Ann. Henri Poincaré *Online First*

https://doi.org/10.1007/s00023-022-01217-6

Annales Henri Poincaré

Recurrent Motions in a Piecewise Linear Oscillator

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Professor Alan Lazer, in memoriam.

Abstract. We study the oscillator $\ddot{x} + n^2 x + h(x) = p(t)$, where h is a piecewise linear saturation function and p is a continuous 2π-periodic forcing. It is shown that there is recurrence if and only if p satisfies the Lazer–Leach condition. This condition relates the n-th Fourier coefficient of $p(t)$ with the maximum of h and was first introduced to characterize the existence of periodic solutions.

Mathematics Subject Classification. 37E40, 37B20, 34C15, 70K40.

1. Introduction

For $n \in \mathbb{N} = \{1, 2, \ldots\}$, consider the forced oscillator model

$$\ddot{x} + n^2 x + h(x) = p(t), \tag{1.1}$$

where $h, p \in C(\mathbb{R})$ are bounded and p is 2π-periodic. The n-th Fourier coefficient of p is given by:

$$\hat{p}_n = \frac{1}{2\pi} \int_0^{2\pi} p(t) e^{-int} \, \mathrm{d}t.$$

In the linear case $h = 0$, it is a well-established fact that solutions of (1.1) are 2π-periodic (and hence bounded) if $\hat{p}_n = 0$, and otherwise unbounded and non-recurrent due to resonance phenomena. In [10], Lazer and Leach studied the case when h has two distinct finite limits $h(\pm\infty) = \lim_{x\to\pm\infty} h(x)$ at infinity and all values of h lie between those limits. They were able to show that (1.1) has a 2π-periodic solution if and only if

$$\pi |\hat{p}_n| < |h(+\infty) - h(-\infty)|. \tag{1.2}$$

Published online: 11 July 2022

Birkhäuser

Later, it was proven in [2] that the negation of this inequality implies that all solutions $x(t)$ satisfy

$$\lim_{|t|\to\infty} \left[x(t)^2 + \dot{x}(t)^2\right] = \infty. \tag{1.3}$$

See also [17] for a previous related work. Results with respect to boundedness were obtained in [14]. There, it was shown that the same condition (1.2) leads to the boundedness of all solutions in the special case where $h = h_L$ with $L > 0$ is the piecewise linear function given by

$$h_L(x) = \begin{cases} L & \text{if } x \geq 1, \\ Lx & \text{if } |x| \leq 1, \\ -L & \text{if } x \leq -1, \end{cases}$$

provided that $p \in C^5(\mathbb{R})$ is 2π-periodic (see also [6] for a related result with a discontinuous h). Moreover, this led to the insight that almost every solution $x(t)$ is Poisson stable, which can be understood as follows in the context of 2π-periodic systems. There is a sequence of integers $\{\sigma_n\}_{n\in\mathbb{Z}}$ with $\sigma_n \to \pm\infty$ as $n \to \pm\infty$ such that

$$|x(t + 2\pi\sigma_n) - x(t)| + |\dot{x}(t + 2\pi\sigma_n) - \dot{x}(t)| \to 0 \quad \text{as } |n| \to \infty,$$

uniformly with respect to $t \in [0, 2\pi]$. In the same year, Liu obtained a similar result for general $h \in C^6(\mathbb{R})$ such that $\lim_{|x|\to\infty} x^k h^{(k)}(x) = 0$ for $1 \leq k \leq 6$, if $p \in C^7(\mathbb{R})$ is 2π-periodic [9]. Recent results for more general nonlinearities can be found in [15,18]. All latter results were obtained by using variants of Moser's small twist theorem. However, the application of any such invariant curve theorem requires a considerable degree of smoothness of either $h(x)$ or $p(t)$. It is an interesting question if any of the nice features of solutions survive if only mild regularity assumptions are made. In the present paper, we investigate this question for the piecewise linear equation:

$$\ddot{x} + n^2 x + h_1(x) = p(t). \tag{1.4}$$

By rescaling $\ddot{x} + n^2 x + h_L(x) = p(t)$, one obtains the function $\tilde{h}_L(x) = \operatorname{sign}(x)$ for $|x| \geq \frac{1}{L}$ and $\tilde{h}_L(x) = Lx$ for $|x| < \frac{1}{L}$. Since the slope L has basically no effect on the dynamics, we have normalized the equation by setting $L = 1$. Besides giving a good starting point for more general nonlinearities, such piecewise linear oscillators are also known in the engineering literature. For example, (1.4) can be considered as a model for an oscillator with stops (see [5] and also [14] for the derivation of (1.4)). Our main result is the following.

Theorem 1.1. *Suppose $p \in C(\mathbb{R})$ is 2π-periodic and satisfies the Lazer–Leach condition $\pi|\hat{p}_n| < 2$. If $x(t) = x(t; \tilde{x}, \tilde{v})$ denotes the solution of (1.4) with the initial condition $x(0) = \tilde{x}$ and $\dot{x}(0) = \tilde{v}$; then, $x(t; \tilde{x}, \tilde{v})$ is Poisson stable for almost every $(\tilde{x}, \tilde{v}) \in \mathbb{R}^2$.*

This theorem is an improvement of Corollary 2.1 in [14]. The statement is also true for the discontinuous limit case.

Theorem 1.2. *Suppose $p \in C(\mathbb{R})$ is 2π-periodic and satisfies the Lazer–Leach condition $\pi|\hat{p}_n| < 2$. Moreover, assume that ∂N is countable, where $N = \{t \in \mathbb{R} : |p(t)| = 1\}$. If $x(t) = x(t; \tilde{x}, \tilde{v})$ denotes the solution of*

$$\ddot{x} + n^2 x + \operatorname{sign}(x) = p(t), \tag{1.5}$$

with the initial condition $x(0) = \tilde{x}$ and $\dot{x}(0) = \tilde{v}$, then $x(t; \tilde{x}, \tilde{v})$ is Poisson stable for almost every $(\tilde{x}, \tilde{v}) \in \mathbb{R}^2$.

Remark 1.3. (a) In Theorem 1.1, the Lazer–Leach condition is not only sufficient but also necessary for recurrence, since all solutions satisfy (1.3) if $\pi|\hat{p}_n| \geq 2$ [2].
(b) Equation (1.5) is basically the case considered in [6]. Note that one first has to define a proper notion of solutions to (1.5) (see Definition 5.1). We also refer the reader to [11] for a discussion of chaos in second-order equations with signum nonlinearities.
(c) Theorems 1.1 and 1.2 show that the set of initial conditions such that (1.3) holds has measure zero. The authors know of no example exhibiting unbounded orbits, provided the assumptions of these theorems are satisfied.

The proof to Theorem 1.1 starts out similar to those of the boundedness results stated above. A twist map $\bar{P} : (\bar{\tau}, v) \mapsto (\bar{\tau}_1, v_1)$ on the annulus $S^1 \times [0, \infty[$ is constructed so that its orbits correspond to large amplitude solutions of (1.4). The construction is done in such a way that $\bar{P}$ is recurrent if and only if the time-2π map associated with (1.4) is recurrent. The lift of $\bar{P}$ has the form

$$\begin{cases} \tau_1 = \tau + 2\pi - \frac{L(\tau)}{v} + R(\tau, v), \\ v_1 = v + L'(\tau) + S(\tau, v), \end{cases}$$

where $L \in C^2$ is a positive 2π-periodic function and $R, S \in C^1$. Our main abstract result (Theorem 3.1) states that any such map is recurrent if R and S satisfy certain bounds. This claim is established by first finding an adiabatic invariant of the system and then applying a refined version of Poincaré's recurrence theorem; a method recently introduced by Dolgopyat [4].

The paper is organized as follows: In Sect. 2, Maharam's recurrence theorem for measure-preserving maps is introduced. In Sect. 3, we find an adiabatic invariant for a family of exact symplectic twist maps. This leads to the proof of Theorem 3.1 stating that this family is recurrent. Section 4 contains the application to Eq. (1.4) and the proof of Theorem 1.1. Finally, the discontinuous equation (1.5) is discussed in the last section.

2. Recurrence

Let $(X, \mathcal{A}, \mu)$ be a measure space and introduce the following useful notation. For $A, B \in \mathcal{A}$, we write

$$A \subset B \quad \text{mod } \mu,$$

if $A \subset B \cup \mathcal{N}$, where $\mathcal{N}$ is a set of measure zero. Now, consider a map $T : X \to X$ which is bi-measurable, that is

$$T^{-1}(A), T(A) \in \mathcal{A} \qquad \text{for all } A \in \mathcal{A}.$$

Such a map T is said to be *measure-preserving*, if

$$\mu(T(A)) = \mu(A) \qquad \text{for all } A \in \mathcal{A}.$$

As a consequence, such a measure-preserving transformation satisfies

$$\mu(T^{-1}(A)) \leq \mu(A) \qquad \text{for all } A \in \mathcal{A},$$

with equality if $A \subset T(X) \mod \mu$.

Remark 2.1. In the literature, there is no unique way of defining the two properties above. In particular, T is often called measure-preserving, if $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{A}$. However, the definition in this work was chosen since it seems to be the most natural in the application to mechanical problems and suchlike.

As a simple example, consider the space $X = [0, \infty[$ equipped with the Lebesgue measure λ and the map $T_1(x) = x+1$. Then, T_1 is measure-preserving in the above sense, but the strict inequality $\lambda(T^{-1}(A)) < \lambda(A)$ holds, e.g., for $A = [0, 2]$ since $T^{-1}(A) = [0, 1]$.

Since T maps X into itself, the iterates $T^n = T^{n-1} \circ T$, where $T^0 = \text{id}$, are well-defined for all $n \in \mathbb{N}$. We call the map T *recurrent*, if for every $A \in \mathcal{A}$ for almost all $x \in A$ there is $n \in \mathbb{N}$ such that $T^n(x) \in A$, that is

$$A \subset \bigcup_{n=1}^{\infty} T^{-n}(A) \mod \mu,$$

where $T^{-n}(A)$ denotes the pre-image under T^n. In other words, the set of points in A not returning to A has measure zero. Since T is measure-preserving, also any (iterated) pre-image of this set has measure zero. Hence, T is even *infinitely recurrent*, i.e., for almost all $x \in A$ there is an increasing sequence $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $T^{n_k}(x) \in A$ for all $k \in \mathbb{N}$.

In the case of a finite measure-space, the famous Poincaré recurrence theorem characterizes the relation between measure-preserving and recurrent maps. We will use it in the following form.

Lemma 2.2. *Let $(X, \mathcal{A}, \mu)$ be a measure space such that $\mu(X) < \infty$ and suppose $T : X \to X$ is measure-preserving. Then, T is recurrent.*

Unfortunately, the situation is less clear if the space has infinite measure. However, the statement of the recurrence theorem stays valid if there exists a set $\mathcal{M}$ of finite measure which acts as some kind of bottleneck. This is described in the following generalization of Lemma 2.2 due to Maharam, see [12], which also recently got some attention in the context of twist maps by Dolgopyat [3].

Lemma 2.3 (Maharam's recurrence theorem). *Let $(X, \mathcal{A}, \mu)$ be a measure space and suppose $T : X \to X$ is measure-preserving. If there exists a set $\mathcal{M} \in \mathcal{A}$ with $\mu(\mathcal{M}) < \infty$, such that*

$$X \subset \bigcup_{n=1}^{\infty} T^{-n}(\mathcal{M}) \mod \mu,$$

then T is recurrent.

Proof. The "time of first return" $r(x) = \min\{k \in \mathbb{N} : T^k(x) \in \mathcal{M}\}$ is well-defined for almost all $x \in X$ by assumption. In particular, it can be shown that there is a set Γ of measure zero such that the induced map $S : \mathcal{M}\backslash\Gamma \to \mathcal{M}$ given by $S(x) = T^{r(x)}(x)$ is well-defined and satisfies $S(\mathcal{M}\backslash\Gamma) \subset \mathcal{M}\backslash\Gamma$. Moreover, S is measure-preserving, and hence, one can apply the Poincaré recurrence theorem to see that S is also recurrent. Now, let $A \in \mathcal{A}$ be a measurable set in X and for $k \in \mathbb{N}$ consider the sets

$$A_k = \{x \in A : r(x) = k\}.$$

Moreover, define $B_k = T^k(A_k) \subset \mathcal{M}$. Since S is recurrent, we have

$$B_k \subset \bigcup_{n=1}^{\infty} S^{-n}(B_k) \subset \bigcup_{n=1}^{\infty} T^{-n}(B_k) \mod \mu.$$

From this, it follows

$$T^{-k}(B_k) \subset \bigcup_{n=1}^{\infty} T^{-(n+k)}(B_k) \mod \mu.$$

Since $A_k \subset T^{-k}(B_k)$ and $\mu(A_k) = \mu(B_k)$, we know that $T^{-k}(B_k) = A_k$ up to a set of measure zero. This in turn implies

$$A_k \subset \bigcup_{n=1}^{\infty} T^{-n}(A_k) \mod \mu.$$

Finally, taking the union over all $k \in \mathbb{N}$ shows that almost every point in A returns to A. □

There are two drawbacks to Lemma 2.3. On the one hand, such a set $\mathcal{M}$ does not exist for every recurrent measure-preserving transformation, as already a trivial example like the identity shows. On the other hand, even when it does exist, it can be hard to find. In the following section, we will introduce a class of measure-preserving transformations for which the construction of $\mathcal{M}$ can be done explicitly.

Remark 2.4. If X is σ-finite, the following observation can be made. A measure-preserving map $T : X \to X$ is recurrent if and only if there is a covering $\{X_j\}_{j\in\mathbb{N}}$ of X and a collection of sets $\{\mathcal{M}_j\}_{j\in\mathbb{N}}$ with $\mu(\mathcal{M}_j) < \infty$ such that for all $j \in \mathbb{N}$ we have $T(X_j) \subset X_j$ and $X_j \subset \cup_{n=1}^{\infty} T^{-n}(\mathcal{M}_j) \mod \mu$.

Note that there are several more generalizations to the Poincaré recurrence theorem, and depending on the situation one might choose the appropriate version. For example, if $\{X_j\}_{j\in\mathbb{N}}$ is a covering of X with $\mu(X_j) < \infty$ and for every fixed $j \in \mathbb{N}$ the measure-preserving map T satisfies

$$\lim_{n\to\infty} \frac{1}{n}\mu\left(\bigcup_{k=1}^{n} T^k(X_j)\right) = 0,$$

then T is also recurrent (see [8]). For a more thorough discussion of maps preserving an infinite measure, we refer the reader to [1,3].

3. Exact Symplectic Twist Maps

We identify the circle S^1 with the quotient space $\mathbb{R}/2\pi\mathbb{Z}$. With a small abuse of notation, $C^n(S^1)$ denotes the space of n-times continuously differentiable functions $F : \mathbb{R} \to \mathbb{R}$ that are 2π-periodic. Sometimes, we will not differentiate between a map $\bar{F} : S^1 \to S^1$ and its lift satisfying

$$F(\theta + 2\pi) = F(\theta) \mod 2\pi.$$

For $v_* > 0$, consider the space $M_{v_*} = S^1 \times [v_*, \infty[$ equipped with the absolutely continuous measure $\mu = v\, d\bar{\theta} \otimes \mathrm{d}v$. In this chapter, we will mostly study maps:

$$\bar{f} : M_{v_*} \to S^1 \times [0, \infty[, \quad (\bar{\theta}, v) \mapsto (\bar{\theta}_1, v_1),$$

and we will use the same convention as above regarding its lift on the universal cover $\mathbb{M}_{v_*} = \mathbb{R} \times [v_*, \infty[$. Moreover, assume $\bar{f}$ is a C^1-diffeomorphism with respect to its image. We say $\bar{f}$ satisfies the twist condition if

$$\frac{\partial \bar{\theta}_1}{\partial v} \neq 0, \qquad \text{in } M_{v_*}.$$

Furthermore, suppose there is a function $\eta = \eta(\bar{\theta}, v)$ in $C^1(M_{v_*})$ such that

$$\mathrm{d}\eta = v_1^2 \mathrm{d}\bar{\theta}_1 - v^2 \mathrm{d}\bar{\theta}, \tag{3.1}$$

where we use the usual abuse of notation $\mathrm{d}\bar{\theta}_1 = \frac{\partial\bar{\theta}_1}{\partial\bar{\theta}}\mathrm{d}\bar{\theta} + \frac{\partial\bar{\theta}_1}{\partial v}\mathrm{d}v$. Then, $\bar{f}$ is called *exact symplectic twist map*. There are two direct consequences: First, the map is symplectic in the sense that

$$v_1 \mathrm{d}\bar{\theta}_1 \wedge \mathrm{d}v_1 = v \mathrm{d}\bar{\theta} \wedge \mathrm{d}v. \tag{3.2}$$

And second, given any $v > v_*$ denote by C_v the embedded circle $S^1 \times \{v\}$. Then,

$$0 = \int_{C_v} \mathrm{d}\eta = \int_{S^1} \left(v_1^2 \frac{\partial \bar{\theta}_1}{\partial \bar{\theta}} - v^2 \right) \mathrm{d}\bar{\theta}. \tag{3.3}$$

Let us also introduce the class $\mathcal{F}_{\mathrm{u}}(m)$ of continuous functions $F : M_{v_*} \to \mathbb{R}$, for some $v_* > 0$, such that $\{v^m F(\cdot, v)\}_{v \geq v_*}$ converges uniformly as $v \to \infty$. Since we want to analyze the recurrence properties of $\bar{f}$, we need to make sure that the forward iterates $\bar{f}^n$ are well-defined for $n \in \mathbb{N}$. To this end, let $\mathcal{D}_1 = M_{v_*}$ and $\mathcal{D}_{n+1} = \bar{f}^{-1}(\mathcal{D}_n)$ for $n \in \mathbb{N}$. Then,

$$\mathcal{D} = \bigcap_{n=1}^{\infty} \mathcal{D}_n \subset M_{v_*}$$

denotes the set of initial condition $(\bar{\theta}_0, v_0)$, for which the complete forward orbit $(\bar{\theta}_n, v_n) := \bar{f}^n(\bar{\theta}_0, v_0)$, $n \in \mathbb{N}$, is well-defined. Moreover, the restricted map

$$\bar{f} : \mathcal{D} \to \mathcal{D}$$

is bi-measurable, since $\bar{f}$ is a diffeomorphism and also preserves the measure μ, due to (3.2). Moreover, this restricted version of $\bar{f}$ is a self-map, which is necessary for the application of Lemma 2.3. Note that possibly $\mathcal{D} = \emptyset$. The so-called escaping set of $\bar{f}$ is given by:

$$\mathcal{E} = \left\{ (\bar{\theta}_0, v_0) \in \mathcal{D} : \lim_{n\to\infty} v_n = \infty \right\}$$

and clearly f is non-recurrent on $\mathcal{E}$. Its complement $\mathcal{D}\backslash\mathcal{E}$ on the other hand can be covered by the measurable sets:

$$\mathcal{B}_m = \left\{ (\bar{\theta}_0, v_0) \in \mathcal{D} : \liminf_{n\to\infty} v_n \le m \right\}, \quad m \in \mathbb{N}.$$

Since every orbit starting in $\mathcal{B}_m$ eventually has to enter the set $S^1 \times [0, m+1]$, Lemma 2.3 can be applied to the restricted map $\bar{f} : \mathcal{B}_m \to \mathcal{B}_m$. It follows easily that $\bar{f}$ is recurrent on $\mathcal{D}\backslash\mathcal{E}$. Therefore, proving that $\bar{f} : \mathcal{D} \to \mathcal{D}$ is recurrent is equivalent to showing $\mu(\mathcal{E}) = 0$. This is the subject of our main abstract result. The strategy of the proof will be to apply Lemma 2.3 with $X = \mathcal{E}$ and $T = \bar{f}$.

Theorem 3.1. *Consider the twist map $\bar{f} : M_{v_*} \to S^1 \times [0, \infty[$, of which the lift f is given by*

$$\begin{cases} \theta_1 = \theta + 2\pi - \frac{L(\theta)}{v} + R(\theta, v), \\ v_1 = v + L'(\theta) + S(\theta, v), \end{cases}$$

where $L \in C^2(S^1)$, $L > 0$, $R, S \in C^1(M_{v_*})$, $R \in \mathcal{F}_u(2)$, $S \in \mathcal{F}_u(1)$ and

$$\sup_{(\theta,v)\in\mathbb{M}_{v_*}} v^{\nu_2} |\partial^\nu R(\theta, v)| < \infty, \qquad \lim_{v\to\infty} v^{\nu_2} \partial^\nu R(\theta, v) = 0,$$

for every $\theta \in \mathbb{R}$ and $\nu = (\nu_1, \nu_2)$ with $|\nu| = 1$. Moreover, assume that $\bar{f}$ is one-to-one and exact symplectic in the sense that there is a function $\eta \in C^1(M_{v_*})$ with $\mathrm{d}\eta = v_1^2 \mathrm{d}\bar{\theta}_1 - v^2 \mathrm{d}\bar{\theta}$. Then $\mu(\mathcal{E}) = 0$, where

$$\mathcal{E} = \left\{ (\bar{\theta}_0, v_0) \in M_{v_*} : \lim_{n\to\infty} v_n = \infty \right\}$$

denotes the escaping set of $\bar{f}$.

Remark 3.2. Under the stronger assumptions $L \in C^6(S^1)$, $R, S \in C^5(M_{v_*})$ and

$$\sup_{(\theta,v)\in\mathbb{M}_{v_*}} v^{2+\nu_2}|\partial^\nu R(\theta,v)| < \infty, \qquad \sup_{(\theta,v)\in\mathbb{M}_{v_*}} v^{1+\nu_2}|\partial^\nu S(\theta,v)| < \infty,$$

for any multi-index $\nu = (\nu_1, \nu_2)$ with $|\nu| \leq 5$, KAM-theory is applicable and shows the boundedness of all orbits. See [14] for a suitable invariant curve theorem and its application to a map of the type under consideration.

In the proof, we will need the following auxiliary lemma, which is basically a variant of Lemma 4.1 in [7].

Lemma 3.3. *Consider a map $\bar{f} : D \to S^1 \times [0,\infty[$, $(\bar{\theta}, v) \mapsto (\bar{\theta}_1, v_1)$, where $D \subset S^1 \times [0,\infty[$. Let $\rho(\bar{\theta}, v) = v + \beta(\bar{\theta})$ with $\beta \in C(S^1)$. Moreover, suppose there is $v_* > 0$ such that for all $(\bar{\theta}, v) \in D \cap M_{v_*}$ we have*

$$|\rho(\bar{f}(\bar{\theta}, v)) - \rho(\bar{\theta}, v)| \leq \frac{\delta(v)}{v}, \tag{3.4}$$

where $\delta : [v_*, \infty[\to [0,\infty[$ is a decreasing function with $\lim_{v\to\infty} \delta(v) = 0$. Then, there is a set $\mathcal{M} \subset S^1 \times [0,\infty[$ with $\mu(\mathcal{M}) < \infty$ such that every unbounded orbit of $\bar{f}$ enters $\mathcal{M}$.

Proof. Let $(\rho_j)_{j\in\mathbb{N}} \subset [2v_*, \infty[$ be an increasing sequence with $\lim_{j\to\infty} \rho_j = \infty$ such that

$$\rho_1 > \frac{1}{2\rho_1} + \|\beta\|_\infty \qquad \text{and} \qquad \delta\left(\tfrac{\rho_j}{2}\right) < 2^{-(j+1)}$$

for all $j \in \mathbb{N}$. Now, define

$$\mathcal{M} = \bigcup_{j\in\mathbb{N}} \mathcal{M}_j, \qquad \mathcal{M}_j = \rho^{-1}\left(\left(\rho_j - \frac{1}{2^j\rho_j}, \rho_j + \frac{1}{2^j\rho_j}\right)\right).$$

Then, $\mathcal{M}_j \subset S^1 \times [0,\infty[$, and moreover, we have

$$\begin{aligned}\mu(\mathcal{M}_j) &= \int_0^{2\pi} \int_{\rho_j - \frac{1}{2^j\rho_j} - \beta(\bar{\theta})}^{\rho_j + \frac{1}{2^j\rho_j} - \beta(\bar{\theta})} v \,\mathrm{d}v \,\mathrm{d}\bar{\theta} \\ &= \int_0^{2\pi} 2^{-j}\left(1 - \frac{\beta(\bar{\theta})}{\rho_j}\right) \mathrm{d}\bar{\theta} \\ &\leq 2^{-j+1}\pi\left(1 + \frac{\|\beta\|_\infty}{2v_*}\right).\end{aligned}$$

In particular, this implies $\mu(\mathcal{M}) < \infty$.

Fix some $(\bar{\theta}_0, v_0) \in D$ such that the corresponding complete forward orbit $(\bar{\theta}_n, v_n)$ is unbounded. Moreover, select $j_0 \in \mathbb{N}$ such that

$$\rho_{j_0} > 2\|\delta\|_\infty v_*^{-1} + 2\|\beta\|_\infty. \tag{3.5}$$

Since $\limsup_{n\to\infty} \rho(\bar{\theta}_n, v_n) = \infty$, there is $N \in \mathbb{N}$ so that $\rho(\bar{\theta}_N, v_N) > \rho_{j_0}$. Thus, $\rho(\bar{\theta}_N, v_N)$ lies in the interval $]\rho_{j_0}, \rho_{j_1}]$ for some $j_1 > j_0$. Since the orbit is unbounded, there must be a first index $K > N$ such that $\rho(\bar{\theta}_K, v_K) \notin]\rho_{j_0}, \rho_{j_1}]$. But this cannot happen without the orbit entering either $\mathcal{M}_{j_0}$ or $\mathcal{M}_{j_1}$. First consider the case that $\rho(\bar\theta_K, v_K) > \rho_{j_1} \geq \rho(\bar\theta_{K-1}, v_{K-1})$. Then, using (3.4) and (3.5) yields

$$\begin{aligned}
v_{K-1} &= \rho(\bar\theta_{K-1}, v_{K-1}) - \beta(\bar\theta_{K-1}) \\
&\geq \rho(\bar\theta_K, v_K) - \frac{\delta(v_{K-1})}{v_{K-1}} - \beta(\bar\theta_{K-1}) \\
&> \rho_{j_1} - \|\delta\|_\infty v_*^{-1} - \|\beta\|_\infty \\
&> \tfrac{\rho_{j_1}}{2}.
\end{aligned}$$

From this, it follows

$$|\rho(\bar\theta_K, v_K) - \rho_{j_1}| \leq |\rho(\bar\theta_K, v_K) - \rho(\bar\theta_{K-1}, v_{K-1})| \leq \frac{\delta(v_{K-1})}{v_{K-1}} < \frac{2\delta\left(\frac{\rho_{j_1}}{2}\right)}{\rho_{j_1}} < \frac{1}{2^{j_1}\rho_{j_1}}.$$

Thus, $(\bar\theta_K, v_K) \in \mathcal{M}_{j_1}$. In the other case, $\rho(\bar\theta_{K-1}, v_{K-1}) > \rho_{j_0} \geq \rho(\bar\theta_K, v_K)$, we have

$$v_{K-1} > \rho_{j_0} - \beta(\bar\theta_{K-1}) > \tfrac{\rho_{j_0}}{2}.$$

Then, $(\bar\theta_K, v_K) \in \mathcal{M}_{j_0}$ follows analogously. □

Now, we are in position to prove the main result of this section.

Proof of Theorem 3.1. As the first step, we perform the change of variables $\Phi : (\theta, v) \mapsto (\tau, r)$ defined by

$$\tau(\theta) = \gamma \int_0^\theta \frac{1}{L(s)^2}\,\mathrm{d}s, \qquad r(\theta, v) = \gamma^{-\frac{1}{2}} L(\theta) v,$$

where

$$\gamma = 2\pi \left(\int_0^{2\pi} \frac{1}{L(s)^2}\,\mathrm{d}s\right)^{-1} > 0.$$

The constant γ is chosen such that

$$\tau(\theta + 2\pi) = \tau(\theta) + 2\pi.$$

Since $\tau'(\theta) = \frac{\gamma}{L(\theta)^2} > 0$, the map $\tau \in C^3(S^1, S^1)$ is a diffeomorphism. Hence, also Φ is a diffeomorphism with regard to its image. The Taylor expansion of τ implies

$$\tau(\theta_1) = \tau(\theta_1 - 2\pi) = \tau(\theta) - \frac{\gamma}{L(\theta) v} + R_1(\theta, v), \tag{3.6}$$

where

$$\begin{aligned}
R_1(\theta, v) &= \tfrac{\gamma R(\theta, v)}{L(\theta)^2} + \int_\theta^{\theta_1 - 2\pi} (\theta_1 - 2\pi - s)\tau''(s)\,\mathrm{d}s \\
&= \tfrac{\gamma R(\theta, v)}{L(\theta)^2} + (\theta_1 - 2\pi - \theta)^2 \int_0^1 (1-\lambda)\tau''((1-\lambda)\theta + \lambda(\theta_1 - 2\pi))\,\mathrm{d}\lambda \\
&= \tfrac{\gamma R(\theta, v)}{L(\theta)^2} + \left(R(\theta, v) - \tfrac{L(\theta)}{v}\right)^2 \int_0^1 (1-\lambda)\tau''\left(\theta + \lambda\left(R(\theta, v) - \tfrac{L(\theta)}{v}\right)\right)\mathrm{d}\lambda.
\end{aligned}$$

Then $R_1 \in \mathcal{F}_{\mathrm{u}}(2)$, since $R \in \mathcal{F}_{\mathrm{u}}(2)$ and

$$\lim_{v\to\infty} v^2(\theta_1 - 2\pi - \theta)^2 \int_0^1 (1-\lambda)\tau''((1-\lambda)\theta + \lambda(\theta_1 - 2\pi))\,\mathrm{d}\lambda = L(\theta)^2\frac{\tau''(\theta)}{2}$$

holds uniformly in θ. Moreover, a direct calculation shows that also the derivatives have the same asymptotics, i.e., for $\nu = (\nu_1, \nu_2)$ with $|\nu| = 1$ we have

$$\sup_{(\theta,v)\in\mathbb{M}_{v_*}} v^{\nu_2}|\partial^\nu R_1(\theta, v)| < \infty, \qquad \lim_{v\to\infty} v^{\nu_2}|\partial^\nu R_1(\theta, v)| = 0.$$

Similarly, the Taylor expansion of L yields

$$L(\theta_1) = L(\theta_1 - 2\pi) = L(\theta) + L'(\theta)\left(R(\theta, v) - \tfrac{L(\theta)}{v}\right) + I(\theta, v)$$

with $I \in \mathcal{F}_{\mathrm{u}}(2)$. Altogether, this yields

$$\begin{aligned} L(\theta_1)v_1 &= \left(L(\theta) - \tfrac{L(\theta)L'(\theta)}{v} + L'(\theta)R(\theta, v) + I(\theta, v)\right)\left(v + L'(\theta) + S(\theta, v)\right) \\ &= L(\theta)v + S_1(\theta, v), \end{aligned}$$

where $S_1 \in \mathcal{F}_{\mathrm{u}}(1)$. On $\Phi(\mathbb{M}_{v_*})$, we define

$$R_2(\tau, r) = R_1\left(\Phi^{-1}(\tau, r)\right) \quad \text{and} \quad S_2(\tau, r) = \gamma^{-\frac{1}{2}} S_1\left(\Phi^{-1}(\tau, r)\right).$$

Then, the lift $g = \Phi \circ f \circ \Phi^{-1}$ of the transformed twist map $\bar{g}$ is given by:

$$\begin{cases} \tau_1 = \tau + 2\pi - \frac{\sqrt{\gamma}}{r} + R_2(\tau, r), \\ r_1 = r + S_2(\tau, r), \end{cases}$$

with $R_2 \in \mathcal{F}_{\mathrm{u}}(2)$ and $S_2 \in \mathcal{F}_{\mathrm{u}}(1)$. Moreover, writing $\Phi^{-1}(\tau, r) = (\theta(\tau), v(\tau, r))$ we get

$$\begin{aligned} \frac{\partial R_2}{\partial \tau} &= \frac{\partial R_1}{\partial \theta}(\theta, v)\theta' + \frac{\partial R_1}{\partial v}(\theta, v)\frac{\partial v}{\partial \tau} \\ &= \frac{\partial R_1}{\partial \theta}(\theta, v)\frac{L^2(\theta)}{\gamma} - \frac{\partial R_1}{\partial v}(\theta, v)\frac{vL(\theta)L'(\theta)}{\gamma}. \end{aligned}$$

In particular, it follows

$$\sup_{(\tau,r)\in\Phi(\mathbb{M}_{v_*})} \left|\frac{\partial R_2}{\partial \tau}(\tau, r)\right| < \infty, \quad \text{and} \quad \lim_{r\to\infty} \frac{\partial R_2}{\partial \tau}(\tau, r) = 0,$$

for any $\tau \in \mathbb{R}$. Also, note that $\bar{g}$ is again one-to-one and

$$r_1^2\mathrm{d}\bar{\tau}_1 - r^2\mathrm{d}\bar{\tau} = \mathrm{d}\hat{\eta}$$

holds for $\hat{\eta} = \eta \circ \Phi^{-1}$. Therefore, the new map is an exact symplectic twist map as well. Hence, Lemma 3.3 can be applied to $\bar{g}$ if a suitable adiabatic invariant $\rho(\bar{\tau}, r)$ can be found. In order to construct ρ, let

$$\alpha(\tau) = \lim_{r\to\infty} rS_2(\tau, r)$$

be the uniformly continuous and 2π-periodic limit. Due to the fact that $\bar{g}$ is exact symplectic, we know by (3.3) that

$$\int_0^{2\pi} \left(r_1^2\frac{\partial \tau_1}{\partial \tau} - r^2\right)\mathrm{d}\tau = 0$$

holds for any fixed $r > r_* = \gamma^{-\frac{1}{2}} \max_{\theta\in\mathbb{R}} L(\theta)v_*$. Furthermore, we have

$$r_1(\tau,r)^2 \frac{\partial \tau_1}{\partial \tau}(\tau,r) - r^2 = r^2 \frac{\partial R_2}{\partial \tau}(\tau,r) + 2rS_2(\tau,r) + W(\tau,r),$$

where $W = S_2^2 + \frac{\partial R_2}{\partial \tau}(2rS_2 + S_2^2)$. In particular, W is bounded on $\mathbb{M}_{r_*}$ and $W(\tau,r) \to 0$, as $r \to \infty$. Since R_2 is 2π-periodic in τ, we get

$$\int_0^{2\pi} (2rS_2(\tau,r) + W(\tau,r))\,\mathrm{d}\tau = 0.$$

Sending $r \to \infty$ yields $\int_0^{2\pi} \alpha(\tau)\,\mathrm{d}\tau = 0$ by the dominated convergence theorem. Now, the sought adiabatic invariant can be defined as $\rho(\tau,r) = r + \beta(\tau)$, where

$$\beta(\tau) = \gamma^{-\frac{1}{2}} \int_0^{\tau} \alpha(s)\,\mathrm{d}s.$$

Note that $\beta \in C^1(S^1)$, because α is purely periodic. With a similar argument as before, it follows

$$\begin{aligned}
\beta(\tau_1) &= \beta(\tau) + \gamma^{-\frac{1}{2}} \int_{\tau}^{\tau_1 - 2\pi} \alpha(s)\,\mathrm{d}s \\
&= \beta(\tau) + \gamma^{-\frac{1}{2}}(\tau_1 - 2\pi - \tau) \int_0^1 \alpha((1-\lambda)\tau + \lambda(\tau_1 - 2\pi))\,\mathrm{d}\lambda \\
&= \beta(\tau) + \left(\frac{R_2(\tau,r)}{\sqrt{\gamma}} - \frac{1}{r}\right) \int_0^1 \alpha((1-\lambda)\tau + \lambda(\tau_1 - 2\pi))\,\mathrm{d}\lambda \\
&= \beta(\tau) - \frac{1}{r}\int_0^1 \alpha((1-\lambda)\tau + \lambda(\tau_1 - 2\pi))\,\mathrm{d}\lambda + S_3(\tau,r),
\end{aligned}$$

with $S_3 \in \mathcal{F}_{\mathrm{u}}(2)$. From this, we obtain

$$\begin{aligned}
\rho(\tau_1,r_1) &= r + S_2(\tau,r) + \beta(\tau) - \frac{1}{r}\int_0^1 \alpha((1-\lambda)\tau + \lambda(\tau_1 - 2\pi))\,\mathrm{d}\lambda + S_3(\tau,r) \\
&= \rho(\tau,r) + \frac{1}{r}\left(rS_2(\tau,r) - \int_0^1 \alpha((1-\lambda)\tau + \lambda(\tau_1 - 2\pi))\,\mathrm{d}\lambda + rS_3(\tau,r)\right) \\
&= \rho(\tau,r) + \frac{S_4(\tau,r)}{r},
\end{aligned}$$

where $S_4(\tau,r) \to 0$ uniformly as $r \to \infty$. Therefore, one can find a decreasing function $\delta : [r_*,\infty[\to \mathbb{R}$ with $|S_4(\tau,r)| \le \delta(r)$ on $\mathbb{M}_{r_*}$, such that $\delta(r) \to 0$, as $r \to \infty$. Thus, we have shown that all conditions of Lemma 3.3 are satisfied. The application yields a set $\mathcal{M}$ with $\mu(\mathcal{M}) < \infty$ such that every unbounded orbit of $\bar{g}$ enters $\mathcal{M}$. But since $\limsup_{n\to\infty} r_n = \infty$ holds if and only if $\limsup_{n\to\infty} v_n = \infty$, this means that every unbounded orbit of $\bar{f}$ enters $\mathcal{M}' = \Phi^{-1}(\mathcal{M})$. In particular, this implies

$$\mathcal{E} \subset \bigcup_{n=1}^{\infty} \bar{f}^{-n}(\mathcal{M}') \mod \mu.$$

Finally, due to the fact that $\mu\left(\Phi^{-1}(\mathcal{M})\right) = \mu(\mathcal{M}) < \infty$, we can apply Lemma 2.3 to the restricted map $\bar{f} : \mathcal{E} \to \mathcal{E}$ and deduce its recurrence. However, orbits starting in $\mathcal{E}$ are by definition non-recurrent and thus $\mu(\mathcal{E}) = 0$. □

4. A Piecewise Linear Oscillator

In this section, we prove our main result, Theorem 1.1. As indicated in the introduction, we start by constructing a twist map suitable for the application of Theorem 3.1, such that its orbits correspond to large-amplitude solutions of

$$\ddot{x} + n^2 x + h_1(x) = p(t). \tag{4.1}$$

In a second step, we then show that the recurrence of this twist map implies Poisson stability of almost every solution.

To this end, suppose x is a solution of (4.1) such that there are $\tau \in \mathbb{R}$ and $v > 0$ with $x(\tau) = 0$ and $\dot{x}(\tau) = v$. Then, x is also a solution of the integral equation

$$x(t) = v\frac{\sin n(t-\tau)}{n} + \int_\tau^t [p(s) - h_1(x(s))]\frac{\sin n(t-s)}{n}\,\mathrm{d}s, \tag{4.2}$$

and the derivative is given by

$$\dot{x}(t) = v\cos n(t-\tau) + \int_\tau^t [p(s) - h_1(x(s))]\cos n(t-s)\,\mathrm{d}s. \tag{4.3}$$

Given any time span $T > 0$, it follows from these formulas that $x(t)/v$ is arbitrary close to $(\sin n(t-\tau))/n$ in $C^2[\tau, \tau+T]$ for large values of v. In particular, one can find $v_* > 0$ with the following property. If $v > v_*$, then $x(t)$ has $2n$ consecutive non-degenerate zeros

$$\tau = \tau_0 < \tau_1 < \cdots < \tau_{2n} = \tau'$$

and crosses the line $x = (-1)^i$ twice in each interval (τ_i, τ_{i+1}). We denote these crossings by $\tau_i^* < {}^*\tau_{i+1}$ and write

$$v_i = \dot{x}(\tau_i), \quad v_i^* = \dot{x}(\tau_i^*), \quad {}^*v_i = \dot{x}({}^*\tau_i),$$

for the corresponding velocities. For $i = 0, \ldots, 2n-1$, each of the three maps

$$(\tau_i, v_i) \to (\tau_i^*, v_i^*) \to ({}^*\tau_{i+1}, {}^*v_{i+1}) \to (\tau_{i+1}, v_{i+1})$$

can be described in terms of a forced linear oscillator. The arguments in Proposition 2.2 and Proposition 2.3 of [13] show that these maps are of class C^1 and exact symplectic in the sense of (3.1). Since the induced function

$$\bar{P} : M_{v_*} \to S^1 \times [0, \infty[, \qquad \bar{P}(\bar{\tau}, v) = (\bar{\tau}', v') = (\bar{\tau}_{2n}, v_{2n}),$$

can be decomposed into $6n$ such maps, also $\bar{P} \in C^1(M_{v_*})$ is exact symplectic. The map $\bar{P}$ is one-to-one due to the unique solvability of the corresponding initial value problem. Following the computations in Section 7 of [14], it can be seen that for $p \in C(S^1)$ the associated lift $P : \mathbb{M}_{v_*} \to \mathbb{R} \times [0, \infty[$ has the form

$$\begin{cases} \tau' = \tau + 2\pi - (1/nv)L_1(\tau) + R_1(\tau, v), \\ v' = v + L_2(\tau) + R_2(\tau, v), \end{cases} \tag{4.4}$$

where

$$L_1(\tau) = 2\pi\Im(e^{in\tau}\hat{p}_n) + 4, \qquad L_2(\tau) = 2\pi\Re(e^{in\tau}\hat{p}_n),$$

and $R_1 \in \mathcal{F}^1(2)$, $R_2 \in \mathcal{F}^0(1)$. Here, $\mathcal{F}^k(r)$ denotes the space of functions $F(\tau, v)$ such that $F \in C^k(\mathbb{M}_{v_*})$ for some $v_* > 0$ and

$$\sup_{(\tau,v)\in\mathbb{M}_{v_*}} v^{r+\nu_2}|\partial^\nu F(\tau, v)| < \infty$$

for every multi-index $\nu = (\nu_1, \nu_2)$ with $|\nu| \leq k$. We define $\mathcal{F}^k_{\mathrm{u}}(r)$ to be the subspace $\mathcal{F}^k(r)\cap\mathcal{F}_{\mathrm{u}}(r)$, i.e., all $F \in \mathcal{F}^k(r)$ such that $\{v^r F(\cdot, v)\}_{v\geq v_*}$ converges uniformly as $v \to \infty$. Throughout the computations in [14], one can in fact replace the space $\mathcal{F}^k(r)$ by $\mathcal{F}^k_{\mathrm{u}}(r)$ with some obvious adjustments. This leads to the conclusion that $R_1 \in \mathcal{F}^1_{\mathrm{u}}(2)$ and $R_2 \in \mathcal{F}_{\mathrm{u}}(1)$. The Poincaré map of the discontinuous oscillator discussed in the next section has an expansion of the same form. This is shown in full detail in [16]. Finally, note that for $L_1 \in C^2(S^1)$ we have $L_1' = nL_2$ and also the condition $L_1 > 0$ is guaranteed by (1.2). In total, $\bar{P}$ satisfies all assumptions of Theorem 3.1 and therefore the escaping set

$$\mathcal{E}_P = \left\{(\bar{\tau}, v) \in M_{v_*} : (\bar{\tau}_j', v_j') = \bar{P}^j(\bar{\tau}, v) \in M_{v_*} \forall j \in \mathbb{N} \text{ and } \lim_{j\to\infty} v_j' = \infty\right\}$$

has measure zero.

Going back to the question of Poisson stability, we denote by $x(t) = x(t; \tilde{x}, \tilde{v})$ the solution of (4.1) satisfying the initial condition $x(0) = \tilde{x}$ and $\dot{x}(0) = \tilde{v}$. Thus, the time-2π map of (4.1) is given by

$$\Pi : \mathbb{R}^2 \to \mathbb{R}^2, \qquad (\tilde{x}, \tilde{v}) \mapsto (x(2\pi; \tilde{x}, \tilde{v}), \dot{x}(2\pi; \tilde{x}, \tilde{v})).$$

It can be shown that Π preserves the 2-dimensional Lebesgue measure λ. We now prove that it is also recurrent. To this end, consider $x(t) = x(t; \tilde{x}, \tilde{v})$ for some $(\tilde{x}, \tilde{v}) \in \mathbb{R}^2$. The solution of the unperturbed linear system $\ddot{z} + n^2 z = 0$ satisfying the same initial condition $z(0) = \tilde{x}$, $\dot{z}(0) = \tilde{v}$ is given by $z(t) = \hat{r}\frac{\sin n(t-\hat{\tau})}{n}$ for some $\hat{\tau} \in \mathbb{R}$ and $\hat{r} = \sqrt{n^2\tilde{x}^2 + \tilde{v}^2}$. Furthermore, $x(t)$ also solves the integral equation

$$x(t) = \hat{r}\frac{\sin n(t - \hat{\tau})}{n} + \int_0^t [p(s) - h_1(x(s))]\frac{\sin n(t - s)}{n}\,\mathrm{d}s. \tag{4.5}$$

Again, $x(t)/\hat{r}$ is close to $(\sin n(t - \hat{\tau}))/n$ in $C^2[0, 4\pi]$ for large values of $\hat{r}$. Let $r(t) = \sqrt{n^2x(t)^2 + \dot{x}(t)^2}$. Then, one can infer from (4.5) that there is a constant $C_p > 0$ (depending on $\|p\|_\infty$) such that

$$|r(t) - r(0)| \leq C_p, \qquad \text{for } t \in [0, 4\pi]. \tag{4.6}$$

Thus, if $\hat{r} = r(0) > v_*+C_p$, then $r(t) > v_*$ holds for all $t \in [0, 4\pi]$. In particular, there is a unique first $\tau \geq 0$ such that $x(\tau) = 0$ and $v = \dot{x}(\tau) > v_*$. Let S be the induced map

$$S : \mathbb{R}^2\backslash E \to \mathbb{M}_{v_*}, \qquad (\tilde{x}, \tilde{v}) \mapsto (\tau, v),$$

where

$$E = \{(a, b) \in \mathbb{R}^2 : \sqrt{n^2a^2 + b^2} \leq v_* + C_p\}.$$

S is a diffeomorphism with respect to its image and the inverse map can be obtained by plugging $t = 0$ into (4.2) and (4.3). For a given solution $x(t) =$

$x(t;\tilde{x},\tilde{v})$ define $r_n = r(2\pi j)$ for $j \in \mathbb{N}$. Then, the escaping set E_Π of the map Π is given by

$$\mathcal{E}_\Pi = \left\{ (\tilde{x},\tilde{v}) \in \mathbb{R}^2 : \lim_{j\to\infty} r_j = \infty \right\}.$$

It can be shown by the same argument as in Sect. 3 that the restricted map $\Pi : \mathbb{R}^2 \backslash \mathcal{E}_\Pi \to \mathbb{R}^2 \backslash \mathcal{E}_\Pi$ is recurrent. Thus, it remains to show that $\lambda(\mathcal{E}_\Pi) = 0$. Suppose $(\tilde{x},\tilde{v}) \in \mathcal{E}_\Pi$. In view of (4.6), this means $\lim_{t\to\infty} r(t) = \infty$. Let $m \in \mathbb{N}$ be such that $\Pi^j(\tilde{x},\tilde{v}) \in \mathbb{R}^2 \backslash E$ for all $j \geq m$. Moreover, set $(\tau_0', v_0') = S\left(\Pi^m(\tilde{x},\tilde{v})\right)$ and denote its corresponding orbit by $(\tau_j', v_j') = P^j(\tau_0', v_0')$. Then, clearly $\lim_{j\to\infty} v_j' = \lim_{j\to\infty} r(\tau_j') = \infty$ so that $\iota\left(S\left(\Pi^m(\tilde{x},\tilde{v})\right)\right) \in \mathcal{E}_P$, where $\iota : \mathbb{M}_{v_*} \to M_{v_*}$ denotes the covering map $\iota(\tau, v) = (\bar{\tau}, v)$. This leads to the inclusion

$$\mathcal{E}_\Pi \subset \bigcup_{m=0}^{\infty} \Pi^{-m}\left(S^{-1}(\iota^{-1}(\mathcal{E}_P))\right),$$

which in turn implies $\lambda(\mathcal{E}_\Pi) = 0$. In summary, we have shown that Π is recurrent. Due to the symmetry of the problem, the same is true for the inverse map Π^{-1}. Now, the Poisson stability of almost every solution $x(t;\tilde{x},\tilde{v})$ follows from the fact that the corresponding flow is Lipschitz-continuous on $\mathbb{R}^2$.

5. The Discontinuous Case

Consider the piecewise linear oscillator

$$\ddot{x} + n^2 x + \operatorname{sign}(x) = p(t), \tag{5.1}$$

where $p \in C(S^1)$. Let $N = \{t \in \mathbb{R} : |p(t)| = 1\}$ and suppose the set ∂N of its boundary points is countable. The goal of this section is to proof Theorem 1.2, that is to show that almost every solution of (5.1) is Poisson stable. But first we have to give the following

Definition 5.1. We say a function $x \in C^1(I)$ with $I =]\alpha,\beta[\subset \mathbb{R}$ is a *solution* of (5.1) if it satisfies the following conditions:

(i) $\dot{x}(t) \neq 0$ if $t \in Z$, where $Z = \{t \in \mathbb{R} : x(t) = 0\}$,
(ii) $x \in C^2(I\backslash Z)$ and x satisfies (5.1) on $I\backslash Z$.

Moreover, we say a solution is *global* if $I = \mathbb{R}$.

Between two consecutive zeros, any such solution must coincide with the solution of the corresponding linear problem. Thus given $(\tau, v) \in \mathbb{R}^2$, let $y_\pm(t) = y_\pm(t;\tau,v)$ be the unique solution of

$$\ddot{y} + n^2 y \pm 1 = p(t), \quad y(\tau) = 0, \quad \dot{y}(\tau) = v.$$

The functions $(t,\tau,v) \mapsto y_\pm(t;\tau,v), \dot{y}_\pm(t;\tau,v)$ are both in $C^1(\mathbb{R}^3)$. Moreover, note that $y_\pm(t)$ also solves the integral equation

$$y(t) = \frac{v}{n}\sin(n(t-\tau)) + \int_\tau^t (p(s) \mp 1)\,\frac{\sin(n(t-s))}{n}\,\mathrm{d}s. \tag{5.2}$$

In the following, we discuss properties of the solution $y_+(t)$. Its counterpart $y_-(t)$ can be dealt with completely analogously. It can be shown that all solutions of the linear equations are either oscillatory or of constant sign [13]. In particular, if $v \neq 0$ there is a unique time $\hat{\tau} > \tau$ and a corresponding velocity $\hat{v}$ such that

$$y_+(\hat{\tau}) = 0, \quad y_+(t) \neq 0 \quad \forall t \in]\tau, \hat{\tau}[, \quad \dot{y}_+(\hat{\tau}) = \hat{v}. \tag{5.3}$$

Therefore, we can define the map

$$S_+ : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \times (\mathbb{R}_- \cup \{0\}), \qquad S_+(\tau, v) = (\hat{\tau}, \hat{v}),$$

where $\mathbb{R}_+ =]0, \infty[$ and $\mathbb{R}_- =]-\infty, 0[$. This mapping is well-defined, one-to-one and satisfies

$$S_+(\tau + 2\pi, v) = S_+(\tau, v) + (2\pi, 0) \qquad \forall \tau \in \mathbb{R}.$$

Let $\Sigma_+ = \{(\tau, v) \in \mathbb{R} \times \mathbb{R}_+ : \hat{v} = 0\}$. The map S_+ can have discontinuities on Σ_+. On the open set $(\mathbb{R} \times \mathbb{R}_+) \backslash \Sigma_+$ however, the implicit function theorem can be applied to the equation $y_+(\hat{\tau}; \tau, v) = 0$. This way one obtains a function $\hat{\tau} = \hat{\tau}(\tau, v)$ in $C^1\left((\mathbb{R} \times \mathbb{R}_+) \backslash \Sigma_+\right)$. Also, $\hat{v}(\tau, v) = \dot{y}_+(\hat{\tau}; \tau, v)$ is in that class. Since the same argument can be applied to the inverse, this shows that S_+ restricted to $(\mathbb{R} \times \mathbb{R}_+) \backslash \Sigma_+$ is a diffeomorphism with respect to its image. Moreover, S_+ is symplectic in the sense of (3.1) on this domain (see Proposition 2.2 in [13]). Next we will show that Σ_+ has measure zero. To this end, define

$$\mathcal{N}_+ = \{\hat{\tau} \in \mathbb{R} : (\hat{\tau}, 0) \in S_+(\Sigma_+)\}.$$

Fix some $\hat{\tau}_* \in \mathcal{N}_+$ and let $(\tau_*, v_*) = S_+^{-1}(\hat{\tau}_*, 0)$. The equation $y_+(\tau; \hat{\tau}, 0) = 0$ can be solved implicitly for τ around $\tau = \tau_*$ and $\hat{\tau} = \hat{\tau}_*$. This yields an open interval $I_{\hat{\tau}_*}$ containing $\hat{\tau}_*$ and a function $\tau = \tau_{\hat{\tau}_*}$ of class $C^1(I_{\hat{\tau}_*})$ such that $\tau(\hat{\tau}_*) = \tau_*$ and

$$y_+(\tau(\hat{\tau}); \hat{\tau}, 0) = 0, \qquad \text{for } \hat{\tau} \in I_{\hat{\tau}_*}.$$

Hence, the map $T = T_{\hat{\tau}_*}$ defined by

$$T : I_{\hat{\tau}_*} \to \mathbb{R}^2, \quad \hat{\tau} \mapsto (\tau(\hat{\tau}), \dot{y}_+(\tau(\hat{\tau}); \hat{\tau}, 0)),$$

is also C^1 and $\lambda(T(I_{\hat{\tau}_*})) = 0$. We have $T_{\hat{\tau}_*}(\hat{\tau}_*) = S_+^{-1}(\hat{\tau}_*, 0)$. Note however that $T_{\hat{\tau}_*}(\hat{\tau}) = S_+^{-1}(\hat{\tau}, 0)$ does not have to hold for $\hat{\tau}$ in a neighborhood of $\hat{\tau}_*$. But we still have $\Sigma_+ = S_+^{-1}(\mathcal{N}_+ \times \{0\}) \subset \bigcup_{\hat{\tau}_* \in \mathcal{N}_+} T_{\hat{\tau}_*}(I_{\hat{\tau}_*})$. So if one can extract a countable sub-covering, then clearly $\lambda(\Sigma_+) = 0$ follows.

First suppose $\hat{\tau}_* \in \mathcal{N}_+ \cap N_+$, where $N_+ = \{t \in \mathbb{R} : p(t) = 1\}$. If $\hat{\tau}_*$ would be in the interior of N_+, then $y_+(t) = 0$ holds in a neighborhood of $\hat{\tau}_*$. But this contradicts the minimality condition in (5.3). Thus, $\hat{\tau}_* \in \mathcal{N}_+ \cap \partial N_+$. By assumption, this set is countable and hence $\lambda(S_+^{-1}((\mathcal{N}_+ \cap N_+) \times \{0\})) = 0$.

Now, assume $\hat{\tau}_* \in \mathcal{N}_+ \backslash N_+$, that is $p(\hat{\tau}_*) \neq 1$. Then, $y_+(t; \hat{\tau}_*, 0)$ has a strict local extremum in $\hat{\tau}_*$. Due to the continuous dependence on initial condition, one can in fact find $\epsilon > \delta > 0$ such that $\hat{\tau} \in]\hat{\tau}_* - \delta, \hat{\tau}_* + \delta[$ implies $y_+(t; \hat{\tau}, 0) \neq 0$ for $t \in [\hat{\tau}_* - \epsilon, \hat{\tau}_* + \epsilon] \backslash \{\hat{\tau}\}$. Moreover, since $v_* > 0$, one can find a neighborhood U of (τ_*, v_*) such that $y_+(t; \tau, v) \neq 0$ if $t \in]\tau, \hat{\tau}_* - \epsilon[$ for all $(\tau, v) \in U$. By decreasing $\delta > 0$ if necessary, one can assume that $]\hat{\tau}_* - \delta, \hat{\tau}_* + \delta[\subset I_{\hat{\tau}_*}$ and

$T(]\hat{\tau}_* - \delta, \hat{\tau}_* + \delta[) \subset U$. Then, $T_{\hat{\tau}_*}(\cdot) = S_+^{-1}(\cdot, 0)$ on $]\hat{\tau}_* - \delta, \hat{\tau}_* + \delta[$. In particular, it follows that $\mathcal{N}_+ \backslash N_+$ is open and that $S_+^{-1}(\cdot, 0) \in C^1(\mathcal{N}_+ \backslash N_+, \mathbb{R} \times \mathbb{R}_+)$. Thus, $\lambda(S_+^{-1}((\mathcal{N}_+ \backslash N_+) \times \{0\})) = 0$.

In summary, we have shown that Σ_+ has measure zero. Using $y_-(t)$ instead of $y_+(t)$ in (5.3), one can define the successor map

$$S_- : \mathbb{R} \times \mathbb{R}_- \to \mathbb{R} \times (\mathbb{R}_+ \cup \{0\}), \qquad S_-(\tau, v) = (\hat{\tau}, \hat{v}),$$

and the set $\Sigma_- = S_-^{-1}(\mathbb{R} \times \{0\})$. Again, S_- restricted to $(\mathbb{R} \times \mathbb{R}_-) \backslash \Sigma_-$ is a symplectic diffeomorphism with respect to its image and Σ_- has measure zero. Now, define $\Sigma_\pm^1 = \Sigma_\pm$ and $\Sigma_\pm^r = S_\pm^{-1}(\Sigma_\mp^{r-1})$ for $r \geq 2$. Then $\Sigma_\pm^r$ consists of those points $(\tau_0, v_0) \in \mathbb{R} \times \mathbb{R}_\pm$ such that the corresponding orbit (τ_j, v_j) satisfies $v_j \neq 0$ for $j = 0, \ldots, r-1$ and $v_r = 0$. Finally, define

$$\Sigma = \bigcup_{r \in \mathbb{N}} (\Sigma_+^r \cup \Sigma_-^r) \cup (\mathbb{R} \times \{0\}),$$

then Σ has measure zero and every $(\tau_0, v_0) \in \mathbb{R}^2 \backslash \Sigma$ leads to a complete forward orbit $(\tau_j, v_j)_{j \in \mathbb{N}_0}$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, that never touches the line $v = 0$. In particular, the map

$$P : (\mathbb{R} \times \mathbb{R}_+) \backslash \Sigma \to \mathbb{R} \times [0, \infty[, \qquad P(\tau_0, v_0) = (S_- \circ S_+)^n(\tau_0, v_0) = (\tau_{2n}, v_{2n}),$$

is well-defined. Analogously to the last section, one can show that P has an expansion of the form (4.4) and moreover satisfies all conditions necessary for the application of Theorem 3.1. Hence, the corresponding twist map $\bar{P}$ is recurrent. Since $\bar{P}$ is recurrent for almost all $(\bar{\tau}, v) \in S^1 \times \mathbb{R}_+$, we have $\lim_{j \to \infty} \tau_j = \infty$ for almost all orbits (τ_j, v_j) starting in $\mathbb{R}^2 \backslash \Sigma$. This leads to the following observation.

Lemma 5.2. *For almost every $(\tilde{x}, \tilde{v}) \in \mathbb{R}^2$, there exists a global solution $x(t) = x(t; \tilde{x}, \tilde{v})$ of (5.1) with initial condition $x(0) = \tilde{x}$, $\dot{x}(0) = \tilde{v}$.*

Proof. Let $\Omega_r \subset (\mathbb{R} \backslash \{0\}) \times \mathbb{R}$ be the set of initial condition leading to solutions $x(t) = x(t; \tilde{x}, \tilde{v})$ such that $x(t) \neq 0$ for $t > 0$. Then, $x \in C^2([0, \infty[)$ since it solves the linear problem. Similar to $S_\pm$, we define

$$\tilde{S}_\pm : (\mathbb{R}_\pm \times \mathbb{R}) \backslash \Omega_r \to \mathbb{R} \times (\mathbb{R}_\mp \cup \{0\}), \qquad \tilde{S}_\pm(\tilde{x}, \tilde{v}) = (\hat{\tau}, \hat{v}),$$

where again $\hat{\tau} > 0$ denotes the first zero of $x(t; \tilde{x}, \tilde{v})$ to the right and $\hat{v}$ is the corresponding velocity. Let $\tilde{\Sigma}_\pm = \{(\tilde{x}, \tilde{v}) \in \mathbb{R}_\pm \times \mathbb{R} : \hat{v} = 0\}$. These sets have measure zero since we have $\tilde{\Sigma}_\pm \subset \gamma_\pm(\mathbb{R})$ for the C^1-map $\gamma_\pm(\hat{\tau}) = (y_\pm(0; \hat{\tau}, 0), \dot{y}_\pm(0; \hat{\tau}, 0))$. Moreover, $\tilde{S}_\pm \in C^1\big((\mathbb{R}_\pm \times \mathbb{R}) \backslash (\Omega_r \cup \tilde{\Sigma}_\pm)\big)$ are diffeomorphisms with respect to their images and thus also the sets $\tilde{S}_\pm^{-1}(\Sigma)$ have measure zero. Therefore, $x \in C^1([0, \infty[)$ is a solution in the sense of Definition 5.1 for almost all $(\tilde{x}, \tilde{v})$, since almost every $(\hat{\tau}_0, \hat{v}_0) \in \mathbb{R}^2 \backslash \Sigma$ leads to a complete forward orbit $(\hat{\tau}_j, \hat{v}_j)_{j \in \mathbb{N}_0}$ such that $\hat{\tau_j} \to \infty$, $x(\hat{\tau}_j) = 0$ and $\dot{x}(\hat{\tau}_j) = \hat{v}_j \neq 0$. Now, the assertion follows by repeating the whole argument for the set Ω_l of initial condition producing solutions such that $x(t) \neq 0$ for $t < 0$. ☐

We have shown that there is a set Γ of measure zero such that all initial condition in $\mathbb{R}^2\backslash\Gamma$ lead to global solutions of (5.1). In particular, the time-2π map $\Pi : \mathbb{R}^2\backslash\Gamma \to \mathbb{R}^2\backslash\Gamma$ is well-defined. We will demonstrate that this map is also measure-preserving. To this end, we keep the notation introduced in the proof of Lemma 5.2. Given $(\tilde{x}, \tilde{v}) \in (\mathbb{R}_\pm \times \mathbb{R})\backslash(\Omega_r \cup \Gamma)$, let $\tilde{S}_\pm(\tilde{x}, \tilde{v}) = (\hat{\tau}_0, \hat{v}_0)$, then there is an infinite series of non-degenerate consecutive zeros $(\hat{\tau}_j)_{j\in\mathbb{N}_0}$ of $x(t; \tilde{x}, \tilde{v})$. Moreover, let $\hat{\tau}_0 = \infty$ if $(\tilde{x}, \tilde{v}) \in \Omega_r$. We define the sets

$$A_j^\pm = \big\{(\tilde{x}, \tilde{v}) \in (\mathbb{R}_\pm \times \mathbb{R})\backslash\Gamma : j = \min\{i \in \mathbb{N}_0 : \hat{\tau}_i \geq 2\pi\}\big\},$$

where the index j counts the number of zeros in the interval $[0, 2\pi]$. Clearly, $\mathbb{R}^2\backslash\Gamma = \bigcup_{j\in\mathbb{N}_0}(A_j^+ \cup A_j^-)$. Moreover, the sets Ω_r and $A_j^\pm$ are measurable. For Ω_r, this follows from the fact that $(\mathbb{R}_\pm \times \mathbb{R})\backslash(\Omega_r \cup \tilde{\Sigma}_\pm)$ is open, so that Ω_r differs from a Borel set only by a set of measure zero. In the case of $A_j^\pm$ with $j \in \mathbb{N}$, consider the maps $g_j(\tilde{x}, \tilde{v}) = (\hat{\tau}_j, \hat{v}_j)$. These maps are well-defined and continuous almost everywhere on $\mathbb{R}^2\backslash\Omega_r$ and hence measurable. Thus also

$$A_j^\pm = (\mathbb{R}_\pm \times \mathbb{R}) \cap g_j^{-1}([2\pi, \infty[\times\mathbb{R})\backslash g_{j-1}^{-1}([2\pi, \infty[\times\mathbb{R})$$

is measurable. The argument for $j = 0$ is similar. On $A_0^\pm$ the map Π is just the time-2π map of a linear oscillator and thus it preserves the 2-dimensional Lebesgue measure λ. For $A_j^\pm$ with $j \in \mathbb{N}$ we again consider the maps g_j. Without loss of generality, let $(\tilde{x}, \tilde{v}) \in A_j^+$, where $j = 2k$ with $k \in \mathbb{N}$. Then

$$(\hat{\tau}_j, \hat{v}_j) = (S_+ \circ S_-)^k(\tilde{S}_+(\tilde{x}, \tilde{v})).$$

$\tilde{S}_+$ restricted to $(\mathbb{R}_+ \times \mathbb{R})\backslash(\Omega_r \cup \tilde{\Sigma}_+)$ is a diffeomorphism with respect to its image and the inverse is given by $\tilde{S}_+^{-1}(\hat{\tau}, \hat{v}) = (y_+(0; \hat{\tau}, \hat{v}), \dot{y}_+(0; \hat{\tau}, \hat{v}))$. Considering formula (5.2), one easily derives $\det D\tilde{S}_+^{-1}(\hat{\tau}, \hat{v}) = -\hat{v}$ for the Jacobian determinant. This implies that we have $\lambda(B) = \mu(\tilde{S}_+(B))$ for any measurable set $B \subset (\mathbb{R}_+ \times \mathbb{R})\backslash(\Omega_r \cup \tilde{\Sigma}_+)$, where $\mu = v\,\mathrm{d}\tau \otimes \mathrm{d}v$. Furthermore, the maps $S_\pm$ are exact symplectic in the sense of (3.1) on the relevant domain and therefore preserve the measure μ. It follows that the sets

$$B_j^+ = \{(\tilde{x}, \tilde{v}) \in A_j^+ : \hat{\tau}_j = 2\pi\}$$

have measure zero. Finally, note that for $(\tilde{x}, \tilde{v}) \in A_j^+\backslash B_j^+$ we have

$$\Pi(\tilde{x}, \tilde{v}) = \tilde{S}_+^{-1}(\hat{\tau}_j - 2\pi, \hat{v}_j).$$

In view of the argument above, the latter identity shows that Π preserves the two-dimensional Lebesgue measure also on $A_j^\pm$ with $j \geq 1$ and hence on all of $\mathbb{R}^2\backslash\Gamma$. Analogously to the continuous case, it now follows that Π is recurrent and that almost every solution $x(t; \tilde{x}, \tilde{v})$ is Poisson stable.

Acknowledgements

Funding was provided by Deutsche Forschungsgemeinschaft (Grant No. SF-B/TRR 191).

Funding Information Open Access funding enabled and organized by Projekt DEAL.

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Communicated by Dmitry Dolgopyat.
Received: February 1, 2022.
Accepted: June 26, 2022.