On multivariate heavy-tailed risk modelling in insurance and finance

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Doctoral dissertation

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Abstract

Companies in the financial and insurance sector have to face a large amount and a large variety of risks. Some of these risks are easy to estimate and model, but there also exist risks that are unpredictable and hard to understand. Financial crises in the past show that commonly used tools do not always suffice to model such large risks.

In the insurance sector, for instance, the occurrence of unexpectedly large claims can easily cause large losses that result in the insolvency of a company. Catastrophic events such as earthquakes, floods, pandemics and terror or cyber-attacks create big damages which lead to exceptionally large costs for insurance and reinsurance companies. Observations and data from the past are not always sufficient to predict such extremal events in the future since, for example, climate change affects the occurrence of natural disasters such as forest fires and storms whereas the development of information technology enables cyber crime. On financial markets, return rates of portfolios have similar properties. Most often changes in return rates are small and follow expected trends, but it is not uncommon that stock market crashes happen.

Therefore, one has to establish models that consider smaller claim sizes as well as large ones caused by extremal events. Mathematically, heavy-tailed distributions meet these requirements. Heavy-tailed distributions are distributions whose tail is not exponentially bounded and thus rare events with a large impact have a substantially higher probability than similar events modelled by light-tailed distributions such as the normal distribution.

Usually, insurance companies operate in different lines of business, offer products of different types of insurances or operate in different regions. Hence, the companies do not only have to be aware of the risks in every single line of business but also in their interactions and dependence. For instance, a catastrophic event like a natural disaster might not only affect a single line of business, but can cause large claims in different types of insurances at the same time. A

thunderstorm with heavy rain can break down electricity, fell trees, flood buildings, and cause other damage in buildings and infrastructure. Furthermore, the natural disaster might also cause injuries and deaths so it may affect different non-life insurance contracts such as home-owner policies and forest insurances as well as life insurance contracts. If the insurance company would visualise each of these types of contracts in a different model, extreme events would occur in every model, but a possible dependence of the extreme events can not be observed. Often, insolvency of a company is due to the fact that extreme losses in one line of business occur simultaneously with extreme losses in other lines of business. Therefore, the insurance company is not only interested in the behaviour of one-dimensional risks but also in multivariate models with heavy-tailed increments.

Other examples for applications of multivariate heavy-tailed distributions in insurance and finance are return rates of different stocks in finance to model portfolios, different types of insurances in insurance contracts and supervision of companies operating in the financial or insurance sector. In such a model for insurance supervision, each dimension could represent the risk reserve process of one insurance company. In a multivariate model of an insurance company, viewing the yearly net payout as a stochastic process with underlying heavy-tailed random vectors permits us to model and analyse the long-term behaviour of the solvency of the company.

This work aims to help to understand how one can model the long-term activities of an insurance company in markets where large losses are possible and investment returns from different industry sectors can collapse at the same time. The work intends to analyse the nature and order of magnitudes of risks, as well as their impacts on different lines of business.

List of articles

The dissertation consists of this introduction and the following three articles. The articles are referenced in the text by Roman numerals [I]-[III].

[I] Hägele, M. Precise asymptotics of ruin probabilities for a class of multivariate heavy-tailed distributions. *Statistics & Probability Letters*, (2020), Volume 166: 108871.
DOI:10.1016/j.spl.2020.108871

[II] Hägele M., Lehtomaa J. Large deviations for a class of multivariate heavy-tailed risk processes used in insurance and finance. *Journal of Risk and Financial Management*, (2021), Volume 14: 202.
DOI:10.3390/jrfm14050202

[III] Hägele M., Lehtomaa J. On the identification of the riskiest directional components from multivariate heavy-tailed data. *Submitted*, (2022)

Author's contribution

Following the tradition in mathematical literature, the authors of Articles II and III are listed alphabetically. Article [I] is an independent work of the author. In Article [II], the author made fundamental contributions to all parts of the manuscript. In Article [III], the author made fundamental contributions to all parts of the manuscript. None of the articles have been included in other dissertations.

Contents

Chapter 1

Heavy-tailed random variables

1.1 Introduction

Mathematically, risks appearing in the sector of finance and insurance are often modelled by random walks and their generalisations. A random walk is a stochastic process $(S_n)_{n=1}^{\infty}$,

$$S_n := X_1 + X_2 + \cdots + X_n,$$

where S_n is the sum of independent and identically distributed real-valued increments $X_1, X_2, \ldots$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In the classical theory, one assumes that the random variables are regular in the sense that the occurrence of extremal observations can be neglected. However, many real-life applications do not follow these assumptions but extremal events like financial crises on stock markets or natural disasters in catastrophe insurance are observed from time to time. Therefore, so-called "heavy-tailed" distributions, which model extremal events in a more accurate way, play a central role in such applications.

1.1.1 Definition

Heavy-tailed distributions are distributions whose tail functions converge much slower to zero than the tail functions of light-tailed distributions such as the normal distribution. We denote the tail function of a random variable with distribution function F by $\overline{F}(x) := 1 - F(x)$. The name heavy-tailed often refers to the right-hand tail of the distributions, so that the tail function of a heavy-tailed distribution exceeds the tail function of a light-tailed distribution from some threshold onwards. In this dissertation, heavy tail refers to the right-hand tail if nothing else is mentioned. Mathematically, heavy-tailed distributions are most commonly defined through the non-existence of the moment-generating function for any positive value.

Definition 1.1.1. A random variable X is light-tailed from the right, if $\mathbb{E}(e^{sX}) < \infty$ for some $s > 0$. Otherwise, we call X a heavy-tailed random variable.

The condition of the finite moment generating function for some positive argument is usually called Cramér's condition in honour of the Swedish mathematician and actuary Harald Cramér, who was one of the first mathematicians to study random walks in the context of insurance. He introduced the classical large deviations theory, the study of rare events, which deals with light-tailed distributions that fulfil Cramér's condition.

Lemma 1.1.2. *Let X be a random variable for which Cramér's condition holds. Then, the right-hand tail of the random variable has an exponential upper bound*

$$\mathbb{P}(X > x) \leq c_1 e^{-c_2 x},$$

where $c_1, c_2 > 0$ are constant.

Proof. Cramér's condition implies $\mathbb{E}(e^{sX})$ for some $s > 0$. Therefore, it holds that

$$\mathbb{E}(e^{sX}) \geq \mathbb{E}(e^{sX} \mathbb{1}(X > x)) \geq e^{sx} \mathbb{P}(X > x),$$

where $\mathbb{1}$ denotes the indicator function $\mathbb{1}(X > x) = 1$, if $X > x$ and $\mathbb{1}(X > x) = 0$ elsewhere. Rearranging the inequality yields the claim. □

For heavy-tailed distributions, a similar upper bound of the tail function does not exist.

The following example shows important examples of heavy-tailed distributions.

Example 1.1.3. (i) The distribution on the non-negative real axis $\mathbb{R}_{\geq 0}$ defined by the tail function

$$\overline{F}(x) = \left(\frac{K}{K + x} \right)^{\alpha}$$

with scale parameter $K > 0$ and shape parameter $\alpha > 0$ is called Pareto distribution.

(ii) The Cauchy distribution on $\mathbb{R}$ is defined by the density function

$$f(x) = \frac{K}{\pi((x-a)^2 + K^2)}$$

with scale parameter $K > 0$ and position parameter $a \in \mathbb{R}$.

(iii) The density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left(-\frac{(\log(x) - \mu)^2}{2\sigma^2} \right)$$

with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$ defines the lognormal distribution on the positive real axis $\mathbb{R}_+$. If a positive random variable X has a lognormal distribution with the parameters above, $\log(X)$ has a normal distribution with parameters μ and σ.

(iv) The Weibull distribution on $\mathbb{R}_{\geq 0}$ with shape parameter β in the open interval $(0, 1)$ is given by the tail function

$$\overline{F}(x) = \exp\left(-\left(\frac{x}{K}\right)^{\beta} \right)$$

where $K > 0$ is some scale parameter.

Additionally, generalisations of the lognormal distributions satisfying for some $p > 1$

$$\mathbb{P}(X > x) \sim e^{-(\log(x))^p}, \qquad \text{as } x \to \infty, \tag{1.1}$$

where $f \sim g$ refers to $\lim f(x)/g(x) = 1$, as $x \to \infty$, are heavy-tailed. Throughout the thesis, we call these distributions lognormal type distributions.

The fact that the moment generating function of heavy-tailed distributions does not exist does not imply that heavy-tailed distributions do not have finite moments at all. Many heavy-tailed distributions, like the Pareto distribution with a shape parameter greater than one, have a finite expected value and there are even heavy-tailed distributions with finite moments of all orders such as the lognormal distribution or the Weibull distribution.

1.2 Overview of one-dimensional heavy-tailed distributions

Instead of studying the class of heavy-tailed distributions as a whole, one often focuses on a suitable subclass of heavy-tailed distributions whose members share the same probabilistic features. Examples of such commonly studied subclasses are the class of subexponential distributions or its subclass, the class of regularly varying distributions.

1.2.1 Subexponential distributions

The class of subexponential distributions is one of the most important subclasses of heavy-tailed distributions. Subexponential distributions are defined by the asymptotic behaviour of the convolution of the tail functions: the probability that the sum of two subexponentially distributed random variables exceeds some large threshold is asymptotically equivalent with twice the probability that one of the random variables exceeds this threshold.

Definition 1.2.1. The distribution of a positive random variable X with distribution function F belongs to the class of subexponential distributions $\mathscr{S}$ if

$$\lim_{x\to\infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = 2. \tag{1.2}$$

Here, F^{*2} denotes the convolution of the distribution function F with itself which is the distribution of the sum of two i.i.d. random variables with distribution function F.

The class of subexponential distributions was first introduced in [14] and the name refers to the fact that the tail functions of subexponential distributions decay slower than the tail function of the exponential distribution. The distribution class is a very rich class in the sense that it includes distributions with regularly varying tails like Pareto distributions as well as distributions with lighter, but still heavy, tails that have finite moments of all orders including Weibull distributions and lognormal distributions.

A survey of the basic properties of subexponential distributions can be found in [36] and the more recent book [29]. Below, we give an overview of the properties of subexponential distributions that are most important for our work.

The asymptotic relation in Definition 1.2.1 also holds in a more general way. Exchanging the convolution of the tail function with its correspondent n-fold convolution, the limit of the asymptotic relation (1.2) is n. The more general asymptotic relation is also equivalent to the asymptotic relation

$$\mathbb{P}(X_1 + \cdots + X_n > x) \sim \mathbb{P}(\max(X_1, \ldots, X_n) > x), \tag{1.3}$$

as $x \to \infty$, where $X_1, \ldots, X_n$ are i.i.d. random variables with distribution function F. Asymptotic relation (1.3) is often referred to as the *principle of a single* *big jump*. Its interpretation is that the only significant way a sum of independent random variables $X_1 + \cdots + X_n$ can exceed some large threshold x is that one of the random variables itself exceeds the level.

Subexponentiality is preserved under tail equivalence for non-negative random variables. Tail equivalence means that if F is a subexponential distribution satisfying $\overline{F}(x) \sim \overline{G}(x)$, as $x \to \infty$ for some distribution function G, it follows that G is subexponential, too. Another important property is the long-tailedness.

Definition 1.2.2. A distribution F belongs to the class of long-tailed distributions $\mathscr{L}$ if for any $y \in \mathbb{R}$ fixed

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1. \tag{1.4}$$

While every subexponential distribution fulfils the long-tail property, the converse is not true in general so $\mathscr{S} \subsetneq \mathscr{L}$. See, for instance, Section 3.1 and 3.7 in [29].

The long-tail property plays an important role in extending the definition of subexponential distributions from positive random variables to random variables on the entire real axis $\mathbb{R}$. Most commonly, one says that a distribution F with support on the whole real line $\mathbb{R}$ is subexponential, if F^+ with $F^+(x) = F(x)$ if $x \geq 0$ and $F^+(x) = 0$ if $x < 0$ is a subexponential distribution. Lemma 3.4 in [29] shows that this definition is equivalent to the fact that F is subexponential and in addition F has a long-tail defined in (1.4).

Subexponential distributions can be characterised by their risk or hazard functions which we study more precisely in Section 2.3.1. The following lemma, which is stated in Theorem 2 in [73] gives a sufficient condition for subexponentiality in terms of the risk function.

Lemma 1.2.3. *If for some distribution F on $\mathbb{R}_+$ the following three conditions hold, F is subexponential.*

(i) The risk function $h(x) := -\log(\overline{F}(x))$ is concave for all x greater than some $x_0 > 0$.

(ii) There exists a function $0 < g(x) \to \infty$ with $x - g(x) \to \infty$ as $x \to \infty$ such that

$$\lim_{x\to\infty} \frac{\overline{F}(x-g(x))}{\overline{F}(x)} = 1.$$

(iii) Additionally,

$$\lim_{x\to\infty} h(x)e^{-h(g(x))} = 0,$$

where g is as in Condition (ii).

For distributions with a concave risk function, we can modify the sufficient conditions in the following way.

Corollary 1.2.4. *Let X be a random variable satisfying*

$$-\log(\mathbb{P}(X > x)) = h(x)$$

for all x greater than some $x_0 > 0$ where $h(x)$ is a concave function such that there exists $\alpha < 1$ satisfying

$$\lim_{x\to\infty} \frac{h(x)}{x^\alpha} = 0. \tag{1.5}$$

Then, the random variable X is subexponential.

Proof. We prove Corollary 1.2.4 by showing that all conditions in Lemma 1.2.3 hold. Condition (i) in Lemma 1.2.3 is immediately valid. To show Condition (ii) and (iii), we take $g(x) = \sqrt{x/h(x)}$. Then, it follows $g(x) \to \infty$ and $x - g(x) \to \infty$, as $x \to \infty$. Without loss of generality, we can assume $h(0) \geq 0$. If h does not fulfil this condition, one can easily construct a function $\tilde{h}$ with $\tilde{h}(x) = h(x)$ for $x \geq x_0$ and $\tilde{h}(0) \geq 0$. Thus, h is a subadditive function and it holds that

$$\begin{aligned}
\lim_{x\to\infty} \frac{\mathbb{P}(X > x - g(x))}{\mathbb{P}(X > x)} &= \lim_{x\to\infty} \exp\left(-h\left(\left(1 - \frac{g(x)}{x}\right)x\right) + h(x)\right) \\
\leq \quad \lim_{x\to\infty} \exp\left(-\left(1 - \frac{g(x)}{x}\right)h(x) + h(x)\right) &= \lim_{x\to\infty} \exp\left(\frac{g(x)}{x}h(x)\right) \\
= \quad \lim_{x\to\infty} \exp(g(x)^{-1}) = 1.
\end{aligned}$$

The corresponding lower bound of the limit follows directly by definition so Condition (ii) holds. Due to the assumptions on h, the term $\exp(-h(\sqrt{(x/h(x)}))$ converges faster to 0 than $h(x)$ to ∞, as $x \to \infty$, so Condition (iii) holds which concludes the proof. □

In Section 2.3.1, we show that Pareto distributions, Weibull distributions with parameter $\beta \in (0, 1)$ and lognormal type distributions where (1.1) holds with an equation belong to the class of subexponential distributions according to Corollary 1.2.4.

Another characterisation for the class of subexponential distributions gives the following lemma, which can be found in Appendix A3 in [27]. It relies on the hazard rate or mortality intensity q.

Lemma 1.2.5. *Let F be an absolutely continuous distribution function with density f. Furthermore, let the hazard rate*

$$q(x) := \frac{\mathrm{d}}{\mathrm{d}x}(-\log(\overline{F}(x))) = \frac{f(x)}{\overline{F}(x)}$$

be decreasing to 0.

(i) Then, $F \in \mathscr{S}$ if and only if

$$\lim_{x\to\infty} \int_0^x e^{yq(x)} f(y)\mathrm{d}y = 1.$$

(ii) If the function $x \mapsto e^{xq(x)} f(x)$ is integrable on the non-negative real axis, then $F \in \mathscr{S}$.

Applying Lemma 1.2.5, one can show that Weibull distributions with shape parameter $\beta \in (0, 1)$ as well as the lognormal type distribution defined by (1.1) belong to the class of subexponential distributions. Here, we show the subexponentiality for distributions of the lognormal type for which (1.1) holds with an equality for large enough values.

Example 1.2.6. (i) Weibull distribution: Let X be a random variable with tail function $\overline{G}(x) = e^{-x^\beta}$, where $x \geq 0$ and $0 < \beta < 1$. Then $g(x) = \beta x^{\beta-1} e^{-x^\beta}$ and $q(x) = \beta x^{\beta-1}$. Now, $q'(x) = \beta(\beta - 1)x^{\beta-2} < 0$ implies that $q(x)$ is a decreasing function. It is easy to obtain that $x \mapsto e^{xq(x)} g(x) = \beta x^{\beta-1} e^{(\beta-1)x^\beta}$ is integrable on $[0, \infty)$ and thus G is a subexponential distribution by Lemma 1.2.5.

(ii) Lognormal-type distribution: Let X be a positive random variable with tail function $\overline{F}(x) = e^{-(\log(x))^p}$, where $p > 1$. Then, the probability density function is

$$f(x) = \frac{p(\log(x))^{p-1}}{x} e^{-(\log(x))^p}$$

and the hazard rate is $q(x) = \frac{p(\log(x))^{p-1}}{x}$ which is a decreasing function. One can see that

$$x \mapsto e^{xq(x)} f(x) = \frac{p(\log(x))^{p-1}}{x} e^{\left(\frac{p}{\log(x)} - 1\right)(\log(x))^p}$$

is integrable on the positive real line so Lemma 1.2.5 implies the subexponentiality of the distribution.

(iii) Let X be a positive random variable with tail function $\overline{F}(x) = e^{-x^\beta(\log(x))^p}$ for large enough x, where $0 < \beta < 1$ and $p > 0$. Then, the probability density function is

$$f(x) = \left(\beta + \frac{p}{\log(x)}\right) x^{\beta-1}(\log(x))^p e^{-x^\beta(\log(x))^p}$$

and the hazard function $q(x) = \left(\beta + \frac{p}{\log(x)}\right) x^{\beta-1}(\log(x))^p$. Computing the derivative of the hazard function, it is easy to obtain that the function is decreasing to 0. Furthermore, the mapping

$$x \mapsto e^{xq(x)} f(x) = \left(\beta + \frac{p}{\log(x)}\right) x^{\beta-1}(\log(x))^p e^{\left(\beta + \frac{p}{\log(x)} - 1\right)x^\beta(\log(x))^p}$$

is integrable on the non-negative real line and thus by Lemma 1.2.5 the distribution is subexponential.

For other characterisations of subexponential distributions, see Lemma 1, Proposition 3 and Theorem 5 in [7] and Corollary 2 in [73].

When calculating ruin probabilities, one needs the subexponentiality of the integrated tail distribution of the claim size distribution. We define the integrated tail distribution F_I of the distribution F with positive expectation μ as

$$F_I(x) = \frac{1}{\mu}\int_0^x \overline{F}(y)\mathrm{d}y, \qquad x \geq 0. \tag{1.6}$$

The following lemma gives sufficient conditions for the integrated tail distribution F_I to belong to the class $\mathscr{S}$. The conditions are based on the hazard rate. For proof, see Lemma 1.4.6 in [27].

Lemma 1.2.7. *If $\lim_{x\to\infty} q(x) = 0$, $\lim_{x\to\infty} xq(x) = \infty$ and one of the following conditions holds, then $F_I \in \mathscr{S}$:*

(i) The hazard rate q is regularly varying with index $\alpha \in [-1, 0)$.

(ii) The hazard rate q is decreasing and slowly varying and in addition the function $-\log(\overline{F}(x)) - xq(x)$ is regularly varying with index 1.

We define regularly varying distributions in Section 1.2.2. Regularly varying functions are defined in a similar way, that is, a function f is regularly varying with index α if

$$\lim_{x\to\infty} \frac{f(tx)}{f(x)} = t^{\alpha} \tag{1.7}$$

for any $t > 0$. If $\alpha = 0$, the function is slowly varying. We show that the distributions in (i) and (ii) in Example 1.2.6 fulfil these conditions.

Example 1.2.8. (i) Weibull distribution: Let F be as in Example 1.2.6(i). It is easy to obtain that $q(x) \to 0$ and $xq(x) \to \infty$, as $x \to \infty$. Moreover, q is regularly varying with $\beta - 1$ and thus Condition (i) in Lemma 1.2.7 holds and thus the integrated tail distribution of a Weibull distribution belongs to the class of subexponential distributions.

(ii) Lognormal type distribution: Let F be as in Example 1.2.6(ii). Then, $q(x) = \frac{p(\log(x))^{p-1}}{x} \to 0$ and $xq(x) \to \infty$, as $x \to \infty$. Furthermore, for any $t > 0$

$$\lim_{x\to\infty} \frac{q(tx)}{q(x)} = \lim_{x\to\infty} \frac{(\log(tx))^{p-1}}{t(\log(x))^{p-1}} = t^{-1}$$

so the hazard rate is regularly varying with index -1 and Condition (i) in Lemma 1.2.7 holds. Thus, the integrated tail distribution of a lognormal type distribution is subexponential.

1.2.2 Regularly varying distributions

An important and well-studied subclass of subexponential distributions are regularly varying distributions including distributions with power law tails. A distribution belongs to the class of regularly varying distributions, if its tail function is regularly varying in the sense of Equation (1.7). This definition of regular variation was introduced and studied by the mathematician Jovan Karamata in [45, 46, 47].

Definition 1.2.9. The distribution of a positive random variable X with distribution function F has a regularly varying tail, if for any $t > 0$

$$\lim_{x\to\infty} \frac{\overline{F}(tx)}{\overline{F}(x)} = t^{\alpha}$$

for some regular variation index $\alpha \geq 0$. We write $X \in \mathrm{RV}_{\alpha}$.

The class of distributions with regularly varying tails only includes distributions that do not have finite moments of all orders. In general, the regular variation index indicates which moments are finite: Moments of an order smaller than the regular variation index are finite, whereas moments of greater order than α are infinity. If the regular variation index is a natural number, the moment of this order might be finite or not.

Important examples for distributions with regularly varying tails are Pareto distributions. Furthermore, Cauchy distributions, Fréchet distributions, Burr distributions, loggamma distributions and α-stable distributions with $\alpha > 2$ belong to the class of regularly varying distributions.

Many properties and results of the asymptotic behaviour of regularly varying distributions are presented in the monographs [70, 69] by Sidney Resnick. The most important properties include the properties of subexponential distributions as well as the fact that the tail function of every regularly varying distribution can be represented as

$$\overline{F}(x) = x^{-\alpha} L(x),$$

where $\alpha > 0$ is the regular variation index and L is a slowly varying function satisfying $\lim_{x\to\infty} L(tx)/L(x) = 1$ for every $t > 0$. Regularly varying distributions are studied a lot in the literature due to their central role in extreme value analysis.

Chapter 2

Multivariate distributions

In practice, companies in the financial sector do not only deal with univariate heavy-tailed distributions but use multivariate heavy-tailed models. For instance, an insurance company can model their payout process as a multivariate random walk, where each component of a random vector represents a different line of business or a particular type of insurance. The risks of each line of business could be presented in a one-dimensional model, but these models would not reflect the dependence between several lines of business. Therefore, one needs multivariate models where components of the random vectors might not be independent of each other but depend on the same events.

This dissertation focuses on modelling and analysing such multivariate risks. In the models, we are specifically interested in rare events of multivariate random walks and their asymptotic probabilities. Throughout the dissertation, we consider d-dimensional multivariate random walks $(\mathbf{S}_n)_{n=1}^{\infty}$, where

$$\mathbf{S}_n := \mathbf{X}_1 + \cdots + \mathbf{X}_n$$

is generated by independent and identically distributed heavy-tailed increments $\mathbf{X}_i \in \mathbb{R}^d$. We look at the extremal behaviour of the multivariate random walks and estimate the probabilities of extremal events.

For heavy-tailed distributions, large deviations of the sum of independent random variables are often due to one random variable taking a large value as the principle of a single big jump suggests. A similar behaviour can be observed in the multivariate case, as well. If the sum of i.i.d. heavy-tailed random vectors with expectation in the origin is far away from the origin, the reason is often that one random vector dominates the behaviour of the sum. Therefore, such extremal events can not be neglected, but have to be taken into account in models with multivariate heavy-tailed distributions.

It turns out that studying multivariate models is not as straightforward as studying one-dimensional models. Often, families of multivariate distributions have to be defined in a much more complex way than one-dimensional distribution families and it is not enough to characterise them by their marginal distributions. Tools and methods that worked well in the one-dimensional case are insufficient and extending concepts to a multivariate setting, properties that hold in the one-dimensional case may not hold in a multivariate generalisation any more or they hold only partially. In this chapter, we introduce different concepts and models for multivariate heavy-tailed distributions. Chapter 3 investigates ruin probabilities in a multivariate setting and Chapter 4 focuses on results from large deviations theory. In Chapter 5, we have a closer look at the differences in tail behaviour in different directions.

In the multivariate case, we denote vectors by bold symbols and random vectors by bold upper-case symbols. Components of vectors are denoted by upper indices, so a d-dimensional vector $\mathbf{x}$ can be written as $\mathbf{x} = (x^1, \dots, x^d)$. The origin will be denoted by $\mathbf{0}$ and relation symbols of vectors, for instance, $\mathbf{x} < \mathbf{y}$ are understood componentwise. For the positive orthant $[\mathbf{0}, \boldsymbol{\infty})^d$ we use the additional notation $\mathbb{R}^d_{\geq \mathbf{0}}$, $\mathbb{R}^d_+ := (0,\infty)^d$ denotes the positive orthant without axes and we set $\overline{\mathbb{R}^d} := [-\boldsymbol{\infty}, \boldsymbol{\infty}]^d$. For the unit sphere of $\mathbb{R}^d$ we use the notation $\mathbb{S}^{d-1} := \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| = 1\}$, where $\| \cdot \|$ denotes the used norm, which depends on the model. The norm defines the metric and thus also open and closed sets.

2.1 Multivariate regular variation

Perhaps the most commonly studied class of multivariate heavy-tailed distributions are multivariate regularly varying distributions, which were introduced in [68]. Multivariate regular variation turns out to be an important concept because of its role in multivariate extreme value theory. There, multivariate regular variation is used as the characterisation of certain maximum domains of attraction.

2.1.1 Definition and properties

The definition of multivariate regular variation extends the one-dimensional definition in a natural way such that many properties of one-dimensional regularly varying distributions find a correspondent multivariate extension. Usually, one defines multivariate regular variation on the positive orthant, but the definition can also be extended to the entire space $\mathbb{R}^d$. The book [69] provides an overview over the essential properties of multivariate regularly varying distributions and introduces standard statistical tools.

There exist several equivalent formulations of the definition of multivariate regular variation, see Theorem 2 in [68]. Here, we state the definition that is based on the concept of vague convergence. We denote the σ-algebra, a collection of measurable subsets, of the Borel sets of a space Σ as $\mathcal{B}(\Sigma)$. A measure that is finite on compact sets is called Radon measure.

Definition 2.1.1. Let Σ be a locally compact Hausdorff space and $\mu, \mu_n, n \in \mathbb{N}$ Radon measures on $\mathcal{B}(\Sigma)$. Then, μ_n is vague convergent to μ if for all continuous functions with compact support f it holds that

$$\lim_{n\to\infty} \int_\Sigma f \mathrm{d}\mu_n = \int_\Sigma f \mathrm{d}\mu.$$

Definition 2.1.2. Suppose there exists a nonnull Radon measure μ that is not identically zero and not degenerated at a point on the σ-algebra $\mathcal{B}(\overline{\mathbb{R}^d}\backslash\{\mathbf{0}\})$ of the Borel-sets of $\overline{\mathbb{R}^d}\backslash\{\mathbf{0}\}$ with $\mu(\overline{\mathbb{R}^d}\backslash\mathbb{R}^d) = 0$. Then, a random vector $\mathbf{X}$ on $\mathbb{R}^d$ is multivariate regularly varying, if there exists a sequence (a_n) of positive real numbers with $a_n \to \infty$, as $n \to \infty$, for which

$$n\mathbb{P}(a_n^{-1}\mathbf{X} \in \cdot) \xrightarrow{v} \mu(\cdot), \tag{2.1}$$

as $n \to \infty$, where v denotes vague convergence. Then, we also write $\mathbf{X} \in \mathrm{RV}((a_n), \mu, \overline{\mathbb{R}^d}\backslash\{\mathbf{0}\})$.

We call μ the limit measure. The positive sequence (a_n) can always be chosen as the inverse distribution function or quantile function of the norm of the random vector $\mathbf{X}$ at $\frac{n-1}{n}$, so one can choose a_n as

$$G^{-1}\left(1 - \frac{1}{n}\right) = \inf\left\{s \in \mathbb{R}_{>0} : G(s) \geq 1 - \frac{1}{n}\right\}, \tag{2.2}$$

where G is the distribution function of the norm of the random vector $\mathbf{X}$ denoted by $\|\mathbf{X}\|$. Hence, the choice of the sequence (a_n) might depend on the choice of the norm. In general, one can take any norm but for some examples a certain norm might be more convenient than others. Choosing (a_n) as in (2.2), it holds that $n\mathbb{P}(\|\mathbf{X}\| > a_n) \to 1$, as $n \to \infty$. If $\mathbf{X} \in \mathrm{RV}((a_n), \mu, \overline{\mathbb{R}^d}\backslash\{\mathbf{0}\})$, it follows that

$$\frac{\mathbb{P}(\mathbf{X} \in u\cdot)}{\mathbb{P}(\|\mathbf{X}\| > u)} \xrightarrow{v} c\mu(\cdot) \qquad \text{in } \mathcal{B}(\overline{\mathbb{R}^d}\backslash\{\mathbf{0}\}), \tag{2.3}$$

as $n \to \infty$ where the set in the numerator of the left-hand side of (2.3) is defined in (2.4). The constant c depends on the choice of the positive sequence (a_n).

With this notation, similarly to one-dimensional regular variation, we are able to define a regular variation index in the multivariate case. For a set $B \subset \overline{\mathbb{R}^d} \backslash \{\mathbf{0}\}$ bounded away from the origin, we define

$$uB := \{\mathbf{x} : \mathbf{x} = u\mathbf{b}, \mathbf{b} \in B\}. \tag{2.4}$$

Then, if $\mathbf{X} \in \mathrm{RV}((a_n), \mu, \overline{\mathbb{R}^d} \backslash \{\mathbf{0}\})$, there exists $\alpha > 0$ such that

$$\mu(uB) = u^{-\alpha}\mu(B) \qquad \text{for all } u > 0 \text{ and } B \subset \overline{\mathbb{R}^d} \backslash \{\mathbf{0}\}.$$

We write $\mathbf{X} \in \mathrm{RV}(\alpha, \mu)$ and call α the regular variation index.

Another way of defining multivariate regular variation is through weak convergence to some probability measure σ and the regular variation index α: There exists some index $\alpha > 0$ and a probability measure σ of the set $\mathcal{B}(\mathbb{S}^{d-1})$ such that for all $x > 0$

$$\frac{\mathbb{P}(\|\mathbf{X}\| > ux, \mathbf{X}/\|\mathbf{X}\| \in \cdot)}{\mathbb{P}(\|\mathbf{X}\| > u)} \xrightarrow{w} x^{-\alpha}\sigma(\cdot) \qquad \text{in } \mathcal{B}(\mathbb{S}^{d-1}), \tag{2.5}$$

as $u \to \infty$, where $\mathbb{S}^{d-1} := \{\mathbf{x} : \|\mathbf{x}\| = 1\}$ is the unit sphere of $\mathbb{R}^d$ and $\xrightarrow{w}$ denotes weak convergence. For a review of weak convergence, see Chapter 3 in [69]. The definition using weak convergence is equivalent to Definition 2.1.2. For proof, see, for instance, [53]. Writing the random vector in polar coordinates as in (2.5), we call α the tail index of $\mathbf{X}$ and the probability measure σ is called spectral or angular measure of $\mathbf{X}$. The spectral or angular measure is defined on the unit sphere $\mathbb{S}^{d-1}$, whose shape depends on the choice of the norm.

Instead of the sequence (a_n) in Definition 2.1.2, it is also possible to define multivariate regular variation by a scaling function $b(t) \in \mathrm{RV}_{1/\alpha}$ and a suitable limit measure ν. With this equivalent definition, multivariate regular variation is denoted by $\mathrm{MRV}(\alpha, b(t), \nu, \overline{\mathbb{R}^d})$. We use this notation in Definition 2.1.3, the definition of hidden regular variation.

If the random vector $\mathbf{X}$ is regularly varying with index $\alpha > 0$, every linear combination of the random vector is a one-dimensional regularly varying random variable with the same index $\alpha > 0$, see Theorem 1.1 in [8]. In particular, every component of the random vector is regularly varying with index $\alpha > 0$. This means that Definition 2.1.2 and its equivalents imply tail equivalence for the distribution tails of the components. However, in applications often the components of the random vectors do not have the same index α, but the marginal distributions might be regularly varying with different regular variation indices. In this case, we speak of non-standard regular variation. Applying theoretical results of regular variation to non-standard regularly varying observations can be done after transforming the data to the standard case.

2.1.2 Asymptotic dependence

For multivariate regularly varying distributions, one is usually interested in whether the components are asymptotically independent or not. If the regularly varying components are asymptotically independent, extremal events are concentrated around the axes because usually extremal outcomes do not occur in several components simultaneously due to the independence of the components for large outcomes. If $\mathbf{X} = (X^1, X^2)$ is a two-dimensional random vector with identically distributed components, the components are said to be asymptotically independent, if

$$\lim_{x \to \infty} \mathbb{P}(X^1 > x | X^2 > x) = 0,$$

see [66]. Although, components might be dependent on each other, they can still be asymptotically independent of each other. Asymptotic independence reduces risks in applications since for asymptotically dependent components the simultaneous occurrence of extremal events in several components involves large risks that can not be neglected.

There are many different ways in which the components can depend on each other asymptotically. The most commonly discussed types of dependence are asymptotic full and asymptotic strong dependence. For an overview on asymptotic full and strong dependence, see [21]. Roughly speaking, if the components are asymptotically fully dependent on each other, the extremal events are concentrated on the neighbourhood of a single ray that is the subspace spanned by a single vector. In this case, one would usually scale the components such that the ray is the diagonal. If the components are asymptotically strongly dependent, the extremal events can be observed inside a cone spanned by certain linear combinations of the dependent components. As in the case of asymptotic full dependence, one would typically transform the components in such a way that the cone is mirrored along the diagonal ray. For full asymptotic dependence the multivariate regular variation limit measure is concentrated on a single ray, whereas for strong asymptotic dependence the limit measure concentrates on a cone.

2.1.3 Hidden regular variation

An important concept for models with multivariate regular variation is hidden regular variation [66, 58, 54]. A random vector that is regularly varying can additionally have hidden regular variation for example in the case of asymptotic independence. If the components of the random vector turn out to be asymptotically independent, one analyses whether there are some regularly varying features in the positive quadrant after reducing the axes from the set.

Definition 2.1.3. A multivariate random vector $\mathbf{X}$ has hidden regular variation if it is regularly varying ($\mathbf{X} \in \text{MRV}(\alpha, b(t)\nu, \overline{\mathbb{R}^d})$) and in addition there exists a non-decreasing function $b_0(t) \to \infty$ with $b(t)/b_0(t) \to \infty$ as $t \to \infty$ and a positive radon measure ν_0 on $\mathbb{R}^d_+$ the positive orthant with removed axes such that

$$t\mathbb{P}\left(\frac{\mathbf{X}}{b_0(t)} \in \cdot\right) \xrightarrow{v} \nu_0$$

on $\mathbb{R}^d_+$. Then, there exists some regular variation index $\alpha_0 \geq \alpha$ such that $b_0(t) \in RV_{1/\alpha_0}$ and $\mathbf{X} \in \text{MRV}(\alpha_0, b_0(t), \nu_0, \mathbb{R}^d_+)$.

As in Definition 2.1.3, hidden regular variation is often considered when the components of multivariate heavy-tailed data are asymptotic independent, but it is not restricted to that case.

In [21], Das and Resnick extend the definition of hidden regular variation for distributions in $\mathbb{R}^2_+$ also to the case where the components are not independent of each other. If the components possess asymptotic full or strong dependence, hidden regular variation can occur in the space that remains after removing the ray or the cone of the limit measure with its neighbourhood. For an exact definition, see Definition 3.1 in [21].

2.2 Multivariate subexponential distributions

For a multivariate counterpart of subexponential distributions, no definition has been established in the literature. However, there exist at least three different approaches for defining multivariate subexponentaility which have their own advantages over the others.

2.2.1 Definition using vague convergence

The first definition introduced in Cline and Resnick [15] uses a similar approach as the definition of multivariate regular variation and defines multivariate subexponential distributions using vague convergence. Therefore, we consider the compactified Euclidean space $E = [-\infty, \infty]^d \backslash \{-\boldsymbol{\infty}\}$, the closed space $\overline{\mathbb{R}^d}$ where the point $-\boldsymbol{\infty}$ is removed. Hence, relatively compact sets are sets bounded away from $-\boldsymbol{\infty}$, the point with $-\infty$ in every component. Similarly to the extension of positive subexponentially distributed random variables to subexponential random variables on the entire real axis, also the extension to multivariate distributions needs some kind of long-tailedness.

Definition 2.2.1. For a multivariate function $\mathbf{b}(t)$, where $\mathbf{b}^i(t) \to \infty$ as $t \to \infty$ for all components $i = 1, \ldots, d$, a probability distribution F is said to be long-tailed if

$$tF(\mathbf{b}(t) + \cdot) \xrightarrow{v} \nu, \tag{2.6}$$

as $t \to \infty$, where $\xrightarrow{v}$ denotes vague convergence. Here, ν is a finite measure on E concentrated on the points $\{-\infty, \infty\}^d \backslash \{-\boldsymbol{\infty}\}$ such that for all components $i = 1, \ldots, d$ it holds that $\nu(\mathbf{x} \in E : x^i = \infty) > 0$.

The definition of multivariate subexponentiality in terms of vague convergence uses this extension of the long-tail property to the multivariate setting. As in the one-dimensional setting, the symbol $*$ denotes convolutions.

Definition 2.2.2. The distribution function F is subexponential if

$$tF * F(\mathbf{b}(t) + \cdot) \xrightarrow{v} 2\nu,$$

on E as $t \to \infty$ with respect to the same measure ν and function $\mathbf{b}$.

A distribution function F is then multivariate subexponential in the sense of Definition 2.2.2, if and only if it has the multivariate long-tail property (2.6) and its marginal distributions all belong to the class of one-dimensional subexponential distributions.

This definition of multivariate subexponential distributions extends many properties of one-dimensional subexponential distributions in a natural way. For instance, distributions with support on the positive quadrant that are regularly varying belong to the class of multivariate subexponential distributions according to (2.2.2).

2.2.2 Subexponentiality along lines

The approach by Omey [62] defines multivariate subexponentiality in a more direct way for distributions with support in the positive orthant $\mathbb{R}^d_+$.

Definition 2.2.3. A random vector has a subexponential distribution, if for all $\mathbf{x} > \mathbf{0}, \mathbf{x} \neq \boldsymbol{\infty}$,

$$\lim_{t\to\infty} \frac{\overline{F^{*2}}(t\mathbf{x})}{\overline{F}(t\mathbf{x})} = 2.$$

This approach defines multivariate subexponentiality along lines through the origin of the form $t\mathbf{x}$. Instead of lines, one can define subexponentiality also along curves or along regions. The curves are defined by $\mathbf{c}(t) * \mathbf{x}$, where $*$ denotes the dot product and $\mathbf{c}^i(t) \to \infty$, as $t \to \infty$ for all components of $\mathbf{c}$. For more details on these variations of the definition, see [62].

All variations of the definition introduced by Omey assume subexponential marginals. In addition, a multivariate long-tail property holds. The correspondent multivariate long-tail property of Definition 2.2.3 is the following.

Definition 2.2.4. For every $\mathbf{x} > \mathbf{0}, \mathbf{x} \neq \boldsymbol{\infty}$ and $\mathbf{a} \in [0, \infty)^d$, it holds that

$$\lim_{t\to\infty} \frac{\overline{F}(t\mathbf{x} - \mathbf{a})}{\overline{F}(t\mathbf{x})} = 1,$$

as $t \to \infty$.

Theorem 7 and Corollary 11 in [62] show that a distribution is subexponential according to Definition 2.2.3 if and only if the marginal distributions are subexponential and the multivariate long-tail property in Definition 2.2.4 holds. Although Definition 2.2.3 implies subexponentiality of the marginal distribution, it does not guarantee the one-dimensional subexponentiality of any linear combination of the random vector as in the case of multivariate regular variation. For an example, see Example 3.1 in [71].

With Definition 2.2.3, the principle of a single big jump in a multivariate setting is shown in [63].

2.2.3 Definition motivated by applications to ruin probabilities

The definition by Samorodnitsky and Sun introduced in [71] is motivated by applications to ruin probabilities. In order to extend one-dimensional subexponentiality to a multivariate setting, ruin probabilities are considered and a one-dimensional distribution is defined using a so called ruin set. In applications such a ruin set could be, for instance, the set that the sum of the components exceeds a threshold or the set where at least one of the components exceeds the threshold. Mathematically, a ruin set is an open increasing set A such that A^c

is convex and the origin does not belong to the closure $\overline{A}$ of A. Here, increasing set means that if $\mathbf{x} \in A$ also $\mathbf{x} + \mathbf{a} \in A$ for all $\mathbf{a} \in [0, \infty)^d$. For a d-dimensional distribution F with support on the positive quadrant $(0, \infty)^d$, one defines then

$$F_A(t) := 1 - F(tA), \qquad t \geq 0,$$

which is a probability distribution function on $[0, \infty)$. We use this one-dimensional distribution function to define multivariate subexponentiality.

Definition 2.2.5. A multivariate distribution is subexponential with respect to the ruin set A, if the corresponding one-dimensional distribution $F_A(t)$ is subexponential. The distribution F is subexponential, if $F_A(t)$ is subexponential for all A in

$$\mathcal{R} := \{A \in \mathbb{R}^d : A \text{ is an open increasing set}, A^c \text{ is convex}, \mathbf{0} \notin \overline{A}\}$$

the set of ruin sets.

The following picture shows three examples of ruin sets in $\mathbb{R}^2$, which can be easily extended to higher dimensions.

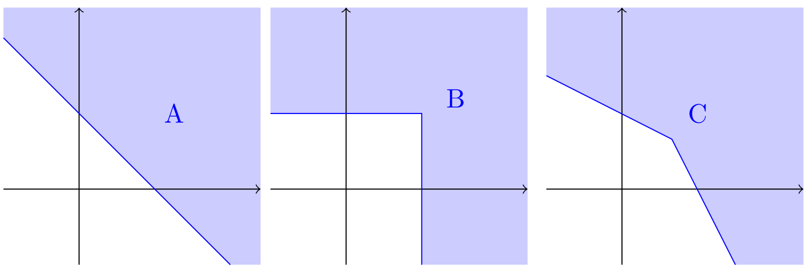

Figure 2.1: Examples of ruin sets in $\mathbb{R}^2$

In the model where each line of business of an insurance company is represented by one component, the half space in $\mathbb{R}^d$, $\{\mathbf{x} : x^1 + \cdots + x^d > 1\}$ or Set A in Figure 2.1 refer to the case where ruin occurs if the sum of losses of all lines of business exceed some threshold. This can be interpreted such that the initial capital of the insurance company can be freely moved from one line of business to another to cover the losses. In insurance supervision where every insurance company is represented by one component, Set B in Figure 2.1 or in higher dimensions the set $\{x^1 > 1\} \cup \cdots \cup \{x^d > 1\}$ is associated with the case that the losses of one insurance company exceed the threshold. Thus, the initial capital of the insurance company in question does not suffice to cover the costs and the insurance company gets insolvent. As a consequence, insurance supervision fails. Set C in Figure 2.1 again has a natural interpretation in the model where different lines of business are represented by different components. The set represents the case where transferring capital from one line of business to another line of business is possible, but always creates some costs. Ruin probabilities for this framework are studied in [41].

Similarly to Definitions 2.2.2 and 2.2.3, also Definition 2.2.5 implies one-dimensional subexponentiality of the marginals. Moreover, it follows from Definition 2.2.5 that every linear combination with non-negative coefficients of a multivariate subexponential vector $\mathbf{X}$ is subexponential itself. In addition, it holds the following multivariate long-tail property.

Definition 2.2.6. For any $\mathbf{a} \in \mathbb{R}^d$, it holds that

$$\lim_{u\to\infty} \frac{F(uA + \mathbf{a})}{F(uA)} = 1.$$

Furthermore, the class of distributions for which F_A is subexponential for every $A \in \mathcal{R}$ includes the class of multivariate regularly varying distributions.

All presented definitions of multivariate subexponentiality have in common that they imply subexponential marginal distributions. Additionally, every approach presented above defines a corresponding long-tail property, but these do not coincide with each other.

2.3 Polar coordinates

In models for multivariate heavy-tailed data such as the framework of multivariate regular variation, one often uses polar coordinates to distinguish between the angular components of random vectors and its heavyness. In this dissertation, we usually write the random vectors using polar coordinates as the product

$$\mathbf{X} = R\boldsymbol{\Theta} \tag{2.7}$$

of the random variable R and the random vector $\boldsymbol{\Theta}$. In this representation, the random variable R indicates the length of the random vector $R := \|\mathbf{X}\|$ or the distance from the origin and the random vector $\boldsymbol{\Theta} := \mathbf{X}/\|\mathbf{X}\|$ its angle or direction. The assumptions on the random variable R and the random vector $\boldsymbol{\Theta}$ vary in the different articles as well as the exact definition of length and direction. Representation (2.7) ensures that we can study the heavy-tailedness of the random vectors in terms of the heavy-tailedness of the one-dimensional random variable R.

This product model defined by (2.7) suggests studying the probabilities of sets of the form

$$V_{r,S} := \left\{ \mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| > r, \frac{\mathbf{x}}{\|\mathbf{x}\|} \in S \right\}, \tag{2.8}$$

where $r \in \mathbb{R}_+$ and S is a subset of the unit sphere $\mathbb{S}^{d-1}$. These truncated cones $V_{r,S}$ generate the same σ-algebra as the usual Borel σ-algebra of the Euclidean space $\mathbb{R}^d$ and can thus be used to state any probability of the random vector $\mathbf{X}$ in terms of its length represented by the random variable R and its direction indicated by the random vector $\boldsymbol{\Theta}$.

2.3.1 Risk function

One way to define the distribution of the length R of the random vector $\mathbf{X}$ is to study the risk or hazard function $h(x) := -\log(\mathbb{P}(R > x))$ and its asymptotic behaviour. In the literature, the risk or hazard function is often denoted by $R(x)$, but we avoid this notation and use $h(x)$ instead to prevent confusion with the random variable R itself.

If and only if R is heavy-tailed with risk function h, it holds that

$$\liminf_{x \to \infty} \frac{h(x)}{x} = 0.$$

This is an immediate consequence of the definition of heavy-tailed distributions, see, for instance, Lemma 5 in [18].

The risk function indicates which moments are finite through the moment index $\kappa := \sup\{s \geq 0 : \mathbb{E}((R^+)^s) < \infty\}$. For a proof of Lemma 2.3.1, see, for example, the proof of Lemma 3.6 in [5].

Lemma 2.3.1. *If R is heavy-tailed, its moment index κ is*

$$\kappa = \liminf_{x \to \infty} \frac{h(x)}{\log(x)}.$$

It turns out that if R is a heavy-tailed random variable, its risk function can be approximated asymptotically from below by a smooth, concave function which is called natural scale. In large deviations theory, the natural scale and risk functions are natural ways to describe the behaviour of extremal events of random walks.

Lemma 2.3.2. *Let R be a heavy-tailed random variable with risk function h. Then, there exists a non-negative, strictly increasing concave function $\tilde{h} : [0,\infty) \to [0,\infty)$ that satisfies $\tilde{h}(0) = 0$ and $\lim_{x\to\infty} \frac{\tilde{h}(x)}{x} = 0$. Furthermore, it holds that*

$$\liminf_{x\to\infty} \frac{h(x)}{\tilde{h}(x)} = 1$$

which is equivalent to the condition

$$\sup\left\{s \geq 0 : \mathbb{E}(e^{s\tilde{h}(R)})\right\} = 1.$$

This concave function, the natural scale, is also subadditive. For proof of Lemma 2.3.2, see the proof of Theorem 2.1 in [49].

The following example states the risk function of some common subexponential distributions. The lognormal type distribution is defined in (1.1).

Example 2.3.3.
(i) Pareto distribution: If $\mathbb{P}(R > x) = x^{-\alpha}$ for some $\alpha > 0$ and x large enough, the risk function converges to $h(x) = \alpha \log(x)$.

(ii) Weibull distribution: Assume $\mathbb{P}(R > x) = e^{-x^{\beta}}$ for some $\beta \in (0,1)$ and x large enough. Then, the risk function is $h(x) = x^{\beta}$ for x large enough.

(iii) Lognormal type distribution: Let $\mathbb{P}(R > x) = e^{-\log(x)^q}$ for some $q > 1$ and x large enough. Then, the risk function is $h(x) = \log(x)^q$ for large x.

Applying Corollary 1.2.4 to the risk functions determined in Example 2.3.3, one can easily see that all risk functions are convex and fulfil Condition (1.5). Hence, all distributions of Example 2.3.3 belong to the class of subexponential distributions.

Studying the dominating tail behaviour of multivariate distributions in article [III], we compare the risk function of the distribution in different cones with each other. Example 2.3.3 shows that the ratio of the risk functions of two Pareto distributions with different tail index is a positive constant smaller or greater than one. The ratio of the risk functions of two different distribution types such as Pareto and Weibull or Weibull and Lognormal type is either zero or infinity. This indicates that these different distribution types belong to a different class of order of magnitude considering the logarithmic tail.

2.3.2 Distribution of length R

In the context of ruin probabilities, the assumptions on the random variable R are usually stated in terms of integrated tails instead of the risk function. The integrated tail of the random variable R with distribution function F is defined as

$$\overline{F}_I(u) := \min\left(1, \int_u^\infty \mathbb{P}(R > v)\mathrm{d}v\right). \tag{2.9}$$

We use a similar assumption in [I], the article investigating ruin probabilities in a multivariate setting. In [II] and [III], the assumptions on the length R of the random vector are defined in terms of the risk function.

In articles [I] and [II], we focus on distributions that are heavy-tailed, but have a lighter tail than the tail of regular varying distributions. We formulate this restriction in [I] by the condition $\overline{F}_I(\gamma u) = o(\overline{F}_I(u))$ for all $\gamma > 1$ and in [II] by the conditions $h(x) = o(x)$ and $\log(x) = o(h(x))$, as $x \to \infty$. These additional conditions exclude regularly varying distributions, but hold, for instance, for heavy-tailed Weibull distributions and lognormal or lognormal-type distributions that possess finite moments of all orders. The assumption on R in article [III] admits a larger class of distributions.

2.3.3 Distribution of direction and dependence structure

The assumptions on the distribution of the directional component, the random vector $\mathbf{\Theta}$ vary between the different articles. In general, we assume that probability mass can occur in any direction, but we do not necessarily assume any uniform distribution of the random vector $\mathbf{\Theta}$.

For multivariate models an important feature is the dependence between components of the random vector. The assumed dependence structure differs in the different articles.

Article [I] analyses a model where the dominating tail behaviour is concentrated on a subset θ of the positive orthant, which means that for large values of R observations belong most likely to the subset θ whereas observations closer to the origin are not restricted to any set of directions. This assumption resembles strong asymptotic dependence as it is used in the context of multivariate regular variation. However, article [I] focuses on distributions with lighter tails than power laws.

In article [II], we study asymptotically spherical and elliptical distributions which are introduced in Section 2.3.4. Spherical distributions have a similar tail behaviour in every direction. The tail behaviour of elliptical distributions in different directions differs only by a constant. We assume asymptotic independence in the sense that the limit

$$\lim_{x\to\infty} \mathbb{P}(\mathbf{\Theta} \in S | R > x) = \mathbb{P}(\mathbf{\Theta} \in S)$$

exists for any measurable subset S of the support of the random vector $\boldsymbol{\Theta}$.

Article [III] presents an algorithm to find the set that dominates the tail behaviour of the random vector. This approach is applicable to models with different tail dependence structures and does not require any strong assumptions on the directions of the random vectors.

2.3.4 Elliptical distributions

For applications where observations are not concentrated on the positive orthant, the framework of elliptical distributions turns out to be useful. In a bivariate setting, a random vector $\mathbf{X} = (X^1, X^2)$ is said to be elliptically distributed if

$$\mathbf{X}^T = \boldsymbol{\mu} + R\Lambda\mathbf{U}$$

where $\boldsymbol{\mu} \in \mathbb{R}^2$ is a location vector, $R > 0$ is a random variable, Λ is a (2×2)-dimensional deterministic matrix and $\mathbf{U}$ is a two-dimensional random vector independent of R. Usually one assumes that $\mathbf{U}$ is uniformly distributed on the unit sphere $\mathbb{S}^{d-1}$ generated by the L_2-norm and the matrix Λ is such that $\Lambda\Lambda^T$ is a non-negative definite symmetric matrix. The use of the Euclidean norm guarantees that the matrix Λ transforms observations from the unit sphere to an ellipsoid. In the literature, heavy-tailed behaviour of elliptical distributions is often modelled using regular variation, see, for instance, [48, 40].

For asymptotically elliptical distributions in [II], we assume that $\boldsymbol{\mu} = \mathbf{0}$ and define the directional component of the random vector on the boundary Ω of an ellipsoid as $\boldsymbol{\Theta} = \Lambda\mathbf{U}$. Moreover, we do not need the assumption of uniformly distributed normed random vectors, but it is enough to assume that any subset of the unit sphere with positive Lebesque measure has positive probability mass.

Chapter 3

Ruin probabilities and review of [I]

3.1 Ruin probabilities

Classical ruin theory studies the risk process of an insurance company. Common models used in ruin theory are the Cramer-Lundberg model, a compound Poisson risk model, and the Sparre-Andersen model. These models consider incoming premiums and outgoing claims and represent them by compounded random variables. Ruin probabilities are then defined as the probability that the risk process exceeds a large threshold which represents the initial capital.

The classical Cramer-Lundberg theorem for large claims considers random variables with subexponentially distributed integrated tails. It studies the Cramer-Lundberg model with the total claim amount process defined as

$$S(t) = \begin{cases} \sum_{i=1}^{N(t)} X_i, & N(t) > 0, \\ 0, & N(t) = 0, \end{cases}$$

where the claim sizes $(X_i)_{i \in \mathbb{N}}$ are i.i.d. positive random variables with finite mean μ and distribution function F and the number of claims $N(t)$ is a Poisson process. More precisely, the number of claims it is defined as the sum of exponentially distributed i.i.d. inter-arrival times with finite mean $1/\lambda$. In this model, the claim sizes and the inter-arrival times are assumed to be independent of each other and the integrated tail function (2.9) of the claim sizes X_i

is assumed to belong to the class of subexponential distributions. One is then interested in the risk process

$$U(t) = u + ct - S(t),$$

for $t \geq 0$ where $u \geq 0$ denotes the initial capital and $c > 0$ denotes the premium income rate. In this setting, the ruin probability is the probability that the initial capital together with the premium income are not enough to cover the total claim amount. Mathematically, this is the probability $\mathbb{P}(U(t) < 0 \text{ for some } t)$, where $t \geq 0$. The classical result in the one-dimensional case as stated in Theorem 3.1.1 presents the relation between the ruin probability in infinte time and the integrated tail distribution $F_I(x) = \frac{1}{\mu}\int_0^x \overline{F}(y)\mathrm{d}x$ for $x \geq 0$. For proof of the classical result see Theorem 1.3.6 in [27].

Theorem 3.1.1. *Under the assumptions of the Cramer-Lundberg model with net profit condition $\rho := c/(\mu\lambda) - 1 > 0$ where the integrated tail of the claim size distribution F is subexponentially distributed, it holds that*

$$\mathbb{P}(U(t) < 0 \textit{ for some } t) \sim \rho^{-1}\overline{F}_I(u),$$

as $u \to \infty$.

The constant ρ in the net profit condition is also called the safety loading. Remark that the subexponentiality assumption does not refer to the distribution of the claim size process, but to the integrated tail distribution of the claim size distribution. For many subexponential distributions such as regularly varying distributions and Weibull distributions, also the integrated tail distribution is subexponential. However, this is not true in general, but requires additional conditions, see Lemma 1.4.6 in [27] and Theorem 4.32 in [29].

Literature on ruin probabilities in the univariate setting considers problems under different assumptions on risk processes. Book [2] gives a broad overview of the topic. In the multivariate setting, significant contributions can be found in the light-tailed case in [16] and [17]. For multivariate heavy-tailed distributions, ruin probabilities were studied mostly under the assumption of multivariate regular variation. Theorem 3.1 in [43] states a multivariate extension of the result in a multivariate heavy-tailed setting. To state the theorem, we restrict the sets B to the space $\mathbb{R}^d \backslash K_{\mathbf{c}}^\delta$, for some $\delta > 0$ and a vector $\mathbf{c} > \mathbf{0}$ where $K_{\mathbf{c}}^\delta := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}/\|\mathbf{x}\| + \mathbf{c}/\|\mathbf{c}\|| < \delta\}$ to avoid sets that can be hit by drifting into the direction of the vector $-\mathbf{c}$. Here, we use $\|\cdot\|$ for the norm of a vector. Then, we set $\mu^*(B) := \int_0^\infty \mu(cv + B_{\mathbf{c}})\mathrm{d}v$ and $B_{\mathbf{c}} := \{\mathbf{x} + t\mathbf{c}, \mathbf{x} \in B, t \geq 0\}$ for any set $B \in \mathcal{B}(\mathbb{R}^d \backslash K_{\mathbf{c}}^\delta)$ that is any Borel set on the space $\mathbb{R}^d \backslash K_{\mathbf{c}}^\delta$.

Theorem 3.1.2. *Let $\boldsymbol{X}$ be a multivariate regularly varying random vector with regular variation index $\alpha > 1$ and probability measure μ, so $\boldsymbol{X} \in \mathrm{MRV}(\alpha, \mu)$. Furthermore, assume $\mathbb{E}(\boldsymbol{X}) = \boldsymbol{0}$ and $\boldsymbol{c} > \boldsymbol{0}$. Then, for any set $A \in \mathcal{B}(\mathbb{R}^d \backslash K_{\boldsymbol{c}}^{\delta})$ bounded away from $\boldsymbol{0}$, it holds that*

$$\begin{aligned}\mu^*(A^\circ) &\leq \liminf_{u\to\infty} \frac{\mathbb{P}(\boldsymbol{S}_n - n\boldsymbol{c} \in uA \text{ for some } n \geq 1)}{u\mathbb{P}(\|\boldsymbol{X}\| > u)} \\ &\leq \limsup_{u\to\infty} \frac{\mathbb{P}(\boldsymbol{S}_n - n\boldsymbol{c} \in uA \text{ for some } n \geq 1)}{u\mathbb{P}(\|\boldsymbol{X}\| > u)} \leq \mu^*(\overline{A}),\end{aligned}$$

where $\boldsymbol{S}_n = \boldsymbol{X}_1 + \cdots + \boldsymbol{X}_n$ as before, A° denotes the interior and $\overline{A}$ the closure of the set A.

In applications, the random vector $\mathbf{X}$ could be interpreted as the difference between some sufficient premium and the total claim amount. The fixed vector $-\mathbf{c}$ would then correspond to a safety loading and together with the sufficient premium it would determine the premium rate of the insurance company. In the one-dimensional setting, Theorem 3.1.2 coincides with the classical result of Theorem 3.1.1. The multivariate counterpart of the integrated tail function in Theorem 3.1.2 is given in terms of the norm of the random vector as $u\mathbb{P}(\|\mathbf{X}\| > u)$. This term coincides in the univariate setting with the integrated tail function.

Other results under the assumption of multivariate regular variation can be found, for instance, in [52, 10] considering finite-time ruin probabilities and [42] for infinite-time ruin probabilities.

3.2 Precise asymptotics of ruin probabilities

In [I], we study asymptotic approximations of ruin probabilities in a multivariate setting. We derive ruin probabilities for random walks where the increments of the random walk are closely connected to multivariate subexponentiality as it is presented in Definition 2.2.5. Therefore, we write $\mathbf{X}$ as the product of a heavy-tailed random variable R and a random vector $\boldsymbol{\theta}$ supported by the unit sphere according to the L_1-norm. Assuming that the random variable R is subexponenitally distributed, the increments of the random walk admit dependence between components.

The main result presented in [I] considers random vectors with lighter tails than regularly varying tails, but is similar to the result in Theorem 3.1.2 for regularly varying distributions.

Writing $\mathbf{X} = R\boldsymbol{\theta}$, setting $\mathbf{c} := -\mathbb{E}(\mathbf{X})$ and

$$A := \left\{ \mathbf{x} \in \mathbb{R}^d : \sum_{k=1}^{d} x^k > 1 \right\},$$

the assumptions on $\mathbf{X}$ are:

(A1) R is subexponentially distributed with distribution function F.

(A2) $\overline{F}_I(\gamma u) = o(\overline{F}_I(u))$ for all $\gamma > 1$.

(A3) There exists a set Θ with $\overline{\Theta} \subset \mathbb{S}^{d-1}_{>0}$ such that

$$\lim_{h \to \infty} \mathbb{P}\left(\boldsymbol{\theta} \in \Theta^{\varepsilon} | R > h\right) = 1 \quad \text{for all } \varepsilon > 0, \tag{3.1}$$

where $\overline{\Theta}$ denotes the closure of the set Θ and Θ^{δ} is defined in (3.2) below.

(A4) $\mathbf{c} = -\mathbb{E}(\mathbf{X}) > \mathbf{0}$.

(A5) $\int_0^\infty \mathbb{P}(\mathbf{X} \in A + v\mathbf{c})\mathrm{d}v < \infty$.

(A6) The distribution defined by $H(u) = \max\left(0, 1 - \int_0^\infty \mathbb{P}(\mathbf{X} \in uA + v\mathbf{c})\mathrm{d}v\right)$ is subexponential.

We define the δ-swelling Θ^{δ} of the set Θ in the subset of $\mathbb{S}^{d-1}$ in the positive orthant by

$$\Theta^{\delta} := \left\{ \mathbf{x} \in \mathbb{S}^{d-1}_{>0} : |\mathbf{x} - \mathbf{y}| < \delta \text{ for some } \mathbf{y} \in \Theta \right\}. \tag{3.2}$$

Applying the model to an insurance company, we can interpret the distribution of $\mathbf{X}$ as the yearly net-payout, the difference between the collected premium and the total claim amount. In this setting, Assumption (A4) corresponds to a positive safety loading and ensures that the stochastic process drifts to the negative orthant. Assumption (A3) excludes the possibility of asymptotically independent components. In a two-dimensional setting, Assumption (A3) can be seen as an analogue to full or strong asymptotic dependence in the framework of multivariate regular variation depending on the shape of the set Θ. Finally, Assumption (A6) can be understood as an extension of the integrated tail function to a multivariate setting where one integrates along the set $uA+v\mathbf{c}$.

Assumption (A2) does not hold for regularly varying distributions of R as the following example shows.

Example 3.2.1. Let R be Pareto distributed with $\mathbb{P}(R > x) = x^{-\alpha}$ and $\alpha > 1$. Then, the integrated tail distributions is

$$\int_u^\infty x^{-\alpha} \mathrm{d}x = \frac{u^{1-\alpha}}{\alpha - 1}.$$

For any $\gamma > 1$, it holds that

$$\lim_{u\to\infty} \frac{\overline{F}_I(\gamma u)}{\overline{F}_I(u)} = \lim_{u\to\infty} \frac{(\gamma u)^{1-\alpha}}{\alpha - 1} \left(\frac{u^{1-\alpha}}{\alpha - 1} \right)^{-1} = \gamma^{1-\alpha} > 0$$

so Assumption (A2) does not hold.

Under Assumptions (A1) - (A6), we derive a multivariate generalisation of the ruin probability result for the probability that the random walk $\mathbf{S}_n$ hits the truncated cone V_{u,Θ^δ} defined in (2.8).

Theorem 3.2.2. *Set $\delta > 0$ such that for all $\boldsymbol{y} \in \Theta^\delta$ it holds for all components that $y^k > \delta/(4+\delta)$. Set V_{u,Θ^δ} as in* (2.8) *and assume (A1) - (A6) hold. Then, it holds that*

$$\lim_{u\to\infty} \frac{\mathbb{P}(\boldsymbol{S}_n \in V_{u,\Theta^\delta} \text{ for some } n \geq 1)}{\frac{1}{\|\boldsymbol{c}\|} \int_u^\infty \mathbb{P}(R > v) \mathrm{d}v} = 1. \tag{3.3}$$

Theorem 3.2.2 is a multivariate extension of the classical ruin probability result for random vectors fulfilling Assumptions (A1) - (A6). The integrated tail of the claim size distribution in the classical result presented in Theorem 3.1.1 corresponds in the multivariate case to the integrated tail of the length of the random vectors $\mathbf{X}$. A partial reverse result of Theorem 3.2.2 holds.

Theorem 3.2.3. *Suppose (A1), (A2), (A4) - (A6) hold and $\lim_{h\to\infty} \mathbb{P}(\boldsymbol{\theta} \in \mathbb{S}^{d-1}_{>0} | R > h) = 1$. Assume further that the limit in (3.1) exists for all $\Theta^\delta \subset \mathbb{S}^{d-1}_{>0}$ and that $u^2 \overline{F}(u) = o(1)$. Then, there exists some set Θ for which Equation (3.3) is equivalent to Condition (3.1).*

Chapter 4

Large deviations theory and review of [II]

4.1 Large deviations theory

The theory of large deviations deals with rare or "unlikely" events. It examines how fast or at which rate the probabilities of such rare events decay and offers techniques to estimate other properties of these rare events such as their frequency and most likely manner of occurrence. A large deviation principle characterises the limiting behaviour of a family of probability measures in terms of a rate function. In the terminology of large deviations theory, a function I is said to be a rate function if it is lower semi-continuous. A rate function I is good if the sub-level sets $\{\mathbf{x} \in \mathbb{R}^d : f(\mathbf{x}) \geq \alpha\}$ are compact for all $\alpha \in \mathbb{R}$. Intuitively, the rate function is a function that quantifies the probabilities of rare events. A large deviations principle is then defined as follows:

Definition 4.1.1. A distribution family $\{P_n\}$ fulfils a large deviations principle with rate function I, if

$$\liminf_{n\to\infty} n^{-1} \log(P_n(G)) \geq - \inf_{x\in G} I(x)$$

for all open sets $G \subset \mathbb{R}^d$ and

$$\limsup_{n\to\infty} n^{-1} \log(P_n(F)) \geq - \inf_{x\in F} I(x) \tag{4.1}$$

for all compact sets $F \subset \mathbb{R}^d$. If Inequality (4.1) holds for all closed sets $F \subset \mathbb{R}^d$, the distribution family $\{P_n\}$ fulfils a full large deviations principle with rate function I.

Book [22] gives an overview of large deviations theory focusing more on the case of light-tailed distributions. Large deviations principles in heavy-tailed frameworks have been studied, for instance, in [39, 32] and precise large deviations in [59, 55] and the monograph [12].

In a one-dimensional setting, a full large deviations principle for distributions with a regularly varying risk function was shown in Theorem 2.7 in [49].

Theorem 4.1.2. *Suppose X is a random variable for which $\mathbb{E}(X) = 0$, the random variable $X^- = \max(0, -X)$ is light-tailed and the risk function $-\log(\mathbb{P}(X > x))$ is regularly varying with index $\alpha \in (0, 1)$. Then,*

$$\lim_{n\to\infty} \frac{-\log(\mathbb{P}(S_n > na))}{-\log(\mathbb{P}(X > n))} = \begin{cases} 0, & a < 0 \\ a^\alpha, & a \geq 0 \end{cases}$$

and the process $\{S_n/n\}$ satisfies a full large deviations principle with good rate function

$$I(x) = \begin{cases} \infty, & x < 0 \\ x^\alpha, & x \geq 0 \end{cases}$$

and the risk function as scale.

In [II], we extend this one-dimensional result to spherical and elliptical distributions where the tail decreases at a similar rate as the tail of the random variable in Theorem 4.1.2.

In the multivariate setting, large deviations are often examined under assumptions close to multivariate regular variation as in [61, 57]. [43] studies precise asymptotics in the setting of regular variation.

4.2 Large deviations for asymptotically elliptical risk processes

In [II], we study the random walks generated by the random vectors $\mathbf{X} = R\mathbf{U}$ or $\mathbf{X} = R\boldsymbol{\Theta}$, where R has a moderate heavy tail. For the random vectors, we assume in the spherical framework that $\mathbf{U}$ is supported by the $d-1$ dimensional unit sphere and in the elliptical case $\boldsymbol{\Theta}$ is supported by the boundary of an ellipsoid. This ellipsoid is centred at the origin and we denote its boundary by

Ω. We represent the length of the random vector using the L_2-norm because it treats ellipsoids as such. The moderate heavy tail distributions of the random variable R include Weibull distributions with parameter $\beta \in (0,1)$ and lognormal type distributions defined by the asymptotic relation (1.1). The central assumptions on R and $\mathbf{U}$ are:

(A1) The tail function of the random variable R satisfies

$$-\log(\mathbb{P}(R > x)) \sim h(x), \tag{4.2}$$

as $x \to \infty$, where $h(x)$ is an increasing and concave function such that

(i) $h(x) = o(x)$ and

(ii) $\log(x) = o(h(x))$, as $x \to \infty$.

(A2) The random vector $\mathbf{U} \in \mathbb{S}^{d-1}$ has a distribution on the d-dimensional unit sphere $\mathbb{S}^{d-1}$. Let $S \subset \mathbb{S}^{d-1}$ be a subset with positive Lebesgue measure. Then, we assume that $\mathbb{P}(\mathbf{U} \in S) > 0$. In addition, $\mathbf{U}$ is assumed to be asymptotically independent of the random variable R in the sense that the limit

$$\lim_{x\to\infty} \mathbb{P}(\mathbf{U} \in S | R > x) = \mathbb{P}(\mathbf{U} \in S)$$

exists and $\mathbb{E}(R\mathbf{U}) = \mathbf{0}$.

The following example shows that Assumption (A1) does not hold in general for regularly varying random variables.

Example 4.2.1. A Pareto distributed random variable R with regular variation index $\alpha > 0$ has the risk function $h(x) = \alpha \log(x)$. Thus, the risk function is concave and $h(x) = o(x)$ holds, but $h(x)$ decreases at the same rate as $\log(x)$.

Moreover, the fact that $\log(x) = o(h(x))$ implies that $\mathbb{E}(R^s) < \infty$ for all $s > 0$, see Lemma 2.3.1. Thus, the random variable R has finite moments of all orders and Assumption (A1) does not hold for regularly varying random variables. Lemma 1.2.3 ensures that R has a subexponential distribution if its risk function is equal to the concave function h.

To derive large deviations principles for asymptotically spherical and elliptical distributions, we have to investigate the asymptotic behaviour of the norm of the random walk generated by random vectors $R\mathbf{U}$. The following theorem shows the asymptotic relation between the norm of the random walk and the length R of a single random vector.

Theorem 4.2.2. *Let $a > 0$ be a fixed number. Suppose the increment of the random walk $\{\boldsymbol{S}_n\}$ is of the form $\boldsymbol{X} = R\boldsymbol{U}$, where assumptions (A1) and (A2) hold. Then,*

$$\lim_{n\to\infty} \frac{\log(\mathbb{P}(\|\boldsymbol{S}_n\|_2 > na))}{h(na)} = -1.$$

The asymptotic relation in Theorem 4.2.2 yields a full large deviations principle for asymptotically spherical distributions. Applying the contraction principle, we can extend that full large deviations principle to asymptotically elliptical distributions. Therefore, we have to modify the assumption on the random vector.

(A2') The d-dimensional random vector $\boldsymbol{\Theta}$ is distributed on an ellipse or ellipsoid Ω centred at the origin fulfilling $\mathbb{E}(R\boldsymbol{\Theta}) = \boldsymbol{0}$. It holds for every set $S \subset \Omega$ with positive Lebesgue measure on Ω that $\mathbb{P}(\boldsymbol{\Theta} \in S) > 0$, and $\boldsymbol{\Theta}$ is asymptotically independent of the random variable R in the sense that $\lim_{x\to\infty} \mathbb{P}(\boldsymbol{\Theta} \in S | R > x) = \mathbb{P}(\boldsymbol{\Theta} \in S)$ exists.

In Theorem 4.2.3, we denote by B° the interior of the set B and $\overline{B}$ means its closure.

Theorem 4.2.3. *Let $\boldsymbol{X} = R\boldsymbol{\Theta}$ where R and $\boldsymbol{\Theta}$ fulfil assumptions (A1) and (A2'), $a > 0$ be a fixed number and the limit*

$$\lim_{x\to\infty} \frac{h(ax)}{h(x)}$$

exists. Let A be a symmetric, positive definite $d \times d$ matrix and define $\Lambda : \mathbb{R}^d \to \mathbb{R}^d$ by $\Lambda(\boldsymbol{x}) = A\boldsymbol{x}$ such that Λ maps $\mathbb{S}^{d-1}$ to Ω. Then, the process $\{\boldsymbol{S}_n/n\}$ satisfies the large deviations principle with scale h, so for all Borel sets $B \subset \mathbb{R}^d$

$$\begin{aligned} -\inf_{\boldsymbol{y}\in\Lambda^{-1}(B^\circ)} I(\boldsymbol{y}) &\le \liminf_{n\to\infty} \frac{\log(\mathbb{P}(\boldsymbol{S}_n/n \in B))}{h(n)} \\ &\le \limsup_{n\to\infty} \frac{\log(\mathbb{P}(\boldsymbol{S}_n/n \in B))}{h(n)} \le -\inf_{\boldsymbol{y}\in\Lambda^{-1}(\overline{B})} I(\boldsymbol{y}) \end{aligned}$$

where

$$I(\boldsymbol{x}) = \begin{cases} \lim_{n\to\infty} \frac{h(n\|\boldsymbol{x}\|_2)}{h(n)}, & \text{if } \boldsymbol{x} \neq \boldsymbol{0} \\ 0, & \text{if } \boldsymbol{x} = \boldsymbol{0} \end{cases}$$

and $\Lambda^{-1} : \mathbb{R}^d \to \mathbb{R}^d$ with $\Lambda^{-1}(\boldsymbol{x}) = A^{-1}\boldsymbol{x}$.

If R has a Weibull distribution with $\beta \in (0,1)$, the rate function in Theorem 4.2.3 is $I(\mathbf{x}) = \|\mathbf{x}\|_2^\beta$ and hence good. For a lognormal type distribution, the rate function is constant everywhere except at the origin.

As an application of Theorem 4.2.3, we study optimal Quota Share risk sharing strategies. Quota Share is a widely used reinsurance concept, where the insurance company and the reinsurance company agree to share a random risk X with a fixed ratio $q \in [0,1]$. One contract partner pays the share qX whereas the other contract partner pays the remaining part $(1-q)X$. We derive optimal ratios $q_1, \dots, q_d$ for Quota Share contracts of d-dimensional insurance portfolios under the assumption of elliptical risk processes from the point of view of both, the insurance company and its counterpart the reinsurer.

Remark 4.2.4. In the proof of Lemma 1 in [II] there are two typing errors. The inequality in the calculation should be less or equal and in the last sentence we mean the corresponding lower bound instead of the corresponding upper bound. Lemma 1 remains true.

Chapter 5

Multivariate heavy-tailed data and review of [III]

5.1 Overview of the analysis of multivariate heavy-tailed data

Analysing multivariate heavy-tailed data, a broadly used approach is to study at first the norm of the observations using tools for one-dimensional data. In this way, one gets an intuition about the dominating tail behaviour of the multivariate distribution. Knowing the asymptotic behaviour of the norm of the random variables helps to find a suitable model. However, the distribution of the norm of the random vector does only reflect the heaviness of the tail of the normed vector, but it does not tell anything about the asymptotic dependence structure between the individual components. Usually, one wants to know whether the tail behaviour depends on the direction and if so, how it changes in different directions. Mathematically, one is interested in the asymptotic distribution of the directional component on the unit sphere.

Multiple different models have been introduced to analyse multivariate data with heavy-tailed features. Many of them rely on the assumption of multivariate regular variation [69], which is introduced in Section 2.1. Beside the concepts of full and strong asymptotic dependence [21], copulas [28, 76] and other methods including estimating methods for the limit measure [26], its support [50, 21], the tail dependence coefficient [30] and the tail dependence function [64, 72] are used to describe the tail dependence of heavy-tailed data.

In addition to the framework of multivariate regular variation, other multivariate models such as elliptical distributions [48, 40] briefly introduced in Section 2.3.4 and parametric models [1] are applied to multivariate observations to understand their underlying distributions. Elliptical distributions assume a uniform distribution of the normed random vectors. Due to this property, they are used to explain heavy-tailed features in applications where observations occur in all directions such as financial data representing return rates of stock portfolios.

In the framework of multivariate regular variation, one wants to verify that the multivariate data with the assumption on independent and identically distributed observations is due to an underlying generalized Pareto distribution. Therefore, one would take the one-dimensional data gained from the norm of the observations and proceed in analysing the mean excess plot [19]. The mean excess plot is a diagnostic tool that takes thresholded data and visualises the mean excess function $\mathbb{E}(X - u|X > u)$. A linear-looking mean excess plot with positive slope indicates generalized Pareto assumptions, see [34, 35]. If the one-dimensional data turns out to be regularly varying, a hill plot [24] or other statistical methods like QQ-estimators and plots [69] help to identify the regular variation indices of the marginal distributions. Via rank transform [68, 38], the data is transformed from non-standard regular variation to the standard case, where the angular measure can be estimated and tail dependence can be characterised.

For multivariate heavy-tailed models not necessarily based on regular variation, [III] presents a method to find the set of directions that contains the dominating tail behaviour of the multivariate distribution.

5.2 Identifying the riskiest directional components

For multivariate heavy-tailed data, the question is not only how the dominating tail behaves and how thick the tail is, but the directional component also plays a role. One needs to understand what produces the largest sources of risks to find ways to reduce riskiness.

Insurance companies with different lines of business or offering insurance contracts including different policies would like to know where the major risks lie in order to find a suitable reinsurance strategy. Companies and stakeholders on financial markets might be interested in which directions extremal observation of their portfolios occur to find good hedging strategies.

In [III], we write the random vector $\mathbf{X} = R\mathbf{\Theta}$ using polar coordinates as in Equation (2.7) in Section 2.3. The length of $\mathbf{X}$ is determined by the random variable R and its directional component by the random vector $\mathbf{\Theta}$ on the unit sphere. If the conditional distribution exists, we can present the conditional risk or hazard function for $x \geq 0$ as

$$h(x, \boldsymbol{\theta}) = -\log(\mathbb{P}(R > x \mid \mathbf{\Theta} = \boldsymbol{\theta})),$$

where $h(x, \boldsymbol{\theta})$ is a positive and increasing function for fixed $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$.

The main focus of [III] is to detect the set of directions that dominates the tail behaviour of the norm of the random vector. Mathematically we are interested in set S defined by

$$\left\{\mathbf{v} \in \mathbb{S}^{d-1} \colon \lim_{x\to\infty} \frac{\log(\mathbb{P}(R > x | \mathbf{\Theta} \in B(\mathbf{v}, \varepsilon)))}{\log(\mathbb{P}(R > x))} = 1,\, \mathbb{P}(\mathbf{\Theta} \in B(\mathbf{v}, \varepsilon)) > 0,\, \forall \varepsilon > 0\right\}, \tag{5.1}$$

where $B(\mathbf{v}, \varepsilon)$ denotes the ε-environment of the vector $\mathbf{v} \in \mathbb{S}^{d-1}$. We call set S the set of the most risky directions or the minimal set that dominates the tail behaviour of the random vector $\mathbf{X}$.

In the main results, we use the following assumptions:

(A1) R is a positive random variable with right-unbounded support. For any $k \in \mathbb{R}$, it holds that $\mathbb{P}(R > k) > 0$ and R is heavy-tailed in the sense that $\lim_{k\to\infty} -\log(\mathbb{P}(R > k))/k = 0$.

(A2) $\mathbf{\Theta}$ is a random vector on the unit sphere $\mathbb{S}^{d-1}$ such that the quantity

$$\mathbb{P}(\mathbf{\Theta} \in A | R > k)$$

remains constant for all $k > k_0$ where $k_0 > 0$ is a fixed number that does not depend on the Borel set $A \subset \mathbb{S}^{d-1}$. In particular, the limiting probability distribution of $\mathbf{\Theta} \mid R > k$ exists, as $k \to \infty$. In fact, the limiting distribution on $\mathbb{S}^{d-1}$ is reached once $k > k_0$.

(A3) The limit $\lim_{k\to\infty} g(k, A)$ exists in $[0, \infty]$ for all Borel sets $A \subset \mathbb{S}^{d-1}$, where $g : (k_R, \infty) \times \mathcal{B}(\mathbb{S}^{d-1}) \to \mathbb{R}$ is defined as

$$g(k, A) = \frac{\log(\mathbb{P}(R > k, \mathbf{U} \in A))}{\log(\mathbb{P}(R > k))} \tag{5.2}$$

and $k_R = \sup\{k : \mathbb{P}(R > k) = 1\}$.

(A4) Firstly, we assume that there exists an open set in S^c. Secondly, we assume that S in (5.1) can be written as

$$S = \overline{T_1} \cup T_2,$$

where T_1 is an open subset (possibly empty) of $\mathbb{S}^{d-1}$ and T_2 is a finite collection of individual points (possibly empty) of $\mathbb{S}^{d-1}$. We assume that each point in T_2 contains positive probability mass of the limit distribution of $\mathbf{\Theta} \,|\, R > k$, as $k \to \infty$.

One of the main results of [III] shows how the set of the most risky directions S can be defined in terms of the limit of the ratio of risk functions

$$g(k, A) := \frac{\log(\mathbb{P}(R > k, \mathbf{\Theta} \in A))}{\log(\mathbb{P}(R > k))},$$

where $k > 0, A \subset \mathbb{S}^{d-1}$. To state the result, we define the collection $\mathcal{A}$ of testing sets $A \subset \mathbb{S}^{d-1}$ as follows. A set A is an element of $\mathcal{A}$ if A is a finite union of open balls such that for all $\mathbf{x} \in A^c$ and for all $\varepsilon > 0$ the open ball $B(\mathbf{x}, \varepsilon)$ contains an open ball B that belongs to A^c. In particular, this guarantees that A^c does not contain any isolated points.

Theorem 5.2.1. *Let $\boldsymbol{X} = R\mathbf{\Theta} \in \mathbb{R}^d, d \geq 2$ be such that Assumptions (A1)-(A4) hold. Set*

$$\tilde{S} = \cap \left\{ A \in \mathcal{A} : \lim_{k\to\infty} g(k, A) < \lim_{k\to\infty} g(k, A^c) \right\}. \tag{5.3}$$

Then, $S = cl(\tilde{S})$, where S is as in (5.1). Furthermore, for all $\delta > 0$,

$$\lim_{k\to\infty} \frac{\log\left(\mathbb{P}(R > k, \mathbf{\Theta} \in \tilde{S}^\delta)\right)}{\log(\mathbb{P}(R > k))} = 1 \tag{5.4}$$

and

$$\lim_{k\to\infty} \frac{\log\left(\mathbb{P}(R > k, \mathbf{\Theta} \in (\tilde{S}^\delta)^c)\right)}{\log(\mathbb{P}(R > k))} > 1. \tag{5.5}$$

Set S in Theorem 5.2.1 is minimal in the sense that it is the smallest set that dominates the tail behaviour of the random vector in a logarithmic scale since it is the intersection of all sets $A \in \mathcal{A}$ for which $\lim_{x\to\infty} g(x, A) < \lim_{x\to\infty} g(x, A^c)$ holds.

Besides studying set S and its properties, [III] discusses an idea to find the minimal set S of the most risky directions in practical applications. This approach is based on the following corollary of Theorem 5.2.1.

Corollary 5.2.2. *Let $\boldsymbol{X} = R\boldsymbol{\Theta}$ be such that Assumptions (A1)-(A4) hold and $A \subset \mathbb{S}^{d-1}$ is a Borel set. If there exists $\delta > 0$ such that $\tilde{S}^{\delta}$, the δ-swelling of the set $\tilde{S}$, is a subset of A, it holds that*

$$\lim_{k\to\infty} g(k, A) < \lim_{k\to\infty} g(k, A^c).$$

We define the empirical counterpart of the function $g(k, A)$ as

$$\hat{g}(k, A) := \frac{\log\left(\sharp\{\mathbf{x}_i : \|\mathbf{x}_i\| > k, \frac{\mathbf{x}_i}{\|\mathbf{x}_i\|} \in A\}/n\right)}{\log\left(\sharp\{\mathbf{x}_i : \|\mathbf{x}_i\| > k\}/n\right)},$$

where k is the threshold to identify the tail and $A \subset \mathbb{S}^{d-1}$ and suppose a fixed tolerance threshold $c > 0$ and large sample size n.

Corollary 5.2.2 indicates that if $\hat{g}(k, A) > 1 + c$ for a set A, it gives evidence for A being in the complement of S and if $\hat{g}(k, A) > 1 + c$ we gain evidence for A containing at least some subset of S. In [III], we apply this idea to two-dimensional simulated data with heavier Pareto distributed radial components and lighter Weibull distributed components. Therefore, we divide $\mathbb{R}^2$ into half spaces A, A^c by taking some vector $\mathbf{v}$ and increasing the radius r of the closed ball $\overline{B}(\mathbf{v}, r)$ on $\mathbb{S}^{d-1}$ until at least half of the observations belong to the ball $\overline{B}(\mathbf{v}, r)$ and set $A = \overline{B}(\mathbf{v}, r)$. For these sets A and A^c, we check whether the empirical functions $\hat{g}(k, A)$ and $\hat{g}(k, A^c)$ are greater or less than $1 + c$ and reject all sets for which $\hat{g}(k, A) > 1 + c$. Splitting $\mathbb{R}^2$ in different pairs of half spaces A_i, A_i^c, we get a preliminary estimate for S by taking the complement of the union of the rejected sets.

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