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Parallels and Divergencies: Gödel and von Neumann*

I. A WEAK CLAIM

John von Neumann (1903–1957) and Kurt Gödel (1906–1974) are two towering figures of 20th century science, contributing in particular to mathematics in exceptionally significant ways that had a lasting impact on modern mathematics. Their life and scientific careers had many parallels and their research interests overlapped. But their philosophical views about sciences, especially about the nature and foundations of mathematics were very different. The aim of this paper is to highlight some parallels and what appears to be a correlation between the divergences of their philosophical positions and differences in their scientific research and career. Correlation is not causation in general. So no simplistic claim is formulated here about either scientific research determining the nature of a philosophical position or about their philosophical views setting their research agenda. Rather, on their example it should become clear that scientific research and philosophical views are intertwined and they both enfold conditioned both by personal traits and by a broader social context, some of which the paper indicates.

II. BEFORE THE 1930 KÖNIGSBERG ENCOUNTER

Both von Neumann and Gödel were born in the Austro-Hungarian Empire: Gödel in Brünn/Brno, Bohemia; von Neumann in Budapest, Hungary. Their life and careers have been reviewed by several biographers (see Feferman 1986, Buldt et al. 2006. Chap. B), especially Köhler 2006a and Köhler 2006b for Gödel, and Macrae 1992, Aspray 1990, Rédei 2005 for von Neumann). Below I rely on these sources when it comes to recalling some episodes from their life and career.

The families they were born into were both well-to-do, headed by a father working successfully in textile industry (Gödel's father) and in banking (von Neumann's father). The financial security provided by the family environments

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made it possible to realize talents, first by receiving solid elementary education, and, subsequently, allowing to benefit from the higher education provided by universities in the German speaking segment of the European university system: Gödel studied in Vienna, von Neumann in Berlin and in Zürich.

Gödel, after considering physics as a field of study, finally chose to study mathematics in Vienna University. His teacher was Hans Hahn, a major figure in functional analysis ("Hahn-Banach Theorem"). Although von Neumann was already a reasonably trained mathematician at the time of graduating from high school, which was due to private tutoring he had received from a university professor, he enrolled in the chemical engineering program in the *Eidgenössische Technische Hochshule* in Zürich. Simultaneously, he registered as a PhD student in mathematics in Budapest. Gödel's PhD (1930) was in logic, proving completeness of first order logic, von Neumann's PhD (1926) presented a new axiomatization of set theory. Both PhD's were major contributions to logic and mathematics, respectively – a very similar start of their academic careers.

In Vienna Gödel was in touch with the philosophers in the Vienna Circle from 1926 but he distanced himself intellectually from this circle because "he had developed strong philosophical views of his own which were, in large part almost diametrically opposed to the views of the logical positivists" (Feferman 1986. 4). According to Gödel's reply to a questionnaire, he had embraced a realist philosophy of mathematics by 1925 (Gödel 1986. 37). Such a philosophy of mathematics was in sharp contrast to the logicist understanding of the nature of mathematics adopted by the Vienna Circle. So, from 1931 Gödel started abandoning the Vienna Circle meetings and from 1933 he stopped attending completely (Köhler 2006a).

Von Neumann did not have contacts to the Vienna Circle – or to any significant philosophical school – during this time. His interest in philosophy was weak at best, and at that time was restricted to the rather internal, technical issues of the Hilbert program. He hoped to be able to help to show that the Hilbert program can succeed. Accordingly, von Neumann was regarded as the major representative of the formalist understanding of mathematics. But to the extent this classification of von Neumann's view is correct, it is only so by qualification: he was a moderate formalist, emphasizing the importance of the intuitive content behind the concepts in formal axiomatization. This is expressed already in his axiomatization of set theory (von Neumann 1928):

> We begin with describing the system to be axiomatized and with giving the axioms. This will be followed by a brief clarification of the meaning of the symbols and axioms […]. It goes without saying that in axiomatic investigations as ours, expressions such as "meaning of a symbol" or "meaning of an axiom" should not be taken literally: these symbols and axioms do not have a meaning at all (in principle at least), they only represent (in more or less complete manner) certain concepts of the untenable "naive set theory". Speaking of "meaning" we always intend the meaning of the concepts taken from "naive set theory". (Taub 1961. 344, translation from Rédei and Stöltzner 2006.)

After receiving his PhD von Neumann went to Göttingen to work as Hilbert's assistant; apparently with the intention of continuing his work on the Hilbert problem (von Neumann 1927d); however, in Göttingen his interest turned to the mathematical foundations of quantum mechanics.

The publication of the three foundational papers on quantum mechanics (von Neumann 1927a, von Neumann 1927c, von Neumann 1927b) marks a significant deviation of von Neumann's scientific interest from that of Gödel. Not just in the sense that von Neumann's attention gets diverted from the problems of mathematical logic and foundations of mathematics to the foundations of physics – while Gödel was working on his dissertation on completeness –; but, more importantly, the pure mathematical problems von Neumann solves in these papers (first and foremost the spectral theory of unbounded selfadjoint operators defined on an abstract Hilbert space) are obviously directly motivated by the problem situation in the sciences (physics). This type of mathematical work, which is growing out from the empirical sciences, is uncharacteristic of Gödel – a divergence between von Neumann and Gödel about which more will be said below, and which is already present at this early stage of their career. This is due to some extent to the contingent fact that Göttingen was a major center of theoretical physics where the newest results of the emerging quantum mechanics were followed and Hilbert happened to be lecturing on the foundations of quantum theory in 1926.

Working on foundations of physics in Göttingen von Neumann also had to deal with a problem which, to the best of my knowledge, Gödel did not address systematically: the problem of the nature of the axiomatic approach in the context of *empirical* sciences. The problem of how to carry out an axiomatization of an empirical science, which goes back to Hilbert's 6th problem *Mathematical Problems. Lecture delivered before the International Congress of Mathematicians at Paris in 1900* (Hilbert 1976; see also Wightman 1976 and Corry 1997), and which is very different from the problem of axiomatization within mathematics and logic. Von Neumann addresses this problem explicitly first in his joint publication with Hilbert and Nordheim (Hilbert et al. 1927). The position they work out is a characteristic mixture of formal axiomatics and informal but explicit stipulations linking mathematics to empirical postulates. This position was dubbed "opportunistic soft axiomatics" in the papers Stöltzner 2001, Stöltzner 2004, Rédei and Stöltzner 2006, Rédei 2005, where the details of this concept can be found, together with an illustration of this sort of axiomatization by the example of (non-relativistic) quantum mechanics as systematized by von Neumann in his book von Neumann 1932. What is relevant from the perspective of the comparison of Gödel's and von Neumann's views is that for von Neumann this sort of "soft" axiomatization is, again, directly motivated by the problem situation in empirical science (physics). Furthermore, this concept takes into account the actual practice of creating mathematical models of physical phenomena.

III. THE 1930 KÖNIGSBERG ENCOUNTER

The world lines of Gödel and von Neumann crossed the first time at the Königsberg conference in 1930, and their meeting coincided with the well-known substantial turn in the history of logic and hence philosophy of mathematics: it was during this conference that Gödel announced his first incompleteness theorem the first time in public. The main events at (and right after) the conference are described in Sieg's introductory comments (Gödel 2003. 329–335) to the von Neumann–Gödel correspondence (see also Köhler 2006a). The essential points are the following: von Neumann, after learning from Gödel the existence of undecidable propositions, proved the second incompleteness theorem independently and reported on this to Gödel in a letter dated November 20, 1930. But by then Gödel had also arrived at this result and had in fact submitted his paper containing this result on November 17. Von Neumann, acknowledging Gödel's priority, did not wish to publish on the matter (von Neumann's letter to Gödel, November 29, 1930. Gödel 2003. 339–340).

From the perspective of parallels and divergences between von Neumann and Gödel the remarkable aspect of the von Neumann-Gödel exchange right after the Königsberg conference is that they sharply disagreed on the philosophical significance of the second incompleteness theorem: von Neumann declared:

> Thus, I think that your result has solved negatively the foundational question: there is no rigorous justification for classical mathematics. What sense to attribute to our hope, according to which it is de facto consistent, I do not know – but in my view that does not change the completed fact. (Von Neumann to Gödel, November 29, 1930. Gödel 2003. 339–340.)

Von Neumann held this position consistently from the moment of discovery of the second incompleteness theorem and he expressed it unambiguously several times: in a letter to Carnap in which he discusses the publication of the Königsberg talks and also in a letter to his Hungarian friend, the physics professor in Budapest, Rudolf Ortvay. The relevant passages from these letters are as follows:

To Ortvay:

> Gödel's results mean that there is no "complete" axiomatic system, not even in mathematics, and I believe that there is actually no other consistent interpretation of this complex of questions. (Von Neumann to Ortvay, July 18, 1939. Rédei 2005.)

To Carnap:

> (1) Gödel has shown the unrealizability of Hilbert's program.[1]
> (a) There is no more reason to reject intuitionism (if one disregards the aesthetic issue, which in practice will also for me be the decisive factor).
> Therefore I consider the state of the foundational discussion in Königsberg to be outdated, for Gödel's fundamental discoveries have brought the question to a completely different level. (I know that Gödel is much more careful in the evaluation of his results, but in my opinion on *this* point he does not see the connections correctly). (Von Neumann to Carnap June 7, 1931. Rédei 2005; also see Mancosu 1999.)

Gödel disagreed with this interpretation; at least initially, in 1931:

> I wish to note expressly that Theorem XI [the second incompleteness theorem] does not contradict Hilbert's formalistic viewpoint. For this viewpoint presupposes only the existence of a consistency proof in which nothing but finitary means of proof is used, and it is conceivable that there exists finitary proofs that *cannot* be expressed in the formalism of P [Russell's Principia plus the Peano axioms]. (Gödel 1931, Gödel 1986. 195.)

The disagreement between Gödel and von Neumann is explained by their different interpretations of intuitionism and finitism: for von Neumann these were essentially the same from the start whereas Gödel regarded finitism a narrower concept. Identifying finitism with the Hilbert program Gödel thus came into agreement with von Neumann's evaluation of the significance of the second incompleteness theorem in 1933 (see Sieg's description for more details about Gödel's changing position and eventual agreement with von Neumann's interpretation of the second incompleteness theorem; Gödel 2003. 332).

Although Gödel's and von Neumann's views on the interpretation of the second incompleteness theorem converged eventually, they diverged in the more informal philosophical conclusions they had drawn from the failure of the Hilbert program. The divergence was both explicit and tacit: it got formulated explicitly as a Platonist philosophy of mathematics in the philosophical works of Gödel and it led to an empiricist concept of mathematics in the philosophical reflections by von Neumann; furthermore, it manifested in a tacit manner in the different types of mathematical research they carried out.

[1] Von Neumann's footnote: "I would like to emphasize: nothing in Hilbert's aims is false. Could they be carried out then it would follow from them absolutely what he claims. But they cannot be carried out, this I know only since September 1930."

IV. DIVERGENT CONCLUSIONS FROM THE INCOMPLETENESS THEOREM

Von Neumann never tried to write philosophy systematically[2], Gödel did. In fact, from about 1943, "[…] Gödel devoted himself almost entirely to the philosophy of mathematics and then to general philosophy and metaphysics" (Feferman 1986. 13). By that time Gödel was at the Institute of Advanced Study (IAS) in Princeton – just like von Neumann. Von Neumann got appointed in 1933, soon after the IAS had been established. Gödel visited IAS three times before settling there permanently in 1940. It was during those visits that Gödel found the proofs of relative independence in ZF of the axiom of choice (1935) and continuum hypothesis (1937) – Gödel's other two major contributions to mathematics. Von Neumann was fully aware of these achievements and he played a crucial role in arranging Gödel's permanent appointment to IAS, when Gödel desperately tried to leave Asutria in 1939: he urged IAS to try to secure a special visa for Gödel. In a letter to Veblen von Neumann writes:

> The claim may be made with perfect justification that Gödel is unreplaceable for our educational program. Indeed Gödel is absolutely irreplaceable; he is the only mathematician alive about whom I would dare to make this statement. He represents a very important branch of mathematics, formal logics, in which he outranks everybody else to a much higher degree than usually happens in any other branch of mathematics. Indeed, the entire modern development of formal logics concerning "undecidable questions", the solution of the famous "continuum hypothesis", and quite unexpected connections between this field and other parts of mathematics, are his entirely individual contribution. Besides, the ouvre of his scientific achievements is obviously still in steep ascent, and more is to be expected from him in the future. I am convinced that salvaging him from the wreck of Europe is one of the great single contributions anyone could make to science at this moment. (Von Neumann to Veblen September 27, 1939. Rédei 2005.)

The expectation expressed in von Neumann's letter about further major contributions to mathematics by Gödel were not really met. It has been found puzzling why from the early 1940s Gödel's interest changed to philosophy from mathematics, where he proved so brilliant (Köhler 2006b). It sure is part of the answer that Gödel, by nature being an introverted person, needed congenial stimulus, discussions with colleagues, and it was unfortunate that the most suitable colleague to exchange ideas with, namely von Neumann, was mainly away

[2] His reservation to write philosophical papers came to the surface when he declined an invitation to a philosophy conference when the invitation was coupled with the expectation of writing up his contribution in form of a paper. (See von Neumann's letter to Ernest Nagel December 9, 1953. Rédei 2005.)

from IAS doing war-work (Köhler 2006b). But this divergence between Gödel and von Neumann was not simply a contingent, unfortunate circumstance caused by war. It was already a consequence of a difference in attitudes towards mathematics – a difference in philosophy of mathematics.

Gödel embraced a realist-platonist concept of mathematics. Perhaps the most important (Köhler 2006a) articulation of his platonistic philosophy is his paper prepared for the Gibbs Lecture in 1951. Gödel intended to publish this paper; however, the paper remained a hand-written manuscript that only got published in 1995 (Gödel 1951). One of the main claims of this paper is that mathematics is *incompletable*, *inexhaustible*. The main argument in favor of this claim uses the second incompleteness theorem:

> It is *this* theorem which makes the incompletability of mathematics particularly evident. For *it makes it impossible that someone should set up a certain well-defined system of axioms and rules and consistently make the following assertion about it: All of these axioms and rules I perceive (with mathematical certitude) to be correct, and moreover I believe that they contain all of mathematics*. If someone makes such a statement he contradicts himself. For if he perceives the axioms under consideration to be correct, he also perceives (with the same certainty) that they are consistent. Hence he has a mathematical insight not derivable from his axioms. (Gödel 1951. 309; emphasis in original.)

From this incompletability argument Gödel draws the following "disjunctive conclusion" (Gödel 1951. 310):

> *Either mathematics is incompletable in this sense, that its evident axioms can never be comprised in a finite rule, that is to say, the human mind (even within the realm of pure mathematics) infinitely surpasses the power of any finite machine, or else there exist absolutely unsolvable diophantine problems*… (Gödel 1951. 310; emphasis in original.)

The further consequence of this (non-exclusive) disjunction is a non-mechanistic, non-materialistic concept of the human mind (if one takes the first component of the disjunction). The second component of the disjunction "…seems to disprove the view that mathematics is only our own creation…" (Gödel 1951. 311) because

> So this alternative seems to imply that mathematical objects and facts (or at least *something* in them) exist objectively and independently of our mental acts and decisions, that is to say […] some form or other of Platonism or "realism" as to the mathematical objects. (Gödel 1951. 211–312.)

On the basis of the position that mathematics is not a human creation Gödel also criticizes the logical positivists concept of mathematics ("logicism", Gödel calls it “conventionalism”), but his criticism is not an outright rejection. He acknowledges that the logicist position is right about claiming that mathematics does not state anything about the physical world because mathematical statements are true “…already owing to the meaning of the terms occurring in it, irrespectively of the world of real things” (Gödel 1951. 320).

> What is wrong, however, is that the meaning of these terms (that is, the concepts they denote) is asserted to be something man-made and consisting merely in semantical conventions. The truth, I believe, is that these concepts form an objective reality of their own, which we cannot create of change, but only perceive and describe. (Gödel 1951. 320.)

Since von Neumann did not write papers on philosophy of mathematics proper, one has to interpret the nature of his mathematical research and rely on his semi-popular writings to get a picture of how he saw the features of mathematics. The major source in this connection is his 1947 paper (von Neumann 1961), in which he addresses philosophical questions about mathematics, in particular the consequences of the second incompleteness theorem.

Von Neumann’s first main conclusion from the second incompleteness theorem is that the concept of mathematical rigor is not something that one can establish once and for all. Rather, he regards it as historically changeable. There is no absolute, fixed notion of precision, clarity and exactness:

> Whatever philosophical or epistemological preferences anyone may have in this respect, the mathematical fraternities’ actual experiences with its subject give little support to the assumption of the existence of an a priori concept of mathematical rigor (von Neumann 1961. 6).

From the changeability of the concept of mathematical rigor von Neumann draws another conclusion, which is very characteristic for his concept of mathematics: he thinks that “the variability of the concept of rigor shows that something else besides mathematical abstraction must enter into the makeup of mathematics” (von Neumann 1961. 4). What is this “something else”? von Neumann is very careful in his answer. He says that the case “in favor of the empirical nature of this extra content” is strong – without claiming that such a position is defensible without reasonable doubt. But the whole spirit and trust of his paper von Neumann 1961 – and his activity as a mathematician – clearly indicate that this is the position he thinks is the right one:

> The most vitally characteristic fact about mathematics is, in my opinion, its quite peculiar relationship to the natural sciences, or, more generally, to any science which interprets experience on a higher than purely descriptive level (von Neumann 1961. 4).

A specific aspect of this relationship is that

> It is undeniable that some of the best inspirations in mathematics – *in those parts of it which are as pure mathematics as one can imagine* – have come from the natural sciences (von Neumann 1961. 2; my emphasis).

Von Neumann gives geometry and calculus as "monumental" examples for mathematical theories that have empirical origins – but he could have mentioned many more. In fact, von Neumann's own mathematical activity enfolded in a way that it is itself evidence for the truth of his claim: his early work on functional analysis mentioned earlier grew out of the problem of mathematical modeling of quantum phenomena; his work on ergodic theory originates in Boltzmann's work on classical statistical mechanics; the theory of von Neumann algebras ("rings of operators") emerged in the context of general quantum theory partly out of the need to decompose (factorise) quantum systems into subsystems; the theory of continuous geometry has its origins in quantum logic. These areas are part of pure mathematics; yet, they clearly originate in problem situations in physics. But von Neumann's position is even broader in the sense that he regards fields other than physics as a potential source of mathematical concepts and knowledge. Economics is the prime example: for von Neumann it was the source and motivation to develop game theory. Paying close attention to sciences, formulating mathematical concepts, isolating structures, grasping the content of the scientific situation in terms of axioms and investigating their consequences, is very characteristic of von Neumann's mathematical research, and it is in full harmony with his picture of what is essential about mathematics. In this he diverges from Gödel significantly. Gödel typically did not work on mathematical problems arising from the sciences, physics in particular. When he worked on problems related to physics, the motivation came not from physics proper but rather from philosophy: for instance in Gödel's work on relativity theory, when he proved that the Einstein equations admit a solution in which closed time-like curves appear (Gödel 1949a), the motivation, according to his own account, was his desire to clarify the relation between relativity theory and Kant's philosophy of time (Gödel 1949b. 274) (see also Stöltzner 2006. 289).

This is not to say that Gödel regarded mathematics and physics as completely separate. In his criticism of Carnap's logicist position about meaninglessness of mathematical statements he writes:

> If it is argued that mathematical propositions have no content because, by themselves, they imply nothing about experiences, the answer is that the same is true of laws of nature. For laws of nature without mathematics or logic imply as little about experiences as mathematics without laws of nature. (Gödel 1953. 360.)

In an example following the above quotation Gödel describes how mathematics actually does add genuine content to natural laws (see Stöltzner 2006. 293) for further, current examples elaborating this idea). Yet, it is fair to say that Gödel, unlike von Neumann, did not consider physics (empirical sciences more generally) as a crucial source of mathematics, without which the nature of mathematics cannot be understood.

In one respect the von Neumann and Gödel concepts of mathematics are parallel though. Both think that mathematics is not an arbitrary creation: von Neumann's position entails that mathematical content is coming to us from the natural and social world mediated through natural and social sciences, physics and economics in particular. Gödel would not say this, but to the extent mathematics is not (fully) our creation, he regards it as allowing a somewhat empiricist position much like in connection with physics:

> This whole consideration incidentally shows that the philosophical implications of the mathematical facts explained do not lie entirely on the side of rationalistic or idealistic philosophy, but that in one respect they favor an empiricist viewpoint.[3] (Gödel 1951. 313.)

Gödel even goes as far as drawing the conclusion that, as a result

> If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics (Gödel 1951. 313).

It is perhaps too strong to characterize this parallel between von Neumann and Gödel's pictures of mathematics by saying that "Through the rejection of conventionalism (by Gödel) the strict limit between empirical and mathematical truths disappears" (Stöltzner 2006. 292) but it is clear that neither Gödel nor von Neumann regarded mathematics as subjective or arbitrary.

Another idea formulated by von Neumann underscores the importance of empirical origin of mathematics. Von Neumann acknowledges that once the mathematical concepts needed to form a mathematical model of an extra-mathematical phenomenon has been obtained, they take on their own life, they develop internally and after a while the resulted mathematics gets so far from its origin that those origins are hard to trace or recognize. Von Neumann mentions specifically the axiom of choice and the continuum hypothesis, precisely the two major problems to which Gödel made substantial contributions in the 1930s, as examples of issues to which mathematics has been led following its internal

[3] Gödel's footnote: "To be more precise, it suggests that the situation in mathematics is not so very different from that in the natural sciences" (Gödel 1951. 313).

development (von Neumann 1961). This von Neumann regards as a perfectly normal process that is part of the ordinary workings of mathematics. There is however, in von Neumann's view, also a danger lurking in this internal development:

> As a mathematical discipline travels far from its empirical source, or still more, if it is a second and third generation only indirectly inspired by ideas coming from "reality" it is beset with very grave dangers. It becomes more and more purely aestheticizing, more and more purely I'art pour I'art. This need not be bad, if the field is surrounded by correlated subjects, which still have closer empirical connections […] But there is a grave danger that the subject will develop along the line of least resistance, that the stream, so far from its source, will separate into a multitude of insignificant branches, and that the discipline will become a disorganized mass of details and complexities. In other words, at a great distance from its empirical source, or after much "abstract" inbreeding, a mathematical subject is in danger of degeneration. […] whenever this stage is reached, the only remedy seems to me to be the rejuvenating return to the source: the re-injection of more or less directly empirical ideas. (Von Neumann 1961. 9.)

V. CLOSING COMMENTS

The just described divergence in their views about the nature of mathematics would probably have had the consequence that Gödel and von Neumann would not have interacted too much at IAS even if von Neumann had not been increasingly involved in war work during the 1940s as the war was raging on. In this connection one could mention that von Neumann did not interact too much with the another prominent member of IAS, Einstein, either – in spite the fact that one would expect this, given their major roles and interest in foundations and philosophy of quantum mechanics. (See Rédei 2011 for mentioning an episode of interaction between them on foundations of quantum mechanics.) It seems that a more substantial cooperation between Einstein and von Neumann was hindered by Einstein being *less* mathematically minded than ideal for a fruitful exchange of ideas with von Neumann, whereas Gödel was *more* inspired by pure mathematics than ideal for a useful and more intensive intellectual contact between Gödel and von Neumann at that time.

After the war the careers of Gödel's and von Neumann diverged wildly: von Neumann got involved more and more in applied research (computer development) and government advising, which culminated in his appointment as Atomic Energy Commissioner (1954). In this change of von Neumann's career other philosophical considerations, unrelated to philosophy of mathematics, played a role: von Neumann thought that scientists should get involved in government-military advising for moral reasons (Rédei 2005. Introduction). Gödel remained fully within the protected walls of academia and continued his work, mainly pursuing philosophy. What started in the Austro-Hungarian Empire as very similar academic careers and which developed subsequently through overlapping scientific interests thus ended in the U.S.A. as radically different. In the divergence philosophical differences about the abstract epistemological nature of mathematics and its relation to sciences played a role, and conversely: these views were shaped by the scientific research they carried out.

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