A study on monomial ideals and Specht ideals

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Contents

Preface

Combinatorial commutative algebra lies at the intersection of two more established fields, commutative algebra and combinatorics. In commutative algebra, Cohen-Macaulay and Gorenstein properties, local cohomologies, Castelnuovo-Mumford regularities and Hilbert series are important objects. One of the purposes of this field is to investigate the relationship between these commutative algebraic properties and combinatorial objects. For example, the important events in combinatorial commutative algebra was R. Stanley's proof([36]) of the upper bound conjecture for the simplicial sphere, based on early work by M. Hoschster and G. Reisner. The problem can be defined in combinatorial and geometric terms, but the method of the proof makes full use of commutative ring theory. In this proof, Stanley uses the Stanley-Reisner ring of the simplicial complex.

Monomial ideals are an important subject in combinatorial commutative algebra. Standard methods in combinatorial commutative algebra for studying homological and enumerative problems about arbitrary monomial ideals are to reduce to squarefree or Borel fixed cases. Borel fixed ideals are monomial ideals of a polynomial ring fixed under the action of upper triangular matrices, and it play an important role in Gröbner basis theory and many related areas, since they appear as the generic initial ideals of homogeneous ideals. Moreover, Borel fixed ideals are strongly stable ideals, when characteristic is 0. On the other hand, any monomial ideal is reduced to a squarefree monomial ideal by (standard) polarization.

Edge ideals are also known to be an important object in combinatorial commutative algebra. The study of edge ideals was started by Villarreal in [43]. An edge ideal is a quadratic squarefree monomial ideal associated with a finite simple graph. By using edge ideals, the relationship between ring-theoretic and graph-theoretic properties has been actively investigated. These studies include the characterization of Cohen–Macaulay and Gorenstein properties.

A Specht module is one of the important representations of symmetric groups. It is studied by W. Specht in 1935. The Specht modules form a complete set of irreducible representations of the symmetric group, in characteristic 0. Such modules are vector spaces spanned by Specht polynomials, which can be constructed combinatorially. We can also consider ideals generated by Specht polynomials. This is called the Specht ideal. Such ideals are known to be related to combinatorial commutative algebra, subspace arrangements, equivariant cohomologies of Springer fibers and symmetric system of equations.

The organization of this doctoral thesis is as follows; it consists of three chapters.

In Chapter 1, we study the Alexander duality for strongly stable ideals. In [46], Yanagawa constructed the alternative polarization b-pol(I) of a strongly stable ideal I. Note that b-pol(I) is the squarefree monomial ideal. On the other hand, the Alexander duality for squarefree monomial ideals is a very powerful tool in the Stanley–Reisner ring theory.

In this thesis, we construct the Alexander dual of strongly stable ideal, and as one of its applications, we describe the formula of the Hilbert series of the local cohomology modules of the quotient ring by a strongly stable ideal using its irreducible decomposition. On the other hand, we show that strongly stable property is characterized by its irreducible decomposition.

In Chapter 2, we study the edge-weighted edge ideals. In this chapter, we consider a finite simple graph. The edge-weighted edge ideal of an edge-weighted graph was introduced in [28]. They also investigated unmixedness and Cohen-Macaulayness of these ideals, in the case that a graph is a cycle, a tree or a complete graph. The purpose of this thesis is to continue this research on a Cohen–Macaulay very-well covered graph. In particular, we characterize unmixed and Cohen-Macaulay properties of edge-weighted edge ideals of Cohen–Macaulay very well-covered graphs. Our results can be seen as generalizations of the results concerning the Cohen-Macaulay property of usual edge ideals of very well-covered graphs.Another kind of generalization of edge ideals is considered in [17, 29, 30]. Indeed, [29] introduced the vertex-weighted edge ideal of an oriented graph. In this paper, we provide the counterexample of the conjecture[29, Conjecture 53].

In Chapter 3, we study the (Castelnuovo-Mumford) regularity and the Hilbert series of Specht ideals for some partitions. A Specht ideal I^{Sp}_λ for a partition λ is an ideal generated by the Specht polynomials of λ. For the partition $\lambda = (n-d, d)$ or $(d, d, 1)$, Yanagawa show that I^{Sp}_λ is a radical ideal over any field, and the quotient ring of these ideals are Cohen–Macaulay using by a result of Etingof et al. [13], which concerns the characteristic 0 case. In addition, in [22], results on the Cohen–Macaulay property of $R/I^{\mathrm{Sp}}_{(n-d,d)}$ are proved without using the results of Etingof et al. The paper [44] computes the Betti numbers of Specht ideals for hook type partitions, it means that we know its Hilbert series in this case.

In this thesis, we compute the Hilbert series of a quotient ring by a Specht ideal of $(n-d, d)$ and $(d, d, 1)$. We also prove that the Hilbert series of these Specht ideals is independent of the characteristic of the field, using the theory of Gröbner basis. The main tool in this calculation is the recursive formulas between Specht ideals when considering the number of variables. As an application, we compute the regularity $\mathrm{reg}(R/I^{\mathrm{Sp}}_\lambda)$, when $R/I^{\mathrm{Sp}}_\lambda$ is Cohen–Macaulay.

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Chapter 1

Strongly stable ideals

1.1 Introduction

This chapter is based on the author's paper [38] with Kohji Yanagawa. *Strongly stable ideals* are monomial ideals defined by a simple condition, and they appear as the generic initial ideals of homogeneous ideals in the characteristic 0 case (so they are also called *Borel fixed ideals* in this case). In a positive characteristic case, the generic initial ideal for any homogeneous ideal is the Borel fixed ideal, but a Borel fixed ideal is not necessarily strongly stable. However, any strongly stable ideal is always Borel fixed.

One of standard methods in combinatorial commutative algebra for treating homological and combinatorics problems about arbitrary monomial ideals is to reduce to the squarefree or Borel-fixed case. In particular, (standard) polarization is often used as a method to reduce general monomial ideals to squarefree monomial ideals.

Extending an idea of [26], Yanagawa([46]) constructed the *alternative polarization* b-pol(I) of a strongly stable ideal I. We briefly explain this notion here. Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K. For a monomial ideal I, $G(I)$ denotes the set of minimal monomial generators of I. If $I \subset S$ is a strongly stable ideal with $\deg(\mathsf{m}) \leq d$ for all $\mathsf{m} \in G(I)$, we consider a larger polynomial ring $\widetilde{S} = K[\, x_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq d\,]$ with the surjection $f : \widetilde{S} \ni x_{i,j} \longmapsto x_i \in S$. Then we can construct a squarefree monomial ideal b-pol$(I) \subset \widetilde{S}$ (if there is no danger of confusion, we will simply write $\widetilde{I}$ for b-pol(I)) satisfying the conditions $f(\widetilde{I}) = I$ and $\beta^{\widetilde{S}}_{i,j}(\widetilde{I}) = \beta^{S}_{i,j}(I)$ for all i, j, where $\beta_{i,j}$ stands for the graded Betti number. The alternative polarization is much more compatible with operations for strongly stable ideals than the standard polarization.

On the other hand, the Alexander duality for squarefree monomial ideals is a very powerful tool in the Stanley–Reisner ring theory. For a squarefree monomial ideal $I \subset S$, $I^\vee \subset S$ denotes its Alexander dual. There is a one to one correspondence between the elements of $G(I)$ and the irreducible components of $I^\vee$. Let $\widetilde{S}' = K[\, y_{i,j} \mid 1 \leq i \leq d, 1 \leq j \leq n\,]$ be a polynomial ring with the isomorphism

$(-)^{\mathsf{t}} : \widetilde{S} \ni x_{i,j} \longmapsto y_{j,i} \in \widetilde{S}'$. For a strongly stable ideal I, there is a strongly stable ideal $I^* \subset K[y_1, \dots, y_d]$ with $\text{b-pol}(I^*) = (\text{b-pol}(I)^\vee)^{\mathsf{t}}$. Clearly, the correspondence $I \longleftrightarrow I^*$ should be considered as the Alexander duality for strongly stable ideals.

After we finished an earlier version of [38], we were informed that, in Fløystad [15, §6], the above duality has been constructed using the notion of generalized (co-)letterplace ideals. Each approach has each advantage. The paper [15] treats the duality in a much wider context, but if one starts from the generator set $G(I)$, our construction is more direct (Proposition 1.31 and Theorem 1.23 give a simple procedure to compute I^* from $G(I)$). We will give a complete proof of the existence of the duality, since we will re-use ideas of the proof in later sections.

The outline of the paper is as follows. Section 2 is mainly devoted to the proof of the existence of the dual I^*. If I is a Cohen–Macaulay strongly stable ideal, $\widetilde{S}/\widetilde{I}$ is the Stanley–Reisner ring of a ball or a sphere (a ball in most cases), and its canonical module can be easily described. In Section 3, we show the formula

$$H(H^i_{\mathfrak{m}}(S/I), \lambda^{-1}) = \sum_{j \in \mathbb{Z}} \frac{\beta_{i-j,n-j}(I^*)\lambda^j}{(1-\lambda)^j}$$

on the Hilbert series of the local cohomology module $H^i_{\mathfrak{m}}(S/I)$. This is more or less a consequence of a classical result [10], and we will improve this formula later.

In Section 4, we discuss the relation to the notion of a *squarefree strongly stable ideal*, which is a squarefree analog of a strongly stable ideal. For a strongly stable ideal $I \subset S$, Aramova et al [1] constructed a squarefree strongly stable ideal $I^\sigma \subset T = K[x_1, \dots, x_N]$ with $N \gg 0$. The class of squarefree strongly stable ideals is closed under the (usual) Alexander duality of T, so our duality can be constructed through I^σ. However, without $\text{b-pol}(I)$, it is hard to compare the algebraic properties of I^* with those of I.

In Section 5, we give a procedure to construct the irreducible decomposition of $\text{b-pol}(I)$ from that of a strongly stable ideal I. As corollaries, we will give formulas on the arithmetic degree $\text{adeg}(S/I)$ and $H(H^i_{\mathfrak{m}}(S/I), \lambda)$ from the irredundant irreducible decomposition

$$I = \bigcap_{\mathbf{a} \in E} \mathfrak{m}^{\mathbf{a}}$$

with $E \subset \mathbb{Z}_{>0} \cup (\mathbb{Z}_{>0})^2 \cup \cdots \cup (\mathbb{Z}_{>0})^n$. Here, for $\mathbf{a} = (a_1, \dots, a_t) \in (\mathbb{Z}_{>0})^t$ with $t \leq n$, $\mathfrak{m}^{\mathbf{a}}$ denotes the irreducible ideal $(x_1^{a_1}, \dots, x_t^{a_t})$ of S. In this situation, set $t(\mathbf{a}) := t$, $e(\mathbf{a}) := a_t$, and $w(\mathbf{a}) := n - \sum_{i=1}^t a_i$. Then we have

$$\text{adeg}(S/I) = \sum_{\mathbf{a} \in E} e(\mathbf{a})$$

and

$$H(H^i_{\mathfrak{m}}(S/I), \lambda^{-1}) = \left(\sum_{\substack{\mathbf{a} \in E, \\ t(\mathbf{a}) = n-i}} (\lambda^{w(\mathbf{a})} + \lambda^{w(\mathbf{a})+1} + \cdots + \lambda^{w(\mathbf{a})+e(\mathbf{a})-1}) \right) / (1-\lambda)^i.$$

Section 6 gives additional results on the irreducible decompositions of strongly stable ideals. While a strongly stable ideal I is characterized by the "left shift property" on $G(I)$, Theorem 1.35 states that it is also characterized by the "right shift property" on the irreducible components of I.

1.2 The construction of the Alexander duality for strongly stable ideals

In this section, we define the Alexander duality for strongly stable ideals using the alternative polarization. As applications, we show that the alternative polarization of a Cohen-Macaulay strongly stable ideal is the Stanley–Reisner ideal of a ball or a sphere, and give a description of its canonical module.

First, we introduce the convention and notation used throughout the paper. For a positive integer n, set $[n] := \{1, \ldots, n\}$. Let $S := K[x_1, \ldots, x_n]$ be a polynomial ring over a field K, and $\mathfrak{m} = (x_1, \ldots, x_n)$ the unique graded maximal ideal of S. For a monomial ideal $I \subset S$, $G(I)$ denotes the set of minimal monomial generators of I. We say an ideal $I \subset S$ is *strongly stable*, if it is a monomial ideal, and the condition that $\mathsf{m} \in G(I)$, $x_i | \mathsf{m}$ and $j < i$ imply $(x_j/x_i) \cdot \mathsf{m} \in I$ is satisfied.

Let d be a positive integer, and set

$$\widetilde{S} := K[\, x_{i,j} \,|\, 1 \leq i \leq n, 1 \leq j \leq d\,].$$

Note that

$$\Theta := \{x_{i,1} - x_{i,j} \,|\, 1 \leq i \leq n,\ 2 \leq j \leq d\,\} \subset \widetilde{S}$$

forms a regular sequence with the isomorphism $\widetilde{S}/(\Theta) \cong S$ induced by $\widetilde{S} \ni x_{i,j} \longmapsto x_i \in S$.

Definition 1.1. For a monomial ideal $I \subset S$, a *polarization* of I is a squarefree monomial ideal $J \subset \widetilde{S}$ satisfying the following conditions.

(1) Through the isomorphism $\widetilde{S}/(\Theta) \cong S$, we have $\widetilde{S}/(\Theta) \otimes_{\widetilde{S}} \widetilde{S}/J \cong S/I$.

(2) Θ forms a $\widetilde{S}/J$-regular sequence.

For $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, $x^{\mathbf{a}}$ denotes the monomial $\prod_{i=1}^n x_i^{a_i} \in S$. For a monomial $x^{\mathbf{a}} \in S$ with $\deg(x^{\mathbf{a}}) \leq d$, set

$$\operatorname{pol}(x^{\mathbf{a}}) := \prod_{1 \leq i \leq n} x_{i,1} x_{i,2} \cdots x_{i,a_i} \in \widetilde{S}.$$

If $I \subset S$ is a monomial ideal with $\deg(\mathsf{m}) \leq d$ for all $\mathsf{m} \in G(I)$, then it is well-known that

$$\operatorname{pol}(I) := (\, \operatorname{pol}(\mathsf{m}) \,|\, \mathsf{m} \in G(I)\,)$$

is a polarization of I, which is called the *standard polarization.*

Any monomial $\mathsf{m} \in S$ has a unique expression

$$\mathsf{m} = \prod_{i=1}^{e} x_{\alpha_i} \quad \text{with} \quad 1 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_e \le n. \tag{1.2.1}$$

If $e\,(= \deg(\mathsf{m})) \le d$, we set

$$\text{b-pol}(\mathsf{m}) := \prod_{i=1}^{e} x_{\alpha_i, i} \in \widetilde{S}. \tag{1.2.2}$$

As another expression, for a monomial $x^{\mathbf{a}} \in S$ with $\deg(x^{\mathbf{a}}) \le d$, set $b_i := \sum_{j=1}^{i} a_j$ for each $i \ge 1$ and $b_0 = 0$. Then

$$\text{b-pol}(x^{\mathbf{a}}) = \prod_{\substack{1 \le i \le n \\ b_{i-1}+1 \le j \le b_i}} x_{i,j} \in \widetilde{S}.$$

For a monomial ideal $I \subset S$ with $\deg(\mathsf{m}) \le d$ for all $\mathsf{m} \in G(I)$ (in the sequel, we always assume this condition), set

$$\text{b-pol}(I) := (\, \text{b-pol}(\mathsf{m}) \mid \mathsf{m} \in G(I)\,) \subset \widetilde{S}.$$

See the beginning of Example 1.6 below.

In [46], Yanagawa showed the following.

Theorem 1.2 ([46, Theorem 3.4])**.** *If $I \subset S$ is a strongly stable ideal, then* b-pol(I) *gives a polarization of I.*

In the rest of the paper, the next fact is frequently used without comment.

Lemma 1.3. *Let $I \subset S$ be a strongly stable ideal. For a monomial $\mathsf{m} \in S$ with* $\deg(\mathsf{m}) \le d$*, $\mathsf{m} \in I$ if and only if* b-pol$(\mathsf{m}) \in$ b-pol(I)*.*

Proof. The necessity is shown in [46, Lemma 3.1], and the sufficiency is an easy exercise. □

An *irreducible monomial ideal* of S is an ideal of the form $(\, x_i^{a_i} \mid a_i > 0)$ for some $\mathbf{a} \in \mathbb{N}^n$. A presentation of a monomial ideal I as an intersection $I = \bigcap_{i=1}^{r} Q_i$ of irreducible monomial ideals is called an *irreducible decomposition.* An intersection $I = \bigcap_{i=1}^{r} Q_i$ is *irredundant*, if none of the ideals Q_i can be omitted in this presentation. Any monomial ideal has a unique irredundant irreducible decomposition $I = \bigcap_{i=1}^{r} Q_i$. In this case, each Q_i is called an *irreducible component* of I. If I is a squarefree monomial ideal, then the irreducible components are nothing other than the associated primes.

If $I \subset S$ is a squarefree monomial ideal (equivalently, S/I is the Stanley–Reisner ring of some simplicial complex), then the irreducible components of I are of the form $\mathfrak{m}^F := (x_i \mid i \in F)$ for some $F \subset [n]$, and the ideal

$$I^\vee := \Big(\prod_{i \in F} x_i \mid \mathfrak{m}^F \text{ is an irreducible component of } I \Big)$$

called the *Alexander dual* of I. Then we have $I^{\vee\vee} = I$. This duality is very important in the Stanley–Reisner ring theory. See, for example, [10, 23].

Lemma 1.4. *For a monomial ideal $I \subset S$, the following conditions are equivalent.*

(1) I is strongly stable.

(2) $\text{b-pol}(I) \subset \widetilde{S}$ has an irreducible decomposition $\bigcap_{s=1}^r P_s$ satisfying the following property.

(∗) For each s, there is a positive integer t_s, and integers $\gamma_i^{\langle s \rangle}$ for $1 \le i \le t_s$ such that $P_s = (\, x_{i,\gamma_i^{\langle s \rangle}} \mid 1 \le i \le t_s \,)$ and $1 \le \gamma_1^{\langle s \rangle} \le \gamma_2^{\langle s \rangle} \le \cdots \le \gamma_{t_s}^{\langle s \rangle}$.

Proof. (1) ⇒ (2): This is shown already in [46, Remark 3.3].

(2) ⇒ (1): For a contradiction, assume that $\widetilde{I} := \text{b-pol}(I)$ satisfies the condition (∗) but I is not strongly stable. Then it is easy to see that there is some $\mathsf{m} = x^{\mathbf{a}} \in G(I)$ such that $x_{j+1} \,|\, \mathsf{m}$ and $(x_j/x_{j+1}) \cdot \mathsf{m} \notin I$ for some $j < n$. Then we have $\text{b-pol}((x_j/x_{j+1}) \cdot \mathsf{m}) \notin \text{b-pol}(I)$, and it implies that $\text{b-pol}((x_j/x_{j+1}) \cdot \mathsf{m}) \notin P_s = (x_{1,\gamma_1^{\langle s \rangle}}, x_{2,\gamma_2^{\langle s \rangle}}, \dots, x_{t_s,\gamma_{t_s}^{\langle s \rangle}})$ for some s. As before, set $b_0 := 0$ and $b_i := \sum_{j=1}^i a_j$ for $i \ge 1$. Since

$$\text{b-pol}(\mathsf{m}) = \prod_{\substack{1 \le i \le n \\ b_{i-1}+1 \le j \le b_i}} x_{i,j},$$

we have $\gamma_i^{\langle s \rangle} \notin \{\, b_{i-1}+1, \dots, b_i \,\}$ for all $i \ne j, j+1$, $\gamma_j^{\langle s \rangle} \notin \{\, b_{j-1}+1, \dots, b_j+1 \,\}$, and $\gamma_{j+1}^{\langle s \rangle} \notin \{\, b_j+2, \dots, b_{j+1} \,\}$. Here we have $\text{b-pol}(\mathsf{m}) \in \text{b-pol}(I) \subset P_s$, and it implies $\gamma_{j+1}^{\langle s \rangle} = b_j + 1$. Since $\gamma_j^{\langle s \rangle} \le \gamma_{j+1}^{\langle s \rangle} (= b_j+1)$ and $\gamma_j^{\langle s \rangle} \notin \{\, b_{j-1}+1, \dots, b_j+1 \,\}$, we have $\gamma_j^{\langle s \rangle} \le b_{j-1}$. If $j \ge 2$, combining $\gamma_{j-1}^{\langle s \rangle} \le \gamma_j^{\langle s \rangle} (\le b_{j-1})$ with $\gamma_{j-1}^{\langle s \rangle} \notin \{\, b_{j-2}+1, \dots, b_{j-1} \,\}$, we have $\gamma_{j-1}^{\langle s \rangle} \le b_{j-2}$. Repeating this argument, we have $\gamma_1^{\langle s \rangle} \le b_0$. Since $\gamma_1^{\langle s \rangle} \ge 1$ and $b_0 = 0$, this is a contradiction. □

Let $\widetilde{S}' := K[\, y_{i,j} \,|\, 1 \le i \le d, 1 \le j \le n \,]$ be a polynomial ring with the ring isomorphism $(-)^{\mathsf{t}} : \widetilde{S} \to \widetilde{S}'$ defined by $\widetilde{S} \ni x_{i,j} \longmapsto y_{j,i} \in \widetilde{S}'$.

Theorem 1.5 (c.f. [15])**.** *Let $I \subset S$ be a strongly stable ideal. Then there exists a strongly stable ideal $I^* \subset S' := K[y_1, \dots, y_d]$ such that* $\text{b-pol}(I^*) = (\text{b-pol}(I)^\vee)^{\mathsf{t}}$.

Proof. As before, set $\widetilde{I} :=$ b-pol(I). There is a one to one correspondence between the irreducible components of $\widetilde{I}$ and the elements of $G(\widetilde{I}^\vee)$. If the irrdundant irreducible decomposition of $\widetilde{I}$ is given by

$$\widetilde{I} = \bigcap_{s=1}^{r} (\, x_{i,\gamma_i^{\langle s \rangle}} \mid 1 \leq i \leq t_s \,) \subset \widetilde{S},$$

then we have

$$(\widetilde{I}^\vee)^{\mathsf{t}} = \Big(\prod_{i=1}^{t_s} y_{\gamma_i^{\langle s \rangle},i} \mid 1 \leq s \leq r \Big) \subset \widetilde{S}'.$$

Since $\gamma_1^{\langle s \rangle} \leq \gamma_2^{\langle s \rangle} \leq \cdots \leq \gamma_{t_s}^{\langle s \rangle}$ by Lemma 1.4, we have b-pol$(I^*) = (\widetilde{I}^\vee)^{\mathsf{t}}$ for

$$I^* = \Big(\prod_{i=1}^{t_s} y_{\gamma_i^{\langle s \rangle}} \mid 1 \leq s \leq r \Big) \subset S'.$$

There also exists a one to one correspondence between the irreducible components of $\widetilde{I}^\vee$ and the elements of $G(\widetilde{I})$, equivalently, the elements of $G(I)$. If the monomial m in (1.2.1) belongs to $G(I)$, the irreducible component of $\widetilde{I}^\vee$ given by m is of the form $(x_{\alpha_1,1}, x_{\alpha_2,2}, \ldots, x_{\alpha_e,e})$ by the expression (1.2.2). Then the corresponding irreducible component of $(\widetilde{I}^\vee)^{\mathsf{t}}$ ($=$ b-pol(I^*)) is $(y_{1,\alpha_1}, \ldots, y_{e,\alpha_e}) \subset \widetilde{S}'$. Since $\alpha_1 \leq \cdots \leq \alpha_e$, I^* is strongly stable by Lemma 1.4. □

The above theorem gives a duality between strongly stable ideals $I \subset S = K[x_1, \ldots, x_n]$ whose generators have degree at most d and strongly stable ideals $I^* \subset S' = K[y_1, \ldots, y_d]$ whose generators have degree at most n.

Example 1.6. For a strongly stable ideal $I = (x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3)$, we have

$$\begin{aligned}
\text{b-pol}(I) &= (\, x_{1,1}x_{1,2},\ x_{1,1}x_{2,2},\ x_{1,1}x_{3,2},\ x_{2,1}x_{2,2},\ x_{2,1}x_{3,2} \,) \\
&= (\, x_{1,1},\ x_{2,1} \,) \cap (\, x_{1,1},\ x_{2,2},\ x_{3,2} \,) \cap (\, x_{1,2},\ x_{2,2},\ x_{3,2} \,) \\
\text{b-pol}(I)^\vee &= (\, x_{1,1}x_{2,1},\ x_{1,1}x_{2,2}x_{3,2},\ x_{1,2}x_{2,2}x_{3,2} \,) \\
(\text{b-pol}(I)^\vee)^{\mathsf{t}} &= (\, y_{1,1}y_{1,2},\ y_{1,1}y_{2,2}y_{2,3},\ y_{2,1}y_{2,2}y_{2,3} \,),
\end{aligned}$$

hence the dual strongly stable ideal is given by

$$I^* = (\, y_1^2,\ y_1y_2^2,\ y_2^3 \,).$$

On the other hand, if we use the standard polarization, we have

$$\begin{aligned}
\text{pol}(I) &= (\, x_{1,1}x_{1,2},\ x_{1,1}x_{2,1},\ x_{1,1}x_{3,1},\ x_{2,1}x_{2,2},\ x_{2,1}x_{3,1} \,) \\
&= (\, x_{1,1}, x_{2,1} \,) \cap (\, x_{1,1}, x_{2,2}, x_{3,1} \,) \cap (\, x_{1,2}, x_{2,1}, x_{3,1} \,) \\
\text{pol}(I)^\vee &= (\, x_{1,1}x_{2,1},\ x_{1,1}x_{2,2}x_{3,1},\ x_{1,2}x_{2,1}x_{3,1} \,).
\end{aligned}$$

Here $(\text{pol}(I)^\vee)^{\mathsf{t}} = (y_{1,1}y_{1,2},\ y_{1,1}y_{1,3}y_{2,2},\ y_{1,2}y_{1,3}y_{2,1})$ can not be the standard or altarnative polarization of any ideal.

The next two results are implicitly contained in Fløystad [15]. However they are stated in the context of the preceding papers [16, 8], where the words "letterplace ideal" and "coletterplace ideals" are used in the narrow sense (see Remark 1.8 below).

Proposition 1.7. *If $I \subset S$ is a strongly stable ideal with $\sqrt{I} = \mathfrak{m}$, then* b-pol(I) *(more precisely,* b-pol$(I)^{\mathsf{t}}$*) is the letterplace ideal $L(\mathcal{J}; d, [n])$ in the sense of [8]. Here $\mathcal{J}$ is an order ideal of* $\mathrm{Hom}([n], [d])$. *Conversely, any letterplace ideal $L(\mathcal{J}; d, [n])$ arises in this way from a strongly stable ideal I with $\sqrt{I} = \mathfrak{m}$.*

Proof. If $I \subset S$ is a strongly stable ideal with $\sqrt{I} = \mathfrak{m}$, then the dual $I^* \subset S' = K[y_1, \dots, y_d]$ is a strongly stable ideal whose minimal generators all have degree n. As shown in [16, §6.1], b-pol(I^*) is a co-letterplace ideal $L([n], d; \mathcal{J})$ for some order ideal $\mathcal{J} \subset \mathrm{Hom}([n], [d])$. Then the Alexander dual of b-pol(I^*), which coincides with b-pol$(I)^{\mathsf{t}}$, is the letterplace ideal $L(\mathcal{J}; d, [n])$ by definition.

The second assertion follows from the fact that any co-letterplace ideal $L([n], d; \mathcal{J})$ is the b-pol$(-)$ of some strongly stable ideal whose generators all have degree n. □

Remark 1.8. In [15], Fløystad generalized the notions of a (co-)letterplace ideal so that b-pol(I) of any strongly stable ideal I belongs to these classes (one of the crucial points is considering an order ideal $\mathcal{J}$ in $\mathrm{Hom}([n], \mathbb{N})$, not in $\mathrm{Hom}([n], [d])$). Through this idea, he gave the duality.

For a monomial $x^{\mathbf{a}} \in S$ with $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$, set $\nu(x^{\mathbf{a}}) := \max\{\, i \mid a_i > 0 \,\}$. It is well-known that if I is strongly stable, then

$$\mathrm{proj}-\dim_S(S/I) = \max\{\, \nu(\mathsf{m}) \mid \mathsf{m} \in G(I) \} \quad \text{and} \quad \mathrm{ht}(I) = \max\{\, i \mid x_i \in \sqrt{I}\,\}.$$

Hence, for a strongly stable ideal I with $\mathrm{ht}(I) = c$, S/I is Cohen–Macaulay if and only if $\nu(\mathsf{m})c$ for all $\mathsf{m} \in G(I)$, if and only if $\mathsf{m} \in K[x_1, \dots, x_c]$ for all $\mathsf{m} \in G(I)$. Of course, $\widetilde{S}/$ b-pol(I) is Cohen–Macaulay if and only if so is S/I.

Corollary 1.9. *Let $(0) \neq I \subset S$ be a Cohen–Macaulay strongly stable ideal, and set $\widetilde{I} :=$* b-pol(I). *Then $\widetilde{S}/\widetilde{I}$ is the Stanley–Reisner ring of a ball or a sphere. More precisely, if $n \geq 2$, then $\widetilde{S}/\widetilde{I}$ is the Stanley–Reisner ring of a ball.*

If $n = 1$, then $I = (x^e)$ for some $e \leq d$. Hence $\widetilde{I} = (x_{1,1}x_{1,2} \cdots x_{1,e})$, and $\widetilde{S}/\widetilde{I}$ is the Stanley–Reisner ring of a sphere (resp. ball) if $e = d$ (resp. $e < d$).

Proof. First, assume that $\sqrt{I} = \mathfrak{m}$. In this case, $\widetilde{I}$ is a letterplace ideal $L(\mathcal{J}; d, [n])$ by Proposition 1.7, and the assertion follows from [8, Theorem 5.1] (note that the poset $[n]$ is an antichain if and only if $n = 1$).

If $\sqrt{I} \neq \mathfrak{m}$ (equivalently, $c := \mathrm{ht}(I) < n$), then we have $I = JS$ for a strongly stable ideal $J \subset K[x_1, \dots, x_c]$ with $\sqrt{J} = (x_1, \dots, x_c)$. Moreover, the simplicial complex associated with $\widetilde{I}$ is the cone over the one associated with b-pol(J). So the assertion can be reduced to the first case. □

For $x^{\mathbf{a}} \in S$ with $\deg(x^{\mathbf{a}}) \le d$ and $l := \nu(x^{\mathbf{a}})$, set

$$\mu(x^{\mathbf{a}}) := \Big(\prod_{i=1}^{l-1} x_{i,b_i+1}\Big) \cdot \text{b-pol}(x^{\mathbf{a}}),$$

where $b_i := \sum_{j=1}^{i} a_j$ for each i as before. In [27], R. Okazaki and Yanagawa constructed a minimal $\widetilde{S}$-free resolution $\widetilde{P}_\bullet$ of b-pol(I) of a strongly stable ideal I. If S/I is a Cohen-Macaulay ring of codimension c, the "last" term $\widetilde{P}_c$ of the minimal free resolution is isomorphic to

$$\bigoplus_{\substack{\mathsf{m}\in G(I)\\ \nu(\mathsf{m})=c}} \widetilde{S}(-\deg(\mu(\mathsf{m}))).$$

We also set

$$\widetilde{X} := \prod_{\substack{1\le i\le n\\ 1\le j\le d}} x_{i,j}$$

and

$$\omega(\mathsf{m}) := \widetilde{X}/\mu(\mathsf{m})$$

for $\mathsf{m} \in G(I)$.

Corollary 1.10. *Let $(0) \ne I \subset S$ be a Cohen–Macaulay strongly stable ideal with $\mathrm{ht}(I) = c$, and set $\widetilde{I} :=$ b-pol(I). Then the canonical module $\omega_{\widetilde{S}/\widetilde{I}}$ is isomorphic to the ideal of $\widetilde{S}/\widetilde{I}$ generated by (the image of) $\{\,\omega(\mathsf{m}) \mid \mathsf{m} \in G(I), \nu(\mathsf{m}) = c\,\}$.*

Proof. By Corollary 1.9, $\widetilde{S}/\widetilde{I}$ is the Stanley–Reisner ring of a ball or a sphere. Recall that, for the Stanley–Reisner ring $K[\Delta]$ of a simplicial sphere Δ, $K[\Delta]$ itself is the multigraded canonical module of $K[\Delta]$ (see [4, Corollary 5.6.5]). If Δ is a simplicial ball, then the boundary $\partial\Delta$ is a sphere. Hence the ideal of $K[\Delta]$ generated by all squarefree monomials associated with the faces $\Delta \setminus \partial\Delta$ is a canonical module of $K[\Delta]$ by [4, Theorem 5.7.2]. Anyway, the canonical module $\omega_{\widetilde{S}/\widetilde{I}}$ is isomorphic to a multigraded ideal of $\widetilde{S}/\widetilde{I}$. Since $\omega_{\widetilde{S}/\widetilde{I}} = \mathrm{Ext}^c_{\widetilde{S}}(\widetilde{S}/\widetilde{I}, \omega_{\widetilde{S}})$ and $\omega_{\widetilde{S}}$ is isomorphic to the principal ideal $(\widetilde{X})$ of $\widetilde{S}$, $\omega_{\widetilde{S}/\widetilde{I}}$ is a quotient of

$$\mathrm{Hom}_{\widetilde{S}}(\widetilde{P}_c, \omega_{\widetilde{S}}) \cong \bigoplus_{\substack{\mathsf{m}\in G(I)\\ \nu(\mathsf{m})=c}} \widetilde{S}(-\deg(\omega(\mathsf{m}))).$$

So we are done. □

For a Cohen–Macaulay strongly stable ideal I, the canonical module $\omega_{S/I}$ of S/I itself is isomorphic to $\omega_{\widetilde{S}/\widetilde{I}} \otimes_{\widetilde{S}} \widetilde{S}/(\Theta)$ and Θ forms a $(\omega_{\widetilde{S}/\widetilde{I}})$-regular sequence, where

$\Theta = \{x_{i,1} - x_{i,j} \mid 1 \le i \le n,\ 2 \le j \le d\}$. However, $\omega_{S/I}$ is not isomorphic to an ideal of S/I in general.

We also remark that [8, Corollary 4.3] gives a description of the canonical module of the quotient ring of a letterplace ideal, and it also works in the case of Corollary 1.10. However, our description is much simpler in this case.

1.3 The Hilbert series of $H^i_{\mathfrak{m}}(S/I)$

In this section, for a strongly stable ideal I, we show that the Hilbert series of $H^i_{\mathfrak{m}}(S/I)$ can be described by the irreducible decomposition of b-pol(I).

Let $R = K[x_1, \dots, x_m]$ be a polynomial ring. For a $\mathbb{Z}$-graded R-module M, $H(M, \lambda)$ denotes the Hilbert series $\sum_{i\in\mathbb{Z}}(\dim_K M_i)\lambda^i$ of M. Let ω_R denote the graded canonical module $R(-m)$ of R.

The following must be a fundamental formula on the Alexander duality of Stanley–Reisner ring theory, but we cannot find any reference.

Lemma 1.11. *Let $R = K[x_1, \dots, x_m]$ be a polynomial ring, and $I \subset R$ a squarefree monomial ideal. Then we have*

$$H(\operatorname{Ext}^{m-i}_R(R/I, \omega_R), \lambda) = \sum_{j\ge 0} \frac{\beta_{i-j,m-j}(I^\vee)\lambda^j}{(1-\lambda)^j}.$$

Here $I^\vee \subset R$ is the Alexander dual of I, and $\beta_{p,q}(I^\vee)$ is the graded Betti number of $I^\vee$, that is, the dimension of $[\operatorname{Tor}^R_p(I^\vee, K)]_q$.

Proof. For $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{N}^m$, the vector $\overline{\mathbf{a}} = (\overline{a}_1, \dots, \overline{a}_m) \in \mathbb{N}^m$ is defined by

$$\overline{a}_i = \begin{cases} 1 & \text{if } a_i \ge 1, \\ 0 & \text{if } a_i = 0. \end{cases}$$

By [45, Theorem 2.6], $\operatorname{Ext}^i_R(R/I, \omega_R)$ is a squarefree module. Hence we have $[\operatorname{Ext}^i_R(R/I, \omega_R)]_{\mathbf{a}} = 0$ for all $\mathbf{a} \in \mathbb{Z}^m \setminus \mathbb{N}^m$, and

$$[\operatorname{Ext}^i_R(R/I, \omega_R)]_{\mathbf{a}} \cong [\operatorname{Ext}^i_R(R/I, \omega_R)]_{\overline{\mathbf{a}}}$$

for all $\mathbf{a} \in \mathbb{N}^m$. Furthermore, it is well-known (cf., [45, Theorem 3.4]) that

$$[\operatorname{Ext}^i_R(R/I, \omega_R)]_{\overline{\mathbf{a}}} \cong [\operatorname{Tor}^R_{m-|\overline{\mathbf{a}}|-i}(\widetilde{I}^\vee, K)]_{\mathbf{1}-\overline{\mathbf{a}}}.$$

Here we set $\mathbf{1} := (1, \dots, 1) \in \mathbb{N}^m$, and $|\mathbf{b}| := \sum_{i=1}^m b_i$ for $\mathbf{b} = (b_1, \dots, b_m) \in \mathbb{N}^m$. It is also well-known that $[\operatorname{Tor}^R_i(\widetilde{I}^\vee, K)]_{\mathbf{a}} \ne 0$ for $\mathbf{a} \in \mathbb{Z}^m$ implies $\mathbf{a}$ is a 0-1 vector.

So we have

$$\dim_K[\operatorname{Ext}^{m-i}_R(R/I, \omega_R)]_0 = \beta_{i,m}(I^\vee)$$

and

$$\begin{aligned}
\dim_K[\operatorname{Ext}_R^{m-i}(R/I,\omega_R)]_l &= \sum_{j=1}^{l} \sum_{\substack{\mathbf{a}\in\mathbb{N}^m \\ |\mathbf{a}|=l, |\overline{\mathbf{a}}|=j}} \dim_K[\operatorname{Ext}_R^{m-i}(R/I,\omega_R)]_{\mathbf{a}} \\
&= \sum_{j=1}^{l} \sum_{\substack{\mathbf{a}\in\mathbb{N}^m \\ \mathbf{a}=\overline{\mathbf{a}}, |\mathbf{a}|=j}} \binom{l-1}{l-j} \dim_K[\operatorname{Ext}_R^{m-i}(R/I,\omega_R)]_{\mathbf{a}} \\
&= \sum_{j=1}^{l} \sum_{\substack{\mathbf{a}\in\mathbb{N}^m \\ \mathbf{a}=\overline{\mathbf{a}}, |\mathbf{a}|=j}} \binom{l-1}{l-j} \dim_K[\operatorname{Tor}_{i-j}^R(I^\vee,K)]_{\mathbf{1}-\mathbf{a}} \\
&= \sum_{j=1}^{l} \binom{l-1}{l-j} \beta_{i-j,m-j}(I^\vee)
\end{aligned}$$

for $l>0$. So the assertion follows from the following computation

$$\begin{aligned}
\sum_{j\geq 0} \frac{\beta_{i-j,m-j}(I^\vee)\lambda^j}{(1-\lambda)^j} &= \beta_{i,m}(I^\vee) + \sum_{j\geq 1} \left\{ \beta_{i-j,m-j}(I^\vee)\lambda^j \cdot \sum_{p\geq 0} \binom{j+p-1}{p} \lambda^p \right\} \\
&= \beta_{i,m}(I^\vee) + \sum_{l\geq 1} \left\{ \sum_{j=1}^{l} \binom{l-1}{l-j} \beta_{i-j,m-j}(I^\vee) \right\} \lambda^l \\
&= \dim_K[\operatorname{Ext}_R^{m-i}(R/I,\omega_R)]_0 + \sum_{l\geq 1} \dim_K[\operatorname{Ext}_R^{m-i}(R/I,\omega_R)]_l \cdot \lambda^l,
\end{aligned}$$

where $l := j+p$. □

Corollary 1.12. *For a strongly stable ideal $I \subset S$ with $\widetilde{I} := \text{b-pol}(I)$, we have*

$$H(\operatorname{Ext}_{\widetilde{S}}^{nd-i}(\widetilde{S}/\widetilde{I}, \omega_{\widetilde{S}}), \lambda) = \sum_{j\geq 0} \frac{\beta_{i-j,nd-j}(I^*)\lambda^j}{(1-\lambda)^j}.$$

Proof. The assertion follows from Lemma 1.11 (applying to $\widetilde{I} \subset \widetilde{S}$) and the equality $\beta_{p,q}(\widetilde{I}^\vee) = \beta_{p,q}(I^*)$. □

Theorem 1.13. *Let $I \subset S$ be a strongly stable ideal. Then the Hilbert series of the local cohomology module $H_{\mathfrak{m}}^i(S/I)$ can be described as follows.*

$$H(H_{\mathfrak{m}}^i(S/I), \lambda^{-1}) = \sum_{j\in\mathbb{Z}} \frac{\beta_{i-j,n-j}(I^*)\lambda^j}{(1-\lambda)^j}.$$

Proof. Set $\Theta := \{\, x_{i,1} - x_{i,j} \mid 1 \le i \le n,\ 2 \le j \le d \,\}$. By the full statement of [46, Theorem 3.4], if $\operatorname{Ext}^i_{\widetilde{S}}(\widetilde{S}/\widetilde{I}, \widetilde{S}) \neq 0$, then Θ forms an $\operatorname{Ext}^i_{\widetilde{S}}(\widetilde{S}/\widetilde{I}, \widetilde{S})$-regular sequence. Hence we have

$$[\widetilde{S}/(\Theta) \otimes_{\widetilde{S}} \operatorname{Ext}^{n-i}_{\widetilde{S}}(\widetilde{S}/\widetilde{I}, \omega_{\widetilde{S}})](nd-n) \cong \operatorname{Ext}^{n-i}_S(S/I, \omega_S)$$

and

$$\begin{aligned} H(\operatorname{Ext}^{n-i}_S(S/I, \omega_S), \lambda) &= \lambda^{n-nd} \cdot H(\widetilde{S}/(\Theta) \otimes_{\widetilde{S}} \operatorname{Ext}^{n-i}_{\widetilde{S}}(\widetilde{S}/\widetilde{I}, \omega_{\widetilde{S}}), \lambda) \\ &= \lambda^{n-nd}(1-\lambda)^{nd-n} \cdot H(\operatorname{Ext}^{n-i}_{\widetilde{S}}(\widetilde{S}/\widetilde{I}, \omega_{\widetilde{S}}), \lambda) \\ &= \lambda^{n-nd}(1-\lambda)^{nd-n} \sum_{j \ge 0} \frac{\beta_{nd-n+i-j,nd-j}(I^*)\lambda^j}{(1-\lambda)^j}, \end{aligned}$$

where the last equality follows from Corollary 1.12. Replacing j by $nd - n + j$, we have

$$\begin{aligned} H(H^i_{\mathfrak{m}}(S/I), \lambda^{-1}) &= H(\operatorname{Ext}^{n-i}_S(S/I, \omega_S), \lambda) \\ &= \lambda^{n-nd}(1-\lambda)^{nd-n} \sum_{j \ge n-nd} \frac{\beta_{i-j,n-j}(I^*)\lambda^{nd-n+j}}{(1-\lambda)^{nd-n+j}} \\ &= \sum_{j \ge n-nd} \frac{\beta_{i-j,n-j}(I^*)\lambda^j}{(1-\lambda)^j}. \end{aligned}$$

Here the first equality follows from the fact that $H^i_{\mathfrak{m}}(S/I)$ is the graded Matlis dual of $\operatorname{Ext}^{n-i}_S(S/I, \omega_S)$. □

Corollary 1.14. *Let $I \subset S$ be a strongly stable ideal. Then S/I is a Cohen-Macaulay ring if and only if I^* has a linear resolution.*

Proof. Follows from Theorem 1.13, or from [10, Theorem 3]. □

Corollary 1.15. *Let I be a strongly stable ideal. If the irredundant irreducible decomposition of* b-pol(I) *is of the form*

$$\text{b-pol}(I) = \bigcap_{s=1}^{r} (x_{i,\gamma_i^{\langle s\rangle}} \mid 1 \le i \le t_s\,) \subset \widetilde{S}, \tag{1.3.1}$$

then we have

$$H(H^i_{\mathfrak{m}}(S/I), \lambda^{-1}) = \frac{\sum_{j\ge1} \#\{s \in [r] \mid t_s = n-i,\ \gamma^{\langle s\rangle}_{t_s} = j\,\}\lambda^{i-j+1}}{(1-\lambda)^i}.$$

Proof. By the additivity of the statement, it suffices to compute how an irreducible component

$$P_s = (x_{i,\gamma_i^{\langle s\rangle}} \mid 1 \le i \le t_s\,)$$

of b-pol(I) contributes to the Hilbert series $H(H^i_{\mathfrak{m}}(S/I), \lambda^{-1})$. For simplicity, set $\gamma = \gamma^{<s>}_{t_s}$ and $t = t_s$. This component gives

$$\prod_{i=1}^{t} y_{\gamma_i^{<s>}} \in G(I^*).$$

By the Eliahou-Kervaire formula ([12]), the contribution of P_s to the Betti numbers of I^* is

$$\begin{cases} 0 & \text{if } j \neq t, \\ \binom{\gamma-1}{i} & \text{if } j = t, \end{cases}$$

for $\beta_{i,i+j}(I^*)$, equivalently,

$$\begin{cases} 0 & \text{if } n-i \neq t, \\ \binom{\gamma-1}{i-j} & \text{if } n-i = t, \end{cases}$$

for $\beta_{i-j,n-j}(I^*)$. Hence, by Theorem 1.13, P_s concerns $H^i_{\mathfrak{m}}(S/I)$ if and only if $i = n - t$. Moreover, if $i = n - t$, the contribution to $H(H^i_{\mathfrak{m}}(S/I), \lambda^{-1})$ is the following

$$\begin{aligned}
\sum_{j=i-\gamma+1}^{i} \frac{\binom{\gamma-1}{i-j}\lambda^j}{(1-\lambda)^j} &= \frac{\sum_{j=i-\gamma+1}^{i}(1-\lambda)^{i-j}\binom{\gamma-1}{i-j}\lambda^j}{(1-\lambda)^i} \\
&= \frac{\sum_{k=0}^{\gamma-1}(1-\lambda)^k\binom{\gamma-1}{k}\lambda^{i-k}}{(1-\lambda)^i} \qquad \text{(here } k = i-j\text{)} \\
&= \frac{\left(\sum_{k=0}^{\gamma-1}(1-\lambda)^k\binom{\gamma-1}{k}\lambda^{\gamma-1-k}\right)\lambda^{i-\gamma+1}}{(1-\lambda)^i} \\
&= \frac{((1-\lambda)+\lambda)^{\gamma-1}\lambda^{i-\gamma+1}}{(1-\lambda)^i} \\
&= \frac{\lambda^{i-\gamma+1}}{(1-\lambda)^i}.
\end{aligned}$$

So the proof is completed. □

Example 1.16. For the strongly stable ideal I in Example 1.6, b-pol(I) has two height 3 irreducible components $P_2 = (x_{1,1}, x_{2,2}, x_{3,2})$ and $P_3 = (x_{1,2}, x_{2,2}, x_{3,2})$. Clearly, $\gamma_3^{\langle 2\rangle} = \gamma_3^{\langle 3\rangle} = 2$ in the above notation. Hence we have $H(H^0_{\mathfrak{m}}(S/I), \lambda^{-1}) = 2\lambda^{-2+1} = 2\lambda^{-1}$ by Corollary 1.14.

In Section 5, we will give a procedure to construct the irreducible decomposition of b-pol(I) from that of I itself. After this, we will return to the Hilbert series of $H^i_{\mathfrak{m}}(S/I)$. See Corollary 1.29 below.

1.4 Relation to squarefree strongly stable ideals

We say an ideal $I \subset S$ is *squarefree strongly stable*, if it is a squarefree monomial ideal and the condition that $\mathsf{m} \in G(I),\ x_i \mid \mathsf{m},\ j < i$ and $x_j \not| \mathsf{m}$ imply $(x_j/x_i)\cdot \mathsf{m} \in I$ is satisfied. For our study on (squarefree) strongly stable ideals, the dimension of the ambient ring $S = K[x_1, \dots, x_n]$ is not important. So we consider the following equivalence relation. For monomial ideals $I \subset S_{(n)} := K[x_1, \dots, x_n]$ and $J \subset S_{(m)} := K[x_1, \dots, x_m]$, the relation $I \equiv J$ holds if the following condition is satisfied.

- Without loss of generality, we may assume that $n \leq m$. Then regarding $S_{(n)}$ as a subring of $S_{(m)}$ in the natural way, we have $G(I) = G(J)$.

For a monomial $\mathsf{m} \in S$ of the form (1.2.1), set

$$\mathsf{m}^\sigma := \prod_{i=1}^{e} x_{\alpha_i + i - 1} \in T,$$

where $T = K[x_1, \dots, x_N]$ is a polynomial ring with $N \gg 0$. Aramova et al. [1] showed that if $I \subset S$ is a strongly stable ideal then

$$I^\sigma := (\, \mathsf{m}^\sigma \mid \mathsf{m} \in G(I)\,) \subset T$$

is squarefree strongly stable. Conversely, any squarefree strongly stable ideal is of the form I^σ for some strongly stable ideal I.

Let $I \subset S$ be a strongly stable ideal, and $\widetilde{I} := \text{b-pol}(I) \subset \widetilde{S}$ its alternative polarization. For

$$\Theta_1 := \{\, x_{i,j} - x_{i+1,j-1} \mid 1 \leq i < n, 1 < j \leq d\,\},$$

we have an isomorphism $\widetilde{S}/(\Theta_1) \cong T = K[x_1, \dots, x_N]$ with $N = n + d - 1$ induced by $\widetilde{S} \ni x_{i,j} \longmapsto x_{i+j-1} \in T$. As shown in [46, §4], we have

(1) Through the isomorphism $\widetilde{S}/(\Theta_1) \cong T$, we have $\widetilde{S}/(\Theta_1) \otimes_{\widetilde{S}} \widetilde{S}/\widetilde{I} \cong T/I^\sigma$.

(2) Θ_1 forms a $\widetilde{S}/\widetilde{I}$-regular sequence.

Theorem 1.17. *Let I be a strongly stable ideal. If the irredundant irreducible decomposition of* b-pol(I) *is of the form* (1.3.1)*, then we have*

$$I^\sigma = \bigcap_{s=1}^{r} (\, x_{\gamma_i^{\langle s \rangle} + i - 1} \mid 1 \leq i \leq t_s\,) \subset T.$$

Proof. As above, set $\widetilde{I} := \text{b-pol}(I)$. Since both $\widetilde{S}/\widetilde{I}$ and T/I^σ are reduced, and

$$\widetilde{S}/(\, x_{i,\gamma_i^{\langle s \rangle}} \mid 1 \leq i \leq t_s\,) \otimes_{\widetilde{S}} \widetilde{S}/(\Theta_1) \cong T/(\, x_{\gamma_i^{\langle s \rangle} + i - 1} \mid 1 \leq i \leq t_s\,),$$

it suffices to show that all associated primes of T/I^σ ($\cong \widetilde{S}/\widetilde{I} \otimes_{\widetilde{S}} \widetilde{S}/(\Theta_1)$) come from those of $\widetilde{S}/\widetilde{I}$.

As shown in [46, Theorem 3.2], $\widetilde{S}/\widetilde{I}$ is sequentially Cohen-Macaulay, that is, if $\mathrm{Ext}^c_{\widetilde{S}}(\widetilde{S}/\widetilde{I}, \omega_{\widetilde{S}}) \neq 0$ then it is a Cohen-Macaulay module of codimension c. From [42, pp.349–351], we see that

$$(\text{the number of height } c \text{ associated primes of } \widetilde{I}) = \deg(\mathrm{Ext}^c_{\widetilde{S}}(\widetilde{S}/\widetilde{I}, \widetilde{S}))$$

and

$$(\text{the number of height } c \text{ associated primes of } I^\sigma) = \deg(\mathrm{Ext}^c_T(T/I^\sigma, T)).$$

By the same argument as the proof of [46, Theorem 3.4], we can show that Θ_1 forms an $\mathrm{Ext}^c_{\widetilde{S}}(\widetilde{S}/\widetilde{I}, \widetilde{S})$-regular sequence (see also [46, Proposition 4.1]). Hence

$$\widetilde{S}/(\Theta_1) \otimes_{\widetilde{S}} \mathrm{Ext}^c_{\widetilde{S}}(\widetilde{S}/\widetilde{I}, S) \cong \mathrm{Ext}^c_T(T/I^\sigma, T),$$

and we have

$$\deg(\mathrm{Ext}^c_{\widetilde{S}}(\widetilde{S}/\widetilde{I}, \widetilde{S})) = \deg(\mathrm{Ext}^c_T(T/I^\sigma, T)).$$

So we are done. □

Corollary 1.18. *If I is a strongly stable ideal, we have*

$$(I^\sigma)^\vee \equiv (I^*)^\sigma,$$

where $\equiv$ is the relation defined above.

Proof. If the irredundant irreducible decomposition of b-pol(I) is given as in (1.3.1), then both $(I^\sigma)^\vee$ and $(I^*)^\sigma$ are equal to

$$\Big(\prod_{i=1}^{t_s} x_{\gamma_i^{\langle s\rangle}+i-1} \mid 1 \leq s \leq r\Big).$$

More precisely, $(I^*)^\sigma$ should be an ideal with variables $y_1, y_2 \ldots$, but this is not essential. □

The Alexander duals of squarefree strongly stable ideals already appeared in an earlier paper [19] (of course, they knew that these are squarefree strongly stable again). However, the algebraic relation between I and I^σ is not clear, if one does not know b-pol(I).

Example 1.19. Consider the strongly stable ideal $I = (x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3)$ of Example 1.6. Then

$$I^\sigma = (\, x_1x_2,\, x_1x_3,\, x_1x_4,\, x_2x_3,\, x_2x_4\,) = (\, x_1,\, x_2\,) \cap (\, x_1,\, x_3,\, x_4\,) \cap (\, x_2,\, x_3,\, x_4\,)$$

and hence $(I^\sigma)^\vee = (x_1x_2, x_1x_3x_4, x_2x_3x_4)$.

On the other hand, since $I^* = (y_1^2, y_1y_2^2, y_2^3)$, we have $(I^*)^\sigma = (y_1y_2, y_1y_3y_4, y_2y_3y_4)$.

1.5 The irreducible components of I and b-pol(I)

In this section, for a strongly stable ideal I, we give a procedure to construct the irreducible decomposition of b-pol(I) from that of I itself. As corollaries, we give formulas on the arithmetic degree $\operatorname{adeg}(S/I)$ and the Hilbert series of $H^i_{\mathfrak{m}}(S/I)$.

Definition 1.20. For $\mathbf{a} = (a_1, \ldots, a_t) \in (\mathbb{Z}_{>0})^t$, set

$$\Psi(\mathbf{a}) := \left\{ (b_1, \ldots, b_{t-1}, c) \in (\mathbb{Z}_{>0})^t \;\middle|\; \begin{array}{l} b_i = \Big(\sum_{j=1}^{i} a_j\Big) - i + 1 \quad \text{for} \quad i < t, \\ b_{t-1} \le c \le b_{t-1} + a_t - 1 \end{array} \right\}.$$

Here, if $t = 1$, then we set $1 \le c \le a_1$.

Remark 1.21. In the above situation, we have $|\Psi(\mathbf{a})| = a_t$. Moreover, for $\mathbf{b} = (b_1, , \ldots, b_{t-1}, c) \in \Psi(\mathbf{a})$, we have $1 \le b_1 \le \cdots \le b_{t-1} \le c$.

Example 1.22. If $\mathbf{a} = (3, 2, 1, 2)$, then $\Psi(\mathbf{a}) = \{ (3, 4, 4, 4), (3, 4, 4, 5) \}$.

For $\mathbf{a} = (a_1, \ldots, a_t) \in (\mathbb{Z}_{>0})^t$ with $t \le n$, set $\mathfrak{m}^{\mathbf{a}} := (x_1^{a_1}, \ldots, x_t^{a_t}) \subset S$. If $(0) \ne I \subset S$ is a strongly stable ideal, then an irreducible component of I is of the form $\mathfrak{m}^{\mathbf{a}}$ for some $\mathbf{a} \in (\mathbb{Z}_{>0})^t$. Hence there is some

$$E \subset \mathbb{Z}_{>0} \cup (\mathbb{Z}_{>0})^2 \cup \cdots \cup (\mathbb{Z}_{>0})^n$$

such that

$$I = \bigcap_{\mathbf{a} \in E} \mathfrak{m}^{\mathbf{a}} \tag{1.5.1}$$

is the irredundant irreducible decomposition.

For $\mathbf{b} = (b_1, \ldots, b_t) \in \Psi(\mathbf{a})$, we set

$$\widetilde{\mathfrak{m}}^{\mathbf{b}} := (x_{1,b_1}, x_{2,b_2}, \ldots, x_{t,b_t}) \subset \widetilde{S}.$$

Theorem 1.23. *Let I be a strongly stable ideal whose irredundant irreducible decomposition is given by* (1.5.1). *Set $\Psi(E) := \bigcup_{\mathbf{a} \in E} \Psi(\mathbf{a})$. Then*

$$\text{b-pol}(I) = \bigcap_{\mathbf{b} \in \Psi(E)} \widetilde{\mathfrak{m}}^{\mathbf{b}} \tag{1.5.2}$$

is the irredundant irreducible decomposition.

It is easy to see that $\Psi(E) = \bigsqcup_{\mathbf{a} \in E} \Psi(\mathbf{a})$. We will implicitly use this fact in the arguments below.

To prove the theorem, we need some preparation. Let I be a strongly stable ideal whose irredundant irreducible decomposition is given by (1.5.1). We decompose E into three parts $E_0 = \{ (a_1, \ldots, a_t) \in E \mid t < n \}$, $E_1 = \{ (a_1, \ldots, a_n) \in E \mid a_n = 1 \}$ and $E_2 = \{ (a_1, \ldots, a_n) \in E \mid a_n \ge 2 \}$.

Lemma 1.24. *With the above notation, $I : x_n$ is a strongly stable ideal (not necessarily minimally) generated by*

$$\{\, \mathsf{m} \in G(I) \mid x_n \textit{ does not divide } \mathsf{m} \,\} \cup \{\, \mathsf{m}/x_n \mid \mathsf{m} \in G(I), x_n \textit{ divides } \mathsf{m} \,\}.$$

Moreover, its irredundant irreducible decomposition is given by

$$I : x_n = \Big(\bigcap_{\mathbf{a}\in E_0} \mathfrak{m}^{\mathbf{a}}\Big) \cap \Big(\bigcap_{\mathbf{a}\in E_2} \mathfrak{m}^{\mathbf{a}-\mathbf{e}_n}\Big), \tag{1.5.3}$$

where $\mathbf{e}_n$ is the n-th unit vector $(0, 0, \dots, 1) \in \mathbb{Z}^n$.

Proof. The first and second assertions are clear. To see the last assertion, note that

$$I : x_n = \Big(\bigcap_{\mathbf{a}\in E} \mathfrak{m}^{\mathbf{a}}\Big) : x_n = \bigcap_{\mathbf{a}\in E} (\mathfrak{m}^{\mathbf{a}} : x_n),$$

and

$$\mathfrak{m}^{\mathbf{a}} : x_n = \begin{cases} \mathfrak{m}^{\mathbf{a}} & \text{if } \mathbf{a} \in E_0, \\ S & \text{if } \mathbf{a} \in E_1, \\ \mathfrak{m}^{\mathbf{a}-\mathbf{e}_n} & \text{if } \mathbf{a} \in E_2. \end{cases}$$

So (1.5.3) holds. Since there is no inclusion among $\mathfrak{m}^{\mathbf{a}}$ for $\mathbf{a} \in E_0$ and $\mathfrak{m}^{\mathbf{a}-\mathbf{e}_n}$ for $\mathbf{a} \in E_2$, the decomposition (1.5.3) is irredundant. □

Set

$$\overline{I} := (\, \mathsf{m} \in G(I) \mid x_n \text{ does not divide } \mathsf{m} \,).$$

For $\mathbf{a} = (a_1, \dots, a_t) \in E$, set

$$\varphi(\mathbf{a}) = \begin{cases} \mathbf{a} & \text{if } t < n, \\ (a_1, \dots, a_{n-1}) & \text{if } t = n. \end{cases} \tag{1.5.4}$$

Lemma 1.25. *With the above notation, we have the following.*

(1) $\overline{I}$ is a strongly stable ideal, and

$$\overline{I} = \bigcap_{\mathbf{a}\in E} \mathfrak{m}^{\varphi(\mathbf{a})}$$

is a (possibly redundant) irreducible decomposition.

(2) For $\mathbf{a} \in E_1$, $\mathfrak{m}^{\varphi(\mathbf{a})}$ is an irreducible component of $\overline{I}$.

Proof. (1) Easy.

(2) For a contradiction, assume that $\mathfrak{m}^{\varphi(\mathbf{a})}$ for $\mathbf{a} \in E_1$ is not an irreducible component. Then there is some $\mathbf{a}' \in E \setminus \{\mathbf{a}\}$ such that $\mathfrak{m}^{\varphi(\mathbf{a}')} \subset \mathfrak{m}^{\varphi(\mathbf{a})}$. Since $\mathbf{a} \in E_1$, we have $\mathfrak{m}^{\mathbf{a}'} \subset \mathfrak{m}^{\mathbf{a}}$, and this is a contradiction. □

Next we will study how to recover a strongly stable ideal I from $I : x_n$ and $\overline{I}$. Let

$$I : x_n = \bigcap_{\mathbf{a}\in F} \mathfrak{m}^{\mathbf{a}} \qquad \text{and} \qquad \overline{I} = \bigcap_{\mathbf{a}\in G} \mathfrak{m}^{\mathbf{a}} \tag{1.5.5}$$

be the irredundant irreducible decompositions. Decompose F into

$$F_0 = \{\, (a_1, \ldots, a_t) \in F \mid t < n \,\} \qquad \text{and} \qquad F_1 := (F \setminus F_0) \subset (\mathbb{Z}_{>0})^n,$$

and set $\varphi(F) := \{\, \varphi(\mathbf{a}) \mid \mathbf{a} \in F \,\}$, where φ is the function defined in (1.5.4). By Lemmas 1.24 and 1.25, if $\mathbf{a} \in G \setminus \varphi(F)$, then $\mathbf{a}$ is of the from $(a_1, \ldots, a_{n-1})$ and $\mathfrak{m}^{\mathbf{a}\oplus\mathbf{e}_n}$ is an irreducible component of I, where we set $\mathbf{a} \oplus \mathbf{e}_n := (a_1, \ldots, a_{n-1}, 1)$.

Lemma 1.26. *With the above notation, we have the irredundant irreducible decomposition*

$$I = \Bigl(\bigcap_{\mathbf{a}\in F_0} \mathfrak{m}^{\mathbf{a}}\Bigr) \cap \Bigl(\bigcap_{\mathbf{a}\in F_1} \mathfrak{m}^{\mathbf{a}+\mathbf{e}_n}\Bigr) \cap \Bigl(\bigcap_{\mathbf{a}\in G\setminus\varphi(F)} \mathfrak{m}^{\mathbf{a}\oplus\mathbf{e}_n}\Bigr).$$

Proof. Easily follows from Lemmas 1.24 and 1.25. □

The proof of Theorem 1.23. We prove the theorem by double induction on n and

$$d(I) := \sum_{\mathsf{m}\in G(I)} \deg(\mathsf{m}).$$

Let I be a strongly stable ideal. We may assume that x_n divides some $\mathsf{m} \in G(I)$. In fact, if this is not the case, we can replace I by $I \cap K[x_1, \ldots, x_{n-1}]$, and the induction works. Under this assumption, both $d(I : x_n)$ and $d(\overline{I})$ are smaller than $d(I)$. By the induction hypothesis, if $I : x_n$ and $\overline{I}$ have irreducible decompositions of the form (1.5.5), we have irreducible decompositions

$$\text{b-pol}(I : x_n) = \bigcap_{\mathbf{b}\in\Psi(F)} \widetilde{\mathfrak{m}}^{\mathbf{b}} \qquad \text{and} \qquad \text{b-pol}(\overline{I}) = \bigcap_{\mathbf{b}\in\Psi(G)} \widetilde{\mathfrak{m}}^{\mathbf{b}}.$$

In the sequel, for $\mathbf{a} = (a_1, \ldots, a_n) \in (\mathbb{Z}_{>0})^n$, consider the vector $(b_1, \ldots, b_n)$ with $b_i = \Bigl(\sum_{j=1}^{i} a_j\Bigr) - i + 1$ for $i = 1, \ldots, n$. In this case,

$$\Psi(\mathbf{a}) = \{\, (b_1, \ldots, b_{n-1}, c) \mid b_{n-1} \le c \le b_n \,\} \tag{1.5.6}$$

and

$$\Psi(\mathbf{a} + \mathbf{e}_n) = \Psi(\mathbf{a}) \cup \{\, (b_1, \ldots, b_{n-1}, b_n + 1) \,\}.$$

Set $\widetilde{\mathbf{a}} := (b_1, \ldots, b_{n-1}, b_n + 1)$.

For $\mathbf{a} = (a_1, \ldots, a_{n-1}) \in (\mathbb{Z}_{>0})^{n-1}$, we have

$$\Psi(\mathbf{a} \oplus \mathbf{e}_n) = \{\, (b_1, \ldots, b_{n-1}, b_{n-1}) \,\},$$

where $b_i = \left(\sum_{j=1}^{i} a_j\right) - i + 1$ for $i = 1, \dots, n-1$. Set $\widehat{\mathbf{a}} := (b_1, \dots, b_{n-1}, b_{n-1})$.

By Lemma 1.26, it is enough to show

$$\text{b-pol}(I) = \Big(\bigcap_{\mathbf{b}\in\Psi(F_0)} \widetilde{\mathfrak{m}}^{\mathbf{b}}\Big) \cap \Big(\bigcap_{\substack{\mathbf{a}\in F_1\\ \mathbf{b}\in\Psi(\mathbf{a}+\mathbf{e}_n)}} \widetilde{\mathfrak{m}}^{\mathbf{b}}\Big) \cap \Big(\bigcap_{\mathbf{a}\in G\setminus\varphi(F)} \widetilde{\mathfrak{m}}^{\widehat{\mathbf{a}}}\Big).$$

Since

$$\begin{aligned}\text{(the right hand side)} &= \Big(\bigcap_{\mathbf{b}\in\Psi(F)} \widetilde{\mathfrak{m}}^{\mathbf{b}}\Big) \cap \Big(\bigcap_{\mathbf{a}\in F_1} \widetilde{\mathfrak{m}}^{\widetilde{\mathbf{a}}}\Big) \cap \Big(\bigcap_{\mathbf{a}\in G\setminus\varphi(F)} \widetilde{\mathfrak{m}}^{\widehat{\mathbf{a}}}\Big)\\ &= \text{b-pol}(I : x_n) \cap \Big(\bigcap_{\mathbf{a}\in F_1} \widetilde{\mathfrak{m}}^{\widetilde{\mathbf{a}}}\Big) \cap \Big(\bigcap_{\mathbf{a}\in G\setminus\varphi(F)} \widetilde{\mathfrak{m}}^{\widehat{\mathbf{a}}}\Big),\end{aligned}$$

it suffices to show that

$$\text{b-pol}(I) = \text{b-pol}(I : x_n) \cap \Big(\bigcap_{\mathbf{a}\in F_1} \widetilde{\mathfrak{m}}^{\widetilde{\mathbf{a}}}\Big) \cap \Big(\bigcap_{\mathbf{a}\in G\setminus\varphi(F)} \widetilde{\mathfrak{m}}^{\widehat{\mathbf{a}}}\Big). \tag{1.5.7}$$

First, we will prove the inclusion $\subset$ of (1.5.7). Since $\text{b-pol}(I) \subset \text{b-pol}(I : x_n)$, it suffices to show that

$$\text{b-pol}(\mathsf{m}) \in \Big(\bigcap_{\mathbf{a}\in F_1} \widetilde{\mathfrak{m}}^{\widetilde{\mathbf{a}}}\Big) \cap \Big(\bigcap_{\mathbf{a}\in G\setminus\varphi(F)} \widetilde{\mathfrak{m}}^{\widehat{\mathbf{a}}}\Big) \tag{1.5.8}$$

for all $\mathsf{m} \in G(I)$.

Take an arbitrary $\mathbf{a} \in F_1$, and set $\widetilde{\mathbf{a}} = (b_1, \dots, b_{n-1}, b_n + 1)$ as above. Since $\text{b-pol}(\mathsf{m}) \in \text{b-pol}(I : x_n)$, we have $\text{b-pol}(\mathsf{m}) \in \widetilde{\mathfrak{m}}^{\mathbf{b}}$ for all $\mathbf{b} \in \Psi(\mathbf{a})$. Recall the description (1.5.6) of $\Psi(\mathbf{a})$. If x_n does not divide m, there exists some $1 \le i \le n-1$ such that $x_{i,b_i} \mid \text{b-pol}(\mathsf{m})$. Hence $\text{b-pol}(\mathsf{m}) \in \widetilde{\mathfrak{m}}^{\widetilde{\mathbf{a}}}$. If x_n divides m, then it can be possible that x_{i,b_i} does not divide $\text{b-pol}(\mathsf{m})$ for any $1 \le i \le n-1$. Note that $\mathsf{m}/x_n \in I : x_n$ and $\text{b-pol}(\mathsf{m}/x_n) \in \widetilde{\mathfrak{m}}^{\mathbf{b}}$ for all $\mathbf{b} \in \Psi(\mathbf{a})$. Hence, we have $x_{n,b_n} \mid \text{b-pol}(\mathsf{m}/x_n)$ in this case. It implies that $x_{n,b_n+1} \mid \text{b-pol}(\mathsf{m})$, and hence $\text{b-pol}(\mathsf{m}) \in \widetilde{\mathfrak{m}}^{\widetilde{\mathbf{a}}}$.

Next, take an arbitrary $\mathbf{a} \in G \setminus \varphi(F)$, and set $\widehat{\mathbf{a}} = (b_1, \dots, b_{n-1}, b_{n-1})$ as above. Set $e := \deg_{x_n}(\mathsf{m})$, where $\deg_{x_i}(-)$ stands for the degree with respect to the variable x_i. Then $\mathsf{n} := \mathsf{m} \cdot (x_{n-1}/x_n)^e \in \overline{I}$, and hence $\text{b-pol}(\mathsf{n}) \in \text{b-pol}(\overline{I}) \subset \widetilde{\mathfrak{m}}^{\mathbf{b}}$ for $\mathbf{b} := (b_1, \dots, b_{n-1}) \in \Psi(\mathbf{a})$. It follows that $\text{b-pol}(\mathsf{m}) \in \widetilde{\mathfrak{m}}^{\widehat{\mathbf{a}}}$. In fact, if $x_{i,b_i} \mid \text{b-pol}(\mathsf{n})$ for some $i < n-1$, then $x_{i,b_i} \mid \text{b-pol}(\mathsf{m})$. If $x_{n-1,b_{n-1}} \mid \text{b-pol}(\mathsf{n})$, then either $x_{n-1,b_{n-1}}$ or $x_{n,b_{n-1}}$ divides $\text{b-pol}(\mathsf{m})$. Now we have shown (1.5.8).

Next, we will prove the inclusion $\supset$ of (1.5.7). To do this, it suffices to show that

$$\text{b-pol}(\mathsf{m}) \notin \Big(\bigcap_{\mathbf{a}\in F_1} \widetilde{\mathfrak{m}}^{\widetilde{\mathbf{a}}}\Big) \cap \Big(\bigcap_{\mathbf{a}\in G\setminus\varphi(F)} \widetilde{\mathfrak{m}}^{\widehat{\mathbf{a}}}\Big)$$

for $\mathsf{m} \in G(I : x_n) \setminus I$. Since $\mathsf{m} \notin I$, there is some $\mathbf{a} \in F_1$ with $\mathsf{m} \notin \mathfrak{m}^{\mathbf{a}+\mathbf{e}_n}$, or some $\mathbf{a} \in G \setminus \varphi(F)$ with $\mathsf{m} \notin \mathfrak{m}^{\mathbf{a}\oplus\mathbf{e}_n}$. If $\mathsf{m} \notin \mathfrak{m}^{\mathbf{a}+\mathbf{e}_n}$, then $x_{i,j} |$ b-pol(m) implies $j \leq \sum_{k=1}^{i} \deg_{x_k}(\mathsf{m}) \leq \sum_{k=1}^{i}(a_k - 1) = \left(\sum_{k=1}^{i} a_k\right) - i = b_i - 1$ for $i \leq n-1$, and $j \leq b_n$ for $i = n$. It means that b-pol$(\mathsf{m}) \notin \widetilde{\mathfrak{m}}^{\widetilde{\mathbf{a}}}$. Similarly, $\mathsf{m} \notin \mathfrak{m}^{\mathbf{a}\oplus\mathbf{e}_n}$ implies b-pol$(\mathsf{m}) \notin \widetilde{\mathfrak{m}}^{\widehat{\mathbf{a}}}$. Now we have shown that (1.5.2) holds

It remains to show that there is no inclusion among ideals $\widetilde{\mathfrak{m}}^{\mathbf{b}}$ for $\mathbf{b} \in \Psi(E)$, but this is easy. □

Example 1.27. Consider a strongly stable ideal $I = (x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3^2)$, which is a slight modification of the one in Example 1.6. From the irreducible decomposition $I = (x_1, x_2) \cap (x_1^2, x_2, x_3) \cap (x_1, x_2^2, x_3^2)$, let us construct the decomposition of b-pol(I). Theorem 1.23 states that (x_1, x_2) yields $(x_{1,1}, x_{2,1})$, (x_1^2, x_2, x_3) yields $(x_{1,2}, x_{2,2}, x_{3,2})$, but (x_1, x_2^2, x_3^2) yields $(x_{1,1}, x_{2,2}, x_{3,2})$ and $(x_{1,1}, x_{2,2}, x_{3,3})$. Now we get the irreducible decomposition

$$\text{b-pol}(I) = (x_{1,1}, x_{2,1}) \cap (x_{1,2}, x_{2,2}, x_{3,2}) \cap (x_{1,1}, x_{2,2}, x_{3,2}) \cap (x_{1,1}, x_{2,2}, x_{3,3}).$$

The next result concerns the *arithmetic degree* $\operatorname{adeg}(S/I)$ of S/I. For the basics of this notion, consult [42, §1]. However, following [24], we use the refinement $\operatorname{adeg}_i(S/I)$ of $\operatorname{adeg}(S/I)$ for $0 \leq i \leq \dim S/I$, which measures the contribution of the dimension i components of I. Hence $\operatorname{adeg}(S/I) = \sum_{i \geq 0} \operatorname{adeg}_i(S/I)$.

Corollary 1.28. *Let I be a strongly stable ideal with the irreducible decomposition (1.5.1). For $(a_1, \dots, a_t) \in E$ (recall that $a_t > 0$), set $t(\mathbf{a}) := t$ and $e(\mathbf{a}) := a_t$. Then we have*

$$\operatorname{adeg}_i(S/I) = \sum_{\substack{\mathbf{a} \in E \\ t(\mathbf{a}) = n-i}} e(\mathbf{a})$$

for each i. Hence,

$$\operatorname{adeg}(S/I) = \sum_{\mathbf{a} \in E} e(\mathbf{a})$$

and

$$\deg(S/I) = \sum_{\substack{\mathbf{a} \in E \\ t(\mathbf{a}) = \operatorname{ht}(I)}} e(\mathbf{a}).$$

Proof. Set $\widetilde{I} :=$ b-pol(I). By an argument similar to the proof of Theorem 1.17, we have

$$\begin{aligned} \operatorname{adeg}_{n-c}(S/I) &= \deg(\operatorname{Ext}_S^c(S/I, S)) \\ &= \deg(\operatorname{Ext}_{\widetilde{S}}^c(\widetilde{S}/\widetilde{I}, \widetilde{S})) \\ &= \text{the number of codimension } c \text{ associated primes of } \widetilde{I}. \end{aligned}$$

Take $\mathbf{a} \in E$ with $t(\mathbf{a}) = c$. Then $\mathbf{a}$ yields $e(\mathbf{a})$ irreducible components of $\widetilde{I}$ of codimension c. Any codimension c component of $\widetilde{I}$ is given in this way, and they are all distinct. So we are done. □

Corollary 1.29. *Let I be a strongly stable ideal with the irreducible decomposition (1.5.1). For $\mathbf{a} = (a_1, \dots, a_t) \in E$ (recall that $a_t > 0$), set $t(\mathbf{a}) := t$, $w(\mathbf{a}) := n - \sum_{i=1}^{t} a_i$, and $e(\mathbf{a}) := a_t$. Then the Hilbert series of the local cohomology module $H^i_{\mathfrak{m}}(S/I)$ is given by*

$$H(H^i_{\mathfrak{m}}(S/I), \lambda^{-1}) = \left(\sum_{\substack{\mathbf{a}\in E,\\ t(\mathbf{a})=n-i}} (\lambda^{w(\mathbf{a})} + \lambda^{w(\mathbf{a})+1} + \cdots + \lambda^{w(\mathbf{a})+e(\mathbf{a})-1}) \right) /(1-\lambda)^i.$$

Proof. For $\mathbf{a} = (a_1, \dots, a_t) \in E$ with $|\mathbf{a}| := \sum_{i=1}^{t} a_i$, $\Psi(\mathbf{a})$ is the set

$$\{\, (b_1, b_2, \dots, b_{t-1}, c) \mid |\mathbf{a}| - t - a_t + 2 \le c \le |\mathbf{a}| - t + 1 \,\}$$

with a_t elements. Here $b_i = (\sum_{j=1}^{i} a_j) - i + 1$ for each i, while this value is not important now. By Theorem 1.23, for $\mathbf{b} \in \Psi(\mathbf{a})$, $\widetilde{\mathfrak{m}}^{\mathbf{b}}$ is an irreducible component of b-pol(I), and any irreducible component is given in this way.

As we have shown in the proof of Corollary 1.15, for $\mathbf{b} = (b_1, b_2, \dots, b_{t-1}, c) \in \Psi(\mathbf{a})$, the component $\widetilde{\mathfrak{m}}^{\mathbf{b}}$ contributes to the Hilbert series of $H^i_{\mathfrak{m}}(S/I)$ if and only if $i = n - t$. If $i = n - t$, the contribution is $\lambda^{i-c+1}/(1-\lambda)^i$. Here, the numerator equals $\lambda^{n-t-c+1}$, and the exponent $n - t - c + 1$ moves in the range

$$\begin{array}{rcccl} n - t - (|\mathbf{a}| - t + 1) + 1 & \le & n - t - c + 1 & \le & n - t - (|\mathbf{a}| - t - a_t + 2) + 1 \\ w(\mathbf{a}) & \le & n - t - c + 1 & \le & w(\mathbf{a}) + e(\mathbf{a}) - 1. \end{array}$$

Hence the contribution of $\mathbf{a} \in E$ to $H(H^i_{\mathfrak{m}}(S/I), \lambda^{-1})$ is

$$\begin{cases} 0 & \text{if } i \ne n - t(\mathbf{a}), \\ \dfrac{\lambda^{w(\mathbf{a})} + \lambda^{w(\mathbf{a})+1} + \cdots + \lambda^{w(\mathbf{a})+e(\mathbf{a})-1}}{(1-\lambda)^i} & \text{if } i = n - t(\mathbf{a}). \end{cases}$$

So we are done. □

Example 1.30. This is a continuation of Example 1.27. For the strongly stable ideal $I = (x_1, x_2) \cap (x_1^2, x_2, x_3) \cap (x_1, x_2^2, x_3^2) \subset K[x_1, x_2, x_3]$, let $\mathbf{a}$ and $\mathbf{b}$ denote the exponent vectors $(2, 1, 1)$ and $(1, 2, 2)$ of the height 3 components, respectively. With the notation of Corollaries 1.28 and 1.29, we have $w(\mathbf{a}) = -1, e(\mathbf{a}) = 1, w(\mathbf{b}) = -2$ and $e(\mathbf{b}) = 2$. Hence we have $\operatorname{adeg}_0(S/I) = e(\mathbf{a}) + e(\mathbf{b}) = 3$. Similarly, $H(H^0_{\mathfrak{m}}(S/I), \lambda^{-1}) = \lambda^{-2} + 2\lambda^{-1}$, where the contributions of the components (x_1^2, x_2, x_3) and (x_1, x_2^2, x_3^2) are λ^{-1} and $\lambda^{-2} + \lambda^{-1}$, respectively.

1.6 Remarks on irreducible components of strongly stable ideals

In this section, we collect a few remarks on the irreducible decompositions of strongly stable ideals. These results are only loosely related to the alternative polarization and Alexander duality, but they are useful in actual computation.

For $\mathbf{a} = (a_1, \dots, a_t) \in \mathbb{N}^t$ with $a_t > 0$ and $\mathbf{b} = (b_1, \dots, b_t) \in (\mathbb{Z}_{>0})^t$, we set

$$\widehat{\mathbf{a}} := (a_1 + 1, a_2 + 1, \dots, a_{t-1} + 1, a_t) \in (\mathbb{Z}_{>0})^t,$$

and

$$\widetilde{\mathbf{b}} := (b_1 - 1, b_2 - 1, \dots, b_{t-1} - 1, b_t) \in \mathbb{N}^t.$$

Note that this notation is different from that in the previous section.

For a monomial $x^{\mathbf{a}} \in S$ with $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$, recall that $\nu(x^{\mathbf{a}}) = \max\{\, i \mid a_i > 0 \,\}$. For a monomial ideal I, set $\nu(I) := \max\{\, \nu(x^{\mathbf{a}}) \mid x^{\mathbf{a}} \in G(I) \,\}$. If I is strongly stable, then it is well-known that

$$\begin{aligned} \nu(I) &= \max\{\, \mathrm{ht}(\mathfrak{m}^{\mathbf{a}}) \mid \mathfrak{m}^{\mathbf{a}} \text{ is an irreducible component of } I \,\} \\ &= \max\{\, l \mid \mathfrak{m}^{\mathbf{a}} \text{ is an irreducible component of } I \text{ for some } \mathbf{a} \in (\mathbb{Z}_{>0})^l \,\}. \end{aligned}$$

Proposition 1.31. *Let $I \subset S$ be a strongly stable ideal with $l := \nu(I)$. For $x^{\mathbf{a}} \in G(I)$ with $\nu(x^{\mathbf{a}}) = l$, $\mathfrak{m}^{\widehat{\mathbf{a}}}$ is an irreducible component of I. Conversely, any irreducible component of height l arises in this way. More precisely, if $\mathfrak{m}^{\mathbf{b}}$ is an irreducible component of I with $\mathrm{ht}(\mathfrak{m}^{\mathbf{b}}) = l$ (in other words, $(\mathbf{b} \in \mathbb{Z}_{>0})^l$), then $x^{\widetilde{\mathbf{b}}} \in G(I)$.*

Proof. Take $x^{\mathbf{a}} \in G(I)$ with $\nu(x^{\mathbf{a}}) = l$. Since $x^{\mathbf{a}}/x_l \notin I$, there is an irreducible component $\mathfrak{m}^{\mathbf{b}}$ with $x^{\mathbf{a}}/x_l \notin \mathfrak{m}^{\mathbf{b}}$. Clearly, $\mathbf{b} \in (\mathbb{Z}_{>0})^l$ now. We will show that $\mathbf{b} = \widehat{\mathbf{a}}$. Since $x^{\mathbf{a}} \in I \subset \mathfrak{m}^{\mathbf{b}}$ and $x^{\mathbf{a}}/x_l \notin \mathfrak{m}^{\mathbf{b}}$, we have $b_i > a_i$ for all $i \leq l-1$, and $a_l = b_l$. If $b_i > a_i + 1$ for some $i \leq l-1$, then $(x_i/x_l) \cdot x^{\mathbf{a}} \notin \mathfrak{m}^{\mathbf{b}}$, and this is a contradiction. Therefore, $b_i = a_i + 1$ holds for all $i \leq l-1$, and we have $\mathbf{b} = \widehat{\mathbf{a}}$.

Conversely, we assume that $\mathfrak{m}^{\mathbf{b}}$ is an irreducible component of I with $\mathrm{ht}(\mathfrak{m}^{\mathbf{b}}) = l$. First, we will show that $\mathsf{m} := x^{\widetilde{\mathbf{b}}} \in I$. For a contradiction, assume that $\mathsf{m} \notin I$. Then there is an irreducible component $\mathfrak{m}^{\mathbf{c}}$ of I with $\mathsf{m} \notin \mathfrak{m}^{\mathbf{c}}$. Then we have $c_i \geq b_i$ for all $i < l$, and $c_l > b_l$ (if $\mathbf{c} \in (\mathbb{Z}_{>0})^l$). It follows that $\mathfrak{m}^{\mathbf{c}} \subsetneq \mathfrak{m}^{\mathbf{b}}$. This is a contradiction.

Next we will show that $\mathsf{m} \in G(I)$. Since we have shown that $\mathsf{m} \in I$, there is $x^{\mathbf{c}} \in G(I)$ which divides m. Clearly, $c_i \leq b_i - 1$ for all $i \leq l-1$, $c_l \leq b_l$, and $c_i = 0$ for all $i > l$. Since $x^{\mathbf{c}} \in \mathfrak{m}^{\mathbf{b}}$, we have $\nu(x^{\mathbf{c}}) = l$ and $c_l = b_l > 0$. Moreover, since $(x_i/x_l) \cdot x^{\mathbf{c}} \in \mathfrak{m}^{\mathbf{b}}$ for all $i < l$, we have $c_i = b_i - 1$ for all $i \leq l-1$. Hence we have $\widetilde{\mathbf{b}} = \mathbf{c}$, and $\mathsf{m} = x^{\widetilde{\mathbf{b}}} = x^{\mathbf{c}} \in G(I)$. □

Corollary 1.32. *Let $I \subset S$ be a strongly stable ideal, and set $l := \nu(I)$. Then we have*

$$\mathrm{adeg}_{n-l}(S/I) = \sum_{\substack{x^{\mathbf{a}} \in G(I) \\ \nu(x^{\mathbf{a}}) = l}} a_l$$

Proof. By Corollary 1.28 and Proposition 1.31, the assertion follows. □

Remark 1.33. Let $I \subset S$ be a strongly stable ideal with $l := \nu(I)$. If S/I is Cohen–Macaulay (equivalently, $l = \mathrm{ht}(I)$), then Proposition 1.31 directly gives the irreducible decomposition of I. If $l > \mathrm{ht}(I)$, then we consider

$$I : x_l^\infty := \{\, f \in S \mid x_l^q f \in I \text{ for } q \gg 0 \,\}.$$

This is a strongly stable ideal again, and the intersection of the irreducible components of I whose heights are less than l. Moreover, $G(I : x_l^\infty)$ can be easily computed from $G(I)$. Therefore, combining this operation with Proposition 1.31, we can compute the irreducible decomposition of I.

Example 1.34. For the strongly stable ideal $I = (x_1^3, x_1^2x_2, x_1x_2^2, x_1x_2x_3^2, x_1^2x_3^2)$, the generators $x_1x_2x_3^2$ and $x_1^2x_3^2$ yield (x_1^2, x_2^2, x_3^2) and (x_1^3, x_2, x_3^2), respectively. Next, consider the strongly stable ideal $I' := I : x_3^\infty = (x_1^2, x_1x_2)$, and it has an irreducible component (x_1^2, x_2) given by x_1x_2, which is also an irreducible component of I itself. Finally, $I' : x_2^\infty = (x_1)$ itself is an irreducible component of I. Hence the irreducible decomposition of I is

$$I = (x_1) \cap (x_1^2, x_2) \cap (x_1^2, x_2^2, x_3^2) \cap (x_1^3, x_2, x_3^2).$$

Theorem 1.35. *Let $I \subset S$ be a monomial ideal with the irredundant irreducible decomposition* (1.5.2)*. (Note that the irreducible decomposition of a strongly stable ideal is always in this form.) Then the following are equivalent.*

(1) I is strongly stable.

(2) If $\mathbf{a} = (a_1, \dots, a_t) \in E \cap (\mathbb{Z}_{>0})^t$, $a_i > 1$ and $i < j \le t$, then there is some $\mathbf{b} \in E$ such that $\mathfrak{m}^{\mathbf{a}-\mathbf{e}_i+\mathbf{e}_j} \supset \mathfrak{m}^{\mathbf{b}}$, where $\mathbf{e}_i \in \mathbb{N}^t$ is the i-th unit vector.

Proof. (1) ⇒ (2): For a contradiction, assume that a strongly stable ideal I does not satisfy (2). Then, for each $\mathbf{b} \in E$, we have $\mathfrak{m}^{\mathbf{a}-\mathbf{e}_i+\mathbf{e}_j} \not\supset \mathfrak{m}^{\mathbf{b}}$, and we can take a monomial $\mathsf{m}_{\mathbf{b}} \in G(\mathfrak{m}^{\mathbf{b}})$ with $\mathsf{m}_{\mathbf{b}} \notin \mathfrak{m}^{\mathbf{a}-\mathbf{e}_i+\mathbf{e}_j}$ (of course, $\mathsf{m}_{\mathbf{b}} = x_k^{b_k}$ for some $k \in [n]$). Let m be the least common multiple of $\{\mathsf{m}_{\mathbf{b}} \mid \mathbf{b} \in E\}$. Since $\mathsf{m} \in \bigcap_{\mathbf{b}\in E} \mathfrak{m}^{\mathbf{b}} = I \subset \mathfrak{m}^{\mathbf{a}}$ and $\mathsf{m} \notin \mathfrak{m}^{\mathbf{a}-\mathbf{e}_i+\mathbf{e}_j}$, the degree $\deg_{x_k}(\mathsf{m})$ with respect to x_k is

$$\begin{cases} < a_k & (\text{if } k \ne i, j), \\ = a_j & (\text{if } k = j), \\ < a_i - 1 & (\text{if } k = i). \end{cases}$$

So we have $(x_i/x_j)\cdot\mathsf{m} \notin \mathfrak{m}^{\mathbf{a}}$, and hence $(x_i/x_j)\cdot\mathsf{m} \notin I$. It contradicts the assumption that I is strongly stable and $\mathsf{m} \in I$.

(2) ⇒ (1): For a contradiction, we assume that I satisfies (2) but it is not strongly stable. Then there are some $\mathsf{m} \in G(I)$ and some $i \ge 2$ such that x_i divides m and $(x_{i-1}/x_i)\cdot\mathsf{m} \notin \mathfrak{m}^{\mathbf{a}}$ for some $\mathbf{a} = (a_1, \dots, a_t) \in E$. Then it is easy to see that $a_{i-1} > 1$ and $t \ge i$. By (2), we have $\mathfrak{m}^{\mathbf{a}-\mathbf{e}_{i-1}+\mathbf{e}_i} \supset \mathfrak{m}^{\mathbf{b}}$ for some $\mathbf{b} \in E$. Since $(x_{i-1}/x_i)\cdot\mathsf{m} \notin \mathfrak{m}^{\mathbf{a}}$, we have $\mathsf{m} \notin \mathfrak{m}^{\mathbf{a}-\mathbf{e}_{i-1}+\mathbf{e}_i}$. It contradicts that $\mathsf{m} \in I \subset \mathfrak{m}^{\mathbf{b}}$. □

Example 1.36. For a strongly stable ideal $I = (x_1^2, x_1x_2, x_2^3, x_1x_3, x_2^2x_3) \subset K[x_1, x_2, x_3]$, we have the irreducible decomposition

$$I = (x_1, x_2^2) \cap (x_1^2, x_2, x_3) \cap (x_1, x_2^3, x_3).$$

We consider the irreducible component $\mathfrak{m}^{\mathbf{a}} = (x_1, x_2^3, x_3)$ with $\mathbf{a} = (1, 3, 1)$. Clearly,

$$\mathfrak{m}^{\mathbf{a}-\mathbf{e}_2+\mathbf{e}_3} = \mathfrak{m}^{(1,2,2)} = (x_1, x_2^2, x_3^2) \supset (x_1, x_2^2),$$

where (x_1, x_2^2) is an irreducible component.

Next consider the ideal $J = (x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1x_3)$ with the irreducible decomposition

$$J = (x_1, x_2^3) \cap (x_1^2, x_2^2, x_3) \cap (x_1^3, x_2, x_3).$$

For the irreducible component $\mathfrak{m}^{\mathbf{a}}$ with $\mathbf{a} = (2, 2, 1)$, we have

$$\mathfrak{m}^{\mathbf{a}-\mathbf{e}_2+\mathbf{e}_3} = \mathfrak{m}^{(2,1,2)} = (x_1^2, x_2, x_3^2) \not\supseteq (x_1, x_2^3), (x_1^3, x_2, x_3),$$

and J is not strongly stable. Of course, we can check this directly. In fact, we have $\mathsf{m} := x_1x_3 \in J$, but $(x_2/x_3) \cdot \mathsf{m} = x_1x_2 \notin J$.

Chapter 2

Edge ideals

2.1 Introduction

This chapter is based on the author's paper [35] with S.A.Seyed Fakhari–N.Terai–S.Yassemi. In this chapater, a graph means a simple graph without loops, multiple edges, and isolated vertices. Let G be a graph with vertex set $V(G) = \{x_1, \dots, x_n\}$ and with edge set $E(G)$. Suppose $w : E(G) \longrightarrow \mathbb{Z}_{>0}$ is an edge weight on G. We write G_w for the pair (G, w) and call it an *edge-weighted graph.* Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field K.

For a graph G, $I(G)$ is called the (ordinary) *edge ideal* of G, which is defined as

$$I(G) = \big(x_i x_j \,|\, x_i x_j \in E(G)\big).$$

The study of edge ideals was started by Villarreal in [43]. The *(edge-weighted) edge ideal* of an edge-weighted graph G_w was introduced in [28] and it is defined as

$$I(G_w) = \big((x_i x_j)^{w(x_i x_j)} \,|\, x_i x_j \in E(G)\big),$$

(by abusing the notation, we identify the edges of G with quadratic squarefree monomials of S). For example, we consider the edge weight w such that $w(e) = 1$, for any edge $e \in E(G)$. Then $I(G_w)$ is an (ordinary) edge ideal. Paulsen and Sather-Wagstaff [28] studied the primary decomposition of these ideals. They also investigated unmixedness and Cohen-Macaulayness of these ideals, in the case that G is a cycle, a tree or a complete graph. The aim of this paper is to continue this study. In Section 2.3, we characterize unmixed and Cohen-Macaulay properties of edge-weighted edge ideals of very well-covered graphs (see Section 2.2 for the definition of very well-covered graphs). Our results can be seen as generalizations of the results concerning the Cohen-Macaulay property of usual edge ideals of very well-covered graphs (see e.g., [5, 7, 6, 18]). For other aspects of ring-theoretic study of very well-covered graphs, see e.g., [2, 20, 21, 34].

Another kind of generalization of edge ideals is considered in [17, 29, 30]. Indeed, Pitones, Reyes and Toledo [29] introduced the *vertex-weighted edge ideal* of an oriented graph as follows. Let $\mathcal{D} = (V(\mathcal{D}), E(\mathcal{D}))$ be an oriented graph with $V(\mathcal{D}) = \{x_1, \ldots, x_n\}$, and let $w : V(\mathcal{D}) \longrightarrow \mathbb{Z}_{>0}$ be a *vertex weight* on $\mathcal{D}$. Set $w_j := w(x_j)$. The *vertex-weighted edge ideal* of $\mathcal{D}$ is defined as

$$I(\mathcal{D}) = \left(x_i x_j^{\omega_j} \mid x_i x_j \in E(\mathcal{D})\right).$$

Pitones, Reyes and Toledo proposed the following conjecture.

Conjecture 2.1. [29, Conjecture 53] Let $\mathcal{D}$ be a vertex-weighted oriented graph and let G be its underlying graph. If $I(\mathcal{D})$ is unmixed and $S/I(G)$ is Cohen-Macaulay, then $S/I(\mathcal{D})$ is Cohen-Macaulay.

In Section 2.4, we provide counterexamples for this conjecture.

We close this introduction by mentioning that unmixed and Cohen-Macaulay properties of vertex-weighted edge ideals of vertex-weighted oriented very well-covered graphs are studied by Pitones, Reyes and Villarreal [30].

2.2 Preliminaries

In this section, we provide the definitions and basic facts which will be used in the next sections. We refer to [9] and [42] for detailed information.

Let G be a graph with vertex set $V(G) = \{x_1 \ldots, x_n\}$ and with edge set $E(G)$. For every integer $1 \leq i \leq n$, the *degree* of x_i, denoted by $\deg_G x_i$, is the number of edges of G which are incident to x_i. For $F \subseteq E(G)$ we denote $(V(G), E(G) \setminus F)$ by $G - F$. For a family F of 2-element subsets of $V(G)$ the graph $(V(G), E(G) \cup F)$ is denoted by $G + F$. A subset $C \subseteq V(G)$ is a *vertex cover* of G if every edge of G is incident with at least one vertex in C. A vertex cover C of G is called *minimal* if there is no proper subset of C which is a vertex cover of G. A subset A of $V(G)$ is called an *independent set* of G if no two vertices of A are adjacent. An independent set A of G is *maximal* if there exists no independent set which properly includes A. Observe that C is a minimal vertex cover of G if and only if $V(G) \setminus C$ is a maximal independent set of G. A subset $M \subseteq E(G)$ is a *matching* if $e \cap e' = \emptyset$, for every pair of edges $e, e' \in M$. If every vertex of G is incident to an edge in M, then M is a *perfect matching* of G. A graph G without isolated vertices is said to be *very well-covered* if $|V(G)|$ is an even integer and every maximal independent subset of G has cardinality $|V(G)|/2$.

A graph G is called *Cohen-Macaulay* if $S/I(G)$ is a Cohen-Macaulay ring. An ideal $I \subset S$ is *unmixed* if the associated primes of S/I have the same height. It is well known that I is unmixed if S/I is a Cohen-Macaulay ring. A graph G is called *unmixed* if the minimal vertex covers of G have the same size. It can be easy seen that G is an unmixed graph if and only if $I(G)$ is an unmixed ideal. Also, note that $\operatorname{ht} I(G)$ is equal to the cardinality of the minimum vertex covers of G.

We denote the set of minimal monomial generators of a monomial ideal I by $\operatorname{Gens}(I)$.

We introduce polarization according to [37]. Let I be a monomial ideal of $S = \mathbb{K}[x_1, \dots, x_n]$ with minimal generators $u_1, \dots, u_m$, where $u_j = \prod_{i=1}^{n} x_i^{a_{i,j}}$, $1 \le j \le m$. For every i with $1 \le i \le n$, let $a_i = \max\{a_{i,j} \mid 1 \le j \le m\}$, and suppose that

$$T = \mathbb{K}[x_{11}, x_{12}, \dots, x_{1a_1}, x_{21}, x_{22}, \dots, x_{2a_2}, \dots, x_{n1}, x_{n2}, \dots, x_{na_n}]$$

is a polynomial ring over the field $\mathbb{K}$. Let I^{pol} be the squarefree monomial ideal of T with minimal generators $u_1^{\mathrm{pol}}, \dots, u_m^{\mathrm{pol}}$, where $u_j^{\mathrm{pol}} = \prod_{i=1}^{n} \prod_{k=1}^{a_{i,j}} x_{ik}$, $1 \le j \le m$. The monomial u_j^{pol} is called the *polarization* of u_j, and the ideal I^{pol} is called the *polarization* of I. It is well known that polarization preserves the height of ideals. In particular, I is an unmixed ideal if and only if I^{pol} is an unmixed ideal.

Finally, we recall the concept of Serre's condition. Let I be a monomial ideal of S. For a positive integer k, the ring S/I satisfies the Serre's condition (S_k) if

$$\operatorname{depth}(S/I)_{\mathfrak{p}} \ge \min\{\dim(S/I)_{\mathfrak{p}}, k\}$$

for every $\mathfrak{p} \in \operatorname{Spec}(S/I)$.

Lemma 2.2. *[33, Lemma 3.2.1] The following two conditions are equivalent.*

1. *S/I satisfies the Serre's condition (S_k).*
2. *For every integer i with $0 \le i < \dim S/I$, the inequality*

$$\dim \operatorname{Ext}_S^{n-i}(S/I, S) \le i - k$$

holds, where the dimension of the zero module is defined to be $-\infty$.

2.3 Edge-weighted edge ideals of very well-covered graphs

In this section, we study the unmixed and Cohen-Macaulay properties of edge-weighted edge ideals of very well-covered graphs. We first recall some known facts about the structure of very well-covered graphs. It is obvious that every vertex of a very well-covered graph G belongs to a maximal independent set of cardinality $|V(G)|/2$. Thus, we conclude the following result from [15, Theorem1.2].

Lemma 2.3. *Let G be a very well-covered graph. Then G has a perfect matching.*

By the above lemma, we may assume that the vertices of the very well-covered graph G are labeled such that the following condition is satisfied.

(*) $V(G) = X \cup Y$, $X \cap Y = \emptyset$, where $X = \{x_1, \dots, x_h\}$ is a minimal vertex cover of G and $Y = \{y_1, \dots, y_h\}$ is a maximal independent set of G such that $\{x_1y_1, \dots, x_hy_h\} \subset E(G)$.

Following the notations of condition (*), for the rest of this section, we set $S = K[x_1, \dots, x_h, y_1, \dots, y_h]$. For later use, we recall the following characterization of very well-covered graphs.

Proposition 2.4. *[7, Proposition 2.3] and [25, Theorem 2.9] Let G be a graph with $2h$ vertices, which are not isolated. Assume that the vertices of G are labeled such that condition (*) is satisfied. Then G is very well-covered if and only if the following conditions hold:*

(i) if z_ix_j, $y_jx_k \in E(G)$, then $z_ix_k \in E(G)$ for distinct indices i, j and k and for $z_i \in \{x_i, y_i\}$;

(ii) if $x_iy_j \in E(G)$, then $x_ix_j \notin E(G)$.

Remark 2.5. Let G be a very well-covered graph with $2h$ vertices and let w be an edge weight on G. Moreover, assume that the vertices of G are labeled in such a way that condition (*) is satisfied. Fix an integer i with $1 \leq i \leq h$. As $x_iy_i \in E(G)$, we deduce that every minimal prime $\mathfrak{p}$ of $I(G_w)$ contains one of the variables x_i, y_i. On the other hand, the height of $\mathfrak{p}$ is equal to h. Therefore, $\mathfrak{p}$ can not contain both x_i and y_i.

We are now ready to state and prove the first main result of this paper, which characterizes edge-weighted very well-covered graphs with unmixed edge ideal.

Theorem 2.6. *Let G be a very well-covered graph with $2h$ vertices and let w be an edge weight on G. Moreover, assume that the vertices of G are labeled in such a way that condition (*) is satisfied. Then $I(G_w)$ is unmixed if and only if the following conditions hold:*

(i) if $x_iz_j \in E(G)$, then $w(x_iz_j) \leq w(x_iy_i)$ and $w(x_iz_j) \leq w(x_jy_j)$ for distinct indices i, j, and for any vertex $z_j \in \{x_j, y_j\}$;

(ii) if z_ix_j and y_jx_k are edges of G, then $w(z_ix_k) \leq w(z_ix_j)$ and $w(z_ix_k) \leq w(y_jx_k)$ for distinct indices i, j, k and $z_i \in \{x_i, y_i\}$, or for $j \neq i = k$ and $z_i = y_i$.

Proof. Set $J := I(G_w)^{\mathrm{pol}}$.

Suppose $I(G_w)$ is unmixed. Then J is an unmixed ideal of height h. It follows from Remark 2.5 that for every integer i with $1 \leq i \leq h$, any minimal prime of J contains exactly one variable whose first index is i. We first prove condition (i). Assume that $x_iz_j \in E(G)$. Set $a := w(x_iz_j)$ and $b := w(x_iy_i)$. As

$$x_{i1}x_{i2}\cdots x_{ia}z_{j1}z_{j2}\cdots z_{ja} \in J,$$

there is a minimal prime $\mathfrak{p}_1$ of J with $x_{ia} \in \mathfrak{p}_1$. By contradiction, suppose $a > b$. It follows from

$$x_{i1}x_{i2}\cdots x_{ib}y_{i1}y_{i2}\cdots y_{ib} \in J$$

that at least one of the variable $x_{i1}, x_{i2}, \ldots, x_{ib}, y_{i1}, y_{i2}, \ldots, y_{ib}$ belongs to $\mathfrak{p}_1$. Therefore, $\mathfrak{p}_1$ contains two variables with first index i, which is a contradiction. Hence, $a \leq b$.

Now, set $c := w(x_jy_j)$ and suppose $a > c$. As

$$x_{i1}x_{i2}\cdots x_{ia}z_{j1}z_{j2}\cdots z_{ja} \in J,$$

there is a minimal prime $\mathfrak{p}_2$ of J with $z_{ja} \in \mathfrak{p}_2$. Also, it follows from

$$x_{j1}x_{j2}\cdots x_{jc}y_{j1}y_{j2}\cdots y_{jc} \in J$$

that at least one of the variable $x_{j1}, x_{j2}, \ldots, x_{jc}, y_{j1}, y_{j2}, \ldots, y_{jc}$ belongs to $\mathfrak{p}_2$. Therefore, $\mathfrak{p}_2$ contains two variables with first index j, which is a contradiction. Hence, $a \leq c$.

Next, we prove condition (ii). Assume that z_ix_j and $y_jx_k \in E(G)$. Since G is unmixed, it follows from Proposition 2.4 that $z_ix_k \in E(G)$ (this is trivially true, if $i = k$ and for $z_i = y_i$). Set $d := w(z_ix_k)$, $e := w(z_ix_j)$, $f := w(y_jx_k)$. Suppose $d > e$. Since $w(z_ix_k) = d$, it follows that

$$z_{i1}z_{i2}\cdots z_{i(d-1)}x_{k1}x_{k2}\cdots x_{kf} \notin J.$$

Thus, there is a minimal prime $\mathfrak{p}_3$ of J with

$$z_{i1}z_{i2}\cdots z_{i(d-1)}x_{k1}x_{k2}\cdots x_{kf} \notin \mathfrak{p}_3.$$

Hence, neither of the variables $z_{i1}, z_{i2}, \ldots, z_{i(d-1)}, x_{k1}, x_{k2}, \ldots, x_{kf}$ belongs to $\mathfrak{p}_3$. Then we deduce from

$$z_{i1}z_{i2}\cdots z_{ie}x_{j1}x_{j2}\cdots x_{je}, y_{j1}y_{j2}\cdots y_{jf}x_{k1}x_{k2}\cdots x_{kf} \in J$$

that $x_{js}, y_{jt} \in \mathfrak{p}_3$, for some positive integers s and t. This is a contradiction, as no minimal prime of J can contain both of x_{js} and y_{jt}. Thus, $d \leq e$.

Suppose $d > f$. Since

$$z_{i1}z_{i2}\cdots z_{ie}x_{k1}x_{k2}\cdots x_{k(d-1)} \notin J,$$

there is a minimal prime $\mathfrak{p}_4$ of J which contains neither of the variables

$$z_{i1}, z_{i2}, \ldots, z_{ie}, x_{k1}, x_{k2}, \ldots, x_{k(d-1)}.$$

It follows from

$$z_{i1}z_{i2}\cdots z_{ie}x_{j1}x_{j2}\cdots x_{je}, y_{j1}y_{j2}\cdots y_{jf}x_{k1}x_{k2}\cdots x_{kf} \in J$$

that $x_{j\ell}, y_{jr} \in \mathfrak{p}_4$, for some positive integers ℓ and r. This is again a contradiction. Therefore, $d \leq f$.

We now prove the reverse implication. Suppose conditions (i) and (ii) hold and assume by contradiction that $I(G_w)$ is not unmixed. Hence, J is not an unmixed ideal. Thus, there is a minimal prime $\mathfrak{p}$ of J such that $x_{jp}, y_{jq} \in \mathfrak{p}$, for some integers $j, p, q \geq 1$. As above set $c := w(x_j y_j)$. Note that if $x_{jp}, y_{jq} \in \mathfrak{p}$, then we have $x_{js}, y_{jt} \notin \mathfrak{p}$ for $s \neq p$ and $t \neq q$, since $\mathfrak{p}$ is a minimal prime of the polarization J of $I(G_w)$. See, e.g., the proof of Lemma 1.1 in [40].

Since we have $x_{j1}x_{j2}\cdots x_{jc}y_{j1}y_{j2}\cdots y_{jc} \in J \subseteq \mathfrak{p}$ and $x_{jp}, y_{jq} \in \mathfrak{p}$, it follows that $p \leq c$ or $q \leq c$. First we consider the case $q \leq c$. Assume $p > c$. Since $\mathfrak{p}$ is a minimal prime of J and $x_{jp} \in \mathfrak{p}$, there is $i \neq j$ and $z_i \in \{x_i, y_i\}$ such that $z_i x_j \in E(G)$ and $w(z_i x_j) \geq p$ (otherwise x_{jp} does not belong to any minimal prime of J). Then by (i), we have $c \geq w(z_i x_j) \geq p > c$, which is a contradiction. Hence $p \leq c$. Next we consider the case $p \leq c$. Suppose $q > c$. Since $\mathfrak{p}$ is a minimal prime of J and $y_{jq} \in \mathfrak{p}$, there is $k \neq j$ such that $y_j x_k \in E(G)$ with $w(y_j x_k) \geq q$. Then by (i) we have $c \geq w(y_j x_k) \geq q > c$, which is a contradiction. Therefore, $q \leq c$. We have then shown that both $p \leq c$ and $q \leq c$.

Now we show that there is $\ell \neq j$ and $z_\ell \in \{x_\ell, y_\ell\}$ such that $z_\ell x_j \in E(G)$ with $\alpha := w(z_\ell x_j) \geq p$ and

$$z_{\ell 1}, z_{\ell 2}, \ldots, z_{\ell \alpha} \notin \mathfrak{p}.$$

As seen before, since $\mathfrak{p}$ is a minimal prime of J and $x_{jp} \in \mathfrak{p}$, there is $\ell \neq j$ and $z_\ell \in \{x_\ell, y_\ell\}$ such that $z_\ell x_j \in E(G)$ with $\alpha \geq p$. Suppose there does not exist ℓ such that

$$z_{\ell 1}, z_{\ell 2}, \ldots, z_{\ell \alpha} \notin \mathfrak{p}.$$

Then for any $z_m x_j \in E(G)$, there exists $\beta_m \leq \alpha_m := w(z_m x_j)$ such that $z_{m\beta_m} \in \mathfrak{p}$. Then we have

$$z_{m1}z_{m2}\cdots z_{m\alpha_m}x_{j1}x_{j2}\cdots x_{j\alpha_m} \in \mathfrak{p}' := (\mathrm{Gens}(\mathfrak{p}) \setminus \{x_{jp}\}).$$

Since any minimal monomial generator of J which is not divided by x_{jp} belongs to $\mathfrak{p}'$, we have $J \subset \mathfrak{p}'$, which contradicts the fact that $\mathfrak{p}$ is a minimal prime of J. Similarly, there is $r \neq j$ such that $y_j x_r \in E(G)$ with $\beta := w(y_j x_r) \geq q$ and

$$x_{r1}, x_{r2}, \ldots, x_{r\beta} \notin \mathfrak{p}.$$

By Proposition 2.4, $z_\ell x_r \in E(G)$. Set $\gamma := w(z_\ell x_r)$. It follows from condition (ii) that $\gamma \leq \alpha$ and $\gamma \leq \beta$. Thus,

$$z_{\ell 1}, z_{\ell 2}, \ldots, z_{\ell \gamma}, x_{r1}, x_{r2}, \ldots, x_{r\gamma} \notin \mathfrak{p}.$$

This contradicts

$$z_{\ell 1}z_{\ell 2}\cdots z_{\ell\gamma}x_{r1}x_{r2}\cdots x_{r\gamma} \in J.$$

Hence, $I(G_w)$ is an unmixed ideal. □

Remark 2.7. Let G be a very well-covered graph and let w be an edge weight, such that $I(G_w)$ is an unmixed ideal. Assume that the vertices of G are labeled in such a way that condition (*) is satisfied. It follows from Theorem 2.6 that if $x_i y_j, x_j y_i \in E(G)$, then $w(x_i y_i) = w(x_j y_j) = w(x_i y_j) = w(x_j y_i)$.

Example 2.8. Let G be the graph with vertex set $V(G) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and the edge set

$$E(G) = \big\{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_1, y_2), (x_1, y_3), (x_2, y_3)\big\}.$$

then the edge ideal of G is

$$I(G) = (x_1y_1, x_2y_2, x_3y_3, x_1y_2, x_1y_3, x_2y_3),$$

and G is very well–covered, and consider edge–weighted edge ideals

$$I(G_{w_1}) = (x_1^3y_1^3, x_2^2y_2^2, x_3^3y_3^3, x_1^2y_2^2, x_1y_3, x_2^2y_3^2),$$
$$I(G_{w_2}) = (x_1^2y_1^2, x_2^2y_2^2, x_3^3y_3^3, x_1^3y_2^3, x_1y_3, x_2^2y_3^2).$$

Then $I(G_{w_1})$ is unmixed, since w_1 satisfies the assumption of Theorem 2.6. However, $I(G_w)$ is not unmixed, because w_2 does not satisfy the assumption of Theorem 2.6. In fact, $w_2(x_1y_2) > w_2(x_2y_2)$.

Our next goal is to provide a characterization of Cohen-Macaulay edge-weighted edge ideals. First we summarize the known results concerning the Cohen-Macaulay property of a (non-weighted) very well-covered graph.

Lemma 2.9. *[7, Lemma 3.5] Let G be an unmixed graph with $2h$ vertices, which are not isolated, and assume that the vertices of G are labeled such that condition (*) is satisfied. If G is a Cohen-Macaulay graph, then there exists a suitable simultaneous change of labeling on both $\{x_i\}_{i=1}^h$ and $\{y_i\}_{i=1}^h$ (i.e., we relabel $(x_{i_1}, \dots, x_{i_h})$ and $(y_{i_1}, \dots, y_{i_h})$ as $(x_1, \dots, x_h)$ and $(y_1, \dots, y_h)$ at the same time), such that $x_iy_j \in E(G)$ implies $i \leq j$.*

Hence, for a Cohen-Macaulay very well-covered graph G satisfying condition (*), we may assume that

(**) $x_iy_j \in E(G)$ implies $i \leq j$.

Now we recall a Cohen-Macaulayness criterion for very well-covered graphs. See also [5, Theorem 6.3] and [6, Theorem 2.3] for different characterizations.

Theorem 2.10. *[7, Theorem 0.3] Let G be a graph with $2h$ vertices, which are not isolated and assume that the vertices of G are labeled such that conditions (*) and (**) are satisfied. Then the following conditions are equivalent:*

1. *G is Cohen-Macaulay;*
2. *G is unmixed;*
3. *The following conditions hold:*

(i) if $z_i x_j, y_j x_k \in E(G)$, then $z_i x_k \in E(G)$ for distinct indices i, j, k and for $z_i \in \{x_i, y_i\}$;

(ii) if $x_i y_j \in E(G)$, then $x_i x_j \notin E(G)$.

In order to study the Cohen-Macaulay property of edge-weighted edge ideal of very well-covered graphs, we introduce an operator which allows us to construct a new weighted very well-covered graph from a given one.

Let G_w be a weighted very well-covered graph with $n = 2h$ vertices and assume that the vertices of G are labeled such that condition (*) is satisfied. For any $k \in [h] := \{1, \dots, h\}$, set

$$N_k := \{i \in [h] : x_i y_k \in E(G)\} \setminus \{k\},$$

and define the *base graph* $O_k(G)$ as follows

$$O_k(G) := G - \{x_i y_k : i \in N_k\} + \{x_i x_k : i \in N_k\}.$$

Now we define the weight w' on $O_k(G)$ by

$$w'(e) = \begin{cases} w(x_i y_k) & \text{if } e = x_i x_k,\ i \in N_k \\ w(e) & \text{otherwise.} \end{cases}$$

Finally, we set

$$O_k(G_w) := O_k(G)_{w'}.$$

We are now ready to prove the second main result of this paper.

Theorem 2.11. *Let G be a Cohen-Macaulay very well-covered graph and let w be an edge weight on G. Then the following conditions are equivalent:*

1. *$I(G_w)$ is an unmixed ideal;*

2. *$S/I(G_w)$ is a Cohen-Macaulay ring.*

Proof. The implication (2) $\Longrightarrow$ (1) is well known. So, we prove (1) implies (2). As G is a Cohen-Macaulay very well-covered graph, we may assume that conditions (*) and (**) are satisfied. In particular, $|V(G)| = 2h$, for some $h \geq 1$. It follows from the unmixedness of $I(G_w)$ that the height of every associated prime of $S/I(G_w)$ is h. Using Remark 2.5, for every $\mathfrak{p} \in \operatorname{Ass} S/I(G_w)$ and for every integer k with $1 \leq k \leq h$, exactly one of x_k and y_k belongs to $\mathfrak{p}$.

We use induction on $m := \sum_{i=1}^{h} \deg_G y_i \geq h$. For $m = h$, the assertion follows from [28, Theorem 5.7]. Hence, suppose $m > h$. Then there exists an integer k with $1 \leq k \leq h$ such that $\deg y_k \geq 2$. By contradiction, assume that $S/I(G_w)$ is not Cohen-Macaulay. Set $G'_{w'} := O_k(G_w)$, where

$$w'(e) = \begin{cases} w(x_i y_k) & \text{if } e = x_i x_k,\ i \in N_k \\ w(e) & \text{otherwise.} \end{cases}$$

Using Theorem 2.6, one can easily check that $I(G'_{w'})$ is an unmixed ideal. By induction $S/I(G'_{w'})$ is Cohen-Macaulay. Therefore,

$$(S/I(G_w))/(x_k - y_k) \cong (S/I(G'_{w'}))/(x_k - y_k)$$

is Cohen-Macaulay since y_k is a leaf of $G'_{w'}$ and hence $x_k - y_k$ is regular on $S/I(G'_{w'})$. As $S/I(G_w)$ is not Cohen-Macaulay, $x_k - y_k$ is not regular on $S/I(G_w)$. Hence,

$$x_k - y_k \in \bigcup_{\mathfrak{p} \in \operatorname{Ass} S/I(G_w)} \mathfrak{p}.$$

Thus, there exists an associated prime ideal $\mathfrak{p}$ of $S/I(G_w)$ such that $x_k - y_k \in \mathfrak{p}$. Consequently, $x_k, y_k \in \mathfrak{p}$. This is a contradiction and proves that $S/I(G_w)$ is Cohen-Macaulay. □

It is well known (and easy to prove) that every unmixed bipartite graph is very well-covered (for example, it follows from [42, Theorem 7.1.8 and Lemma 7.4.18]). Hence, as an immediate consequence of Theorem 2.11, we obtain the following corollary.

Corollary 2.12. *Let G be a Cohen-Macaulay bipartite graph and let w be an edge weight on G. Then the following conditions are equivalent:*

1. *$I(G_w)$ is an unmixed ideal;*

2. *$S/I(G_w)$ is a Cohen-Macaulay ring.*

2.4 Examples

Let $\mathcal{D}$ be a vertex-weighted oriented graph and let G be its underlying graph. As we mentioned in Section 2.1, Pitones, Reyes and Toledo conjectured that $S/I(\mathcal{D})$ is Cohen-Macaulay, if $I(\mathcal{D})$ is unmixed and $S/I(G)$ is Cohen-Macaulay (see Conjecture 2.1). The following example shows that Conjecture 2.1 is not true.

Example 2.13. Let $\mathbb{K}$ be a field with char($\mathbb{K}$) = 0 and let $\mathcal{D}$ be the oriented graph with vertex set $V(\mathcal{D}) = \{x_1, \ldots, x_{11}\}$ and the edge set

$$\begin{aligned} E(\mathcal{D}) = \{&(x_1,x_3),(x_1,x_4),(x_7,x_1),(x_1,x_{10}),(x_1,x_{11}),(x_2,x_4),(x_2,x_5),\\ &(x_2,x_8),(x_2,x_{10}),(x_2,x_{11}),(x_3,x_5),(x_3,x_6),(x_3,x_8),(x_3,x_{11}),\\ &(x_4,x_6),(x_4,x_9),(x_4,x_{11}),(x_7,x_5),(x_5,x_9),(x_{11},x_5),(x_6,x_8),\\ &(x_6,x_9),(x_9,x_7),(x_7,x_{10}),(x_8,x_{10})\}. \end{aligned}$$

Consider the weight functions

$$w_1(x_i) = \begin{cases} 1 & \text{if } i \neq 11 \\ 2 & \text{if } i = 11, \end{cases}$$

and

$$w_2(x_i) = \begin{cases} 1 & \text{if } i \neq 7 \\ 2 & \text{if } i = 7. \end{cases}$$

For $i = 1, 2$, let $\mathcal{D}_i$ be the vertex-weighted oriented graph obtained from $\mathcal{D}$ by considering the weight function w_i. Then

$$\begin{aligned} I(\mathcal{D}_1) =& (x_1x_3, x_1x_4, x_1x_7, x_1x_{10}, x_1x_{11}^2, x_2x_4, x_2x_5, x_2x_8, x_2x_{10}, x_2x_{11}^2, \\ & x_3x_5, x_3x_6, x_3x_8, x_3x_{11}^2, x_4x_6, x_4x_9, x_4x_{11}^2, x_5x_7, x_5x_9, x_5x_{11}, \\ & x_6x_8, x_6x_9, x_7x_9, x_7x_{10}, x_8x_{10}), \end{aligned}$$

and

$$\begin{aligned} I(\mathcal{D}_2) =& (x_1x_3, x_1x_4, x_1x_7, x_1x_{10}, x_1x_{11}, x_2x_4, x_2x_5, x_2x_8, x_2x_{10}, x_2x_{11}, \\ & x_3x_5, x_3x_6, x_3x_8, x_3x_{11}, x_4x_6, x_4x_9, x_4x_{11}, x_5x_7, x_5x_9, x_5x_{11}, \\ & x_6x_8, x_6x_9, x_7^2x_9, x_7x_{10}, x_8x_{10}). \end{aligned}$$

Let G be the underlying graph of $\mathcal{D}$. The edge ideal $I(G)$ of G comes from the triangulation of the real projective plane and it is known that $S/I(G)$ is Cohen-Macaulay, as $\mathrm{char}(\mathbb{K}) \neq 2$ (see for example [42, Exercise 6.3.65]). It is known that $S/I(G)$ is Cohen-Macaulay. However, for $i = 1, 2$, as a *Macaulay2* computation shows, $I(\mathcal{D}_i)$ is unmixed but not Cohen-Macaulay, disproving Conjecture 2.1. We show that $S/I(\mathcal{D}_1)$ satisfies the Serre's condition (S_2), while $S/I(\mathcal{D}_2)$ does not. Using *Macaulay2* we know that $\operatorname{depth} S/I(\mathcal{D}_i) = 2$ for $i = 1, 2$. Since for $i = 1, 2$, $\dim S/I(\mathcal{D}_i) = 3$, the quotient ring $S/I(\mathcal{D}_i)$ satisfies the (S_2) condition if and only if

$$\dim \operatorname{Ext}_S^9(S/I(\mathcal{D}_i), S) = \dim \operatorname{Ext}_S^{11-2}(S/I(\mathcal{D}_i), S) \leq 2 - 2 = 0,$$

by Lemma 2.2. With *Macaulay2*, one can check that $\dim \operatorname{Ext}_S^9(S/I(\mathcal{D}_1), S) = 0$ and $\dim \operatorname{Ext}_S^9(S/I(\mathcal{D}_2), S) = 1$.

The following example provides counterexamples for the edge-weighted version of Conjecture 2.1.

Example 2.14. Let $\mathbb{K}$ be a field with $\mathrm{char}(\mathbb{K}) = 0$ and let G be the same graph as in Example 2.13. Consider the following edge-weighted edge ideals.

$$\begin{aligned} I(G_{w_1}) =& (x_1x_3, x_1x_4, x_1x_7, x_1x_{10}, x_1x_{11}, x_2x_4, x_2x_5, x_2x_8, x_2x_{10}, x_2x_{11}, \\ & x_3x_5, x_3x_6, x_3x_8, x_3x_{11}, x_4x_6, x_4x_9, x_4x_{11}, x_5x_7, x_5x_9, x_5x_{11}, \\ & x_6x_8, x_6x_9, x_7x_9, x_7x_{10}, x_8^2x_{10}^2). \\ I(G_{w_2}) =& (x_1^2x_3^2, x_1^2x_4^2, x_1^2x_7^2, x_1^2x_{10}^2, x_1^2x_{11}^2, x_2^2x_4^2, x_2^2x_5^2, x_2^2x_8^2, x_2^2x_{10}^2, x_2^2x_{11}^2, \\ & x_3^2x_5^2, x_3^2x_6^2, x_3^2x_8^2, x_3^2x_{11}^2, x_4^2x_6^2, x_4^2x_9^2, x_4^2x_{11}^2, x_5^2x_7^2, x_5^2x_9^2, x_5^2x_{11}^2, \\ & x_6^2x_8^2, x_6^2x_9^2, x_7^2x_9^2, x_7^2x_{10}^2, x_8x_{10}). \end{aligned}$$

Then $S/I(G)$ is Cohen-Macaulay. However, a *Macaulay2* computation shows that $I(G_{w_1})$ is unmixed, but $S/I(G_{w_1})$ does not satisfy the Serre's condition (S_2), hence it is not Cohen-Macaulay. On the other hand, $I(G_{w_2})$ is unmixed and $S/I(G_{w_2})$ satisfies the Serre's condition (S_2) condition, but it is not Cohen-Macaulay.

Chapter 3

Specht ideals

3.1 Introduction

This chapter is based on the author's paper [39] with Kohji Yanagawa. For a positive integer n, a *partition* of n is a sequence $\lambda = (\lambda_1, \dots, \lambda_l)$ of integers with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 1$ and $\sum_{i=1}^{l} \lambda_i = n$. The *Young tableau* of shape λ is a bijection from $[n] := \{1, 2, \dots, n\}$ to the set of boxes in the Young diagram of λ. For example, the following is a tableau of shape $(4, 2, 1)$.

3	5	1	7
6	2		
4			

(3.1.1)

Let $\mathrm{Tab}(\lambda)$ be the set of Young tableaux of shape λ. If $\lambda = (\lambda_1, \dots, \lambda_l)$, then we simply write as $\mathrm{Tab}(\lambda_1, \dots, \lambda_l)$. We say a tableau T is *standard*, if all columns (resp. rows) are increasing from top to bottom (resp. from left to right). Let $\mathrm{SYT}(\lambda)$ (or $\mathrm{SYT}(\lambda_1, \dots, \lambda_l)$) be the set of standard tableaux of shape $\lambda = (\lambda_1, \dots, \lambda_l)$.

Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K, λ a partition of n, and T a Young tableau of shape λ. If the j-th column of T consists of $j_1, j_2, \dots, j_m$ in the order from top to bottom, then

$$f_T(j) := \prod_{1 \leq s < t \leq m} (x_{j_s} - x_{j_t}) \in R$$

(if the j-th column has only one box, then we set $f_T(j) = 1$). The *Specht polynomial* f_T of T is given by

$$f_T := \prod_{j=1}^{\lambda_1} f_T(j).$$

For example, if T is the tableau (3.1.1), then $f_T = (x_3 - x_6)(x_3 - x_4)(x_6 - x_4)(x_5 - x_2)$.

The symmetric group $\mathfrak{S}_n$ acts on the vector space V_λ spanned by $\{ f_T \mid T \in \mathrm{Tab}(\lambda)\}$. An $\mathfrak{S}_n$-module of this form is called a *Specht module*, and very important in the theory of symmetric groups, especially in the characteristic 0 case. Over any field of characteristic 0 the Specht modules are irreducible, and form a complete list of irreducible representations of the symmetric group. Here, we remark that $\{f_T \mid T \in \mathrm{SYT}(\lambda)\}$ forms a basis of V_λ.

In [47], Yanagawa studied the *ideal*

$$I_\lambda^{\mathrm{Sp}} := (\, f_T \mid T \in \mathrm{Tab}(\lambda))$$

of R. Yanagawa has $\mathrm{ht}(I_\lambda^{\mathrm{Sp}}) = \lambda_1$ by [47, Proposition 2.3]. The main result of [47] states the following.

Theorem 3.1 ([47, Proposition 2.8 and Corollary 4.4])**.** *If $R/I_\lambda^{\mathrm{Sp}}$ is Cohen–Macaulay, then one of the following conditions holds.*

(1) $\lambda = (n-d, 1, \dots, 1)$,

(2) $\lambda = (n-d, d)$,

(3) $\lambda = (d, d, 1)$.

If $\mathrm{char}(K) = 0$*, the converse is also true.*

The case (1) is treated in the joint paper [44] , and it is shown that $R/I^{\mathrm{Sp}}_{(n-d,1,\dots,1)}$ is Cohen–Macaulay over any K. To prove the last assertion of the above theorem for the cases (2) and (3), we first show that I_λ^{Sp} is a radical ideal (at least, in these cases) over any K, and use a result of Etingof et al. [13], which concerns the characteristic 0 case. In addition, in [22], results on the Cohen–Macaulayness of $R/I^{\mathrm{Sp}}_{(n-d,d)}$ are proved without using the results of Etingof et al. In particular, the results for the positive characteristic case are also given.

The paper [44] computes the Betti numbers of $R/I^{\mathrm{Sp}}_{(n-d,1,\dots,1)}$, it means that we know its Hilbert series in this case. In the present paper, we compute the Hilbert series

$$H(R/I_\lambda^{\mathrm{Sp}}, t) := \sum_{i \in \mathbb{N}} \dim_K [R/I_\lambda^{\mathrm{Sp}}]_i \cdot t^i$$

in the cases (2) and (3) of Theorem 3.1. The main tool for computation is the following recursive formulas as graded S-modules

$$(R/I^{\mathrm{Sp}}_{(n-d,d)})/(S/I^{\mathrm{Sp}}_{(n-d-1,d)}) \cong \bigoplus_{m \geq 1} (S/I^{\mathrm{Sp}}_{(n-d,d-1)})(-m). \tag{3.1.2}$$

for $n - d > d \geq 2$, and

$$(R/I^{\mathrm{Sp}}_{(d,d)})/(S/I^{\mathrm{Sp}}_{(d-1,d-1,1)}) \cong \bigoplus_{m \geq 1} (S/I^{\mathrm{Sp}}_{(d,d-1)})(-m). \tag{3.1.3}$$

as graded S-modules, for $n = 2d \geq 4$. Here we set $S = K[x_1, \dots, x_{n-1}]$. Since $(n-d-1, d)$ is a partition of $n-1$, $I^{\mathrm{Sp}}_{(n-d-1,d)}$ is an ideal of S. The same is true for other partitions of $n-1$.

As an application, we have the following.

Theorem 3.2. *If* $R/I^{\mathrm{Sp}}_{(n-d,d)}$ *is Cohen-Macaulay (e.g., when* $\mathrm{char}(K) = 0$*), then we have* $\mathrm{reg}(R/I^{\mathrm{Sp}}_{(n-d,d)}) = d$ *for* $d \geq 2$*, and if* $R/I^{\mathrm{Sp}}_{(d,d,1)}$ *is Cohen-Macaulay (e.g., when* $\mathrm{char}(K) = 0$*), then* $\mathrm{reg}(R/I^{\mathrm{Sp}}_{(d,d,1)}) = d+1$*. Hence* $I^{\mathrm{Sp}}_{(d,d,1)}$ *has a* $(d+2)$*-linear resolution in this case.*

Since $R/I^{\mathrm{Sp}}_{(d,d,1)}$ (not $S/I^{\mathrm{Sp}}_{(d-1,d-1,1)}$) does not appear in the above recursion formulas, these formulas are not enough. So we use [31, Theorem 3.2] for $R/I^{\mathrm{Sp}}_{(d,d,1)}$. However, this result assumes the Cohen–Macaulay property, so we have to show the following.

Theorem 3.3. *Hilbert series of* $R/I^{\mathrm{Sp}}_{(n-d,d)}$ *and* $R/I^{\mathrm{Sp}}_{(d,d,1)}$ *do not depend on* $\mathrm{char}(K)$*.*

We prove this (essentially) in §3 using the Gröbner basis argument.

However, Our paper [39] was submitted, the authors were informed that minimal free resolutions of $R/I^{\mathrm{Sp}}_{(n-d,d)}$ for $1 \leq d \leq n/2$ in $\mathrm{char}(K) = 0$, and the Hilbert series of their rings had been studied by Berkesch Zamaere, Griffeth, and Sam [3]. More precisely, [3] determined the $\mathfrak{S}_n$-module structure of $\mathrm{Tor}_i^R(K, R/I^{\mathrm{Sp}}_{(n-d,d)})$. (They called $I^{\mathrm{Sp}}_{(n-d,d)}$ the "$(d+1)$-equal ideal". Of course, this name comes from the decomposition (3.4.1) below.) However we do not use results of [3], and they do not prove that the Hilbert series does not depend on $\mathrm{char}(K)$. Moreover, the recursive formulas (3.1.2) and (3.1.3) is not appeared in their paper.

3.2 Main theorem and related arguments

For the definition and basic properties of Specht ideals I^{Sp}_λ, consult the previous section. Here we just remark that the Cohen–Macaulay-ness of $R/I^{\mathrm{Sp}}_\lambda$ actually depends on $\mathrm{char}(K)$. For example, $R/I^{\mathrm{Sp}}_{(n-3,3)}$ is Cohen–Macaulay if and only if $\mathrm{char}(K) \neq 2$. The same is true for $R/I^{\mathrm{Sp}}_{(2,2,1)}$. See [47, Theorem 5.3]. If $\mathrm{char}(K) = 2$, *Macaulay2* computation shows that $R/I^{\mathrm{Sp}}_{(2,2,1)}$ and $R/I^{\mathrm{Sp}}_{(n-3,3)}$ for $n \leq 10$ do not satisfy even Serre's (S_2) condition. So the (S_2) condition of $R/I^{\mathrm{Sp}}_\lambda$ also depends on $\mathrm{char}(K)$ (recall that the (S_2)-ness of the Stanley–Reisner ring $K[\Delta]$ does not depend on $\mathrm{char}(K)$). *Macaulay2* computation also shows that $R/I^{\mathrm{Sp}}_{(4,4)}$ is not Cohen–Macaulay, if $\mathrm{char}(K) = 2, 3$. See [47, Conjecture 5.5]. To the authors' best knowledge, examples of $R/I^{\mathrm{Sp}}_\lambda$ satisfying the (S_2) condition are Cohen–Macaulay.

We regard $S = K[x_1, \ldots, x_{n-1}]$ as a subring of $R = K[x_1, \ldots, x_n]$. If μ is a partition of $n-1$, then the Specht ideal I^{Sp}_μ is an ideal of S.

Theorem 3.4. *Hilbert series of* $R/I^{\mathrm{Sp}}_{(n-d,d)}$ *and* $R/I^{\mathrm{Sp}}_{(d,d,1)}$ *do not depend on* $\mathrm{char}(K)$*. Furthermore, the Hilbert series of* $R/I^{\mathrm{Sp}}_{(n-d,d)}$ *is given by*

$$H(R/I^{\mathrm{Sp}}_{(n-d,d)}, t) = \frac{1 + h_1 t + h_2 t^2 + \cdots + h_d t^d}{(1-t)^d}$$

with

$$h_i = \begin{cases} \binom{n-d+i-1}{i} & \text{if } 1 \le i \le d-1, \\ \binom{n-1}{d-2} & \text{if } i = d. \end{cases}$$

Similarly, when $n = 2d+1$, the Hilbert series of $R/I^{\mathrm{Sp}}_{(d,d,1)}$ is given by

$$H(R/I^{\mathrm{Sp}}_{(d,d,1)}, t) = \frac{1 + h_1 t + h_2 t^2 + \cdots + h_{d+1} t^{d+1}}{(1-t)^{d+1}}$$

with

$$h_i = \binom{d+i-1}{i}$$

for all $1 \le i \le d+1$.

The proof of Theorem 3.2 (assuming Theorem 3.4). For a Cohen–Macaulay graded ring R/I of dimension d whose Hilbert series is given by

$$H(R/I, t) = \frac{1 + h_1 t + h_2 t^2 + \cdots + h_s t^s}{(1-t)^d}$$

with $h_s \neq 0$, it is well-known that $\operatorname{reg}(R/I) = s$. So the assertion follows from Theorem 3.4. □

3.3 The initial monomials of Specht polynomials

In this section, we will give Gröbner basis theoretic results, which can be used to show the Hilbert series do not depend $\operatorname{char}(K)$. See [11, §15] for notions and results of the Gröbner basis theory. Here we consider the lexicographic order on R with $x_n \succ x_{n-1} \succ \cdots \succ x_1$. Let $\operatorname{in}(f)$ be the initial monomial of $0 \neq f \in R$.

Consider a tableau

$$T = \begin{array}{|c|c|c|c|c|c|c|} \hline i_1 & i_2 & \cdots & i_d & i_{d+1} & \cdots & i_{n-d} \\ \hline j_1 & j_2 & \cdots & j_d \\ \cline{1-4} \end{array} \in \operatorname{Tab}(n-d, d).$$

Since the permutation of i_k and j_k only changes the sign of f_T, we may assume that $i_k < j_k$ for all $1 \le k \le d$. Then we have $\operatorname{in}(f_T) = x_{j_1} x_{j_2} \cdots x_{j_d}$.

The following lemma holds for a general partition λ, and must be well-known to specialists. Since we could not find appropriate references, we give a quick proof for the reader's convenience.

Lemma 3.5. *For a partition $\lambda = (n-d, d)$, we have the following.*

(1) For distinct $T, T' \in \mathrm{SYT}(\lambda)$, we have $\operatorname{in}(f_T) \neq \operatorname{in}(f_{T'})$.

(2) Let $V_\lambda \subset R$ be the Specht module of shape λ. For $0 \neq f \in V_\lambda$, there is a unique $T \in \mathrm{SYT}(\lambda)$ such that $\mathrm{in}(f) = \mathrm{in}(f_T)$.

Proof. (1) If $\mathrm{in}(f_T) = \mathrm{in}(f_{T'})$ holds for $T, T' \in \mathrm{SYT}(\lambda)$, then the second rows of T and T' are same. It means that $T = T'$.

(2) It is well-known that $\{f_T \mid T \in \mathrm{SYT}(\lambda)\}$ forms a basis of V_λ. Hence the assertion follows from (1). □

In the rest of this section, we assume that $n = 2d$. Let $\mathfrak{m}^{\langle d+1\rangle}$ be the ideal of R generated by all squarefree monomials of degree $d+1$, and set

$$J^{\mathrm{Sp}}_{(d,d)} := I^{\mathrm{Sp}}_{(d,d)} + \mathfrak{m}^{\langle d+1\rangle}.$$

For $\mathbf{a} \in \mathbb{N}^n$, set $x^{\mathbf{a}} := \prod_{i\in[n]} x_i^{a_i} \in R$. For $f = \sum_{\mathbf{a}\in\mathbb{N}^n} c_{\mathbf{a}} x^{\mathbf{a}} \in R$ ($c_{\mathbf{a}} \in K$), we call

$$\mathrm{trm}(f) := \sum_{x^{\mathbf{a}} \notin \mathfrak{m}^{\langle d+1\rangle}} c_{\mathbf{a}} x^{\mathbf{a}} \in R$$

the *trimmed form* of f. For example, if $d = 2$ and $f = x_1x_4^2 - 2x_2x_3^2 + 3x_1x_3x_4 - x_2x_3x_4$, then we have $\mathrm{trm}(f) = x_1x_4^2 - 2x_2x_3^2$.

For $F \subset [n]$ with $\#F =: c \leq d$, let $\mathrm{Tab}_F(d, d-c)$ be the set of Young tableaux of shape $(d, d-c)$ with the letter set $[n] \setminus F$. For example, if $n = 8$ (i.e., $d = 4$) and $F = \{1, 6\}$, then

2	8	7	3
4	5		

is an element of $\mathrm{Tab}_F(4, 2)$. For the convention, set $\mathrm{Tab}_\emptyset(d, d) := \mathrm{Tab}(d, d)$. If $\#F = d$, then $T \in \mathrm{Tab}_F(d, 0)$ consists of a single row, and we have $f_T = 1$.

For a subset $F \subset [n]$, set $x^F := \prod_{i\in F} x_i \in R$ and $x^{2F} := \prod_{i\in F} x_i^2 \in R$. For a monomial $x^{\mathbf{a}} \in R$, set $\mathrm{supp}(x^{\mathbf{a}}) := \{\, i \mid a_i > 0\}$.

Lemma 3.6. *Let $x^{\mathbf{a}} \in R$ be a monomial with $F := \mathrm{supp}(x^{\mathbf{a}})$, and set $c := \#F$. If $\mathrm{trm}(x^{\mathbf{a}} f_T) \neq 0$ for $T \in \mathrm{Tab}(d, d)$, then we have $c \leq d$, and there is some $T' \in \mathrm{Tab}_F(d, d-c)$ such that*

$$\mathrm{trm}(x^{\mathbf{a}} f_T) = x^{\mathbf{a}} x^F f_{T'}.$$

In prticular, we have

$$\mathrm{trm}(x^F f_T) = x^{2F} f_{T'}.$$

The converse also holds, that is, any $x^{\mathbf{a}} x^F f_{T'}$ for $T' \in \mathrm{Tab}_F(d, d-c)$ equals $\mathrm{trm}(x^{\mathbf{a}} f_T)$ for some $T \in \mathrm{Tab}(d, d)$.

Proof. Without loss of generality, we may assume that $F = \{1, 2, \ldots, c\}$. Note that nonzero terms of f_T are ($\pm$ of) squarefree monomials of degree d. Hence if $\mathrm{trm}(x^{\mathbf{a}} f_T) \neq 0$, then T is of the form

i_1	$\cdots$	i_{d-c}	i_{d-c+1}	i_{d-c+2}	$\cdots$	i_d
j_1	$\cdots$	j_{d-c}	1	2	$\cdots$	c

after a suitable column permutation and permutations of the two boxes in the same columns (f_T is stable under these permutations up to sign). Moreover,

$T' :=$

i_1	$\cdots$	i_{d-c}	i_{d-c+1}	i_{d-c+2}	$\cdots$	i_d
j_1	$\cdots$	j_{d-c}				

satisfies the expected condition.

The last assertion can be proved in a similar way. □

Theorem 3.7. *With the above situation,*

$$\left(\bigcup_{F \subset [n]} \{ \mathrm{trm}(x^F f_T) \mid T \in \mathrm{Tab}(d,d) \} \setminus \{0\} \right) \cup G(\mathfrak{m}^{\langle d+1 \rangle}) \tag{3.3.1}$$

forms a Gröbner basis of $J^{\mathrm{Sp}}_{(d,d)}$. Here $G(\mathfrak{m}^{\langle d+1 \rangle})$ is the set of squarefree monomials of degree $d+1$.

Proof. Take $F_1, F_2 \subset [n]$ and $T_1, T_2 \in \mathrm{Tab}(d,d)$ with $\varphi_1 := \mathrm{trm}(x^{F_1} f_{T_1}) \neq 0$ and $\varphi_2 := \mathrm{trm}(x^{F_2} f_{T_2}) \neq 0$. We have some $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ such that the least common multiple of $\mathrm{in}(\varphi_1)$ and $\mathrm{in}(\varphi_2)$ coincides with $x^{\mathbf{a}} \mathrm{in}(\varphi_1) = x^{\mathbf{b}} \mathrm{in}(\varphi_2)$. Note that $x^{\mathbf{a}}$ and $x^{\mathbf{b}}$ need not be squarefree, and $x^{\mathbf{a}} \mathrm{in}(\varphi_1) = x^{\mathbf{b}} \mathrm{in}(\varphi_2)$ might belong to $\mathfrak{m}^{\langle d+1 \rangle}$. These phenomena make the following argument a bit complicated.

We set $\psi := x^{\mathbf{a}} \varphi_1 \pm x^{\mathbf{b}} \varphi_2$, where we take $\pm$ to cancel the initial terms. By Buchberger's criterion ([11, Theorem 15.8]), it suffices to show that ψ can be reduced to 0 modulo (3.3.1) by the division algorithm. To do this, it suffices to show that $\mathrm{trm}(\psi)$ can be reduced to 0 modulo

$$\bigcup_{F \subset [n]} \{ \mathrm{trm}(x^F f_T) \mid T \in \mathrm{Tab}(d,d) \} \setminus \{0\}. \tag{3.3.2}$$

If $\mathrm{trm}(\psi) \neq 0$, then at least one of the following conditions holds

(i) $\mathrm{trm}(x^{\mathbf{a}}\varphi_1) \neq 0$, equivalently, for $G_1 := \mathrm{supp}(x^{\mathbf{a}}x^{F_1})$, x^{G_1} divides some non-zero term of f_{T_1}.

(ii) $\mathrm{trm}(x^{\mathbf{b}}\varphi_2) \neq 0$, equivalently, for $G_2 := \mathrm{supp}(x^{\mathbf{b}}x^{F_2})$, x^{G_2} divides some non-zero term of f_{T_2}.

Assume that only (i) holds. Since (ii) is not satisfied now, we have

$$\mathrm{trm}(\psi) = \mathrm{trm}(x^{\mathbf{a}}\varphi_1) = \mathrm{trm}(x^{\mathbf{a}}x^{F_1}f_{T_1}) = x^{\mathbf{a}}x^{F_1}x^{G_1}f_{T_1'}$$

for some $T_1' \in \mathrm{Tab}_{G_1}(d, d-c_1)$ by Lemma 3.6, where $c_1 := \#G_1$. Hence $\mathrm{trm}(\psi)$ belongs to

$$\langle\, \mathrm{trm}(x^{\mathbf{a}}x^{F_1}f_T) \mid T \in \mathrm{Tab}(d,d)\,\rangle = x^{\mathbf{a}}x^{F_1}x^{G_1}\langle\, f_{T'} \mid T' \in \mathrm{Tab}_{G_1}(d, d-c_1)\,\rangle.$$

The subset

$$\{\, \mathrm{trm}(x^{G_1}f_T) \mid T \in \mathrm{Tab}(d,d)\,\} \setminus \{0\} \tag{3.3.3}$$

of (3.3.2) spans the subspace

$$V_1 := \langle\, \mathrm{trm}(x^{G_1}f_T) \mid T \in \mathrm{Tab}(d,d)\,\rangle = x^{2G_1}\langle\, f_{T'} \mid T' \in \mathrm{Tab}_{G_1}(d, d-c_1)\,\rangle,$$

which is actually the Specht module of shape $(d, d-c_1)$. In fact, the symmetric group $\mathfrak{S}_{[n]\setminus G_1}$ acts on V_1. The multiplication $\times(x^{\mathbf{a}}x^{F_1}/x^{G_1})$ gives a bijection from V_1 to $x^{\mathbf{a}}x^{F_1}x^{G_1}\langle\, f_{T'} \mid T' \in \mathrm{Tab}_{G_1}(d, d-c_1)\,\rangle$, to which $\mathrm{trm}(\psi)$ belongs, and this bijection preserves the monomial order. Hence $\mathrm{trm}(\psi)$ can be reduced to 0 modulo (3.3.3) by Lemma 3.5 (2). The case when only (ii) holds can be proved in the same way.

Next consider the case when both (i) and (ii) are satisfied. Set

$$V_1 := \langle\, \mathrm{trm}(x^{\mathbf{a}}x^{F_1}f_T) \mid T \in \mathrm{Tab}(d,d)\,\rangle \text{ and } V_2 := \langle\, \mathrm{trm}(x^{\mathbf{b}}x^{F_2}f_T) \mid T \in \mathrm{Tab}(d,d)\,\rangle.$$

Clearly, $\mathrm{trm}(x^{\mathbf{a}}\varphi_1) \in V_1$ and $\mathrm{trm}(x^{\mathbf{b}}\varphi_2) \in V_2$. If a monomial $x^{\mathbf{e}}$ appears as a non-zero term of $f \in V_1$ (resp. $f \in V_2$), then we have $\prod_{e_i \geq 2} x_i^{e_i-1} = x^{\mathbf{a}}x^{F_1}$ (resp. $\prod_{e_i \geq 2} x_i^{e_i-1} = x^{\mathbf{b}}x^{F_2}$). Hence if $x^{\mathbf{a}}x^{F_1} \neq x^{\mathbf{b}}x^{F_2}$, then $V_1 \cap V_2 = \{0\}$. Therefore, either $V_1 = V_2$ or $V_1 \cap V_2 = \{0\}$ holds. If $V_1 = V_2$, then we have $\mathrm{trm}(x^{\mathbf{a}}\varphi_1), \mathrm{trm}(x^{\mathbf{b}}\varphi_2) \in V_1$, and hence $\mathrm{trm}(\psi) \in V_1$. So the situation is essentially the same as the previous cases, and $\mathrm{trm}(\psi)$ is reduced to 0 modulo (3.3.3). If $V_1 \cap V_2 = \{0\}$, then we have

$$\mathrm{trm}(\psi) = \mathrm{trm}(x^{\mathbf{a}}\varphi_1) \pm \mathrm{trm}(x^{\mathbf{b}}\varphi_2) \in V_1 + V_2 = V_1 \oplus V_2,$$

and no terms of $\mathrm{trm}(x^{\mathbf{a}}\varphi_1)$ and $\mathrm{trm}(x^{\mathbf{b}}\varphi_2)$ are canceled. Hence $\mathrm{trm}(\psi)$ can be reduced to 0 modulo the subset

$$(\{\, \mathrm{trm}(x^{G_1}f_T) \mid T \in \mathrm{Tab}(d,d)\} \cup \{\, \mathrm{trm}(x^{G_2}f_T) \mid T \in \mathrm{Tab}(d,d)\}) \setminus \{0\}$$

of (3.3.2). In fact, the "V_1-part" and the "V_2-part" can be reduced to 0 individually. □

Corollary 3.8. *The Hilbert function of $J^{\mathrm{Sp}}_{(d,d)}$ does not depend on* $\mathrm{char}(K)$.

Proof. In general, a homogeneous ideal $I \subset R$ and its initial ideal $\mathrm{in}(I) := (\mathrm{in}(f) \mid 0 \neq f \in I)$ have the same Hilbert function, and the Hilbert function of a monomial ideal (e.g., $\mathrm{in}(I)$) does not depend on $\mathrm{char}(K)$. Since the Gröbner basis of $J^{\mathrm{Sp}}_{(d,d)}$ does not depend on $\mathrm{char}(K)$ by Theorem 3.7, the assertion follows. □

3.4 The proof of the main theorem and some examples

To prove Theorem 3.4, we can extend the base field. So we assume that $\#K = \infty$ in this section.

For a subset $\emptyset \neq F \subset [n]$, set $P_F := (x_i - x_j \mid i, j \in F) \subset R$. Clearly, this is a prime ideal with $\mathrm{ht}(P_F) = \#F - 1$. By [47, Theorem 3.1], $I^{\mathrm{Sp}}_{(n-d,d)}$ is a radical ideal, and hence we have

$$I^{\mathrm{Sp}}_{(n-d,d)} = \bigcap_{\substack{F \subset [n] \\ \#F = n-d+1}} P_F \tag{3.4.1}$$

by [47, Proposition 2.4].

Lemma 3.9 ([47])**.** *For $f \in R$, the following are equivalent.*

(1) $f \in I^{\mathrm{Sp}}_{(n-d,d)}$.

(2) Take $\mathbf{a} = (a_1, \ldots, a_n) \in K^n$. If there is a subset $F \subset [n]$ with $\#F = n - d + 1$ such that $a_i = a_j$ for all $i, j \in F$, then we have $f(\mathbf{a}) = 0$.

Proof. Easily follows from (3.4.1). □

Lemma 3.10. *If $n - d > d \geq 2$, then we have*

$$I^{\mathrm{Sp}}_{(n-d,d)} \cap S = I^{\mathrm{Sp}}_{(n-d-1,d)}.$$

Moreover, as graded S-modules, we have

$$(R/I^{\mathrm{Sp}}_{(n-d,d)})/(S/I^{\mathrm{Sp}}_{(n-d-1,d)}) \cong \bigoplus_{m \geq 1} (S/I^{\mathrm{Sp}}_{(n-d,d-1)})(-m). \tag{3.4.2}$$

Proof. For a subset $F \subset [n-1]$, we set $P'_F := (x_i - x_j \mid i, j \in F) \subset S$. It is easy to check that $P_F \cap S = P'_{F \setminus \{n\}}$ for $F \subset [n]$. For $F \subset [n-1]$ and $i \in F$, set $F' := (F \setminus \{i\}) \cup \{n\}$. Then we have $\mathrm{ht}(P_F) = \mathrm{ht}(P_{F'})$ but $(P_{F'} \cap S) \subsetneq (P_F \cap S)$, unless $F = \{i\}$ (in this case, $P_F = (0)$). Hence we have

$$\begin{aligned} I^{\mathrm{Sp}}_{(n-d,d)} \cap S &= \Bigg(\bigcap_{\substack{F \subset [n] \\ \#F = n-d+1}} P_F \Bigg) \cap S \\ &= \bigcap_{\substack{F \subset [n] \\ \#F = n-d+1}} (P_F \cap S) \\ &= \bigcap_{\substack{F \subset [n], n \in F \\ \#F = n-d+1}} (P_F \cap S) \\ &= \bigcap_{\substack{F \subset [n-1] \\ \#F = n-d}} P'_F \\ &= I^{\mathrm{Sp}}_{(n-d-1,d)}. \end{aligned}$$

Hence we have shown the first assertion, and we see that $S/I^{\mathrm{Sp}}_{(n-d-1,d)}$ can be seen as an S-submodule of $R/I^{\mathrm{Sp}}_{(n-d,d)}$ in the natural way. To show the second assertion, set $M := (R/I^{\mathrm{Sp}}_{(n-d,d)})/(S/I^{\mathrm{Sp}}_{(n-d-1,d)})$. This is a graded S-module.

We denote the image of $x_n^m \in R$ in M for $m \geq 1$ by $\overline{x_n^m}$. First, we show that $I^{\mathrm{Sp}}_{(n-d,d-1)} \subset (0 :_M \overline{x_n^m})$. It suffices to show that $f_T \overline{x_n^m} = 0$, equivalently, $f_T x_n^m \in S + I^{\mathrm{Sp}}_{(n-d,d)}$ for all

$$T = \begin{array}{|c|c|c|c|c|c|c|} \hline i_1 & i_2 & \cdots & i_{d-1} & i_d & \cdots & i_{n-d} \\ \hline j_1 & j_2 & \cdots & j_{d-1} \\ \cline{1-4} \end{array} \in \mathrm{Tab}(n-d, d-1).$$

We can prove this by induction on m. In fact, if we set

$$T' = \begin{array}{|c|c|c|c|c|c|c|c|} \hline i_1 & i_2 & \cdots & i_{d-1} & i_d & i_{d+1} & \cdots & i_{n-d} \\ \hline j_1 & j_2 & \cdots & j_{d-1} & n \\ \cline{1-5} \end{array} \in \mathrm{Tab}(n-d, d),$$

we have $f_{T'} = f_T \cdot (x_{i_d} - x_n)$, and hence $f_T x_n = f_T x_{i_d} - f_{T'} \in S + I^{\mathrm{Sp}}_{(n-d,d)}$ (note that $x_{i_d} \in S$). If $m > 1$, then

$$f_T x_n^m = (f_T x_n) x_n^{m-1} = (f_T x_{i_d} - f_{T'}) x_n^{m-1} = (f_T x_n^{m-1}) x_{i_d} - f_{T'} x_n^{m-1}.$$

Here $f_{T'} x_n^{m-1} \in I^{\mathrm{Sp}}_{(n-d,d)}$, and $f_T x_n^{m-1} \in S + I^{\mathrm{Sp}}_{(n-d,d)}$ by the induction hypothesis. Hence $(f_T x_n^{m-1}) x_{i_d} \in S + I^{\mathrm{Sp}}_{(n-d,d)}$, and

$$f_T x_n^m = (f_T x_n^{m-1}) x_{i_d} - f_{T'} x_n^{m-1} \in S + I^{\mathrm{Sp}}_{(n-d,d)}.$$

Next we show that

$$M \cong \bigoplus_{m \geq 1} S \cdot \overline{x_n^m}, \tag{3.4.3}$$

as S-modules. Here $S \cdot \overline{x_n^m}$ is the S-submodule generated by $\overline{x_n^m} \in M$, and it does not mean this is a free S-module. To do this, assume that

$$\sum_{i=1}^{l} f_i \overline{x_n^i} = 0$$

for some $l \geq 1$ and $f_1, \dots, f_l \in S$. Then we have

$$\sum_{i=1}^{l} f_i x_n^i \in S + I^{\mathrm{Sp}}_{(n-d,d)}.$$

Hence there is some $f_0 \in S$ such that

$$\sum_{i=0}^{l} f_i x_n^i \in I^{\mathrm{Sp}}_{(n-d,d)}.$$

Take $\mathbf{a} = (a_1, \dots, a_{n-1}) \in K^{n-1}$. If there is a subset $F \subset [n-1]$ with $\#F = n-d+1$ such that $a_i = a_j$ for all $i, j \in F$, then we have

$$\sum_{i=0}^{l} f_i(\mathbf{a}) b^i = 0$$

for all $b \in K$ by Lemma 3.9. Since $\#K = \infty$ now, we have

$$\sum_{i=0}^{l} f_i(\mathbf{a}) x_n^i = 0.$$

Hence we have $f_i(\mathbf{a}) = 0$ for all i, and $f_i \in I^{\mathrm{Sp}}_{(n-d,d-1)}$ by Lemma 3.9. Since we have shown that $I^{\mathrm{Sp}}_{(n-d,d-1)} \overline{x_n^i} = 0$ for $i \geq 1$, we get the direct sum (3.4.3). Moreover, the above argument also shows that $I^{\mathrm{Sp}}_{(n-d,d-1)} \supset (0 :_M \overline{x_n^i})$. Hence we have $I^{\mathrm{Sp}}_{(n-d,d-1)} = (0 :_M \overline{x_n^i})$, and $S \cdot \overline{x_n^i} \cong S/I^{\mathrm{Sp}}_{(n-d,d-1)}$. So we are done. $\square$

If $n = 2d + 1$, $I^{\mathrm{Sp}}_{(d,d,1)} \subset R$ is a radical ideal by [47, Theorem 4.2], and we have

$$I^{\mathrm{Sp}}_{(d,d,1)} = \bigcap_{\substack{F \subset [n] \\ \#F = d+1}} P_F \tag{3.4.4}$$

by [47, Proposition 2.4].

Lemma 3.11. *If $n = 2d \geq 4$, then we have*

$$I^{\mathrm{Sp}}_{(d,d)} \cap S = I^{\mathrm{Sp}}_{(d-1,d-1,1)}.$$

Moreover, as graded S-modules, we have

$$(R/I^{\mathrm{Sp}}_{(d,d)})/(S/I^{\mathrm{Sp}}_{(d-1,d-1,1)}) \cong \bigoplus_{m\geq 1}(S/I^{\mathrm{Sp}}_{(d,d-1)})(-m). \tag{3.4.5}$$

Proof. The assertion follows from the argument similar to the proof of Lemma 3.10, while we use (3.4.4) this time. □

Corollary 3.12. *If $n - d > d \geq 2$, then we have*

$$H(R/I^{\mathrm{Sp}}_{(n-d,d)}, t) = H(S/I^{\mathrm{Sp}}_{(n-d-1,d)}, t) + \frac{t}{1-t} H(S/I^{\mathrm{Sp}}_{(n-d,d-1)}, t). \tag{3.4.6}$$

Similarly, if $n = 2d$ and $d \geq 2$, we have

$$H(R/I^{\mathrm{Sp}}_{(d,d)}, t) = H(S/I^{\mathrm{Sp}}_{(d-1,d-1,1)}, t) + \frac{t}{1-t} H(S/I^{\mathrm{Sp}}_{(d,d-1)}, t). \tag{3.4.7}$$

Proof. The first assertion follows from (3.4.2). In fact, we have

$$\begin{aligned} H(R/I^{\mathrm{Sp}}_{(n-d,d)}, t) &= H(S/I^{\mathrm{Sp}}_{(n-d-1,d)}, t) + \sum_{m=1}^{\infty} t^m \cdot H(S/I^{\mathrm{Sp}}_{(n-d,d-1)}, t) \\ &= H(S/I^{\mathrm{Sp}}_{(n-d-1,d)}, t) + \frac{t}{1-t} H(S/I^{\mathrm{Sp}}_{(n-d,d-1)}, t). \end{aligned}$$

Similarly, the second assertion follows from (3.4.5). □

Now we can start the proof of the main theorem.

The proof of Theorem 3.4. We prove the assertion by induction on n. Note that we also have to prove that the Hilbert series of $R/I^{\mathrm{Sp}}_{(n-d,d)}$ and $R/I^{\mathrm{Sp}}_{(d,d,1)}$ do not depend on $\mathrm{char}(K)$.

It is easy to see that $R/I^{\mathrm{Sp}}_{(n-1,1)} \cong K[X]$, and its Hilbert series is $1/(1-t)$. So the assertion holds in this case. Similarly, $R/I^{\mathrm{Sp}}_{(1,1,1)}$ is a hypersurface ring of degree 3, and its Hilbert series is $(1+t+t^2)/(1-t)^2$. So the assertion also holds in this case.

We assume that the statement holds for $n-1$. If n is an odd number $2d+1$, we first treat $I^{\mathrm{Sp}}_{(d,d,1)}$. Recall that $I^{\mathrm{Sp}}_{(d,d)} \subset K[x_1, \dots, x_{n-1}]$, and let $\pi : R \to S$ ($\cong R/(x_n)$) be the natural surjection. Clearly, we have $S/\pi(I^{\mathrm{Sp}}_{(d,d,1)}) \cong R/(I^{\mathrm{Sp}}_{(d,d,1)} + (x_n))$. As shown in [47, §2], x_n is $R/I^{\mathrm{Sp}}_{\lambda}$-regular for any non-trivial partition λ of n. So we can recover the Hilbert series of $R/I^{\mathrm{Sp}}_{(d,d,1)}$ from that of $S/\pi(I^{\mathrm{Sp}}_{(d,d,1)})$.

Since $\pi(I^{\mathrm{Sp}}_{(d,d,1)}) = I^{\mathrm{Sp}}_{(d,d)} \cap \mathfrak{m}^{\langle d+1\rangle}$ by [47, Lemma 2.10], we have the short exact sequence

$$0 \to S/\pi(I^{\mathrm{Sp}}_{(d,d,1)}) \longrightarrow S/I^{\mathrm{Sp}}_{(d,d)} \oplus S/\mathfrak{m}^{\langle d+1\rangle} \longrightarrow S/J^{\mathrm{Sp}}_{(d,d)} \longrightarrow 0. \tag{3.4.8}$$

The Hilbert series of $S/I^{\mathrm{Sp}}_{(d,d)}$ (resp. $S/J^{\mathrm{Sp}}_{(d,d)}$) do not depend on $\mathrm{char}(K)$ by the induction hypothesis (resp. Corollary 3.8). Since $\mathfrak{m}^{\langle d+1\rangle}$ is a monomial ideal, its Hilbert series is also characteristic free. Hence $H(S/\pi(I^{\mathrm{Sp}}_{(d,d,1)}), t)$ does not depend on $\mathrm{char}(K)$ by (3.4.8). So the same is true for $H(R/I^{\mathrm{Sp}}_{(d,d,1)}, t)$.

So we may assume that $\mathrm{char}(K) = 0$, then $R/I^{\mathrm{Sp}}_{(d,d,1)}$ is Cohen-Macaulay. The number of minimal generators of $I^{\mathrm{Sp}}_{(d,d,1)}$ is $\#\,\mathrm{SYT}(d,d,1)$, and we have

$$\#\,\mathrm{SYT}(d,d,1) = \frac{(2d+1)!}{(d+2)d!(d+1)(d-1)!} = \binom{2d+1}{d+2}$$

by the hook formula (c.f., [32, Theorem 3.10.2]). Since $R/I^{\mathrm{Sp}}_{(d,d,1)}$ is Cohen-Macaulay and $\mathrm{ht}(I^{\mathrm{Sp}}_{(d,d,1)}) = d$, $I^{\mathrm{Sp}}_{(d,d,1)}$ has a $(d+2)$-linear resolution by [31, Theorem 3.2]. Hence,

$$H(R/I^{\mathrm{Sp}}_{(d,d,1)}, t) = \frac{\sum_{i=0}^{d+1} \binom{i+d-1}{i} t^i}{(1-t)^{d+1}}$$

by [42, Exercises 5.3.16]. So the assertion holds in this case.

Next, we consider the Hilbert series of $R/I^{\mathrm{Sp}}_{(n-d,d)}$ with $n-d > d \geq 2$ (without the assumption that $\mathrm{char}(K) = 0$). By the induction hypothesis, $H(S/I^{\mathrm{Sp}}_{(n-d-1,d)}, t)$ and $H(S/I^{\mathrm{Sp}}_{(n-d,d-1)}, t)$ do not depend on $\mathrm{char}(K)$, and we have

$$H(S/I^{\mathrm{Sp}}_{(n-d-1,d)}, t) = \frac{1 + h'_1 t + h'_2 t^2 + \cdots + h'_d t^d}{(1-t)^d}$$

with

$$h'_i = \begin{cases} \binom{n-d+i-2}{i} & \text{if } 1 \leq i \leq d-1, \\ \binom{n-2}{d-2} & \text{if } i = d, \end{cases}$$

and

$$H(S/I^{\mathrm{Sp}}_{(n-d,d-1)}, t) = \frac{1 + h''_1 t + h''_2 t^2 + \cdots + h''_{d-1} t^{d-1}}{(1-t)^{d-1}}$$

with

$$h''_i = \begin{cases} \binom{n-d+i-1}{i} & \text{if } 1 \leq i \leq d-2, \\ \binom{n-2}{d-3} & \text{if } i = d-1. \end{cases}$$

Since $\dim R/I^{\mathrm{Sp}}_{(n-d,d)} = d$, the denominator of the Hilbert series of $H(R/I^{\mathrm{Sp}}_{(n-d,d)}, t)$ is $(1-t)^d$, and the numerator is

$$1 + (h'_1 + 1)t + (h'_2 + h''_1)t^2 + \cdots + (h'_{d-1} + h''_{d-2})t^{d-1} + (h'_d + h''_{d-1})t^d$$

by (3.4.6). Now, we have

$$h'_1 + 1 = (n-d-1) + 1 = n-d,$$

$$h'_i + h''_{i-1} = \binom{n-d+i-2}{i} + \binom{n-d+i-2}{i-1} = \binom{n-d+i-1}{i}$$

for $2 \le i \le d-1$, and

$$h'_d + h''_{d-1} = \binom{n-2}{d-2} + \binom{n-2}{d-3} = \binom{n-1}{d-2}.$$

So the assertion holds in this case.

Next, assuming that $n = 2d$, we consider the Hilbert series of $R/I^{\mathrm{Sp}}_{(d,d)}$ with $d \ge 2$. By the induction hypothesis, $H(S/I^{\mathrm{Sp}}_{(d,d-1)}, t)$ does not depend on $\mathrm{char}(K)$, and we have

$$H(S/I^{\mathrm{Sp}}_{(d,d-1)}, t) = \frac{1 + h''_1 t + h''_2 t^2 + \cdots + h''_{d-1} t^{d-1}}{(1-t)^{d-1}}$$

with

$$h''_i = \begin{cases} \binom{d+i-1}{i} & \text{if } 1 \le i \le d-2, \\ \binom{2d-2}{d-3} & \text{if } i = d-1. \end{cases}$$

We have already shown that

$$H(S/I^{\mathrm{Sp}}_{(d-1,d-1,1)}, t) = \frac{1 + h'_1 t + h'_2 t^2 + \cdots + h'_d t^d}{(1-t)^d}$$

with

$$h'_i = \binom{d+i-2}{i}.$$

Since $\dim R/I^{\mathrm{Sp}}_{(d,d)} = d$, the denominator of the Hilbert series of $H(R/I^{\mathrm{Sp}}_{(d,d)}, t)$ is $(1-t)^d$, and the numerator is

$$1 + (h'_1 + 1)t + (h'_2 + h''_1)t^2 + \cdots + (h'_{d-1} + h''_{d-2})t^{d-1} + (h'_d + h''_{d-1})t^d$$

by (3.4.7). Now, we have

$$h'_1 + 1 = (d-1) + 1 = d,$$

$$h'_i + h''_{i-1} = \binom{d+i-2}{i} + \binom{d+i-2}{i-1} = \binom{d+i-1}{i}$$

for $2 \le i \le d-1$, and

$$h'_d + h''_{d-1} = \binom{2d-2}{d} + \binom{2d-2}{d-3} = \binom{2d-2}{d-2} + \binom{2d-2}{d-3} = \binom{2d-1}{d-2}.$$

So we are done. □

Example 3.13. [47, Proposition 5.1, Proposition 5.2] We state the case of $d = 2$ as an example. For $R/I^{\mathrm{Sp}}_{(n-2,2)}$,

$$H(R/I^{\mathrm{Sp}}_{(n-2,2)}, t) = \frac{1 + (n-2)t + t^2}{(1-t)^2},$$

and $\operatorname{reg}(R/I^{\mathrm{Sp}}_{(n-2,2)}) = 2$. This case has already been studied in [47]. In particular, it has been proved that $R/I^{\mathrm{Sp}}_{(n-2,2)}$ is Gorenstein over any field K. See [47, Proposition 5.2]

In [22], they have the following result about the Cohen-Macaulay property of Specht ideals $I^{\mathrm{Sp}}_{(n-d,d)}$.(They have also studied some cases with positive characteristic.)

Theorem 3.14. *[22, Theorem 1.2] Let K be any field of $p = \operatorname{char}(K) \geq 0$, and fix positive integers n, d satisfying $n \geq 2d$. If $p = 0$ or $p \geq d$, then $R/I^{\mathrm{Sp}}_{(n-d,d)}$ is Cohen-Macaulay.*

Corollary 3.15. *Let K be any field of $p = \operatorname{char}(K) \geq 0$. If $p = 0$ or $p \geq d$, then* $\operatorname{reg}(R/I^{\mathrm{Sp}}_{(n-d,d)}) = d$.

Proof. We assume that $p = 0$ or $p \geq d$. By Theorem 3.14, $R/I^{\mathrm{Sp}}_{(n-d,d)}$ is Cohen-Macaulay. Hence, by using Theorem 3.2, we get $\operatorname{reg}(R/I^{\mathrm{Sp}}_{(n-d,d)}) = d$. □

Example 3.16. We consider the Specht ideal $I^{\mathrm{Sp}}_{(5,4)}$. If $\operatorname{char}(K) = 0$ or $\operatorname{char}(K) \geq 5$, then

$$\operatorname{reg}(R/I^{\mathrm{Sp}}_{(5,4)}) = 4$$

This is known from Corollary 3.15.

Remark 3.17. Let $p = \operatorname{char}(K) \geq 0$. In [47, Conjecture 5.5], Yanagawa conjectures that $R/I^{\mathrm{Sp}}_{(d,d,1)}$ is Cohen–Macaulay if and only if $p = 0$ or $\operatorname{char}(K) \geq d + 1$. We assume that this conjecture is true. Then $\operatorname{reg}(R/I^{\mathrm{Sp}}_{(d,d,1)}) = d + 1$, if $p = 0$ or $\operatorname{char}(K) \geq d + 1$, by using Theorem 3.2.

Example 3.18. We consider the Specht ideal $I^{\mathrm{Sp}}_{(3,3,1)}$. If $\operatorname{char}(K) = 0$, then

$$\operatorname{reg}(R/I^{\mathrm{Sp}}_{(3,3,1)}) = 4$$

This is known from Corollary 3.2. Note that if [47, Conjecture 5.5] is true, then $\operatorname{reg}(R/I^{\mathrm{Sp}}_{(3,3,1)}) = 4$, when $\operatorname{char}(K) \geq 5$.

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