ON THE EXERCISE OF AMERICAN QUANTO OPTIONS

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Abstract

American option pricing is an important and engaging area of financial economics, particularly so in the presence of negative interest rates. Quanto options offer major international hedging/investment opportunities. We provide a comprehensive description of the optimal exercise policies associated with American quanto options. We show that a non-standard exercise policy characterized by a double continuation region may be optimal in the presence of non–positive domestic interest rates. We study empirical examples of finite-maturity American quanto options for which a double continuation region surrounding a non-empty early exercise region exists even if the infinite-maturity early exercise region is empty and the value of the infinite-maturity option is unbounded. Under the assumptions underpinning such empirical examples, we carefully characterize the existence, the monotonicity properties and the close-to-maturity behavior of the upper and lower critical prices.

KEYWORDS: Quanto Options; American Options; Valuation; Optimal Exercise; Negative Interest Rates; FX Markets.

1 Introduction

As options markets offer vital services and signals to hedgers, investors, and financial decision makers, American/European-type option pricing remains an extremely active research area in the financial economics literature (e.g. Almeida and Freire (2021), Amaya, Bégin, and Gauthier (2021), Aramonte,

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Jahan-Parvar, Rosen, and Schindler (2021), Bakshi, Cao, and Zhong (2021), Barro and Liao (2021), Battauz and Rotondi (2022), De Donno, Palmowski and Tumilewicz (2020), Figlewski (2021), Golez and Goyenko (2021), Jeon and Kim (2021), Jing, Li, and Ma (2021), Lee, Han and Lee (2021), Yu (2021), Wang (2021a), Wang (2021b), Wang, Wang and Shao (2022), and Wang and Zhang (2022)). Amid the impressive growth of the FX options markets (see for example James, Fullwood, and Billington (2015)), the appeal of quanto options comes from their ability of opening up hedging/investment opportunities in international markets as they are derivative contracts written on an underlying foreign security. European Quanto options can be exercised at maturity only and have been carefully studied (e.g. Ng, Li, and Chan (2013), Kim, Lee, Mittnik, and Park (2015), Li, Zhang, and Liu (2018), and Fallahgoul, Kim, Fabozzi, and Park (2019)). American quanto options can be exercised during their whole life and have been much less investigated.

We fill this gap by using a parsimonious diffusive model to provide an exhaustive characterization of the optimal exercise policies of American quanto put options. We show that they depend on the payoff structure as well as on the interplay between the domestic and the foreign riskless interest rates. Importantly, we highlight that, in the presence of a domestic non-positive interest rate (as for instance the Euro denominated or the Yen denominated markets in recent years) and of a foreign positive interest rate (as for instance the US Dollar denominated market), these options can exhibit unusual optimal exercise policies. As the sign of the domestic interest rate changes from positive to negative, a non-standard double continuation region can appear: immediate option exercise is optimally postponed not only if the contract is not enough in-the-money but also, unusually, if the contract is too much in-the-money. This is because the urge to defer exercise due to the negative domestic rate overwhelms the incentive of immediate exercise due to a drift toward the out-of-the-money region, if the distance from the out-of-the-money region is comfortably huge.

We provide numerical examples of Euro-denominated American quanto put options on US stocks in which a non-standard double continuation region appears in the binomial approximation. Interestingly, these finite-maturity American quanto put options do exhibit a non-empty exercise region, even if the corresponding perpetual American quanto put options have an empty exercise region and an unbounded value.

Motivated by such numerical examples, we markedly generalize and extend the findings of Battauz, De Donno and Sbuelz (2015) by working out the existence, the monotonicity properties and the at-maturity asymptotics of the upper and lower critical prices for American options under the milder assumption of non-emptiness of the early exercise region at some date during the life of the options involved.

The rest of the paper is organized as follows. Section 2 discusses different types of American quanto options in a lognormal currency market. Section 3 analyzes the optimal exercise policies of these options and their interplay with the sign of the riskless interest rates. Section 4 provides numerical examples as well as market-calibrated examples of the non-standard double continuation region. Section 5 concludes.

2 American quanto options in a lognormal currency market

We consider a frictionless continuous-time market, modeled through a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ satisfying the usual assumptions (in the sense of Definitions I.1.2 and I.1.3 in Jacod and Shiryaev (2003)). Let $B_d(t) = e^{r_d t}$ be the domestic riskless bond price, where r_d is the constant *domestic* riskless interest rate. Denote with $B_f(t) = e^{r_f\, t}$ the foreign riskless bond price, where r_f is the constant *foreign* riskless interest rate and with $S_f = \{S_f(t)\}_{t\in[0,T]}$ the foreign risky security price described by

$$\frac{dS_f(t)}{S_f(t)} = \mu_f dt + \sigma_f \, dW^P(t)$$

where W^P is the $\Re^2$-Brownian motion under the historical probability measure P with respect to the filtration $\mathcal{F}$ and σ_f is the $\Re^2-$vector of volatilities of the foreign security. Let G_f be the cumulative gain process obtained by buying 1 unit of the the foreign risky security at the initial date $t = 0$. If the foreign security pays a continuous dividend yield q_f and the dividend $q_f S_f(t)dt$ is continuously reinvested in the security S_f, the value of the cumulative gain process at t is

$$G_f(t) = e^{q_f\, t}\, S_f(t)$$

and its differential

$$dG_f(t) = e^{q_f\, t} dS_f(t) + q_f S_f(t)dt = G_f(t)\left((\mu_f + q_f)\, dt + \sigma_f \, dW^P(t)\right). \tag{2.1}$$

The two markets are connected via the *foreign to domestic exchange rate* X. If, for instance, we pick the Euro market as the domestic one and the US market as the foreign one, X is the *dollar to euros* exchange rate. Assume that X is lognormal[1] and driven by

$$\frac{dX(t)}{X(t)} = \mu_X dt + \sigma_X \, dW^P(t)$$

under the historical probability measure P (σ_X is an $\Re^2-$vector). For sake of notation we will denote the scalar product of two vectors $\sigma_1, \sigma_2 \in \Re^2$ with $\sigma_1\sigma_2$. The following proposition describes the domestic risk neutral distribution of the assets in the market and the forward rate.

Proposition 2.1 *The domestic risk neutral dynamics of the foreign risky security price is*

$$\frac{dS_f(t)}{S_f(t)} = \mu_f^Q dt + \sigma_f \, dW^Q(t)$$

where W^Q *is a* $\Re^2-$*Brownian motion,* Q *denotes the domestic risk neutral measure and*

$$\mu_f^Q = r_f - q_f - \sigma_f\sigma_X.$$

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[1] For continuous-time models with currency returns predictability see e.g. Pavlova and Rigobon (2007), and Bakshi, Carr, and Wu (2008). Bakshi and Panayotov (2013) investigate the payoff predictability of currency carry trades. Karolyi and Wu (2021) study the pricing of currency risk factors across global stock returns.

The exchange rate is

$$\frac{dX(t)}{X(t)} = \mu_X^Q dt + \sigma_X \, dW^Q(t)$$

with $\mu_X^Q = r_d - r_f$*. The correlation between the foreign security* S_f *and the exchange rate* X *is*

$$\rho = \frac{\sigma_f \cdot \sigma_X}{\|\sigma_f\| \, \|\sigma_X\|}.$$

The domestic-denominated foreign security price $S_f^*(t) = S_f(t)X(t)$ *is driven by*

$$dS_f^*(t) = S_f^*(t) \left[(r_d - q_f) \; dt + \left(\sigma_X \; + \sigma_f \right) dW^Q(t) \right].$$

The forward exchange rate F_0*, i.e. the number of units of domestic currency settled at* $t = 0$ *to receive one unit of the foreign currency at the maturity* T *is*

$$F_0 = E^Q \left[X(T) \right] = X(0) \; e^{\mu_X^Q T} = X_0 e^{\left(r_d - r_f \right) T}.$$

Proof. Let $B_f^*(t) = B_f(t)X(t)$. B_f^* and S_f^* are both *risky domestic securities.* Then, exploiting the integration by parts' formula and observing that the covariation between B_f and X is 0, we have

$$\begin{aligned} dB_f^*(t) = d\left(B_f(t)X(t) \right) = X(t)dB_f(t) + B_f(t)dX(t) = \\ = B_f^*(t) \left[\left(r_f + \mu_X^Q \right) dt + \sigma_X \; dW^Q(t) \right] \end{aligned}$$

and no-arbitrage implies $r_f + \mu_X^Q = r_d$, delivering $\mu_X^Q = r_d - r_f$. Consider the cumulative gain process denominated in domestic currency $G_f^*(t) = G_f(t)X(t)$. The differential of G_f in Equation (2.1) can be rewritten with respect to the domestic risk neutral measure Q as

$$dG_f(t) = dG_f(t) = e^{q_f \; t} dS_f(t) + q_f S_f(t)dt = G_f(t) \left(\left(\mu_f^Q + q_f \right) dt + \sigma_f \; dW^Q(t) \right).$$

Ito formula implies

$$\begin{aligned} dG_f^*(t) = d(G_f(t)X(t)) = X(t)dG_f(t) + G_f(t)dX(t) + \sigma_f G_f(t)\sigma_X X(t)dt = \\ = G_f^*(t) \left[\left(\mu_f^Q + q_f + \mu_X^Q + \sigma_f \sigma_X \right) dt + \left(\sigma_X \; + \sigma_f \right) dW^Q(t) \right]. \end{aligned}$$

No-arbitrage implies that G_f^* discounted at the rate r_d is a $Q-$martingale, i.e. the domestic-risk neutral drift of G_f^* equals r_d. This delivers the equation $\mu_f^Q + q_f + \mu_X^Q + \sigma_f \sigma_X = r_d$, which implies $\mu_f^Q = r_f - q_f - \sigma_f \sigma_X$. The equation for S_f^* becomes

$$\begin{aligned} dS_f^*(t) &= S_f^*(t) \left[\left(\mu_f^Q + \mu_X^Q + \sigma_f \sigma_X \right) dt + \left(\sigma_X \; + \sigma_f \right) dW^Q(t) \right] \\ &= S_f^*(t) \left[\left(r_f - q_f - \sigma_f \sigma_X + r_d - r_f + \sigma_f \sigma_X \right) dt + \left(\sigma_X \; + \sigma_f \right) dW^Q(t) \right] \\ &= S_f^*(t) \left[(r_d - q_f) \; dt + (\sigma_X \; + \sigma_f) \, dW^Q(t) \right] \end{aligned}$$

The *forward exchange rate* F_0, i.e. the number of units of domestic currency settled at $t = 0$ to receive one unit of the foreign currency at the maturity T, is determined by $\mathbf{E}^Q \left[e^{-r_d T} \left(1 \cdot X(T) - F_0 \right) \right] = 0$, that implies the following relation between the forward and the spot exchange rate in terms of the domestic and foreign interest rates: $F_0 = \mathbf{E}^Q\left[X(T)\right] = X(0)\, e^{\mu_X^Q T} = X_0 e^{\left(r_d - r_f\right)T}$. $\square$

Quanto options are domestically negotiated derivatives written on a foreign security. The simplest quanto options are call and put options on foreign stocks or indexes. The strike price can be expressed either in domestic or in foreign currency. The foreign-currency denominated payoff is converted into the domestic currency either via a fixed exchange rate (as the forward exchange rate or the spot exchange rate at inception) or a via floating one (the spot exchange rate prevailing at the exercise date). We focus on American quanto put options, and provide now a list of them under various contract specifications. Via the American put-call symmetry, our results do also apply to symmetric American quanto call options (see for instance Proposition 3.1 in Battauz, De Donno and Sbuelz (2015)).

In Equation (2.2), the time t instantaneous payoff has a foreign-denominated strike K_f and is converted in domestic currency at the floating exchange rate at exercise:

$$V(t) = \sup_{t \leq \tau \leq T} \mathbf{E}^Q\left[e^{-r_d(\tau - t)}\, X(\tau)\left(K_f - S_f(\tau)\right)^+ \middle| \mathcal{F}_t\right] \tag{2.2}$$

In this case the American quanto option coincides with the foreign American option, converted in domestic currency at the current floating exchange rate. Thus early exercise is optimal for the American quanto put option if the foreign underlying risky security enters the optimal early exercise region of the American put option (see Proposition 2.3).

But American quanto options are more appealing to investors, if the currency risk is reduced. This goal is achieved by settling a domestic denominated strike price K_d as in (2.3) , where the foreign security is converted in domestic currency at the floating exchange rate at exercise,

$$V(t) = \sup_{t \leq \tau \leq T} \mathbf{E}^Q\left[e^{-r_d(\tau - t)}\left(K_d - X(\tau)S_f(\tau)\right)^+ \middle| \mathcal{F}_t\right] \tag{2.3}$$

or, as in Equation (2.4), where the strike price is denominated in domestic currency and the exchange rate for conversion at any exercise date is fixed at its initial spot level

$$V(t) = \sup_{t \leq \tau \leq T} \mathbf{E}^Q\left[e^{-r_d(\tau - t)}\left(K_d - X_0 S_f(\tau)\right)^+ \middle| \mathcal{F}_t\right]. \tag{2.4}$$

A similar payoff structure in Equation (2.5) maintains the domestic-denominated strike price and fixes the exchange rate at the initial forward level:

$$V(t) = \sup_{t \leq \tau \leq T} \mathbf{E}^Q\left[e^{-r_d(\tau - t)}\left(K_d - F_0 S_f(\tau)\right)^+ \middle| \mathcal{F}_t\right]. \tag{2.5}$$

In another popular version of the quanto option, the strike price is denominated in the foreign currency and the payoff converted at the initial spot exchange rate

$$V(t) = \sup_{t \leq \tau \leq T} \mathbf{E}^Q\left[e^{-r_d(\tau - t)}\, X_0 \cdot \left(K_f - S_f(\tau)\right)^+ \middle| \mathcal{F}_t\right] \tag{2.6}$$

2.1 American put options

We carefully outline the optimal exercise results for American put options that we will later employ to construct a template for classifying and characterizing the optimal exercise policy of American quanto put options. Let B^Q be a one-dimensional $Q-$Brownian motion and denote with

$$v(t,s;\mu,\sigma,\delta,K)=\sup_{0\le\Theta\le T-t}\mathbf{E}^Q\left[e^{-\delta\Theta}\left(K-s\cdot\exp\left(\left(\mu-\frac{\sigma^2}{2}\right)\Theta+\sigma B^Q(\Theta)\right)\right)^+\right] \tag{2.7}$$

the time$-t$ value of an American put option on a lognormal security with drift μ, volatility σ, interest rate δ, strike price K and maturity T. The drift can be expressed as drift $\mu=\delta-q$, where q is the dividend yield. Throughout our analysis we assume $\sigma>0$.

Denote by

$$v_\infty(s;\mu,\sigma,\delta,K)=\sup_{0\le\Theta\le\infty}\mathbf{E}^Q\left[e^{-\delta\Theta}\left(K-s\cdot\exp\left(\left(\mu-\frac{\sigma^2}{2}\right)\Theta+\sigma B^Q(\Theta)\right)\right)^+\right] \tag{2.8}$$

the value of the perpetual American put option. Obviously,

$$(K-s)^+\le v(t,s;\mu,\sigma,\delta,K)\le v_\infty(s;\mu,\sigma,\delta,K),\ \text{for all } t\in[0,T],$$

no matter of the parameters' values. Hence, if there exists an optimal early exercise opportunity for the perpetual put option, this is also the case for the finite-maturity one.

In Theorem 2.2 we provide a comprehensive description of the optimal early exercise region for American put options on a lognormal underlying asset in case of positive, zero, and negative interest rates. The resulting asymptotic behavior of the critical price at maturity depends also on the interplay with the underlying risk neutral drift (for an economic discussion on the critical price near maturity see Battauz at al. (2022)).

In particular, Theorem 2.2, Point 1, focuses on the standard case of a positive interest rate δ. When $\delta>0$, it is well known that there exists a constant critical price that triggers optimal early exercise for the perpetual option and the American put option value is finite (see also Broadie and Detemple (1996) and (2004), Detemple (2012), and Chan (2020)). On the contrary, when the interest rate δ is negative, the perpetual American put option may have an infinite value. Assumption (2.13) in Theorem 2.2, Point 4, ensures that the perpetual put option has a finite value and displays optimal early exercise opportunities above (resp. below) an upper (resp. lower) constant critical price (see also Proposition 2.2 in Battauz, De Donno and Sbuelz (2015)). As a consequence, the finite maturity put option does also have optimal early exercise opportunities above (resp. below) an upper (resp. lower) critical price. However, Assumption (2.13) is not satisfied in many practical examples, that display optimal early exercise opportunities in the finite-maturity case only (see Section 4). Therefore, in Theorem 2.2 Point 5, we extend the results of Theorems 2.3 and Theorem 2.4 in Battauz, De Donno and Sbuelz (2015) to describe the monotonicity properties and the asymptotics of the upper and lower critical prices at maturity under the milder condition of the existence of *some* optimal early exercise opportunity during the option life.

Theorem 2.2 *1. If $\delta > 0$, early exercise is optimal for the perpetual American put option when the underlying price $S(t) \leq S^c_\infty$, where S^c_∞ is the (constant) critical price of the perpetual American put option,*

$$S^c_\infty = -\frac{\alpha}{1-\alpha}K < K, \qquad A = \frac{(S^c_\infty)^{1-\alpha}}{-\alpha} > 0, \tag{2.9}$$

and

$$\alpha = \frac{-\left(\mu - \frac{\sigma^2}{2}\right) - \sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\delta\sigma^2}}{\sigma^2} < 0. \tag{2.10}$$

The perpetual put value is

$$v_\infty(s;\mu,\sigma,\delta,K) = \begin{cases} As^\alpha & \text{for } s > S^c_\infty \\ \\ K - s & \text{for } s \leq S^c_\infty. \end{cases} \tag{2.11}$$

Moreover, if $0 \leq \mu = \delta - q \leq \delta$ the finite-maturity critical price is monotonically increasing and such that

$$\lim_{t\to T} S^c(t) = K$$

with

$$\lim_{t\to T} \frac{K - S^c(t)}{\sigma K\sqrt{(T-t)\ln\frac{\gamma}{(T-t)}}} = 1 \text{ if } \mu > 0,$$

where $\gamma = \frac{\sigma^2}{8\pi\mu^2}$, and

$$\lim_{t\to T} \frac{K - S^c(t)}{\sigma K\sqrt{(T-t)\ln\frac{1}{(T-t)}}} = \sqrt{2} \text{ if } \mu = 0.$$

If $\mu = \delta - q < 0 < \delta$ the finite-maturity critical price is monotonically increasing and such that

$$S^c(T^-) = \lim_{t\to T} S^c(t) = \frac{\delta}{q}K < K$$

with

$$\lim_{t\to T} \frac{S^c(T^-) - S^c(t)}{S^c(T^-)\sigma\sqrt{(T-t)}} = -y^*,$$

where $y^* \approx -0.638$ is the number such that the function

$$\phi(y) = \sup_{0\leq\Theta\leq 1} \mathbb{E}\left[\int_0^\Theta (y + B(s))\,ds\right] = 0 \tag{2.12}$$

for all $y \leq y^*$ *and* $\phi(y) > 0$ *for all* $y > y^*$. *The function*[2] ϕ *defined in Equation* (2.12) *is a supremum over all stopping times* Θ *with values in* $[0,1]$.

2. *If* $\delta \leq 0$, *and* $\mu \leq 0$ *i.e.* $q \geq 0$, *then early exercise is never optimal, and the value of the American put option coincides with the European one.*

3. *If* $\delta = 0$, *and* $\mu - \frac{\sigma^2}{2} > 0$, *then* α *in Equation* (2.10) *becomes* $\alpha = \frac{-2\left(\mu - \frac{\sigma^2}{2}\right)}{\sigma^2} < 0$. *There exists a unique critical price* $S^c(t) \leq S^c_\infty$, *with* S^c_∞ *defined in Equation* (2.9), $S^c(t)$ *is monotonically increasing and*

$$S^c(t) - K \sim -K\sigma\sqrt{(T-t)\ln\frac{\sigma^2}{8\pi(T-t)\mu^2}} \text{ as } t \to T$$

4. *If* $\delta < 0$,

$$\mu - \frac{\sigma^2}{2} > 0, \text{ and } \left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\delta\sigma^2 > 0, \tag{2.13}$$

then the perpetual American put option value v_∞ *is*

$$v_\infty(x) = \begin{cases} A_l \cdot x^{\xi_l} & \text{for } x \in (0; l_\infty) \\ K - x & \text{for } x \in [l_\infty; u_\infty] \\ A_u \cdot x^{\xi_u} & \text{for } x \in (u_\infty; +\infty) \end{cases} \tag{2.14}$$

where $\xi_u < \xi_l$ *are the negative solutions of the equation*

$$\frac{1}{2}\sigma^2\xi^2 + \left(\mu - \frac{\sigma^2}{2}\right)\xi - \delta = 0, \tag{2.15}$$

The critical prices are

$$l_\infty, u_\infty = K\frac{\xi_i}{\xi_i - 1} \quad \text{for } i = l, u \tag{2.16}$$

and the constant A_l *and* A_u *are given by*

$$A_l = -\frac{(l_\infty)^{1-\xi_l}}{\xi_l} \text{ and } A_u = -\frac{(u_\infty)^{1-\xi_u}}{\xi_u}. \tag{2.17}$$

There exist a lower critical price $l(t)$ *and an upper critical price* $u(t)$ *such that*

$$\frac{\delta K}{\delta - \mu} \leq l(t) < u(t) \leq K \tag{2.18}$$

such that the finite-maturity American put option is optimally exercised at t *if* $S(t) \in [l(t), u(t)]$ *and optimally continued if* $S(t) < l(t)$ *or* $S(t) > u(t)$. *Moreover,* $u(t)$ *is monotonically increasing and*

$$u(t) - K \sim -K\sigma\sqrt{(T-t)\ln\frac{\sigma^2}{8\pi(T-t)\mu^2}} \text{ for } t \to T \tag{2.19}$$

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[2] For more details on y^* and the function ϕ see Section 2.1 in Lamberton, D., & Villeneuve, S. (2003).

The lower free boundary is monotonically decreasing and satisfies

$$l(t) - \frac{\delta K}{\delta - \mu} \sim \frac{\delta K}{\delta - \mu}\left(-y^{*}\sigma\sqrt{(T-t)}\right) \quad \text{for } t \to T \tag{2.20}$$

where $y^{*} \approx -0.638$ *is the number defined in Equation* (2.12).

5. *If* $\delta < 0$, *and there exists* $\overline{x} > 0$ *such that the finite-maturity American put option is optimally exercised at* $\overline{t} \in (0,T)$ *if* $S\left(\overline{t}\right) = \overline{x}$, *then the segment with extremes*

$$l(t) = \inf\left\{s \geq 0 : v(t,s;\mu,\sigma,\delta,K) = (K-s)^{+}\right\} \tag{2.21}$$
$$u(t) = \sup\left\{s \geq 0 : v(t,s;\mu,\sigma,\delta,K) = (K-s)^{+}\right\} \wedge K \tag{2.22}$$

is non-empty for any $t \in \left[\overline{t},T\right]$. *The option is optimally exercised at any* $t \geq \overline{t}$ *whenever* $S(t) \in [l(t), u(t)]$. *The lower (resp. the upper) free boundary is monotonically decreasing (resp. increasing) for any* $t \geq \overline{t}$. *The lower and the upper free boundaries satisfy the inequality* (2.18) *for any* $t \geq \overline{t}$ *as well as the asymptotics* (2.20) *and* (2.19).

Proof. For the perpetual put option results see Battauz, De Donno, and Sbuelz (2012) and (2015).

Consider the finite-maturity case. When $\delta > 0$ and $0 < \mu = \delta - q < \delta$ or $\mu < 0 < \delta$, the asymptotics of the critical price $S^{c}(t)$ as $t \to T$ are determined by Evans, Kuske, and Keller (2002), and further improved by De Marco, and Henry-Labordère (2017). For the case $\delta > 0$ and $0 = \mu$ the asymptotics are provided in Theorem 3 in Lamberton and Villeneuve (2003). Hence Point 1 and 3 are proved.

If $\delta \leq 0$, and $q = \delta - \mu \geq 0$, then Jensen inequality implies that for any $0 < \Theta \leq T - t$

$$\begin{aligned}
&\mathbf{E}^{Q}\left[e^{-\delta\Theta}\left(K - s\cdot\exp\left(\left(\mu - \frac{\sigma^{2}}{2}\right)\Theta + \sigma B^{Q}(\Theta)\right)\right)^{+}\right] \\
&\quad \geq e^{-\delta\Theta}\left(K - s\cdot e^{\mu\Theta}\right)^{+} \\
&\quad = \left(Ke^{-\delta\Theta} - s\cdot e^{-q\Theta}\right)^{+} \\
&\quad \geq \left(K - s\cdot e^{-q\Theta}\right)^{+} \text{ since } e^{-\delta\Theta} \geq 1 \\
&\quad \geq (K-s)^{+} \text{ since } e^{-q\Theta} \leq 1.
\end{aligned}$$

The inequality in the fourth line is strict if $\delta < 0$, and in this case

$$\mathbf{E}^{Q}\left[e^{-\delta\Theta}\left(K - s\cdot\exp\left(\left(\mu - \frac{\sigma^{2}}{2}\right)\Theta + \sigma B^{Q}(\Theta)\right)\right)^{+}\right] > (K-s)^{+}.$$

If $\delta = 0$, for any constant Θ the European put pricing function

$$v_{e}(s) = \mathbf{E}^{Q}\left[e^{-\delta\Theta}\left(K - s\cdot\exp\left(\left(\mu - \frac{\sigma^{2}}{2}\right)\Theta + \sigma B^{Q}(\Theta)\right)\right)^{+}\right] = Ke^{-\delta\Theta}\mathcal{N}(-d_{2}) - se^{-q\Theta}\mathcal{N}(-d_{1})$$

with $d_{1,2} = \frac{1}{\sigma\sqrt{\Theta}}\left(\ln\left(\frac{s}{K}\right) + \left(\delta - q \pm \frac{1}{2}\sigma^2\right)\Theta\right)$ is convex with respect to s and hence it is always above its tangent line computed at $s \to 0^+$. Since the slope of the tangent line computed at $s \to 0^+$ is $-e^{-q\Theta}\mathcal{N}(-d_1) \to -e^{-q\Theta} \geq -1$ for $q \geq 0$, this implies that $v_e(s) > (K-s)^+$ for all $s > 0$ when $\delta = 0$. Thus, when $\delta \leq 0$ and $q = \delta - \mu \geq 0$, the immediate put payoff at t is strictly dominated by the continuation value at any future $0 < \Theta \leq T - t$, and exercise is optimal at T only.

In the limiting case $\delta = 0$, if $\mu \leq 0$ i.e. $q \geq 0$, then early exercise is never optimal, and the value of the American put option coincides with the European one. Indeed, if $\delta = 0$ and $\mu \leq 0$ Equation (2.15) does not admit any negative solution. Hence Point 2 follows.

If $\delta < 0$, and $q = \delta - \mu < 0$, then early exercise may be optimal even in the perpetual case. The proof follows by Proposition 2.2, and the geometry and the asymptotics for the finite-maturity critical prices can be retrieved by Theorems 2.3 and 2.4 in Battauz, De Donno and Sbuelz (2015). Point 4 follows.

Consider now Point 5. If there exists $\overline{x} > 0$ such that the finite-maturity American put option is optimally exercised at $\overline{t} \in (0,T)$ if $S(\overline{t}) = \overline{x}$, then early exercise is optimal for all $t \geq \overline{t}$ and $S(t) = \overline{x}$ since

$$(K - \overline{x})^+ \leq v(t, \overline{x}; \mu, \sigma, \delta, K) \leq v(\overline{t}, \overline{x}; \mu, \sigma, \delta, K) = (K - \overline{x})^+,$$

where the first inequality follows by the payoff value dominance of the American option, and the second inequality as the American option value is decreasing with respect to time t. Since $v(t, 0; \mu, \sigma, \delta, K) = Ke^{-\delta(T-t)} > K = (K-0)^+$, as $\delta < 0$, it follows that $l(t) > 0$ for all $t \in (\overline{t}, T)$. The monotonicity of $l(t)$ and $u(t)$ for $t \geq \overline{t}$ follows by the fact that the American option value is decreasing with respect to time t (see also Battauz, De Donno and Sbuelz (2015)). The remaining part of Point 5 follows by the proof of Theorems 2.3 and 2.4 in Battauz, De Donno and Sbuelz (2015) restricted to $t \in [\overline{t}, T]$. $\square$

Remark 2.1 *We comment here Assumption* (2.13). *Intuitively, a sufficient condition for the existence of optimal early exercise of the perpetual put with value* v_∞ *is*

$$\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\delta\sigma^2 > 0. \tag{2.23}$$

Condition (2.23) *ensures that the function* $v_\infty(x) = As^\alpha$ *has at least one tangency point with the immediate put payoff in the extreme(s) of the early exercise region. This condition is always satisfied when* $\delta \geq 0$, *no matter of the sign of* μ. *Therefore the perpetual American put option admits the representation of Equation* (2.11), *and it is optimally exercised when the underlying is below the unique critical price* S^c_∞. *If* $\delta < 0$, *Condition* (2.23) *is not always true. Battauz, De Donno and Sbuelz (2015) show in Proposition 2.2 (see also Battauz, De Donno, and Sbuelz (2012)) that a sufficient condition for the existence of optimal early exercise of the perpetual put* v_∞ *is* (2.13). *This conditions ensures the existence of the two real negative roots of the tangency equation* (2.15).

The sufficient condition (2.13) may not be true, even if binomial approximations show that the finite-maturity American put option displays optimal early exercise opportunities. These cases satisfy a necessary condition for early exercise established in Proposition 2.5 in Battauz, De Donno and Sbuelz (2015). For the ease of the reader, we state the necessary condition (2.24) here below.

Condition 2.1 ***(necessary condition for early exercise, negative interest rate).*** *If* $\delta < 0$ *and* $\mu > 0$ *a necessary condition for the optimal exercise of the finite-maturity American put option at* $t \in [0; T)$ *is*

$$\mathcal{N}^{-1}\left(e^{\delta(T-t)}\right) - \mathcal{N}^{-1}\left(e^{(\delta-\mu)(T-t)}\right) \geq \sigma\sqrt{T-t}, \tag{2.24}$$

where $\mathcal{N}^{-1}(\cdot)$ *denotes the inverse of the standard normal cumulative distribution function.*

2.2 From American put to American quanto put options

We introduce now a template to classify and characterize the optimal exercise policy of American quanto put options in terms of American put options for various combination in the level, sign and hierarchy of the key parameters r_d, r_f and the volatility vectors σ_f and σ_X. In the next propositions we rewrite the American quanto put options in terms of American put options on a lognormal (onedimensional) security. We start from the option defined in Equation (2.2), whose behavior is unaffected by r_d.

Proposition 2.3 *Consider the American quanto put option defined in Equation* (2.2)*. Then*

$$\begin{aligned} V(t) &= \sup_{t\leq\tau\leq T} \mathbf{E}^Q\left[e^{-r_d(\tau-t)}\, X(\tau)\left(K_f - S_f(\tau)\right)^+ \middle| \mathcal{F}_t\right] = \\ &= X(t)\cdot v(t, S_f(t); r_f - q_f, \|\sigma_f\|, r_f, K_f), \end{aligned}$$

i.e. the American quanto put option price coincides with the foreign American put option price converted at the current spot exchange rate. Therefore, early exercise is optimal at t if $S_f(t)$ *is in the early exercise region of the foreign American put option* $v(t, S_f(t); r_f - q_f, \|\sigma_f\|, r_f, K_f)$ *as described in Theorem 2.2 with* $\delta = r_f$, $\mu = r_f - q_f$, $\sigma = \|\sigma_f\|$, *and* $K = K_f$.

Proof. Let $\widetilde{N}(t) = X(t)e^{(r_f-r_d)t}$ the numeraire (see Battauz (2002)) associated to the equivalent probability measure Q^N, whose density is

$$\frac{dQ^N}{dQ} = \frac{\widetilde{N}(T)}{\widetilde{N}(0)}.$$

Bayes' theorem implies that

$$\begin{aligned}
V(t) &= \sup_{t\leq\tau\leq T} \mathbf{E}^{Q}\left[e^{-r_d(\tau-t)}\ X(\tau)\left(K_f - S_f(\tau)\right)^+\middle|\,\mathcal{F}_t\right] \\
&= \sup_{t\leq\tau\leq T} \frac{\mathbf{E}^{Q^N}\left[\frac{dQ}{dQ^N}e^{-r_d(\tau-t)}\ X(\tau)\left(K_f - S_f(\tau)\right)^+\middle|\,\mathcal{F}_t\right]}{\mathbf{E}^{Q^N}\left[\frac{dQ}{dQ^N}\middle|\,\mathcal{F}_t\right]} \\
&= \sup_{t\leq\tau\leq T} \frac{\mathbf{E}^{Q^N}\left[\frac{X(0)}{X(\tau)e^{\left(r_f-r_d\right)\tau}}e^{-r_d(\tau-t)}\ X(\tau)\left(K_f - S_f(\tau)\right)^+\middle|\,\mathcal{F}_t\right]}{\frac{X(0)}{X(t)e^{\left(r_f-r_d\right)t}}} \\
&= X(t) \sup_{t\leq\tau\leq T} \mathbf{E}^{Q^N}\left[e^{-r_f(\tau-t)}\ \left(K_f - S_f(\tau)\right)^+\middle|\,\mathcal{F}_t\right].
\end{aligned}$$

The factor

$$\sup_{t\leq\tau\leq T} \mathbf{E}^{Q^N}\left[e^{-r_f(\tau-t)}\ \left(K_f - S_f(\tau)\right)^+\middle|\,\mathcal{F}_t\right]$$

is the foreign price of the American put option on S_f if the Q^N-drift of S_f coincides with the foreign riskless interest rate r_f. Indeed, Girsanov theorem implies that the process

$$dW^N(t) = -\sigma_X\ dt\ +\ dW^Q(t)$$

is a $2-$*dimensional* Q^N *Brownian motion.* Therefore

$$\begin{aligned}
\frac{dS_f(t)}{S_f(t)} &= \mu_f^Q dt + \sigma_f\ dW^Q(t) \\
&= \left(r_f - q_f - \sigma_f\sigma_X\right)dt + \sigma_f\ \left(dW^N(t) + \sigma_X\ dt\right) \\
&= \left(r_f - q_f\right)dt + \sigma_f\ dW^N(t),
\end{aligned}$$

and thus the Q^N-distribution of S_f coincides with its foreign risk-neutral distribution. The American quanto option coincides with

$$V(t) = X(t)\cdot v(t, S_f(t); r_f - q_f, \|\sigma_f\|, r_f, K_f),$$

and the rest of the proposition follows by applying Theorem 2.2 with $\delta = r_f$, $\mu = r_f - q_f$, $\sigma = \|\sigma_f\|$, and $K = K_f$. □

In the previous proposition we have shown that the price of the American quanto put option in Equation (2.2) coincides with the foreign American put option price converted at the current spot exchange rate. The result has also an intuitive financial justification. In fact, if the domestic investor buys the option (2.2), she has the right to get the foreign-denominated payoff $(K_f - S_f(\tau))^+$ whenever exercised at τ with $t \leq \tau \leq T$. The domestic denominated value of the payoff $(K_f - S_f(\tau))^+$ exercised at τ is $X(\tau)\left(K_f - S_f(\tau)\right)^+$. The same right is obtained by entering a long position on the foreign put with $(K_f - S_f(\tau))^+$ at exercise date τ, whose price at time t in domestic currency units is $X(t)\cdot v(t, S_f(t); r_f, \|\sigma_f\|, r_f, K_f)$.

In Proposition 2.3 we have characterized the optimal exercise policies for the American quanto option defined in Equation (2.2) , whose behavior depends only on the foreign riskless rate r_f.

In the what follows we focus on American quanto put options defined in Equations (2.3) , (2.4) , (2.5) , (2.6), whose behavior depends on *both* the domestic rate r_d and the foreign rate r_f.

Our first step consists in reducing American quanto put options (2.3) , (2.4) , (2.5) and (2.6) to American put options. This characterization, which is done in the following lemma, will allow us to work out the optimal exercise policies for American quanto options in Propositions 3.1 and 3.2.

Lemma 2.1 *The no-arbitrage price of the option* (2.3) *can be computed as*

$$V(t) = v(t, X(t)S_f(t); r_d - q_f, \|\sigma_X + \sigma_f\|, r_d, K_d)$$

The option in (2.4) *can be computed as*

$$V(t) = v(t, X_0 \cdot S_f(t); \mu_f^Q, \|\sigma_f\|, r_d, K_d)$$

and the option (2.5) *as*

$$V(t) = v(t, F_0 \cdot S_f(t); \mu_f^Q, \|\sigma_f\|, r_d, K_d),$$

and (2.6) *as*

$$\begin{aligned} V(t) &= v(t, X_0 \cdot S_f(t); \mu_f^Q, \|\sigma_f\|, r_d, X_0 \cdot K_f) \\ &= X_0 v(t, S_f(t); \mu_f^Q, \|\sigma_f\|, r_d, K_f). \end{aligned}$$

Proof. The underlying of the option in Equation (2.3) is the lognormal $S_f^*(t) = X(t)S_f(t)$, whose domestic risk neutral drift is r_d and whose volatility vector is $\sigma_X + \sigma_f$ as from Equation (2.1) . This yields the formula for (2.3) . In the remaining options (2.4) and (2.5) the underlying S_f is denominated in the domestic currency by using the constant initial spot exchange rate X_0 in option (2.4) , and the constant initial forward exchange rate F_0 in option (2.5) . Thus the domestic risk neutral underlying drift is μ_f^Q, its volatility is σ_f and its initial level is multiplied by the constant X_0 in option (2.4) , and the constant initial forward exchange rate F_0 in option (2.5) . In the last option (2.6) , both the strike price and the foreign security are denominated in the domestic currency by using the constant initial spot exchange rate. Therefore, the domestic risk neutral underlying S_f drift is μ_f^Q, its volatility is σ_f , and both the initial underlying value and the strike price are then multiplied by X_0. Because of the put payoff homogeneity, we then obtain $V(t) = v(t, X_0 \cdot S_f(t); \mu_f^Q, \|\sigma_f\|, r_d, X_0 \cdot K_f)$ $= X_0 v(t, S_f(t); \mu_f^Q, \|\sigma_f\|, r_d, K_f)$. $\square$

3 American quanto options and the interplay with the sign of the riskless rates

In order to describe in detail the optimal exercise policies for options (2.3), that is with domestic strike price and floating exchange rate, options (2.4), that is with domestic strike price and fixed exchange rate, options (2.5), that is with domestic strike price and forward exchange rate, and options (2.6), that is with foreign strike price and fixed exchange rate. We distinguish two main cases.

- Case 1: $r_d \geq 0$.

Case 1 is the traditional assumption on non-negative domestic interest rate. We solve the problem in Proposition 3.1.

- Case 2: $r_d < 0$.

The case of a negative domestic interest rate is definitely interesting due to the persistence of negative interest rates in recent years in the European and in the Japanese markets. We address Case 2 in Proposition 3.2.

We start with the analysis of the optimal exercise policy under Case 1.

Proposition 3.1 *Suppose $r_d \geq 0$. Then*

1. *For the option (2.3) there exists a critical price at t, $S^c(t)$ such that early exercise is optimal at t if*

$$X(t)S_f(t) \leq S^c(t).$$

If $r_d > q_f$ then

$$\lim_{t\to T} S^c(t) = K_d$$

and

$$\lim_{t\to T} \frac{K_d - S^c(t)}{\|\sigma_f + \sigma_X\| K_d\sqrt{(T-t)\ln\frac{\gamma}{(T-t)}}} = 1 \text{ where } \gamma = \frac{\|\sigma_f + \sigma_X\|^2}{8\pi\left(r_d - q_f\right)^2}.$$

If $q_f = r_d$

$$\lim_{t\to T} \frac{K_d - S^c(t)}{\|\sigma_f + \sigma_X\| K_d\sqrt{(T-t)\ln\frac{1}{(T-t)}}} = \sqrt{2}.$$

If $q_f > r_d > 0$ we have that

$$S^c(T^-) = \lim_{t\to T} S^c(t) = \frac{r_d}{q_f}K_d < K_d$$

with

$$\lim_{t\to T} \frac{S^c(T^-) - S^c(t)}{S^c(T^-)\|\sigma_f + \sigma_X\|\sqrt{(T-t)}} = y^*,$$

where $y^* \approx -0.638$ is defined in (2.12).

2. *For the option* (2.4) *there exists a critical price at* t, $S^c(t)$, *such that early exercise is optimal at* t *if*
$$X_0 S_f(t) \leq S^c(t).$$
Set $q_* = q_f + r_d - r_f + \sigma_f \sigma_X$. *If* $r_d > q_*$ *i.e.* $q_f - r_f + \sigma_f \sigma_X < 0$ *then*
$$\lim_{t \to T} S^c(t) = K_d$$
with
$$\lim_{t \to T} \frac{K_d - S^c(t)}{\|\sigma_f\| K_d \sqrt{(T-t) \ln \frac{\gamma}{(T-t)}}} = 1 \text{ where } \gamma = \frac{\|\sigma_f\|^2}{8\pi \left(r_f - q_f - \sigma_X \sigma_f\right)^2}.$$
If $q_* = r_d$,
$$\lim_{t \to T} \frac{K_d - S^c(t)}{\|\sigma_f\| K_d \sqrt{(T-t) \ln \frac{1}{(T-t)}}} = \sqrt{2}$$
If $q_* > r_d > 0$, *i.e.* $q_f - r_f + \sigma_f \sigma_X > 0$, *we have that*
$$S^c(T^-) = \lim_{t \to T} S^c(t) = \frac{r_d}{q_*} K_d < K_d$$
with
$$\lim_{t \to T} \frac{S^c(T^-) - S^c(t)}{S^c(T^-) \|\sigma_f\| \sqrt{(T-t)}} = y^*,$$
where $y^* \approx -0.638$ *is defined in* (2.12) *.*

3. *For the option* (2.5) *there exists a critical price at* t, $S^c(t)$, *such that early exercise is optimal at* t *if*
$$F_0 S_f(t) \leq S^c(t).$$
The limits and the asymptotics for $S^c(t)$ *at maturity coincide with the ones described for the option* (2.4) *in Point 2.*

4. *For the option* (2.6) *there exists a critical price at* t, $S^c(t)$, *such that early exercise is optimal at* t *if*
$$S_f(t) \leq S^c(t).$$
The limits and the asymptotics for $S^c(t)$ *at maturity coincide with the ones described for the option* (2.4) *in Point 2.*

Proof. The proof follows by applying Theorem 2.2 and Lemma 2.1. In particular Point 1 follows with
$$\delta = r_d, \ \mu = r_d - q_f, \ \sigma = \|\sigma_f + \sigma_X\|, \ K = K_f.$$
Points 2, 3 and 4 follow with
$$\delta = r_d, \ \mu = r_f - q_f - \sigma_X \sigma_f, \ \sigma = \|\sigma_f\|, \ K = K_f. \qquad \square$$

We now focus on Case 2. Importantly, as the sign of the domestic interest rate becomes negative, a non-standard double continuation region can appear: immediate option exercise is optimally postponed not only if the contract is not enough in-the-money but also, unconventionally, if the contract is too much in-the-money. If the option is excessively in-the-money, immediate exercise is optimally delayed because the incentive to postpone the option exercise due to the negative domestic rate prevails over the incentive to exercise immediately created by a positive underlying risk-neutral drift μ_f^Q. Such a drift toward the out-of-the-money region becomes relatively negligible if the distance from the out-of-the-money region is vast.

Proposition 3.2 *Suppose* $r_d < 0$.

1. *Consider the option* (2.3). *Its underlying drift is* $r_d - q_f < 0$ *and the American quanto option is optimally exercised at maturity only.*

2. *Consider the option* (2.4). *If* $\mu_f^Q = r_f - q_f - \sigma_f \sigma_X < 0$, *then the American quanto option is optimally exercised at maturity only.*

   If $\mu_f^Q - \frac{\|\sigma_f\|^2}{2} = r_f - q_f - \sigma_f \sigma_X - \frac{\|\sigma_f\|^2}{2} > 0$ *and*

$$\left(r_f - q_f - \sigma_f \sigma_X - \frac{\|\sigma_f\|^2}{2} \right)^2 + 2 r_d \|\sigma_f\|^2 > 0 \tag{3.1}$$

   holds (or, resp., if there exists $\overline{x} > 0$ *such that the finite-maturity American quanto put option* (2.4) *is optimally exercised at* $\overline{t} \in (0, T)$ *when* $X_0 S_f(\overline{t}) = \overline{x}$*), then there exist two critical prices at* t, $l(t) < u(t)$, *such that early exercise is optimal at* t *if*

$$l(t) \leq X_0 S_f(t) \leq u(t),$$

   and continuation is optimal at t *if*

$$X_0 S_f(t) < l(t) \text{ or } X_0 S_f(t) > u(t),$$

   for all $t \in [0, T]$ *(resp. for all* $t \in \left[\overline{t}, T\right]$*). Moreover*

$$u(t) - K_d \sim -K_d \|\sigma_f\| \sqrt{(T - t) \ln \frac{\|\sigma_f\|^2}{8\pi (T - t) \left(r_f - q_f - \sigma_f \sigma_X \right)^2}}.$$

   For $t \longrightarrow T$, *the lower free boundary satisfies*

$$l(t) - \frac{r_d K_d}{r_d - \left(r_f - q_f - \sigma_f \sigma_X \right)} \sim \frac{r_d K_d}{r_d - \left(r_f - q_f - \sigma_f \sigma_X \right)} \left(-y^* \|\sigma_f\| \sqrt{(T - t)} \right),$$

   where $y^* \approx -0.638$ *is defined in* (2.12).

3. *Consider the option* (2.5). *If* $\mu_f^Q = r_f - q_f - \sigma_f \sigma_X < 0$, *then the American quanto option is optimally exercised at maturity only.*

   If $\mu_f^Q = r_f - q_f - \sigma_f \sigma_X > 0$ *and* (3.1) *holds (or, resp., if there exists* $\overline{x} > 0$ *such that the finite-maturity American quanto put option* (2.5) *is optimally exercised at* $\overline{t} \in (0, T)$ *when* $F_0 S_f(\overline{t}) = \overline{x}$*), then there exist two critical prices at* t, $l(t) < u(t)$, *such that early exercise is optimal at* t *if*

$$l(t) \leq F_0 S_f(t) \leq u(t),$$

   and continuation is optimal at t *if*

$$F_0 S_f(t) < l(t) \text{ or } F_0 S_f(t) > u(t),$$

   for all $t \in [0, T]$ *(resp. for all* $t \in \left[\overline{t}, T\right]$*). The limits and the asymptotics of the critical prices at maturity coincide with the ones computed for the option* (2.4).

4. *Consider the option* (2.6). *If* $\mu_f^Q = r_f - q_f - \sigma_f \sigma_X < 0$, *then the American quanto option is optimally exercised at maturity only.*

   If $\mu_f^Q = r_f - q_f - \sigma_f \sigma_X > 0$ *and* (3.1) *holds (or, resp., if there exists* $\overline{x} > 0$ *such that the finite-maturity American quanto put option* (2.6) *is optimally exercised at* $\overline{t} \in (0, T)$ *when* $S_f(\overline{t}) = \overline{x}$*), then there exist two critical prices at* t, $l(t) < u(t)$, *such that early exercise is optimal at* t *if*

$$l(t) \leq S_f(t) \leq u(t),$$

   and continuation is optimal at t *if*

$$S_f(t) < l(t) \text{ or } S_f(t) > u(t),$$

   for all $t \in [0, T]$ *(resp. for all* $t \in \left[\overline{t}, T\right]$*). The limits and the asymptotics of the critical prices at maturity coincide with the previous ones computed for the options* (2.4) *and* (2.5).

Proof. The proof follows by applying Theorem 2.2 and Lemma 2.1, as explained in the proof of Proposition 3.1. □

We observe that the critical prices described in Propositions 3.1 and 3.2 for the American quanto options of Equations (2.3), (2.4) and (2.5) are all expressed in domestic currency. Early exercise occurs at t if $S_f(t)$, converted in the domestic currency according to the payoff definition, enters the early exercise region determined by the (domestic) critical price. On the contrary, for the American quanto option of Equation (2.6) the critical price is expressed in foreign currency units.

Finite maturity American quanto options may display optimal early exercise opportunities, even if the corresponding perpetual American quanto options do not admit finite perpetual free boundaries. This happens when assumption (3.1) is not verified, but the necessary condition translating condition (2.24) for the different payoff's specifications holds true. In the next proposition we state the condition for the quanto options.

Proposition 3.3 ***(Necessary condition for early exercise of American quanto options when*** $r_d < 0$ ***and*** $\mu_f^Q = r_f - q_f - \sigma_f \sigma_X > 0$***)*** *A necessary condition for the optimal exercise of the finite-maturity American quanto put option* (2.4), (2.5) *and* (2.6) *is*

$$\mathcal{N}^{-1}\left(e^{r_d T}\right) - \mathcal{N}^{-1}\left(e^{\left(r_d - \mu_f^Q\right)T}\right) \geq \|\sigma_f\| \sqrt{T}. \tag{3.2}$$

where $\sigma_f = \|\sigma_f\|$.

Proof. The necessary condition (2.24) found in Proposition 2.5 in Battauz, De Donno and Sbuelz (2015) requires the European put option

$$v_e(t, x; \mu, \sigma, \delta, K) = K e^{-\delta(T-t)} \mathcal{N}(\bar{z}) - x e^{(\mu-\delta)(T-t)} \mathcal{N}\left(\bar{z} - \sigma\sqrt{(T-t)}\right), \tag{3.3}$$

with $\mathcal{N}(y)$ denoting the distribution function of a standard normal random variable, and $\bar{z} = \left(\ln \frac{K}{x} - \left(\mu - \frac{\sigma^2}{2}\right)(T-t)\right) \frac{1}{\sigma\sqrt{T-t}}$, to fall below the immediate payoff at t for some values of the underlying. For the quanto options this corresponds to the existence of x_m such that

$$v_e(t, x_m; \mu_f^Q, \|\sigma_f\|, r_d, K) = (K - x_m)^+$$

where $K = K_d$ and $x_m = X_0 \cdot S_f(t)$ for the American quanto option (2.4) , $K = K_d$ and $x_m = F_0 \cdot S_f(t)$ for the American quanto option (2.5) , and $K = K_f$ and $x_m = S_f(t)$ for the American quanto option (2.6) . Then the remaining part of the proof of Proposition 2.5 in Battauz, De Donno and Sbuelz (2015) follows. Assumption (3.2) is necessary for the existence of optimal exercise opportunities at date $t = 0$. If early exercise is optimal at any date $t \in [0, T]$ for some x_m, it is also optimal for all future dates for the same x_m, as the American quanto options (2.4) , (2.5) , and (2.6) are decreasing with respect to t. □

4 Numerical Examples

In this section we provide examples of American quanto options to show how the domestic interest rate contributes in shaping their free boundaries. To streamline our analysis we focus on the American quanto option with payoff (2.6), that can be reduced to the American put option

$$\begin{aligned} V(t) &= \sup_{t \leq \tau \leq T} \mathbf{E}^Q \left[e^{-r_d(T-t)} X_0 \cdot (K_f - S_f(\tau))^+ \middle| \mathcal{F}_t \right] \\ &= X_0 v(t, S_f(t); \mu_f^Q, \|\sigma_f\|, r_d, K_f), \end{aligned}$$

following Lemma 2.1. We first introduce an example with a positive interest rate, and then move to the case of a domestic negative interest rate. Option prices are computed via binomial approximation (see Hull, 2018), setting the upwards and downwards coefficients

$$u = e^{\|\sigma_f\|\sqrt{\Delta t}}, \quad d = e^{-\|\sigma_f\|\sqrt{\Delta t}}$$

and the risk-neutral probability of an upward movement

$$\mathbf{q} = \frac{e^{\mu_f^Q \Delta t} - d}{u - d}$$

We fix

$$r_d = +0.90\%, \quad r_f = 2\%, \quad X(0) = 0.94, \ \|\sigma_X\| = 7.8\%$$
$$\|\sigma_f\| = 10\%, \quad \rho = -1\%, \ \text{ and } q_f = 0,$$

that deliver

$$\mu_f^Q = r_f - q_f - \rho \|\sigma_f\| \|\sigma_X\| = 2.008\%$$

For an American quanto put option (2.6) deeply in-the-money at inception with $S_f(0) = 0.5$ and $K_f = 1$, maturity $T = 6$ months and $N = 125$ time steps we obtain

$$\Delta t = 0.004, \quad u = 1.006, \quad d = 0,994, \quad \text{and } \mathbf{q} = 50.47\%.$$

We take an initial in-the-money underlying value because we want to investigate what happens within our binomial model in the deeply in-the-money region during the option life. In Figure 1 the binomial tree S_f is delimited by the grey diamonds. The horizontal axis represents calendar time in year units, whereas the vertical axis represents the underlying asset value. We compute the upper (and unique) free boundary by taking at any t the maximum underlying value within the early exercise region at t, namely

$$u(t) = \max\left(S_f(t) : X_0 \left(K_f - S_f(t)\right)^+ = V(t)\right),$$

among the binomial realizations of $S_f(t)$ at t. The upper free boundary is plotted in Figure 1 with the blue dots, starting after 103 consequent upwards movements, where the continuation region begins. The asymptotic approximation for the free boundary at maturity obtained in Proposition 3.1 is plotted with blue stars and is very closed to the binomial upper free boundary. The flat perpetual boundary,valued 0.78, is dotted with blue circles in Figure 1. As the initial underlying value $S_f(0)$ is below both the perpetual and the finite maturity free boundary, the initial value of both the perpetual and the finite maturity binomial option is $X_0 \cdot (K_f - S_f(0))^+ = 0.94 \cdot 0.5 = 0.47$.

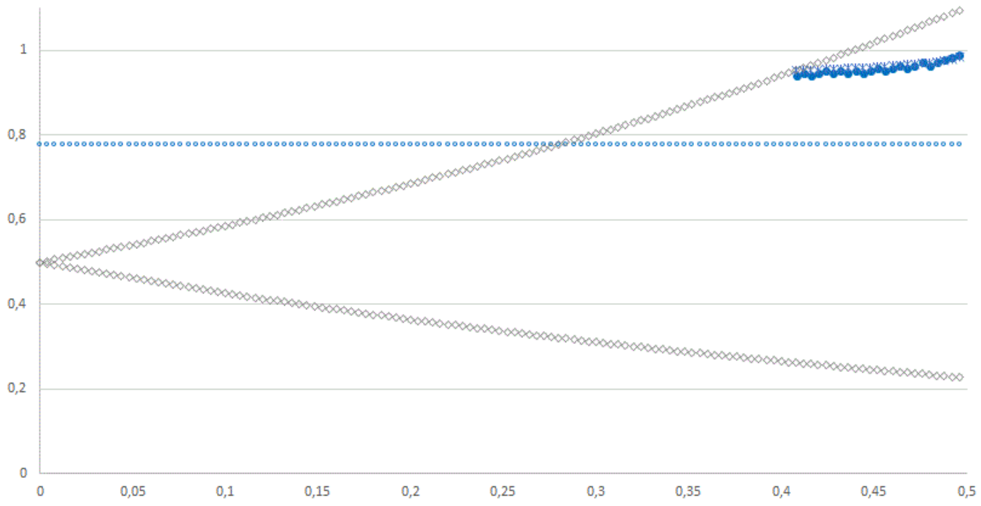

Figure 1. The free boundary when r_d is positive. The grey diamonds delimit the stock binomial tree. Blue dots denote the binomial free boundary, and blue stars its asymptotic approximation. Blue circles denote the perpetual boundary.

We then assign to the domestic riskless interest rate a opposite negative value, $r_d = -0.90\%$, keeping all the other parameters unchanged.

In this case, conditions (3.1) and (3.2) are met. A double continuation region appears. Its existence is remarkable, because it violates the usual property of down-connectedness of the exercise region of put options, that has been established in quite general settings (see Detemple and Tian (2002)). The perpetual lower and upper free boundaries are, resp., 0.45 and 0.68. Again, the price of the perpetual option coincides with its immediate payoff 0.47. The underlying binomial tree is unchanged, as the domestic riskless interest rate $r_d = -0.90\%$ does not enter the domestic risk-neutral dynamics of the foreign risky security. The binomial upper (resp. lower) free boundary is computed by taking at any t the maximum (resp. the minimum) underlying value within the early exercise region at t, namely

$$u(t) = \max\left(S_f(t) : X_0\left(K_f - S_f(t)\right)^+ = V(t)\right)$$
$$l(t) = \min\left(S_f(t) : X_0\left(K_f - S_f(t)\right)^+ = V(t)\right)$$

among the binomial realizations of $S_f(t)$ at t. In Figure 2 the binomial tree is delimited by grey diamonds. The standard part of the continuation region appears in the upper region of the tree after 103 upwards movements, that push the American Quanto put option towards the out-of-the-money region. The non-standard part of the continuation region appears in the very deeply in-the-money region, below the perpetual lower free boundary, after 73 downwards movements. In Figure 2 we plot with blue (resp. red) dots the upper (resp. lower) binomial free boundary. The asymptotic approximation for the upper (resp. lower) free boundary obtained in Proposition 3.2 are plotted with blue (resp. red) stars. The flat perpetual upper (resp. lower) boundary is plotted with blue (resp. red) circles. As the initial value $S_f(0)$ is within the early exercise region, the initial price of the finite-maturity option coincides with its immediate payoff 0.47.

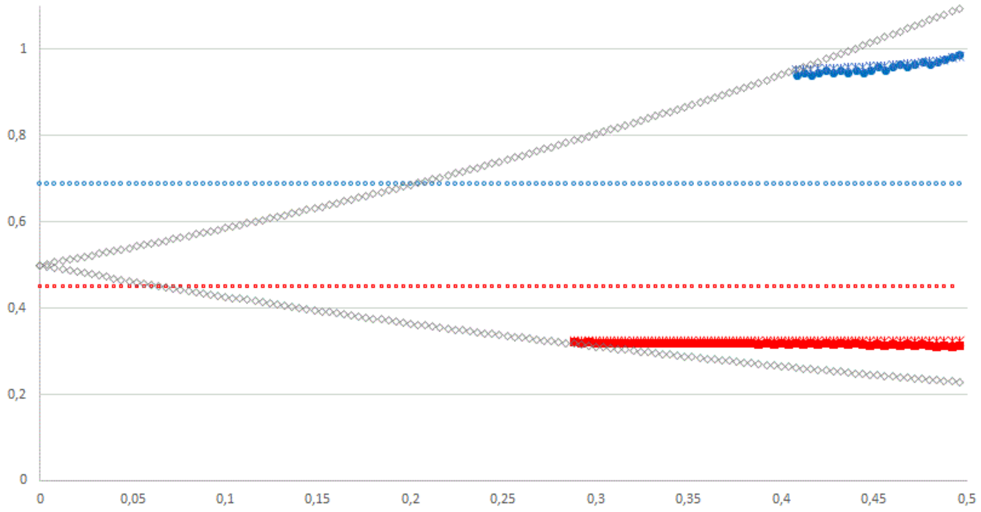

Figure 2. The double free boundaries when r_d is negative. Grey diamonds delimit the stock binomial tree. Blue (resp.red) dots denote the binomial upper (lower) free boundary, and blue (red) stars the asymptotic approximations. Blue (red) circles denote the perpetual upper (lower) boundary.

Finally, we focus on US underlying assets, calibrating our parameters over the period December, 15th, 2015 to December, 15th, 2016, and evaluating Quanto options on December 14th 2016 (data source: Bloomberg). The euro yield curve is negative and the US yield curve is positive, thus fitting into assumptions of 3.2. In particular, on December 14th 2016, the euro and the US yield curves are reported in Table 1.

Table 1: The EU and US Yield Curves, December 14th 2016

Maturity	EU Bund Percent. Yield	US Percent. Yield
30d	-0.97%	0.64%
90d	-0.91%	0.54%
180d	-0.80%	0.69%
270d	-0.80%	0.78%
360d	-0.80%	0.89%

(Source: Bloomberg)

Since our model allows us only constant interest rates, we fix the rates at the intermediate level $r_d = -0.80\%$ for the euro, and $r_f = 0.69\%$ for the USD. On the same date, the exchange spot rate $X(0) = 0.94$ (the inverse eurodollar is $1/0.94 = 1.06$). The volatility of the exchange rate X is $\|\sigma_X\| = 7.8\%$, evaluated as the annualized standard deviation of the one-year time series of daily log-returns of the exchange rate $\left\{ln\left(\frac{X(t+\Delta t)}{X(t)}\right)\right\}$ ending up on December 14th 2016.

We select US stocks from different sectors, Johnson and Johnson's (JNJ), Microsoft (MSFT), Amazon (AMZN), and Apple (AAPL.0). The stocks display different levels of volatility and correlation with the exchange rate X over the period under investigation; they are reported in the first two rows of Table 2. The third row displays the stock's risk neutral drift $\mu_f^Q = r_f - q_f - \rho\,\|\sigma_f\|\,\|\sigma_X\|$, under the assumption $q_f = 0$. The correlation has been estimated from the historical covariance between the one-year time series of daily log-returns of the exchange rate $\left\{ln\left(\frac{X(t+\Delta t)}{X(t)}\right)\right\}$ and the one-year time series of daily log-returns of the US stock price $\left\{ln\left(\frac{S_f(t+\Delta t)}{S_f(t)}\right)\right\}$ ending up on December 14th 2016.

Table 2: Stock parameters values

	JNJ	MSFT	AMZN	AAPL.0
$\|\sigma_f\|$	14%	23%	30%	13.70%
ρ	-0.5%	6%	$-1,7\%$	-0.09%
μ_f^Q	0.7%	0.6%	0.7%	0.7%

The correlation is slightly negative, but in the Microsoft case, where is positive. Interestingly, Assumption (3.1) that ensures the boundedness of the perpetual American quanto option is never true. On the contrary, the necessary finite- maturity condition (3.2) is satisfied over the 6 months option maturity. Interestingly, all American quanto put options (2.6) with 6 months maturity on these stocks display a non-standard deeply in the money continuation region in the binomial approximation. For all the examples of Table 2 we fix the maturity date $T = 0.5$ (in years) and we apply a moneyness normalization by setting the in-the-money initial values $S_f(0) = 1$ and $K_f = 1.15$, to be able to reach the deeply in the money continuation region within our binomial approximation. With $N = 125$ time steps we obtain for the JNJ stock

$$\Delta t = 0.004, \quad u = 1.009, \quad d = 0,992, \quad \text{and } \mathbf{q} = 49.94\%.$$

In Figure 3 the binomial tree for the JNJ stock is delimited by the grey diamonds. The blue (resp. red) dots denote the upper (resp. the lower) binomial free boundary. As from Proposition 3.2, the upper free boundary converges at maturity to the strike price $K_f = 1.15$. The left-limit of the lower free boundary at T is 0.6. The asymptotic approximation for the upper (resp. lower) free boundary obtained in Proposition 3.2 are plotted with blue (resp. red) stars. The asymptotic approximations for both the upper and lower free boundary are very closed to the binomial boundaries over the entire option life. There exist no perpetual constant barriers in this case, as Assumption (3.1) is violated and the perpetual option is unbounded.

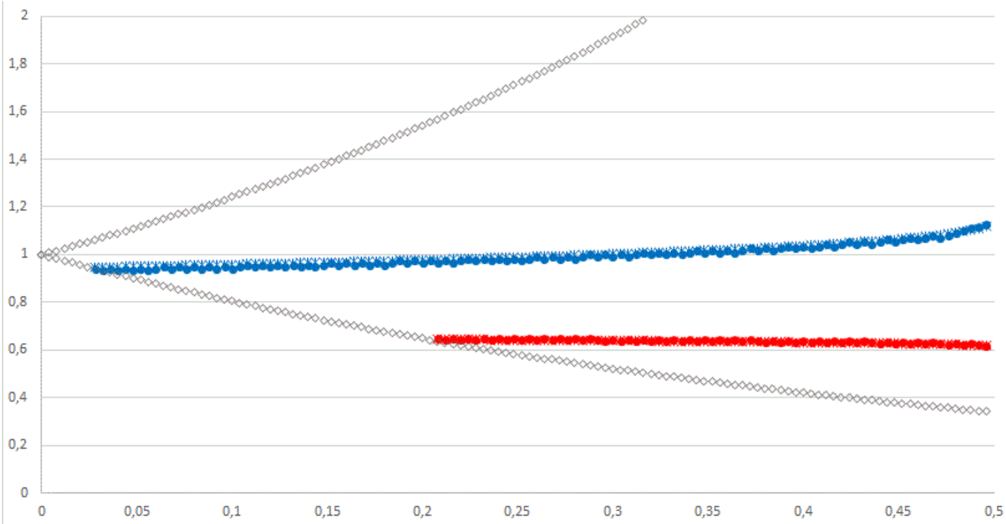

Figure 3. Quanto put option on JNJ stock. Grey diamonds delimit the stock binomial tree. The early exercise region is delimited between the dotted blue and red lines (the blue and red stars denote the asymptotic approximations provided in Proposition 3.2).

For the MSFT stock we obtain very similar results, plotted in Figure 4 with the same legend. In this case we obtain

$$\Delta t = 0.004, \quad u = 1,015, \quad d = 0,985, \quad \text{and } \mathbf{q} = 49,70\%.$$

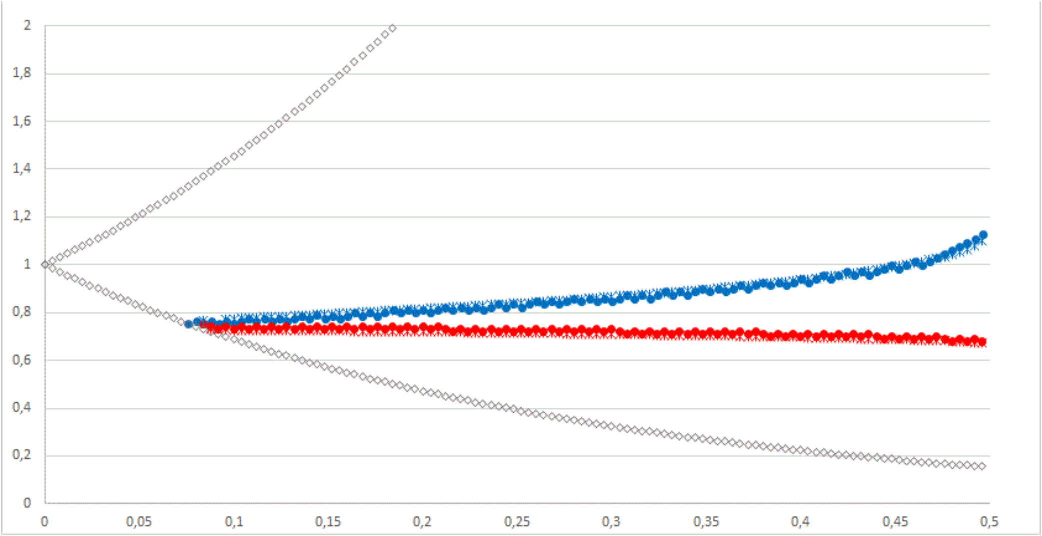

Figure 4. Quanto put option on MSFT stock. Grey diamonds delimit the stock binomial tree. The early exercise region is delimited between the dotted blue and red lines (the blue and red stars denote the asymptotic approximations provided in Proposition 3.2).

In Figure 5 we plot the Quanto put option on the AMZN stock. In this case we have

$$\Delta t = 0.004, \quad u = 1.019, \quad d = 0,981, \quad \text{and } \mathbf{q} = 49.60\%.$$

Because the stock has a higher volatility compared to the previous cases (around 30%), the binomial approximation quickly grows out of the plot area. Nevertheless, the asymptotic approximations for both the upper and lower free boundary are very closed to the binomial boundaries over the entire option life.

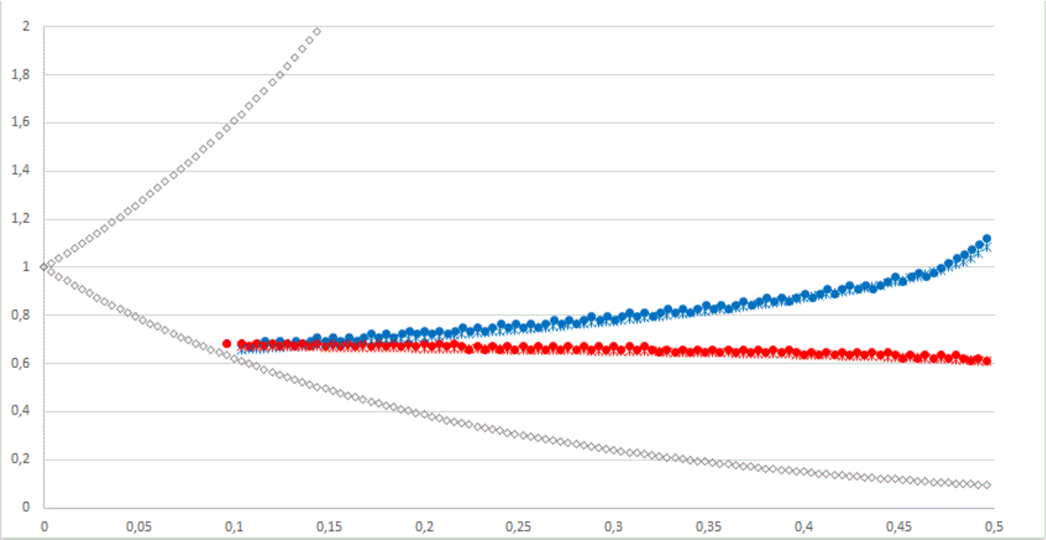

Figure 5. Quanto put option on AMZN stock. Grey diamonds delimit the stock binomial tree. The early exercise region is delimited between the dotted blue and red lines (the blue and red stars denote the asymptotic approximations provided in Proposition 3.2).

In figure 6 we plot our results for the AAPL stock. In this case

$$\Delta t = 0.004, \quad u = 1.009, \quad d = 0.991, \quad \text{and } \mathbf{q} = 49.94\%.$$

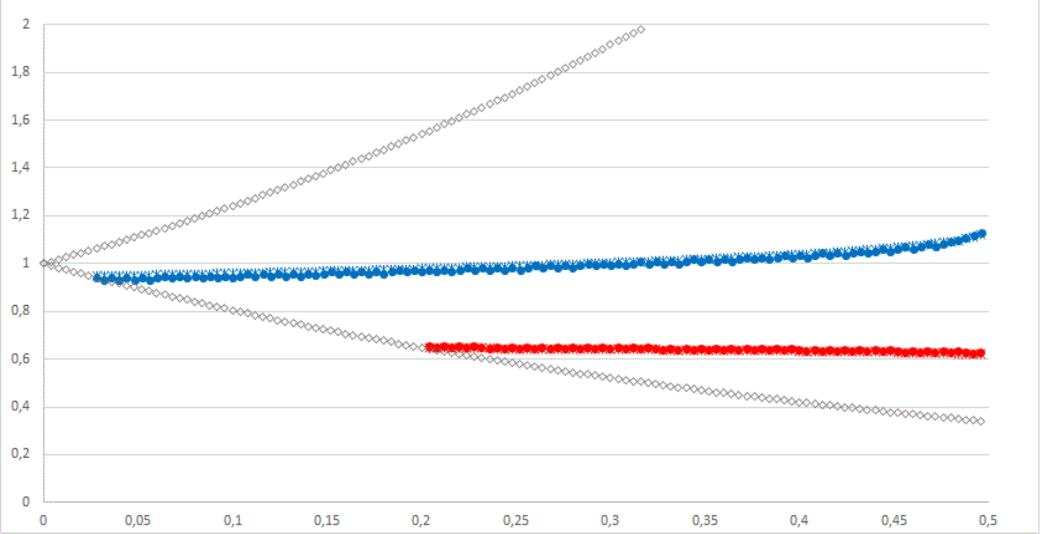

Figure 6. Quanto put option on AAPL stock. Grey diamonds delimit the stock binomial tree. The early exercise region is delimited between the dotted blue and red lines (the blue and red stars denote the asymptotic approximations provided in Proposition 3.2).

The empirical examples depicted in Figures 3-6 show that the asymptotic approximation for both the upper and the lower critical price work well over the option's entire life. The JNJ case (Figure 3) and the AAPL case (Figure 6) are characterized by low levels of the volatility $\|\sigma_f\|$, which result in rather wide early exercise regions. By contrast, the MSFT case (Figure 4) and the AMZN case (Figure 5) are associated with high levels of the volatility $\|\sigma_f\|$, which imply less conspicuous early exercise region. The biggest effect of the increased volatility is on the upper free boundary, which decreases and becomes more concave around maturity. The lower free boundary is relatively more stable and flat.

These examples show how a non-standard double continuation region appears for finite maturity American quanto put options under actual market circumstances. Interestingly, when the maturity of these options tends to infinite, the value of the perpetual American quanto put options becomes unbounded, as the incentive to infinite postponement caused by negative domestic interest rates prevails. Hence, in the perpetual case, early exercise is never optimal, the early exercise region is empty, and there is no free boundary.

5 Conclusion

American option pricing is a dynamic research domain of financial economics. Quanto options offer important financial services as they disclose international hedging/investment opportunities. In a continuous-time, diffusive currency market model we study the interplay of the signs of the domestic and the foreign riskfree rates with the optimal exercise policies for American quanto options. In particular, we show that a negative domestic riskless rate (as in the European and the Japanese markets over recent years) can lead to the existence of a non-standard double continuation region for American quanto options written on a foreign risky security in a positive foreign interest rate environment.

We provide market-based examples of finite-maturity American quanto put options that exhibit a double continuation region surrounding a non-empty early exercise region even if the perpetual early exercise region is empty and the value of the corresponding perpetual option is unbounded. In such empirical examples, we compute accurate asymptotic approximations for the free boundaries at maturity.

We significantly generalize and extend the results of Battauz, De Donno and Sbuelz (2015) by determining the existence, the monotonicity properties and the close-to-maturity behavior of the upper and lower critical prices for American options under the softer assumption of non-emptiness of the early exercise region at some date during the option contract life.

Our current results open interesting avenues for future research, like the introduction of additional sources of risk (e.g. Bakshi, Cao, and Chen (1997), and Cao, Bakshi, and Chen (2015)). Exploring the impact on the optimal exercise policy of American quanto options of systematic jump risk is particularly interesting as, among other things, it can imply a richer parametric structure for the risk-adjusted drift of the underlying foreign security price for the domestic investor. We leave these issues open for future investigation.

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