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Do Secondary School Students' Strategies in Solving Permutation and Combination Problems Change with Instruction?

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Abstract This work is part of an investigation conducted in Italy, which aims to explore the effects of instruction on secondary school students' combinatorial reasoning. We gave a questionnaire adapted from Navarro-Pelayo's research to two groups of students with and without instruction on combinatorics in order to analyse the students' performances and the strategies used in their solutions, as well as the effect of instruction on the same. We present the results obtained in two permutation and two combination problems (each in the distribution and selection models). Permutation problems were found easier than combination problems, selection problems were found easier after instruction, and the instruction group obtained better results. We found differences in the main strategies used in both groups: enumeration and dividing a problem in parts was more common in the no-instruction group. The instruction group frequently relied on the use of a formula and the product rule.

Résumé Ce travail s'inscrit dans le cadre d'une enquête faite en Italie dont le but consiste à explorer les effets d'une formation portant sur le raisonnement combinatoire donnée aux élèves du secondaire. Afin d'analyser les performances et les stratégies de résolution des élèves ainsi que l'impact de l'administration d'une formation sur ces aspects, nous avons soumis un questionnaire inspiré du travail de recherche effectué par Navarro-Pelayo à deux groupes d'élèves, l'un à qui on a donné une formation en combinatoire et l'autre à qui aucune formation n'a été fournie. Nous présentons les résultats obtenus à deux problèmes de combinaison ainsi qu'à deux problèmes de permutation (chacun s'inscrivant dans

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des modèles de distribution et de sélection). Les problèmes de permutation se sont avérés plus faciles que ceux relevant de la combinatoire, ce fut de même pour les problèmes de sélection après la formation et le groupe qui a bénéficié de celle-ci a obtenu de meilleurs résultats. Nous avons constaté des différences dans les stratégies principales utilisées dans les deux groupes: le groupe sans formation a eu davantage recours à l'énumération et à la division du problème en parties que l'autre groupe alors que ce dernier a souvent utilisé une formule et la règle du produit.

Keywords Combinatorial reasoning · Permutation problems · Combination problems · Solving strategies · Secondary students

Introduction

Combinatorics is the art of counting (Hart, 1992) and a component of discrete mathematics. According to Batanero et al. (1997b), combinatorics deals with problems that involve a finite number of elements (discrete sets) with which we perform different operations. Some of these operations only modify the set structure (e.g. a permutation of its elements), while others change the set composition (taking a sample). Piaget and Inhelder (1951) considered combinatorial reasoning as a prerequisite for understanding the notions of randomness and probability. In addition, combinatorics is an area where students can develop and exercise problem-solving strategies, as well as mathematical processes such as generalization and recursive thinking (Kapur, 1970). Moreover, it is possible to find adequate problems to teach combinatorics at very early ages (Fischbein, 1975; Kapur, 1970).

Combinatorics is taught in the Italian school curriculum (MIUR, 2010), combinatorics in connection to probability. Combinatorial reasoning is needed to form the sample space in probability problems and understand compound experiments and some discrete distributions (e.g. the binomial distribution). According to Batanero et al. (1997a), to apply the classical definition of probability, as the ratio between all the favourable cases of a given event and all the possible cases in an experiment, strongly relies on combinatorial reasoning. Moreover, the study of the frequentist definition of probability is supported by simulation, which is also related to combinatorics, because urn models are used to simulate real-world experiments.

Piaget and Inhelder (1951) assumed that combinatorial reasoning spontaneously develops with age. However, Fischbein (1975) proved that not all adults acquire the capacity to solve simple combinatorial problems without specific instruction. Moreover, combinatorics is a field that many students find difficult because they do not correctly recognize the problem data, fail to select an appropriate notation, or find it difficult to perceive the problem as a set of partial problems (Hadar & Hadass, 1981).

Despite the relevance of combinatorics and the need to develop related competence in all citizens (Gal, 2002), not much attention is given to the teaching of combinatorics, especially within the Italian school system. Moreover, most research on students' combinatorial reasoning deals with small children (e.g. English 1991, 2005) or university students (Godino et al., 2005). Even when some researcher, such as Navarro-Pelayo (1994), assessed secondary school students' difficulties in solving combinatorial problems, none of this research considered Italian students.

To fill this gap, the aim of this research is to analyse the performances (in terms of correct solutions) and the strategies used by a sample of Italian secondary school students in solving two permutation and two combination problems. We also compare the results obtained by the groups of students with and without instruction.

Background

We based our research on the work by Fischbein and Gazit (1988), who analysed the combinatorial capacity of children from the 10th year of age and proved that they learn to solve combinatorial problems when they receive a specific instruction. These authors also investigated the effect of several task variables on the difficulty found by the students when solving combinatorial problems. These variables were:

- Type of combinatorial operation (permutations, combinations, and arrangement, with or without repetition), where permutations were the hardest.
- Type of elements to be combined; numbers and letters are easier than objects or people because, in the latter, it is more difficult to decide if the order of elements is relevant.
- Dimension of the problem: the total number of configurations (combinations or permutations) to form.

Navarro-Pelayo (1994) built a questionnaire considering the different combinatorial operations, problem dimension, and types of elements to be combined. The author also included in her questionnaire, as a new variable, the implicit combinatorial model, theoretically described by Dubois (1984) who, however, did not study its effect. Dubois suggested that every elementary combinatorial problem (here defined as not compound) belongs to one of three possible schemes: selection, distribution, or partition. In the selection model, a sample of *n* elements is drawn from a set of *m* (usually distinct) objects. In the distribution model, we distribute a set of *n* objects in *m* cells. We have different possibilities in this model; these depend on whether the objects or the cells are identical, and the number of objects to place into each container. The third model consists in dividing *n* objects into *m* subsets. Whereas there is a biunivocal correspondence between each partition and distribution problem (for each partition problem, there is only one distribution problem and vice versa), the selection problems cannot always be translated to partition or distribution problems and vice versa.

Navarro-Pelayo (1994) reported the errors produced by 720 secondary students (14–15 years old) when solving different combinatorial problems. Half of the students completed the questionnaire before instruction in combinatorics and the remaining after instruction. The author studied the effect of instruction on students' performance and errors. In her research, the proposed combination problems showed an overall lower percentage of correct answers with respect to permutation problems. Batanero et al. (1997b) re-analysed data from Navarro-Pelayo (1994) and found a statistically significant difference in the students' performances after instruction, as well as not much difference in the problem difficulty between the three combinatorial models before instruction while, after teaching, the selection problems were easier.

Other authors analysed the strategies used by students when solving combinatorial problems. English (2005) described enumeration, or explicit listing of all possible combinatorial configurations to be formed, according to the problem statement. This strategy was used in her research by small children when solving simple combinatorial problems. The author distinguished between a-systematic and systematic enumeration. A-systematic enumeration consists of a random or incomplete forming of the configurations, by selecting the elements to be combined or permuted mostly in a trial-and-error procedure.

Systematic enumeration involves the cyclic selection and fixing of one or more elements (for example, the first element in a permutation) and combining them with all the other elements, as well as repeating this procedure until all the configurations are listed. Adults also use enumerative strategies. Thus, Lockwood and Gibson (2016) worked with 42 undergraduate students and concluded that even creating partial lists of the set of outcomes led to significant improvement in the students' performance in solving combinatorial problems. The authors also suggested that instruction should facilitate systematic enumeration processes in the students. Roa (2000) performed a research study with 91 undergraduates in their 5th year of mathematics degree, adapting Navarro-Pelayo's (1994) questionnaire. He observed, in general, better performances in his sample than those from secondary school students; however, the distribution problems still were hard for the university students. The author found that students still used enumeration by the end of their university studies and described the following different strategies used by these students:

- Applying a combinatorial formula: the students recognized and applied the combinatorial operation solving the problem.
- Reference to another problem: the students transformed the problem in another equivalent situation, which was used to obtain the solution.
- Decomposition into sub-problems: the students divided the original problem into several combinatorial problems of smaller dimension, each of which was solved independently and whose solutions were combined together to solve the initial problem.
- Applying the arithmetical rules of sum, product, and quotient: the students did not remember the combinatorial formula and solved the problem using arithmetical principles in order to reconstruct the formula.
- Using a tree diagram: the students built a tree diagram as a support in producing all the configurations. This method facilitates the solution of the problem, according to Fischbein and Gazit (1988).

In our investigation, we analyse all the strategies used by students in the sample and their effectiveness to obtain a correct solution. We also investigate the differences between students' solutions and approaches in the groups with and without instruction. Secondary school students' strategies were only analysed by Batanero et al. (1997b) in a sample of 17 Spanish students; the authors only indicated that, after instruction, these students preferred the use of formulas to solve the problems.

In later research (Roa, 2000; Godino et al., 2005), the authors interviewed a reduced group of students to analyse their solving procedure in order to discriminate the strategies used by successful and poor problem solvers.

Those who solved 12 or 13 of the 13 problems correctly identified the problem data and used the following strategies: translated the problem to an equivalent situation, subdivided the problem, used the addition and product rule, and applied systematics enumeration. Some of these schemes failed for the poor problem solvers (students solving less than half the problems in the questionnaire), who incorrectly translated the problem, did not identify the formula needed, or used a-systematic enumeration.

In a previous paper (Lamanna et al., 2021), we analysed the performance and strategies of a sample of secondary school students with and without instruction in solving two permutation problems. In order to complement previous research, in this paper, we focus on the strategies used by the same sample of secondary school students and expand the analysis carried out by producing a deeper study of students' strategies and its effectiveness to get a correct solution. We also compare strategies and performances in permutation and combination problems and the differences between students with and without instruction.

These strategies were studied in the case of undergraduates by Roa (2000). Neither Batanero et al. (1997b) nor Navarro-Pelayo (1994) focused on secondary school students' strategies. Lamanna et al. (2022) developed an exploratory study of secondary school students' solving strategies in combination problems.

Method

Participants and Setting

The sample in our study was non-random and consisted of 115 secondary school Italian students from different school grades and specialties, 64 of which had not received instruction on combinatorics and 51 students who received instruction on the subject. In order to obtain a sample of students as heterogeneous as possible, we included seven groups of students from five different schools receiving instruction by six different teachers. The theoretical level of mathematics of these students was homogeneous, since in primary schools and lower secondary schools the curriculum is uniform. Students' preparation changes with upper secondary school, depending on the speciality they studied. We included groups of students from scientific and human sciences high schools: six groups of science students—three that already had received instruction and three groups without instruction—and a group of human sciences students who had not received instruction. Since combinatorics is a topic that does not need mathematical prerequisites (Kapur, 1970), we assumed that students with no previous instruction in the theme could perform similarly, regardless of the school speciality they were attending.

The sample of students with no previous instruction in combinatorics included groups of students in grades 10, 11, and 12 (15–18 years old). Most students in the instruction group were in grade 12 (17–18 years old), although three grade 10 students (15-year-olds) studied combinatorics in extracurricular courses. The instruction was in general formal, based on the use of formulas to solve combinatorial problems. The exception was one group of 12 students who approached the topic in a constructivist way, through exercises and examples, constructing combinatorial rules before being taught a formula. In the different school specialties, combinatorics was presented as part of the school segment of probability and statistics and as a tool to solve all kinds of problems.

Tasks Proposed to the Students

The students completed a questionnaire of 13 open-ended problems (in the Italian language) adapted from the questionnaire used in Navarro-Pelayo (1994) and Batanero et al. (1997b). Depending on the availability of the school, students had 60 or 90 min to complete the questionnaire. Time did not represent an issue, considering that most students completed the questionnaire in less time. In this work, we focus on the following items; the first two items correspond to permutation, and the remaining are combination problems:

Item 1 Andrew, Burt, Charles, and Dan (A, B, C, and D in short) decide to have lunch at the school cafeteria, and after taking the tray, they line up to get the food. Obviously, each of them would like to be the first! We want to write all the possible ways they can get in line. For example, if A is the first, B second, C third, and D fourth, we write ABCD. How many ways can the boys be lined up?

Item 2 At the amusement park, there is a new game: in a pool, there are boats numbered with the digits 2, 4, and 7. The game consists of selecting a boat and writing its number. Then, without replacing the first boat back into the pool, a second one is selected, and its number is written. Finally, the last boat is caught. How many different three-digit numbers can be obtained with this method? For example, we could get the number 724.

Item 3 Supposing we have three identical letters, we want to place them into four different-coloured envelopes: yellow, blue, red, and green. It is only possible to introduce one letter in each different envelope. How many ways can the three identical letters be placed into the four different envelopes? For example, we could introduce a letter into the yellow envelope, another into the blue envelope, and the last one into the green envelope.

Item 4 School students must elect their representatives. Five students are the candidates: Elisabeth, Ferdinand, George, Lucy, and Mary. In how many different ways can three of the five candidates be chosen? For example, Elisabeth, Mary, and George could be elected.

Item 1 is a distribution problem (Dubois, 1984) of medium dimension (i.e. requiring between 12 and 24 configurations), in which students are asked to distribute people in a row and the solution is $P_4 = 4! = 24$. Item 2 is a selection small-dimensional problem (since the solution requires less than 10 combinatorial configurations) in which students are asked to select objects (boats), and the solution is $P_3 = 3! = 6$. Items 3 and 4 are small-dimensional; item 3 is a distribution problem whose solution is $C_{4,3} = \frac{4!}{3!(4-3)!} = 4$, and item 4 is a selection problem and its solution is $C_{5,3} = \frac{5!}{3!(5-3)!} = 10$.

Analysis

After collecting the written responses of each student, we performed a content analysis (Neuendorf, 2017). We coded each solution through numerical vectors and included all the information related to the problem (e.g. student ID, group, class, solution correctness, and strategy). We added qualitative data for the strategy, for example, correct formulas with arithmetical error or wrong terminology for adequate formula. We also took into account the cases in which students used more than one strategy. For example, the response displayed in Fig. 1 used systematic enumeration and the rule of product, also known as the fundamental counting principle (Godino et al., 2005). Consequently, the sum of percentages in all types of the strategies performed by students may add more than 100%. Moreover, since not all the strategies were correct, the sum of strategies producing adequate solution may add less than 100%.

We performed different analyses, after the data were coded:

- A first analysis focussed mainly on the correctness of the problem solution. We classified each item solution as correct if the student identified the total number of configurations using the correct formula or arithmetic operations or else if he provided a complete list of all the possible configurations.
- A second analysis described the strategy used by the student and whether that strategy produced a correct solution. We considered all the methods used in the same problem by each student, even if one of these procedures was support for another procedure.
- We also compared the correctness and strategies by item features, student group, and previous research.

Results

Item Difficulty

In Table 1, we present the percentages of correct responses in each item and group and compare them to results obtained by Navarro-Pelayo (1994). Contrary to Fischbein and Gazit (1988), permutation problems were simpler than combination problems in both samples before and after instruction. Henceforth, we use simpler and harder-to-compare percentages of correct answers, whereas easier problems are the tasks on which students produced a higher number of correct answers. The

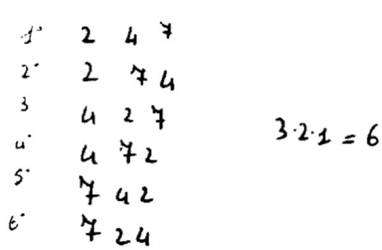


Fig. 1 Response to item 2 with two different strategies

Table 1 Percentage correct responses per item and group and comparison with results from Navarro-Pelayo (1994)

	No instruction		Instruction	
	N. Pelayo ($N=348$)	Lamanna ($N=64$)	N. Pelayo ($N=352$)	Lamanna ($N=51$)
Item 1 P_4 distribution	23.9	43.8	71.0	74.5
Item 2 P_3 selection	77.2	59.5	80.7	84.3
Item 3 $C_{4,3}$ distribution	26.9	39.1	26.7	51.0
Item 4 $C_{5,3}$ selection	22.5	12.5	46.0	54.9

most challenging item before instruction in our sample was item 4, where many students failed to get the solution because they produced incomplete lists of configurations or performed incorrect calculations.

In agreement with Fischbein and Gazit (1988) and Navarro-Pelayo (1994), there was a general improvement after instruction in all the items. This improvement was still more visible in item 4, where the item statement is familiar to the students because they have learnt combinations using the selection model and have solved other similar problems in the classroom.

The differences in the proportion of success between the instruction and no-instruction groups were statistically significant in the Z test ($Z=40.54$, 38.99, 13.31, and 64.59 in items 1 to 4; all the p values smaller than 0.0001). The trend was similar to Navarro-Pelayo (1994), where item 2 was also the easiest. However, in her research, there was an improvement in the instruction group in all the items except for item 3.

To compare the effect of the combinatorial operation and combinatorial model on the students' success, we present in Table 2 the average percentages of correct responses in distribution and selection, and permutation and combination problems.

Contrary to Fishbein and Gazit (1988), permutation was generally simpler than combination in both groups with and without instruction ($Z=32.83$, and $Z=51.72$, statistically significant in the Z test for difference of proportions between permutation and combinations, with p values smaller than 0.0001). Navarro-Pelayo (1994) also reported this result; the author attributed the difference to the lower age of students in the research by Fishbein and Gazit (1988). In Navarro-Pelayo (1994), the distribution model was more challenging in the groups with and without instruction; however, in our study, it was easier with instruction ($Z=6.57$, statistically significant in the Z test for difference of proportions). There was not much difference in the difficulty of both models with instruction (result not statistically significant in the Z test for difference of proportions).

Table 2 Average percentage correct responses by combinatorial model and operation per group and comparison with results from Navarro-Pelayo (1994)

	No instruction		Instruction	
	N. Pelayo ($N=348$)	Lamanna ($N=64$)	N. Pelayo ($N=352$)	Lamanna ($N=51$)
Distribution	25.4	41.5	48.9	62.8
Selection	49.9	36.0	63.4	69.6
Permutation	50.6	51.7	75.9	79.4
Combination	24.7	25.8	36.4	53.0

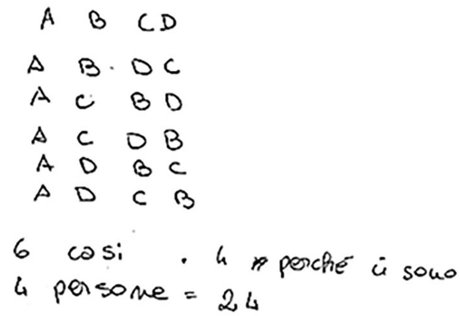


Fig. 2 Systematic enumeration and product rule in a correct solution to item 1

Students' Strategies

We also analysed the students' strategies, basing our work on previous research by English (2005), Godino et al. (2005), and Roa (2000). Below, we describe the strategies identified in the students' solutions to the items and include examples to clarify the different categories.

Enumeration Producing a list of all the configurations, which is systematic or a-systematic. In systematic enumeration, an algorithmic procedure consists in cyclically fixing elements (pivoting of variables). The listing procedure is finished when all the possible cases are provided. Students producing systematic enumeration usually also reduce the item dimension by fixing the element in one position (usually the first position) and then generalize the partial solution by applying the product rule to finish solving the problem (see an example in Fig. 2). This strategy usually leads to the correct solution to the problem and was widely employed by the students without instruction.

Hadar and Hadass (1981) pointed out the difficulty of many students in performing a systematic enumeration. Some of our students also used a-systematic listing. This strategy, also identified by Fischbein and Gazit (1988), consists of a random listing of configurations, through a trial-and-error process, without adopting a rule or a proper systematic method. Usually, this procedure leads to a non-complete list or a wrong list in which some configurations repeat. These students usually failed to enumerate all the configurations, and produced small lists of configurations, which were not generalized to obtain the correct solution (see an example in Fig. 3). Roa (2000) also found this strategy in some undergraduates.

Sum or Product Rule Another strategy applied by some participants was using some arithmetical rules. The product rule or fundamental principle of counting (Godino et al., 2005) states that, given a situation that can occur in k concatenated phases, each of which can respectively occur in $x_1, x_2, \dots, x_k$ distinct manners, then the situation can occur in $x_1 \cdot x_2 \cdot \cdots \cdot x_k$ different ways.

The product rule, also identified by Roa (2000), sometimes was used in connection with enumeration (as seen in Fig. 2). Other students used this strategy in isolation. For example, in Fig. 4, the student

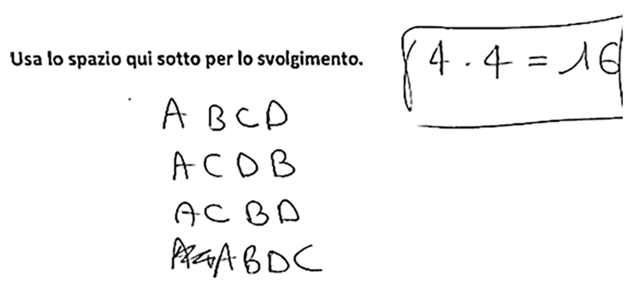


Fig. 3 A-systematic enumeration leading to an incorrect solution to item 1

Il primo della fila può essere uno qualsiasi di loro quattro.
Il secondo uno dei rimanenti tre.
Il terzo uno dei rimanenti due.
Il quarto l' ultimo rimasto.
I ragazzi possono essere allineati in 4×3×2×1 = 24 modi.

Fig. 4 Product rule in a correct solution to item 1

analyses the possibilities for every place of the row to be formed. He explains that (translated from Italian), "we can place any of the four students in the first position, one of the remaining three in the second, one of the remaining two in the third place, and the fourth should be the remaining student". The student then finished the problem through the product rule by justifying that "we can arrange the four boys in $4\times3\times2\times1=24$ ways". The students obtained the solution through the Cartesian product of the possible sets of elements chosen to form the configurations.

In items 2, 3, and 4, a few students used the addition, or sum, rules (Godino et al., 2005). Given k different manners $a_1, a_2, \dots, a_k$ mutually incompatible in which a specific situation may appear and which can be produced respectively in $x_1, x_2, \dots, x_k$ different ways, then there are $x_1 + x_2 + \cdots + x_k$ distinct manners in which the initial event may occur. In Fig. 5, we present an example of this strategy in item 2. The student fixes an element in the first position and analyses the number of possible configurations with this choice. The student explains (translated from Italian), "there are two possible numbers if you select 2 as the first digit". The student repeats this reasoning by fixing in the first place the digits 4 and 7; he then computes the solution to the original problem by adding the number of configurations obtained in each sub-problem.

Formula Some students use a formula for a combinatorial operation to solve the problem (permutation or combination, depending on the item). Many students in the instruction group applied the correct formula. To get the adequate response, the student needs to analyse the problem and, when needed, translate the problem into one of the models he learnt. Then, the students have to identify the parameters and correctly remember and apply the formula. We present an example in Fig. 6, where the student used a scheme to understand the situation.

Reference to Another Problem To solve the problems, the students must model the situation to be solved; along this process, it is possible that they recognize a model used to solve a different question or that they found in familiar examples or situations. Some students in the instruction group used this strategy by indicating in their solution that the problem was equivalent to another problem. In Fig. 7, a student states that problem 1 is analogous to calculating the different anagrams of a word (a word with all the original letters differently placed). The student applies the formula of permutations that he recognizes as the way to compute the number of anagrams in a word with n letters.

Fig. 5 Addition rule in a correct solution to item 2

2 possibili numeri se si pesca il 2 come primo
" " " " " " 4 " "
" " " " " " 7 " "
6 totali

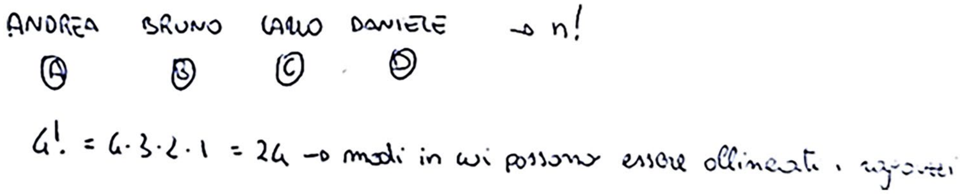


Fig. 6 Formula in a correct response to item 1

Sub-Problem Decomposition Some students reduce the task dimension by decomposing it in several problems to be solved independently, then obtain partial solutions to answer the original problem. This strategy is linked to other procedures. In Fig. 2, we can observe an example of multiple methods used in a hierarchical way. The student fixes the variable A in the first position to reduce the problem dimension. He solves the sub-problems through enumeration, which appeared as a supporting procedure, instead of being the principal strategy involved in the global solution. Once the sub-problems are solved, the student generalizes the result to solve the original problem. The student explains (translated from Italian) that "6 cases times 4, because there are four people".

Tree Diagram According to Fischbein (1975), the tree diagram supports visualization and construction of combinatorial configurations; in spite of this, like enumeration, this technique is not always taught to students. This explains the low percentage of students who solved the problems using tree diagrams; moreover, many students produced incorrect representations. However, a few students correctly used a tree diagram in order to compute all the configurations requested in the problem. We reproduce an example in Fig. 8, where we can observe the correct use of this representation in a solution of item 2. The student fixes the first element of the configuration and creates a tree diagram, by selecting systematically the remaining components. Once all the elements are listed, the student repeats the process to produce all the configurations by choosing a different initial element.

Finally, other students presented a confused explanation or used strategies not explicitly described in the literature, producing random calculations or proposing an unjustified answer. In this paper, all the solutions described as "other" represent unjustified calculations, not explained by arithmetical rules or solutions providing only a number with no further explanations.

In Tables 3 and 4, we present the percentages of use of each strategy in the permutation and combination problems, respectively. For each item and group, we computed percentages in two columns: in the first column, we display the global percent of students who applied a given strategy and, in the second column, the percentage of students who used the procedure and obtained a correct solution to the problem. These percentages are computed with respect to the number of students in each group that provided a solution to the problem. Given that some students used more than one strategy in their solutions, the sum of percentages can be higher than 100%. Since not all the students using a method obtained the correct response, the sum of percentages of correct use of strategies can also be smaller than 100%.

The students in the instruction group employed a more heterogeneous distribution of strategies than their counterparts; this is due to the fact that they learnt some new procedures (i.e. formula, tree diagram, and arithmetical rules of combinatorics). The data in the table suggest a change of strategies

Fig. 7 Reference to another problem in a correct solution to item 1

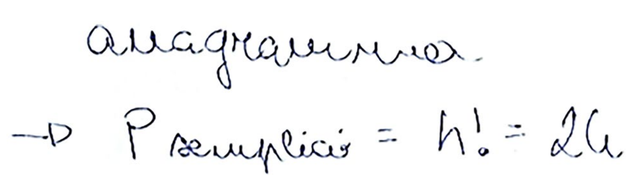
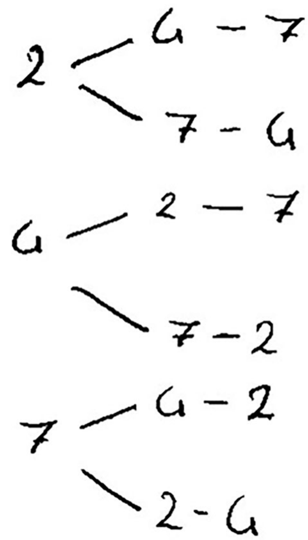


Fig. 8 Solution of item 2 through a tree diagram



with instruction: while enumeration, mostly systematic, was the most frequent procedure used by the students with no instruction to solve the problems, this strategy changed toward use of the permutation or combination formulas and the arithmetic rule of product in the instruction group. Undergraduate students with high mathematical preparation in Roa's (2000) research used with similar frequency enumeration (27.2%), formula (30.9), or both (21.0%) for item 3. They applied more formulas (50%) than enumeration (15.3%) or both (18.1%) for item 4, which is similar to what happened in our study in the instruction group.

Furthermore, even though students of the group without instruction relied on enumeration, a few of them spontaneously developed a different strategy (i.e. arithmetical rules) but failed to reach a correct solution. It is worth noting that even though an arithmetical rule did not always lead to a correct solution, many students with no instructions produced a partially correct solution, that is to say, a strategy correctly developed but failing to result in the adequate answer. For example, in item 1, some students usually started by suggesting that the first student could be chosen in four different ways, the second in

Table 3 Percentages of global and correct use of each strategy by item and group in permutation problems

Strategy	Instruction ($N=51$)				No instruction ($N=64$)			
	Item 1 P_4 Distribution		Item 2 P_3 Selection		Item 1 P_4 Distribution		Item 2 P_3 Selection	
	Global	Correct	Global	Correct	Global	Correct	Global	Correct
Systematic enumeration	12.8	10.6	39.6	35.4	43.1	43.1	60.0	50.9
A-systematic enumeration			2.1	2.1	32.8		12.7	5.5
Sum rule			4.2	2.1			7.3	3.6
Product rule	51.1	34.0	45.8	41.7	25.9	3.4	23.6	12.7
Formula	61.7	61.7	33.3	33.3				
Reference to another problem	6.4	6.4	2.1	2.1	1.7		1.8	
Sub-problem decomposition	6.4	6.4	2.1		36.2	31.0	12.7	5.5
Tree diagram			4.2	4.2				
Other					1.7	1.7		

Table 4 Percentages of global and correct use of each strategy by item and group in combination problems

Strategy	Instruction ($N=51$)				No instruction ($N=64$)			
	Item 3 $C_{4,3}$ Distribution		Item 4 $C_{5,3}$ Selection		Item 3 $C_{4,3}$ Distribution		Item 4 $C_{5,3}$ Selection	
	Global	Correct	Global	Correct	Global	Correct	Global	Correct
Systematic enumeration	21.3	19.1	17.0	8.5	45.5	30.9	75.4	12.3
A-systematic enumeration	10.6	6.4	4.3	2.1	5.5		1.8	
Sum rule	8.5	8.5			10.9	9.1	5.3	
Product rule	29.8		14.9	2.1	25.5		21.1	1.8
Formula	29.8	19.1	57.4	46.8				
Reference to another problem	6.4	6.4	6.4	4.3				
Sub-problem decomposition			2.1		7.3		10.5	
Tree diagram	2.1		2.1		16.4	5.5	1.8	
Quotient rule	6.4	6.4	10.6	6.4				
Other			2.1		3.6	1.8		

three different ways, and so on; however, after this correct reasoning, they did not manage to apply the multiplication principle.

We also note the scarce use of general problem-solving strategies, such as referring to a different problem (mainly applied in the instruction group) or subdividing the problem into parts (in the no-instruction group). Despite the relevance of the tree diagram (Fischbein, 1975), there was scarce use of this tool with few correct applications of the same.

When comparing the permutation problems (Table 3) with the combination problems (Table 4), we observe that students of the instruction group usually solved the permutation problems correctly regardless of their strategy, with less success in the combination problems. We also notice a scarcer development of correct strategies in combination problems in both groups. The percentages of correct use of strategies in the combination problems were smaller than in the case of permutations, which explains why these problems showed, in our research, a lower percentage of correct answers than permutation problems.

The difference in percentages of global and correct use of strategies was higher in the combination tasks, in particular, systematic enumeration in item 4 in the no-instruction group and use of formula in the instruction group. Note that use of a formula was general in permutation problems within the instruction group and was less frequent in combination problems where, many times, its application was incorrect. Thus, for our students, it was easier to identify the combinatorial operation in the permutation problems we posed to them than in the combination problems. A possible explanation is that in permutations just a parameter intervenes, while combinations depend on two parameters that can be confused by the students. There was also scarcer use of the product rule in combination problems; for item 3, no student with instruction correctly applied this rule. Roa (2000) did not use permutation problems in his questionnaire, so that we cannot compare the strategies in the permutation problems with ones developed by university students.

The dimension of the problem also played a role in the students' choice of strategy, especially in the case of permutations: in medium-dimensional items, students in the instruction group mainly used formulas or product rules, and students without instruction tended to rely on enumerations or developed multiple strategies. On the other hand, in item 2, which was small-dimensional, students who received instruction often used multiple methods. We can identify two multiple-strategy behaviours depending on how they used single strategies:

- We observed this strategical behaviour mostly in the instruction group in item 2. In Fig. 1, the student solved the problem through enumeration but then provided an explicit calculation that probably was considered more related to what he learnt in the classroom. We interpret that this behaviour emerged from the student's need to find a confirmation of the results they spontaneously obtained developing a non-official solving strategy (in this case, enumeration). We noticed this kind of strategical behaviour in the group of students with instruction, specifically for item 2. Mashiach and Zaslavsky (2004) also reported the tendency to use verification strategies to check the results in combinatorial problem.
- Other multiple strategies were linked together in the solution, where a primary method relied on a second or more. These procedures were sequentially developed, and there was a hierarchy of the different strategies involved, starting from the primary strategy and then chaining all the following supporting procedures. We present an example in Fig. 2, where the reduction to a sub-problem was supported by enumeration and product rule. This strategical conduct was generally found within the no-instruction group, specifically for item 1, the medium-dimensional problem. The students of the no-instruction group also intended to adapt their solution to item 2. Anyhow, after fixing the first variable, they noticed that there were few configurations to be listed and then solved the problem directly through enumeration. We observed this behaviour in many students without instruction and coming from different groups, but not in the students with instruction.

There was a similar trend in combination problems: students of the instruction group tended to provide a double solution to the problem (again employing enumeration and product rule or formula). In contrast, students of the group without instruction mostly relied on enumerative strategies and failed to get to a correct solution when using a different procedure. Finally, there were no clear differences in strategies between the selection and distribution problems.

Discussion and Conclusions

In this paper, we discussed the results of performance and strategies in a sample of secondary school Italian students when solving permutation and combination problems in the models of distribution and selection (Dubois, 1984).

In agreement with Fischbein and Gazit (1988) with children in Israel and with Batanero et al. (1997b) and Navarro-Pelayo (1994) with secondary Spanish students, instruction was useful in increasing the competence of the tested Italian students in solving combinatorial problems in our research. Students' performances (in terms of correctly solved problems) increased with statistically significant differences in all the items in the group with instruction, even though some difficulties remained, and some of the students with instruction were not always able to get to the correct solution. When comparing the items in our research belonging to different combinatorial models (Dubois, 1984), the distribution model resulted in higher percentages of correct answers in the group without instruction, while there was not much difference between the models within the group of students with instruction, differently with respect to Navarro-Pelayo's (1994) results. Contrary to what was found in 1988 by Fischbein and Gazit with primary school children and in agreement with Navarro-Pelayo (1994), our students got more correct solutions to permutations than combination problems.

The study of strategies used by students revealed a variety of them, still more numerous in the instruction group. In spite of a (mostly) formal instruction based on the learning of formulas, and after acquiring new procedures, some students of the group with instruction continued to rely on enumeration, even though sometimes they combined enumeration with formulas and arithmetical rules, providing a double solution to a problem. Comparing our results with the ones presented in Roa (2000) with undergraduate students, we notice that the distribution of strategies for secondary school Italian students was different, shifting toward a higher use of enumeration in the students in our research. However, when we focus only on the group with instruction, the distribution of strategies was more similar to what was observed with university students with higher mathematical preparation that, consequently, did not seem to play a key role in the development of solving strategies.

Students without instruction in combinatorics spontaneously approached the problem with initial strategies (e.g. subdividing a problem in parts) that they not always developed correctly. However, they showed correct combinatorial and problem-solving intuitions (Fischbein, 1975) that are worthwhile to encourage in learning the topic. Part of these strategies tended to disappear with instruction, as they were scarcely employed in this group. Our interpretation is that instruction plays a key role in the development of solving strategies in combinatorics either reinforcing or suppressing some of the spontaneously developed procedures of students, and that this depends on how the teaching is carried on. Finally, we highlight the scarce use of the sum rule and the tree diagram, in spite of Fischbein and Gazit (1988) considering the tree diagram a productive resource in combinatorics.

Despite the moderate size of our sample, this exploratory research adds new knowledge to previous research, since it provides new insights on the solving strategies students activate when solving permutation and combination problems. These strategies were only studied with undergraduate students by Roa (2000) but not with secondary school Italian students. Moreover, we added information about the percentage of strategies producing correct results, change in strategies with instruction, and strategy patterns observed in distribution and selection problems, as well as in permutation and combination problems. We also found that most of the correct answers come from the use of either systematic enumeration or formulas, because other different procedures were not always productive to reach the correct solution.

These results can be used by new researchers who continue exploring the combinatorial reasoning of students in other countries or with different problems. Teachers can also benefit from knowing in advance the difficulties in the different types of combinatorial problems, as well as the procedures that the students tend to activate in solving these problems. This knowledge would allow the teacher to better implement the teaching of the topic, by encouraging the productive strategies that students spontaneously develop before instruction and introducing different techniques to help the students improve their procedures.

However, further and deeper research is still needed, for example, widening the dimension of the sample and including new qualitative analysis, aiming to better understand the mechanisms that lay behind the choice of a solving strategy and characterize how this mechanism evolves after instruction.

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Declarations

Ethical Approval All procedures performed in our study involving human participants were in accordance with the ethical standards of the institutional and/or national research committee and with the 1964 Helsinki declaration and its later amendments or comparable ethical standards. On behalf of all the authors, the corresponding author states that there is no conflict of interest.

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