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Discrete Optimization

A branch-and-price approach for the continuous multifacility monotone ordered median problem

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A B S T R A C T

In this paper, we address the Continuous Multifacility Monotone Ordered Median Problem. The goal of this problem is to locate p facilities in $\mathbb{R}^d$ minimizing a monotone ordered weighted median function of the distances between given demand points and its closest facility. We propose a new branch-and-price procedure for this problem, and three families of matheuristics based on: solving heuristically the pricer problem, aggregating the demand points, and discretizing the decision space. We give detailed discussions of the validity of the exact formulations and also specify the implementation details of all the solution procedures. Besides, we assess their performances in an extensive computational experience that shows the superiority of the branch-and-price approach over the compact formulation in medium-sized instances. To handle larger instances it is advisable to resort to the matheuristics that also report rather good results.

1. Introduction

In the last years, a lot of attention has been paid to the discrete aspects of location theory and a large body of literature has been published on this topic (see, e.g., Beasley, 1985; Elloumi, Labbé, & Pochet, 2004; Espejo, Marín, Puerto, & Rodríguez-Chía, 2009; García, Labbé, & Marín, 2010; Marín, Nickel, Puerto, & Velten, 2009; Marín, Nickel, & Velten, 2010; Puerto, Ramos, & Rodríguez-Chía, 2013; Puerto & Tamir, 2005). One of the reasons of this flourish is the recent development of integer programming and the success of MIP solvers. In spite of that, the reader might notice that the mathematical origins of this theory emerged very close to some classical continuous problems such as the well-known Fermat-Torricelli or Weber problem, and the Simpson problem (see, e.g., Drezner & Hamacher, 2002; Laporte, Nickel, & Saldanha da Gama, 2015; Nickel & Puerto, 2005). However, the continuous counterparts of location problems have been mostly analyzed and solved using geometric constructions, valid on the plane and the three dimensional space, that are difficult to extend when the dimensions grow or the problems are slightly modified to include some side constraints (Blanco & Gázquez, 2021; Blanco, Puerto, & Ponce, 2017; Carrizosa, Conde, Muñoz, & Puerto, 1995; Carrizosa, Muñoz Márquez, & Puerto, 1998; Fekete, Mitchell, & Beurer, 2005; Nickel, Puerto, & Rodríguez-Chía, 2003; Puerto & Rodríguez-Chía, 2011). These problems, although very interesting, quickly fall within the field of global optimization and they become very hard to solve. Even those problems that might be considered *easy*, as for instance the classical Weber problem with Euclidean norms, are most of the times solved with algorithms (as the Weiszfeld algorithm, Weiszfeld, 1937), whose convergence is still unknown (Chandrasekaran & Tamir, 1989). Moreover, most problems studied in continuous location assume that a single facility is to be located, since their multifacility counterparts lead to difficult non-convex problems (Manzour-al Ajdad, Torabi, & Eshghi, 2012; Blanco, 2019; Blanco, ElHaj BenAli, & Puerto, 2014; Brimberg, 1995; Carrizosa et al., 1998; Mallozzi, Puerto, & Rodríguez-Madrena, 2019; Puerto, 2020; Rosing, 1992; Valero-Franco, Rodríguez-Chía, & Espejo, 2013).

Motivated by the recent advances on discrete multifacility location problems with ordered median objectives (Deleplanque, Labbé, Ponce, & Puerto, 2020; Espejo, Puerto, & Rodriguez-Chía,

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2021; Fernández, Pozo, & Puerto, 2014; Labbé, Ponce, & Puerto, 2017; Marín, Ponce, & Puerto, 2020), and the available results on conic optimization (Blanco et al., 2014; Puerto, 2020), we analyze here a family of difficult continuous multifacility location problems with ordered median objectives and distances induced by a general family of norms. These problems gather the essential elements of discrete and continuous location analysis, making their solution a challenging question.

In this paper, we develop an *ad hoc* branch-and-price algorithm for solving this general family of continuous location problems. The continuous multifacility Weber problem has been already studied using branch-and-price methods (Krau, 1997; du Merle, Villenueve, Desrosiers, & Hansen, 1999; Righini & Zaniboni, 2007; Venkateshan & Mathur, 2015). In addition, in discrete location, these techniques have also been applied to the p-median problem (Avella, Sassano, & Vasilev, 2007, see, e.g.,). However, to the best of our knowledge, a branch-and-price approach for location problems with ordered median objectives has only been developed for the discrete version in Deleplanque et al. (2020) beyond a multisource hyperplanes application (Blanco, Japón, Ponce, & Puerto, 2021).

Our goal in this paper is to analyze the *Continuous Multifacility Monotone Ordered Median Problem* (MFMOMP, for short), in which we are given a finite set of demand points, $\mathcal{A}$, and the goal is to find the optimal location of p new facilities such that: (1) each demand point is allocated to a single facility; and (2) the measure of the goodness of the solution is an ordered weighted aggregation of the distances of the demand points to their closest facility (see, e.g., Nickel & Puerto, 2005). We consider a general framework for the problem, in which the demand points (and the new facilities) lie in $\mathbb{R}^d$, the distances between points and facilities are polyhedral- or ℓ_τ-norms for $\tau \geq 1$, and the ordered median functions are assumed to be defined by non-decreasing monotone weights. These problems are analyzed in Blanco, Puerto, & ElHaj BenAli (2016), in which the authors provide a Mixed Integer Second Order Cone Optimization (MISOCO) reformulation of the problem able to solve, for the first time, problems of small to medium size (up to 50 demand points), using off-the-shelf solvers.

The family of problems under analysis has a broad range of applications in different fields. On the one hand, continuous location has been proven to be an adequate tool in case the services to be located are sensors, surveillance cameras, etc., that are allowed to be flexibly positioned in the space. Also, multifacility location problems can be seen as a unified modelling tool to extend classical clustering algorithms, as the k-means or k-median approaches, or more general approaches (Blanco et al., 2021). The use of ordered median objective functions determines, at the same time, the positions of the *optimal* location of the services balancing equity and efficiency of the list of distances from the demand points to their closest facilities (see, e.g., Aouad & Segev, 2019; Calvino, López-Haro, Muñoz-Ocaña, Puerto, & Rodríguez-Chía, 2022; Espejo et al., 2009; Fourour & Lebbah, 2020; Muñoz-Ocaña et al., 2020; Ogryczak, Perny, & Weng, 2011; Olender & Ogryczak, 2019; Tamir, 2001). The connection between discrete location and its continuous counterpart has been a topic of study since the introduction of the continuous problem (Cooper, 1963; Kalczynski, Brimberg, & Drezner, 2021). Thus, the extension of some facility location problems that have been analyzed in a discrete space (voting, exam qualifications, etc.) to the continuous framework, is a topic of interest in the Location Science field (Drezner & Nickel, 2009; Espejo et al., 2009; Ponce, Puerto, Ricca, & Scozzari, 2018). We also refer the reader to Bruno, Genovese, & Improta (2014), Drezner & Hamacher (2002), Love, Morris, & Wesolowsky (1988), Mirchandani & Francis (1990), and the references therein to find more applications in the fields of industry, urban or regional planning, clustering, mobile location, commerce, public service facilities, or transport facilities.

Our contribution in this paper is to introduce a new set partitioning-like (with side constraints) reformulation for this family of problems that allows us to develop a branch-and-price algorithm for solving it. This approach gives rise to a decomposition of the original problem into a master problem (set partitioning with side constraints), and a pricing problem that consists of a special form of the maximal weighted independent set problem combined with a single facility location problem. We compare this new strategy with the one obtained by solving Mixed Integer Nonlinear Programming (MINLP) formulations using standard solvers. Our results show that it is worth using the new reformulation since it allows us to solve larger instances and reduce the gap when the time limit is reached. Moreover, we also exploit the structure of the branch-and-price approach to develop some new matheuristics for the problem that provide good quality feasible solutions for fairly large instances of several hundreds of demand points.

The paper is organized in six sections and two appendixes. Section 2 formally describes the problem considered in this paper, namely the MFMOMP, and develops MISOCO formulations for it. Section 3 is devoted to present the new set partitioning-like formulation and all the details of the branch-and-price algorithm proposed to solve it. There, we present how to obtain initial variables for the restricted master problem, we discuss and formulate the pricing problem and set properties for handling it, and describe the branching strategies and variable selection rules implemented in our algorithm. The next section, namely Section 4, deals with some heuristic algorithms proposed to provide solutions for large-sized instances. In this section, we also describe how to solve heuristically the pricing problem which gives rise to a matheuristic algorithm consisting of applying the branch-and-price algorithm but solving the pricing problem only heuristically. Obviously, since in this case the optimality of the pricing problem is not guaranteed, we cannot ensure optimality for the solution of the master problem, although we always obtain feasible solutions. In addition, we also present another aggregation heuristic based on clustering strategies that allows us to provide bounds for the problem. Finally, a third heuristic, based on discretizing the space, is developed. Section 5 reports the results of an exhaustive computational study with real-world instances of different nature. There, we compare the standard formulations with the branch-and-price approach and also with the heuristic algorithms. The paper ends with some conclusions in Section 6. Finally, Appendix A reports the details of the computational experiment for different norms showing the usefulness and generality of our approach, and Appendix B shows the computational results disaggregated by different parameters of the instances.

2. The continuous multifacility monotone ordered median problem

In this section, we describe the problem under study and fix the notation for the rest of the paper.

We are given a set of n demand points in $\mathbb{R}^d$, $\mathcal{A} = \{a_1, \ldots, a_n\} \subset \mathbb{R}^d$, and $p \in \mathbb{N}$ ($p > 0$). Our goal is to find p new facilities located in $\mathbb{R}^d$ that minimize a function of the closest distances from the demand points to the new facilities. We denote the index sets of demand points and facilities by $I = \{1, \ldots, n\}$ and $J = \{1, \ldots, p\}$, respectively. Several elements are involved when finding the *best* p new facilities to provide service to the n demand points. In what follows, we describe them:

- *Closeness Measure:* Given a demand point a_i, $i \in I$, and a facility $x \in \mathbb{R}^d$, we use norm-based distances to measure the point-to-facility closeness. Thus, we consider the following measure for the distance between a_i and x:

$$\delta_i(x) = \|a_i - x\|,$$

Table 1
Examples of Ordered Median aggregation functions.

Notation	λ-vector	Name
W	$(1,\dots,1)$	p-median
C	$(0,\dots,0,1)$	p-center
K	$(0,\dots,0,\overbrace{1,\dots,1}^{k})$	k-center
D	$(\alpha,\dots,\alpha,1)$	centdian
S	$(\alpha,\dots,\alpha,\overbrace{1,\dots,1}^{k})$	k-entdian
A	$(0=\frac{0}{n-1},\frac{1}{n-1},\frac{2}{n-1},\dots,\frac{n-2}{n-1},\frac{n-1}{n-1}=1)$	ascendant

where $\|\cdot\|$ is a norm in $\mathbb{R}^d$. In particular, we will assume that the norm is polyhedral or an ℓ_τ-norm (with $\tau \geq 1$), i.e., $\delta_i(x) = \left(\sum_{l=1}^{d} |a_{il}-x_l|^\tau\right)^{\frac{1}{\tau}}$.

- *Allocation Rule*: Given a set of p new facilities, $\mathcal{X} = \{x_1,\dots,x_p\} \subset \mathbb{R}^d$, and a demand point a_i, $i \in I$, once all the distances between a_i and x_j $(j \in J)$ are calculated, one has to allocate the point to a single facility. As usual in the literature, we assume that each point is allocated to its closest facility, i.e., the closeness measure between a_i and $\mathcal{X}$ is:

$$\delta_i(\mathcal{X}) = \min_{x\in\mathcal{X}} \delta_i(x),$$

and the facility $x \in \mathcal{X}$, reaching such a minimum is the one where a_i is allocated to (in case of ties among facilities, a random assignment is performed).
- *Aggregation of Distances*: Given the set of demand points $\mathcal{A}$, the distances $\{\delta_i(\mathcal{X}) : i \in I\} = \{\delta_1,\dots,\delta_n\}$ must be aggregated abusing of notation, and unless necessary, we will avoid the dependence of $\mathcal{X}$ in the δ-values). To this end, we use the family of ordered median criteria. Given $\lambda \in \mathbb{R}^n_+$ the λ-ordered median function is defined as:

$$\mathrm{OM}_\lambda(\mathcal{A};\mathcal{X}) = \sum_{i\in I} \lambda_i\, \delta_{(i)}, \tag{OM}$$

where $(\delta_{(i)})_{(i\in I)}$ is a permutation of $(\delta_i)_{(i\in I)}$ such that $\delta_{(1)} \leq \cdots \leq \delta_{(n)}$. Some particular choices of λ-weights are shown in Table 1. Note that most of the classical continuous location problems can be cast under this *ordered median* framework, e.g., the multisource Weber problem, $\lambda = (1,\dots,1)$, or the multisource p-center problem, $\lambda = (0,\dots,0,1)$.

Summarizing all the above considerations, given a set of n demand points in $\mathbb{R}^d$, $\mathcal{A} = \{a_1,\dots,a_n\} \subset \mathbb{R}^d$ and $\lambda \in \mathbb{R}^n_+$ (with $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$), the Continuous Multifacility Monotone Ordered Median Problem (**MFMOMP**$_\lambda$) can be stated as:

$$\min_{\mathcal{X}=\{x_1,\dots,x_p\}\subset\mathbb{R}^d} \mathrm{OM}_\lambda(\mathcal{A};\mathcal{X}). \tag{$\mathbf{MFMOMP}_\lambda$}$$

Observe that the problem above is $\mathcal{NP}$-hard since the multisource p-median problem is just a particular instance of (**MFMOMP**$_\lambda$) where $\lambda = (1,\dots,1)$ (see Sherali & Nordai, 1988). In the following result we provide a suitable Mixed Integer Second Order Cone Optimization (MISOCO) formulation for the problem.

Theorem 1. *Let $\|\cdot\|$ be an ℓ_τ-norm in $\mathbb{R}^d$, where $\tau = \frac{r}{s}$ with $r,s \in \mathbb{N}\setminus\{0\}$, $r > s$ and $\gcd(r,s) = 1$ or a polyhedral norm. Then, (**MFMOMP**$_\lambda$) can be formulated as a MISOCO problem.*

Proof. First, assume that $\{\delta_i(\mathcal{X}) : i \in I\} = \{\delta_1,\dots,\delta_n\}$ are given. Then, sorting the elements and multiplying them by the λ-weights can be equivalently written as the following assignment problem (see Blanco et al., 2014; Blanco et al., 2016), whose dual problem (right side) allows to compute the value of the ordered median function:

$$\sum_{k\in I}\lambda_k\delta_{(k)} = \max \sum_{i,k\in I}\lambda_k\delta_i\sigma_{ik} \quad = \quad \min \sum_{i\in I}u_i + \sum_{k\in I}v_k$$
$$\text{s.t.} \sum_{k\in I}\sigma_{ik} = 1,\ \forall i\in I, \qquad \text{s.t.}\ u_i + v_k \geq \lambda_k\delta_i,\ \forall i,k\in I,$$
$$\sum_{i\in I}\sigma_{ik} = 1,\ \forall k\in I, \qquad u_i, v_k \in \mathbb{R},\ \forall i,k\in I.$$
$$\sigma_{ik}\in[0,1],\ \forall i,k\in I.$$

Now, we can embed the above representation of the ordered median aggregation of $\delta_1,\dots,\delta_n$, into (**MFMOMP**$_\lambda$). On the other hand, we have to represent the allocation rule (closest distances). This family of constraints is given by

$$\delta_i = \min_{j\in J}\|a_i - x_j\|,\ \forall i\in I.$$

In order to represent it, we use the following set of decision variables: $w_{ij} = 1$ if a_i is allocated to facility j, $w_{ij} = 0$ otherwise, $\forall i \in I, j \in J$; in addition, z- and r-variables are auxiliary variables.

Then, a *Compact* formulation for (**MFMOMP**$_\lambda$) is:

$$\min \sum_{i\in I}u_i + \sum_{k\in I}v_k$$
$$\text{s.t.}\ u_i + v_k \geq \lambda_k r_i,\ \forall i,k\in I, \tag{C1}$$
$$z_{ij} \geq \|a_i - x_j\|,\ \forall i\in I, j\in J, \tag{C2}$$
$$r_i \geq z_{ij} - M(1-w_{ij}),\ \forall i\in I, j\in J, \tag{C3}$$
$$\sum_{j\in J}w_{ij} = 1,\ \forall i\in I, \tag{C4}$$
$$x_j\in\mathbb{R}^d,\ \forall j\in J, \tag{C5}$$
$$w_{ij}\in\{0,1\},\ \forall i\in I, j\in J, \tag{C6}$$
$$z_{ij}\geq 0,\ \forall i\in I, j\in J, \tag{C7}$$
$$r_i \geq 0,\ \forall i\in I, \tag{C8}$$

where (C_3) allows to compute the distance between the points and its closest facility and (C_4) assures single allocation of points to facilities. Here M is a big enough constant $M > \max_{i,k\in I}\|a_i - a_k\|$.

Finally, in case $\|\cdot\|$ is the $\ell_{\frac{r}{s}}$-norm, constraint (C_2), as already proven in Blanco et al. (2014), can be rewritten as:

$$t_{ijl} + a_{il} - x_{jl} \geq 0,\ \forall i\in I, j\in J, l = 1,\dots,d,$$
$$t_{ijl} - a_{il} + x_{jl} \geq 0,\ \forall i\in I, j\in J, l = 1,\dots,d,$$
$$t_{ijl}^r \leq \xi_{ijl}^s z_{ij}^{r-s},\ \forall i\in I, j\in J, l = 1,\dots,d,$$
$$\sum_{l=1}^d \xi_{ijl} \leq z_{ij},\ \forall i\in I, j\in J,$$
$$t_{ijl}, \xi_{ijl} \geq 0,\ \forall i\in I, j\in J, l = 1,\dots,d,$$

where t_{ijl} are auxiliary variables that allow to model the absolute values $|a_{il} - x_{jl}|$ being a_{il} and x_{jl} the lth coordinates of the demand point a_i and the facility x_j, respectively. The variables ξ_{ijl} are also auxiliary variables that allow to adequately represent the $\ell_{\frac{r}{s}}$-norm (see Blanco et al., 2014, for further details on this representation).

If $\|\cdot\|$ is a polyhedral norm, then (C_2) is equivalent to:

$$\sum_{l=1}^{g} e_{gl}(a_{il} - x_{jl}) \leq z_{ij},\quad \forall i\in I, j\in J, e\in \mathrm{Ext}_{\|\cdot\|^\circ},$$

where $\text{Ext}_{\|\cdot\|^o} = \{e^o_1, \ldots, e^o_g\}$ are the extreme points of the unit ball of the dual norm of $\|\cdot\|$ (see, e.g., Nickel & Puerto, 2005; Ward & Wendell, 1985).

The final compact formulation depends on the norm, but in any case, we have a MISOCO reformulation for (**MFMOMP**$_\lambda$). □

Note that (**MFMOMP**$_\lambda$) is an extension of the single-facility ordered median location problem (see, e.g., Blanco et al., 2014), where apart from finding the location of p new facilities, the allocation patterns between demand points and facilities are also determined. In the rest of the paper, we will exploit the combinatorial nature of the problem by means of a set partitioning-like formulation which is based on the following observation:

Proposition 1. *Any optimal solution of (**MFMOMP**$_\lambda$) is characterized by p pairs*
$(S_1, x_1), \ldots, (S_p, x_p)$ with $S_j \subset I$ and $x_j \in \mathbb{R}^d$, $\forall j \in J$, such that:

1. $\bigcup_{j\in J} S_j = I$.
2. $S_j \cap S_{j'} = \emptyset,\ j, j' \in J : j \neq j'$.
3. *For each* $j \in J$, $x_j \in \arg\min_{x\in\{x_1,\ldots,x_p\}} \|a_i - x\|,\ \forall i \in S_j$.
4. $(x_1, \ldots, x_p) \in \arg\min_{y_1,\ldots,y_p} \sum_{j\in J}\sum_{i\in S_j} \lambda_{(i)} \|a_i - y_j\|$, *where* $(i) \in I$ *such that* $\|a_i - y_j\|$ *is the* (i)*th smallest element in* $\{\|a_i - y_j\| : i \in I, j \in J\}$.

From the structure of the optimal solutions of (**MFMOMP**$_\lambda$) described in Proposition 1, we can conclude that there exists a finite candidate set of admissible solutions of this problem given by the different partitions of I in p subsets and one of their associated p best facilities, as defined in Proposition 1 (4). In addition, if the demand points $\mathcal{A}$ are non-collinear and $\tau > 1$ the solution of the problem in Proposition 1 (4) is unique; otherwise, we can always restrict the choices of $x_1, \ldots, x_p$ to the extreme points of the set of optimal solutions which is finite. From the above discussion, we conclude that there exists a finite dominating set of candidates, that we will denote as $\mathcal{FDS}$, to optimal solutions of (**MFMOMP**$_\lambda$).

From now on, we will call a pair (S, x) with $S \subset I$ and $x \in \mathbb{R}^d$ a *suitable pair if*

1. There exist $(S_2, x_2), \ldots, (S_p, x_p)$ such that $\bigcup_{j=2}^{p} S_j = I \setminus S$, $S_j \cap S_{j'} = \emptyset$ for $j, j' \in \{2, \ldots, p\} : j \neq j'$, $x_j \in \mathbb{R}^d$, $j = 2, \ldots, p$.
2. $(S, x), (S_2, x_2), \ldots, (S_p, x_p) \in \mathcal{FDS}$.

In words, a suitable pair is any pair (S, x) that can be part of a candidate solution of (**MFMOMP**$_\lambda$) within the set $\mathcal{FDS}$. By the finiteness of the sets of admissible solutions, it also follows that the number of suitable pairs is finite as well.

3. A set partitioning-like formulation

The compact formulation shown in the previous section is affected by the size of p and d, and it exhibits the same limitations as many other compact formulations for continuous location models even without ordering constraints. For this reason, in the following we propose an alternative set partitioning-like formulation (du Merle & Vial, 2002; du Merle et al., 1999) for (**MFMOMP**$_\lambda$).

Let $S \subset I$ be a subset of demand points that are assigned to the same facility. Let $R = (S, x)$ be a suitable pair composed by a subset $S \subset I$ and a facility $x \in \mathbb{R}^d$. We denote by δ^R_i the contribution of demand point $i \in S$ in the subset with respect to the facility x. Finally, for each suitable pair $R = (S, x)$ we define the variable

$$y_R = \begin{cases} 1 & \text{if subset } S \text{ is selected and its associated facility is } x, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $\mathcal{R} = \{(S, x) : \text{suitable pairs}, S \subset I \text{ and } x \in \mathbb{R}^d\}$.

The set partitioning-like formulation is given by the following master problem (MP):

$$\min \sum_{i\in I} u_i + \sum_{k\in I} v_k \tag{MP_1}$$

$$\text{s.t.} \sum_{R=(S,x)\in\mathcal{R}: i\in S} y_R = 1,\ \forall i \in I, \tag{MP_2}$$

$$\sum_{R\in\mathcal{R}} y_R = p, \tag{MP_3}$$

$$u_i + v_k \geq \lambda_k \sum_{R=(S,x)\in\mathcal{R}: i\in S} \delta^R_i y_R,\ \forall i, k \in I, \tag{MP_4}$$

$$y_R \in \{0, 1\},\ \forall R \in \mathcal{R}, \tag{MP_5}$$

$$u_i, v_k \in \mathbb{R},\ \forall i, k \in I. \tag{MP_6}$$

The objective function (MP_1) and constraints (MP_4) give the correct ordered median function of the distances from the demand points to the closest facility (see Section 2). Constraints (MP_2) ensure that all demand points appear in exactly one set S in each feasible solution. Exactly p facilities are open due to constraint (MP_3). Finally, (MP_5) define the variables as binary.

The reader might notice that this formulation has an exponential number of variables, and therefore in the following we describe the necessary elements to address its solution by means of a branch-and-price scheme, namely:

1. *Initial Pool of Variables:* Generation of initial feasible solutions induced by a set of initial subsets of demand points (and their costs).
2. *Pricing Problem:* In set partitioning problems, instead of solving initially the problem with the whole set of variables, the variables have to be incorporated *on-the-fly* by solving adequate pricing subproblems derived from previously computed solutions until the optimality of the solution is guaranteed. The pricing problem is derived from the relaxed version of the master problem and using the strong duality properties of the induced Linear Programming Problem.
3. *Branching:* The rule that creates new nodes of the branch-and-bound tree when a fractional solution is found at a node of the search tree. We have adapted the Ryan and Foster branching rule to our problem.
4. *Stabilization*: The convergence of column generation approaches can be sometimes erratic since the values of dual variables in the first iterations might oscillate, leading to variables of the master problem that will never appear in the optimal solution of the problem. Stabilization tries to avoid that behaviour.

In what follows, we describe how each of the above items is implemented in our proposal.

3.1. Initial variables

In the solution process of the set partitioning-like formulation using a branch-and-price approach, it is convenient to generate an initial pool of variables before starting solving the problem. The adequate selection of these initial variables might help to reduce the CPU time required to solve the problem. We apply an iterative strategy to generate this initial pool of y-variables. In the first iteration, we randomly generate p positions for the facilities. The demand points are then allocated to their closest facilities, and at most p subsets of demand points are generated. We incorporate the variables associated with these subsets to the master problem

(MP). In the next iterations, instead of generating p new facilities, we keep those with more associated demand points, and randomly generate the remainder. After a fixed number of iterations, an initial set of columns is generated to define the restricted master problem, and also an upper bound of our problem. Since the optimal position of the facilities belongs to a bounded set contained in the rectangular hull of the demand points, the random facilities are generated in the smallest hyperrectangle containing $\mathcal{A}$.

3.2. The pricing problem

To apply the column generation procedure we restrict and relax (MP), in the following restricted relaxed master problem (RRMP).

$$
\begin{aligned}
\rho^*_{\text{MP}} := \min \quad & \sum_{i\in I} u_i + \sum_{k\in I} v_k & \textbf{Dual Multipliers} \qquad \text{(RRMP)}\\
\text{s.t.} \quad & \sum_{R=(S,x)\in\mathcal{R}_0 : i\in S} y_R \geq 1,\ \forall i\in I, & \alpha_i \geq 0\\
& -\sum_{R\in\mathcal{R}_0} y_R \geq -p, & \gamma \geq 0\\
& u_i + v_k - \lambda_k \sum_{R=(S,x)\in\mathcal{R} : i\in S} \delta_i^R y_R \geq 0,\\
& \quad \forall i,k\in I, & \epsilon_{ik} \geq 0\\
& y_R \geq 0,\ \forall R\in\mathcal{R}_0,\\
& u_i, v_k \in \mathbb{R},\ \forall i,k\in I,
\end{aligned}
$$

where $\mathcal{R}_0 \in \mathcal{R}$ represents the initial pool of columns used to initialize the set partitioning-like formulation (MP). Constraints (MP_2) and (MP_3) are slightly modified from equations to inequalities in order to get nonnegative dual multipliers. This transformations keeps the equivalence with the original formulation since coefficients affecting the y-variables in constraint (MP_4) are nonnegative. The notation for the dual variables associated with each family of constraints is written in the right column $(\alpha, \gamma, \epsilon)$.

The value of the distances is unknown beforehand because the location of facilities can be anywhere in the continuous space. Hence, its determination requires solving continuous optimization problems.

By strong duality, the objective value of the continuous relaxation (RRMP), can be obtained as:

$$
\begin{aligned}
\rho^*_{\text{MP}} = \max \quad & \sum_{i\in I} \alpha_i - p\gamma & \text{(Dual RRMP)}\\
\text{s.t.} \quad & \sum_{i\in I} \epsilon_{ik} = 1,\ \forall k\in I,\\
& \sum_{k\in I} \epsilon_{ik} = 1,\ \forall i\in I,\\
& \sum_{i\in S} \alpha_i - \gamma - \sum_{i\in S}\sum_{k\in I} \delta_i^R \lambda_k \epsilon_{ik} \leq 0,\ \forall R=(S,x)\in\mathcal{R}_0,\\
& \alpha_i, \gamma, \epsilon_{ik} \geq 0,\ \forall i,k\in I.
\end{aligned}
$$

Hence, for any variable y_R in the master problem, its reduced cost is

$$c_R - z_R = -\sum_{i\in S} \alpha_i^* + \gamma^* + \sum_{i\in S}\sum_{k\in I} \delta_i^R \lambda_k \epsilon_{ik}^*,$$

where $(\alpha^*, \gamma^*, \epsilon^*)$ is the dual optimal solution of the current (RRMP).

To certify optimality of the relaxed problem one has to check implicitly that all the reduced costs for the variables not currently included in the (RRMP) are nonnegative. Otherwise, new variables must be added to the pool of columns. This can be done solving the so-called pricing problem.

The pricing problem consists of finding the minimum reduced cost among the variables that have not yet been included in the pool. That is, we have to find the set $S \subset I$ and the position of the facility x (its coordinates) which minimizes the reduced cost.

For a given set of dual multipliers, $(\alpha^*, \gamma^*, \epsilon^*) \geq 0$, the problem to be solved is

$$
\begin{aligned}
\min_{\substack{S\subset I\\ x\in\mathbb{R}^d}} \quad & -\sum_{i\in S} \alpha_i^* + \gamma^* + \sum_{i\in S}\sum_{k\in I} \delta_i^S \lambda_k \epsilon_{ik}^*\\
\text{s.t.} \quad & \delta_i^S \geq \|x - a_i\|,\ \forall i\in S.
\end{aligned}
$$

The above formal problem can be reformulated by means of a mixed integer program. We define variables $w_i = 1,\ i\in I$ if the demand point belongs to S, and zero otherwise. We also define variables $r_i,\ i\in I$ to represent the distance from demand point i to facility x and zero in case $w_i = 0$. Finally, $z_i,\ i\in I$ are auxiliary variables to represent the distances from demand point i to facility x in any case.

$$\min \quad -\sum_{i\in I} \alpha_i^* w_i + \gamma^* + \sum_{i\in I} c_i r_i \tag{3}$$

$$\text{s.t.} \quad z_i \geq \|x - a_i\|,\ \forall i\in I, \tag{4}$$

$$r_i + M(1 - w_i) \geq z_i,\ \forall i\in I, \tag{5}$$

$$w_i \in \{0,1\},\ \forall i\in I, \tag{6}$$

$$z_i, r_i \geq 0,\ \forall i\in I, \tag{7}$$

where M is a big enough constant ($M > \max\{\|a_i - a_{i'}\| : i, i' \in I, i \neq i'\}$) and $c_i = \sum_{k\in I} \lambda_k \epsilon_{ik}^*,\ i\in I$.

Objective function (10) is the minimum reduced cost associated with the optimal solution of the pricing problem. Constraints (11) define the distances. As in Section 2, this family of constraints is defined *ad hoc* for a given norm. Constraints (12) set correctly the r-variables. Finally, constraints (13) and (14) are the domain of the variables.

As it has been shown in the proof of Theorem 1, the above problem can be formulated as a MISOCO problem in case of polyhedral or ℓ_τ-norms.

The so-called *Farkas pricing* should be adequately defined in case the feasibility of (RRMP) is not ensured. That strategy allows one to detect such an infeasibility by means of solving a pricing problem similar to (RRMP). However, we avoid the use of the Farkas pricing applying the following strategy: (a) we introduce in the firstly solved master problem the initial pool of variables as described in Section 3.1; and (b) since the feasibility of the master problem might be lost along the branching process of our branch-and-price approach, we add an *artificial* variable $y_{(I,x_0)}$ whose local lower bound is never set to zero and with $\delta_i^{(I,x_0)}$ being a big enough value. This strategy allows us to assure that (MP_2) is satisfied by this variable, and the overall master problem is always feasible.

When the pricing problem is optimally solved, one can obtain a theoretical lower bound even if more variables must be added. The following remark explains how the result is applied to our particular problem.

Remark 1. Desrosiers & Lübecke (2005) provide theoretical lower bounds for binary programming problems that are embedded into branch-and-price approaches, in case the number of binary variables that can take value one is upper bounded. In our case, the number of y-variables in (MP) that take value one is exactly p. Thus, one can compute a lower bound for (MP) as:

$$LB = z_{\text{RRMP}} + p \min_{S,x} \bar{c}_{(S,x)}, \tag{15}$$

where z_{RRMP} is the objective value of any of the relaxed problems (RRMP) and $\bar{c}_{(S,x)}$ is the reduced cost of the variable defined by (S, x).

It is important to remark that this bound can be computed at each node of the branch-and-bound tree. The bounds are particularly useful at the root node since they may help to accelerate the optimality certification, or for large instances where the linear relaxation is not solved within the time limit.

Observe also that for adding a variable to the master problem, it suffices to find one variable y_R with negative reduced cost. This search can be performed by solving exactly the pricing problem, although that might have a high computational load. Alternatively, one could also solve heuristically the pricing problem, hoping for variables with negative reduced costs. In what follows, this approach will be called the *heuristic pricer*. The key observation is to check if a candidate facility is promising to this end.

Given the coordinates of a facility, x, we construct a set of demand points, S, compatible with the conditions of the node of the branch-and-bound tree by allocating demand points in S to x whenever the reduced cost $c_{(S,x)} - z_{(S,x)} = \gamma^* + \sum_{i\in S} e_i < 0$, where $e_i = -\alpha_i^* + \sum_{k\in I} \delta_i(x)\lambda_k \epsilon_{ik}^*$. In that case, the variable $y_{(S,x)}$ is candidate to be added to the pool of columns. Here, we detail how the heuristic pricer algorithm is implemented at the root node. For deeper nodes in the branch-and-bound tree we refer the reader to Section 3.3.

For the root node, there is a very easy procedure to solve this problem, just selecting the negative ones, i.e., we define $S = \{i \in I : e_i < 0\}$ and, in case $c_{(S,x)} - z_{(S,x)} < 0$, the variable $y_{(S,x)}$ could be added to the problem. Additionally, the region where the facility is generated can be significantly reduced, in particular to the hyperrectangle defined by demand points with negative e_i.

In both exact and heuristic pricer, we use multiple pricing, i.e., several columns are added to the pool at each iteration, if possible. In the exact pricer, we take advantage that the solver saves different solutions besides the optimal one, so it might provide us more than one column with negative reduced cost. In the heuristic pricer, we add the best variables in terms of reduced cost as long as their associated reduced costs are negative.

3.3. Branching

When the relaxed (MP) is solved, but the solution is not integer, the next step is to define an adequate branching rule to explore the searching tree. In this problem, we apply an adaptation of the Ryan and Foster branching rule (Ryan & Foster, 1981). Given a solution with fractional y-variables in a node, it might occur that

$$0 < \sum_{R=(S,x)\in\mathcal{R}: i_1, i_2 \in S} y_R < 1, \text{ for some } i_1, i_2 \in I,\ i_1 < i_2. \tag{16}$$

Provided that this happens, in order to find an integer solution, we create the following branches from the current node:

- **Left branch:** i_1 and i_2 must be served by different facilities.

$$\sum_{R=(S,x)\in\mathcal{R}: i_1, i_2 \in S} y_R = 0.$$

- **Right branch:** i_1 and i_2 must be served by the same facility.

$$\sum_{R=(S,x)\in\mathcal{R}: i_1, i_2 \in S} y_R = 1.$$

Remark 2. The above information is easily translated to the pricing problem adding one constraint to each one of the branches: 1) $w_{i_1} + w_{i_2} \leq 1$ for the left branch; and 2) $w_{i_1} = w_{i_2}$ for the right branch.

It might also happen that being some y_R fractional, $\sum_{R=(S,x)\in\mathcal{R}: i_1, i_2 \in S} y_R$ is integer for all $i_1, i_2 \in I, i_1 < i_2$. The following result allows us to use this branching rule and provides a procedure to recover a feasible solution encoded in the current solution of the node.

Theorem 2. *If $\sum_{R=(S,x)\in\mathcal{R}: i_1, i_2 \in S} y_R \in \{0, 1\}$, for all $i_1, i_2 \in I$, such that $i_1 < i_2$, then there exists an integer feasible solution of (MP) with the same objective function value.*

Proof. Let $\mathcal{X}_S$ be the set of all facilities which are part of a variable $y_{(S,x)}$ belonging to the pool of columns. We define $\mathcal{X}_S$ for all used partitions S. First, it is proven in Barnhart, Johnson, Nemhauser, Savelsbergh, & Vance (1998) that, under the hypothesis of the theorem, the following expression holds for any set S in a partition,

$$\sum_{x\in\mathcal{X}_S} y_{(S,x)} \in \{0, 1\}.$$

If $\sum_{x\in\mathcal{X}_S} y_{(S,x)} = 0$, then $y_{(S,x)} = 0$, for all $x \in \mathcal{X}_S$, because of the nonnegativity of the variables. However, if

$$\sum_{x\in\mathcal{X}_S} y_{(S,x)} = 1, \tag{17}$$

$y_{(S,x)}$ could be fractional, for some $x \in \mathcal{X}_S$.

Observe that, currently, the distance associated with demand point $i \in S$ in the problem is

$$\delta_i^S = \sum_{x\in\mathcal{X}_S} y_{(S,x)} \delta_i(x).$$

Thus, from the above we construct a new facility x^* for S.

$$x_l^* = \sum_{x\in\mathcal{X}_S} y_{(S,x)} x_l, \ \forall l = 1, \ldots, d, \tag{18}$$

so that $\delta_i(x^*) \leq \delta_i^S$, $\forall i \in S$.

Indeed, by the triangular inequality and by (17),

$$\delta_i(x^*) = \|x^* - a_i\| = \|\sum_{x\in\mathcal{X}_S} y_{(S,x)}(x - a_i)\| \leq \sum_{x\in\mathcal{X}_S} y_{(S,x)} \|x - a_i\| = \delta_i^S,$$

for all $i \in S$. The inequality being strict unless $x - a_i$, for all $x \in \mathcal{X}_S$, are collinear.

Finally, we have constructed the variable $y_{(S,x^*)} = 1$ as part of a feasible integer solution of the master problem (MP). Therefore, it ensures that either the solution is binary or there exists a binary feasible solution with the same objective function value. □

Among all the possible choices of pairs i_1, i_2 verifying (16), we propose to select the one provided by the following rule:

$$\arg \max_{\substack{i_1, i_2: \\ 0 < \sum_{R=(S,x)\in\mathcal{R}: i_1, i_2 \in S} y_R < 1}} \left\{ \theta \min \left\{ \sum_{\substack{R=(S,x)\in\mathcal{R}: \\ i_1, i_2 \in S}} y_R, 1 - \sum_{\substack{R=(S,x)\in\mathcal{R}: \\ i_1, i_2 \in S}} y_R \right\} + \frac{1-\theta}{\|a_{i_1} - a_{i_2}\|} \right\}. \tag{θ-rule}$$

This rule uses the most fractional y-solution, but also pays attention to the pairs of demand points which are close to each other in the solution, assuming they will be part of the same variable with value one at the optimal solution. It has been successfully applied in a related Discrete Ordered Median Problem (Deleplanque et al., 2020). The parameter θ is chosen in [0,1], where for $\theta = 0$, the closest demand points among the pairs with fractional sum will be selected, while for $\theta = 1$, the most fractional branching will be applied.

The above branching rule affects the heuristic pricer procedure, since not all $S \subset I$ are compatible with the branching conditions leading to a node. In case that we have to respect some branching decisions, the pricing problem gains complexity. Therefore, we develop a greedy algorithm which generates heuristic variables respecting the branching decision in the current node. This algorithm uses the information from the branching rule to build the new variable to add.

The candidate set S is built by means of a greedy algorithm similar to the one presented in Sakai, Togasaki, & Yamazaki (2003). First, we construct a graph of incompatibilities $G = (V, E)$, with V and E defined as follows: for each maximal subset of demand points $i_1 < i_2 < \cdots < i_m$, that according to the branching rule have to be assigned to the same subset, we include a vertex v_{i_1} with weight $\omega_{i_1} = \sum_{i \in \{i_1,\ldots,i_m\}} e_i$; next, for each $v_i, v_{i'} \in V$, such that i and i' cannot be assigned to the same subset at the current node, we define $\{v_i, v_{i'}\} \in E$. The subset S minimizing the reduced cost for a given x can be calculated solving the Maximum Weighted Independent Set Problem over G. The algorithm solves this problem heuristically applying the GGWMIN selection vertex rule proposed by Sakai et al. (2003).

3.4. Convergence

Due to the huge number of variables that might arise in column generation procedures, it is very important checking the possible degeneracy of the algorithm. Accelerating the convergence has been traditionally afforded by means of stabilization techniques. In recent papers, it has been shown how heuristic pricers avoid degeneracy (e.g., Benati, Ponce, Puerto, & Rodríguez-Chía, 2022; Blanco et al., 2021). Stabilization and heuristic pricers have in common that both do not add in the first iterations variables with the minimum associated reduced cost. This idea has been empirically shown to accelerate convergence (see, e.g., du Merle et al., 1999; Pessoa, Uchoa, de Aragão, & Rodrigues, 2010).

For the sake of readability, all the computational analysis is included in Section 5. There, the reader can see how our heuristic pricer needs less variables to certify optimality than the exact pricer for medium- and large-sized instances, therefore, accelerating the convergence.ad

4. Matheuristic approaches

(**MFMOMP$_\lambda$**) is an $\mathcal{NP}$-hard combinatorial optimization problem, and both the compact formulation and the proposed branch-and-price approach are limited by the number of demand points (n) and facilities (p) to be considered. Actually, as we will see in Section 5, the two exact approaches are only capable of solving, optimally, small- and medium-sized instances. In this section, we derive three different matheuristic procedures, capable to handle larger instances in reasonable CPU times. The first approach is based on using the branch-and-price scheme but solving only heuristically the pricing problem. The second is an aggregation based-approach that will also allow us to derive theoretical error bounds on the approximation. A third heuristic based on discretizing the space is proposed.

4.1. Heuristic pricer

The matheuristic procedure described here has been successfully applied in the literature. See, e.g., Albornoz & Zamora (2021); Benati et al. (2022), Deleplanque et al. (2020), and the references therein. Recall that our pricing problem is $\mathcal{NP}$-hard. In order to avoid the exact procedure for large-sized instances, where not even a single iteration could be solved exactly, we propose a matheuristic. It consists of solving each pricing problem heuristically. The inconvenience of doing that is that we do not have a theoretic lower bound during the process. Nevertheless, for instances where the time limit is reached, we are able to visit more nodes in the branch-and-bound tree which could allow us to obtain better incumbent solutions than the unfinished exact procedure.

4.2. Aggregation schemes

The second matheuristic approach that we propose is based on applying aggregation techniques to the input data (the set of demand points). This type of approaches has been successfully applied to solve large-scale continuous location problems (see Blanco & Gázquez, 2021; Blanco et al., 2021; Blanco, Puerto, & Salmerón, 2018; Current & Schilling, 1990; Daskin, Haghani, Khanal, & Malandraki, 1989; Irawan, 2016).

Let $\mathcal{A} = \{a_1, \ldots, a_n\} \subset \mathbb{R}^d$ be a set of demands points. In an aggregation procedure, the set $\mathcal{A}$ is replaced by a multiset $\mathcal{A}' = \{a'_1, \ldots, a'_n\}$, where each point a_i in $\mathcal{A}$ is assigned to a point a'_i in $\mathcal{A}'$. In order to be able to solve (**MFMOMP$_\lambda$**) for $\mathcal{A}'$, the cardinality of the different elements of $\mathcal{A}'$ is assumed to be smaller than the cardinality of $\mathcal{A}$, and then, several a_i might be assigned to the same a'_i.

Once the points in $\mathcal{A}$ are aggregated into $\mathcal{A}'$, the procedure consists of solving (**MFMOMP$_\lambda$**) for the demand points in $\mathcal{A}'$. We get a set of p optimal facilities for the aggregated problem, $\mathcal{X}' = \{x'_1, \ldots, x'_p\}$, associated with its objective value $\mathrm{OM}_\lambda(\mathcal{A}'; \mathcal{X}')$. These positions can also be evaluated in the original objective function of the problem for the demand points $\mathcal{A}$, $\mathrm{OM}_\lambda(\mathcal{A}; \mathcal{X}')$. The following result allows us to get upper bound of the error incurred when aggregating demand points.

Theorem 3. *Let $\mathcal{X}^*$ be the optimal solution of (**MFMOMP$_\lambda$**) and $\Delta = \max\limits_{i=1,\ldots,n} \|a_i - a'_i\|$. Then*

$$|\mathrm{OM}_\lambda(\mathcal{A}; \mathcal{X}^*) - \mathrm{OM}_\lambda(\mathcal{A}; \mathcal{X}')| \leq 2\Delta \sum_{i=1}^{n} \lambda_i. \tag{19}$$

Proof. By the triangular inequality and the monotonicity and sublinearity of the ordered median function we have that $\mathrm{OM}_\lambda(\mathcal{A}; \mathcal{X}) \leq \mathrm{OM}_\lambda(\mathcal{A}'; \mathcal{X}) + \mathrm{OM}_\lambda(\mathcal{A}'; \mathcal{A})$ for all $\mathcal{X} = \{x_1, \ldots, x_p\} \subset \mathbb{R}^d$. Since $\Delta \geq \|a_i - a'_i\|$ for all $i \in I$ we get that $|\mathrm{OM}_\lambda(\mathcal{A}; \mathcal{X}) - \mathrm{OM}_\lambda(\mathcal{A}'; \mathcal{X})| \leq \Delta \sum_{i \in I} \lambda_i$ for all $\mathcal{X} = \{x_1, \ldots, x_p\} \subset \mathbb{R}^d$. Applying (Geoffrion, 1977, Theorem 5), we get that $|\mathrm{OM}_\lambda(\mathcal{A}; \mathcal{X}^*) - \mathrm{OM}_\lambda(\mathcal{A}'; \mathcal{X}')| \leq \Delta \sum_{i \in I} \lambda_i$, and then:

$$|\mathrm{OM}_\lambda(\mathcal{A}; \mathcal{X}^*) - \mathrm{OM}_\lambda(\mathcal{A}; \mathcal{X}')| \leq |\mathrm{OM}_\lambda(\mathcal{A}; \mathcal{X}^*) - \mathrm{OM}_\lambda(\mathcal{A}'; \mathcal{X}')| + |\mathrm{OM}_\lambda(\mathcal{A}'; \mathcal{X}') - \mathrm{OM}_\lambda(\mathcal{A}; \mathcal{X}')| \leq 2\Delta \sum_{i \in I} \lambda_i.$$

□

There are different strategies to reduce the dimensionality by aggregating points. In our computational experiments we consider two differentiated approaches: the *k-Means Clustering* (KMEANS) and the *Pick The Farthest* (PTF). In KMEANS, we replace the original points by the centroids. Alternatively, in PTF, an initial random demand point from $\mathcal{A}$ is chosen and the rest are selected as the farthest demand point from the last one chosen, until a predefined number of points is reached (Daskin et al., 1989).

4.3. Discretization

We propose a third heuristic algorithm, based on an adaptation of similar heuristics applied to a different location problem

Table 2
Average number of pricer iterations, variables and time using the combined heuristic and exact pricers or only using the exact pricer.

n	Heurvar	Iterations		Vars	Time
		Exact	Total		
20	FALSE	13	13	2189	64.92
	TRUE	4	23	2219	18.02
30	FALSE	15	15	2827	1034.97
	TRUE	3	60	2856	191.84
40	FALSE	50	50	4713	9086.33
	TRUE	13	136	4511	2229.21

(see Gamal & Salhi, 2003; Hansen, Mladenović, & Taillard, 1998), which consists of solving a discrete version of our problem, also known as the Discrete Ordered Median Problem (DOMP for short) (Nickel & Puerto, 2005). In this matheuristic, the potential facilities are chosen among the demand points to solve the DOMP with the solution methods developed in Deleplanque et al. (2020).

This approach produces suboptimal solutions since the feasible domain of the DOMP is a discrete set contained in the solution space of (**MFMOMP**$_\lambda$). Discretizing the continuous space is a heuristic technique that has been previously exploited in the literature for the Uncapacitated Multisource Weber Problem (see, e.g., Brimberg, Drezner, Mladenovi, & Salhi, 2014; Brimberg, Hansen, Mladenović, & Taillard, 2000; Drezner, Brimberg, Mladenović, & Salhi, 2016; Gamal & Salhi, 2003; Hansen et al., 1998). Here, we adapt those algorithms to (**MFMOMP**$_\lambda$). Firstly, we reduce the solution space to the demand points. Later, the resulting DOMP is solved heuristically by a GRASP algorithm to have a good upper bound and by a branch-price-and-cut procedure to obtain a lower bound and improve the upper bound. Note that the obtained upper bound is valid for the original (**MFMOMP**$_\lambda$), whereas the lower bound is not. The reader can see in Section 5.2 that, for large-sized instances, this methodology provides rather good results.

5. Computational study

In order to compare the performance of our branch-and-price and our matheuristic approaches, we report the results of our computational experiments. We consider different sets of instances used in the location literature with size ranging from 20 to 654 demand points in the plane. In all of them, the number of facilities to be located, p, ranges in $\{2, 5, 10\}$ and we solve the instances for the λ-vectors in Table 1, {W, C, K, D, S, A}. We set $k = \frac{n}{2}$ for the k-center and k-entdian, and $\alpha = 0.9$ for the centdian and k-entdian.

For the sake of readability, we restrict the computational study of this document to ℓ_1-norm based distances. However, the reader can find extensive computational results for other norms in Appendix A and Appendix B.

The models were coded in C and solved with SCIP v.7.0.2 (Gamrath et al., 2020) using as optimization solver SoPlex 5.0.2 in a Mac OS Catalina with a Core Intel Xeon W clocked at 3.2 GHz and 96 GB of RAM memory.

5.1. Computational performance of the branch-and-price procedure

In this section we report the results for our branch-and-price approach based on the classical dataset provided by Eilon, Watson-Gandy, & Christofides (1971). From this dataset, we randomly generate five instances with sizes $n \in \{20, 30, 40, 45\}$ and we also consider the entire complete original instance with $n = 50$. Together with the number of facilities p and the different ordered weighted median functions (`type`), a total of 378 instances has been considered.

Firstly, concerning convergence (Section 3.4), each line in Table 2 shows the average results of 45 instances, five for each type of ordered median objective function to be minimized {W, D, S} and $p \in \{2, 5, 10\}$, solved to optimality. The results has been split by size (n) and by `Heurvar`: `FALSE` when only the exact pricer is used; `TRUE` if the heuristic pricer is used and the exact pricer is called when it does not provide new columns to add. The reader can see a significant reduction of the CPU time (`Time`) caused by a decrease of the number of calls to the exact pricer (`Exact`) even though the number of total iterations (`Total`) increases. Additionally, a second effect is that the necessary number of variables to certify optimality (`Vars`) is slightly less when the heuristic is applied for $n = 40$. Hence, we will use the heuristic pricer for the rest of the experiments.

Secondly, we have tuned the values of θ for the branching rule (*theta – rule*) for each of the objective functions (different values for the λ-vector) based in our computational experience. In Table 3, we show the average gap at termination of the above-mentioned 378 instances when we apply our branch-and-price approach fixing a time limit of 2 h.

Therefore we set $\theta = 0$ for the center problem (C), $\theta = 0.5$ for the k-center problem (K), and $\theta = 1$ for the p-median (W), centdian (D), k-entdian (S), and ascendant problems (A). Recall that when we use $\theta = 0$, we are selecting a pure distance branching rule. In contrast, when $\theta = 1$, we select the most fractional variable. On the other hand, when $\theta = 0.5$, we use a hybrid selection between the two extremes of the (*theta – rule*). In the following, the above fixed parameters will be used in the computational experiments for exact and matheurisitic methods.

The average results obtained for the Eilon et al. (1971) instances, with a CPU time limit of 2 h, are shown in Table 4. There, for each combination of n (size of the instance), p (number of facilities to be located) and type (ordered median objective function to be minimized), we provide the average results for ℓ_1-norm with a comparison between the compact formulation (C) (Compact) and the branch-and-price approach (B&P). The table is organized as follows: the first column gives the CPU time in seconds needed to solve the problem (`Time`) and within parentheses the number of unsolved instances (`#Unsolved`), i.e., those for which the lower and upper bound do not coincide within the time limit; the second column shows the gap at the root node; the third one gives the gap at termination, i.e., the remaning MIP gap in percentage (`GAP(%)`) when the time limit is reached, 0.00 otherwise; in the fourth column we show the number of variables (`Vars`) needed to solve the problem; in the fifth column we show the number

Table 3
`GAP(%)` for ℓ_1-norm, Eilon et al. (1971) instances.

type	$\theta = 0.0$	$\theta = 0.1$	$\theta = 0.3$	$\theta = 0.5$	$\theta = 0.7$	$\theta = 0.9$	$\theta = 1.0$
W	0.04	0.04	0.04	0.04	0.04	0.04	**0.02**
C	**27.94**	28.34	28.29	28.47	28.64	28.74	28.19
K	12.83	12.63	12.80	**12.46**	12.73	13.15	12.88
D	0.09	0.07	0.09	0.09	0.09	0.09	**0.02**
S	0.11	0.14	0.14	0.14	0.14	0.13	**0.10**
A	7.73	7.66	7.69	7.71	7.64	7.73	**7.33**

Table 4
Results for Eilon et al. (1971) instances for ℓ_1-norm.

n	type	p	Time (#Unsolved)				GAProot(%)		GAP(%)		Vars		Nodes		Memory (MB)	
			Compact		B&P		Compact	B&P	Compact	B&P	Compact	B&P	Compact	B&P	Compact	B&P
20	W	2	**1.59**	**(0)**	22.90	(0)	93.92	**0.00**	0.00	0.00	**224**	2131	9518	**1**	**4**	103
		5	1588.99	(1)	**8.34**	**(0)**	100.00	**0.00**	3.38	**0.00**	**470**	2408	10967305	**1**	1278	**49**
		10	–	(5)	**3.95**	**(0)**	100.00	**0.46**	43.84	**0.00**	**880**	2127	19785215	**2**	3425	**28**
	C	2	**0.06**	**(0)**	237.96	(4)	78.92	**22.59**	**0.00**	10.78	**224**	97635	**7**	4652	**4**	2239
		5	**12.58**	**(0)**	–	(5)	100.00	**29.46**	**0.00**	17.16	**470**	15251	40379	**18660**	**12**	464
		10	**511.69**	**(2)**	1831.83	(4)	100.00	**37.64**	**7.59**	20.28	**880**	4243	7928195	**21617**	725	**160**
	K	2	**0.35**	**(0)**	1412.69	(1)	91.43	**7.55**	**0.00**	1.42	**224**	37917	**630**	670	**3**	953
		5	**243.88**	**(0)**	404.99	(3)	100.00	**15.40**	**0.00**	3.85	**470**	9363	657827	**6642**	**77**	279
		10	32.22	(4)	**3156.63**	**(2)**	100.00	**18.53**	36.95	**3.26**	**880**	4071	12150962	**9244**	2265	**111**
	D	2	**2.18**	**(0)**	30.36	(0)	93.78	**0.03**	**0.00**	0.00	**224**	2135	9222	**1**	**5**	108
		5	1535.82	(1)	**12.18**	**(0)**	100.00	**0.00**	6.69	**0.00**	**470**	2401	8972062	**1**	1225	**49**
		10	5030.79	(4)	**6.88**	**(0)**	100.00	**0.46**	48.19	**0.00**	**880**	2127	15660031	**4**	2798	**28**
	S	2	**2.24**	**(0)**	54.23	(0)	93.77	**0.16**	0.00	0.00	**224**	2119	7677	**1**	**5**	106
		5	1238.87	(1)	**15.75**	**(0)**	100.00	**0.06**	4.40	**0.00**	**470**	2401	8141244	**2**	745	**50**
		10	–	(5)	**7.61**	**(0)**	100.00	**0.53**	50.12	**0.00**	**880**	2126	16072018	**5**	2835	**28**
	A	2	**0.85**	**(0)**	783.95	(1)	91.63	**4.45**	**0.00**	0.35	**224**	16973	1340	**400**	**4**	738
		5	**411.21**	**(0)**	2304.77	(0)	100.00	**10.18**	**0.00**	0.00	**470**	7405	878975	**1697**	**126**	222
		10	60.27	(4)	**883.79**	**(1)**	100.00	**17.10**	31.87	**1.73**	**880**	3288	10637608	**2721**	1723	**79**
30	W	2	**139.91**	**(0)**	526.92	(0)	93.86	**0.00**	0.00	0.00	**334**	3142	787145	**1**	**38**	260
		5	–	(5)	**64.66**	**(0)**	100.00	**0.00**	52.05	**0.00**	**700**	2963	17382888	**1**	8647	**109**
		10	–	(5)	**19.51**	**(0)**	100.00	**0.00**	76.26	**0.00**	**1310**	2472	12250097	**1**	4692	**55**
	C	2	**0.11**	**(0)**	39.44	(4)	79.19	**21.41**	**0.00**	15.46	**334**	125429	**66**	931	**8**	1443
		5	**30.64**	**(0)**	1564.58	(4)	100.00	**31.68**	**0.00**	22.73	**700**	30216	69019	**2817**	**19**	389
		10	**4212.55**	**(3)**	–	(5)	100.00	**34.18**	**16.67**	27.51	**1310**	12928	9619002	**6027**	1823	**190**
	K	2	**4.44**	**(0)**	409.69	(4)	90.88	**8.65**	**0.00**	7.58	**334**	45846	8511	**147**	**10**	1696
		5	2956.65	(4)	**5199.43**	**(3)**	100.00	**12.01**	17.79	**5.82**	**700**	18893	12169516	**815**	2570	**534**
		10	–	(5)	**2740.67**	**(4)**	100.00	**18.84**	69.60	**12.31**	**1310**	7416	9299590	**2992**	3105	**187**
	D	2	**201.28**	**(0)**	454.39	(0)	93.77	**0.00**	0.00	0.00	**334**	3087	757445	**1**	**49**	258
		5	–	(5)	**65.46**	**(0)**	100.00	**0.00**	57.16	**0.00**	**700**	2957	9914066	**1**	7439	**111**
		10	–	(5)	**21.25**	**(0)**	100.00	**0.00**	79.34	**0.00**	**1310**	2464	10108803	**1**	4631	**55**
	S	2	**203.04**	**(0)**	370.63	(0)	93.68	**0.00**	**0.00**	0.00	**334**	3184	566382	**1**	**41**	263
		5	–	(5)	**160.85**	**(0)**	100.00	**0.03**	56.47	**0.00**	**700**	2963	9283122	**2**	7054	**112**
		10	–	(5)	**42.86**	**(0)**	100.00	**0.09**	79.91	**0.00**	**1310**	2469	9530286	**3**	4686	**56**
	A	2	**21.89**	**(0)**	3640.13	(2)	91.15	**4.46**	**0.00**	3.26	**334**	12721	26764	**41**	**12**	845
		5	5403.72	(4)	**2750.01**	**(3)**	100.00	**7.76**	28.99	**2.60**	**700**	8615	8044660	**188**	2288	**357**
		10	–	(5)	**804.71**	**(4)**	100.00	**13.38**	70.51	**6.64**	**1310**	5529	7159232	**1465**	2364	**165**
40	W	2	4028.70	(4)	**1675.34**	**(0)**	93.79	**0.01**	12.34	**0.00**	**444**	5211	26828725	**1**	2515	**645**
		5	–	(5)	**1647.86**	**(0)**	100.00	**0.02**	67.11	**0.00**	**930**	4028	12240990	**3**	10977	**229**
		10	–	(5)	**348.57**	**(0)**	100.00	**0.09**	81.57	**0.00**	**1740**	4001	7841923	**2**	4267	**125**
	C	2	**0.25**	**(0)**	–	(5)	75.52	**30.52**	**0.00**	29.73	**444**	136451	**237**	259	**15**	1541
		5	**116.02**	**(0)**	–	(5)	100.00	**42.30**	**0.00**	41.65	**930**	27041	195892	**158**	**42**	224
		10	**3022.45**	**(4)**	–	(5)	100.00	**36.47**	**31.47**	33.88	**1740**	12733	7126207	**667**	2024	**110**
	K	2	**58.78**	**(0)**	–	(5)	90.67	**14.52**	**0.00**	14.52	**444**	14164	93918	**11**	**27**	897
		5	–	(5)	–	(5)	100.00	**21.45**	56.58	**21.44**	**930**	10132	6803627	**28**	5632	**360**
		10	–	(5)	–	(5)	100.00	**19.04**	75.08	**17.71**	**1740**	8823	5436226	**280**	2606	**198**
	D	2	5908.68	(4)	**436.48**	**(1)**	93.67	**0.02**	15.22	**0.01**	**444**	5669	16542227	**2**	3164	**709**
		5	–	(5)	**855.62**	**(1)**	100.00	**0.11**	68.93	**0.08**	**930**	4094	7984937	**2**	10233	**233**
		10	–	(5)	**331.54**	**(0)**	100.00	**0.07**	83.85	**0.00**	**1740**	4004	5704188	**2**	4413	**126**
	S	2	4977.44	(4)	**429.96**	**(1)**	93.60	**0.47**	14.33	**0.47**	**444**	5195	12853124	**1**	2430	**657**
		5	–	(5)	**2159.56**	**(1)**	100.00	**0.14**	70.18	**0.02**	**930**	4082	7715457	**4**	9805	**233**
		10	–	(5)	**615.35**	**(0)**	100.00	**0.17**	84.62	**0.00**	**1740**	3999	5409994	**4**	4687	**126**
	A	2	**533.79**	**(0)**	–	(5)	90.85	**8.30**	**0.00**	8.19	**444**	6506	455652	**3**	**48**	769
		5	–	(5)	–	(5)	100.00	**14.76**	58.65	**14.17**	**930**	5538	3684557	**10**	3409	**331**
		10	–	(5)	–	(5)	100.00	**12.35**	74.52	**10.21**	**1740**	6285	4403838	**161**	2000	**214**
45	W	2	–	(5)	**483.59**	**(1)**	94.05	**0.04**	27.06	**0.02**	**499**	7219	24989615	**2**	5854	**1085**
		5	–	(5)	**1745.55**	**(2)**	100.00	**0.32**	71.65	**0.27**	**1045**	4855	11473640	**4**	11171	**374**
		10	–	(5)	**635.43**	**(0)**	100.00	**0.03**	83.54	**0.00**	**1955**	4239	5717627	**1**	3767	**168**
	C	2	**0.46**	**(0)**	–	(5)	74.99	**39.01**	**0.00**	38.99	**499**	109398	628	**104**	**17**	1364
		5	**144.75**	**(0)**	–	(5)	100.00	**40.69**	**0.00**	40.62	**1045**	20483	215522	**31**	**44**	176
		10	–	(5)	–	(5)	100.00	**32.80**	37.16	**31.54**	**1955**	14204	6469050	**219**	1915	**110**
	K	2	**342.18**	**(0)**	–	(5)	91.23	**16.93**	**0.00**	16.93	**499**	10490	497310	**4**	**59**	845
		5	–	(5)	–	(5)	100.00	**22.98**	64.55	**22.98**	**1045**	6631	5434589	**11**	6520	**295**
		10	–	(5)	–	(5)	100.00	**16.68**	77.74	**16.48**	**1955**	8738	4555667	**78**	2681	**220**
	D	2	–	(5)	**364.11**	**(1)**	93.96	**0.02**	29.64	**0.02**	**499**	6473	14042725	**1**	6338	**973**
		5	–	(5)	**1744.42**	**(2)**	100.00	**0.17**	73.98	**0.11**	**1045**	4731	7322624	**3**	10301	**365**
		10	–	(5)	**667.13**	**(0)**	100.00	**0.02**	84.73	**0.00**	**1955**	4231	5228591	**1**	4875	**169**
	S	2	–	(5)	**623.35**	**(2)**	93.87	**0.10**	28.98	**0.09**	**499**	7260	10776521	**2**	4776	**1093**
		5	–	(5)	–	(5)	100.00	**0.72**	76.35	**0.62**	**1045**	4899	7356115	**7**	10281	**378**
		10	–	(5)	**1848.85**	**(0)**	100.00	**0.18**	85.38	**0.00**	**1955**	4258	4378226	**4**	4283	**168**
	A	2	**4681.25**	**(0)**	–	(5)	91.28	**11.43**	**0.00**	11.43	**499**	6849	3057137	**2**	**121**	975
		5	–	(5)	–	(5)	100.00	**17.39**	65.13	**17.17**	**1045**	5476	2139768	**4**	2517	**415**
		10	–	(5)	–	(5)	100.00	**10.28**	76.58	**8.88**	**1955**	6105	3577556	**56**	1976	**244**

(continued on next page)

Table 4 (*continued*)

n	type	p	Time (#Unsolved)				GAProot(%)		GAP(%)		Vars		Nodes		Memory (MB)	
			Compact		B&P		Compact	B&P	Compact	B&P	Compact	B&P	Compact	B&P	Compact	B&P
50	W	2	–	(1)	**331.87**	**(0)**	94.13	**0.00**	34.44	**0.00**	**554**	8094	24416531	**1**	6585	**1464**
		5	–	(1)	**410.87**	**(0)**	100.00	**0.00**	76.08	**0.00**	**1160**	5292	9438723	**1**	9646	**466**
		10	–	(1)	**1005.02**	**(0)**	100.00	**0.00**	84.68	**0.00**	**2170**	4914	5017512	**1**	3878	**225**
	C	2	**0.34**	**(0)**	–	(1)	75.02	**30.33**	**0.00**	30.31	**554**	80356	367	**37**	**22**	1064
		5	**379.06**	**(0)**	–	(1)	100.00	**41.37**	**0.00**	41.37	**1160**	15314	443313	**14**	**137**	143
		10	–	(1)	–	(1)	100.00	**37.49**	46.67	**36.94**	**2170**	12538	5837062	**213**	2937	**98**
	K	2	**1135.78**	**(0)**	–	(1)	91.28	**15.46**	**0.00**	15.46	**554**	10042	1334361	**3**	**84**	926
		5	–	(1)	–	(1)	100.00	**24.86**	68.28	**24.86**	**1160**	6541	4368607	**4**	6095	**324**
		10	–	(1)	–	(1)	100.00	**23.05**	79.98	**23.04**	**2170**	7164	2448072	**15**	1851	**205**
	D	2	–	(1)	**328.07**	**(0)**	94.07	**0.00**	37.30	**0.00**	**554**	8035	12235346	**1**	7096	**1485**
		5	–	(1)	**4430.48**	**(0)**	100.00	**0.08**	78.70	**0.00**	**1160**	5415	5502769	**5**	8618	**485**
		10	–	(1)	**1408.05**	**(0)**	100.00	**0.00**	86.81	**0.00**	**2170**	4914	3617149	**2**	4412	**219**
	S	2	–	(1)	**516.57**	**(0)**	93.95	**0.00**	37.29	**0.00**	**554**	7579	8797114	**1**	4912	**1387**
		5	–	(1)	–	(1)	100.00	**0.57**	79.68	**0.57**	**1160**	5704	5750451	**5**	9004	**508**
		10	–	(1)	**3413.97**	**(0)**	100.00	**0.00**	87.36	**0.00**	**2170**	4962	3126980	**5**	5175	**230**
	A	2	–	(1)	–	(1)	91.49	**10.36**	19.20	**10.36**	**554**	8056	3163853	**2**	**1161**	1369
		5	–	(1)	–	(1)	100.00	**19.06**	67.25	**18.75**	**1160**	5872	2177800	**4**	3290	**542**
		10	–	(1)	–	(1)	100.00	**10.17**	78.48	**9.36**	**2170**	5764	2718050	**16**	1862	**268**
Total Average:			645.10	(229)	**772.80**	**(171)**	96.71	**10.19**	35.81	**7.98**	**897**	13958	6581088	**1111**	3014	**427**

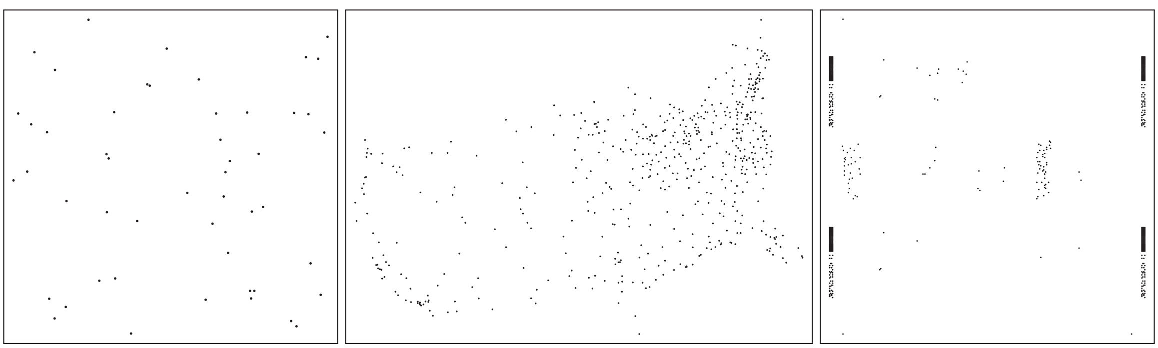
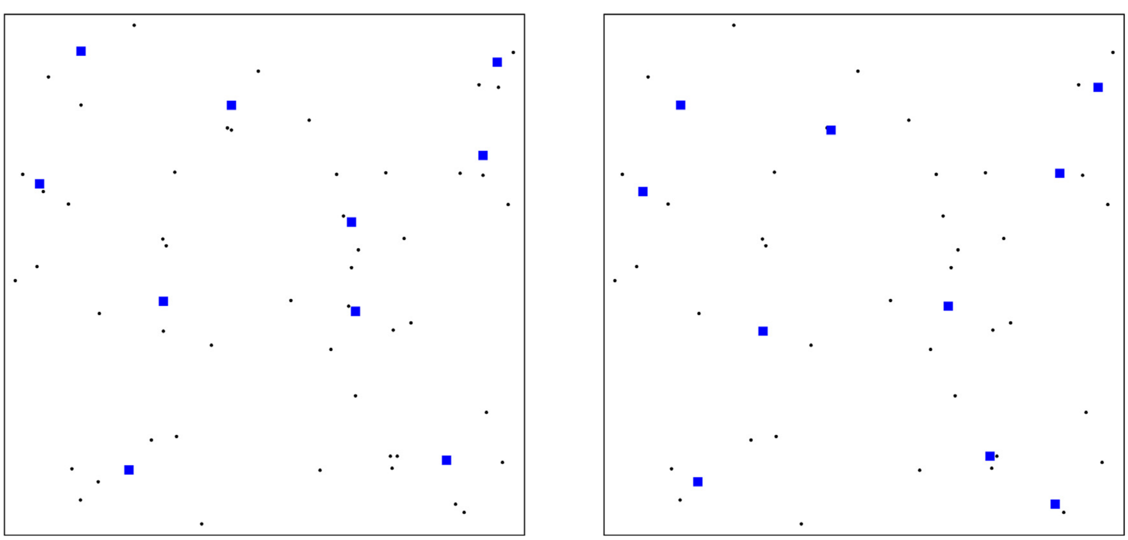


Fig. 1. Demand points of each of the instances that we use in our computational study: $n = 50$, $n = 532$, and $n = 654$ (from left to right).

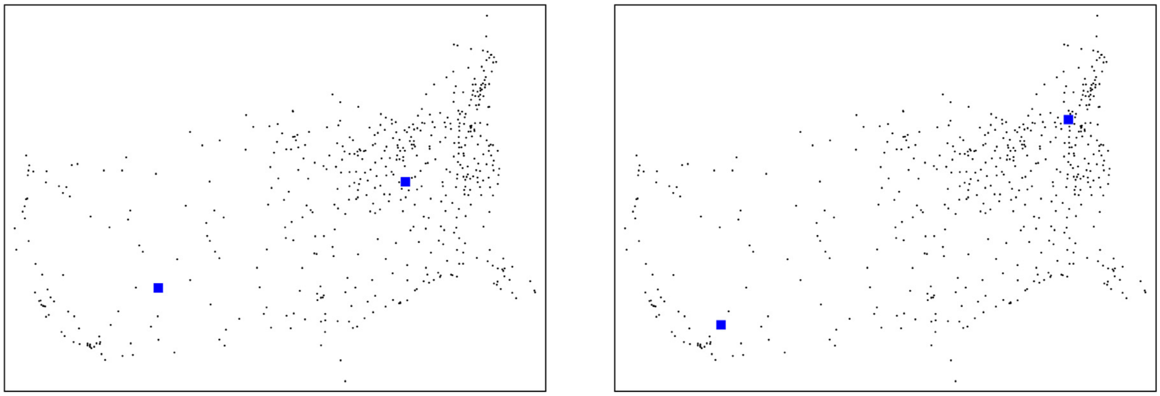


(a) B&P (b) Discretization

Fig. 2. Solutions for $n = 50$ (Eilon et al., 1971), S, $p = 10$, and ℓ_1-norm.

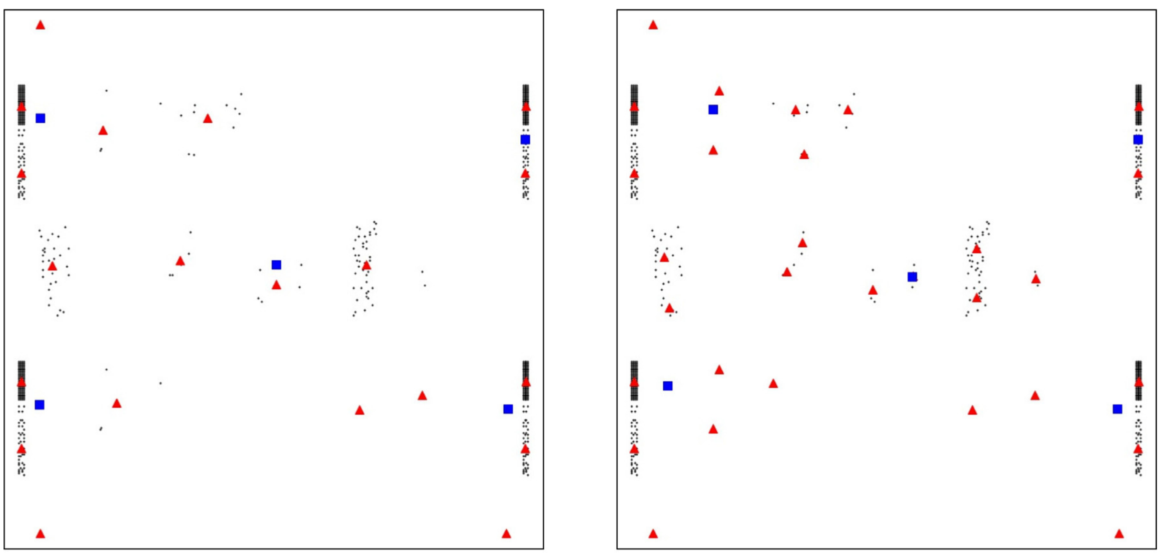
of nodes (Nodes) explored in the branch-and-bound tree; and, in the last one, the RAM memory (Memory (MB)) in Megabytes required during the execution process is reported. Within each column, we highlight in bold the best result between the two formulations, namely Compact or B&P.

The branch-and-price algorithm is able to solve optimally 58 instances more than the compact formulation. However, for some instances (mainly Center and k-center problems or when $p = 2$) the solved instances with the compact formulation need less CPU time. Thus, the first conclusion could be that when p increases de-



(a) Matheur (b) Discretization

Fig. 3. Solutions for $n = 532$ (Padberg & Rinaldi, 1987), C, $p = 2$, and ℓ_1-norm.



(a) KMEANS-20 (b) KMEANS-30

Fig. 4. Solutions for $n = 654$ (Reinelt, 1992), W, $p = 5$, and ℓ_1-norm.

composition techniques become more important because the number of variables is not so dependant of this parameter. The second conclusion from the results is that the branch-and-price is a very powerful tool when the gap at the root node is close to zero which does not happen when a big percentage of the positions of the λ-vector are zeros. Concerning the memory used by the tested formulations, the compact formulation needs bigger branch-and-bound trees to deal with fractional solutions whereas that branch-and-price uses more variables.

Since the average gap at termination for the branch-and-price algorithm is much smaller than the one obtained by the compact formulation (7.98% against 35.81%), we will use decomposition-based algorithms to study medium- and large-sized instances.

5.2. Computational performance of the matheuristics

In this section, we show the performance of our matheuristic procedures. Firstly, we will test them for $n = 50$ (Eilon et al., 1971) where the solutions can be compared with the theoretic bounds provided by the exact method. Secondly, we will compare them using larger instances. Specifically, we use two instances from the TSP library (Reinelt, 1991), which is a well known repository of complex instances for the the TSP and related problems (Goldengorin & Krushinsky, 2011): `att532.tsp` and `p654.tsp`. They contain the coordinates of 532 cities of the continental US (Padberg & Rinaldi, 1987) and 654 points from a drilling problem example (Reinelt, 1992), respectively. The spatial distribution of the demand points for each of the instances that we use is shown in Fig. 1, where one can see the different nature of the datasets that we test.

Tables 5, 6, and 7 present a similar layout. The instances are solved with 18 different configurations of ordered weighted median functions and number of open facilities. Each of these 18 problems has been solved by means of the following strategies: branch-and-price procedure (B&P); the heuristic used to generate initial columns (InitialHeur); the decomposition-based heuristic (Matheur); the aggregation-based approaches described in Section 4.2 (KMEANS-20, KMEANS-30, PTF-20, PTF-30) for $|\mathcal{A}'| = \{20, 30\}$; and the discretization strategy explained in Section 4.3 (Discretization). The reported results are the CPU time and:

1. the gap ($\texttt{GAP}_{LB}(\%)$) which is calculated with respect to the lower bound of the branch-and-price algorithm when the time limit is reached. Thereby, we have a theoretic gap knowing exactly the room for improvement of our heuristics;

Table 5
Heuristic results for instances of $n = 50$, Eilon et al. (1971).

type	p	B&P		InitialHeur		Matheur		KMEANS-20		KMEANS-30		PTF-20		PTF-30		Discretization	
		Time	GAP_{LB}(%)	Time	GAP_{LB}(%)	Time	GAP_{LB}(%)	Time	GAP_{LB}(%)	Time	GAP_{LB}(%)	Time	GAP_{LB}(%)	Time	GAP_{LB}(%)	Time	GAP_{LB}(%)
W	2	331.87	**0.00**	0.00	1.44	134.54	**0.00**	74.60	3.10	5204.85	2.80	35.14	7.58	7200.18	2.50	12.94	1.23
	5	410.87	**0.00**	0.00	11.53	9.51	**0.00**	21.47	12.55	398.93	11.25	16.38	17.88	187.03	7.90	2680.48	1.17
	10	1005.02	**0.00**	0.00	21.57	2.84	**0.00**	4.23	17.16	59.08	8.79	16.11	23.45	57.35	9.88	4.95	3.11
C	2	7200.64	43.49	0.00	43.49	6.61	39.68	7200.16	47.05	7200.52	32.36	7200.21	**31.46**	7200.47	41.39	7205.96	43.11
	5	7200.19	70.56	0.00	73.45	7200.00	**46.66**	7200.09	82.76	7200.10	76.52	7200.09	94.74	7200.09	72.70	7203.66	70.86
	10	7200.19	58.58	0.00	101.43	7200.18	**28.93**	7200.16	104.98	7200.19	73.44	7200.16	154.72	7200.18	70.54	7203.84	87.77
K	2	7200.13	18.28	0.00	21.40	227.64	19.79	3589.02	21.19	7200.40	22.00	7200.35	20.05	7200.57	18.81	7203.55	**16.83**
	5	7200.34	33.09	0.00	33.09	392.07	**22.37**	7200.10	26.84	7200.15	36.08	7200.10	42.99	7200.13	30.43	7203.81	26.59
	10	7200.28	29.94	0.00	41.73	864.26	**11.79**	7200.17	35.38	7200.20	19.91	7200.17	41.97	7200.19	15.93	7202.90	38.36
D	2	328.07	**0.00**	0.00	1.45	130.00	**0.00**	42.82	3.76	5213.35	2.82	18.77	7.70	5105.83	2.47	5730.24	1.25
	5	4430.48	**0.00**	0.00	11.43	10.73	**0.00**	7.87	12.53	162.45	10.95	17.50	16.36	191.88	8.13	4714.13	1.12
	10	1408.05	**0.00**	0.00	21.38	3.06	0.43	4.76	17.00	50.22	10.18	22.26	23.27	135.57	9.07	811.58	3.03
S	2	516.57	**0.00**	0.00	1.62	111.73	**0.00**	25.43	2.97	7200.20	2.59	35.77	7.34	7200.27	2.33	7203.79	1.24
	5	7200.43	**0.57**	0.00	12.28	14.96	0.64	36.68	12.60	498.07	11.81	41.88	18.33	746.57	8.26	7203.80	1.77
	10	3413.97	**0.00**	0.00	21.24	3.22	0.66	10.99	16.86	29.98	7.98	50.52	23.29	114.10	8.97	2756.51	4.06
A	2	7200.39	11.56	0.00	11.56	155.18	10.91	2611.77	**9.05**	7200.15	10.62	7200.41	13.49	7200.09	13.69	7203.98	10.27
	5	7200.40	23.08	0.00	23.61	325.35	**14.00**	7200.10	19.96	7200.16	18.63	7200.11	26.03	7200.13	22.18	7204.38	16.41
	10	7200.24	10.33	0.00	31.65	129.77	**6.29**	7200.17	17.48	7200.21	16.63	7200.17	33.49	7200.21	21.82	7204.03	22.39
Total Average:		4658.23	16.64	0.00	26.97	940.09	**11.23**	3157.25	25.73	4645.51	20.85	3614.23	33.56	4763.38	20.39	5330.81	19.48

Table 6
Heuristic results for instances of $n = 532$, Padberg & Rinaldi (1987).

type	p	B&P		InitialHeur		Matheur		KMEANS-20		KMEANS-30		PTF-20		PTF-30		Discretization	
		Time	GAP_{Best}(%)	Time	GAP_{Best}(%)	Time	GAP_{Best}(%)	Time	GAP_{Best}(%)	Time	GAP_{Best}(%)	Time	GAP_{Best}(%)	Time	GAP_{Best}(%)	Time	GAP_{Best}(%)
W	2	86427.79	3.78	0.04	3.78	86427.79	3.78	214.12	13.36	607.50	6.06	29.81	3.09	807.38	12.08	86465.01	**0.00**
	5	86430.70	3.16	0.04	3.16	86430.70	3.16	15.25	27.80	115.15	21.13	38.83	15.75	296.68	18.38	86417.45	**0.00**
	10	86431.06	1.62	0.04	2.59	86431.06	1.62	7.16	27.35	310.76	35.82	7.42	40.26	51.25	36.48	86423.45	**0.00**
C	2	86406.87	0.08	0.04	0.08	38146.48	**0.00**	86403.24	4.55	86400.09	12.48	86428.73	9.69	86401.42	8.36	86474.06	1.70
	5	86407.73	6.65	0.04	6.65	12456.56	6.65	86400.07	18.75	86400.10	31.08	86400.07	17.39	86400.10	3.09	86427.62	**0.00**
	10	86407.25	28.72	0.04	28.72	86405.98	28.72	15064.77	33.06	86400.16	**0.00**	14624.77	59.63	86400.16	18.58	86460.43	7.32
K	2	86419.13	2.23	0.04	2.23	86419.13	2.23	1601.48	7.51	86406.88	9.01	3974.78	7.09	86403.41	8.29	86445.75	**0.00**
	5	86418.39	2.64	0.04	2.64	86418.39	2.64	86400.08	16.05	86400.12	15.28	86400.09	21.98	86402.98	21.14	86462.28	**0.00**
	10	86418.91	4.83	0.04	4.83	86418.91	4.83	13699.21	83.01	86400.17	9.35	86400.13	37.08	86400.18	31.92	86446.91	**0.00**
D	2	86428.55	3.78	0.04	3.78	86428.55	3.78	234.79	12.45	809.15	5.78	17.49	2.85	206.49	12.99	86448.31	**0.00**
	5	86427.84	3.16	0.04	3.16	86427.84	3.16	35.12	26.28	119.24	21.67	75.10	14.79	241.96	15.48	86466.61	**0.00**
	10	86427.84	1.61	0.04	2.59	86427.84	1.61	8.04	24.60	656.42	23.75	14.80	45.41	37.98	35.53	86464.28	**0.00**
S	2	86428.95	3.71	0.04	3.71	86428.95	3.71	518.28	13.83	7700.72	5.74	49.64	2.85	408.46	12.02	86451.83	**0.00**
	5	86428.30	2.95	0.04	2.95	86428.30	2.95	29.61	26.42	1267.17	20.84	87.31	15.90	661.87	13.14	86427.65	**0.00**
	10	86428.12	2.63	0.04	2.63	86428.12	2.63	19.91	30.84	1681.79	38.23	40.72	44.90	76.69	35.11	86468.68	**0.00**
A	2	86428.30	2.79	0.04	2.79	86428.30	2.79	798.53	4.91	86402.27	2.72	1291.04	7.08	74404.87	11.16	86434.48	**0.00**
	5	86426.24	2.28	0.04	2.28	86426.24	2.28	66616.11	15.03	86400.12	18.50	20039.72	26.95	86400.25	16.98	86419.49	**0.00**
	10	86426.11	4.03	0.04	4.03	86426.11	4.03	17972.81	10.69	86400.17	16.75	86400.13	31.32	86400.18	29.85	86482.04	**0.00**
Total Average:		86423.23	4.48	0.04	4.59	79633.63	4.48	20891.03	22.03	43937.66	16.34	26240.03	22.45	42689.02	18.92	86449.24	**0.50**

Table 7
Heuristic results for instances of $n = 654$, Reinelt (1992).

type	p	B&P Time	B&P $\text{GAP}_{Best}(\%)$	InitialHeur Time	InitialHeur $\text{GAP}_{Best}(\%)$	Matheur Time	Matheur $\text{GAP}_{Best}(\%)$	KMEANS-20 Time	KMEANS-20 $\text{GAP}_{Best}(\%)$	KMEANS-30 Time	KMEANS-30 $\text{GAP}_{Best}(\%)$	PTF-20 Time	PTF-20 $\text{GAP}_{Best}(\%)$	PTF-30 Time	PTF-30 $\text{GAP}_{Best}(\%)$	Discretization Time	Discretization $\text{GAP}_{Best}(\%)$
W	2	86441.18	18.56	0.06	18.56	86441.18	18.56	13.31	8.04	755.99	40.81	32.59	19.19	697.98	40.02	86426.03	**0.00**
	5	86444.11	13.10	0.06	13.68	86444.11	13.10	6.56	45.35	97.79	79.07	7.46	68.54	74.85	76.34	86408.63	**0.00**
	10	86439.34	15.13	0.06	32.85	86439.34	15.13	2.64	101.43	31.49	112.39	1.58	157.47	68.11	133.94	86437.47	**0.00**
C	2	86407.88	4.16	0.07	4.16	86407.32	**0.00**	420.97	8.91	86400.83	3.43	5765.05	**0.00**	86401.06	**0.00**	86436.12	2.27
	5	86407.52	6.54	0.06	8.92	51003.35	4.47	86424.42	32.61	86400.09	**0.00**	86400.15	20.98	86400.09	1.64	86420.51	12.42
	10	86408.46	20.39	0.06	20.39	83212.74	3.23	42425.62	31.02	86400.17	1.36	86400.16	63.01	86400.13	2.00	86419.39	**0.00**
K	2	86424.57	5.30	0.06	5.30	86424.57	5.30	274.39	4.34	40549.49	14.86	365.79	13.07	15942.12	10.65	86404.35	**0.00**
	5	86424.89	11.27	0.06	11.27	86424.89	11.27	854.51	38.53	86400.13	64.01	2868.28	49.15	86400.26	42.60	86441.63	**0.00**
	10	86425.31	32.55	0.06	32.55	86425.31	32.55	6695.50	149.41	86400.19	105.65	13189.85	162.33	86400.14	126.61	86436.97	**0.00**
D	2	86440.98	18.55	0.07	18.55	86440.98	18.55	29.22	6.78	737.65	40.14	23.55	18.85	669.43	39.84	86429.90	**0.00**
	5	86440.05	13.09	0.06	13.67	86440.05	13.09	15.08	46.33	123.71	81.13	6.47	71.68	96.06	82.16	86444.51	**0.00**
	10	86439.96	15.12	0.06	32.79	86439.96	15.12	7.19	99.40	21.26	101.71	3.05	156.24	106.33	133.98	86428.90	**0.00**
S	2	86440.27	17.23	0.07	17.23	86440.27	17.23	19.02	8.52	427.39	37.97	27.17	18.58	515.89	36.72	86424.95	**0.00**
	5	86439.85	12.69	0.07	13.27	86439.85	12.69	9.14	45.19	200.77	77.05	6.68	68.55	143.32	79.93	86421.39	**0.00**
	10	86440.73	15.01	0.06	33.18	86440.73	15.01	6.69	102.11	32.23	108.25	2.44	156.15	107.15	134.09	86437.97	**0.00**
A	2	86439.83	7.51	0.06	7.51	86439.83	7.51	264.51	3.79	11439.37	13.09	416.03	10.15	6083.81	11.98	86432.48	**0.00**
	5	86443.12	9.69	0.06	9.69	86443.12	9.69	256.67	35.94	40907.89	51.62	414.57	43.77	15986.23	42.06	86405.38	**0.00**
	10	86438.72	33.58	0.06	33.58	86438.72	33.58	4315.73	86.09	86400.18	121.59	29377.52	127.07	86400.14	121.93	86407.30	**0.00**
Total Average:		86432.60	14.97	0.06	18.18	84288.13	13.67	7891.18	47.43	34095.92	58.56	12517.13	68.04	31049.62	62.03	86425.77	**0.82**

2. or $\text{GAP}_{Best}(\%)$ as the gap with respect to the best known integer solution. We calculate it for large-sized instances since the branch-and-price provides poor lower bounds even using (15).

In order to obtain Table 5, a time limit of 2 h was fixed for this experiment with $n = 50$. For these instances, B&P and Matheur report the best performance in most of the cases. In general they present less gap and, on average, it is better not wasting the time solving the exact pricer letting the algorithm go further adding columns or branching before certifying optimality. Thus, with the Matheur strategy we obtain an 11.23% of average gap. In fact, this matheuristic finds the optimal solution (certified by the exact method) at least in six instances. Concerning the time, only InitialHeur needs much smaller CPU time, obtaining good quality solutions for some instances. For the Eilon dataset, the aggregation schemes exhibit that the larger the aggregated set the smaller the gap and the larger the CPU time, as expected.

Fig. 2 illustrates the optimal solution (square points) of a particular instance (B&P) and the solution when the solution space is limited to the demand points coordinates (Discretization). One can observe in that figure that although the continuous nature of the problem is not completely captured by the discrete version of the problem, the structure of the clusters of demand points obtained by discretizing the space is similar to the one obtained by the exact approach, being this method an adequate heuristic for larger instances in which the exact approach is not able to certify optimality.

For large-sized problems (Tables 6 and 7) we set the time limit to 24 h. The best solutions are found by the discretization matheuristic except for the center problems (see Fig. 3 where the solutions obtained with the Matheur and the Discretization approaches clearly differ). Among the other strategies, decomposition-based matheuristic stands out, but the improvement from the initial heuristic is null for some cases. The reader may note that in many of these large-sized instances the results of B&P and Matheur coincide. For these instances, the relaxed restricted master problem is big enough to expend in each iteration a considerable amount of time to be solved. Thus, within the time limit, B&P does not solve any pricing subproblem to optimality what makes its results coincide with the ones obtained by Matheur.

Some instances have the best performance using KMEANS-20 or PTF-20 matheuristics. Not much improvement is appreciated taking 30 points instead of 20 for the aggregation method. To find an explanation for that, Fig. 4 depicts the aggregation (triangular points) and the solution for a particular instance. The reader can see how the demand points are concentrated by zones. Adding more points to $\mathcal{A}'$ gives an importance to some aggregated points that does not represent properly the original data of this instance of $n = 654$. In this case, we can see an example for which the aggregation algorithm works better under the *less is more* paradigm.

6. Conclusions

In this work, the Continuous Multifacility Monotone Ordered Median Problem is analyzed. To solve this problem, defined in a continuous space, we have proposed two exact methods, namely a compact formulation and a branch-and-price procedure, using binary variables. Along the paper, we give full details of the branch-and-price algorithm and all its crucial steps: master problem, restricted relaxed master problem, pricing problem, initial pool of columns, feasibility, convergence, and branching.

Moreover, theoretic and empirical results have proven the utility of the obtained lower bound. Using that bound, we have tested three matheuristics that we propose. The decomposition-based heuristics have shown a very good performance on the computational experiments. For large-sized instances, the best known solutions have been obtained reducing the solution space by means of a discretization of the continuous problem.

Among the extensive computational experiments and configurations of the problem, we highlight the usefulness of the branch-and-price approach for medium- to large-sized instances, but also the utility of the compact formulation and the aggregation-based heuristics for small values of p or for some particular ordered weighted median functions.

Further research on the topic includes the design of similar branch-and-price approaches to other continuous facility location and clustering problems. Specifically, the application of set-partitioning column generation methods to hub location and covering problems with generalized upgrading (see, e.g., Blanco & Marín, 2019) where the index set for the y-variables must be adequately defined.

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Appendix A. Computational results for alternative ℓ_τ-norms

In this section, we show the results of our computational experiments for other ℓ_τ-norms, in particular, we have considered $\tau \in \{\frac{3}{2}, 2, 3\}$. We have shown in Theorem 1 the general way to formulate the constraint for general values of τ. In Table A.1, we present the sets of constraints for the τ considered, for all $i \in I$, $j \in J$, $l \in \{1, \ldots, d\}$. Tables A.2 , A.3 , and A.4 report the results.

Table A1
Constraints for different values of τ to represent ℓ_τ-norms.

$\ell_{\frac{3}{2}}$	ℓ_2	ℓ_3
$t_{ijl} \geq a_{il} - x_{jl}$,	$t_{ijl} \geq a_{il} - x_{jl}$,	$t_{ijl} \geq a_{il} - x_{jl}$,
$t_{ijl} \geq -a_{il} + x_{jl}$,	$t_{ijl} \geq -a_{il} + x_{jl}$,	$t_{ijl} \geq -a_{il} + x_{jl}$,
$z_{ij} \geq \sum_{l=1}^{d} \xi_{ijl}$,	$z_{ij}^2 \geq \sum_{l=1}^{d} t_{ijl}^2$,	$z_{ij} \geq \sum_{l=1}^{d} \xi_{ijl}$
$t_{ijl}^2 \leq \psi_{ijl}\xi_{ijl}$,		$t_{ijl}^2 \leq \psi_{ijl} z_{ij}$,
$\psi_{ijl}^2 \leq z_{ij} t_{ijl}$,		$\psi_{ijl}^2 \leq \xi_{ijl} t_{ijl}$,

Appendix B. Aggregated results

In order to show the influence of p, the Ordered Median aggregation function, and the ℓ_τ-norm, we have aggregated the results of the 1512 instances used in Tables 4, A.2, A.3, and A.4 (Tables B.1, B.2, B.3).

Table A2
Results for Eilon et al. (1971) instances for $\ell_{\frac{3}{2}}$-norm.

n	type	p	Time (#Unsolved)				GAProot(%)		GAP(%)		Vars		Nodes		Memory (MB)	
			Compact		B&P		Compact	B&P	Compact	B&P	Compact	B&P	Compact	B&P	Compact	B&P
20	W	2	**75.10**	**(0)**	966.81	(0)	100.00	**0.00**	0.00	0.00	524	**500**	31161	**1**	187	**25**
		5	–	(5)	**332.73**	**(0)**	100.00	**0.00**	58.11	**0.00**	1270	**397**	2323815	**1**	11580	**12**
		10	–	(5)	**152.66**	**(0)**	100.00	**0.00**	97.34	**0.00**	2480	**377**	1280023	**1**	7438	**10**
	C	2	**0.70**	**(0)**	520.73	(4)	100.00	**21.46**	**0.00**	17.11	**524**	13259	**317**	650	**9**	174
		5	**57.47**	**(0)**	74.10	(4)	100.00	**34.21**	**0.00**	33.65	**1270**	1333	15338	**302**	44	**13**
		10	**3863.55**	**(1)**	1270.00	(2)	100.00	**18.56**	**2.45**	17.81	2480	**1638**	1164203	**1936**	1159	**15**
	K	2	**6.23**	**(0)**	1227.76	(2)	100.00	**6.46**	**0.00**	2.58	**524**	7247	3052	**197**	**20**	212
		5	–	(5)	**824.52**	**(3)**	100.00	**10.90**	27.82	**6.21**	**1270**	2719	3205611	**1120**	7563	**57**
		10	–	(5)	**2409.37**	**(3)**	100.00	**17.66**	96.76	**8.78**	2480	**1235**	1093174	**3003**	8496	**24**
	D	2	**63.65**	**(0)**	724.11	(0)	100.00	**0.00**	0.00	0.00	524	**437**	31224	**1**	153	**22**
		5	–	(5)	**284.21**	**(0)**	100.00	**0.00**	65.95	**0.00**	1270	**383**	2204675	**1**	12441	**11**
		10	–	(5)	**163.89**	**(0)**	100.00	**0.08**	95.40	**0.00**	2480	**370**	1140802	**3**	9200	**9**
	S	2	**58.41**	**(0)**	1040.72	(0)	100.00	**0.00**	0.00	0.00	524	**519**	25556	**1**	155	**25**
		5	–	(5)	**423.48**	**(0)**	100.00	**0.11**	59.03	**0.00**	1270	**397**	2398328	**3**	12161	**12**
		10	–	(5)	**175.32**	**(0)**	100.00	**0.34**	98.74	**0.00**	2480	**383**	878394	**4**	8683	**10**
	A	2	**16.20**	**(0)**	2802.48	(1)	100.00	**3.44**	**0.00**	0.35	**524**	2987	5372	**54**	**37**	146
		5	–	(5)	**2131.87**	**(2)**	100.00	**10.91**	47.25	**3.22**	**1270**	2548	1996233	**413**	8655	**69**
		10	–	(5)	**3440.13**	**(1)**	100.00	**11.84**	99.32	**2.01**	2480	**1288**	705456	**2001**	7031	**30**
30	W	2	**3007.98**	**(0)**	854.86	(4)	86.97	**19.38**	**0.00**	19.38	**784**	852	1101778	**1**	2823	**71**
		5	–	(5)	**3389.46**	**(3)**	87.13	**22.60**	79.53	**22.60**	1900	**845**	744808	**1**	12424	**47**
		10	–	(5)	**2960.90**	**(0)**	89.05	**0.00**	88.07	**0.00**	3710	**773**	341608	**2**	3070	**32**
	C	2	**1.27**	**(0)**	110.41	(4)	81.07	**30.05**	**0.00**	26.62	**784**	10754	416	**190**	**15**	137
		5	**311.13**	**(0)**	–	(5)	81.71	**45.91**	**0.00**	44.26	**1900**	3340	57541	**200**	116	**33**
		10	**4.95**	**(4)**	–	(5)	81.93	**45.96**	75.89	**45.96**	3710	**1758**	407909	**231**	1166	**17**
	K	2	**77.50**	**(0)**	19.84	(4)	85.63	**39.16**	**0.00**	38.71	**784**	1708	28803	**5**	115	**80**
		5	–	(5)	**20.48**	**(4)**	85.80	**18.37**	66.83	**17.99**	**1900**	1986	956059	**19**	11102	**61**
		10	–	(5)	**194.29**	**(4)**	86.29	**22.05**	84.71	**21.13**	3710	**1574**	390091	**234**	2707	**42**
	D	2	**2441.14**	**(1)**	300.79	(4)	86.55	**22.86**	**2.74**	22.86	**784**	918	1208553	**1**	3490	**77**
		5	–	(5)	**4831.03**	**(2)**	86.99	**24.04**	74.25	**24.04**	1900	**841**	798302	**3**	10071	**48**
		10	–	(5)	**2546.68**	**(0)**	87.46	**0.05**	86.29	**0.00**	3710	**774**	343890	**2**	4196	**32**
	S	2	**2363.80**	**(0)**	470.41	(4)	86.86	**32.35**	**0.00**	32.35	**784**	895	822524	**1**	2187	**72**
		5	–	(5)	**4534.78**	**(2)**	86.78	**18.98**	76.88	**18.98**	1900	**840**	726885	**1**	10244	**48**
		10	–	(5)	**2666.23**	**(0)**	88.10	**0.04**	87.16	**0.00**	3710	**776**	410399	**5**	4551	**31**
	A	2	**327.89**	**(0)**	93.44	(4)	85.31	**49.21**	**0.00**	49.18	**784**	1259	70967	**2**	365	**104**
		5	–	(5)	**95.22**	**(4)**	86.39	**7.72**	70.59	**7.37**	1900	**1346**	626210	**9**	8166	**74**
		10	–	(5)	**169.25**	**(4)**	86.14	**13.71**	84.74	**10.99**	3710	**1487**	307003	**141**	2647	**54**
40	W	2	–	(5)	–	(5)	100.00	**32.33**	45.10	**32.33**	**1044**	1292	1282229	**1**	15092	**131**
		5	–	(5)	–	(5)	100.00	**67.86**	96.95	**67.86**	2530	**1298**	364182	**1**	8105	**102**
		10	–	(5)	–	(5)	100.00	**92.83**	100.00	**92.83**	4940	**1339**	165306	**1**	2712	**83**
	C	2	**4.22**	**(0)**	–	(5)	100.00	**44.51**	**0.00**	44.51	**1044**	2468	1043	**2**	**20**	35
		5	**3879.78**	**(3)**	–	(5)	100.00	**77.56**	**41.01**	77.56	2530	**2525**	606460	**2**	4626	**27**
		10	–	(5)	–	(5)	100.00	**78.16**	91.15	**78.16**	4940	**2206**	237096	**13**	883	**22**
	K	2	**2889.87**	**(0)**	–	(5)	100.00	**61.39**	**0.00**	61.39	**1044**	2226	614033	**1**	2098	**125**
		5	–	(5)	–	(5)	100.00	**83.30**	92.54	**83.30**	2530	**2213**	451617	**1**	4187	**94**
		10	–	(5)	–	(5)	100.00	**75.79**	100.00	**75.79**	4940	**2169**	163856	**3**	1240	**83**
	D	2	–	(5)	–	(5)	100.00	**32.13**	40.95	**32.13**	**1044**	1529	1548255	**1**	11470	**153**
		5	–	(5)	–	(5)	100.00	**68.37**	99.29	**68.37**	2530	**1310**	360994	**1**	12324	**99**
		10	–	(5)	–	(5)	100.00	**91.94**	100.00	**91.94**	4940	**1357**	141565	**1**	4573	**79**
	S	2	–	(5)	–	(5)	100.00	**30.53**	42.63	**30.53**	**1044**	1443	1109568	**1**	14496	**145**
		5	–	(5)	–	(5)	100.00	**72.37**	98.23	**72.37**	2530	**1299**	358058	**1**	11144	**97**
		10	–	(5)	–	(5)	100.00	**90.94**	100.00	**90.94**	4940	**1362**	176603	**1**	2825	**76**
	A	2	**2886.85**	**(4)**	–	(5)	100.00	**61.20**	**14.49**	61.20	**1044**	1876	1127650	**1**	3383	**212**
		5	–	(5)	–	(5)	100.00	**82.57**	93.62	**82.57**	2530	**1656**	402572	**1**	4383	**134**
		10	–	(5)	–	(5)	100.00	**75.96**	100.00	**75.96**	4940	**1782**	145643	**1**	2437	**114**
45	W	2	–	(5)	–	(5)	100.00	**34.65**	46.87	**34.65**	**1174**	1462	974796	**1**	11024	**145**
		5	–	(5)	–	(5)	100.00	**83.70**	97.65	**83.70**	2845	**1444**	367765	**1**	3877	**126**
		10	–	(5)	–	(5)	100.00	100.00	100.00	100.00	5555	**1522**	122910	**1**	1081	**101**
	C	2	**6.63**	**(0)**	–	(5)	100.00	**57.24**	**0.00**	57.24	**1174**	2438	1160	**1**	**23**	34
		5	**4337.92**	**(3)**	–	(5)	100.00	**84.59**	**49.89**	84.59	2845	**2696**	467399	**1**	1217	**33**
		10	–	(5)	–	(5)	100.00	**64.84**	96.12	**64.84**	5555	**2327**	135558	**3**	578	**24**
	K	2	**6688.87**	**(4)**	–	(5)	100.00	**63.44**	**14.13**	63.44	**1174**	2443	1718920	**1**	6733	**142**
		5	–	(5)	–	(5)	100.00	**71.76**	95.10	**71.76**	2845	**2663**	453635	**1**	3364	**169**
		10	–	(5)	–	(5)	100.00	**91.29**	99.70	**91.29**	5555	**2231**	158192	**1**	781	**97**
	D	2	–	(5)	–	(5)	100.00	**30.83**	48.64	**30.83**	**1174**	1508	933735	**1**	10637	**153**
		5	–	(5)	–	(5)	100.00	**91.29**	98.36	**91.29**	2845	**1470**	374986	**1**	5117	**129**
		10	–	(5)	–	(5)	100.00	100.00	100.00	100.00	5555	**1528**	159745	**1**	1184	**106**
	S	2	–	(5)	–	(5)	100.00	**31.76**	45.32	**31.76**	**1174**	1556	1059244	**1**	8872	**160**
		5	–	(5)	–	(5)	100.00	**86.56**	99.18	**86.56**	2845	**1477**	452467	**1**	4738	**125**
		10	–	(5)	–	(5)	100.00	100.00	100.00	100.00	5555	**1522**	131588	**1**	976	**100**
	A	2	–	(5)	–	(5)	100.00	**60.42**	**25.43**	60.42	**1174**	1992	861937	**1**	4928	**221**
		5	–	(5)	–	(5)	100.00	**84.00**	97.37	**84.00**	2845	**1822**	297805	**1**	4261	**172**
		10	–	(5)	–	(5)	100.00	**93.77**	100.00	**93.77**	5555	**2008**	141643	**1**	845	**175**

(*continued on next page*)

Table A2 (*continued*)

n	type	p	Time (#Unsolved)				GAProot(%)		GAP(%)		Vars		Nodes		Memory (MB)	
			Compact		B&P		Compact	B&P	Compact	B&P	Compact	B&P	Compact	B&P	Compact	B&P
50	W	2	–	(1)	–	(1)	100.00	**36.39**	57.48	**36.39**	**1304**	1749	759779	**1**	11084	**186**
		5	–	(1)	–	(1)	100.00	**88.42**	97.49	**88.42**	3160	**1659**	270228	**1**	3125	**135**
		10	–	(1)	–	(1)	100.00	100.00	100.00	100.00	6170	**1777**	122705	**1**	1843	**127**
	C	2	**6.06**	**(0)**	–	(1)	100.00	**57.05**	**0.00**	57.05	**1304**	2813	879	**1**	**28**	36
		5	–	(1)	–	(1)	100.00	**94.44**	**86.11**	94.44	3160	**2767**	489141	**1**	1274	**41**
		10	–	(1)	–	(1)	100.00	**66.91**	100.00	**66.91**	6170	**2678**	117338	**2**	535	**32**
	K	2	–	(1)	–	(1)	100.00	**66.85**	**22.51**	66.85	**1304**	2554	1624297	**1**	9722	**141**
		5	–	(1)	–	(1)	100.00	**95.48**	**92.64**	95.48	3160	**2458**	501803	**1**	1496	**132**
		10	–	(1)	–	(1)	100.00	**99.57**	100.00	**99.57**	6170	**2418**	128940	**1**	669	**115**
	D	2	–	(1)	–	(1)	100.00	**31.81**	51.21	**31.81**	**1304**	1704	1059488	**1**	10438	**182**
		5	–	(1)	–	(1)	100.00	**91.53**	97.42	**91.53**	3160	**1671**	405884	**1**	1222	**156**
		10	–	(1)	–	(1)	100.00	100.00	100.00	100.00	6170	**1710**	156712	**1**	1360	**127**
	S	2	–	(1)	–	(1)	100.00	**23.29**	56.18	**23.29**	**1304**	2040	757576	**1**	10234	**215**
		5	–	(1)	–	(1)	100.00	**46.71**	98.40	**46.71**	3160	**1772**	404250	**1**	2374	**166**
		10	–	(1)	–	(1)	100.00	100.00	100.00	100.00	6170	**1739**	144453	**1**	764	**131**
	A	2	–	(1)	–	(1)	100.00	**58.75**	**40.71**	58.75	**1304**	2006	519352	**1**	6759	**215**
		5	–	(1)	–	(1)	100.00	**86.23**	95.15	**86.23**	3160	**2175**	396890	**1**	2119	**208**
		10	–	(1)	–	(1)	100.00	100.00	100.00	100.00	6170	**2127**	127758	**1**	672	**183**
Total Average:			1016.47	(282)	**1438.77**	**(277)**	96.64	**44.54**	59.19	**43.82**	2451	**1902**	628495	**143**	4813	**85**

Table A3
Results for Eilon et al. (1971) instances for ℓ_2-norm.

n	type	p	Time (#Unsolved)				GAProot(%)		GAP(%)		Vars		Nodes		Memory (MB)	
			Compact		B&P		Compact	B&P	Compact	B&P	Compact	B&P	Compact	B&P	Compact	B&P
20	W	2	26.03	(0)	**24.22**	**(0)**	100.00	**0.00**	0.00	0.00	**204**	2385	50632	**1**	**41**	120
		5	–	(5)	**20.33**	**(0)**	100.00	**0.00**	52.53	**0.00**	**470**	2504	8992191	**1**	15633	**53**
		10	–	(5)	**9.15**	**(0)**	100.00	**0.00**	89.75	**0.00**	**880**	2178	5306625	**1**	13982	**28**
	C	2	**0.16**	**(0)**	757.86	(4)	100.00	**18.45**	**0.00**	10.76	**204**	45528	**119**	13798	**4**	962
		5	**16.30**	**(0)**	–	(5)	100.00	**32.78**	**0.00**	21.37	**470**	7002	13102	**7867**	**31**	148
		10	**983.51**	**(0)**	2223.12	(3)	100.00	**27.39**	**0.00**	10.97	**880**	3717	650055	**9690**	1569	**75**
	K	2	**5.03**	**(0)**	1648.82	(1)	100.00	**7.02**	**0.00**	1.49	**204**	11126	6787	**349**	**11**	301
		5	4231.23	(4)	**2850.11**	**(2)**	100.00	**13.04**	23.89	**2.37**	**470**	5627	8109266	**1830**	7136	**117**
		10	–	(5)	**1822.27**	**(2)**	100.00	**16.01**	85.45	**6.61**	**880**	2925	4227203	**4287**	11279	**68**
	D	2	25.62	(0)	**20.60**	**(0)**	100.00	**0.00**	0.00	0.00	**204**	2365	45501	**1**	**37**	119
		5	–	(5)	**15.53**	**(0)**	100.00	**0.00**	50.45	**0.00**	**470**	2495	9156476	**1**	11659	**52**
		10	–	(5)	**17.64**	**(0)**	100.00	**0.05**	87.20	**0.00**	**880**	2175	5927899	**2**	15624	**28**
	S	2	**25.31**	**(0)**	28.86	(0)	100.00	**0.00**	**0.00**	0.00	**204**	2445	42178	**1**	**37**	126
		5	–	(5)	**49.89**	**(0)**	100.00	**0.03**	49.72	**0.00**	**470**	2491	9117840	**3**	12277	**53**
		10	–	(5)	**24.45**	**(0)**	100.00	**0.25**	87.53	**0.00**	**880**	2183	5399238	**4**	15145	**29**
	A	2	**15.72**	**(0)**	1663.90	(1)	100.00	**3.21**	**0.00**	0.17	**204**	6400	13975	**133**	**19**	288
		5	–	(5)	**1603.15**	**(2)**	100.00	**9.93**	36.09	**1.74**	**470**	5066	6284524	**703**	9720	**138**
		10	–	(5)	**2420.97**	**(1)**	100.00	**15.86**	87.63	**0.94**	**880**	2987	3478065	**2125**	11095	**60**
30	W	2	1214.67	(0)	**264.29**	**(0)**	86.79	**0.00**	0.00	0.00	**304**	3787	1814638	**1**	559	**339**
		5	–	(5)	**80.27**	**(0)**	86.91	**0.00**	72.61	**0.00**	**700**	2914	3104800	**1**	12866	**107**
		10	–	(5)	**34.14**	**(0)**	87.86	**0.00**	84.94	**0.00**	**1310**	2474	1626671	**1**	18935	**54**
	C	2	**0.43**	**(0)**	104.99	(4)	81.07	**14.80**	**0.00**	11.58	**304**	45929	**422**	1794	**6**	643
		5	**76.89**	**(0)**	503.59	(4)	81.71	**27.87**	**0.00**	20.17	**700**	8380	40315	**1409**	165	**100**
		10	**1.89**	**(4)**	–	(5)	81.93	**38.05**	41.89	**33.92**	**1310**	4996	2709802	**2978**	4156	**69**
	K	2	**54.31**	**(0)**	744.55	(4)	85.50	**6.60**	**0.00**	6.45	**304**	7743	62564	**44**	**58**	368
		5	–	(5)	**5040.71**	**(4)**	86.00	**11.54**	67.36	**8.59**	**700**	5932	2838928	**295**	15993	**170**
		10	–	(5)	–	(5)	87.68	**17.84**	83.29	**13.27**	**1310**	4040	1799130	**1270**	12831	**88**
	D	2	1247.78	(0)	**222.14**	**(0)**	86.38	**0.00**	0.00	0.00	**304**	3810	1731910	**1**	529	**335**
		5	–	(5)	**139.30**	**(0)**	86.94	**0.00**	73.46	**0.00**	**700**	2939	2824067	**1**	12906	**108**
		10	–	(5)	**38.85**	**(0)**	87.51	**0.00**	85.28	**0.00**	**1310**	2480	1519282	**1**	16694	**55**
	S	2	1271.98	(0)	**186.78**	**(0)**	86.69	**0.00**	**0.00**	0.00	**304**	3699	1631950	**1**	518	**324**
		5	–	(5)	**540.52**	**(0)**	87.12	**0.02**	73.42	**0.00**	**700**	2905	3042832	**3**	13666	**109**
		10	–	(5)	**94.57**	**(0)**	87.66	**0.08**	84.50	**0.00**	**1310**	2465	1569091	**2**	13710	**55**
	A	2	**220.34**	**(0)**	3254.28	(3)	85.20	**2.15**	**0.00**	1.69	**304**	4997	136052	**16**	**110**	403
		5	–	(5)	**1745.13**	**(4)**	86.59	**5.37**	71.23	**3.06**	**700**	4216	1909608	**51**	10892	**174**
		10	–	(5)	**1207.09**	**(4)**	86.97	**11.27**	83.28	**6.38**	**1310**	3735	1103129	**465**	10680	**106**

(*continued on next page*)

Table A3 (*continued*)

n	type	*p*	Time (#Unsolved)				GAProot(%)		GAP(%)		Vars		Nodes		Memory (MB)	
			Compact		B&P		Compact	B&P	Compact	B&P	Compact	B&P	Compact	B&P	Compact	B&P
40	W	2	–	(5)	**252.49**	**(0)**	100.00	**0.00**	31.03	**0.00**	**404**	6365	7312331	**1**	5864	**817**
		5	–	(5)	**364.88**	**(0)**	100.00	**0.00**	94.01	**0.00**	**930**	4168	1925927	**1**	11549	**234**
		10	–	(5)	**355.67**	**(0)**	100.00	**0.00**	99.63	**0.00**	**1740**	3922	862095	**1**	8294	**120**
	C	2	**1.55**	**(0)**	–	(5)	100.00	**23.26**	**0.00**	22.73	**404**	29032	1164	**419**	**9**	448
		5	**594.78**	**(0)**	–	(5)	100.00	**36.17**	**0.00**	35.69	**930**	5983	271138	**92**	862	**57**
		10	**4207.08**	**(1)**	–	(5)	100.00	**37.29**	**11.40**	35.81	**1740**	4780	1357589	**266**	5116	**40**
	K	2	**719.29**	**(0)**	–	(5)	100.00	**8.69**	**0.00**	8.69	**404**	5651	701127	**4**	429	**412**
		5	–	(5)	–	(5)	100.00	**16.32**	89.79	**16.31**	**930**	4724	1771444	**4**	11673	**171**
		10	–	(5)	–	(5)	100.00	**16.50**	99.43	**16.47**	**1740**	4836	700210	**44**	13117	**100**
	D	2	–	(5)	**259.07**	**(0)**	100.00	**0.00**	30.59	**0.00**	**404**	6694	6885185	**1**	7289	**865**
		5	–	(5)	**485.52**	**(0)**	100.00	**0.00**	94.27	**0.00**	**930**	4240	2040736	**1**	9260	**238**
		10	–	(5)	**473.11**	**(0)**	100.00	**0.01**	99.90	**0.00**	**1740**	3891	692838	**1**	10352	**119**
	S	2	–	(5)	**691.25**	**(0)**	100.00	**0.02**	29.89	**0.00**	**404**	6224	6075474	**1**	5159	**806**
		5	–	(5)	**244.29**	**(0)**	100.00	**0.00**	94.33	**0.00**	**930**	4120	2196020	**1**	11994	**230**
		10	–	(5)	**1223.93**	**(0)**	100.00	**0.07**	99.90	**0.00**	**1740**	3910	925128	**3**	11056	**120**
	A	2	**3347.72**	**(4)**	–	(5)	100.00	**4.09**	9.28	**4.09**	**404**	5760	2823594	**2**	1885	**763**
		5	–	(5)	–	(5)	100.00	**9.56**	91.43	**9.08**	**930**	4410	1346465	**5**	6365	**258**
		10	–	(5)	–	(5)	100.00	**9.54**	99.86	**8.60**	**1740**	4404	677970	**41**	9029	**141**
45	W	2	–	(5)	**388.20**	**(0)**	100.00	**0.00**	41.73	**0.00**	**454**	9960	4536228	**1**	6272	**1570**
		5	–	(5)	**207.18**	**(0)**	100.00	**0.00**	96.23	**0.00**	**1045**	4741	1544401	**1**	10363	**360**
		10	–	(5)	**282.84**	**(0)**	100.00	**0.00**	100.00	**0.00**	**1955**	4318	494940	**1**	10023	**172**
	C	2	**2.49**	**(0)**	–	(5)	100.00	**22.33**	**0.00**	22.32	**454**	4517	1618	**6**	**13**	67
		5	**398.50**	**(0)**	–	(5)	100.00	**35.75**	**0.00**	35.62	**1045**	4850	162356	**10**	453	**46**
		10	–	(5)	–	(5)	100.00	**35.00**	76.29	**34.76**	**1955**	4710	1628650	**35**	5534	**36**
	K	2	**5287.03**	**(2)**	–	(5)	100.00	**12.13**	**5.66**	12.12	**454**	6140	4935475	**3**	2537	**535**
		5	–	(5)	–	(5)	100.00	**17.38**	93.84	**17.36**	**1045**	5116	2086094	**4**	9338	**226**
		10	–	(5)	–	(5)	100.00	**14.67**	100.00	**14.67**	**1955**	4781	618710	**5**	11268	**120**
	D	2	–	(5)	**358.37**	**(0)**	100.00	**0.00**	40.50	**0.00**	**454**	9220	4961123	**1**	5740	**1483**
		5	–	(5)	**207.01**	**(0)**	100.00	**0.00**	96.12	**0.00**	**1045**	4756	1837767	**1**	8596	**363**
		10	–	(5)	**483.48**	**(0)**	100.00	**0.00**	100.00	**0.00**	**1955**	4310	486390	**1**	9703	**172**
	S	2	–	(5)	**370.88**	**(0)**	100.00	**0.00**	40.04	**0.00**	**454**	9566	4705056	**1**	5430	**1507**
		5	–	(5)	**1332.07**	**(0)**	100.00	**0.20**	95.75	**0.00**	**1045**	4809	2080272	**4**	10194	**370**
		10	–	(5)	**1487.71**	**(0)**	100.00	**0.02**	99.90	**0.00**	**1955**	4315	659990	**3**	10857	**173**
	A	2	–	(5)	–	(5)	100.00	**6.72**	25.73	**6.26**	**454**	6957	2069919	**4**	3364	**1078**
		5	–	(5)	–	(5)	100.00	**11.97**	91.39	**11.12**	**1045**	4895	1615563	**6**	3655	**373**
		10	–	(5)	–	(5)	100.00	**7.59**	100.00	**7.46**	**1955**	4486	483721	**6**	8786	**172**
50	W	2	–	(1)	**456.01**	**(0)**	100.00	**0.00**	48.21	**0.00**	**504**	10414	4777385	**1**	6240	**2007**
		5	–	(1)	**615.74**	**(0)**	100.00	**0.03**	96.50	**0.00**	**1160**	5458	1930033	**3**	6233	**470**
		10	–	(1)	**204.96**	**(0)**	100.00	**0.00**	100.00	**0.00**	**2170**	4971	355938	**1**	7946	**232**
	C	2	**4.78**	**(0)**	–	(1)	100.00	**22.61**	**0.00**	22.61	**504**	4342	2935	**2**	**17**	59
		5	–	(1)	–	(1)	100.00	**34.42**	67.17	**34.37**	**1160**	5142	2940681	**4**	5728	**51**
		10	–	(1)	–	(1)	100.00	**40.27**	83.64	**40.21**	**2170**	5198	1333733	**18**	3970	**41**
	K	2	–	(1)	–	(1)	100.00	**11.24**	18.49	**11.24**	**504**	6665	4998109	**3**	3635	**673**
		5	–	(1)	–	(1)	100.00	**17.43**	91.41	**17.43**	**1160**	5674	3154623	**3**	5249	**284**
		10	–	(1)	–	(1)	100.00	**14.01**	100.00	**13.91**	**2170**	5463	567705	**7**	10827	**156**
	D	2	–	(1)	**405.36**	**(0)**	100.00	**0.00**	47.33	**0.00**	**504**	9675	4367022	**1**	6329	**1789**
		5	–	(1)	**209.77**	**(0)**	100.00	**0.00**	94.73	**0.00**	**1160**	5302	2598004	**1**	4692	**469**
		10	–	(1)	**605.10**	**(0)**	100.00	**0.00**	100.00	**0.00**	**2170**	5000	548086	**1**	7504	**233**
	S	2	–	(1)	**451.37**	**(0)**	100.00	**0.00**	48.39	**0.00**	**504**	10286	3512436	**1**	6039	**1986**
		5	–	(1)	**815.35**	**(0)**	100.00	**0.07**	96.27	**0.00**	**1160**	5236	3055646	**3**	6241	**459**
		10	–	(1)	**204.63**	**(0)**	100.00	**0.01**	100.00	**0.01**	**2170**	4900	386889	**1**	9984	**231**
	A	2	–	(1)	–	(1)	100.00	**5.95**	32.08	**5.36**	**504**	8326	1790159	**4**	2035	**1513**
		5	–	(1)	–	(1)	100.00	**12.91**	91.62	**11.92**	**1160**	5559	1874166	**7**	3311	**483**
		10	–	(1)	–	(1)	100.00	**7.13**	100.00	**6.88**	**2170**	5089	462300	**9**	10198	**231**
Total Average:			673.68	(267)	**557.40**	**(157)**	96.65	**8.44**	53.08	**6.79**	**886**	6179	2347788	**663**	7186	**310**

Table A4
Results for Eilon et al. (1971) instances for ℓ_3-norm.

n	type	p	Time (#Unsolved)				GAProot(%)		GAP(%)		Vars		Nodes		Memory (MB)	
			Compact		B&P		Compact	B&P	Compact	B&P	Compact	B&P	Compact	B&P	Compact	B&P
20	W	2	**151.03**	**(0)**	743.17	(0)	100.00	**0.00**	0.00	0.00	524	**471**	56126	**1**	418	**24**
		5	–	(5)	**267.56**	**(0)**	100.00	**0.00**	62.95	**0.00**	1270	**415**	2700255	**1**	13112	**14**
		10	–	(5)	**133.91**	**(0)**	100.00	**0.00**	98.54	**0.00**	2480	**406**	1259405	**1**	13418	**13**
	C	2	**0.91**	**(0)**	–	(5)	100.00	**30.56**	**0.00**	29.19	**524**	9181	507	**417**	**11**	121
		5	**71.86**	**(0)**	–	(5)	100.00	**35.58**	**0.00**	30.56	**1270**	3061	17235	**1473**	62	**32**
		10	**2142.39**	**(0)**	–	(5)	100.00	**47.17**	**0.00**	47.00	2480	**912**	261767	**553**	879	**8**
	K	2	**16.05**	**(0)**	5363.15	(2)	100.00	**9.14**	**0.00**	2.07	**524**	12397	7592	**424**	**40**	352
		5	–	(5)	**2946.11**	**(3)**	100.00	**14.87**	38.13	**5.02**	**1270**	3466	2839547	**1556**	8104	**75**
		10	–	(5)	**3206.51**	**(2)**	100.00	**8.25**	93.62	**5.18**	2480	**1348**	1021924	**2630**	6013	**23**
	D	2	**125.29**	**(0)**	810.43	(0)	100.00	**0.00**	0.00	0.00	524	**498**	53138	**1**	300	**26**
		5	–	(5)	**230.99**	**(0)**	100.00	**0.00**	60.85	**0.00**	1270	**382**	2732256	**1**	12124	**11**
		10	–	(5)	**112.30**	**(0)**	100.00	**0.09**	99.24	**0.00**	2480	**377**	1083407	**2**	11367	**10**
	S	2	**104.35**	**(0)**	973.36	(0)	100.00	**0.05**	0.00	0.00	**524**	530	45754	**1**	239	**27**
		5	–	(5)	**255.50**	**(0)**	100.00	**0.11**	64.54	**0.00**	1270	**398**	2470849	**2**	10763	**12**
		10	–	(5)	**125.04**	**(0)**	100.00	**0.26**	99.51	**0.00**	2480	**378**	1131966	**4**	11380	**10**
	A	2	**33.54**	**(0)**	2705.24	(1)	100.00	**4.22**	**0.00**	0.15	**524**	3049	11374	**71**	**74**	150
		5	–	(5)	**1648.71**	**(2)**	100.00	**8.18**	42.12	**1.90**	**1270**	2262	2117999	**329**	6961	**64**
		10	–	(5)	**4194.46**	**(1)**	100.00	**13.35**	99.33	**1.41**	2480	**1216**	842028	**2054**	8107	**29**
30	W	2	2404.31	(4)	**435.00**	**(4)**	86.64	**14.53**	**12.54**	14.53	**784**	843	2130171	**1**	10112	**65**
		5	–	(5)	**3934.80**	**(2)**	86.35	**5.57**	77.53	**5.57**	1900	**860**	830494	**1**	12706	**48**
		10	–	(5)	**2479.98**	**(0)**	86.89	**0.03**	86.14	**0.00**	3710	**816**	306492	**3**	4764	**37**
	C	2	**2.30**	**(0)**	1631.39	(4)	81.07	**29.54**	**0.00**	29.53	**784**	7137	**675**	7559	**17**	94
		5	**571.24**	**(0)**	2360.18	(4)	81.71	**48.42**	**0.00**	46.18	**1900**	2388	71418	**386**	212	**27**
		10	**8.07**	**(4)**	1801.17	(4)	81.93	**48.01**	71.16	**47.51**	3710	**1236**	316323	**33**	1003	**10**
	K	2	**211.31**	**(0)**	17.36	(4)	85.39	**24.11**	**0.00**	23.96	**784**	1808	86793	**7**	303	**81**
		5	–	(5)	**27.59**	**(4)**	86.11	**14.05**	63.23	**13.64**	1900	**1707**	1143862	**12**	9716	**52**
		10	–	(5)	**34.09**	**(4)**	85.92	**22.77**	85.05	**22.13**	3710	**1409**	365197	**168**	4413	**35**
	D	2	4446.26	(3)	**3777.09**	**(3)**	86.24	**18.03**	**10.14**	18.03	**784**	946	2197143	**1**	9689	**71**
		5	–	(5)	**3574.66**	**(2)**	86.38	**9.19**	74.99	**9.19**	1900	**853**	958686	**1**	12995	**45**
		10	–	(5)	**2472.72**	**(0)**	86.60	**0.06**	85.43	**0.00**	3710	**796**	375899	**3**	5968	**35**
	S	2	**3256.59**	**(3)**	305.27	(4)	86.54	**17.51**	**9.82**	17.51	**784**	850	1889060	**1**	8632	**70**
		5	–	(5)	**3697.87**	**(1)**	86.19	**1.29**	78.49	**1.23**	1900	**760**	873284	**2**	12851	**40**
		10	–	(5)	**2525.29**	**(0)**	87.92	**0.12**	87.51	**0.00**	3710	**791**	351574	**4**	4490	**32**
	A	2	**1365.87**	**(0)**	61.55	(4)	85.11	**21.30**	**0.00**	21.04	**784**	1141	327100	**3**	1255	**96**
		5	–	(5)	**11.83**	**(4)**	85.51	**7.43**	72.18	**6.75**	1900	**1251**	643473	**8**	9003	**67**
		10	–	(5)	**100.35**	**(4)**	85.84	**15.80**	84.79	**13.75**	3710	**1385**	306764	**101**	4475	**51**
40	W	2	–	(5)	–	(5)	100.00	**26.96**	51.29	**26.96**	**1044**	1248	1344963	**1**	16445	**121**
		5	–	(5)	–	(5)	100.00	**83.14**	95.28	**83.14**	2530	**1267**	482105	**1**	5041	**104**
		10	–	(5)	–	(5)	100.00	**92.94**	100.00	**92.94**	4940	**1350**	188692	**1**	3381	**85**
	C	2	**6.66**	**(0)**	–	(5)	100.00	**38.00**	**0.00**	37.95	**1044**	2502	1762	**5**	**22**	37
		5	**2658.74**	**(2)**	–	(5)	100.00	**81.92**	**39.02**	81.92	2530	**2235**	434526	**1**	3523	**25**
		10	–	(5)	–	(5)	100.00	**75.34**	100.00	**75.34**	4940	**1921**	349858	**12**	1348	**18**
	K	2	**4990.55**	**(1)**	–	(5)	100.00	**50.04**	**3.02**	50.04	**1044**	2202	1651655	**1**	5136	**113**
		5	–	(5)	–	(5)	100.00	**74.12**	97.64	**74.12**	2530	**2018**	479613	**1**	7167	**86**
		10	–	(5)	–	(5)	100.00	**69.68**	100.00	**69.68**	4940	**1934**	210396	**3**	2563	**72**
	D	2	–	(5)	–	(5)	100.00	**20.20**	42.95	**20.20**	**1044**	1247	1505294	**1**	12264	**125**
		5	–	(5)	–	(5)	100.00	**89.82**	98.91	**89.82**	2530	**1248**	463628	**1**	8186	**98**
		10	–	(5)	–	(5)	100.00	**90.59**	100.00	**90.59**	4940	**1350**	153036	**1**	5215	**87**
	S	2	–	(5)	–	(5)	100.00	**25.20**	43.94	**25.20**	**1044**	1323	1360254	**1**	12167	**126**
		5	–	(5)	–	(5)	100.00	**70.38**	96.48	**70.38**	2530	**1294**	462667	**1**	6780	**91**
		10	–	(5)	–	(5)	100.00	**84.09**	100.00	**84.09**	4940	**1334**	139121	**1**	6489	**81**
	A	2	–	(5)	–	(5)	100.00	**24.46**	**22.07**	24.46	**1044**	1810	1460678	**1**	5826	**191**
		5	–	(5)	–	(5)	100.00	**62.67**	99.14	**62.67**	2530	**1646**	345643	**1**	7108	**134**
		10	–	(5)	–	(5)	100.00	**85.25**	100.00	**85.25**	4940	**1767**	155190	**1**	3426	**113**
45	W	2	–	(5)	–	(5)	100.00	**26.30**	54.64	**26.30**	**1174**	1398	1133697	**1**	11416	**134**
		5	–	(5)	–	(5)	100.00	**92.43**	99.51	**92.43**	2845	**1413**	401143	**1**	5947	**113**
		10	–	(5)	–	(5)	100.00	100.00	100.00	100.00	5555	**1469**	153297	**1**	2343	**104**
	C	2	**6.40**	**(0)**	–	(5)	100.00	**46.67**	**0.00**	46.67	**1174**	2326	1132	**2**	**24**	34
		5	**5003.88**	**(3)**	–	(5)	100.00	**75.71**	**60.00**	75.71	2845	**2532**	500043	**1**	5891	**29**
		10	–	(5)	–	(5)	100.00	**75.66**	100.00	**75.64**	5555	**2236**	171358	**7**	722	**22**
	K	2	–	(5)	–	(5)	100.00	**44.43**	**28.39**	44.43	**1174**	2477	1611616	**1**	11832	**146**
		5	–	(5)	–	(5)	100.00	**80.52**	99.07	**80.52**	2845	**2482**	426670	**1**	6214	**157**
		10	–	(5)	–	(5)	100.00	**84.97**	100.00	**84.97**	5555	**2165**	160867	**1**	1944	**90**
	D	2	–	(5)	–	(5)	100.00	**24.77**	54.89	**24.77**	**1174**	1406	1171833	**1**	12252	**135**
		5	–	(5)	–	(5)	100.00	**98.15**	98.67	**98.15**	2845	**1412**	424560	**1**	4188	**113**
		10	–	(5)	–	(5)	100.00	100.00	100.00	100.00	5555	**1466**	152963	**1**	2822	**104**
	S	2	–	(5)	–	(5)	100.00	**23.69**	56.00	**23.69**	**1174**	1479	1069851	**1**	11291	**141**
		5	–	(5)	–	(5)	100.00	**72.98**	99.95	**72.98**	2845	**1450**	514193	**1**	5809	**107**
		10	–	(5)	–	(5)	100.00	100.00	100.00	100.00	5555	**1467**	159205	**1**	2847	**101**
	A	2	–	(5)	–	(5)	100.00	**24.73**	37.56	**24.73**	**1174**	1731	695531	**1**	6228	**178**
		5	–	(5)	–	(5)	100.00	**65.14**	97.05	**65.14**	2845	**1800**	295359	**1**	4504	**160**
		10	–	(5)	–	(5)	100.00	**95.05**	100.00	**95.05**	5555	**2027**	131122	**1**	1477	**172**

(*continued on next page*)

Table A4 (*continued*)

n	type	*p*	Time (#Unsolved)				GAProot(%)		GAP(%)		Vars		Nodes		Memory (MB)	
50	W	2	–	(1)	–	(1)	100.00	**75.57**	**61.16**	75.57	**1304**	1616	1047768	**1**	11081	**161**
		5	–	(1)	–	(1)	100.00	**99.83**	100.00	**99.83**	3160	**1623**	406553	**1**	2969	**141**
		10	–	(1)	–	(1)	100.00	100.00	100.00	100.00	6170	**1709**	186637	**1**	934	**128**
	C	2	**11.71**	**(0)**	–	(1)	100.00	**62.36**	**0.00**	62.36	**1304**	2575	2112	**1**	38	**37**
		5	–	(1)	–	(1)	100.00	**90.14**	**87.76**	90.14	3160	**2577**	540242	**1**	1441	**34**
		10	–	(1)	–	(1)	100.00	**99.12**	100.00	**99.12**	6170	**2587**	112265	**1**	572	**26**
	K	2	–	(1)	–	(1)	100.00	**36.29**	46.27	**36.29**	**1304**	2810	957166	**1**	10628	**164**
		5	–	(1)	–	(1)	100.00	**91.13**	98.89	**91.13**	3160	**2450**	434558	**1**	4069	**127**
		10	–	(1)	–	(1)	100.00	**90.03**	100.00	**90.03**	6170	**2410**	142538	**1**	1231	**108**
	D	2	–	(1)	–	(1)	100.00	**31.05**	63.12	**31.05**	**1304**	1666	878351	**1**	10528	**162**
		5	–	(1)	–	(1)	100.00	**74.10**	100.00	**74.10**	3160	**1695**	285611	**1**	4410	**134**
		10	–	(1)	–	(1)	100.00	100.00	100.00	100.00	6170	**1777**	83410	**1**	545	**129**
	S	2	–	(1)	–	(1)	100.00	**36.71**	60.33	**36.71**	**1304**	1577	983547	**1**	10718	**148**
		5	–	(1)	–	(1)	100.00	**94.34**	99.55	**94.34**	3160	**1635**	583530	**1**	2230	**129**
		10	–	(1)	–	(1)	100.00	100.00	100.00	100.00	6170	**1737**	153870	**1**	775	**135**
	A	2	–	(1)	–	(1)	100.00	**24.67**	48.59	**24.67**	**1304**	2258	514472	**1**	5470	**226**
		5	–	(1)	–	(1)	100.00	**75.58**	100.00	**75.58**	3160	**1998**	403520	**1**	2368	**182**
		10	–	(1)	–	(1)	100.00	**91.23**	100.00	**91.23**	6170	**2051**	117660	**1**	1968	**170**
Total Average:			928.16	(292)	**1680.95**	**(276)**	96.54	**41.27**	61.20	**40.52**	2451	**1819**	711082	**237**	5823	**82**

Table B.1
Results for Eilon et al. (1971) instances disaggregated by *p*.

n	*p*	Time (#Unsolved)				GAProot(%)		GAP(%)		Vars		Nodes		Memory (MB)	
		Compact		B&P		Compact	B&P	Compact	B&P	Compact	B&P	Compact	B&P	Compact	B&P
20	2	**31.53**	**(0)**	995.23	(27)	97.64	**5.78**	**0.00**	3.18	**352**	11676	19115	**909**	**76**	311
	5	598.89	(77)	**645.63**	**(36)**	100.00	**9.40**	31.41	**5.29**	**841**	3424	4014722	**1775**	6812	**86**
	10	1887.11	(100)	**944.99**	**(27)**	100.00	**10.49**	68.18	**5.25**	**1623**	1837	4961903	**2579**	7318	**38**
30	2	**800.60**	**(11)**	782.84	(68)	86.56	**15.67**	**1.47**	14.99	**527**	12187	724493	**448**	1706	**341**
	5	605.00	(98)	**1716.72**	**(59)**	89.26	**13.33**	54.38	**11.70**	**1258**	4621	3293952	**259**	8505	**124**
	10	1688.00	(115)	**1273.04**	**(52)**	89.74	**12.60**	77.60	**10.90**	**2427**	2794	3021553	**672**	5907	**64**
40	2	**1269.05**	**(71)**	637.73	(92)	97.42	**22.37**	**18.71**	22.30	**702**	10587	3899173	**30**	5302	**452**
	5	1285.99	(105)	**920.48**	**(92)**	100.00	**43.95**	76.39	**43.86**	**1674**	4274	2224552	**13**	7266	**162**
	10	3970.16	(115)	**558.03**	**(90)**	100.00	**47.30**	88.85	**46.92**	**3228**	3561	1806857	**63**	4752	**102**
45	2	**1404.32**	**(86)**	417.24	(94)	97.47	**24.07**	27.13	**24.05**	**789**	9011	3575284	**6**	5658	**592**
	5	1528.56	(106)	**914.34**	**(99)**	100.00	**47.27**	79.86	**47.20**	**1883**	4121	2010197	**4**	5773	**216**
	10	–	(120)	**900.91**	**(90)**	100.00	**50.95**	92.38	**50.81**	**3630**	3778	1503232	**18**	3886	**130**
50	2	231.73	(19)	**414.88**	**(18)**	97.50	**26.53**	34.60	**26.50**	**877**	8218	3270850	**3**	5870	**733**
	5	379.06	(23)	**1296.44**	**(19)**	100.00	**49.11**	85.88	**49.05**	**2091**	4208	2014876	**3**	4056	**261**
	10	–	(24)	**1140.29**	**(18)**	100.00	**53.29**	93.65	**53.22**	4033	**3983**	1167240	**13**	3434	**158**
Total Average:		788.01	(1070)	**950.76**	**(881)**	96.64	**26.11**	52.32	**24.78**	**1614**	5964	2567113	**538**	5209	**226**

Table B.2
Results for Eilon et al. (1971) instances disaggregated by type.

n	type	Time (#Unsolved)				GAProot(%)		GAP(%)		Vars		Nodes		Memory (MB)	
		Compact		B&P		Compact	B&P	Compact	B&P	Compact	B&P	Compact	B&P	Compact	B&P
20	W	317.70	(36)	**223.81**	**(0)**	99.49	**0.04**	42.20	**0.00**	**939**	1358	4396856	**1**	6710	**40**
	C	**586.30**	**(3)**	1167.87	(50)	98.24	**29.65**	**0.84**	22.22	**939**	16897	840935	**6801**	376	**368**
	K	208.19	(33)	**2302.80**	**(26)**	99.29	**12.07**	33.55	**4.07**	**939**	8287	2776965	**2663**	4251	**214**
	D	490.31	(35)	**202.43**	**(0)**	99.48	**0.05**	42.83	**0.00**	**939**	1345	3918058	**2**	6411	**39**
	S	246.13	(36)	**264.52**	**(0)**	99.48	**0.16**	42.80	**0.00**	**939**	1364	3810920	**3**	6202	**41**
	A	94.15	(34)	**2244.67**	**(14)**	99.30	**9.39**	36.97	**1.16**	**939**	4622	2247746	**1058**	4463	**168**
30	W	1513.57	(44)	**1106.95**	**(13)**	89.87	**5.18**	52.47	**5.17**	**1404**	1895	3535132	**1**	7636	**102**
	C	**298.00**	**(15)**	1014.47	(52)	84.44	**34.66**	**17.14**	30.95	**1404**	21207	1107742	**2046**	725	**263**
	K	**223.54**	**(39)**	1637.34	(48)	88.77	**18.00**	44.82	**15.96**	**1404**	8338	2429087	**501**	5244	**283**
	D	1618.90	(44)	**1283.18**	**(11)**	89.57	**6.19**	52.42	**6.18**	**1404**	1905	2728170	**1**	7388	**103**
	S	1512.19	(43)	**1267.56**	**(11)**	89.80	**5.88**	52.85	**5.84**	**1404**	1883	2558116	**2**	6886	**101**
	A	**718.27**	**(39)**	1701.10	(44)	88.68	**13.30**	47.19	**11.06**	**1404**	3974	1721747	**207**	4355	**208**
40	W	4028.70	(59)	**774.14**	**(30)**	99.48	**33.01**	72.86	**33.00**	**1868**	2957	5069956	**1**	7853	**233**
	C	**980.10**	**(20)**	–	(60)	97.96	**50.13**	**26.17**	49.58	**1868**	19156	881914	**158**	1541	**215**
	K	**2015.89**	**(41)**	–	(60)	99.22	**42.57**	59.51	**42.46**	**1868**	5091	1589810	**32**	4656	**226**
	D	5908.68	(59)	**461.24**	**(32)**	99.47	**32.77**	72.91	**32.76**	**1868**	3053	3668574	**1**	8229	**244**
	S	4977.44	(59)	**865.44**	**(32)**	99.47	**31.20**	72.88	**31.17**	**1868**	2965	3231789	**2**	8253	**232**
	A	**1271.93**	**(53)**	–	(60)	99.24	**37.56**	63.59	**37.20**	**1868**	3620	1419121	**19**	4108	**281**
45	W	–	(60)	**545.90**	**(33)**	99.50	**36.46**	76.57	**36.45**	**2101**	3670	4325838	**1**	6928	**371**
	C	**631.76**	**(26)**	–	(60)	97.92	**50.86**	**34.95**	50.71	**2101**	14393	812873	**35**	1369	**165**
	K	**2695.65**	**(51)**	–	(60)	99.27	**44.77**	64.85	**44.75**	**2101**	4696	1888145	**9**	5273	**253**
	D	–	(60)	**565.54**	**(33)**	99.50	**37.10**	77.13	**37.10**	**2101**	3542	3091420	**1**	6813	**355**
	S	–	(60)	**1176.85**	**(37)**	99.49	**34.68**	77.24	**34.64**	**2101**	3672	2778561	**2**	6696	**369**
	A	**4681.25**	**(55)**	–	(60)	99.27	**40.71**	68.02	**40.45**	**2101**	3846	1280588	**7**	3555	**361**
50	W	–	(12)	**504.08**	**(6)**	99.51	**41.69**	79.67	**41.68**	**2333**	4106	4060816	**1**	5964	**479**
	C	**80.39**	**(7)**	–	(12)	97.92	**56.37**	**47.61**	56.32	**2333**	11574	985006	**25**	1392	**139**
	K	**1135.78**	**(11)**	–	(12)	99.27	**48.78**	68.21	**48.77**	**2333**	4721	1721732	**3**	4630	**280**
	D	–	(12)	**1231.14**	**(6)**	99.51	**35.71**	79.72	**35.71**	**2333**	4047	2644819	**1**	5596	**464**
	S	–	(12)	**1080.38**	**(7)**	99.50	**33.47**	80.29	**33.47**	**2333**	4097	2304729	**2**	5704	**477**
	A	–	(12)	–	(12)	99.29	**41.84**	72.76	**41.59**	**2333**	4273	1188832	**4**	3434	**466**
Total Average:		788.01	(1070)	**950.76**	**(881)**	96.64	**26.11**	52.32	**24.78**	**1614**	5964	2567113	**538**	5209	**226**

Table B.3
Results for Eilon et al. (1971) instances disaggregated by norm.

n	norm	Time (#Unsolved)				GAProot(%)		GAP(%)		Vars		Nodes		Memory (MB)	
		Compact		B&P		Compact	B&P	Compact	B&P	Compact	B&P	Compact	B&P	Compact	B&P
20	ℓ_1	436.31	(27)	**536.32**	**(21)**	96.86	**9.14**	12.95	**3.27**	**387**	12007	6217790	**3685**	959	**322**
	$\ell_{\frac{3}{2}}$	431.87	(51)	**988.85**	**(22)**	100.00	**7.55**	41.56	**5.10**	**1425**	2112	1027930	**538**	5279	**49**
	ℓ_2	237.07	(49)	**695.93**	**(21)**	100.00	**8.00**	36.12	**3.13**	**518**	6200	3712315	**2266**	6961	**154**
	ℓ_3	330.68	(50)	**1287.62**	**(26)**	100.00	**9.55**	42.16	**6.80**	**1425**	2264	1036285	**529**	5743	**56**
30	ℓ_1	507.49	(51)	**719.49**	**(33)**	96.81	**8.47**	33.60	**5.77**	**582**	16294	6498700	**857**	2749	**394**
	$\ell_{\frac{3}{2}}$	**1149.07**	**(55)**	2365.89	(57)	85.90	**22.91**	48.76	**22.36**	2131	**1818**	519097	**58**	4414	**59**
	ℓ_2	567.61	(54)	**450.17**	**(37)**	85.92	**7.53**	45.62	**5.84**	**771**	6524	1636955	**463**	8071	**200**
	ℓ_3	1098.91	(64)	**2343.41**	**(52)**	85.46	**16.54**	49.94	**16.14**	2131	**1499**	731911	**461**	6256	**53**
40	ℓ_1	895.06	(66)	**941.95**	**(49)**	96.56	**11.16**	44.14	**10.67**	**772**	14886	7295651	**89**	3794	**429**
	$\ell_{\frac{3}{2}}$	**1932.07**	**(77)**	–	(90)	100.00	**67.76**	69.78	**67.76**	2838	**1742**	514263	**2**	5889	**101**
	ℓ_2	1337.71	(70)	**483.36**	**(45)**	100.00	**8.97**	59.71	**8.75**	**1025**	6284	2142580	**49**	7183	**330**
	ℓ_3	**2330.98**	**(78)**	–	(90)	100.00	**63.60**	71.65	**63.60**	2838	**1650**	621616	**2**	6227	**95**
45	ℓ_1	1292.16	(70)	**984.00**	**(58)**	96.63	**11.66**	49.03	**11.45**	**868**	13141	6512940	**30**	4305	**523**
	$\ell_{\frac{3}{2}}$	**1924.73**	**(82)**	–	(90)	100.00	**73.90**	**72.99**	73.90	3191	**1895**	489638	**1**	3902	**123**
	ℓ_2	1374.31	(77)	**568.64**	**(45)**	100.00	**9.10**	66.84	**8.98**	**1151**	5691	1939349	**5**	6785	**490**
	ℓ_3	**1434.25**	**(83)**	–	(90)	100.00	**68.40**	76.98	**68.40**	3191	**1819**	509691	**1**	5431	**113**
50	ℓ_1	505.06	(15)	**1480.61**	**(10)**	96.66	**11.82**	53.46	**11.72**	**966**	11475	5577448	**18**	4265	**634**
	$\ell_{\frac{3}{2}}$	**6.06**	**(17)**	–	(18)	100.00	**74.63**	77.52	**74.63**	3545	**2101**	443749	**1**	3651	**140**
	ℓ_2	4.78	(17)	**440.92**	**(9)**	100.00	**9.23**	73.10	**9.11**	**1278**	6261	2147547	**4**	5899	**632**
	ℓ_3	**11.71**	**(17)**	–	(18)	100.00	**76.23**	81.43	**76.23**	3545	**2042**	435212	**1**	3999	**130**
Total Average:		788.01	(1070)	**950.76**	**(881)**	96.64	**26.11**	52.32	**24.78**	**1614**	5964	2567113	**538**	5209	**226**

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