, D). We start with the toric pair $(\bar{Y}_P, \bar{D})$ and then blow up the mutable cycle of f to obtain a Looijenga pair $(\tilde{Y}_f, \tilde{D})$. A choice of orientation of $\bar{D}$ gives rise to a canonical identification $D_i^{int} \cong \mathbb{C}^\times$ by Proposition 2.1.1. Let m_i correspond to (-1) under this identification. Let $\tilde{\phi} \in \operatorname{Hom}(\tilde{Y}_f, \mathbb{C}^\times)$ be the marked period point of $(\tilde{Y}_f, \tilde{D}, m_i)$ (where we identify m_i with the corresponding point on the strict transform $\tilde{D}$ of D). [GHK14, Lemma 2.8] shows that the marked period point of $(\bar{Y}_P, \bar{D}, m_i)$ is 1, so $\tilde{\phi}_{|\bar{Y}_P} = 1$. f is maximally mutable, so the mutable cycle of f is supported on the points m_i. It follows that in the construction of $(\tilde{Y}_f, \tilde{D})$ we only blow up the points m_i or points on a strict transform of D lying over m_i, so for any exceptional curve E meeting D_i we have that $\tilde{\phi}(E) = \mathcal{O}_{\tilde{D}}(m_i) \otimes \mathcal{O}_{\tilde{D}}(-m_i) = \mathcal{O}_{\tilde{D}}$. Since the exceptional curves together with $\operatorname{Pic}(\bar{Y}_P)$ generate $\operatorname{Pic}(\tilde{Y}_f)$ we conclude that $\tilde{\phi} = 1$ and hence also that $\phi_{\tilde{Y}_f} = \tilde{\phi}_{|\Lambda} = 1$. Finally, recall that in order to pass to (Y_f, D), we perform a composition of toric blowdowns $\pi \colon (\tilde{Y}_f, \tilde{D}) \to (Y_f, D)$. This gives a canonical identification between the lattices $\Lambda_{\tilde{Y}_f}$ and Λ_{Y_f} via π^* and an isomorphism $\pi^* \colon \operatorname{Pic}^0(D) \to \operatorname{Pic}^0(\tilde{D})$. The period points are then the same in the sense that $\pi^* \circ \phi_{Y_f} = \phi_{\tilde{Y}_f} \circ \pi^*$, so we have that $\phi_{Y_f} = 1$ as well. □

Remark 3.2.5. The statement that the period point of (Y_f, D) is 1 is equivalent to the statement that the mixed Hodge structure on $H^2(Y_f \setminus D)$ is *split*, see [Fri16, Proposition 3.12].

Recall from Definition 3.2.1 that Y_f admits a relatively minimal genus 1 fibration $Y_f \xrightarrow{\pi} \mathbb{P}^1$

Proposition 3.2.6. *Suppose that f is of Tveiten class and let $\pi \colon Y_f \to \mathbb{P}^1$ be the associated genus* 1 *fibration. Then the fibre $\pi^*(\infty)$ is of Kodaira type* I_n *for $1 \leq n \leq 9$. In particular, $\pi^*(\infty) = D$, and π admits a section.*

Proof. Suppose first that f is maximally mutable. By Proposition 2.4.4, we have that $\pi^*(\infty) = mD$ for some $m > 0$, and therefore $D^2 = 0$ and hence $[D] \in \Lambda$. Applying the period map, we obtain

$$\phi_{Y_f}([D]) = \mathcal{O}(D)_{|D}$$

Since f is maximally mutable, $\phi_{Y_f} \equiv 1$, so in particular $\mathcal{O}(D)_{|D}$ is trivial. However, [BHPVdV04, III, Lemma 8.3] shows that if $\pi^{-1}(\infty) = mD$ then $\mathcal{O}(D)_{|D}$ is torsion of exact order m. We conclude that $m = 1$. Equivalently, all components of the strict transform of D_P on Y_f must have multiplicity 1. However, this characterization is independent of the choice of mutable cycle, so we conclude that $m = 1$ for any f of Tveiten class. It follows that $-K_{Y_f}$ is the class of a fibre $[F]$. Therefore, any irreducible (-1)-curve E (for example, the exceptional divisor of the last blowup) on Y_f satisfies $E \cdot F = 1$ by adjunction and so E is a section of Y_f. Finally, since the Picard rank of Y_f is 10, we see that $n \leq 9$. □

3.3 Deformation Families of Looijenga pairs and Lattice Theory

In Section 3.2 we have associated to a Laurent polynomial of Tveiten class a strictly negative semidefinite Looijenga pair (Y_f, D). We now want to show that there are exactly 10 deformation families of strictly negative semi-definite Looijenga pairs (Y, D). We assume throughout that (Y, D) is *minimal*, i.e that D does not contain any (-1)-components. We proceed by associating to (Y, D) a root sublattices $A_r \subset E_8$ up to isometry, and then derive the result from the classification of root sublattices of E_8. This result has first appeared in [Fri16]. We give a slightly different, more elementary proof here. See [Dol12, Chapter 8] for a good introduction to lattice theory in the context of Algebraic Geometry.

Throughout this section, a lattice L is an abelian group together with a symmetric nondegenerate bilinear form. A class $\alpha \in L$ is called a *root* if $\alpha^2 = -2$. A negative definite lattice that is generated by its roots is called a *root lattice*. Given a Dynkin diagram of type ADE, we can contruct an associated root lattice as follows: label the vertices as $\alpha_0, \dots, \alpha_n$ and define $R = \mathbb{Z}\alpha_0 \oplus \cdots \oplus \mathbb{Z}\alpha_n$. The intersection product on R is defined by $\alpha_i^2 = -2, \alpha_i \cdot \alpha_j = 1$ or 0, depending on whether there is an edge joining α_i to α_j or not. Given any root lattice R of type ADE and a root basis $\{\beta_0, \dots \beta_n\}$ such that $\beta_i \cdot \beta_j \geq 0$ for $i \neq j$, we may consider the associated Coxeter-Dynkin diagram: This is a graph with $n+1$ vertices and $\beta_i \cdot \beta_j$ edges between vertices i and j. We say that $\{\beta_0, \dots \beta_n\}$ is a *canonical root basis* for R if the associated Coxeter-Dynkin diagram is the Dynkin diagram R.

Let $\mathrm{I}_{1,n}$ be the (up to isomorphism unique) unimodular lattice of signature $(1, n)$. $\mathrm{I}_{1,n}$ has a basis given by classes $e_0, e_1 \dots, e_n$ satisfying $e_0^2 = 1,\ e_i^2 = -1,\ e_i \cdot e_j = 0$ for all $i \neq j$. Any other basis $(v_0, \dots, v_n)$ for $\mathrm{I}_{1,n}$ with the same intersection products will be called an *orthonormal basis*. Define the special element $k_n = -3e_0 + \sum_{i=1}^n e_i$ and define the lattice $(\mathbb{Z}k_n)^\perp \subset \mathrm{I}_{1,n}$. This has a basis of roots given by

$$\alpha_i = e_i - e_{i+1}, \quad 1 \leq i \leq n-1$$
$$\alpha_0 = e_0 - e_1 - e_2 - e_3$$

and the Coxeter-Dynkin diagram associated to this basis is E_n, so that $(\mathbb{Z}k_n)^\perp \cong E_n$. We have the following Lemma from [Dol12, Lemma 8.2.6]

Lemma 3.3.1. *Suppose that $n \geq 3$, and let $(v_0, \dots, v_n)$ be an orthonormal basis for $\mathrm{I}_{1,n}$ such that $-3v_0 + \sum_{i=1}^n v_i = k_n$. Then*

$$\beta_0 = v_0 - v_1 - v_2 - v_3$$
$$\beta_i = v_i - v_{i+1}, \quad 1 \leq i \leq n-1$$

is a canonical root basis for $(\mathbb{Z}k_n)^\perp$.

Proof. By inspection, we see that $\beta_i \cdot k_n = 0$ for all i. Every $v = \sum_{i=1}^n a_i v_i \in k_n^\perp$ must satisfy

$$3a_0 + a_1 + \cdots + a_n = 0 \tag{3.1}$$

We can write

$$v = a_0\beta_0 + (a_0 + a_1)\beta_1 + (2a_0 + a_1 + a_2)\beta_2 + (3a_0 + a_1 + a_2 + a_3)\beta_3 + \cdots + (3a_0 + \sum_{i=1}^{n-1} a_i)\beta_{n-1}$$

using that $3a_0 + \sum_{i=1}^{n-1} a_i = -a_n$ by (3.1) so we see that the β_i form a basis. A quick calculation of intersection products shows that the associated Coxeter-Dynkin diagram is E_n, so the claim follows. $\square$

Given a root lattice R, we denote $O(R)$ the group of isometries of R. The Weyl group $W(R)$ is the subgroup of $O(R)$ generated by reflection in the roots. Explicitly, for a root α the corresponding reflection s_α is defined as $s_\alpha(\beta) = \beta + \langle \beta, \alpha \rangle \alpha$. For a *finite* root lattice of type ADE we have

$$O(R) \cong A(R) \ltimes W(R) \tag{3.2}$$

where $A(R)$ denotes the group of automorphisms of the Dynkin diagram of R. We will be most interested in the lattice $(\mathbb{Z}k_9)^\perp$, in this case we have the following lemma:

Lemma 3.3.2. *Given a canonical root basis $\beta_0, \dots, \beta_8$ for $(\mathbb{Z}k_9)^\perp$ we have an induced isomorphism of lattices*

$$E_9 \cong E_8 \oplus (0)$$

where (0) *denotes the lattice with underlying group $\mathbb{Z}$ and trivial intersection product, generated by k_9.*

Proof. Note first that $k_9^2 = 0$. The roots $\beta_0, \dots, \beta_7$ are linearly independent and have Coxeter-Dynkin diagram E_8, so span a root lattice isomorphic to E_8. Define an injective homomorphism of groups by

$$\varphi\colon \mathbb{Z} \oplus E_8 \to (\mathbb{Z}k_9)^\perp, \quad (nk_9, \alpha) \mapsto nk_9 + \alpha$$

φ preserves the intersection pairing since $k_9 \cdot \beta_i = 0$ for all i. Note that we have

$$-k_9 = 3\beta_0 + 2\beta_1 + 4\beta_2 + 6\beta_3 + 5\beta_4 + 4\beta_5 + 3\beta_6 + 2\beta_7 + \beta_8$$

so that given any $\alpha \in (\mathbb{Z}k_9)^\perp$, we may uniquely write $\alpha = nk_9 + \alpha'$ for some $\alpha' \in E_8$, showing that φ is surjective. □

We now reinterpret these definitions in a geometric context. Let Y be a rational surface with $\mathrm{Pic}(Y) \cong \mathrm{I}_{1,n}$. Recall that a class $\alpha \in \mathrm{Pic}(Y)$ is *numerically exceptional* if $\alpha^2 = \alpha \cdot K_Y = -1$. For later use, we have the following result from [Fri16, Lemma 5.9(ii)]

Lemma 3.3.3. *Any numerically exceptional class on a rational elliptic surface Y is effective.*

Proof. Let $\alpha \in \mathrm{Pic}(Y)$ be numerically exceptional, i.e $\alpha^2 = \alpha \cdot K_Y = -1$ and write L_α for the line bundle associated to α. We have that $\chi(L_\alpha) = 1$ by Riemann Roch and $h^2(L_\alpha) = h^0(L_\alpha^{-1} \otimes K_Y)$ by Serre duality. Observe that $-\alpha + K_Y$ cannot be the class of an effective divisor since $-K_Y$ is nef and $-K_Y \cdot (-\alpha + K_Y) = -1$. So $h^2(L_\alpha) = 0$ and therefore $h^0(L_\alpha) \geq 1$, i.e α is effective. □

We make the following definition

Definition 3.3.4. Let Y be a rational surface with $\mathrm{Pic}(Y) \cong \mathrm{I}_{1,n}$. A *marking* of Y is an ordered orthonormal basis $(v_0, v_1, \dots v_n)$ for $\mathrm{Pic}(Y)$ such that $v_1, \dots v_n$ are numerically exceptional, $v_0^2 = 1$ and $-K_Y = 3v_0 - \sum_{i=1}^n v_i$.

We have the following easy result

Lemma 3.3.5. *Let $(\mathcal{Y}, \mathcal{D})$ be a family of Looijenga pairs over a connected surface S, and suppose that we are given a path $\gamma\colon [0,1] \to S$ and a marking of $Y_{\gamma(0)}$. Then parallel transport along γ induces a marking of $Y_{\gamma(1)}$.*

Proof. Write $s = \gamma(0)$ and $t = \gamma(1)$. Since $\mathrm{Pic}(Y) = H^2(Y)$ for a Looijenga pair (Y, D), parallel transport induces an isometry $\mu\colon \mathrm{Pic}(Y_s) \cong \mathrm{Pic}(Y_t)$, and therefore $v_i' = \mu(v_i),\ 0 \le i \le n$ is an ordered orthonormal basis for $\mathrm{Pic}(Y_t)$. Since (Y_s, D_s) and (Y_t, D_t) are deformation equivalent as Looijenga pairs we have $\mu(K_{Y_s}) = K_{Y_t}$. This shows that numerically exceptional classes map to numerically exceptional classes under μ so that the v_i' give a marking for $\mathrm{Pic}(Y_t)$. □

A marking of Y gives an isomorphism $\mathrm{Pic}(Y) \cong \mathrm{I}_{1,n}$ by sending each v_i to e_i, and under this identification K_Y maps to the class k_n. This induces an isomorphism $K_Y^\perp \cong k_n^\perp$, and under this isomorphism, the canonical root basis $\beta_0, \dots, \beta_n$ induced by the marking on Y maps to the canonical root basis $\alpha_0, \dots, \alpha_n$ for E_n.
Examples of markings arise by representing Y as an iterated blowup of $\mathbb{P}^2$: A *blowing-down structure* on Y is a composition of birational morphisms

$$Y = Y_n \to Y_{n-1} \to \cdots \to Y_1 \to \mathbb{P}^2$$

where each morphism $Y_i \to Y_{i-1}$ is given by blowing up a point on Y_i. A blowing-down structure on Y induces a marking of $\mathrm{Pic}(Y) \cong \mathrm{I}_{1,n}$: Let E_i be the divisor class of the total transform of the i-th exceptional divisor, and H the pullback of the ample generator of $\mathrm{Pic}(\mathbb{P}^2)$. Then $(H, E_1, \dots, E_n)$ is an orthonormal basis, the E_i are numerically exceptional, and it is clear that $-K_Y = 3H - \sum_i E_i$. A marking of Y obtained from a blowing-down structure is called a *geometric marking.*

Suppose now that (Y, D) is a Looijenga pair and write $D = D_0 + \cdots + D_r$. Since $[D] = -K_Y$, we see that a marking of Y induces an isomorphism $D^\perp \cong k_n^\perp$. If the intersection matrix of D is strictly negative semidefinite, we simply say that (Y, D) is strictly negative semidefinite. Such D is either an irreducible nodal 0-curve, or a cycle of (-2)-curves. This implies that $D^2 = 0$ and $D \cdot D_i = 0$ for all i. We have the following easy result, adapted from [Fuj90];

Lemma 3.3.6. *Suppose that (Y, D) is strictly negative semidefinite. Then Y can be given a blowing-down structure consisting of nine blowups.*

Proof. Since D is nef, we must have that $C^2 = D \cdot C - 2 \ge -2$ for every smooth rational curve C on Y, by the adjunction formula. The same statement holds for any blowdown of Y, as any blowdown of Y is either an interior blowdown or a corner blowdown, again by the adjunction formula. It follows that we may blow down Y to $\mathbb{P}^2, \mathbb{F}_0$ or $\mathbb{F}_2$. Every one-point blowup of $\mathbb{F}_0$ is a two point blowup of $\mathbb{P}^2$, so we only need to check the case where the minimal model is $\mathbb{F}_2$. In this case, the unique (-2) curve cannot contain any of the blown-up points, by the inequality above. Since a blowup of $\mathbb{F}_2$ at a point away from the (-2)-curve is a two-point blowup of $\mathbb{P}^2$, we see that Y is a blowup of $\mathbb{P}^2$. Using $K_Y^2 = 0$, Noether's formula gives $\chi(Y) = 12$, so that Y is the blowup of $\mathbb{P}^2$ in nine (possibly infinitely near) points. $\square$

We conclude that in the strictly negative semidefinite case, $D^\perp \cong k_9^\perp$. In particular, a marking of Y gives an embedding

$$\mathbb{Z}D_0 + \cdots + \mathbb{Z}D_r/[D] \to D^\perp/[D] \cong k_9^\perp/k_9 \cong E_8$$

We have an abstract isomorphism $\mathbb{Z}D_0 + \cdots + \mathbb{Z}D_r/[D] \cong A_r$, so that this construction gives a root sublattice

$$L = \varphi(\mathbb{Z}D_0 + \cdots + \mathbb{Z}D_r/[D]) \subset E_8$$

isomorphic to A_r. We summarize all this in the following:

Proposition 3.3.7. *Let (Y, D) be a strictly negative semidefinite Looijenga pair such that D has $r + 1$ components*

- *A marking of Y determines a root sublattice $L \subset E_8$ isomorphic to A_r*
- *A geometric marking of Y determines a root sublattice $L \subset E_8$ isomorphic to A_r together with a root basis for L.*
- *Changing the marking on Y changes L by an element of $W(E_8)$.*

Proof. The first part is clear from the discussion preceding the proposition. Let $\beta_0, \dots \beta_7$ be the canonical root basis induced by the marking, and consider the induced splitting

$$D^\perp = \mathbb{Z}D \oplus K$$

where K is the lattice generated by $\beta_0, \dots \beta_7$. If the marking on Y is geometric, then the class of the last exceptional divisor v_9 is an interior exceptional curve. We have $v_9 \cdot D = 1$ by adjunction. Since D does not have any (-1)-components, v_9 is not a component of D, and therefore meets a unique component D_0, and it follows that the inclusion $\mathbb{Z}D_1 + \cdots + \mathbb{Z}D_r \to D^\perp$ has image contained in K. We obtain an inclusion

$$\mathbb{Z}D_1 + \cdots + \mathbb{Z}D_r \hookrightarrow K \cong E_8$$

where the last isomorphism is determined by the (geometric) marking on Y. The image of the D_i under this composition give a basis for L, and therefore a canonical isomorphism $L \cong A_r$.

Two different markings give rise to two isomorphisms $D^\perp/D \cong E_8$ and therefore differ by an element of $O(E_8)$. By (3.2), we have that $O(E_8) = W(E_8)$, so the final claim follows. $\square$

Given any root sublattice $L \subset E_8$, we write $[L]$ for the equivalence class under the action of $W(E_8)$.

Proposition 3.3.8. *Let (Y, D) and (Y', D') are two strictly negative semidefinite Looijenga pairs. Then the pairs are deformation equivalent iff the associated equivalence class of root sublattice is the same.*

Proof. Suppose first that the pairs are deformation equivalent. Then there exists a family $(\mathcal{Y}, \mathcal{D})/S$ and a path $\gamma\colon [0,1] \to S$ such that $(Y, D) = (\mathcal{Y}_{\gamma(0)}, \mathcal{D}_{\gamma(0)})$ and $(Y', D') = (\mathcal{Y}_{\gamma(1)}, \mathcal{D}_{\gamma(1)})$. Parallel transport induces an isometry $\mu\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(Y')$ which sends D to D' and $D^\perp$ to $D'^\perp$, so we obtain an induced isometry $\mu\colon D^\perp/[D] \to D'^\perp/[D']$. Since Y and Y' are blowups of $\mathbb{P}^2$ in nine points, we may choose (geometric) markings, yielding isomorphisms $\varphi\colon D^\perp/D \cong E_8$ and $\varphi'\colon D'^\perp/D' \cong E_8$. Let $w \in W(E_8) = O(E_8)$ be the element making the diagram

$$\begin{array}{ccc} D^\perp/[D] & \xrightarrow{\mu} & D'^\perp/[D'] \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi'} \\ E_8 & \xrightarrow{w} & E_8 \end{array}$$

commute. Since μ is induced by parallel transport, we have $\mu(D_i) = D'_i$ for all i, so that

$$\mu(\mathbb{Z}D_0 + \cdots + \mathbb{Z}D_r/[D]) = \mathbb{Z}D'_0 + \cdots + \mathbb{Z}D'_r/[D']$$

and therefore $L' = w(L)$, as required.

Conversely, suppose that the equivalence classes of root sublattices associated to (Y, D) and (Y', D') are the same. We will show that for a certain choice of labelings on D and D', we can construct an isometry $\mu\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(Y')$ satisfying the conditions of Proposition 3.1.1, thus showing that the pairs (Y, D) and (Y', D') are deformation equivalent. By 3.3.6, we may choose geometric markings $v_0, \ldots v_9$ for Y and $v'_0, \ldots, v'_9$ for Y', with associated canonical root bases $\beta_0, \ldots \beta_8$ and $\beta'_0, \ldots \beta'_8$. As in Proposition 3.3.7 this determines distinguished components D_0 (resp D'_0) met by the last exceptional divisor. We label the remaining components $D_1, \ldots D_r$ (resp $D'_1, \ldots D'_r$), note that this labeling involves an arbitrary choice of orientation of D. We obtain splittings $D^\perp = \mathbb{Z}D \oplus K$ and $D^\perp = \mathbb{Z}D' \oplus K'$, where K and K' are the lattices generated by $\beta_0, \ldots, \beta_7$ and $\beta'_0, \ldots \beta'_7$, respectively and we obtain inclusions

$$\mathbb{Z}D_1 + \cdots + \mathbb{Z}D_r \hookrightarrow K \xrightarrow[\sim]{\varphi} E_8$$

$$\mathbb{Z}D'_1 + \cdots + \mathbb{Z}D'_r \hookrightarrow K' \xrightarrow[\sim]{\varphi'} E_8$$

Let L and L' be the associated root sublattices of E_8. Define moreover $a_i = \varphi([D_i])$ and $b_i = \varphi'([D'_i])$. By assumption, we have $[L] = [L']$, so there exists $w \in W(E_8)$ such that $w(L) = L'$. The sets $\{w(a_i)\}_{1 \le i \le r}$ and $\{b_i\}_{1 \le i \le r}$ are both root bases for L', so there exist an isometry $\sigma \in O(L')$ mapping each $w(a_i)$ to b_i. The automorphism group of the Dynkin diagram of A_r is $\mathbb{Z}/2\mathbb{Z}$, so up to changing the orientation of D we may assume that $\sigma \in W(L') \subset W(E_8)$ by (3.2). So we may actually find $w \in W(E_8)$ such that $w(a_i) = b_i$ for $1 \le i \le r$. Consider the isometry

$$e\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(Y') \quad v_i \mapsto v'_i$$

Note that e restricts to an isometry $K \to K'$ mapping each β_i to β'_i, so that the top square of (3.3) commutes.

$$\begin{tikzcd} K \arrow[r, "\overset{\varphi}{\sim}"] \arrow[d, "e"] & E_8 \arrow[d, "id"] \\ K' \arrow[r, "\overset{\varphi'}{\sim}"] \arrow[d, "w'"] & E_8 \arrow[d, "w"] \\ K' \arrow[r, "\overset{\varphi'}{\sim}"] & E_8 \end{tikzcd} \tag{3.3}$$

If we set $w' = \varphi'^{-1} \circ w \circ \varphi' \in W(K') \subset \operatorname{Aut}(\operatorname{Pic}(Y'))$, then the bottom square (3.3) also commutes and it follows that the isometry

$$\mu := w' \circ e\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(Y')$$

satisfies

$$\mu(D_i) = \varphi'^{-1} \circ w \circ \varphi(D_i) = \varphi'^{-1}(w(a_i)) = \varphi'^{-1}(b_i) = D'_i$$

for $1 \le i \le r$. We also have that $\mu(-K_Y) = -K_{Y'}$ because w' is a product of reflections in roots $\beta' \in K_{Y'}^{\perp}$, so it follows that $\mu(D_0) = D'_0$ as well, since $D_0 = -K_Y - D_1 - \cdots - D_r$.

The cone C^+ has two connected components and any isometry must either preserve them or exchange them. Note that $[D]$, being the class of a nef divisor, lies on the boundary of C^+ and since $\mu([D]) = [D']$, μ must in fact preserve the cone C^+.

Finally we need to show that μ preserves the cone C^{++}. For this, we need to show that

μ permutes the set of effective numerically exceptional curves. Since μ is an isometry and $\mu(-K_Y) = -K_{Y'}$ it is clear that μ permutes numerically exceptional curves, and we have shown in Lemma 3.3.3 that every such curve is effective.

Summarising, we have shown that with respect to our labeling of D and D' the isometry $\mu\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(Y')$ satisifes $\mu(D_i) = D'_i$ for $0 \leq i \leq r$ and $\mu(C^{++}) = C^{++}$, so by Proposition 3.1.1, (Y, D) and (Y', D') are deformation equivalent. □

We can now finish our proof of Friedman's classification result (c.f [Fri16, Theorem 9.15, Proposition 9.16])

Theorem 3.3.9. *There exist* 10 *deformation classes of strictly negative semidefinite Looijenga pairs* (Y, D).

Proof. Equivalence classes of root sublattice embeddings $A_{r-1} \subset E_8$ up to $W(E_8)$ have been classified, see for example [SS19, Theorem 8.2]. There exist a unique embedding for $0 \leq r \leq 9$, $r \neq 8$, and two nonequivalent embeddings for $r = 8$. In the latter case one embedding is primitive and one imprimitive. It follows by Proposition 3.3.8 that there exist at most 10 deformation classes of such Looijenga pairs (Y, D). To see that there are exactly 10 deformation equivalence classes we simply find representatives for each class: let (Y_{f_n}, D_{f_n}) be the pairs associated to the Laurent polynomials f_n, $1, \ldots, 8, 8', 9$ in Figure 4.1. The pairs with $n \neq 8$ give distinct representatives because D has a different number of components in each case, so it suffices to show that the pairs (Y_{f_8}, D) and $(Y_{f_{8'}}, D)$ are distinct, where

$$f_8 = x + y + \tfrac{1}{xy} + \tfrac{1}{y}$$
$$f_{8'} = x + y + \tfrac{1}{x} + \tfrac{1}{y}$$

Let $U = Y \setminus D$. Then the long exact cohomology sequence of the pair (Y, U) shows that $H_1(U_{f_8}, \mathbb{Z}) = 0$ but $H_1(U_{f_{8'}}, \mathbb{Z}) = \mathbb{Z}/2$, so that the pairs (Y_{f_8}, D) and $(Y_{f_{8'}}, D)$ cannot be deformation equivalent. □

Chapter 4

The classification of T-polygons

In this section, we give a new geometric proof (Corollary 4.2.2), of the classification of T-polygons. We briefly explain the main idea: the results of Chapter 3 construct for each maximally mutable Laurent polynomial f with $P = \mathrm{Newt}(f)$ a T-polygon a strictly negative semidefinite Looijenga pair (Y_f, D) with period point $\phi_Y \equiv 1$. The classification of deformation families 3.3.9 combined with the Torelli theorem 3.1.3 imply that there exist exactly 10 such pairs (Y_f, D). Consequently, our pair (Y_f, D) must be isomorphic to one of our 10 reference pairs, say (Y_{f_n}, D). This gives a diagram (possibly after a corner blowup)

$$\begin{tikzcd} & Y_f = Y_{f_n} \arrow[dl] \arrow[dr] & \\ (\mathbb{C}^\times)^2 \subset \bar{Y}_P \arrow[rr, dashed, "\varphi"] & & \bar{Y}_{P_n} \supset (\mathbb{C}^\times)^2 \end{tikzcd}$$

where the vertical maps are interior blowdowns. The birational map $\varphi\colon (\mathbb{C}^\times)^2 \dashrightarrow (\mathbb{C}^\times)^2$ is volume-preserving (see §4.1)) and satisfies $\varphi^* f = f_n$. If we can show that f and f_n are mutation equivalent, then it follows by Lemma 2.3.2 that P is mutation equivalent to the polygon P_n.
Blanc has proved that any volume-preserving birational map factors as a composition of algebraic mutations, so we have that $\varphi = \varphi_n \circ \cdots \circ \varphi_1$, where the φ_i are algebraic mutations. However, to conclude that f and f_n are mutation-equivalent, we must also have that $\varphi_i \circ \cdots \circ \varphi_1^* f$ is a Laurent polynomial for all i, and this does not follow from Blanc's result.
In §4.1 we prove Theorem 4.1.5 (Theorem B in the introduction) using an adapted version of the Sarkisov algorithm for factorizing birational maps. Theorem 4.1.5 implies Blanc's result and also implies that f and f_n are mutation-equivalent (see Theorem 4.2.1). This section is logically independent from the rest of the thesis, and the result may be of independent interest.
In §4.2, we prove the classification of T-polygons and a few related results, for instance we show that for normalized maximally mutable Laurent polynomials, the classical period determines

the polynomial up to birational equivalence, see Theorem 4.2.3.

4.1 An adapted Sarkisov algorithm

Let Γ be a linear system without fixed components on a surface Y. We define the *multiplicity* $m_p(\Gamma)$ of Γ at $p \in Y$ to be the multiplicity of a general member of Γ at p, and we say that p is a *proper base point* of Γ if $m_p(\Gamma) > 0$. There is a finite sequence of birational maps

$$\pi\colon Y_n \xrightarrow{\pi_n} Y_{n-1} \xrightarrow{\pi_{n-1}} \dots \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} Y_0 = Y$$

such that π_k blows up all proper basepoints of the birational transform $\Gamma_k := (\pi_1 \circ \cdots \circ \pi_k)^{-1}_*(\Gamma)$ and Γ_n is basepoint free. The set of points where π_k^{-1} is undefined is the set of *infinitely near base points of order k*. We define the multiplicity of Γ at an infinitely near base point q of order k to be $m_q(\Gamma_k)$. We say that q lies over $p = (\pi_1 \circ \cdots \circ \pi_k)(q)$ and write $q \mapsto p$.
Let (Y, D) be a Looijenga pair. Since $K_Y + D \sim 0$, there exists a nowhere vanishing volume form Ω on $Y \setminus D$ with simple poles along D, necessarily unique up to scaling.

Definition 4.1.1. Let (Y, D) and (Y', D') be Looijenga pairs, and let $\varphi\colon Y \dashrightarrow Y'$ be a birational map. We say that φ is volume-preserving if there exists a resolution

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "p'"] & \\ Y \arrow[rr, dashed, "\varphi"] & & Y' \end{tikzcd}$$

such that $p^*\Omega = \lambda p'^*\Omega'$ for some $\lambda \in \mathbb{C}^\times$ where Ω is a holomorphic volume form on $Y \setminus D$ with simple poles along D, and similarly for Ω'.

We have the following easy lemma

Lemma 4.1.2. *Let (Y, D) and (Y', D') be Looijenga pairs. A birational morphism $p\colon Y \to Y'$ is volume preserving iff it is a composition of corner blowups, interior blowups, and volume-preserving isomorphisms.*

Proof. If p is an interior or corner blowup, then one computes that $p^*(K_{Y'} + D') = K_Y + D$, showing that p is volume-preserving. On the other hand, we can use [ACn02, Theorem 1.3.5], to factor p as

$$p = p_n \circ \cdots \circ p_1 \circ u$$

where the p_i are point blowups and u is an isomorphism. We calculate that

$$p^*(K_Y + D) = K_{Y'} + u^*\tilde{D} - \sum_{i=1}^{k} u^* E_i$$

where the sum is over all exceptional divisors arising from blowups of points that are not on D (or a strict transform of D), and E_i is the pullback of the class of the corresponding exceptional divisor to Z. Since p is volume preserving, we must have $u^*\tilde{D} = D$ and $k = 0$, showing that u is volume preserving, and the p_i are interior or corner blowups. $\square$

Theorem 4.1.3. *The following are equivalent:*

1. *φ is volume-preserving*
2. *There exists a resolution*

$$\begin{tikzcd} & Z' \arrow[dl, "p"'] \arrow[dr, "p'"] & \\ Y \arrow[rr, dashed, "\varphi"] & & Y' \end{tikzcd}$$

   where p and p' are volume-preserving birational morphisms

3. *φ restricts to a birational map $\varphi\colon D \dashrightarrow D'$ and the morphism that blows up all points (including infinitely near ones) on Y where φ is undefined is a composition of corner and interior blowups, and similarly for Y' and φ^{-1}.*

Proof. (2) $\implies$ (1): By Lemma 4.1.2 we have $p^*(K_Y + D) = K_Z + \tilde{D}$, so that $p^*(\Omega)$ is a nowhere vanishing holomorphic form on $Z \setminus \tilde{D}$ with simple poles along $\tilde{D}$. The same holds true for $p'^*\Omega$, so that $p'^*\Omega = \lambda p^*\Omega$ for some $\lambda \in \mathbb{C}^\times$, i.e φ is volume-preserving.
(1) $\implies$ (2): We have that $p^*(K_Y + D) = K_Z + \tilde{D} - \sum_i E_i$, where the sum is over all exceptional divisors arising from blowups of points that are not on D (or a strict transform of D), and E_i is the pullback of the class of the corresponding exceptional divisor to Z. This means that there exists a holomorphic form un $Z \setminus \tilde{D}$ with simples poles along $\tilde{D}$ and simple zeros along the E_i. Similarly, we have $p'^*(K_{Y'} + D) = K_Z + \tilde{D'} - \sum_i F_i$. Since φ is volume preserving, we must have $\tilde{D} = \tilde{D'}$ and $E_i = F_i$ up to reordering. Note that each E_i is a chain of smooth rational curves of the form $E_i = C_1 + \cdots + C_r$ where each of the C_i is the strict transform of an exceptional divisor arising from a blowup of a point not on D (or a strict transform of D). It follows that

we may successively contract (-1)-curves on Z to obtain a factorization

$$\begin{tikzcd} & Z \arrow[d] \arrow[ddl, "p"'] \arrow[ddr, "p'"] & \\ & Z' \arrow[dl, "q"] \arrow[dr, "q'"'] & \\ Y \arrow[rr, dashed, "\varphi"] & & Y' \end{tikzcd}$$

where $q^*(K_Y + D) = K_Z + \tilde{D}$ and similarly for q', meaning that q and q' are compositions of interior and corner blowups and hence volume-preserving.

(3) $\implies$ (2): By [ACn02, Corollary 1.3.8], there exists a commutative diagram

$$\begin{tikzcd} \tilde{Y} \arrow[r, "u"] \arrow[d, "q"] & \tilde{Y}' \arrow[d, "q'"] \\ Y \arrow[r, dashed, "\varphi"] & Y' \end{tikzcd}$$

where q blows up all points where φ is undefined, q' blows up all points where φ' is undefined and u is a birational isomorphism. Since φ restricts to a birational map $D \dashrightarrow D'$, the isomorphism u must be volume-preserving, so we may take $p = q$ and $p' = q' \circ u$.

(2) $\implies$ (3): It is clear that φ restricts to a birational morphism $D \dashrightarrow D'$. Let q be the morphism that blows up all basepoints of φ. By [ACn02, Theorem 1.3.7, Corollary 1.3.8], there exist morphisms $Z' \to \tilde{Y}$ and $q' \colon \tilde{Y} \to Y'$ making the diagram

$$\begin{tikzcd} & Z' \arrow[d] \arrow[ddl, "p"'] \arrow[ddr, "p'"] & \\ & \tilde{Y} \arrow[dl, "q"] \arrow[dr, "q'"'] & \\ Y \arrow[rr, dashed, "\varphi"] & & Y' \end{tikzcd}$$

commute. By assumption, p is volume-preserving, so that $p^*(K_Y + D) = K_{Z'} + \tilde{D}$ and there exists a nowhere vanishing holomorphic volume $p^*(\Omega)$ on $Z' \setminus \tilde{D}$ with simple poles along $\tilde{D}$. If φ had a basepoint away from D (or a strict transform of D), then $p^*(\Omega)$ would vanish on the corresponding exceptional divisor, so we conclude that there cannot be such basepoints. The same argument shows the corresponding statement for φ^{-1}, as required. □

Given a toric surface S, write ∂S for its toric boundary. We will use the following definition:

Definition 4.1.4. Let $S \to C$ be a $\mathbb{P}^1$-bundle over a curve and $p \in S$ a point. The *elementary transformation* at p is the birational map $\alpha_p \colon S \dashrightarrow S'$ over C given by blowing up p and contracting the strict transform of the fiber through p.

Consider the toric pair $(\mathbb{F}_k, \partial\mathbb{F}_k)$ with projection $\pi\colon \mathbb{F}_k \to \mathbb{P}^1$ and suppose p lies in the interior of one of the torus invariant sections. Then the elementary transformation at p is a volume-preserving map $\alpha_p\colon (\mathbb{F}_k, \partial\mathbb{F}_k) \dashrightarrow (\mathbb{F}_{k\pm 1}, \partial\mathbb{F}_{k\pm 1})$ of toric Looijenga pairs, and we call α_p a mutation.

This terminology stems from the fact that the restriction of the mutation α_p to the big tori gives a map $\varphi\colon T_N \dashrightarrow T_N$ which is an algebraic mutation, as we now explain.
Let $\mathbb{P}$ be the toric variety defined by the fan consisting of the two rays $\mathbb{R}_{\geq 0}v$ and $\mathbb{R}_{\leq 0}v$. The lattice morphism $N \to N/\mathbb{Z}v$ induces a map $\mathbb{P} \to T_{N/\mathbb{Z}v} \cong \mathbb{C}^\times$, giving $\mathbb{P}$ the structure of a $\mathbb{P}^1$-bundle. $\mathbb{P}$ comes with two toric divisors D_+ and D_-, which we think of as sections at 0 and ∞. Since $f \in \mathbb{C}[v^\perp]$, the Laurent polynomial f defines a regular function on $T_{N/\mathbb{Z}v}$, and we write $p_\pm = \pi^{-1}(V(f)) \cap D_\pm$. Let $b_\pm\colon \tilde{\mathbb{P}}_\pm \to \mathbb{P}$ be the blowup of $\mathbb{P}$ at $p_\pm$. It is shown in [GHK15, Lemma 3.2] that $\mu_f\colon \mathbb{P} \dashrightarrow \mathbb{P}$ extends to a regular isomorphism $\tilde{\mathbb{P}}_+ \to \tilde{\mathbb{P}}_-$, so that μ_f is the birational map which blows up p_+ and contracts the strict transform of $\pi^{-1}(V(f))$ to $p_- \subset \mathbb{P}$.
More generally, let P be a Fano polygon, and Σ_P the normal fan of P, and suppose that $\mathbb{R}_{\geq 0}v$ is a ray of Σ_P. Possibly after performing a toric blowup, we may assume that $\mathbb{R}_{\leq 0}v$ is also a ray of Σ_P, so that we have a proper map $Y_P \to \mathbb{P}^1$. Let $D_\pm$ be the toric divisors corresponding to $\mathbb{R}_{\geq 0}v$ and $\mathbb{R}_{\leq 0}v$ and let Q be the Fano polygon obtained by mutating P with respect to the mutation data $(v, \mathrm{Newt}\,(f))$. Then the birational map $\mu_{(u,v)}\colon Y_P \dashrightarrow Y_Q$ is the map which blows up $p_+ = \pi^{-1}(V(f)) \cap D_+$ and contracts the strict transform of $\pi^{-1}(V(f))$.
We now state the main theorem of this chapter

Theorem 4.1.5. *Let (Y, D) be a Looijenga pair with two toric models*

$$\begin{tikzcd} & (Y,D) \arrow[dl, "p"'] \arrow[dr, "p'"] & \\ (\bar{Y}, \bar{D}) \arrow[rr, dashed, "\varphi"] & & (\bar{Y}', \bar{D}') \end{tikzcd}$$

Then φ has a factorization

$$(\bar{Y}, \bar{D}) = (\bar{Y}_0, \bar{D}_0) \xrightarrow{\varphi_1} (\bar{Y}_1, \bar{D}_1) \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_n} (\bar{Y}_n, \bar{D}_n) = (\bar{Y}', \bar{D}')$$

where each of the maps φ_k is a toric blowup, toric blowdown, or a mutation.
Moreover, let $p_k = \varphi_k \circ \dots \varphi_1 \circ p$. Then $p_k\colon (Y, D) \dashrightarrow (\bar{Y}_k, \bar{D}_k)$ extends to a regular map

$$\tilde{p}_k\colon (\widetilde{Y}, \widetilde{D}) \to (\bar{Y}_k, \bar{D}_k)$$

on some corner blowup $(\widetilde{Y}, \widetilde{D})$ of (Y, D).

If $(\bar{Y}, \bar{D})$ is a toric Looijenga pair and $p\colon (Y, D) \to (\bar{Y}, \bar{D})$ is a composition of toric blowdowns and nontoric blowdowns then $p^{-1}((\mathbb{C}^\times)^2)$ is a well-defined torus chart on $U = Y \setminus D$. Conversely, any torus chart arises in this way from a toric model. We conclude:

Corollary 4.1.6. *Any two torus charts $j, j'\colon (\mathbb{C}^\times)^2 \hookrightarrow U$ on a Looijenga pair (Y, D) with $U = Y \setminus D$ are related by a composition of algebraic mutations between torus charts on U.*

We will deduce Theorem 4.1.5 from a modified version of the Sarkisov program for surfaces. We recall the setup, closely following the exposition in [KSC04]. For $k \in \mathbb{Z}_{\geq 0}$, we denote by $\mathbb{F}_k$ the $\mathbb{P}^1$-bundle

$$\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-k)) \to \mathbb{P}^1$$

and we will use the notation $\mathbb{F}$ to denote either $\mathbb{F}_k$ or $\mathbb{P}^2$. It will be convenient to fix bases for $\mathrm{Pic}(\mathbb{F})$: if $\mathbb{F} = \mathbb{P}^2$ we take the hyperplane class H as a basis of $\mathrm{Pic}(\mathbb{F})$. If $\mathbb{F} = \mathbb{F}_k$ we take A and B as a basis for $\mathrm{Pic}(\mathbb{F})$, where A is the fibre class and B is the class of self-intersection $-k$. Note that if $k = 0$, then $\mathbb{F}_0$ is simply $\mathbb{P}^1 \times \mathbb{P}^1$, although the notation $\mathbb{F}_0$ indicates that we have chosen one of the projections $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. The classes A, B satisfy

$$A^2 = 0,\ B^2 = -k,\ AB = 1$$

We will be concerned with birational maps $\varphi\colon \mathbb{F} \dashrightarrow \mathbb{P}^2$. Associated to such φ is the linear system $\Gamma = \varphi_*^{-1}(|H|)$. Since φ is birational, the linear system Γ is *mobile*, meaning that the locus where φ is undefined does not contain any divisors. Since mobile implies nef on a surface we have that

- $\Gamma \subset |nH|$ for $n > 0$ if $\mathbb{F} = \mathbb{P}^2$
- $\Gamma \subset |aA + bB|$ for $b \geq 0, a \geq kb$ if $Y = \mathbb{F}_k$ and $k > 0$
- $\Gamma \subset |aA + bB|$ for $a, b > 0$ if $Y = \mathbb{F}_0$.

We now define the Sarkisov degree of φ.

Definition 4.1.7. Let $\varphi\colon \mathbb{F} \dashrightarrow \mathbb{P}^2$ be a birational map. The Sarkisov degree of φ is defined to be

- $\frac{n}{3}$ if $\mathbb{F} = \mathbb{P}^2$ and $\Gamma = \varphi_*^{-1}(|H|) \subset |nH|$
- $\frac{b}{2}$ if $\mathbb{F} = \mathbb{F}_k$ and $\Gamma = \varphi_*^{-1}(|H|) \subset |aA + bB|$

The Sarkisov degree is defined in order to compare Γ to the canonical class of $\mathbb{F}$. Indeed, let λ be the Sarkisov degree of φ, we see that if $\mathbb{F} = \mathbb{P}^2$ then $\lambda K_{\mathbb{F}} + \Gamma \sim 0$ and if $\mathbb{F} = \mathbb{F}_k$, then

$-K_{\mathbb{F}} = (2+k)A + 2B$, so that $\lambda K_{\mathbb{F}} + \Gamma$ is at least numerically trivial on the fibres of $\mathbb{F}_k \to \mathbb{P}^1$.

[KSC04, Theorem 2.24] shows that any birational map $\varphi\colon \mathbb{F} \dashrightarrow \mathbb{P}^2$ factors as a composition of elementary birational maps called links of type I to IV. The proof proceeds by giving an algorithm to find a link $\alpha\colon \mathbb{F} \dashrightarrow \mathbb{F}'$ such that the birational map $\varphi \circ \alpha^{-1}$ is 'simpler' than φ. We can then factor φ as

$$\mathbb{F} \xrightarrow{\alpha} \mathbb{F}' \overset{\varphi\circ\alpha^{-1}}{\dashrightarrow} \mathbb{P}^2$$

In most cases, 'simpler' will mean that the Sarkisov degree has dropped. In those cases where it remains constant, a secondary invariant that decreases has to be found, see Definition 4.1.12. We cannot apply this result directly because we only want to use links which induce volume-preserving maps $(\bar{Y}, \bar{D}) \dashrightarrow (\bar{Y}', \bar{D}')$ between toric pairs in our factorization. We will call such links *admissible*. The strategy of proof is then exactly the same: for each volume-preserving map $\varphi\colon (\bar{Y}, \bar{D}) \dashrightarrow (\mathbb{P}^2, \partial\mathbb{P}^2)$ we want to show that we can precompose φ with a sequence of admissible links lowering the Sarkisov degree. We recall the definition of the different types of links:

Definition 4.1.8. A Sarkisov link is a birational map of one of the following types:

- A link of type I is the blowup $\epsilon\colon \mathbb{F}_1 \to \mathbb{P}^2$ of a point $p \in \mathbb{P}^2$. The link ϵ is admissible if and only if p is a torus fixed point, i.e iff ϵ is a toric blowup.
- A link of type II is an elementary transformation $\alpha_p\colon \mathbb{F}_k \dashrightarrow \mathbb{F}_{k\pm1}$. The link α_p is admissible if and only if p is either a torus fixed point, or lies in the interior of one of the two torus invariant sections. In the former case, α_p is a toric blowup followed by a toric blowdown, and in the latter case α_p is a mutation.
- A link of type III is the inverse $\epsilon^{-1}\colon \mathbb{P}^2 \dashrightarrow \mathbb{F}_1$ of a point blowup. The link ϵ^{-1} is admissible if and only if p is a torus fixed point, i.e iff ϵ^{-1} is a toric blowdown.
- A link of type IV is the involution $\tau\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ exchanging the two factors. A link of type IV is always admissible.

We see from this that finding a factorization of φ into admissible Sarkisov links is equivalent to factoring φ as a composition of mutations and toric blowups and blowdowns as in Theorem 4.1.5. In order to prove that such a factorisation by admissible links exists we will use the following elementary but crucial property of volume-preserving birational maps.

Proposition 4.1.9. *Let $\varphi\colon (Y, D) \dashrightarrow (\mathbb{P}^2, \partial\mathbb{P}^2)$ be a volume-preserving birational map of Looijenga pairs and let $\Gamma = \varphi_*^{-1}(|H|)$. Then every component D_i of D satisfies*

$$\sum_q m_q(\Gamma) \leq \Gamma \cdot D_i$$

where the sum is over all basepoints q (including infinitely near ones) of Γ such that $q \mapsto p$ for some $p \in D_i^{int}$, where $D_i^{int} := D_i \setminus \bigcup_{j \neq i} D_j$

Proof. Since φ is volume-preserving, Theorem 4.1.3 shows that the morphism $\pi\colon (\tilde{Y}, \tilde{D}) \to (Y, D)$ that blows up all basepoints of φ that lie over D_i^{int} is a composition of interior blowups. Denote $\tilde{\Gamma} = \pi_*^{-1}(\Gamma)$ and let $\tilde{D}_i$ be the strict transform of D_i. We have

$$0 \leq \tilde{\Gamma} \cdot \tilde{D}_i = (\pi^*\Gamma - \sum_q m_q(\Gamma) E_q) \cdot (\pi^* D_i - \sum_q E_q) = \Gamma \cdot D_i - \sum_q m_q(\Gamma)$$

where the E_q are pullbacks of the exceptional divisors to $\tilde{Y}$ and $m_q(\Gamma)$ is the multiplicity of the corresponding basepoint. The required inequality follows. □

We emphasize that Proposition 4.1.9 may be false for a general birational map $\varphi\colon \mathbb{F} \dashrightarrow \mathbb{P}^2$, since φ might have infinitely near base points away from the strict transform of the toric boundary. We recall [KSC04, Lemma 2.26]

Lemma 4.1.10. *Let $\varphi\colon \mathbb{F} \dashrightarrow \mathbb{P}^2$ be a birational map which is not an isomorphism. Then $\Gamma = \varphi_*^{-1}(|H|)$ has a basepoint of multiplicity strictly higher than the Sarkisov degree, unless*

- *$\mathbb{F} = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $\Gamma \subset |aA + bB|$ for $a < b$; or*
- *$\mathbb{F} = \mathbb{F}_1$ and $\Gamma \subset |aA + bB|$ for $\frac{a}{3} < \frac{b}{2}$*

We will use the following repeatedly

Lemma 4.1.11. *Let $p \in \mathbb{F}_k$ and let $\alpha_p\colon \mathbb{F}_k \dashrightarrow \mathbb{F}_{k\pm 1}$ be an elementary transformation. Let $\Gamma \subset |aA + bB|$ be a mobile linear system and let $m = m_p(\Gamma)$. The strict transform $\alpha_{p*}(\Gamma)$ is a mobile linear system contained in*

- *$|(a + b - m)A + bB|$ if p lies on B*
- *$|(a - m)A + bB|$ otherwise*

In particular, for $k = 0$, the strict transform of Γ is always contained in $|(a + b - m)A + bB|$.

Definition 4.1.12. Let $\varphi\colon \mathbb{F} \dashrightarrow \mathbb{P}^2$ be a birational map, and define the *Sarkisov bidegree* to be the pair consisting of the Sarkisov degree, and the integer which is equal to the sum of the multiplicities of all base points (including infinitely near ones) of Γ.

We will prove the factorization Theorem by induction on the Sarkisov bidegree, ordered lexicographically.

Proposition 4.1.13. *Let $\varphi\colon (\mathbb{F}, \partial\mathbb{F}) \dashrightarrow (\mathbb{P}^2, \partial\mathbb{P}^2)$ be a volume-preserving birational map. Suppose that Γ has a basepoint of multiplicity larger than the Sarkisov degree. Then there is a toric pair $(\mathbb{F}', \partial\mathbb{F}')$ and a composition of admissible Sarkisov links $\varphi'\colon (\mathbb{F}, \partial\mathbb{F}) \dashrightarrow (\mathbb{F}', \partial\mathbb{F}')$ such that the Sarkisov bidegree of $\varphi \circ \varphi'^{-1}$ is strictly smaller than the Sarkisov bidegree of φ.*

Proof. As in [KSC04], the proof proceeds by explicitly finding φ' for all possible $\mathbb{F}$.
Suppose first that $\mathbb{F} = \mathbb{P}^2$. We have $\Gamma \subset |nH|$ for some $n > 0$, so the Sarkisov degree equals $\frac{n}{3}$. By assumption, Γ has a base point q of multiplicity $m_q > \frac{n}{3}$. If q is a torus fixed point, let $\varphi'\colon (\mathbb{F}_1, \partial\mathbb{F}_1) \to (\mathbb{P}^2, \partial\mathbb{P}^2)$ be the blowup of q. The composition

$$\mathbb{F}_1 \xrightarrow{\varphi'} \mathbb{P}^2 \xrightarrow{\varphi} \mathbb{P}^2$$

is defined by a linear system $\Gamma' \subset |nA + (n - m_q)B|$ which has Sarkisov degree $\frac{n-m_q}{2} < \frac{n}{3}$.
If q is not a torus fixed point we proceed as follows. Let p be the torus fixed point opposite to the edge containing q, and let $\epsilon\colon \mathbb{F}_1 \to \mathbb{P}^2$ be the blowup of p. On $\mathbb{F}_1$, let F be the fibre passing through q, and consider the elementary transformation α_q obtained by blowing up q and contracting the strict transform of F. Since q does not lie on the negative section of $\mathbb{F}_1$, this elementary transformation is a mutation $\alpha_q\colon (\mathbb{F}_1, \partial\mathbb{F}_1) \dashrightarrow (\mathbb{F}_0, \partial\mathbb{F}_0)$. Finally, switch the choice of ruling on $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Denoting by m_p, m_q the multiplicities of Γ at p, q, we see by applying Lemma 4.1.11 that the composition

$$\mathbb{F}_0 \xrightarrow{\tau} \mathbb{F}_0 \xrightarrow{\alpha_q^{-1}} \mathbb{F}_1 \xrightarrow{\epsilon} \mathbb{P}^2 \xrightarrow{\varphi} \mathbb{P}^2$$

is defined by a linear system contained in $|(n - m_p)A + (n - m_q)B|$, with Sarkisov degree $\frac{n-m_q}{2} < \frac{n}{3}$.
Suppose now that $\mathbb{F} = \mathbb{F}_k$. We have $\Gamma \subset |aA + bB|$ for $b > 0$ and $a \geq kb$ with Sarkisov degree $\frac{b}{2}$. By assumption, Γ has a base point q of multiplicity $m_q > \frac{b}{2}$. If q is contained in either of the two torus-invariant sections, the elementary transformation based at q is admissible: If q is a torus fixed point the elementary transformation is a toric blowup followed by a toric blowdown, and if q is in the interior of a torus invariant section, the elementary transformation is a mutation. The strict transform of Γ is contained in $|(a + b - m_q)A + bB|$ or $|(a - m_q)A + bB|$, depending on whether q lies on B or not, so in either case the Sarkisov degree remains the same. However, since $\Gamma \cdot A = b$, we see that α_q removes a basepoint of multplicity m_q, and creates a new basepoint of multiplicity $b - m_q$ and leaves all other basepoints unchanged. Since $m_q > \frac{b}{2}$, α_q decreases

the Sarkisov bidegree.

If q is in the interior of one of the two torus invariant fibres the algorithm is more involved. Note first that we can assume that there are no base points of multiplicity $> \frac{b}{2}$ outside the interior of the two torus invariant fibres (otherwise we may perform the admissible elementary transformation there and decrease the Sarkisov bidegree). Since $\Gamma \cdot A = b$, Proposition 4.1.9 crucially shows there can be at most one base point of multiplicity $> \frac{b}{2}$ in each of the two torus invariant fibres, so we conclude that the total number of basepoints of multiplicity $> \frac{b}{2}$ is either one or two.

Case A: $\mathbb{F} = \mathbb{F}_0$ and there is exactly one base point q of multiplicity $m > \frac{b}{2}$. Switch the ruling of $\mathbb{F}_0$ so that B becomes the class of a fibre. The elementary transformation $\alpha_q \colon \mathbb{F}_0 \dashrightarrow \mathbb{F}_1$ is a mutation with respect to this ruling and the strict transform of Γ under α_q is contained in $|(b + a - m)A + aB|$ by Lemma 4.1.11. On the resulting surface $\mathbb{F}_1$, contract the unique (-1)-curve. The composition

$$\mathbb{P}^2 \xrightarrow{\epsilon^{-1}} \mathbb{F}_1 \xrightarrow{\alpha_q^{-1}} \mathbb{F}_0 \xrightarrow{\tau} \mathbb{F}_0 \xrightarrow{\varphi} \mathbb{P}^2$$

is defined by a linear system contained in $|(b + a - m)H|$. We use Lemma 4.1.14(1) below to calculate

$$b + a - m < b + \tfrac{1}{b}(b - m)^2 + \tfrac{3}{2}m - m < b + \tfrac{1}{2}(b - m) + \tfrac{m}{2} = \tfrac{3b}{2}$$

where we have used $b - m < \frac{b}{2}$ in the last inequality. The new Sarkisov degree is $\frac{b+a-m}{3} < \frac{b}{2}$ so our operation has decreased the Sarkisov degree.

Case B: $\mathbb{F} = \mathbb{F}_0$ and there are exactly two basepoints p, q of multiplicity $m_p, m_q > \frac{b}{2}$, one in each of the two torus-invariant fibres. Even though we may assume that both p and q are contained in the interior of the torus invariant fibres, it will be convenient for later to work with the weaker assumption that p lies anywhere on the torus invariant fibre, not necessarily in the interior. Switch the ruling on $\mathbb{F}_0$ so that B becomes the class of a fibre. Note that p and q do not lie on the same fibre with respect to the new ruling (otherwise, performing the two elementary transformations at p and q with respect to the old ruling takes us to $\mathbb{F}_2$, and the strict transform of Γ now has no basepoints of multiplicity $> \frac{b}{2}$ contradicting Lemma 4.1.10. Apply the elementary transformation α_p, taking us to $\mathbb{F}_1$, followed by the elementary transformation α_q. Since q is not on the negative section, α_q takes us back to $\mathbb{F}_0$. Finally switch the ruling on $\mathbb{F}_0$ again. Repeated application of Lemma 4.1.11 shows that the composition

$$\mathbb{F}_0 \xrightarrow{\tau} \mathbb{F}_0 \xrightarrow{\alpha_q^{-1}} \mathbb{F}_1 \xrightarrow{\alpha_p^{-1}} \mathbb{F}_0 \xrightarrow{\tau} \mathbb{F}_0 \xrightarrow{\varphi} \mathbb{P}^2$$

is defined by a linear system contained in $|aA + (b + a - m_p - m_q)B|$. A similar calculation to before using Lemma 4.1.14(2) shows that

$$\begin{aligned} b + a - m_p - m_q &< b + \tfrac{1}{b}((b - m_p)^2 + (b - m_q)^2) - b + \tfrac{3}{2}(m_p + m_q) - m_p - m_q \\ &< \tfrac{1}{2}(b - m_p) + \tfrac{1}{2}(b - m_q) + \tfrac{m_p + m_q}{2} = b \end{aligned}$$

where we have used that $b - m_p < \frac{b}{2}$ and $b - m_q < \frac{b}{2}$ in the last inequality. The new Sarkisov degree is $\frac{b+a-m_p-m_q}{2} < \frac{b}{2}$, so our operation has again decreased the Sarkisov degree.

Case C: $\mathbb{F} = \mathbb{F}_k$ for $k > 0$ and Γ has either one basepoint q or two basepoints p, q of multiplicity larger than $\frac{b}{2}$. Performing a sequence of toric blowups and blowdowns on the fibre *not* containing q we arrive at $\mathbb{F}_0$. The composition

$$\mathbb{F}_0 \dashrightarrow \cdots \dashrightarrow \mathbb{F}_k \xrightarrow{\varphi} \mathbb{P}^2$$

is defined by a linear system Γ' contained in $|a'A + bB|$ for some $a' \geq 0$, so the Sarkisov degree is still $\frac{b}{2}$. Observe that this operation only moved around basepoints on the fibre not containing q. In particular, Γ' now has either one basepoint q of multiplicity larger than $\frac{b}{2}$, contained in the interior of a fibre, or two basepoints q, p of multiplicity larger than $\frac{b}{2}$ where q is in the interior of a fibre and p is a torus fixed point. By case A or case B, we can find a composition of admissible links $\mathbb{F}' \dashrightarrow \mathbb{F}_0$ such that the Sarkisov degree of $\mathbb{F}' \dashrightarrow \mathbb{F}_0 \dashrightarrow \mathbb{F}_k \xrightarrow{\varphi} \mathbb{P}^2$ is strictly smaller than $\frac{b}{2}$. This completes the proof. □

Lemma 4.1.14. *Let $\varphi\colon \mathbb{F}_0 \dashrightarrow \mathbb{P}^2$ be a birational map and suppose $\Gamma = \varphi_*^{-1}(|H|) \subset |aA + bB|$ has at most two basepoints of multiplicity larger than $\frac{b}{2}$.*

1. *If Γ has exactly one basepoint q of multiplicity $m > \frac{b}{2}$ then*

$$a < \tfrac{1}{b}(b - m)^2 + \tfrac{3}{2}m$$

2. *If Γ has exactly two basepoints p, q of multiplicity $m_p, m_q > \frac{b}{2}$ then*

$$a < \tfrac{1}{b}((b - m_p)^2 + (b - m_q)^2) - b + \tfrac{3}{2}(m_p + m_q)$$

Proof. We have the following equalities for Γ [KSC04, Exercise 2.8]:

$$\begin{aligned} \sum m_p^2 &= \Gamma^2 - 1 = 2ab - 1 \\ \sum m_p &= \Gamma \cdot (-K_{\mathbb{F}_0}) - 3 = 2a + 2b - 3 \end{aligned}$$

where the sum is over all basepoints p of Γ and m_p denotes the multiplicity of the basepoint p. For (1), this gives

$$2ab-1=\sum m_p^2 = m^2+\sum_{p\neq q} m_p^2 \leq m^2+\tfrac{b}{2}\sum_{p\neq q} m_p \leq m^2+\tfrac{b}{2}(2a+2b-3-m)$$

Simplifying and dividing by b (noting that $b>0$ since φ is birational), we obtain

$$a-\tfrac{1}{b}+\tfrac{3}{2}\leq \tfrac{m^2}{b}+b-\tfrac{m}{2}=\tfrac{1}{b}(b-m)^2+\tfrac{3}{2}m$$

This gives

$$a<a-\tfrac{1}{b}+\tfrac{3}{2}\leq \tfrac{1}{b}(b-m)^2+\tfrac{3}{2}m$$

as required. The proof of (2) is completely analogous so we omit the calculation. □

Proof of Theorem 4.1.5. We first prove the existence of the factorization, which is now entirely analogous to the proof of [KSC04, Theorem 2.24]. Any smooth projective toric surface can be obtained from $\mathbb{P}^2$ by a sequence of toric blowups and blowdowns, so we may assume that $(\bar{Y},\bar{D})=(\mathbb{F},\partial\mathbb{F})$ and $(\bar{Y}',\bar{D}')=(\mathbb{P}^2,\partial\mathbb{P}^2)$, as in the statement of Proposition 4.1.13. Let φ be the induced birational map and $\Gamma=\varphi_*^{-1}(|H|)$. We argue by induction on the Sarkisov bidgree. Proposition 4.1.13 shows that as long as Γ has a base point of multiplicity larger than the Sarkisov degree, we can find a composition of admissible links $\varphi'\colon(\mathbb{F},\partial\mathbb{F})\dashrightarrow(\mathbb{F}',\partial\mathbb{F}')$ such that we can factor φ as

$$\mathbb{F}\xrightarrow{\varphi'}\mathbb{F}'\xrightarrow{\varphi\varphi'^{-1}}\mathbb{P}^2$$

and the Sarkisov bidegree of $\varphi\circ\varphi'^{-1}$ is smaller than the Sarkisov bidegree of φ. We need to show that we can find such a φ' also if Γ does not have a base point of multiplicity larger than the Sarkisov degree. By Lemma 4.1.10, this can only happen if $\mathbb{F}$ is $\mathbb{F}_0$ or $\mathbb{F}_1$. In the former case we must have $a<b$ so we simply switch the ruling on $\mathbb{F}_0$. In the latter case, we must have $\frac{a}{3}<\frac{b}{2}$ so contracting the negative section on $\mathbb{F}_1$ lowers the Sarkisov degree. In both cases, the link is clearly admissible. We conclude that we can factor φ as

$$\mathbb{F}\xrightarrow{\varphi'}\mathbb{F}'\xrightarrow{p}\mathbb{P}^2$$

where $\mathbb{F}\dashrightarrow\mathbb{F}'$ is a composition of admissible Sarkisov links and p is an isomorphism. Since both φ and φ' are volume preserving, so is p, which completes the proof of the first claim.

For the second part, we simply note that the birational maps appearing in our factorization only blow up or blow down torus fixed points and base points of φ, so that p_k is a regular map (after passing to a suitable corner blowup $(\tilde{Y}, \tilde{D})$). $\square$

4.2 The classification of T-polygons

We will deduce the classification of T-polygons from the following theorem

Theorem 4.2.1. *Let P and Q be T-polygons, and let f and g be maximally mutable Laurent polynomials supported on P and Q respectively. If the Looijenga pairs (Y_f, D) and (Y_g, D) are isomorphic then P and Q are mutation equivalent.*

Proof. Possibly after passing to a corner blowup (Y, D) of $(Y_f, D) = (Y_g, D)$, we have a diagram

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ (\mathbb{C}^\times)^2 \subset \bar{Y}_P \arrow[rr, dashed, "\varphi"] & & \bar{Y}_Q \supset (\mathbb{C}^\times)^2 \end{tikzcd}$$

where the vertical maps are toric models. It follows that the induced birational map $\varphi\colon (\mathbb{C}^\times)^2 \dashrightarrow (\mathbb{C}^\times)^2$ is volume-preserving, i.e $\varphi^*\Omega = \lambda\Omega$ for some $\lambda \in \mathbb{C}^\times$, where $\Omega = \frac{dx \wedge dy}{xy}$. In fact, by pairing Ω with the integral generator $\{|x| = 1, |y| = 1\} \in H_2((\mathbb{C}^\times)^2, \mathbb{Z})$ and using the change of variable formula, we see that $\lambda = \pm 1$, and up to composing with the volume-reversing automorphism of $(\mathbb{C}^\times)^2$ given by $(x, y) \mapsto (x, \frac{1}{y})$, we may assume that $\lambda = 1$.
Let p and q be the elliptic fibrations on Y induced by f and g. Since f is maximally mutable, the elliptic surface Y_f has a section s by Proposition 3.2.6. The isomorphism $(Y_f, D) \cong (Y_g, D)$ of Looijenga pairs maps the fibre $p^{-1}(\infty)$ to $q^{-1}(\infty)$. It follows from [CKM88, Lemma 1.5] that the isomorphism maps every fibre of p to a fibre of q. It follows that $\alpha = q \circ s$ is an automorphism of $\mathbb{P}^1$, so that the diagram

$$\begin{tikzcd} & \mathbb{P}^1 \\ Y \arrow[ur, "q"] \arrow[dr, "p"'] & \\ & \mathbb{P}^1 \arrow[uu, "\alpha"'] \end{tikzcd}$$

commutes. Since α fixes ∞ it must be of the form $z \mapsto az + b$ for $a, b \in \mathbb{C}$ and $a \neq 0$. It follows that $\varphi^* g = af + b$. Set $f' := af + b$, which is still maximally mutable and has $\mathrm{Newt}\,(f') = P$. By Theorem 4.1.5, we have a factorization

$$(\bar{Y}, \bar{D}) = (\bar{Y}_n, \bar{D}_n) \xrightarrow{\varphi_n} (\bar{Y}_{n-1}, \bar{D}_{n-1}) \xrightarrow{\varphi_{n-1}} \dots \xrightarrow{\varphi_1} (\bar{Y}_0, \bar{D}_0) = (\bar{Y}', \bar{D}')$$

where each of the φ_k is either a toric blowup, toric blowdown, or mutation. On dense tori

$\bar{Y}_k \setminus \bar{D}_k$, toric blowups and toric blowdowns restrict to the identity, whereas mutations restrict to algebraic mutations. It follows that (after relabeling) $\varphi = \varphi_n \circ \cdots \circ \varphi_1$ factorizes as a composition of algebraic mutations. It remains to show that f' and g are mutation equivalent, i.e that $f_k := (\varphi_k \circ \cdots \circ \varphi_1)^* g$ is a Laurent polynomial for all k. For this we use the second part of Theorem 4.1.5: Set $U = Y \setminus D = Y_f \setminus D_f$, and let $W = q|_U \colon U \to \mathbb{C}$. Since $p_k \colon (Y, D) \dashrightarrow (\bar{Y}_k, \bar{D}_k)$ extends to a regular map

$$\tilde{p}_k \colon (\widetilde{Y}, \widetilde{D}) \to (\bar{Y}_k, \bar{D}_k)$$

on some corner blowup $(\widetilde{Y}, \tilde{D})$ of (Y, D), each of the toric pairs $(\bar{Y}_k, \bar{D}_k)$ appearing in the factorization gives rise to a torus chart $j_k \colon (\mathbb{C}^\times)^2 \to U$. We obtain the following diagram.

$$\begin{tikzcd}
 & U \arrow[rr, "W"] & & \mathbb{C} \\
(\mathbb{C}^\times)^2 \arrow[ur, "j"] \arrow[r, dashed, "\varphi_n"] & \cdots \arrow[r, dashed, "\varphi_1"] & (\mathbb{C}^\times)^2 \arrow[ul, "j'"'] &
\end{tikzcd}$$

By construction, we have $f' = j^* W$ and $g = j'^* W$, and the pullback $f_k := (\varphi_k \circ \cdots \circ \varphi_1)^* g$ is similarly given by restricting W via the torus chart $j_k \colon (\mathbb{C}^\times)^2 \to U$ induced by the toric model $(\bar{Y}_k, \bar{D}_k)$. In particular, f_k is a Laurent polynomial for all k, so that there is a sequence of algebraic mutations mapping $g = f_0 \mapsto f_1 \mapsto \cdots \mapsto f_n = f'$. By Lemma 2.3.2, this induces a sequence of mutations $Q = \mathrm{Newt}(f') \to \cdots \to \mathrm{Newt}(f_n) = P$ which proves that P is mutation-equivalent to Q, as required. □

We can now complete the classfication of T-polygons

Corollary 4.2.2. *There are* 10 *mutation equivalence classes of T-polygons.*

Proof. Let P be any T-polygon, and let f be a maximally mutable Laurent polynomial supported on P. The Looijenga pair (Y_f, D) must be deformation equivalent to one of the pairs (Y_{f_n}, D_{f_n}) by Theorem 3.3.9, where f_n are as in Figure 4.1. Since both pairs have period point 1, the pairs must be isomorphic by Theorem 3.1.3. It now follows from Theorem 4.2.1 that P is mutation equivalent to the polygon P_n in Figure 4.1. □

We now prove Conjecture 1.0.1 for normalized maximally mutable Laurent polynomials f with $\mathrm{Newt}(f)$ a T-polygon.

Theorem 4.2.3. *Let f, g be normalized maximally mutable Laurent polynomials in two variables such that* $\mathrm{Newt}(f)$ *and* $\mathrm{Newt}(g)$ *are T-polygons, and suppose that $\pi_f(t) = \pi_g(t)$. Then f and g are mutation equivalent.*

Proof. Let $P = \mathrm{Newt}(f)$ and $Q = \mathrm{Newt}(g)$. As in the proof of Corollary 4.2.2, $(Y_f, D) \cong (Y_{f_n}, D)$ for some n, and similarly for g. Each of the Looijenga pairs (Y_f, D) and (Y_g, D) has two toric models (possibly after passing to a toric blowup)

$$\begin{tikzcd} & Y_f = Y_{f_n} \arrow[dl] \arrow[dr] & & & Y_{f_m} = Y_g \arrow[dl] \arrow[dr] & \\ \bar{Y}_P & & \bar{Y}_{P_n} & \bar{Y}_{P_m} & & \bar{Y}_Q \end{tikzcd}$$

for some $1 \leq m, n \leq 10$ and we obtain volume preserving maps $\varphi_1, \varphi_2 \colon (\mathbb{C}^\times)^2 \dashrightarrow (\mathbb{C}^\times)^2$ with $\varphi_1^*(f) = af_n + b$ and $\varphi_2^*(g) = a' f_m + b'$, where f_n, f_m are normalised maximally mutable Laurent polynomials associated to one of the 10 T-polygons in Table 4.1. It follows that $\pi_f(t) = \pi_{af_n+b}(t)$. We must have $b = 0$ because both f and f_n have zero constant term. Similarly we have $\pi_g(t) = \pi_{a'f_m}(t)$. Since $\pi_f(t) = \pi_g(t)$ by assumption, we conclude that $\pi_{af_n}(t) = \pi_{a'f_m}(t)$. By calculating the period sequences of $f_1, \dots f_8, f_{8'}, f_9$, one sees that this implies $n = m$, so $Y_f = Y_g$. This means that $\varphi_1^{-1} \circ \varphi_2$ is the birational map induced from the two toric models

$$\begin{tikzcd} & Y_f = Y_g \arrow[dl] \arrow[dr] & \\ (\mathbb{C}^\times)^2 \subset \bar{Y}_P & & \bar{Y}_Q \supset (\mathbb{C}^\times)^2 \end{tikzcd}$$

Take $Y = Y_f = Y_g$, let p the elliptic fibration on Y induced by g and take $W = p|_{Y \setminus D}$. By construction we have $W|_{j((\mathbb{C}^\times)^2)} = \frac{a'}{a} f$ and $W|_{j'((\mathbb{C}^\times)^2)} = g$. Since

$$\mathrm{const}((af)^d)) = a^d \mathrm{const}(f^d)$$

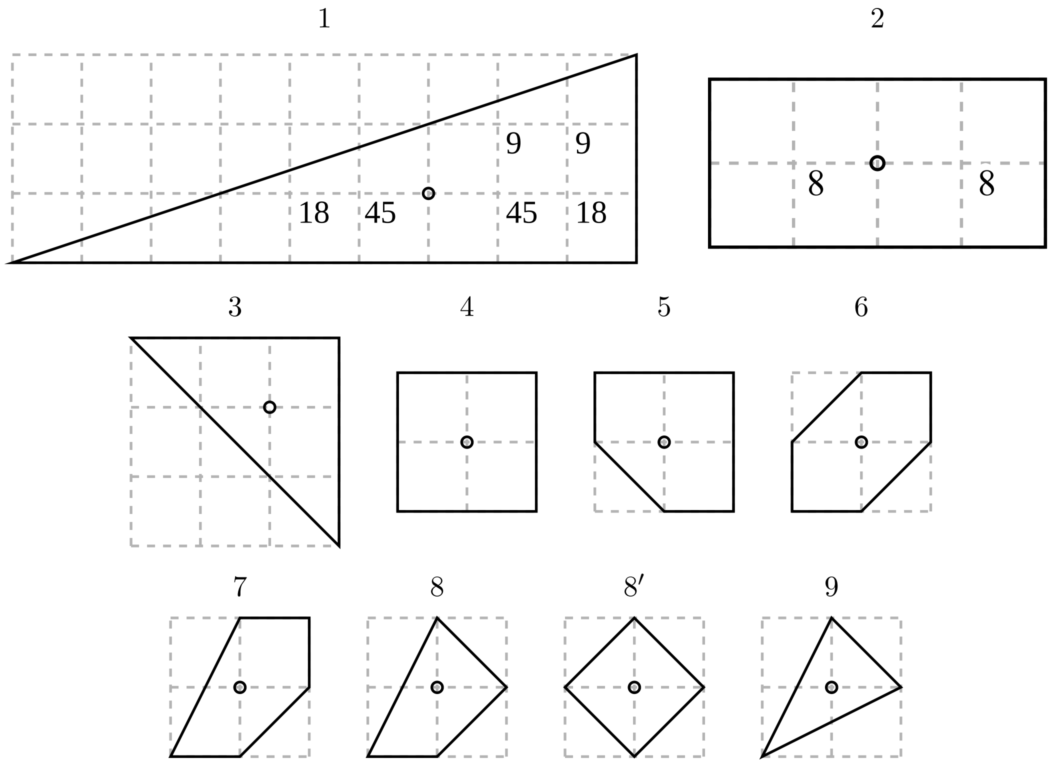
a look at the period sequences of $f_1, \dots f_9$ shows that $\pi_{af_i}(t) = \pi_{a'f_i}(t)$ implies $a = a' = 1$ unless $i = 8'$ or 9.

If $i = 8'$, we must have $\frac{a}{a'} = \pm 1$, and the automorphism of $(\mathbb{C}^\times)^2$ defined by $(x, y) \mapsto (-x, -y)$ pulls back $f_{8'}$ to $-f_{8'}$.

If $i = 9$, we must have $\frac{a}{a'} = \pm\omega$, where $\omega = e^{2\pi i/3}$, and the automorphism of $(\mathbb{C}^\times)^2$ defined by $(x, y) \mapsto (\pm\omega x, \pm\omega y)$ pulls back f_9 to $\pm\omega f_9$.

This shows that (possibly after precomposing j with such an automorphism) we have $W|_{j((\mathbb{C}^\times)^2)} = f$. The same argument as in the proof of Theorem 4.2.1 now shows that f and g are mutation-equivalent, which completes the proof. $\square$

Figure 4.1: The polygons $P_1, \dots P_8, P_{8'}, P_9$ which are representatives for the 10 mutation equivalence classes of T-polygons. The associated unique normalized maximally mutable Laurent polynomials f_i, $i = 1, \dots, 8, 8', 9$ are obtained by assigning binomial coefficients to lattice points on the edges of P_i, 0 to the origin, and the coefficients specified in the figure to the remaining lattice points. Picture taken from [ACC+16]. Below the period sequences of the f_i.



$$\pi_{f_1}(t) = e^{-60t}\sum_{d=0}^{\infty} t^d \frac{(6d)!}{(d!)(2d)!(3d)!} = 1 + 10260t^2 + 2021280t^3 + 618874020t^4 + 184450426560t^5 \dots$$

$$\pi_{f_2}(t) = e^{-12t}\sum_{d=0}^{\infty} t^d \frac{(4d)!}{(d!)^2(2d)!} = 1 + 276t^2 + 6816t^3 + 314532t^4 + 12853440t^5 + 569409360t^6 \dots$$

$$\pi_{f_3}(t) = e^{-6t}\sum_{d=0}^{\infty} t^d \frac{(3d)!}{(d!)^3} = 1 + 54t^2 + 492t^3 + 9882t^4 + 158760t^5 + 2879640t^6 \dots$$

$$\pi_{f_4}(t) = e^{-4t}\sum_{d=0}^{\infty} t^d \frac{(2d)!(2d)!}{(d!)^4} = 1 + 20t^2 + 96t^3 + 1188t^4 + 10560t^5 + 111440t^6 \dots$$

$$\pi_{f_5}(t) = e^{-3t}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} t^{l+m} \frac{(l+2m)!(l+m)!}{(l!)^2(m!)^3} = 1 + 10t^2 + 30t^3 + 270t^4 + 1560t^5 + 11350t^6 \dots$$

$$\pi_{f_6}(t) = \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=\max(a-c,0)}^{a+b} \frac{t^{a+2b+2c+d}(a+2b+2c+d)!}{a!b!c!d!(a+b-d)!(c+d-a)!} = 1 + 6t^2 + 12t^3 + 90t^4 + 360t^5 \dots$$

$$\pi_{f_7}(t) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=\max(l,m)}^{l+m} \frac{t^{l+m+n}(l+m+n)!}{l!m!(l+m-n)!(n-l)!(n-m)!} = 1 + 4t^2 + 6t^3 + 36t^4 + 120t^5 \dots$$

$$\pi_{f_8}(t) = \sum_{l=0}^{\infty}\sum_{m=l}^{\infty} \frac{t^{l+2m}(l+2m)!}{(l!)^2(m-l)!m!} = 1 + 2t^2 + 6t^3 + 6t^4 + 60t^5 + 110t^6 \dots$$

$$\pi_{f_{8'}}(t) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{t^{2l+2m}(2l+2m)!}{(l!)^2(m!)^2} = 1 + 4t^2 + 36t^4 + 400t^6 \dots$$

$$\pi_{f_9}(t) = \sum_{d=0}^{\infty} \frac{t^{3d}(3d)!}{(d!)^3} = 1 + t^3 + 6t^6 + 90t^9 \dots$$

Part II

The Abelian/non-Abelian Correspondence and Gromov–Witten invariants of Blow-ups

Chapter 5

Introduction

Gromov–Witten invariants, roughly speaking, count the number of curves in a projective variety X that are constrained to pass through various cycles. They play an essential role in mirror symmetry, and have been the focus of intense activity in symplectic and algebraic geometry over the last 25 years. Despite this, there are few effective tools for computing the Gromov–Witten invariants of blow-ups. In this paper we improve the situation somewhat: we determine how genus-zero Gromov–Witten invariants change when a smooth projective variety X is blown up in a complete intersection of convex line bundles. In the case where the blow-up $\tilde{X}$ is Fano, a special case of our result gives closed-form expressions for genus-zero one-point descendant invariants of $\tilde{X}$ in terms of invariants of X, and hence determines the small J-function of $\tilde{X}$.

Suppose that $Z \subset X$ is the zero locus of a regular section of a direct sum of convex (or nef) line bundles

$$E = L_0 \oplus \cdots \oplus L_r \to X$$

and that $\tilde{X}$ is the blow-up of X in Z. To determine the genus-zero Gromov–Witten invariants of $\tilde{X}$, we proceed in two steps. First, we exhibit $\tilde{X}$ as the zero locus of a section of a convex (or nef) vector bundle on the bundle of Grassmannians $\mathrm{Gr}(r, E^\vee) \to X$: this is Theorem 5.0.1 below. We then establish a version of the Abelian/non-Abelian Correspondence [CFKS08] that determines the genus-zero Gromov–Witten invariants of such zero loci. This is the Abelian/non-Abelian Correspondence with bundles, for target spaces that are partial flag bundles – see Theorem 5.0.2. It builds on and generalises results by Ciocan-Fontanine–Kim–Sabbah [CFKS08, §6], Brown [Bro14], and Oh [Oh21].

Theorem 5.0.1 (see Proposition 7.3.2 below for a more general result)**.** *Let X be a smooth projective variety, let $E = L_0 \oplus \cdots \oplus L_r \to X$ be a direct sum of line bundles, and let $Z \subset X$ be the zero locus of a regular section s of E. Let $\pi\colon \mathrm{Gr}(r, E^\vee) \to X$ be the Grassmann bundle of* *subspaces and let $S \to \mathrm{Gr}(r, E^\vee)$ be the tautological subbundle. Then the composition*

$$S \hookrightarrow \pi^* E^\vee \xrightarrow{\pi^* s^\vee} \mathcal{O}$$

defines a regular section of $S^\vee$, and the zero locus of this section is the blow-up $\tilde{X} = \mathrm{Bl}_Z X$.

If the line bundles L_i are convex, then the bundle $S^\vee$ is also convex. The fact that $\tilde{X}$ is regularly embedded into $\mathrm{Gr}(r, E^\vee) \cong \mathbb{P}(E)$ (where $\mathbb{P}(E)$ is the projective bundle of lines) is well-known and true for more general blowups, see for example [Ful98, Appendix B8.2] and [Alu10, Lemma 2.1]. However, to apply the Abelian/non-Abelian correpondence, the crucial point is that $\tilde{X}$ is cut out by a regular section of an explicit representation-theoretic bundle (as defined in (6.2)) on $\mathrm{Gr}(r, E^\vee)$. Although this should be well-known to experts, we have been unable to find a reference for this.

To apply Theorem 5.0.1 to Gromov–Witten theory, and to state the Abelian/non-Abelian Correspondence, we will use Givental's formalism [Giv04]. This is a language for working with Gromov–Witten invariants and operations on them, in terms of linear symplectic geometry. We give details in §6.3 below, but the key ingredients are, for each smooth projective variety Y, an infinite-dimensional symplectic vector space $\mathcal{H}_Y$ called the Givental space and a Lagrangian submanifold $\mathcal{L}_Y \subset \mathcal{H}_Y$. Genus-zero Gromov–Witten invariants of Y determine and are determined by $\mathcal{L}_Y$.

We will also consider *twisted* Gromov–Witten invariants [CG07]. These are invariants of a projective variety Y which depend also on a bundle $F \to Y$ and a characteristic class $\mathbf{c}$. For us, this characteristic class will always be the equivariant Euler class (or total Chern class)

$$\mathbf{c}(V) = \sum_{k=0}^{d} \lambda^{d-k} c_k(V) \qquad \text{where } d \text{ is the rank of the vector bundle } V. \tag{5.1}$$

The parameter λ here can be thought of as the generator for the S^1-equivariant cohomology of a point. There is a Lagrangian submanifold $\mathcal{L}_{F_\lambda} \subset \mathcal{H}_Y$ that encodes genus-zero Euler-twisted invariants of Y; the Quantum Riemann–Roch theorem [CG07] implies that

$$\Delta_{F_\lambda} \mathcal{L}_Y = \mathcal{L}_{F_\lambda}$$

where $\Delta_{F_\lambda} \colon \mathcal{H}_Y \to \mathcal{H}_Y$ is a certain linear symplectomorphism. This gives a family of Lagrangian submanifolds $\lambda \mapsto \mathcal{L}_{F_\lambda}$ defined over $\mathbb{Q}(\lambda)$, that is, a meromorphic family of Lagrangian submanifolds parameterised by λ. When F satisfies a positivity condition called convexity, the family $\lambda \mapsto \mathcal{L}_\lambda$ extends analytically across $\lambda = 0$ and the limit $\mathcal{L}_{F_0}$ exists. This limiting submanifold

$\mathcal{L}_{F_0} \subset \mathcal{H}_Y$ determines genus-zero Gromov–Witten invariants of the subvariety of Y cut out by a generic section of F [CG07, Coa14]. Theorem 5.0.1 therefore allows us to determine genus-zero Gromov–Witten invariants of the blow-up $\tilde{X}$, by analyzing the limiting submanifold $\mathcal{L}_{S_0^\vee}$.

Our second main result, Theorem 5.0.2, applies to the Grassmann bundle $\mathrm{Gr}(r, E^\vee) \to X$ considered in Theorem 5.0.1, and more generally to any partial flag bundle $\mathrm{Fl}(E) \to X$ induced by E. Such a partial flag bundle can be expressed as a GIT quotient $A /\!\!/ G$, where G is a product of general linear groups, and so any representation ρ of G on a vector space V induces a vector bundle $V^G \to \mathrm{Fl}(E)$ with fiber V. See §6.2 for details of the construction. We give an explicit family of elements of $\mathcal{H}_{\mathrm{Fl}(E)}$,

$$(t, \tau) \mapsto I_{\mathrm{GM}}(t, \tau, z) \qquad\qquad t \in \mathbb{C}^R \text{ for some } R,\ \tau \in H^\bullet(X) \tag{5.2}$$

defined in terms of genus-zero Gromov–Witten invariants of X and explicit hypergeometric functions, and show that this family, after changing the sign of z, lies on the Lagrangian submanifold that determines Euler-twisted Gromov–Witten invariants of $\mathrm{Fl}(E)$ with respect to V^G.

Theorem 5.0.2 (see Definition 7.2.1 and Theorem 7.2.2)**.** *For all $t \in \mathbb{C}^R$ and $\tau \in H^\bullet(X)$,*

$$I_{\mathrm{GM}}(t, \tau, -z) \in \mathcal{L}_{V^G_\lambda}$$

Under an ampleness condition – which holds, for example, whenever the blow-up $\tilde{X}$ in Theorem 5.0.1 is Fano – the family (5.2) takes a particularly simple form

$$I_{\mathrm{GM}}(t, \tau, z) = z\left(1 + o(z^{-1})\right)$$

and standard techniques in Givental formalism allow us to determine genus-zero twisted Gromov–Witten invariants of $\mathrm{Fl}(E)$ explicitly: see Corollaries 7.2.4 and 7.2.5. Applying this in the setting of Theorem 5.0.1, we recover genus-zero Gromov–Witten invariants of the blow-up $\tilde{X}$ by taking the non-equivariant limit $\lambda \to 0$.

The reader who is focussed on blow-ups can stop reading here, jumping to the end of the Introduction for connections to previous work, §6.2 for basic setup, Corollary 7.2.5 for the key Gromov–Witten theoretic result, and then to §7.4 for worked examples. In the rest of the Introduction, we explain how Theorem 5.0.2 should be regarded as an instance of the Abelian/non-Abelian Correspondence [CFKS08].

The Abelian/non-Abelian Correspondence relates the genus-zero Gromov–Witten theory of quotients $A/\!\!/G$ and $A/\!\!/T$, where A is a smooth quasiprojective variety equipped with the action of a reductive Lie group G, and T is its maximal torus. We fix a linearisation of this action such that the stable and semistable loci coincide and we suppose that the quotients $A/\!\!/G$ and $A/\!\!/T$ are smooth. In our setting the non-Abelian quotient $A/\!\!/G$ will be a partial flag bundle or Grassmann bundle over X, and the Abelian quotient $A/\!\!/T$ will be a bundle of toric varieties over X, that is, a toric bundle in the sense of Brown [Bro14]. To reformulate the Abelian/non-Abelian Correspondence of [CFKS08] in terms of Givental's formalism, however, we pass to the following more general situation. Let W denote the Weyl group of T in G. A theorem of Martin (Theorem 6.1.1 below) expresses the cohomology of the non-Abelian quotient $H^\bullet(A/\!\!/G)$ as a quotient of the Weyl-invariant part of the cohomology of the Abelian quotient $H^\bullet(A/\!\!/T)^W$ by an appropriate ideal, so there is a quotient map

$$H^\bullet(A/\!\!/T)^W \to H^\bullet(A/\!\!/G). \tag{5.3}$$

The Abelian/non-Abelian Correspondence, in the form that we state it below, asserts that this map also controls the relationship between the quantum cohomology of $A/\!\!/G$ and $A/\!\!/T$.

When comparing the quantum cohomology algebras of $A/\!\!/G$ and $A/\!\!/T$, or when comparing the Givental spaces of $A/\!\!/G$ and $A/\!\!/T$, we need to account for the fact that there are fewer curve classes on $A/\!\!/G$ than there are on $A/\!\!/T$. We do this as follows. The Givental space $\mathcal{H}_Y$ discussed above is defined using cohomology groups $H^\bullet(Y;\Lambda)$ where Λ is the Novikov ring for Y: see §6.3. The Novikov ring contains formal linear combinations of terms Q^d where d is a curve class on Y. The quotient map (5.3) induces an isomorphism $H^2(A/\!\!/T)^W \cong H^2(A/\!\!/G)$, and by duality this gives a map $\varrho\colon \mathrm{NE}(A/\!\!/T) \to \mathrm{NE}(A/\!\!/G)$ where NE denotes the Mori cone: see Proposition 6.1.4. Combining the quotient map (5.3) with the map on Novikov rings induced by ϱ gives a map

$$p\colon \mathcal{H}^W_{A/\!\!/T} \to \mathcal{H}_{A/\!\!/G} \tag{5.4}$$

between the Weyl-invariant part of the Givental space for the Abelian quotient and the Givental space for the non-Abelian quotient. Here, and also below when we discuss Weyl-invariant functions, we consider the Weyl group W to act on $\mathcal{H}_{A/\!\!/T}$ through the combination of its action on cohomology classes and its action on the Novikov ring.

We consider now an appropriate twisted Gromov–Witten theory of $A/\!\!/T$. For each root ρ of G, write $L_\rho \to A/\!\!/T$ for the line bundle determined by ρ, and let $\Phi = \oplus_\rho L_\rho$ where the sum runs over all roots. Consider the Lagrangian submanifold $\mathcal{L}_{\Phi_\lambda}$ that encodes genus-zero twisted

Gromov–Witten invariants of $A/\!\!/T$. The bundle Φ is very far from convex, so one cannot expect the non-equivariant limit of $\mathcal{L}_{\Phi_\lambda}$ to exist. Nonetheless, the projection along (5.4) of the Weyl-invariant part of this Lagrangian submanifold does have a non-equivariant limit.

Theorem 5.0.3. *(see Corollary 6.5.4) The limit as $\lambda \to 0$ of $p\Big(\mathcal{L}_{\Phi_\lambda} \cap \mathcal{H}^W_{A/\!\!/T}\Big)$ exists.*

We call this non-equivariant limit the *Givental–Martin cone*[1] $\mathcal{L}_{\mathrm{GM}} \subset \mathcal{H}_{A/\!\!/G}$.

Conjecture 5.0.4 (The Abelian/non-Abelian Correspondence)**.** $\mathcal{L}_{\mathrm{GM}} = \mathcal{L}_{A/\!\!/G}$.

This is a reformulation of [CFKS08, Conjecture 3.7.1]. The analogous statement for twisted Gromov–Witten invariants is the Abelian/non-Abelian Correspondence with bundles; this is a reformulation of [CFKS08, Conjecture 6.1.1]. Fix a representation ρ of G, and consider the vector bundles $V^G \to A/\!\!/G$ and $V^T \to A/\!\!/T$ induced by ρ. Consider the Lagrangian submanifold $\mathcal{L}_{\Phi_\lambda \oplus V^T_\mu}$ that encodes genus-zero twisted Gromov–Witten invariants of $A/\!\!/T$, where for the twist by the root bundle Φ we use the equivariant Euler class (5.1) with parameter λ and for the twist by V^T we use the equivariant Euler class with a different parameter μ. As before, the projection along (5.4) of the Weyl-invariant part of this Lagrangian submanifold has a non-equivariant limit with respect to λ.

Theorem 5.0.5. *(see Theorem 6.5.3) The limit as $\lambda \to 0$ of $p\Big(\mathcal{L}_{\Phi_\lambda \oplus V^T_\mu} \cap \mathcal{H}^W_{A/\!\!/T}\Big)$ exists.*

Let us call this limit the *twisted Givental–Martin cone* $\mathcal{L}_{\mathrm{GM},V^T_\mu} \subset \mathcal{H}_{A/\!\!/G}$.

Conjecture 5.0.6 (The Abelian/non-Abelian Correspondence with bundles)**.** $\mathcal{L}_{\mathrm{GM},V^T_\mu} = \mathcal{L}_{V^G_\mu}$.

As in [CFKS08], the Abelian/non-Abelian Correspondence implies the Abelian/non-Abelian Correspondence with bundles.

Proposition 5.0.7. *Conjectures 5.0.4 and 5.0.6 are equivalent.*

Proof. Conjecture 5.0.4 is the special case of Conjecture 5.0.6 where the vector bundles involved have rank zero. To see that Conjecture 5.0.4 implies Conjecture 5.0.6, observe that the projection of the Quantum Riemann–Roch operator $\Delta_{V^T_\mu}$ under the map (5.4) is $\Delta_{V^G_\mu}$: see Definition 6.4.3. Now apply the Quantum Riemann–Roch theorem [CG07]. □

The following reformulations will also be useful. Given any Weyl-invariant family

$$t \mapsto I(t) \in \mathcal{H}^W_{A/\!\!/T} \qquad \text{of the form} \qquad I(t) = \sum_{d \in \mathrm{NE}(A/\!\!/T)} Q^d I_d(t)$$

[1] We have not emphasised this point, but the Lagrangian submanifolds $\mathcal{L}_Y$, $\mathcal{L}_{F_\lambda}$, etc. are in fact cones [Giv04].

we define its *Weyl modification* $t \mapsto \widetilde{I}(t) \in \mathcal{H}^W_{A/\!\!/T}$ to be

$$\widetilde{I}(t) = \sum_{d \in \mathrm{NE}(A/\!\!/T)} Q^d W_d I_d(t)$$

where W_d is an explicit hypergeometric factor that depends on λ – see (6.14). We prove in Lemma 6.5.1 below that, for a Weyl-invariant family $t \mapsto I(t)$ the image under (5.4) of the Weyl modification $t \mapsto p(\widetilde{I}(t))$ has a well-defined limit as $\lambda \to 0$. We call this limit the *Givental–Martin modification* of $t \mapsto I(t)$ and denote it by $t \mapsto I_{\mathrm{GM}}(t)$; it is a family of elements of $\mathcal{H}_{A/\!\!/G}$. Furthermore, if $t \mapsto I(t)$ satisfies the Divisor Equation in the sense of equation (6.9), then:

- if $t \mapsto I(t)$ is a family of elements of $\mathcal{L}_{A/\!\!/T}$ then $t \mapsto I_{\mathrm{GM}}(t)$ is a family of elements on the Givental–Martin cone $\mathcal{L}_{\mathrm{GM}}$; and
- if $t \mapsto I(t)$ is a family of elements of the twisted cone $\mathcal{L}_{V^T_\mu}$ then $t \mapsto I_{\mathrm{GM}}(t)$ is a family of elements on the twisted Givental–Martin cone $\mathcal{L}_{\mathrm{GM},V^T_\mu}$.

The first statement here is Corollary 6.5.5 with $F' = 0$; the second statement is Corollary 6.5.5. This lets us reformulate the Abelian/non-Abelian Correspondence in more concrete terms.

Conjecture 5.0.8 (a reformulation of Conjecture 5.0.4)**.** *Let $t \mapsto I(t)$ be a Weyl-invariant family of elements of $\mathcal{L}_{A/\!\!/T}$ that satisfies the Divisor Equation. Then the Givental–Martin modification $t \mapsto I_{\mathrm{GM}}(t)$ is a family of elements of $\mathcal{L}_{A/\!\!/G}$.*

Conjecture 5.0.9 (a reformulation of Conjecture 5.0.6)**.** *Let $t \mapsto I(t)$ be a Weyl-invariant family of elements of $\mathcal{L}_{V^T_\mu}$ that satisfies the Divisor Equation. Then the Givental–Martin modification $t \mapsto I_{\mathrm{GM}}(t)$ is a family of elements of $\mathcal{L}_{V^G_\mu}$.*

Let us now specialise to the case of partial flag bundles, as in §6.2 and the rest of the paper, so that $A/\!\!/G$ is a partial flag bundle $\mathrm{Fl}(E) \to X$ and $A/\!\!/T$ is a toric bundle $\mathrm{Fl}(E)_T \to X$. Theorem 5.0.10 below establishes the statement of Conjecture 5.0.8 not for an arbitrary Weyl-invariant family $t \mapsto I(t)$ on $\mathcal{L}_{A/\!\!/T}$, but for a specific such family called the *Brown I-function.* As we recall in Theorems 7.1.1 and 7.1.2, Brown and Oh have defined families $t \mapsto I_{\mathrm{Fl}(E)_T}(t)$ and $t \mapsto I_{\mathrm{Fl}(E)}(t)$, given in terms of genus-zero Gromov–Witten invariants of X and explicit hypergeometric functions, and have shown [Bro14, Oh21] that $I_{\mathrm{Fl}(E)_T}(t) \in \mathcal{L}_{\mathrm{Fl}(E)_T}$ and $I_{\mathrm{Fl}(E)}(t) \in \mathcal{L}_{\mathrm{Fl}(E)}$.

Theorem 5.0.10 (see Proposition 7.1 for details)**.** *The Givental–Martin modification of the Brown I-function $t \mapsto I_{\mathrm{Fl}(E)_T}$ is $t \mapsto I_{\mathrm{Fl}(E)}(t)$.*

The main result of this paper is the analogue of Theorem 5.0.10 for twisted Gromov–Witten invariants. We define a twisted version $t \mapsto I_{V_\mu^T}(t)$ of the Brown I-function and prove:

Theorem 5.0.11 (see Definition 7.2.1 and Corollary 7.2.2 for details)**.**

1. *the twisted Brown I-function $t \mapsto I_{V_\mu^T}(t)$ is a Weyl-invariant family of elements of $\mathcal{L}_{V_\mu^T}$ that satisfies the Divisor Equation;*
2. *the Givental–Martin modification $t \mapsto I_{\mathrm{GM}}(t)$ of this family satisfies $I_{\mathrm{GM}}(t) \in \mathcal{L}_{V_\mu^G}$.*

This establishes the statement of Conjecture 5.0.9, not for an arbitrary Weyl-invariant family, but for the specific such family $t \mapsto I_{V_\mu^T}(t)$. Theorem 5.0.11 follows from the Quantum Riemann–Roch theorem [CG07] together with the results of Brown [Bro14] and Oh [Oh21], using a "twisting the I-function" argument as in [CCIT19].

As we will now explain, Theorem 5.0.10 is quite close to a proof of Conjecture 5.0.8 in the flag bundle case, and similarly Theorem 5.0.11 is close to a proof of Conjecture 5.0.9. We will discuss only the former, as the latter is very similar. Theorem 5.0.10 implies that

$$\text{the Givental–Martin modification } t \mapsto I_{\mathrm{GM}}(t) \text{ lies in } \mathcal{L}_{\mathrm{Fl}(E)} \tag{5.5}$$

for the family $t \mapsto I(t)$ given by the Brown I-function, because the Givental–Martin modification of the Brown I-function is the Oh I-function $t \mapsto I_{\mathrm{Fl}(E)}(t)$. If Oh's I-function were a *big I-function*, in the sense of [CFK16], then Conjecture 5.0.8 would follow. The special geometric properties of the Lagrangian submanifold $\mathcal{L}_Y$ described in [Giv04] and [CCIT09, Appendix B], taking $Y = \mathrm{Fl}(E)$, would then imply that any family $t \mapsto I(t)$ such that $I(t) \in \mathcal{L}_{\mathrm{Fl}(E)}$ can be written as

$$I(t) = I_{\mathrm{Fl}(E)}(\tau(t)) + \sum_\alpha C_\alpha(t,z) z \frac{\partial I_{\mathrm{Fl}(E)}}{\partial \tau_\alpha}(\tau(t)) \tag{5.6}$$

for some coefficients $C_\alpha(t,z)$ that depend polynomially on z and some change of variables $t \mapsto \tau(t)$. Furthermore the same geometric properties imply that any family of the form (5.6) satisfies $I(t) \in \mathcal{L}_{\mathrm{Fl}(E)}$. But $\mathcal{L}_{\mathrm{GM}}$ has the same special geometric properties as $\mathcal{L}_Y$ – it inherits them from the Weyl-invariant part of $\mathcal{L}_{\Phi_\lambda}$ by projection along (5.4) followed by taking the non-equivariant limit – and so if $t \mapsto I_{\mathrm{Fl}(E)}$ is a big I-function then any family of elements $t \mapsto I^\dagger(t)$ on $\mathcal{L}_{\mathrm{GM}}$ can be written as

$$I^\dagger(t) = I_{\mathrm{Fl}(E)}(\tau^\dagger(t)) + \sum_\alpha C^\dagger_\alpha(t,z) z \frac{\partial I_{\mathrm{Fl}(E)}}{\partial \tau_\alpha}(\tau^\dagger(t))$$

That is, $I^\dagger(t)$ can be written in the form (5.6). It follows that $I^\dagger(t) \in \mathcal{L}_{\mathrm{Fl}(E)}$. Applying this with $I^\dagger = I_{\mathrm{GM}}$ from Conjecture 5.0.8 proves that Conjecture; note that we know that the family $t \mapsto I_{\mathrm{GM}}(t)$ here lies in $\mathcal{L}_{\mathrm{GM}}$ by Corollary 6.5.5.

If the Brown and Oh I-functions were big I-functions then Theorem 5.0.10 would continue to hold (with the same proof) and Conjecture 5.0.8 would therefore follow. In reality the Brown and Oh I-functions are only small I-functions, not big I-functions, but Ciocan-Fontanine–Kim have explained in [CFK16, §5] how to pass from small I-functions to big I-functions, whenever the target space is the GIT quotient of a vector space. To apply their argument, and hence prove Conjecture 5.0.8 for partial flag bundles, one would need to check that the Brown I-function arises from torus localization on an appropriate quasimap graph space [CFKM14, §7.2]. The analogous result for the Oh I-function is [Oh21, Proposition 5.1].

Webb has proved a 'big I-function' version of the Abelian/non-Abelian Correspondence for target spaces that are GIT quotients of vector spaces [Web21], and this immediately implies Conjectures 5.0.8 and 5.0.9.

Proposition 5.0.12. *Conjecture 5.0.8 holds when A is a vector space and G acts on A via a representation $G \mapsto \mathrm{GL}(A)$.*

Proof. Combining [Web21, Corollary 6.3.1] with [CFK16, Theorem 3.3] shows that there are big I-functions $t \mapsto I_{A/\!\!/T}(t)$ and $t \mapsto I_{A/\!\!/G}(t)$ such that $I_{A/\!\!/T}(t) \in \mathcal{L}_{A/\!\!/T}$ and $I_{A/\!\!/G}(t) \in \mathcal{L}_{A/\!\!/G}$. Furthermore it is clear from [Web21, equation 62] that the Givental–Martin modification of the Weyl-invariant part of $t \mapsto I_{A/\!\!/T}(t)$ is $t \mapsto I_{A/\!\!/G}(t)$. Now argue as above. $\square$

Connection to Earlier Work

Our formulation of the Abelian/non-Abelian Correspondence very roughly says that, for genus-zero Gromov–Witten theory, passing from an Abelian quotient $A/\!\!/T$ to the corresponding non-Abelian quotient $A/\!\!/G$ is almost the same as twisting by the non-convex bundle $\Phi \to A/\!\!/T$ defined by the roots of G. This idea goes back to the earliest work on the subject, by Bertram–Ciocan-Fontanine–Kim, and indeed our Conjecture is very much in the spirit of the discussion in [BCFK08, §4]. These ideas were given a precise form in [CFKS08], in terms of Frobenius manifolds and Saito's period mapping; the main difference with the approach that we take here is that in [CFKS08] the authors realise the cohomology $H^\bullet(A/\!\!/G)$ as the Weyl-anti-invariant subalgebra of the cohomology of the Abelian quotient $A/\!\!/T$, whereas we realise it as a quotient of the Weyl-invariant part of $H^\bullet(A/\!\!/T)$. The latter approach seems to fit better with Givental's formalism.

Ruan was the first to realise that there is a close connection between quantum cohomology (or more generally Gromov–Witten theory) and birational geometry [Rua99], and the change in Gromov–Witten invariants under blow-up forms an important testing ground for these ideas. Despite the importance of the topic, however, Gromov–Witten invariants of blow-ups have been understood in rather few situations. Early work here focussed on blow-ups in points, and on exploiting structural properties of quantum cohomology such as the WDVV equations and Reconstruction Theorems [Gat96, GP98, Gat01]. Subsequent approaches used symplectic methods pioneered by Li–Ruan [LR01, HLR08, Hu00, Hu01], or the Degeneration Formula following Maulik–Pandharipande [MP06, HHKQ18, CDW20], or a direct analysis of the moduli spaces involved and virtual birationality arguments [Man12, Lai09, AW18]. In each case the aim was to prove 'birational invariance': that certain specific Gromov–Witten invariants remain invariant under blow-up. We take a different approach. Rather than deform the target space, or study the geometry of moduli spaces of stable maps explicitly, we give an elementary construction of the blow-up $\tilde{X} \to X$ in terms that are compatible with modern tools for computing Gromov–Witten invariants, and extend these tools so that they cover the cases we need. This idea – of reworking classical constructions in birational geometry to make them amenable to computations using Givental formalism – was pioneered in [CCGK16], and indeed Lemma E.1 there gives the codimension-two case of our Theorem 5.0.1.

Compared to explicit invariance statements

$$\langle \pi^* \phi_{i_1}, \ldots, \pi^* \phi_{i_n} \rangle^{\tilde{X}}_{0,n,\pi^! \beta} = \langle \phi_{i_1}, \ldots, \phi_{i_n} \rangle^{X}_{0,n,\beta}$$

as in [Lai09, Theorem 1.4], we pay a price for our increased abstraction: the range of invariants for which we can extract closed-form expressions is different (see Corollary 7.2.4) and in general does not overlap with Lai's. But we also gain a lot by taking a more structural approach: our results determine, via a Birkhoff factorization procedure as in [CG07, CFK14], genus-zero Gromov–Witten invariants of the blow-up $\tilde{X}$ for curves of arbitrary degree (not just proper transforms of curves in the base) and with a wide range of insertions that can include gravitational descendant classes. See Remark 7.2.9. Furthermore in general one should not expect Gromov–Witten invariants to remain invariant under blow-ups. The correct statement – cf. Ruan's Crepant Resolution Conjecture [CIT09, CR13, Iri10, Iri09] and its generalisation by Iritani [Iri20] – is believed to involve analytic continuation of Givental cones, and we hope that our formulation here will be a step towards this.

After the first version of this paper appeared on the arXiv, Fenglong You pointed us to the work [LLW17] in which Lee, Lin, and Wang sketch a construction of blow-ups that is very similar to Theorem 5.0.1, and use this to compute Gromov–Witten invariants of blow-ups in complete intersections. The methods they use are different: they rely on a very interesting extension of the Quantum Lefschetz theorem to certain non-split bundles, which they will prove in forthcoming work [LLW]. At first sight, their result [LLW17, Theorem 5.1] is both more general and less explicit than our results. In fact, we believe neither is true. Their theorem as stated applies to blow-ups in complete intersections defined by arbitrary line bundles whereas we require these line bundles to be convex; however, discussions with the authors suggest that both results apply under the same conditions, and the convexity hypothesis was omitted from [LLW17, Theorem 5.1] in error. Furthermore, Lee, Lin, and Wang extract genus-zero Gromov–Witten invariants by combining their generalised Quantum Lefschetz theorem with an inexplicit Birkhoff factorisation procedure whereas we use the formalism of Givental cones. We believe, though, that one can rephrase their argument entirely in terms of Givental's formalism, and after doing so their results become explicit in exactly the same range as ours. The explicit formulas are different, however, and it would be interesting to see if one can derive non-trivial identities from this. Note that Proposition 7.3.2 below is more general than the construction in [LLW17, Section 5]: the fact that we consider Grassmann bundles rather than projective bundles allows us to treat blow-ups in certain degeneracy loci. Combining this with the methods in Section 7.4 allows one to compute genus-zero Gromov–Witten invariants of blow-ups in such degeneracy loci.

One of the most striking features of Givental's formalism is that relationships between higher-genus Gromov–Witten invariants of different spaces can often be expressed as the quantisation, in a precise sense, of the corresponding relationship between the Lagrangian cones that encode genus-zero invariants [Giv04]. Our version of the Abelian/non-Abelian Correspondence hints, therefore, at a higher-genus generalisation. It would be very interesting to develop and prove a higher-genus analog of Conjecture 5.0.4.

Chapter 6

I-functions and Lagrangian cones

6.1 The topology of quotients by a non-Abelian group and its maximal torus

Let G be a complex reductive group acting on a smooth quasi-projective variety A with polarisation given by a linearised ample line bundle L. Let $T \subset G$ be a maximal torus. One can then form the GIT-quotients $A /\!\!/ G$ and $A /\!\!/ T$. We will assume that the stable and semistable points with respect to these linearisations coincide, and that all the isotropy groups of the stable points are trivial; this ensures that the quotients $A /\!\!/ G$ and $A /\!\!/ T$ are smooth projective varieties. The Abelian/non-Abelian Correspondence [CFKS08] relates the genus zero Gromov–Witten invariants of these two quotients. Let $A^s(G)$, and respectively $A^s(T)$, denote the subsets of A consisting of points that are stable for the action of G, and respectively T. The two geometric quotients $A /\!\!/ G$ and $A /\!\!/ T$ fit into a diagram

$$\begin{tikzcd} A/\!\!/T & A^s(G)/T \arrow[l, hook', "j"'] \arrow[d, "q"] \\ & A/\!\!/G \end{tikzcd} \tag{6.1}$$

where j is the natural inclusion and π the natural projection.

A representation $\rho\colon G \to \mathrm{GL}(V)$ induces a vector bundle V_ρ on $A /\!\!/ G$ with fiber V. Explicitly, $V_\rho = (A \times V) /\!\!/ G$ where G acts as

$$g\colon (a, v) \mapsto (ag, \rho(g^{-1})v). \tag{6.2}$$

We call the bundle V_ρ representation-theoretic. Similarly, the restriction $\rho|_T$ of the representation ρ induces a vector bundle V_{ρ_T} over $A /\!\!/ T$. Note that since T is Abelian, V_{ρ_T} splits as a direct sum of line bundles, $V_{\rho_T} = L_1 \oplus \cdots \oplus L_k$ These bundles satisfy

$$j^* V_{\rho_T} \cong q^* V_\rho. \tag{6.3}$$

When the representation $\rho\colon G \to \mathrm{GL}(V)$ is clear from context, we will suppress it from the notation, writing V^G for V_ρ and V^T for V_{ρ_T}.

We will now describe the relationship between the cohomology rings of $A/\!\!/G$ and $A/\!\!/T$, following [Mar00]. Let W be the Weyl group of G. W acts on $A/\!\!/T$ and hence on the cohomology ring $H^\bullet(A/\!\!/T)$. Restricting the adjoint representation $\rho\colon G \to \mathrm{GL}(\mathfrak{g})$ to T, we obtain a splitting $\rho|_T = \oplus_\alpha \rho_\alpha$ into 1-dimensional representations, i.e. characters, of T. The set Δ of characters appearing in this decomposition is the set of roots of G, and forms a root system. Write L_α for the line bundle on $A/\!\!/T$ corresponding to a root α. Fix a set of positive roots Φ^+ and define

$$\omega = \prod_{\alpha \in \Phi^+} c_1(L_\alpha).$$

ω is the fundamental anti-invariant class, it satifies $w \cdot \omega = \mathrm{sgn}(w)\omega$ where sgn is the sign representation of the Weyl group, defined by writing w as a product of k reflections, and declaring $\mathrm{sgn}(w)$ to be ± 1 depending on whether k is even or odd.

Theorem 6.1.1 (Martin)**.** *There is a natural ring homomorphism*

$$H^\bullet(A/\!\!/G) \cong \frac{H^\bullet(A/\!\!/T)^W}{\mathrm{Ann}(\omega)}$$

under which $x \in H^\bullet(A/\!\!/G)$ maps to $\tilde{x} \in H^\bullet(A/\!\!/T)$ if and only if $q^* x = j^* \tilde{x}$.

Theorem 6.1.1 shows that any cohomology class $\tilde{x} \in H^\bullet(A/\!\!/T)^W$ is a lift of a class $x \in H^\bullet(A/\!\!/G)$, with $\tilde{x}$ unique up to an element of $\mathrm{Ann}(\omega)$.

Remark 6.1.2. Note that Martin states this theorem with ω replaced by the product over all roots e. However, it follows from [Kir05, Lemma 7.10] that the annihilators of e and ω on $H^*(A/\!\!/T)^W$ are the same.

Assumption 6.1.3. Throughout this paper, we will assume that the G-unstable locus $A \backslash A^s(G)$ has codimension at least 2.

This implies that elements of $H^2(A/\!\!/G)$ can be lifted uniquely:

Proposition 6.1.4. *Pullback via q gives an isomorphism $H^2(A/\!\!/G) \cong H^2(A/\!\!/T)^W$, and induces a map $\varrho\colon \mathrm{NE}(A/\!\!/T) \to \mathrm{NE}(A/\!\!/G)$ where* NE *denotes the Mori cone.*

Proof. The assumption that $A\backslash A^s(G)$ has codimension at least 2 implies that $A^s(T)/T\backslash A^s(G)/T$ has codimension at least 2, so j induces an isomorphisms $\mathrm{Pic}(A^s(G)/T) \cong \mathrm{Pic}(A^s(T)/T)$ and $H^2(A^s(G)/T) \cong H^2(A^s(T)/T)$. Since q^* always induces an isomorphism between $H^2(A/\!\!/G)$ and $H^2(A^s(G)/T)^W$ [Bor53], the first claim follows. Consequently, the lifting of divisor classes is unique and can be identified with the pullback map $q^*\colon \mathrm{Pic}(A/\!\!/G) \to \mathrm{Pic}(A^s(G)/T)$. By duality, we obtain a map $\varrho : \mathrm{NE}(A/\!\!/T) \to \mathrm{NE}(A/\!\!/G)$. □

Definition 6.1.5. We say that $\tilde{\beta} \in \mathrm{NE}(A/\!\!/T)$ lifts $\beta \in \mathrm{NE}(A/\!\!/G)$ if $\varrho(\tilde{\beta}) = \beta$. Note that any effective β has finitely many lifts.

6.2 Partial flag varieties and partial flag bundles

Notation

We will now specialise to the case of flag bundles and introduce notation used in the rest of the paper. Fix once and for all:

- a positive integer n and a sequence of positive integers $r_1 < \cdots < r_\ell < r_{\ell+1} = n$;
- a vector bundle $E \to X$ of rank n on a smooth projective variety X which splits as a direct sum of line bundles $E = L_1 \oplus \cdots \oplus L_n$.

We write Fl for the partial flag manifold $\mathrm{Fl}(r_1, \ldots, r_\ell; n)$, and $\mathrm{Fl}(E)$ for the partial flag bundle $\mathrm{Fl}(r_1, \ldots, r_\ell; E)$.

Set $N = \sum_{i=1}^{\ell} r_i r_{i+1}$ and $R = r_1 + \cdots + r_\ell$. It will be convenient to use the indexing $\{(1,1), \ldots (1,r_1), (2,1), \ldots, (\ell, r_\ell)\}$ for the set of positive integers smaller or equal than R.

Partial flag varieties and partial flag bundles as GIT quotients

The partial flag manifold Fl arises as a GIT quotient, as follows. Consider $\mathbb{C}^N$ as the space of homomorphisms

$$\bigoplus_{i=1}^{\ell} \mathrm{Hom}\left(\mathbb{C}^{r_i}, \mathbb{C}^{r_{i+1}}\right). \tag{6.4}$$

The group $G = \prod_{i=1}^{\ell} \mathrm{GL}_{r_i}(\mathbb{C})$ acts on $\mathbb{C}^N$ by

$$(g_1, \ldots, g_\ell) \cdot (A_1, \ldots, A_\ell) = (g_2^{-1} A_1 g_1, \ldots, g_\ell^{-1} A_{\ell-1} g_{\ell-1}, A_\ell g_\ell).$$

Let $\rho_i\colon G \to \mathrm{GL}_{r_i}(\mathbb{C})$ be the representation which is the identity on the ith factor and trivial on all other factors. Choosing the linearisation $\chi = \bigotimes_{i=1}^{\ell} \det(\rho_i)$, we have that $\mathbb{C}^N /\!\!/_\chi G$ is the partial flag manifold Fl. More generally, the partial flag bundle also arises as a GIT quotient, of the total space of the bundle of homomorphisms

$$\bigoplus_{i=1}^{\ell-1} \mathrm{Hom}\left(\mathcal{O}^{\oplus r_i}, \mathcal{O}^{\oplus r_{i+1}}\right) \oplus \mathrm{Hom}\left(\mathcal{O}^{\oplus r_\ell}, E\right) \tag{6.5}$$

with respect to the same group G and the same linearisation. $\mathrm{Fl}(E)$ carries ℓ tautological bundles of ranks $r_1, \ldots, r_\ell$, which we will denote $S_1, \ldots, S_\ell$. These bundles restrict to the usual tautological bundles on Fl on each fibre. The bundle S_i is induced by the representation ρ_i.

Definition 6.2.1. Let

$$p_i(t) = t^{r_i} - c_1(S_i^\vee)t^{r_i-1} + \cdots + (-1)^{r_i} c_{r_i}(S_i^\vee)$$

be the Chern polynomial of $S_i^\vee$. We denote the roots of p_i by $H_{i,j}$, $1 \leq j \leq r_i$. The $H_{i,j}$ are in general only defined over an appropriate ring extension of $H^\bullet(\mathrm{Fl}(E), \mathbb{C})$, but symmetric polynomials in the $H_{i,j}$ can be expressed in terms of $c_k(S_i^\vee)$ and therefore give well-defined elements of $H^\bullet(\mathrm{Fl}(E), \mathbb{C})$.

The maximal torus $T \subset G$ is isomorphic to $(\mathbb{C}^\times)^R$. Since E is a direct sum of line bundles, the corresponding Abelian quotient

$$\mathrm{Fl}(E)_T := \mathrm{Hom}\left(\cdots\right) /\!\!/_\chi (\mathbb{C}^\times)^R,$$

where $\mathrm{Hom}\left(\cdots\right)$ is the bundle of homomorphisms (6.5), is a fibre bundle over X with general fibre isomorphic to the toric variety $\mathrm{Fl}_T := \mathbb{C}^N /\!\!/_\chi (\mathbb{C}^\times)^R$. The space $\mathrm{Fl}(E)_T$ also carries natural cohomology classes:

Definition 6.2.2. Let $\rho_{i,j} \colon (\mathbb{C}^\times)^R \to \mathrm{GL}_1(\mathbb{C})$ be the dual of the one-dimensional representation of $(\mathbb{C}^\times)^R$ given by projection to the (i,j)th factor $\mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$; here we use the indexing of the set $\{1, 2, \ldots, R\}$ specified in §6.2. We define $L_{i,j} \in H^2(\mathrm{Fl}_T, \mathbb{C})$ to be the line bundle on $\mathrm{Fl}(E)_T$ induced by $\rho_{i,j}$ and denote its first Chern class by $\tilde{H}_{i,j}$. Similarly, we define $h_{i,j}$ to be the first Chern class of the line bundle on Fl_T induced by the representation $\rho_{i,j}$. Note that $h_{i,j}$ is the restriction of $\tilde{H}_{i,j}$ to a general fibre Fl_T of $\mathrm{Fl}(E)_T$.

Recall that, for a representation ρ of G, the corresponding vector bundle V^T splits as a direct sum of line bundles $F_1 \oplus \cdots \oplus F_k$. It is a general fact that if f is a symmetric polynomial in the $c_1(F_i)$, then f can be written as a polynomial in the elementary symmetric polynomials $e_r(c_1(F_1), \ldots, c_1(F_k))$, that is, in the Chern classes $c_r(V^T)$. By (6.3) we have that $j^* c_r(V^T) =$

$q^* c_r(V^G)$, and so replacing any occurrence of $c_r(V^T)$ by $c_r(V^G)$ gives an expression $g \in H^\bullet(A /\!\!/ G)$ which satisfies $q^* g = j^* f$. That is, f is a lift of g. Applying this to the dual of the standard representation ρ_i of the ith factor of G shows that any polynomial p which is symmetric in each of the sets $\tilde{H}_{i,j}$ for fixed i projects to the same expression in $H^\bullet(\mathrm{Fl}(E))$ with any occurrence of $\tilde{H}_{i,j}$ replaced by the corresponding Chern root $H_{i,j}$. We have the following standard lemma, which we will state without proof.

Lemma 6.2.3. *Let $(\mathbb{C}^\times)^R$ act on $\mathbb{C}^N$, arrange the weights for this action in an $R \times N$-matrix $(m_{i,k})$ and consider $E = L_1 \oplus \cdots \oplus L_N \xrightarrow{\pi} X$ a direct sum of line bundles. Form the associated toric fibration $E /\!\!/ (\mathbb{C}^\times)^R$ with general fibre $\mathbb{C}^N /\!\!/ (\mathbb{C}^\times)^R$ and let h_i (respectively H_i) be the first Chern class of the line bundle on $\mathbb{C}^N /\!\!/ (\mathbb{C}^\times)^R$ (respectively on $E /\!\!/ (\mathbb{C}^\times)^R$ induced by the dual of the representation which is standard on the ith factor of $(\mathbb{C}^\times)^R$ and trivial on the other factors. Then*

- *the Poincaré duals u_k of the torus invariant divisors of the toric variety $\mathbb{C}^N /\!\!/ (\mathbb{C}^\times)^R$ are:*

$$u_k = \sum_{k=1}^{R} m_{i,k} h_i$$

- *the Poincaré duals U_k of the torus invariant divisors of the total space of the toric fibration $E /\!\!/ (\mathbb{C}^\times)^R \xrightarrow{\pi} X$ are:*

$$U_k = \sum_{k=1}^{R} m_{i,k} H_i + \pi^* c_1(L_k)$$

When applying Lemma 6.2.3 to our situation (6.5) it will be convenient to define $H_{\ell+1,j} := \pi^* c_1(L_j^\vee)$. Then the set of torus invariant divisors is

$$H_{i,j} - H_{i+1,j'} \qquad\qquad 1 \le i \le \ell,\ 1 \le j \le r_i,\ 1 \le j' \le r_{i+1}$$

We will also need to know about the ample cone of a toric variety $\mathbb{C}^N /\!\!/ (\mathbb{C}^\times)^R$. This is most easily described in terms of the secondary fan, that is, by the wall-and-chamber decomposition of $\mathrm{Pic}(\mathbb{C}^N /\!\!/ (\mathbb{C}^\times)^R) \otimes \mathbb{R} \cong \mathbb{R}^R$ given by the cones spanned by size $R-1$ subsets of columns of the weight matrix. The ample cone of $\mathbb{C}^N /\!\!/ (\mathbb{C}^\times)^R$ is then the chamber that contains the stability condition χ. Moreover, for a subset $\alpha \subset \{1, \ldots, N\}$ of size R the cone in the secondary fan spanned by the classes u_k, $k \in \alpha$, contains the stability condition (and therefore also the ample cone) iff the intersection $u_\alpha = \bigcap_{k \notin \alpha} u_k$ is nonempty. In this case, $U_\alpha = \bigcap_{k \notin \alpha} U_k$ restricts to a torus fixed point on every fibre and, since E splits as a direct sum of line bundles, U_α is the image of a section of the toric fibration π. We denote this section by s_α. By construction, the torus invariant divisors U_k, $k \in \alpha$, do not meet U_α, so that $s_\alpha^*(U_k) = 0$ for all $k \in \alpha$. For the toric variety Fl_T one can easily write down the set of R-dimensional cones containing $\chi = (1, \dots, 1)$. For each index (i, j), choose some $j' \in \{1, \dots, r_{i+1}\}$. Then the cone spanned by

$$h_{i,j} - h_{i+1,j'} \qquad 1 \leq i < \ell - 1,\ 1 \leq j \leq r_i \qquad h_{\ell,j},\ 1 \leq j \leq r_\ell \tag{6.6}$$

contains χ and every cone containing χ is of that form.

6.3 Givental's Formalism

In this section we review Givental's geometric formalism for Gromov–Witten theory, concentrating on the genus-zero case. The main reference for this is [Giv04]. Let Y be a smooth projective variety and consider

$$\mathcal{H}_Y = H^\bullet(Y, \Lambda)[z, z^{-1}]\!] = \Big\{ \sum_{k=-\infty}^{m} a_i z^i \colon a_i \in H^\bullet(Y, \Lambda),\ m \in \mathbb{Z} \Big\}$$

where z is an indeterminate and Λ is the Novikov ring for Y. After picking a basis $\{\phi_1, \dots, \phi_N\}$ for $H^\bullet(Y; \mathbb{C})$ with $\phi_1 = 1$ and writing $\{\phi^1, \dots, \phi^N\}$ for the Poincaré dual basis, we can write elements of $\mathcal{H}_Y$ as

$$\sum_{i=0}^{m} \sum_{\alpha=1}^{N} q_i^\alpha \phi_\alpha z^i + \sum_{i=0}^{\infty} \sum_{\alpha=1}^{N} p_{i,\alpha} \phi^\alpha (-z)^{-1-i} \tag{6.7}$$

where q_i^α, $p_{i,\alpha} \in \Lambda$. The q_i^α, $p_{i,\alpha}$ then provide coordinates on $\mathcal{H}_Y$. The space $\mathcal{H}_Y$ carries a symplectic form

$$\begin{aligned} \Omega \colon\ & \mathcal{H}_Y \otimes \mathcal{H}_Y \to \Lambda \\ & f \otimes g \to \operatorname{Res}_{z=0}(f(-z), g(z))\, dz \end{aligned}$$

where $(\cdot, \cdot)$ denotes the Poincaré pairing, extended $\mathbb{C}[z, z^{-1}]\!]$-linearly to $\mathcal{H}_Y$. By construction, Ω is in Darboux form with respect to our coordinates:

$$\Omega = \sum_i \sum_\alpha dp_{i,\alpha} \wedge dq_i^\alpha$$

We fix a Lagrangian polarisation of $\mathcal{H}$ as $\mathcal{H}_Y = \mathcal{H}_+ \oplus \mathcal{H}_-$, where

$$\mathcal{H}_+ = H^\bullet(Y; \Lambda)[z], \quad \mathcal{H}_- = z^{-1} H^\bullet(Y; \Lambda)[\![z^{-1}]\!]$$

This polarisation $\mathcal{H}_Y = \mathcal{H}_+ \oplus \mathcal{H}_-$ identifies $\mathcal{H}_Y$ with $T^*\mathcal{H}_+$. We now relate this to Gromov–Witten theory.

Definition 6.3.1. The *genus-zero descendant potential* is a generating function for genus-zero Gromov–Witten invariants:

$$\mathcal{F}_Y^0 = \sum_{n=0}^{\infty} \sum_{d \in \mathrm{NE}(Y)} \frac{Q^d}{n!} t_{i_1}^{\alpha_1} \ldots t_{i_n}^{\alpha_n} \langle \phi_{\alpha_1} \psi^{i_1}, \ldots, \phi_{\alpha_n} \psi^{i_n} \rangle_{0,n,d}$$

Here $\langle \ldots, \rangle_{0,n,d}$ denotes a degree d genus 0 Gromov–Witten invariant with n insertions, t_i^α is a formal variable, $\mathrm{NE}(Y)$ denotes the Mori cone of Y, and Einstein summation is used for repeated lower and upper indices.

After setting

$$t_i^\alpha = q_i^\alpha + \delta_1^i \delta_\alpha^1, \tag{6.8}$$

where δ_i^j denotes the Kronecker delta, we obtain a (formal germ of a) function $\mathcal{F}_Y^0 \colon \mathcal{H}_+ \to \Lambda$.

Definition 6.3.2. The Givental cone $\mathcal{L}_Y$ of Y is the graph of the differential of $\mathcal{F}_Y^0 \colon \mathcal{H}_+ \to \Lambda$:

$$\mathcal{L}_Y = \left\{ (\mathbf{q}, \mathbf{p}) \in T^*\mathcal{H}_+ = \mathcal{H}_+ \oplus \mathcal{H}_- : p_{i,\alpha} = \frac{\partial \mathcal{F}_Y^0}{\partial q_i^\alpha} \right\}$$

Note that $\mathcal{L}_Y$ is Lagrangian by virtue of being the graph of the differential of a function. Moreover, it has the following special geometric properties [Giv04, CCIT09, CG07]

- $\mathcal{L}$ is preserved by scalar multiplication, i.e. it is (the formal germ of) a cone
- the tangent space T_f of $\mathcal{L}_Y$ at $f \in \mathcal{L}_Y$ is tangent to $\mathcal{L}$ exactly along zT_f. This means:

   1. $zT_f \subset \mathcal{L}_Y$
   2. for $g \in zT_f$, we have $T_g = T_f$
   3. $T_f \cap \mathcal{L}_Y = zT_f$

A general point of $\mathcal{L}_Y$ can be written, in view of the dilaton shift (6.8), as

$$\begin{aligned} &-z + \sum_{i=0}^{\infty} t_i^\alpha \phi_\alpha z^i + \sum_{n=0}^{\infty} \sum_{d \in \mathrm{NE}(Y)} \frac{Q^d}{n!} t_{i_1}^{\alpha_1} \ldots t_{i_n}^{\alpha_n} \langle \phi_{\alpha_1} \psi^{i_1}, \ldots, \phi_{\alpha_n} \psi^{i_n}, \phi_\alpha \psi^i \rangle_{0,n+1,d} \phi^\alpha (-z)^{-i-1} \\ =&-z + \sum_{i=0}^{\infty} t_i^\alpha \phi_\alpha z^i + \sum_{n=0}^{\infty} \sum_{d \in \mathrm{NE}(Y)} \frac{Q^d}{n!} t_{i_1}^{\alpha_1} \ldots t_{i_n}^{\alpha_n} \langle \phi_{\alpha_1} \psi^{i_1}, \ldots, \phi_{\alpha_n} \psi^{i_n}, \frac{\phi_\alpha}{-z-\psi} \rangle_{0,n+1,d} \phi^\alpha \end{aligned}$$

Thus knowing $\mathcal{L}_Y$ is equivalent to knowing all genus-zero Gromov–Witten invariants of Y. Setting $t_k^\alpha = 0$ for all $k > 0$, we obtain the *J-function* of Y:

$$J(\tau, -z) = -z + \tau + \sum_{n=0}^{\infty} \sum_{d \in \mathrm{NE(X)}} \frac{Q^d}{n!} \left\langle \tau, \ldots \tau, \frac{\phi_\alpha}{-z-\psi} \right\rangle_{0,n+1,d} \phi^\alpha$$

where $\tau = t_0^1 \phi_1 + \ldots t_0^N \phi_N \in H^\bullet(Y)$. The J-function is the unique family of elements $\tau \mapsto J(\tau, -z)$ on the Lagrangian cone such that

$$J(\tau, -z) = -z + \tau + O(z^{-1}).$$

We will need a generalisation of all of this to twisted Gromov–Witten invariants [CG07]. Let F be a vector bundle on Y and consider the universal family over the moduli space of stable maps

$$\begin{array}{ccc} C_{0,n,d} & \xrightarrow{f} & Y \\ {\scriptstyle \pi}\big\downarrow & & \\ Y_{0,n,d} & & \end{array}$$

Let $\pi_!$ be the pushforward in K-theory. We define

$$F_{0,n,d} = \pi_! f^* F = R^0 \pi_* f^* F - R^1 \pi_* f^* F$$

(the higher derived functors vanish). In general $F_{0,n,d}$ is a class in K-theory and not an honest vector bundle. This means that in order to evaluate a characteristic class $\mathbf{c}(\cdot)$ on $F_{0,n,d}$ we need $\mathbf{c}(\cdot)$ to be *multiplicative* and *invertible*. We can then set

$$\mathbf{c}(F_{0,n,d}) = \mathbf{c}(R^0 \pi_* f^* F) \cup \mathbf{c}(R^1 \pi_* f^* F)^{-1}$$

where $\mathbf{c}(R^i \pi_* f^* F)$ is defined using an appropriate locally free resolution.

Definition 6.3.3. Let F be a vector bundle on Y and let $\mathbf{c}(\cdot)$ be an invertible multiplicative characteristic class. We will refer to the pair $(F, \mathbf{c})$ as twisting data. Define $(F, \mathbf{c})$-twisted Gromov–Witten invariants as

$$\langle \alpha_1 \psi_1^{i_1}, \ldots \alpha_n \psi_n^{i_n} \rangle_{0,n,d}^{F,\mathbf{c}} = \int_{[Y_{0,n,d}]^{\mathrm{vir}} \cap \mathbf{c}(F_{0,n,d})} \mathrm{ev}_1^* \alpha_1 \cup \cdots \cup \mathrm{ev}_n^* \alpha_n \cup \psi_1^{i_1} \cup \cdots \cup \psi_n^{i_n}$$

Any multiplicative invertible characteristic class can be written as $\mathbf{c}(\cdot) = \exp(\sum_{k \geq 0} s_k \,\mathrm{ch}_k(\cdot))$, where ch_k is the kth component of the Chern character and $s_0, s_1, \ldots$ are appropriate coefficients. So we work with cohomology groups $H^\bullet(X, \Lambda_s)$, where Λ_s is the completion of $\Lambda[s_0, s_1, \ldots]$ with respect to the valuation

$$v(Q^d) = \langle c_1(\mathcal{O}(1)), d \rangle, \quad v(s_k) = k+1.$$

Most of the definitions from before now carry over. We have the twisted Poincaré pairing $(\alpha,\beta)^{F,\mathbf{c}} = \int_Y \mathbf{c}(F) \cup \alpha \cup \beta$ which defines the basis $\phi^1, \ldots \phi^N$ dual to our chosen basis $1 = \phi_1, \ldots, \phi_N$ for $H^\bullet(Y)$. The Givental space becomes $\mathcal{H}_Y = H^\bullet(Y, \Lambda_s) \otimes \mathbb{C}[z, z^{-1}]\!]$ with the twisted symplectic form

$$\Omega^{F,\mathbf{c}}(f(z), g(z)) = \mathrm{Res}_{z=0} \big(f(-z), g(z)\big)^{F,\mathbf{c}} dz.$$

This form admits Darboux coordinates as before which give a Lagrangian polarisation of $\mathcal{H}_Y$. Then the twisted Lagrangian cone $\mathcal{L}_{F,\mathbf{c}}$ is defined, via the dilaton shift (6.8), as the graph of the differential of the generating function $\mathcal{F}_Y^{0,F,\mathbf{c}}$ for genus zero *twisted* Gromov–Witten invariants. Finally, just as before, we can define a twisted J-function:

Definition 6.3.4. Given twisting data $(F, \mathbf{c})$ for Y, the twisted J-function is:

$$J_{F,\mathbf{c}}(\tau, -z) = -z + \tau + \sum_{n=0}^{\infty} \sum_{d \in \mathrm{NE}(Y)} \frac{Q^d}{n!} \left\langle \tau, \ldots \tau, \frac{\phi_\alpha}{-z-\psi} \right\rangle_{0,n+1,d}^{F,\mathbf{c}} \phi^\alpha$$

This is once again characterised as the unique family $\tau \mapsto J_{F,\mathbf{c}}(\tau, -z)$ of elements of the twisted Lagrangian cone of the form

$$J_{F,\mathbf{c}}(\tau, -z) = -z + \tau + O(z^{-1})$$

Note that we can recover the untwisted theory by setting $\mathbf{c} = 1$.

In what follows we take $\mathbf{c}$ to be the $\mathbb{C}^\times$-equivariant Euler class (5.1), which is multiplicative and invertible. The $\mathbb{C}^\times$-action here is the canonical $\mathbb{C}^\times$-action on any vector bundle given by rescaling the fibres. We write F_λ for the twisting data $(F, \mathbf{c})$, where F is equipped with the $\mathbb{C}^\times$-action given by rescaling the fibres with equivariant parameter λ. In this setting, Gromov–Witten invariants (and the coefficients s_k) take values in the fraction field $\mathbb{C}(\lambda)$ of the $\mathbb{C}^\times$-equivariant cohomology of a point. Here λ is the hyperplane class on $\mathbb{CP}^\infty$, so that $H^\bullet_{\mathbb{C}^\times}(\{\mathrm{pt}\}) = \mathbb{C}[\lambda]$, and we work over the field $\mathbb{C}(\lambda)$.

Remark 6.3.5. As we have set things up, the twisted cone $\mathcal{L}_{F_\lambda}$ is a Lagrangian submanifold of the symplectic vector space $\big(\mathcal{H}_Y, \Omega^{F_\lambda}\big)$, so as λ varies both the Lagrangian submanifold and the ambient symplectic space change. To obtain the picture described in the Introduction, where all the Lagrangian submanifolds $\mathcal{L}_{F_\lambda}$ lie in a single symplectic vector space $\big(\mathcal{H}_Y, \Omega\big)$, one can identify $\big(\mathcal{H}_Y, \Omega\big)$ with $\big(\mathcal{H}_Y, \Omega^{F_\lambda}\big)$ by multiplication by the square root of the equivariant Euler class of F. See [CG07, §8] for details.

6.4 Twisting the I-function

We will now prove a general result following an argument from [CCIT09]. We say that a family $\tau \mapsto I(\tau)$ of elements of $\mathcal{H}_Y$ *satisfies the Divisor Equation* if the parameter domain for τ is a product $U \times H^2(Y)$ and $I(\tau)$ takes the form

$$I(\tau) = \sum_{\beta \in \mathrm{NE}(Y)} Q^\beta I_\beta(\tau, z)$$

where

$$z\nabla_\rho I_\beta = \big(\rho + \langle \rho, \beta\rangle z\big) I_\beta \qquad \text{for all } \rho \in H^2(Y). \tag{6.9}$$

Here ∇_ρ is the directional derivative along ρ. Let F' be a vector bundle on Y, and consider any family $\tau \mapsto I(\tau) \in \mathcal{L}_{F'_\mu}$ that satisfies the Divisor Equation. Given another vector bundle F which splits as a direct sum of line bundles $F = F_1 \oplus \cdots \oplus F_k$, we explain how to modify the family $\tau \mapsto I(\tau)$ by introducing explicit hypergeometric factors that depend on F. We prove that (1) this modified family can be written in terms of the *Quantum Riemann-Roch operator* and the original family; and (2) the modified family lies on the twisted Lagrangian cone $\mathcal{L}_{F_\lambda \oplus F'_\mu}$.

Definition 6.4.1. Define the element $G(x, z) \in \mathcal{H}_Y$ by

$$G(x, z) := \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} s_{l+m-1} \frac{B_m}{m!} \frac{x^l}{l!} z^{m-1}$$

where B_m are the Bernoulli numbers and the s_k are the coefficients obtained by writing the $\mathbb{C}^\times$-equivariant Euler class (5.1) in the form $\exp\big(\sum_{k\geq 0} s_k \operatorname{ch}_k(\cdot)\big)$.

Remark 6.4.2. Most of the discussion in this section is valid for any invertible multiplicative characteristic class, not just the equivariant Euler class, but we will neither need nor emphasize this.

Definition 6.4.3. Let F be a vector bundle – not necessarily split – and let f_i be the Chern roots of F. Define the *Quantum Riemann-Roch operator*, $\Delta_{F_\lambda}\colon \mathcal{H}_Y \to \mathcal{H}_Y$ as multiplication by

$$\Delta_{F_\lambda} = \prod_{i=1}^{k} \exp(G(f_i, z))$$

Theorem 6.4.4 ([CG07]). *Δ_{F_λ} gives a linear symplectomorphism of $(\mathcal{H}_Y, \Omega_Y)$ with $(\mathcal{H}_Y, \Omega_Y^{F_\lambda})$ such that*

$$\Delta_{F_\lambda}(\mathcal{L}_Y) = \mathcal{L}_{F_\lambda}$$

Since $\Delta_{F_\lambda} \circ \Delta_{F'_\mu} = \Delta_{F_\lambda \oplus F'_\mu}$, it follows immediately that

$$\Delta_{F_\lambda}(\mathcal{L}_{F'_\mu}) = \mathcal{L}_{F_\lambda \oplus F'_\mu}.$$

Lemma 6.4.5. *Let F be a vector bundle and let $f_1, \ldots, f_k$ be the Chern roots of F. Let*

$$D_{F_\lambda} = \prod_{i=1}^{k} \exp\big(-G(z\nabla_{f_i}, z)\big)$$

and suppose that $\tau \mapsto I(\tau)$ is a family of elements of $\mathcal{L}_{F'_\mu}$. Then $\tau \mapsto D_{F_\lambda}(I(\tau))$ is also a family of elements of $\mathcal{L}_{F'_\mu}$.

Proof. This follows [CCIT09, Theorem 4.6]. Let $h = -z + \sum_{i=0}^{m} t_i z^i + \sum_{j=0}^{\infty} p_j(-z)^{-j-1}$ be a point on $\mathcal{H}_Y$. The Lagrangian cone $\mathcal{L}_{F'_\mu}$ is defined by the equations $E_j = 0$, $j = 0, 1, 2, \ldots$ where

$$E_j(h) = p_j - \sum_{n \geq 0} \sum_{d \in \mathrm{NE}(Y)} \frac{Q^d}{n!} t_{i_1}^{\alpha_1} \ldots t_{i_n}^{\alpha_n} \langle \phi_{\alpha_1}\psi^{i_1}, \ldots, \phi_{\alpha_n}\psi^{i_n}, \phi_\alpha \psi^j \rangle_{0,n+1,d} \phi^\alpha$$

We need to show that $E_j(D_{F_\lambda}(I)) = 0$. Note that $D_{F_\lambda}(I) = \prod_{i=1}^{k} \exp(-G(z\nabla_{f_i}, z))I$ depends on the parameters s_i. For notational simplicity assume that $k = 1$, so that

$$D_{F_\lambda}(I) = \exp\big(-G(z\nabla_f, z)\big)I$$

Set $\deg s_i = i+1$. We will prove the result by inducting on degree. Note that if $s_0 = s_1 = \cdots = 0$ then $D_{F_\lambda}(I) = I$ so that $E_j(D_{F_\lambda}(I)) = 0$. Assume by induction that $E_j(D_{F_\lambda}(I))$ vanishes up to degree n in the variables $s_0, s_1, s_2, \ldots$ Then

$$\frac{\partial}{\partial s_i} E_j(D_{F_\lambda}(I)) = d_{D_{F_\lambda}(I)} E_j(z^{-1} P_i(z\nabla_f, z) D_{F_\lambda}(I))$$

where

$$P_i(z\nabla_f, z) = \sum_{m=0}^{i+1} \frac{1}{m!(i+1-m)!} z^m B_m(z\nabla_f)^{i+1-m}$$

By induction there exists $D_{F_\lambda}(I)' \in \mathcal{L}_{F'_\mu}$ such that

$$\frac{\partial}{\partial s_i} E_j(D_{F_\lambda}(I)) = d_{D_{F_\lambda}(I)'} E_j(z^{-1} P_i(z\nabla_f, z) D_{F_\lambda}(I)')$$

up to degree n. But the right hand side of this expression is zero, since the term in brackets lies in the tangent space to the Lagrangian cone. Indeed, applying ∇_f to $D_{F_\lambda}(I_Y)'$ – or to any family lying on the cone – takes it to the tangent space of the cone at the point. And then applying $z\nabla_f$ preserves that tangent space. □

Corollary 6.4.6. *Let $\tau \mapsto I(\tau)$ be a family of elements of $\mathcal{L}_{F'_\mu}$. Then $\tau \mapsto \Delta_{F_\lambda}(D_{F_\lambda}(I(\tau)))$ is a family of elements of $\mathcal{L}_{F_\lambda \oplus F'_\mu}$.*

Proof. This follows immediately by combining 6.4.4 and 6.4.5 □

Corollary 6.4.6 produces a family of elements on the twisted Lagrangian cone $\mathcal{L}_{F_\lambda \oplus F'_\mu}$, but in general it is not obvious whether the nonequivariant limit $\lambda \to 0$ of this family exists. However, in the case when F is split and $\tau \mapsto I(\tau)$ satisfies the Divisor Equation we will show that the family $\Delta_{F_\lambda}(D_{F_\lambda}(I(\tau, -z)))$ is equal to the *twisted I-function* $I_{F'_\mu \oplus F_\lambda}$ given in Definition 6.4.7. This has an explicit expression, which makes it easy to check whether the nonequivariant limit exists. We make the following definitions.

Definition 6.4.7. Let $\tau \mapsto I(\tau)$ be a family of elements of $\mathcal{L}_{F'_\mu}$. Let $F = F_1 \oplus \cdots \oplus F_k$ be a direct sum of line bundles, and let $f_i = c_1(F_i)$. For $\beta \in \mathrm{NE}(Y)$, we define the modification factor

$$M_\beta(z) = \prod_{i=1}^{k} \frac{\prod_{m=-\infty}^{\langle f_i, \beta\rangle} \lambda + f_i + mz}{\prod_{m=-\infty}^{0} \lambda + f_i + mz}$$

The associated *twisted I-function* is

$$I^{\mathrm{tw}}(\tau) = \sum_{\beta \in \mathrm{NE}(Y)} Q^\beta I_\beta(\tau, z) \cdot M_\beta(z)$$

To relate $M_\beta(z)$ to the Quantum Riemann–Roch operator we will need the following Lemma:

Lemma 6.4.8.

$$M_\beta(-z) = \Delta_{F_\lambda}\left(\prod_{i=1}^{k} \exp(-G(f_i - \langle f_i, \beta\rangle z, z))\right)$$

Proof. Define

$$\mathbf{s}(x) = \sum_{k\geq 0} s_k \frac{x^k}{k!}$$

By [CCIT09, equation 13] we have that

$$G(x+z,z) = G(x,z) + \mathbf{s}(x) \tag{6.10}$$

We have that $s_0 = \log(\lambda)$ and $s_k = (-1)^{k-1}(k-1)!/\lambda^k$, so we can rewrite

$$M_\beta(z) = \prod_{i=1}^{k} \frac{\prod_{m=-\infty}^{\langle f_i,\beta\rangle} \lambda + f_i + mz}{\prod_{m=-\infty}^{0} \lambda + f_i + mz} = \prod_{i=1}^{k} \frac{\prod_{m=-\infty}^{\langle f_i,\beta\rangle} \exp[\mathbf{s}(f_i+mz)]}{\prod_{m=-\infty}^{0} \exp[\mathbf{s}(f_i+mz)]}$$

and so

$$\begin{aligned} M_\beta(-z) &= \prod_{i=1}^{k} \exp\left(\sum_{m=-\infty}^{\langle f_i,\beta\rangle} \mathbf{s}(f_i - mz) - \sum_{m=-\infty}^{0} \mathbf{s}(f_i - mz))\right) \\ &= \prod_{i=1}^{k} \exp(G(f_i,z) - G(f_i - \langle f_i,\beta\rangle z, z) \end{aligned}$$

where for the second equality we used (6.10). □

Proposition 6.4.9. *Let $\tau \mapsto I(\tau)$ be a family of elements of $\mathcal{L}_{F'_\mu}$ that satisfies the Divisor Equation, and let $F = F_1 \oplus \cdots \oplus F_k$ be a direct sum of line bundles. Then*

$$I^{\mathrm{tw}} = \Delta_{F_\lambda}(D_{F_\lambda}(I)). \tag{6.11}$$

As a consequence, $\tau \mapsto I^{\mathrm{tw}}(\tau)$ is a family of elements on the cone $\mathcal{L}_{F_\lambda \oplus F'_\mu}$.

Proof. Lemma 6.4.8 shows that

$$I^{\mathrm{tw}}(\tau) = \Delta_{F_\lambda}\left(\sum_{\beta\in \mathrm{NE}(Y)} \prod_{i=1}^{k} \exp(-G(f_i - \langle f_i,\beta\rangle z, z)) I_\beta(\tau,z)\right) \tag{6.12}$$

Applying the Divisor Equation, we can rewrite this as

$$I^{\mathrm{tw}} = \Delta_{F_\lambda}(D_{F_\lambda}(I)) \tag{6.13}$$

as required. The rest is immediate from 6.4.6. □

Proposition 6.4.10. *If the line bundles F_i are nef, then the nonequivariant limit $\lambda \to 0$ of $I^{\mathrm{tw}}(\tau)$ exists.*

Proof. This is immediate from Definition 6.4.7. □

6.5 The Givental–Martin cone

We now restrict to the situation described in the Introduction, where the action of a reductive Lie group G on a smooth quasiprojective variety A leads to smooth GIT quotients $A /\!\!/ G$ and $A /\!\!/ T$. As discussed, the roots of G define a vector bundle $\Phi = \oplus_\rho L_\rho \to Y$, where $Y = A /\!\!/ T$, and we consider twisting data $(\Phi, \mathbf{c})$ for Y where $\mathbf{c}$ is the $\mathbb{C}^\times$-equivariant Euler class. We call the modification factor in this setting the *Weyl modification factor*, and denote it as

$$W_\beta(z) = \prod_\alpha \frac{\prod_{m=-\infty}^{\langle c_1(L_\alpha), \beta\rangle} c_1(L_\alpha) + \lambda + mz}{\prod_{m=-\infty}^{0} c_1(L_\alpha) + \lambda + mz} \tag{6.14}$$

where the product runs over all roots α. For any family $\tau \mapsto I(\tau) = \sum_{\beta \in \mathrm{NE}(Y)} Q^\beta I_\beta(\tau, z)$ of elements of $\mathcal{H}_Y$, the corresponding twisted I-function is

$$I^{\mathrm{tw}}(\tau) = \sum_{\beta \in \mathrm{NE}(Y)} Q^\beta I_\beta(\tau, z) \cdot W_\beta(z) \tag{6.15}$$

Since the roots bundle Φ is not convex, in general the non-equivariant limit $\lambda \to 0$ of I^{tw} will not exist. Recall from (5.4), however, the map $p \colon \mathcal{H}^W_{A/\!\!/T} \to \mathcal{H}_{A/\!\!/G}$.

Lemma 6.5.1. *Suppose that I is Weyl-invariant. Then $p \circ I^{\mathrm{tw}}$ has a well-defined limit as $\lambda \to 0$.*

Proof. The map p is given by the composition of the map on Novikov rings induced by

$$\varrho \colon \mathrm{NE}(A /\!\!/ T) \to \mathrm{NE}(A /\!\!/ G)$$

(see Proposition 6.1.4) with the projection map $H^\bullet(A /\!\!/ T; \mathbb{C})^W \to H^\bullet(A /\!\!/ G; \mathbb{C})$ (see Theorem 6.1.1). Since $I(\tau)$ is Weyl-invariant, $I^{\mathrm{tw}}(\tau)$ is also Weyl invariant and so, after applying ϱ, the coefficient of each Novikov term Q^β in $\tau \mapsto I^{\mathrm{tw}}(\tau)$ lies in $H^\bullet(A /\!\!/ T; \mathbb{C})^W$. The composition $p \circ I^{\mathrm{tw}}$ is therefore well-defined.

The Weyl modification (6.14) contains many factors

$$\frac{c_1(L_\alpha) + \lambda + mz}{-c_1(L_\alpha) + \lambda - mz}$$

which arise by combining the terms involving roots α and $-\alpha$. Such factors have a well-defined limit, -1, as $\lambda \to 0$. Therefore the limit of $p \circ I^{\mathrm{tw}}$ as $\lambda \to 0$ is well-defined if and only if the limit of

$$p\left(\sum_{\beta\in\mathrm{NE}(Y)} Q^\beta I_\beta(\tau,z)\cdot(-1)^{\epsilon(\beta)}\prod_{\alpha\in\Phi^+}\frac{c_1(L_\alpha)\pm\lambda+\langle c_1(L_\alpha),\beta\rangle z}{c_1(L_\alpha)\mp\lambda}\right) \tag{6.16}$$

as $\lambda\to 0$ is well-defined, and the two limits coincide. Here Φ^+ is the set of positive roots of G, and $\epsilon(\beta)=\sum_{\alpha\in\Phi^+}\langle c_1(L_\alpha),\beta\rangle$; cf. [CFKS08, equation 3.2.1]. The limit $\lambda\to 0$ of the denominator terms

$$\prod_{\alpha\in\Phi^+}\big(c_1(L_\alpha)-\lambda\big)$$

in (6.16) is the fundamental Weyl-anti-invariant class ω from the discussion before Theorem 6.1.1. Furthermore

$$\sum_{\beta\in\mathrm{NE}(Y)} Q^\beta I_\beta(\tau,z)\cdot(-1)^{\epsilon(\beta)}\prod_{\alpha\in\Phi^+}\big(c_1(L_\alpha)+\lambda+\langle c_1(L_\alpha),\beta\rangle z\big)$$

has a well-defined limit as $\lambda\to 0$ which, as it is Weyl-anti-invariant, is divisible by ω. The quotient here is unique up to an element of $\mathrm{Ann}(\omega)$, and therefore the projection of the quotient along Martin's map $H^\bullet(A/\!\!/T;\mathbb{C})^W\to H^\bullet(A/\!\!/G;\mathbb{C})$ is unique. It follows that the limit as $\lambda\to 0$ of $p\circ I^{\mathrm{tw}}$ is well-defined. □

Definition 6.5.2. Let $\tau\mapsto I(\tau)$ be a Weyl-invariant family of elements of $\mathcal{H}_Y$ and let I^{tw} denote the twisted I-function as above. We call the nonequivariant limit of $\tau\mapsto p\big(I^{\mathrm{tw}}(\tau)\big)$ the *Givental–Martin modification* of the family $\tau\mapsto I(\tau)$, and denote it by $\tau\mapsto I_{\mathrm{GM}}(\tau)$

Recall that we have fixed a representation ρ of G on a vector space V, and that this induces vector bundles $V^T\to A/\!\!/T$ and $V^G\to A/\!\!/G$. Since the bundle $\Phi\to A/\!\!/T$ is not convex, one cannot expect the non-equivariant limit of $\mathcal{L}_{\Phi_\lambda\oplus V^T_\mu}$ to exist. Nonetheless, the projection along (5.4) of the Weyl-invariant part of $\mathcal{L}_{\Phi_\lambda\oplus V^T_\mu}$ does admit a non-equivariant limit.

Theorem 6.5.3. *The non-equivariant limit $\lambda\to 0$ of $p\left(\mathcal{L}_{\Phi_\lambda\oplus V^T_\mu}\cap\mathcal{H}^W_{A/\!\!/T}\right)$ exists.*

We call this non-equivariant limit the *twisted Givental–Martin cone* $\mathcal{L}_{\mathrm{GM},V^T_\mu}\subset\mathcal{H}^W_{A/\!\!/T}$.

Proof of Theorem 6.5.3. Recall the twisted J-function $J_{V^T_\mu}(\tau,-z)$ from Definition 6.3.4. By [CG07] a general point

$$-z+t_0+t_1z+\cdots+O(z^{-1})$$

on $\mathcal{L}_{V^T_\mu}$ can be written as

$$J_{V^T_\mu}\big(\tau(\mathbf{t}),-z\big)+\sum_{\alpha=1}^{N}C_\alpha(\mathbf{t},z)z\frac{\partial J_{V^T_\mu}}{\partial\tau^\alpha}\big(\tau(\mathbf{t}),-z\big)$$

for some coefficients $C_\alpha(\mathbf{t}, z)$ that depend polynomially on z and some $H^\bullet(A/\!\!/T)$-valued function $\tau(\mathbf{t})$ of $\mathbf{t} = (t_0, t_1, \ldots)$. The Weyl modification $\tau \mapsto I^{\mathrm{tw}}(\tau)$ of $\tau \mapsto J_{V_\mu^T}(\tau, -z)$ satisfies $I^{\mathrm{tw}}(\tau) \equiv J_{V_\mu^T}(\tau, -z)$ modulo Novikov variables, and $I^{\mathrm{tw}}(\tau) \in \mathcal{L}_{\Phi_\lambda \oplus V_\mu^T}$ by Proposition 6.4.9, so a general point

$$-z + t_0 + t_1 z + \cdots + O(z^{-1}) \tag{6.17}$$

on $\mathcal{L}_{\Phi_\lambda \oplus V_\mu^T}$ can be written as

$$I^{\mathrm{tw}}\big(\tau(\mathbf{t})^\dagger, -z\big) + \sum_{\alpha=1}^{N} C_\alpha(\mathbf{t}, z)^\dagger z \frac{\partial I^{\mathrm{tw}}}{\partial \tau^\alpha}\big(\tau(\mathbf{t})^\dagger, -z\big)$$

for some coefficients $C_\alpha(\mathbf{t}, z)^\dagger$ that depend polynomially on z and some $H^\bullet(A/\!\!/T)$-valued function $\tau(\mathbf{t})^\dagger$. Since the twisted J-function is Weyl-invariant, so is $I^{\mathrm{tw}}(\tau)$, and thus if (6.17) is Weyl-invariant then we may take $C_\alpha(\mathbf{t}, z)^\dagger$ to be such that $\sum_\alpha C_\alpha(\mathbf{t}, z)^\dagger \phi_\alpha$ is Weyl-invariant. Projecting along (5.4) we see that a general point

$$-z + t_0 + t_1 z + \cdots + O(z^{-1}) \tag{6.18}$$

on $p\left(\mathcal{L}_{\Phi_\lambda \oplus V_\mu^T} \cap \mathcal{H}^W_{A/\!\!/T}\right)$ can be written as

$$p \circ I^{\mathrm{tw}}\big(\tau(\mathbf{t})^\ddagger, -z\big) + \sum_{\alpha=1}^{N} C_\alpha(\mathbf{t}, z)^\ddagger z \frac{\partial (p \circ I^{\mathrm{tw}})}{\partial \tau^\alpha}\big(\tau(\mathbf{t})^\ddagger, -z\big)$$

for some coefficients $C_\alpha(\mathbf{t}, z)^\ddagger$ that depend polynomially on z and some $H^\bullet(A/\!\!/T)$-valued function $\tau(\mathbf{t})^\ddagger$. Furthermore, since $p \circ I^{\mathrm{tw}}(\tau)$ has a well-defined non-equivariant limit $I_{\mathrm{GM}}(\tau)$, we see that $C_\alpha(\mathbf{t}, z)^\ddagger$ also admits a non-equivariant limit. Hence a general point (6.18) on $p\left(\mathcal{L}_{\Phi_\lambda \oplus V_\mu^T} \cap \mathcal{H}^W_{A/\!\!/T}\right)$ has a well-defined limit as $\lambda \to 0$. □

Corollary 6.5.4. *The non-equivariant limit $\lambda \to 0$ of $p\left(\mathcal{L}_{\Phi_\lambda} \cap \mathcal{H}^W_{A/\!\!/T}\right)$ exists.*

We call this non-equivariant limit the *Givental–Martin cone* $\mathcal{L}_{\mathrm{GM}} \subset \mathcal{H}^W_{A/\!\!/T}$.

Proof. Take the vector bundle V^T in Theorem 6.5.3 to have rank zero. □

Corollary 6.5.5. *If $\tau \mapsto I(\tau)$ is a Weyl-invariant family of elements of $\mathcal{L}_{V_\mu^T}$ that satisfies the Divisor Equation* (6.9) *then the Givental–Martin modification $\tau \mapsto I_{\mathrm{GM}}(\tau)$ is a family of elements of $\mathcal{L}_{\mathrm{GM}, V_\mu^T}$*

Proof. Proposition 6.4.9 implies that $\tau \mapsto I^{\mathrm{tw}}(\tau, -z)$ is a family of elements on $\mathcal{L}_{\Phi_\lambda \oplus V_\mu^T}$. Projecting along (5.4) and taking the limit $\lambda \to 0$, which exists by Lemma 6.5.1, proves the result. □

This completes the results required to state the Abelian/non-Abelian Correspondence (Conjectures 5.0.4 and 5.0.8) and the Abelian/non-Abelian Correspondence with bundles (Conjectures 5.0.6 and 5.0.9).

Chapter 7

The Abelian/non-Abelian Correspondence

7.1 The Work of Brown and Oh

In this section we will review results by Brown [Bro14] and Oh [Oh21], and situate their work in terms of the Abelian/non-Abelian Correspondence (Conjecture 5.0.8). In particular, we show that the Givental–Martin modification of the Brown I-function is the Oh I-function. We freely use the notation introduced in Section 6.2.

Let X be a smooth projective variety. We will decompose the J-function of X, defined in §6.3, into contributions from different degrees:

$$J_X(\tau, z) = \sum_{D \in \mathrm{NE}(X)} J_X^D(\tau, z) Q^D. \tag{7.1}$$

Recall that we have a direct sum of line bundles $E = L_1 \oplus \cdots \oplus L_n \xrightarrow{\pi} X$, and that $\mathrm{Fl}(E) = \mathrm{Fl}(r_1, \ldots, r_\ell, E) = A /\!\!/ G$ is the partial flag bundle associated to E. As in §6.2, we form the toric fibration $\mathrm{Fl}(E)_T = A /\!\!/ T$ with general fibre $\mathbb{C}^N /\!\!/ (\mathbb{C}^\times)^R$. We denote both projection maps $\mathrm{Fl}(E) \to X$ and $\mathrm{Fl}(E)_T \to X$ by π. For the sake of clarity, we will denote homology and cohomology classes on $\mathrm{Fl}(E)_T$ with a tilde and classes on $\mathrm{Fl}(E)$ without. Recall the cohomology classes $\tilde{H}_{\ell+1,j} = -\pi^* c_1(L_j)$ on $\mathrm{Fl}(E)_T$, and $H_{\ell+1,j} = -\pi^* c_1(L_j)$ on $\mathrm{Fl}(E)$. For a fixed homology class $\tilde{\beta}$ on $\mathrm{Fl}(E)_T$ define $d_{\ell+1,j} = \langle -\pi^* c_1(L_j), \tilde{\beta} \rangle$, and for a fixed homology class β on $\mathrm{Fl}(E)$ define $d_{\ell+1,j} = \langle -\pi^* c_1(L_j), \beta \rangle$. We use the indexing of the set $\{1, \ldots, R\}$ defined in Section 6.2, and denote the components of a vector $\underline{d} \in \mathbb{Z}^R$ by $d_{i,j}$. Similarly, we denote components of a vector $\underline{d} \in \mathbb{Z}^\ell$ by d_i.

In [Oh21], the author proves that a certain generating function, the I-function of $\mathrm{Fl}(E)$, lies on the Lagrangian cone for $\mathrm{Fl}(E)$.

Theorem 7.1.1. *Let $\tau \in H^\bullet(X)$, $t = \sum_i t_i c_1(S_i^\vee)$, and define the I-function of $\mathrm{Fl}(E)$ to be*

$$I_{\mathrm{Fl}(E)}(t,\tau,z) =$$
$$e^{\frac{t}{z}} \sum_{\beta \in \mathrm{NE}(\mathrm{Fl}(E))} Q^\beta e^{\langle \beta, t\rangle} \pi^* J_X^{\pi_*\beta}(\tau,z) \sum_{\substack{\underline{d} \in \mathbb{Z}^R: \\ \forall i \sum_j d_{i,j} = \langle \beta, c_1(S_i^\vee)\rangle}} \prod_{i=1}^{\ell} \prod_{j=1}^{r_i} \prod_{j'=1}^{r_{i+1}} \frac{\prod_{m=-\infty}^{0} H_{i,j} - H_{i+1,j'} + mz}{\prod_{m=-\infty}^{d_{i,j}-d_{i+1,j'}} H_{i,j} - H_{i+1,j'} + mz}$$
$$\times \prod_{i=1}^{\ell} \prod_{j \neq j'} \frac{\prod_{m=-\infty}^{d_{i,j}-d_{i,j'}} H_{i,j} - H_{i,j'} + mz}{\prod_{m=-\infty}^{0} H_{i,j} - H_{i,j'} + mz}$$

Then $I_{\mathrm{Fl}(E)}(t,\tau,-z) \in \mathcal{L}_{\mathrm{Fl}(E)}$ for all t and τ.

In [Bro14], the author proves an analogous result for the corresponding Abelian quotient $\mathrm{Fl}(E)_T$.

Theorem 7.1.2. *Let $\tau \in H^\bullet(X)$, $t = \sum_{i,j} t_{i,j}\tilde{H}_{i,j}$, and define the Brown I-function of $\mathrm{Fl}(E)_T$ to be*

$$I_{\mathrm{Fl}(E)_T}(t,\tau,z) =$$
$$e^{\frac{t}{z}} \sum_{\tilde{\beta} \in H_2 \mathrm{Fl}(E)_T} Q^{\tilde{\beta}} e^{\langle \tilde{\beta}, t\rangle} \pi^* J_X^{\pi_*\tilde{\beta}}(\tau,z) \prod_{i=1}^{\ell} \prod_{j=1}^{r_i} \prod_{j'=1}^{r_{i+1}} \frac{\prod_{m=-\infty}^{0} \tilde{H}_{i,j} - \tilde{H}_{i+1,j'} + mz}{\prod_{m=-\infty}^{\langle \tilde{\beta}, \tilde{H}_{i,j} - \tilde{H}_{i+1,j'}\rangle} \tilde{H}_{i,j} - \tilde{H}_{i+1,j'} + mz}$$

Then $I_{\mathrm{Fl}(E)_T}(t,\tau,-z) \in \mathcal{L}_{\mathrm{Fl}(E)_T}$ for all t and τ.

Remark 7.1.3. We have chosen to state Theorem 7.1.2 in a different form than in Brown's original paper. The equivalence of the two versions follows from Lemma 7.1.4 below. The classes $H_{i,j}$ here were denoted in [Bro14] by P_i, and the classes $H_{i,j} - H_{i+1,j'}$ here were denoted there by U_k.

Lemma 7.1.4. *Writing $I_{\mathrm{Fl}(E)_T} = \sum_{\tilde{\beta}} I^{\tilde{\beta}}_{\mathrm{Fl}(E)_T} Q^{\tilde{\beta}}$, any nonzero $I^{\tilde{\beta}}$ must have $\tilde{\beta} \in \mathrm{NE}(\mathrm{Fl}(E)_T)$.*

Proof. To see this we temporarily adopt the notation of Brown and denote the torus invariant divisors by U_k, as in Lemma 6.2.3. Then $I_{\mathrm{Fl}(E)_T}$ takes the form

$$I_{\mathrm{Fl}(E)_T} = \sum_{\substack{\tilde{\beta} \in H_2 \mathrm{Fl}(E)_T: \\ \pi_*\tilde{\beta} \in \mathrm{NE}(X)}} (\dots) \prod_{k=1}^{N} \frac{\prod_{m=-\infty}^{0} U_k + mz}{\prod_{m=-\infty}^{\langle \tilde{\beta}, U_k\rangle} U_k + mz}$$

Let $\alpha \subset \{1, \dots N\}$ be a subset of size R which defines a section of the toric fibration as in

Section 6.2. We have that

$$s_\alpha^* I_{\mathrm{Fl}(E)_T} = (\dots) \prod_{k\in\alpha} \frac{\prod_{m=-\infty}^{0} (0) + mz}{\prod_{m=-\infty}^{\langle\tilde\beta,U_k\rangle} (0) + mz} \prod_{k\notin\alpha} \frac{\prod_{m=-\infty}^{0} s_\alpha^* U_k + mz}{\prod_{m=-\infty}^{\langle\tilde\beta,U_k\rangle} s_\alpha^* U_k + mz}$$

since $s_\alpha^*(U_k) = 0$ if $k \in \alpha$. Therefore, if $\langle\tilde\beta, U_k\rangle < 0$ for some $k \in \alpha$, the numerator contains a term (0) and vanishes. We conclude that any $\tilde\beta \in H_2 \,\mathrm{Fl}(E)_T$ which gives a nonzero contribution to $s_\alpha^* I_{\mathrm{Fl}(E)_T}$ must satisfy the conditions

$$\pi_*\tilde\beta \in \mathrm{NE}(X), \langle\tilde\beta, U_k\rangle \geq 0 \,\forall k \in \alpha.$$

The section s_α gives a splitting $H_2(\mathrm{Fl}(E)_T) = H_2(X) \oplus H_2(\mathrm{Fl}_T)$, via which we may write $\tilde\beta = s_{\alpha_*} D + \iota_* d$ where ι is the inclusion of a fibre. We have

$$\langle\tilde\beta, U_k\rangle = \langle D, s_\alpha^* U_k\rangle + \langle d, \iota^* U_k\rangle = \langle d, \iota^* U_k\rangle \geq 0$$

for all $k \in \alpha$. However, the cone in the secondary fan spanned by the line bundles $\iota^* U_k$ contains the ample cone of Fl_T (see Section 6.2), so this implies $d \in \mathrm{NE}(\mathrm{Fl}_T)$. It follows that any $\tilde\beta$ which gives a nonzero contribution to $s_\alpha^* I_{\mathrm{Fl}(E)_T}$ is effective. We now use the Atiyah-Bott localization formula

$$I_{\mathrm{Fl}(E)_T} = \sum_\alpha s_{\alpha_*}\left(\frac{s_\alpha^* I_{\mathrm{Fl}(E)_T}}{e^\alpha}\right), \quad \text{where } e^\alpha = \prod_{k\notin\alpha} s_\alpha^* U_k$$

where α ranges over the torus fixed point sections of the fibration, to conclude that the same is true for $I_{\mathrm{Fl}(E)_T}$. □

Lemma 7.1.5. *Brown's I-function satisfies the Divisor Equation. That is,*

$$z\nabla_\rho I^{\tilde\beta}_{\mathrm{Fl}(E)_T} = (\rho + \langle\rho, \tilde\beta\rangle z) I^{\tilde\beta}_{\mathrm{Fl}(E)_T}$$

for any $\rho \in H^2(\mathrm{Fl}(E)_T)$.

Proof. Decompose $\rho = \rho_F + \pi^*\rho_B$ into fibre and base part. Basic differentiation and the divisor equation for J_X show that

$$z\nabla_\rho I^{\tilde\beta}_{\mathrm{Fl}(E)_T} = \Big(\rho_F + \langle\rho_F, \tilde\beta\rangle z + (\pi^*\rho_B + \langle\pi^*\rho_B, \tilde\beta\rangle z)\Big)\, e^{t/z} e^{\langle\tilde\beta, t\rangle} \pi^* J_X^{\pi_*\tilde\beta}(\tau, z) \cdot \mathbf{H}$$

where $\mathbf{H}$ is a hypergeometric factor with no dependence on t or τ. The right-hand simplifies to

$$(\rho + \langle\rho, \tilde\beta\rangle z) I^{\tilde\beta}_{\mathrm{Fl}(E)_T}$$

as required. □

Lemma 7.1.6. *If we restrict t to lie in the Weyl-invariant locus $H^2(\mathrm{Fl}(E)_T)^W \subset H^2(\mathrm{Fl}(E)_T)$ then $(t,\tau) \mapsto I_{\mathrm{Fl}(E)_T}(t,\tau,z)$ takes values in $H^\bullet(\mathrm{Fl}(E)_T)^W$.*

Proof. This is immediate from the definition of $I_{\mathrm{Fl}(E)_T}(t,\tau,z)$, in Theorem 7.1.2. □

Proposition 7.1.7. *Restrict t to lie in the Weyl-invariant locus $H^2(\mathrm{Fl}(E)_T)^W \subset H^2(\mathrm{Fl}(E)_T)$ and consider the Brown I-function $(t,\tau) \mapsto I_{\mathrm{Fl}(E)_T}(t,\tau,z)$. The Givental–Martin modification $I_{\mathrm{GM}}(t,\tau)$ of this family is equal to Oh's I-function $I_{\mathrm{Fl}(E)}(t,\tau)$.*

Proof. Lemma 7.1.6 and Lemma 6.5.1 imply that the Givental–Martin modification $I_{\mathrm{GM}}(t,\tau)$ exists. We need to compute it. Note that the restrictions to the fibre of the classes $\tilde{H}_{i,j}$ form a basis for $H^2(\mathrm{Fl}_T)$. Since the general fibre Fl_T of $\mathrm{Fl}(E)_T$ has vanishing first homology, the Leray–Hirsch Theorem gives an identification $\mathbb{Q}[H_2(\mathrm{Fl}(E)_T,\mathbb{Z})] = \mathbb{Q}[H_2(X,\mathbb{Z})][q_{1,1},\ldots,q_{\ell,r_\ell}]$ via the map

$$Q^{\tilde{\beta}} \mapsto Q^{\pi_*\tilde{\beta}} \prod_{i,j} q_{i,j}^{\langle \tilde{H}_{i,j},\tilde{\beta}\rangle} \tag{7.2}$$

By Lemma 7.1.4, the summation range in the sum defining $I_{\mathrm{Fl}(E)_T}$ is contained in $\mathrm{NE}(\mathrm{Fl}(E)_T)$. We can therefore write the corresponding twisted I-function (6.15) as

$$\begin{aligned} I^{\mathrm{tw}}(t,\tau,z) = e^{\frac{t}{z}} \sum_{\substack{D\in\mathrm{NE}(X)\\ \underline{d}\in\mathbb{Z}^R}} Q^D \prod_{i,j} q_{i,j}^{d_{i,j}} e^{t\cdot\underline{d}} \pi^* J_X^D(\tau,z) \prod_{i=1}^{\ell}\prod_{j=1}^{r_i}\prod_{j'=1}^{r_{i+1}} \frac{\prod_{m=-\infty}^{0} \tilde{H}_{i,j}-\tilde{H}_{i+1,j'}+mz}{\prod_{m=-\infty}^{d_{i,j}-d_{i+1,j'}} \tilde{H}_{i,j}-\tilde{H}_{i+1,j'}+mz} \\ \times \prod_{i=1}^{\ell}\prod_{j\neq j'} \frac{\prod_{m=-\infty}^{d_{i,j}-d_{i,j'}} \tilde{H}_{i,j}-\tilde{H}_{i,j'}+\lambda+mz}{\prod_{m=-\infty}^{0} \tilde{H}_{i,j}-\tilde{H}_{i,j'}+\lambda+mz} \end{aligned}$$

where the $t_{i,j}\in\mathbb{C}$, $t=\sum_{i=1}^{\ell}\sum_{j=1}^{r_i} t_{i,j}\tilde{H}_{i,j}$, and $t\cdot\underline{d}=\sum_{i,j} t_{i,j}d_{i,j}$. For the Weyl modification factor we used the fact that the roots of G are given by $\rho_{i,j}\rho_{i,j'}^{-1}$, where the character $\rho_{i,j}$ was defined in section 6.2. By Lemma 7.1.4 the effective summation range for the vector $\underline{d}$ here is contained in the set $S\subset\mathbb{Z}^R$ consisting of $\underline{d}$ such that $\langle\tilde{\beta},\tilde{H}_{i,j}\rangle = d_{i,j}$ for some $\tilde{\beta}\in\mathrm{NE}(\mathrm{Fl}(E)_T)$.

We can identify the group ring $\mathbb{Q}[H_2(\mathrm{Fl}(E))]$ with $\mathbb{Q}[H_2(X,\mathbb{Z})][q_1,\ldots,q_\ell]$ via the map

$$Q^{\beta} \mapsto Q^{\pi_*\beta} \prod_{i} q_i^{\langle c_1(S_i^\vee),\beta\rangle} \tag{7.3}$$

Via (7.2) and (7.3) the map on Mori cones $\varrho : \mathrm{NE}(\mathrm{Fl}(E)_T) \to \mathrm{NE}(\mathrm{Fl}(E))$ becomes

$$Q^D \prod_{i,j} q_{i,j}^{d_{i,j}} \mapsto Q^D \prod_i q_i^{\sum_j d_{i,j}}$$

Restricting t to the Weyl-invariant locus $H^2(\mathrm{Fl}(E)_T)^W$ corresponds to setting $t_{i,j} = t_i$ for all i and j, which gives $e^{t\cdot \underline{d}} = e^{\sum_i t_i d_i}$ where $d_i = \sum_j d_{i,j}$. The identification $H^2(\mathrm{Fl}(E)_T)^W \cong H^2(\mathrm{Fl}(E))$ sends $\sum_{i,j} t_i \tilde{H}_{i,j}$ to $\sum_i t_i c_1(S_i^\vee)$, so projecting along (5.4) and taking the limit as $\lambda = 0$ we obtain

$$e^{\frac{t}{z}} \sum_{\substack{D\in \mathrm{NE}(X)\\ \underline{\delta}\in\mathbb{Z}^\ell}} Q^D \prod_i q_i^{\delta_i} e^{t\cdot\underline{\delta}} \pi^* J_X^D(\tau,z) \sum_{\substack{\underline{d}\in\mathbb{Z}^R:\\ \forall i \sum_j d_{i,j}=\delta_i}} \prod_{i=1}^{\ell}\prod_{j=1}^{r_i}\prod_{j'=1}^{r_{i+1}} \frac{\prod_{m=-\infty}^{0} H_{i,j}-H_{i+1,j'}+mz}{\prod_{m=-\infty}^{d_{i,j}-d_{i+1,j'}} H_{i,j}-H_{i+1,j'}+mz}$$
$$\times \prod_{i=1}^{\ell}\prod_{j\neq j'} \frac{\prod_{m=-\infty}^{d_{i,j}-d_{i,j'}} H_{i,j}-H_{i,j'}+mz}{\prod_{m=-\infty}^{0} H_{i,j}-H_{i,j'}+mz}$$

where now $t = \sum_i t_i c_1(S_i^\vee)$. The effective summation range here is contained in $\mathrm{NE}(\mathrm{Fl}(E))$ by construction. Using (7.3) again we may rewrite this as

$$e^{\frac{t}{z}} \sum_{\beta\in\mathrm{NE}(\mathrm{Fl}(E))} Q^\beta e^{\langle\beta,t\rangle} \pi^* J_X^{\pi_*\beta}(\tau,z) \sum_{\substack{\underline{d}\in\mathbb{Z}^R:\\ \forall i \sum_j d_{i,j}=\langle\beta, c_1(S_i^\vee)\rangle}} \prod_{i=1}^{\ell}\prod_{j=1}^{r_i}\prod_{j'=1}^{r_{i+1}} \frac{\prod_{m=-\infty}^{0} H_{i,j}-H_{i+1,j'}+mz}{\prod_{m=-\infty}^{d_{i,j}-d_{i+1,j'}} H_{i,j}-H_{i+1,j'}+mz}$$
$$\times \prod_{i=1}^{\ell}\prod_{j\neq j'} \frac{\prod_{m=-\infty}^{d_{i,j}-d_{i,j'}} H_{i,j}-H_{i,j'}+mz}{\prod_{m=-\infty}^{0} H_{i,j}-H_{i,j'}+mz}$$

This is $I_{\mathrm{Fl}(E)}(t,\tau,z)$, as required. □

Remark 7.1.8. In view of (6.6), we see that the effective summation range in $I_{\mathrm{Fl}(E)}$ is contained in the subset of vectors satisfying

$$d_{i,j} \geq \min_{j'} d_{i+1,j'}\ , 1\leq i\leq \ell, 1\leq j\leq r_i$$

This will prove useful in calculations in Section 7.4.

7.2 The Abelian/non-Abelian Correspondence with bundles

We are now ready to prove Theorem 5.0.2. Recall from the Introduction that we have fixed a representation $\rho\colon G\to \mathrm{GL}(V)$ where $G=\prod_i \mathrm{GL}_{r_i}(\mathbb{C})$, and that this determines vector bundles $V^G \to \mathrm{Fl}(E)$ and $V^T\to \mathrm{Fl}(E)_T$. Since T is Abelian, V^T splits as a direct sum of line bundles

$$V^T = F_1\oplus\cdots\oplus F_k$$

The Brown I-function gives a family

$$(t,\tau) \mapsto I_{\mathrm{Fl}(E)_T}(t,\tau,-z) \qquad\qquad t \in H^2(\mathrm{Fl}(E)_T)^W,\ \tau \in H^\bullet(X)$$

of elements of $\mathcal{H}_{\mathrm{Fl}(E)_T}$, and Theorem 7.1.2 shows that $I_{\mathrm{Fl}(E)_T}(t,\tau,-z) \in \mathcal{L}_{\mathrm{Fl}(E)_T}$. Twisting by $(F,\mathbf{c})$ where $\mathbf{c}$ is the $\mathbb{C}^\times$-equivariant Euler class with parameter μ gives a twisted I-function, as in Definition 6.4.7, which we denote by

$$(t,\tau) \mapsto I_{V_\mu^T}(t,\tau,-z) \qquad\qquad t \in H^2(\mathrm{Fl}(E)_T)^W,\ \tau \in H^\bullet(X)$$

Applying Proposition 6.4.9 shows that $I_{V_\mu^T}(t,\tau,-z) \in \mathcal{L}_{V_\mu^T}$. Twisting again, by $(\Phi,\mathbf{c}')$ where $\Phi \to \mathrm{Fl}(E)_T$ is the roots bundle from the Introduction and $\mathbf{c}'$ is the $\mathbb{C}^\times$-equivariant Euler class with parameter λ gives a twisted I-function, as in Definition 6.4.7, which we denote by

$$(t,\tau) \mapsto I_{\Phi_\lambda \oplus V_\mu^T}(t,\tau,-z) \qquad\qquad t \in H^2(\mathrm{Fl}(E)_T)^W,\ \tau \in H^\bullet(X)$$

Applying Proposition 6.4.9 again shows that $I_{\Phi_\lambda \oplus V_\mu^T}(t,\tau,-z) \in \mathcal{L}_{\Phi_\lambda \oplus V_\mu^T}$. We now project along (5.4) and take the non-equivariant limit $\lambda \to 0$, obtaining the Givental–Martin modification of $I_{V_\mu^T}$. This is a family

$$(t,\tau) \mapsto I_{\mathrm{GM}}(t,\tau,-z) \qquad\qquad t \in H^2(\mathrm{Fl}(E)_T)^W,\ \tau \in H^\bullet(X)$$

of elements of $\mathcal{H}_{\mathrm{Fl}(E)}$. Explicitly:

Definition 7.2.1 (which is a specialisation of Definition 6.5.2 to the situation at hand)**.**

$$\begin{aligned}
&I_{\mathrm{GM}}(t,\tau,z) = \\
&e^{\frac{t}{z}} \sum_{\beta \in \mathrm{NE}(\mathrm{Fl}(E))} Q^\beta e^{\langle \beta, t\rangle} \pi^* J_X^{\pi_*\beta}(\tau,z) \sum_{\substack{\underline{d} \in \mathbb{Z}^R: \\ \forall i \sum_j d_{i,j} = \langle \beta, c_1(S_i^\vee)\rangle}} \prod_{i=1}^{\ell} \prod_{j=1}^{r_i} \prod_{j'=1}^{r_{i+1}} \frac{\prod_{m=-\infty}^{0} H_{i,j} - H_{i+1,j'} + mz}{\prod_{m=-\infty}^{d_{i,j}-d_{i+1,j'}} H_{i,j} - H_{i+1,j'} + mz} \\
&\qquad\qquad \times \prod_{i=1}^{\ell} \prod_{j \neq j'} \frac{\prod_{m=-\infty}^{d_{i,j}-d_{i,j'}} H_{i,j} - H_{i,j'} + mz}{\prod_{m=-\infty}^{0} H_{i,j} - H_{i,j'} + mz} \prod_{s=1}^{k} \frac{\prod_{m=-\infty}^{f_s \cdot \underline{d}} f_s + \mu + mz}{\prod_{m=-\infty}^{0} f_s + \mu + mz}
\end{aligned}$$

Here $J_X^D(\tau,z)$ is as in (7.1), $f_s \cdot \underline{d} = \sum_{i,j} f_{s,i,j} d_{i,j}$, and $f_s = \sum_{i,j} f_{s,i,j} H_{i,j}$, where

$$c_1(F_s) = \sum_{i=1}^{\ell} \sum_{j=1}^{r_i} f_{s,i,j} \tilde{H}_{i,j}$$

Lemma 6.5.1 shows that this expression is well-defined despite the presence of

$$\omega = \prod_i \prod_{j<j'} (H_{i,j} - H_{i,j'})$$

in the denominator. Corollary 6.5.5 shows that $I_{\mathrm{GM}}(t, \tau, -z) \in \mathcal{L}_{\mathrm{GM}, V^T_\mu}$. Note that $I_{\mathrm{GM}}(t,\tau)$ is *not* the V^G-twist of Oh's I-function $I_{\mathrm{Fl}(E)}$. Indeed V^G need not be a split bundle, so the twist may not even be defined.

Theorem 7.2.2. *Let I_{GM} be as in Definition 7.2.1. Then:*

$$I_{\mathrm{GM}}(t, \tau, -z) \in \mathcal{L}_{V^G_\mu} \qquad \text{for all } t \in H^2(\mathrm{Fl}(E)_T)^W,\ \tau \in H^\bullet(X).$$

Proof. Before projecting and taking the non-equivariant limit, we have

$$I_{\Phi_\lambda \oplus V^T_\mu} = \Delta_{V^T_\mu}\big(D_{V^T_\mu}(I_{\Phi_\lambda})\big)$$

by Proposition 6.11. Projecting along (5.4) gives

$$p \circ I_{\Phi_\lambda \oplus V^T_\mu} = \Delta_{V^G_\mu}\big(D_{V^G_\mu}(p \circ I_{\Phi_\lambda})\big)$$

and taking the limit $\lambda \to 0$, which is well-defined by Lemma 6.5.1, gives

$$I_{\mathrm{GM}} = \Delta_{V^G_\mu}\big(D_{V^G_\mu}(I_{\mathrm{Fl}(E)})\big)$$

by Proposition 7.1. The result now follows from Proposition 6.4.6. □

Exactly the same argument proves:

Corollary 7.2.3. *Let $L \to X$ be a line bundle with first Chern class ρ, and define the vector bundle $F \to \mathrm{Fl}(E)$ to be $F = V^G \otimes \pi^* L$. Let I_{GM} be as in Definition 7.2.1, except that the factor*

$$\prod_{s=1}^{k} \frac{\prod_{m=-\infty}^{f_s \cdot \underline{d}} f_s + \mu + mz}{\prod_{m=-\infty}^{0} f_s + \mu + mz} \qquad \text{is replaced by} \qquad \prod_{s=1}^{k} \frac{\prod_{m=-\infty}^{f_s \cdot \underline{d} + \langle \rho, \pi_* \beta \rangle} f_s + \pi^*\rho + \mu + mz}{\prod_{m=-\infty}^{0} f_s + \pi^*\rho + \mu + mz}$$

Then:

$$I_{\mathrm{GM}}(t, \tau, -z) \in \mathcal{L}_{F_\mu} \qquad \text{for all } t \in H^2(\mathrm{Fl}(E)_T)^W,\ \tau \in H^\bullet(X).$$

The following Corollary gives a closed-form expression for genus-zero Gromov–Witten invariants of the zero locus of a generic section Z of F in terms of invariants of X.

Corollary 7.2.4. *With notation as in Corollary 7.2.3, let Z be the zero locus of a generic section of $F \to \mathrm{Fl}(E)$. Suppose that $-K_Z$ is the restriction of an ample class on $\mathrm{Fl}(E)$ and that $\tau \in H^2(X)$. Then*

$$J_{F_\mu}(t+\tau, z) = e^{-C(t)/z} I_{\mathrm{GM}}(t, \tau, z)$$

where

$$C(t) = \sum_\beta n_\beta Q^\beta e^{\langle \beta, t\rangle}$$

for some constants $n_\beta \in \mathbb{Q}$ and the sum runs over the finite set

$$S = \{\beta \in \mathrm{NE}(\mathrm{Fl}(E)) : \langle -K_{\mathrm{Fl}(E)} - c_1(F), \beta\rangle = 1\}$$

If Z is of Fano index two or more then this set is empty and $C(t) \equiv 0$. Regardless, if the vector bundle F is convex then the non-equivariant limit $\mu \to 0$ of J_{F_μ} exists and

$$J_Z\big(i^* t + i^* \tau, z\big) = i^* J_{F_0}(t+\tau, z)$$

where $i\colon Z \to \mathrm{Fl}(E)$ is the inclusion map.

Proof of Corollary 7.2.4. The statement about Fano index two or more follows immediately from the Adjunction Formula

$$K_Z = \big(K_{\mathrm{Fl}(E)} + c_1(F)\big)\big|_Z$$

We need to show that

$$I_{\mathrm{GM}}(t, \tau, z) = z + t + \tau + C(t) + O(z^{-1}) \tag{7.4}$$

Everything else then follows from the characterisation of the twisted J-function just below Definition 6.3.4, the String Equation

$$J_{F_\mu}(\tau + a, z) = e^{a/z} J_{F_\mu}(\tau, z) \qquad\qquad a \in H^0(\mathrm{Fl}(E))$$

and [Coa14]. To establish (7.4), it will be convenient to set $\deg(z) = \deg(\mu) = 1$, $\deg(\phi) = k$ for $\phi \in H^{2k}(\mathrm{Fl}(E))$, and $\deg(Q^\beta) = \langle -K_X, \beta\rangle$ if $\beta \in H_2(X)$. The degree axiom for Gromov–Witten invariants then shows that $J_X^{\pi_*\beta}$ is homogeneous of degree $\langle K_X, \pi_*\beta\rangle + 1$. Write

$$I_{\mathrm{GM}}(t, \tau, z) = e^{\frac{t}{z}} \sum_{\beta \in \mathrm{NE}(\mathrm{Fl}(E))} Q^\beta e^{\langle \beta, t\rangle} \pi^* J_X^{\pi_*\beta}(\tau, z) \times I_\beta(z) \times M_\beta(z)$$

where

$$M_\beta(z) = \prod_{s=1}^{k} \frac{\prod_{m=-\infty}^{f_s \cdot \underline{d} + \langle \rho, \pi_* \beta \rangle} f_s + \pi^* \rho + \mu + mz}{\prod_{m=-\infty}^{0} f_s + \pi^* \rho + \mu + mz}$$

A straightforward calculation shows that

$$\begin{aligned} I_\beta(z) &= z^{\langle K_{\mathrm{Fl}(E)} - \pi^* K_X, \beta \rangle} i_\beta(z) \\ M_\beta(z) &= z^{\langle c_1(F), \beta \rangle} m_\beta(z) \end{aligned}$$

where $i_\beta(z), m_\beta(z) \in \mathcal{H}_{\mathrm{Fl}(E)}$ are homogeneous of degree 0. It follows that $\pi^* J_X^{\pi_*\beta}(\tau, z) \times I_\beta(z) \times M_\beta(z)$ is homogeneous of degree $\langle K_{\mathrm{Fl}(E)} + c_1(F), \beta \rangle + 1$ which is nonpositive for $\beta \neq 0$ by the assumptions on $-K_Z$. Since $\tau \in H^2(X)$, any negative contribution to the homogenous degree must come from a negative power of z, so that $\pi^* J_X^{\pi_*\beta}(\tau, z) \times I_\beta(z) \times M_\beta(z)$ is $O(z^{-1})$, unless $\beta = 0$ or $\beta \in S$. In the latter case, the expression has homogeneous degree 0 and is therefore of the form $c_0 + \frac{c_1}{z} + O(z^{-2})$ with c_i independent of z and of degree i. Relabeling $n_\beta = c_0$ and expanding I_{GM} in powers of z, we obtain

$$\begin{aligned} I_{\mathrm{GM}}(t, \tau, z) &= \big(1 + tz^{-1} + O(z^{-2})\big)\big(\pi^* J_X^0 \times I_0 \times M_0 + \Big(\sum_{\beta \in S} n_\beta Q^\beta e^{\langle \beta, t \rangle} + O(z^{-1})\Big) + \sum_{0 \neq \beta \notin S} O(z^{-1})\big) \\ &= (z + \tau + t + C(t) + O(z^{-1})) \end{aligned}$$

where $C(t)$ is as claimed. This proves (7.4), and the result follows. □

We restate Corollary 7.2.4 in the case where the flag bundle is a Grassmann bundle, i.e $\ell = 1$, relabelling $H_{1,j} = H_j$, $d_{1,j} = d_j$ and $r_1 = r$. The rest of the notation here is as in §6.2.

Corollary 7.2.5. *Let $V^G \to \mathrm{Gr}(r, E)$ be a vector bundle induced by a representation of G, let $L \to X$ be a line bundle with first Chern class ρ, and let $F = V^G \otimes \pi^* L$. Let Z be the zero locus of a generic section of F. Suppose that F is convex, that $-K_{\mathrm{Gr}(E,r)} - c_1(F)$ is ample, and that $\tau \in H^2(\mathrm{Gr}(r, E))$. Then the non-equivariant limit $\mu \to 0$ of the twisted J-function J_{F_μ} exists and satisfies*

$$J_Z\big(i^* t + i^* \tau, z\big) = i^* J_{F_0}(t + \tau, z)$$

where $i\colon Z \to \mathrm{Gr}(r,E)$ *is the inclusion map. Furthermore*

$$J_{F_0}(t+\tau,z) = e^{\frac{t-C(t)}{z}} \sum_{\beta\in\mathrm{NE}(\mathrm{Gr}(r,E))} Q^\beta e^{\langle\beta,t\rangle}\pi^* J_X^{\pi_*\beta}(\tau,z)$$
$$\sum_{\substack{\underline{d}\in\mathbb{Z}^r:\\ d_1+\cdots+d_r=\langle\beta,c_1(S^\vee)\rangle}} (-1)^{\epsilon(\underline{d})} \prod_{i=1}^r\prod_{j=1}^n \frac{\prod_{m=-\infty}^0 H_i+\pi^*c_1(L_j)+mz}{\prod_{m=-\infty}^{d_i+\langle\pi_*\beta,c_1(L_j)\rangle} H_i+\pi^*c_1(L_j)+mz}$$
$$\times \prod_{i<j}\frac{H_i-H_j+(d_i-d_j)z}{H_i-H_j} \times \prod_{s=1}^k \prod_{m=1}^{f_s\cdot\underline{d}+\langle\rho,\pi_*\beta\rangle} \big(f_s+\pi^*\rho+mz\big) \tag{7.5}$$

Here the Abelianised bundle V^T *splits as a direct sum of line bundles* $F_1\oplus\cdots\oplus F_k$ *with first Chern classes that we write as* $c_1(F_s)=\sum_{i=1}^r f_{s,i}\tilde{H}_i$, $J_X^D(\tau,z)$ *is as in* (7.1), $\epsilon(\underline{d})=\sum_{i<j} d_i-d_j$, $f_s\cdot\underline{d}=\sum_i f_{s,i}d_i$, $f_s=\sum_i f_{s,i}H_i$, *and* $C(t)\in H^0(\mathrm{Gr}(r,E),\Lambda)$ *is the unique expression such that the right-hand side of* (7.5) *has the form* $z+t+\tau+O(z^{-1})$.

Remark 7.2.6. For a more explicit formula for $C(t)$, see Corollary 7.2.4; in particular if Z has Fano index two or greater then $C(t)\equiv 0$. By Remark 7.1.8 the summand in (7.5) is zero unless for each i there exists a j such that $d_i+\langle\pi_*\beta,c_1(L_j)\rangle\geq 0$

Proof of Corollary 7.2.5. We cancelled terms in the Weyl modification factor, as in the proof of Lemma 6.5.1, and took the non-equivariant limit $\mu\to 0$. □

Remark 7.2.7. The relationship between I-functions (or generating functions for genus-zero quasimap invariants) and J-functions (which are generating functions for genus-zero Gromov–Witten invariants) is particularly simple in the Fano case [Giv98] [CFK14, §1.4], and for the same reason Corollary 7.2.4 holds without the restriction $\tau\in H^2(X)$ if $Z\to X$ is relatively Fano[1]. This never happens for blow-ups $\tilde{X}\to X$, however, and it is hard to construct examples where $Z\to X$ is relatively Fano and the rest of the conditions of Corollary 7.2.4 hold. We do not know of any such examples.

Remark 7.2.8. Corollary 7.2.4 gives a closed-form expression for the small J-function of Z – or, equivalently, for one-point gravitional descendant invariants of Z – in the case where Z is Fano. But in general (that is, without the Fano condition on Z) one can use Birkhoff factorization, as in [CG07, CFK14] and [CCIT19, §3.8], to compute any twisted genus-zero gravitional descendant invariant of $\mathrm{Fl}(E)$ in terms of genus-zero descendant invariants of X. The twisting here is with respect to the $\mathbb{C}^\times$-equivariant Euler class and the vector bundle F.

[1]That is, if the relative anticanonical bundle $-K_{Z/X}$ is ample.

Thus Corollary 7.2.4 determines the Lagrangian submanifold $\mathcal{L}_{F_\mu}$ that encodes twisted Gromov–Witten invariants. Applying [Coa14, Theorem 1.1], we see that Corollary 7.2.4 together with Birkhoff factorization allows us to compute any genus-zero Gromov–Witten invariant of the zero locus Z of the form

$$\langle \theta_1 \psi^{i_1}, \dots, \theta_n \psi^{i_n} \rangle_{0,n,d} \tag{7.6}$$

where all but one of the cohomology classes θ_i lie in $\operatorname{im}(i^*) \subset H^\bullet(Z)$ and the remaining θ_i is an arbitrary element of $H^\bullet(Z)$. Here $i\colon Z \to \mathrm{Fl}(E)$ is the inclusion map.

Remark 7.2.9. Applying Remark 7.2.8 to the blow-up $\tilde{X} \to X$ considered in the introduction, we see that Corollary 7.2.4 together with Birkhoff factorization allows us to compute arbitrary invariants of $\tilde{X}$ of the form (7.6) in terms of genus-zero gravitional descendants of X. In this case $\operatorname{im}(i^*) \subset H^\bullet(\tilde{X})$ contains all classes from $H^\bullet(X)$ and also the class of the exceptional divisor.

7.3 The Main Geometric Construction

Let F be a locally free sheaf on a variety X. We denote by $F(x)$ its fibre over x, a vector space over the residue field $\kappa(x)$. A morphism φ of locally free sheaves induces a linear map on fibres, denoted by $\varphi(x)$. We make the following definition:

Definition 7.3.1. Let $\varphi\colon E^m \to F^n$ a morphism of locally free sheaves of rank m and n respectively. The k-th degeneracy locus is the subvariety of X defined by

$$D_k(\varphi) = \left\{ x \in X \colon \operatorname{rk} \varphi(x) \leq k \right\}$$

Note that $D_k(\varphi) = X$ if $k \geq \min\{m, n\}$; if $k = \min\{m, n\} - 1$ we simply call $D_k(\varphi)$ the degeneracy locus of φ.

We have the following results:

- Scheme-theoretically, $D_k(\varphi)$ may be defined as the zero locus of the section $\wedge^k \varphi$; this shows that locally the ideal of $D_k(\varphi)$ is defined by the $(k+1) \times (k+1)$-minors of φ.
- If $E^\vee \otimes F$ is globally generated, then $D_k(\varphi)$ of a generic φ is either empty or has expected codimension $(m-k)(n-k)$, and the singular locus of $D_k(\varphi)$ is contained in $D_{k-1}(\varphi)$. In particular, if φ is generic and $\dim X < (m-k+1)(n-k+1)$, then $D_k(\varphi)$ is smooth [Ott95, Theorem 2.8].
- We may freely assume that $m \geq n$ in what follows, since we can always replace φ with its dual map whose degeneracy locus is the same.

The following result is proved along similar lines to [CCGK16, Lemma E.1].

Proposition 7.3.2. *Let X be a smooth variety, and $\varphi\colon E^m \to F^n$ a generic morphism of locally free sheaves on X. Suppose that $m \geq n$ and write $r = m-n$. Let $Y = D_{n-1}(\varphi)$ be the degeneracy locus of φ, and assume that φ has generically full rank, that Y has the expected codimension $m-n+1$ and that Y is smooth. Let $\pi\colon \mathrm{Gr}(r,E) \to X$ be the Grassmann bundle of E on X, and let S be the tautological subbundle on $\mathrm{Gr}(r,E)$. Then the blow-up $Bl_Y(X)$ of X along Y is a subvariety of $\mathrm{Gr}(r,E)$, cut out as the zero locus of the regular section $s \in \Gamma(\mathrm{Hom}(S,\pi^*F))$ defined by the composition*

$$S \hookrightarrow \pi^*E \xrightarrow{\pi^*\varphi} \pi^*F$$

where the first map is the canonical inclusion.

Proof. We write points in $\mathrm{Gr}(r,E)$ as (p,V), where $p \in X$ and V is a r-dimensional subspace of the fibre $E(p)$. At (p,V), the section s is given by the composition

$$V \hookrightarrow E(p) \xrightarrow{\varphi(p)} F(p)$$

so s vanishes at (p,V) if and only if $V \subset \ker\varphi(p)$.

The statement is local on X, so fix a point $P \in X$ and a Zariski open neighbourhood $U = \mathrm{Spec}(A)$ with trivialisations $E|_U \cong A^m, F|_U \cong A^n$. We will show that the equations of $Z(s) \cap U$ and $Bl_{U\cap Y}U$ agree. Under these identifications φ is given by a $n \times m$ matrix with entries in A. Since φ has generically maximal rank and Y is nonsingular, after changing trivialization and shrinking U if necessary, we may assume that φ is given by the matrix

$$\begin{pmatrix} x_0 & \dots & x_r & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Note that the ideal of the minors of this matrix is just $I = (x_0, \dots x_r)$ and that $x_0, \dots, x_r$ form part of a regular system of parameters around P, so we may assume that $n = 1, m = r+1$. Writing y_i for the basis of sections of $S^\vee$ on $\mathrm{Gr}(r, A^{r+1})$, we see that $Z(s)$ is given by the equation

$$x_0 y_0 + \dots + x_r y_r = 0$$

Under the Plücker isomorphism

$$\mathrm{Gr}(r, A^{r+1}) \to \mathbb{P}(\wedge^r A^{r+1}) \cong U \times \mathbb{P}^r_{y_0,\ldots,y_r}$$

$Z(s)$ maps to the variety cut out by the minors of the matrix

$$\begin{pmatrix} x_0 & \ldots & x_r \\ y_0 & \ldots & y_r \end{pmatrix}$$

i.e the blowup of $Y \cap U$ in U. $\square$

7.4 Examples

We close by presenting three example computations that use Theorems 5.0.1 and 5.0.2, calculating genus-zero Gromov–Witten invariants of blow-ups of projective spaces in various high-codimension complete intersections. Recall, as we will need it below, that if $E \to X$ is a vector bundle of rank n then the anticanonical divisor of $\mathrm{Gr}(r, E)$ is

$$-K_{\mathrm{Gr}(r,E)} = \pi^* \left(-K_X + r(\det E)\right) + n(\det S^\vee) \tag{7.7}$$

where $S \to \mathrm{Gr}(r, E)$ is the tautological subbundle. Recall too that the *regularised quantum period* of a Fano manifold Z is the generating function

$$\widehat{G}_Z(x) = 1 + \sum_{d=2}^{\infty} d! c_d x^d$$

for genus-zero Gromov–Witten invariants of Z, where

$$c_d = \sum_\beta \langle \theta \psi_1^{d-2} \rangle_{0,1,\beta} \qquad \text{for } \theta \in H^{\mathrm{top}}(Z) \text{ the class of a volume form}$$

and the sum runs over effective classes β such that $\langle \beta, -K_Z \rangle = d$.

Example 7.4.1. We will compute the regularised quantum period of $\tilde{X} = \mathrm{Bl}_Y \mathbb{P}^4$ where Y is a plane conic. Consider the situation as in §6.2 with:

- $X = \mathbb{P}^4$
- $E = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-1)$
- $G = \mathrm{GL}_2(\mathbb{C})$, $T = (\mathbb{C}^\times)^2 \subset G$

Then $A/\!\!/G$ is $\mathrm{Gr}(2,E)$, and $A/\!\!/T$ is the $\mathbb{P}^2\times\mathbb{P}^2$-bundle $\mathbb{P}(E)\times_{\mathbb{P}^4}\mathbb{P}(E)\to\mathbb{P}^4$. By Proposition 7.3.2 the zero locus $\tilde{X}$ of a section of $S^\vee\otimes\pi^*(\mathcal{O}(1))$ on $\mathrm{Gr}(2,E)$ is the blowup of $\mathbb{P}^4$ along the complete intersection of two hyperplanes and a quadric. We identify the group ring $\mathbb{Q}[H_2(A/\!\!/T,\mathbb{Z})]$ with $\mathbb{Q}[Q,Q_1,Q_2]$, where Q corresponds to the pullback of the hyperplane class of $\mathbb{P}^4$ and Q_i corresponds to $\tilde{H}_i$. Similarly, we identify $\mathbb{Q}[H_2(A/\!\!/G,\mathbb{Z})]$ with $\mathbb{Q}[Q,q]$, where again Q corresponds to the pullback of the hyperplane class of $\mathbb{P}^4$ and q corresponds to the first Chern class of $S^\vee$.

We will need Givental's formula [Giv96] for the J-function of $\mathbb{P}^4$:

$$J_{\mathbb{P}^4}(\tau,z)=ze^{\tau/z}\sum_{D=0}^{\infty}\frac{Q^De^{D\tau}}{\prod_{m=1}^{D}(H+mz)^5}\qquad\qquad \tau\in H^2(\mathbb{P}^4)$$

In the notation of §6.2, we have $\ell=1$, $r_\ell=r_1=2$, $r_{\ell+1}=3$. We relabel $\tilde{H}_{\ell,j}=\tilde{H}_j$ and $d_{\ell,j}=d_j$. We have that $\tilde{H}_{\ell+1,1}=\tilde{H}_{\ell+1,2}=0$, $\tilde{H}_{\ell+1,3}=\pi^*H$ and $d_{\ell+1,1}=d_{\ell+1,2}=0$, $d_{\ell+1,3}=D$. Write $F=S^\vee\otimes\pi^*\mathcal{O}(1)$. Corollary 7.2.5 and Remark 7.2.6 give

$$\begin{aligned}J_{F_0}(t,\tau,z)=ze^{\frac{t+\tau}{z}}\sum_{D=0}^{\infty}\sum_{d_1=0}^{\infty}\sum_{d_2=0}^{\infty}&\frac{(-1)^{d_1-d_2}Q^Dq^{d_1+d_2}e^{D\tau}e^{(d_1+d_2)t}\prod_{i=1}^{2}\prod_{m=1}^{d_i+D}(H_i+H+mz)}{\prod_{m=1}^{D}(H+mz)^5\prod_{m=1}^{d_1}(H_1+mz)^2\prod_{m=1}^{d_2}(H_2+mz)^2}\\&\times\prod_{i=1}^{2}\frac{\prod_{m=-\infty}^{0}(H_i-H+mz)}{\prod_{m=-\infty}^{d_i-D}(H_i-H+mz)}\frac{(H_1-H_2+z(d_1-d_2))}{H_1-H_2}\end{aligned}$$

To obtain the quantum period we need to calculate the anticanonical bundle of $\tilde{X}$. Equation (7.7) and the adjunction formula give

$$-K_{\tilde{X}}=3H+3\det S^\vee-(2H+\det S^\vee)=H+2\det S^\vee.$$

To extract the quantum period from the non-equivariant limit J_{F_0} of the twisted J-function, we take the component along the unit class $1\in H^\bullet(A/\!\!/G;\mathbb{Q})$, set $z=1$, and set $Q^\beta=x^{\langle\beta,-K_{\tilde{X}}\rangle}$. That is, we set $\lambda=0$, $t=0$, $\tau=0$, $z=1$, $q=x^2$, $Q=x$, and take the component along the unit class, obtaining

$$\begin{aligned}G_{\tilde{X}}(x)=&\sum_{n=0}^{\infty}\sum_{l=n+1}^{\infty}\sum_{m=l}^{\infty}(-1)^{l+m-1}x^{l+2m+2n}\tfrac{(l+n)!(l+m)!(l-n-1)!}{(l!)^5(m!)^2(n!)^2(n-l)!}(n-m)\\&+\sum_{l=0}^{\infty}\sum_{m=l}^{\infty}\sum_{n=l}^{\infty}(-1)^{m+n}x^{l+2m+2n}\tfrac{(l+n)!(l+m)!}{(l!)^5(m!)^2(n!)^2(n-l)!(m-l)!}\Big(1+(n-m)(-2H_n+H_{l+n}-H_{n-l})\Big)\end{aligned}$$

Thus the first few terms of the regularized quantum period are:

$$\widehat{G}_{\tilde{X}}(x) = 1 + 12x^3 + 120x^5 + 540x^6 + 20160x^8 + 33600x^9 + 113400x^{10} + 2772000x^{11} + 2425500x^{12} + \cdots$$

This strongly suggests that $\tilde{X}$ coincides with the quiver flag zero locus with ID 15 in [Kal19], although this is not obvious from the constructions.

Example 7.4.2. We will compute the regularised quantum period of $\tilde{X} = \mathrm{Bl}_Y \mathbb{P}^6$, where Y is a 3-fold given by the intersection of a hyperplane and two quadric hypersurfaces. Consider the situation as in §6.2 with:

- $X = \mathbb{P}^6$
- $E = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1)$
- $G = \mathrm{GL}_2(\mathbb{C})$, $T = (\mathbb{C}^\times)^2 \subset G$

Then $A/\!\!/G$ is $\mathrm{Gr}(2, E)$, and $A/\!\!/T$ is the $\mathbb{P}^2 \times \mathbb{P}^2$-bundle $\mathbb{P}(E) \times_{\mathbb{P}^6} \mathbb{P}(E) \to \mathbb{P}^6$. By Proposition 7.3.2 the zero locus $\tilde{X}$ of a section of $S^\vee \otimes \pi^*(\mathcal{O}(2))$ on $\mathrm{Gr}(2, E)$ is the blowup of $\mathbb{P}^6$ along the complete intersection of a hyperplane and two quadrics. We identify the group ring $\mathbb{Q}[H_2(A/\!\!/T, \mathbb{Z})]$ here with $\mathbb{Q}[Q, Q_1, Q_2]$, where Q corresponds to the pullback of the hyperplane class of $\mathbb{P}^6$ and Q_i corresponds to $\tilde{H}_i$. Similarly, we identify $\mathbb{Q}[H_2(A/\!\!/G, \mathbb{Z})]$ with $\mathbb{Q}[Q, q]$, where again Q corresponds to the pullback of the hyperplane class of $\mathbb{P}^6$ and q corresponds to the first Chern class of $S^\vee$.

The J-function of $\mathbb{P}^6$ is [Giv96]:

$$J_{\mathbb{P}^6}(\tau, z) = ze^{\tau/z} \sum_{D=0}^{\infty} \frac{Q^D e^{D\tau}}{\prod_{m=1}^{D}(H + mz)^7} \qquad \tau \in H^2(\mathbb{P}^6)$$

In the notation of §6.2, we have $\ell = 1$, $r_\ell = r_1 = 2$, $r_{\ell+1} = 3$. We relabel $\tilde{H}_{\ell,j} = \tilde{H}_j$ and $d_{\ell,j} = d_j$. We have that $\tilde{H}_{\ell+1,1} = \tilde{H}_{\ell+1,2} = 0$, $\tilde{H}_{\ell+1,3} = -\pi^* H$ and $d_{\ell+1,1} = d_{\ell+1,2} = 0$, $d_{\ell+1,3} = -D$. Write $F = S^\vee \otimes \pi^* \mathcal{O}(2)$. Corollary 7.2.5 and Remark 7.2.6 give

$$J_{F_0}(t, \tau, z) = ze^{\frac{t+\tau}{z}} \sum_{D=0}^{\infty} \sum_{d_1=-D}^{\infty} \sum_{d_2=-D}^{\infty} \frac{Q^D q^{d_1+d_2} e^{D\tau} e^{(d_1+d_2)t}}{\prod_{m=1}^{D}(H + mz)^7} \prod_{i=1}^{2} \frac{\prod_{m=-\infty}^{0}(H_i + mz)^2}{\prod_{m=-\infty}^{d_i}(H_i + mz)^2} \times \prod_{i=1}^{2} \frac{\prod_{m=1}^{d_i+2D}(H_i + 2H + mz)}{\prod_{m=1}^{d_i+D}(H_i + H + mz)} (-1)^{d_1-d_2} \frac{(H_1 - H_2 + z(d_1 - d_2))}{H_1 - H_2}$$

Again we will need the anticanonical bundle of $\tilde{X}$, which by (7.7) and the adjunction formula is

$$-K_{\widetilde{X}} = 9H + 3\det(S^*) - (4H + \det(S^*)) = 5H + 2\det(S^*).$$

To extract the quantum period from J_{F_0}, we take the component along the unit class $1 \in H^\bullet(A/\!\!/G;\mathbb{Q})$, set $z = 1$, and set $Q^\beta = x^{\langle \beta, -K_{\tilde{X}}\rangle}$. That is, we set $\lambda = 0$, $t = 0$, $\tau = 0$, $z = 1$, $q = x^2$, $Q = x^5$, and take the component along the unit class, obtaining

$$G_{\tilde{X}}(x) = \sum_{D=0}^{\infty}\sum_{d_1=0}^{\infty}\sum_{d_2=0}^{\infty}(-1)^{d_1+d_2}x^{5D+2d_1+2d_2}\frac{(d_1+2D)!(d_2+2D)!}{(D!)^7(d_1!)^2(d_2!)^2(d_1+D)!(d_2+D)!} \times \Big(1 + (d_1 - d_2)(-2H_{d_1} + H_{d_1+2D} - H_{d_1+D})\Big)$$

The first few terms of the regularized quantum period are:

$$\widehat{G}_{\tilde{X}}(x) = 1 + 480x^5 + 5040x^7 + 4082400x^{10} + 119750400x^{12} + 681080400x^{14} + \cdots$$

Example 7.4.3. We will compute the regularised quantum period of $\tilde{X} = \mathrm{Bl}_Y\,\mathbb{P}^6$, where Y is a quadric surface given by the intersection of 3 generic hyperplanes and a quadric hypersurface. Consider the situation as in §6.2 with:

- $X = \mathbb{P}^6$
- $E = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2)$
- $G = \mathrm{GL}_3(\mathbb{C})$, $T = (\mathbb{C}^\times)^3 \subset G$

Then $A/\!\!/G$ is $\mathrm{Gr}(3,E)$, and $A/\!\!/T$ is $\mathbb{P}(E) \times_{\mathbb{P}^6} \mathbb{P}(E) \times_{\mathbb{P}^6} \mathbb{P}(E) \to \mathbb{P}^6$. By Proposition 7.3.2 the zero locus $\tilde{X}$ of a section of $S^\vee \otimes \pi^*(\mathcal{O}(1))$ on $\mathrm{Gr}(3,E)$ is the blowup of $\mathbb{P}^6$ along the complete intersection of three hyperplanes and a quadric. We identify the group ring $\mathbb{Q}[H_2(A/\!\!/T,\mathbb{Z})]$ with $\mathbb{Q}[Q,Q_1,Q_2,Q_3]$, where Q corresponds to the pullback of the hyperplane class of $\mathbb{P}^6$ and Q_i corresponds to $\tilde{H}_i$. Similarly, we identify $\mathbb{Q}[H_2(A/\!\!/G,\mathbb{Z})]$ with $\mathbb{Q}[Q,q]$, where again Q corresponds to the pullback of the hyperplane class of $\mathbb{P}^6$ and q corresponds the first Chern class of $S^\vee$.

In the notation of §6.2, we have $\ell = 1, r_\ell = r_1 = 3, r_{\ell+1} = 4$. We relabel $\tilde{H}_{\ell,j} = \tilde{H}_j$ and $d_{\ell,j} = d_j$. We have that $\tilde{H}_{\ell+1,1} = \tilde{H}_{\ell+1,2} = \tilde{H}_{\ell+1,3} = 0$, $\tilde{H}_{\ell+1,4} = -\pi^*2H$ and $d_{\ell+1,1} = d_{\ell+1,2} =$

$d_{\ell+1,3} = 0$, $d_{\ell+1,4} = -2D$. Write $F = S^\vee \otimes \pi^* \mathcal{O}(1)$. Corollary 7.2.5 and Remark 7.2.6 give

$$J^{F_0}(t,\tau,z) = ze^{\frac{t+\tau}{z}} \sum_{D=0}^{\infty} \sum_{d_1=-2D}^{\infty} \sum_{d_2=-2D}^{\infty} \sum_{d_3=-2D}^{\infty} \frac{Q^D q^{d_1+d_2+d_3} e^{D\tau} e^{(d_1+d_2+d_3)t}}{\prod_{m=1}^{D}(H+mz)^7}$$
$$\times \prod_{i=1}^{3} \frac{\prod_{m=-\infty}^{0}(H_i+mz)^3}{\prod_{m=-\infty}^{d_i}(H_i+mz)^3} \prod_{i=1}^{3} \frac{1}{\prod_{m=1}^{d_i+2D}(H_i+2H+mz)} \prod_{i=1}^{3} \frac{\prod_{m=-\infty}^{d_i+D}(H_i+H+mz)}{\prod_{m=-\infty}^{0}(H_i+H+mz)}$$
$$\times \frac{(H_1-H_2+z(d_1-d_2))}{H_1-H_2} \frac{(H_1-H_3+z(d_1-d_3))}{H_1-H_3} \frac{(H_2-H_3+z(d_2-d_3))}{H_2-H_3}$$

Arguing as before,

$$-K_{\widetilde{X}} = 11H + 4\det(S^*) - (3H + \det(S^*)) = 8H + 3\det(S^*).$$

To extract the quantum period from J_{F_0}, we set $\lambda = 0$, $t = 0$, $\tau = 0$, $z = 1$, $q = x^3$, $Q = x^8$, and take the component along the unit class. The first few terms of the regularised quantum period are:

$$\widehat{G}_{\tilde{X}}(x) = 1 + 108x^3 + 17820x^6 + 5040x^8 + 5473440x^9 + 56364000x^{11} + 1766526300x^{12}$$
$$+ 117076459500x^{14} + 672012949608x^{15} + \cdots$$

Remark 7.4.4. Strictly speaking the use of Theorem 5.0.2 in the examples just presented was not necessary. Whenever the base space X is a projective space, or more generally a Fano complete intersection in a toric variety or flag bundle, then one can replace our use of Theorem 5.0.2 (but not Theorem 5.0.1) by [CFKS08, Corollary 6.3.1]. However there are many examples that genuinely require both Theorem 5.0.1 and Theorem 5.0.2: for instance when X is a toric complete intersection but the line bundles that define the center of the blow-up do not arise by restriction from line bundles on the ambient space. (For a specific such example one could take X to be the three-dimensional Fano manifold $\mathrm{MM}_{3\text{–}9}$: see [CCGK16, §62].) For notational simplicity we chose to present examples with $X = \mathbb{P}^N$, but the approach that we used applies without change to more general situations.

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