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# Operads, Symmetric Monoidal Categories, and Localizations

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**Operads, Symmetric Monoidal Categories, and Localizations**

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## DEDICATION

For Steffen Poltak,

who never doubted that I could do this and who always knew what I needed before I did.

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**ABSTRACT**

An operad can be thought of as a collection of operations, each with a finite number of inputs and a single output, along with a composition rule. We prove that the category of operads in an appropriate concrete symmetric monoidal category $\mathcal{V}$ is equivalent to a subcategory of symmetric monoidal categories enriched in $\mathcal{V}$. Though versions of this result have appeared previously in the literature, we prove that a more restrictive subcategory is needed to construct the equivalence. Our subcategory has the advantage that its objects share important properties with the historical precursor to operads, PROPs.

We also review a localization construction for operads, called the tree hammock localization. Using the above equivalence, we compare this construction to the hammock localization for categories. We believe that these two localization constructions should be suitably equivalent, and present ongoing work on this conjecture using simplicial categories and $\infty$-categories.

# CHAPTER 1

## Introduction

An operad $\mathcal{O}$ in a suitable symmetric monoidal category $\mathcal{V}$ can be thought of as a collection of operations, each with $n$ inputs and a single output, along with a symmetric action and structure maps that give an associative and unital composition. We primarily consider operads in a concrete symmetric monoidal $\mathcal{V}$, and our main result is

**Theorem 1.0.1.** *The category of operads in $\mathcal{V}$ is equivalent to a subcategory of symmetric monoidal categories enriched in $\mathcal{V}$, called $\mathsf{SM}_{\mathsf{O}}$.*

Versions of this theorem have appeared in [BBP+18, Proposition 3.1] and [MZZ20, Theorem 10.10]. However, both sources only partially describe the necessary properties for the objects in $\mathsf{SM}_{\mathsf{O}}$, and do not specify the morphisms. We give a complete definition of the objects and morphisms and carefully construct the functors giving the equivalence.

One major advantage of our description of $\mathsf{SM}_{\mathsf{O}}$ is that the objects clearly fit the definition of Mac Lane's PROPs [Lan65]. Though the name operad was coined by May [May72], they were initially described by Boardman and Vogt as "PROPs in standard form" [Vog98], or PROPs that can be uniquely determined by the morphisms from $n$ to 1, the symmetric action, and the composition. Our equivalence therefore can be considered in two ways: it gives both the requirements to identify a symmetric monoidal category as representing an operad, and it gives the conditions for a PROP to be able to be given in standard form.

Theorem 1.0.1 allows us to use symmetric monoidal categories to create constructions for operads. In particular, one motivating idea for this theorem was to study a localization construction for operads, on which the second half of this dissertation focuses. A priori, there is no reason to expect that any of the operations in an operad are invertible. However, it is useful to investigate when certain 1-ary operations are. For example, to understand when a conformal field theory in the sense of [Seg04] behaves stably, one wants to understand when the action defined by the torus in the surface operad is invertible.

For a category $\mathcal{C}$ and a collection of its morphisms $\mathcal{W}$, Dwyer and Kan constructed the hammock localization [DK80a], a simplicial category equipped with a functor from $\mathcal{C}$ where the morphisms in $\mathcal{W}$ induce weak homotopy equivalences on the hom-objects. Similarly, for an operad $\mathcal{O}$ and a submonoid of its 1-ary operations, $\mathcal{W}$, Basterra et. al. constructed the tree hammock localization [BBP+18], a simplicial operad equipped with a map of operads from $\mathcal{O}$ where the operations in $\mathcal{W}$ induce weak homotopy equivalences on the simplicial sets of $n$-ary operations.

Naively, it seems that instead of using the tree hammock localization, we instead could have applied Theorem 1.0.1, by taking the hammock localization of the symmetric monoidal category associated to $\mathcal{O}$. However, in general the hammock localization of a symmetric monoidal category is not itself symmetric monoidal, and thus not in $\mathsf{SM}_{\mathsf{O}}$. Hence we could not use Theorem 1.0.1 to translate this localization back to operads.

Instead, Basterra et. al. construct a full functor $R$ of simplicial categories from the hammock localization to the tree hammock localization [BBP+18, 5.4]. We believe that $R$ is a DK-equivalence and review potential methods for proving this claim. Once proven, this will imply that the tree hammock localization inherits several nice properties of the hammock localization, such as homotopy invariance, and it will give sufficient conditions for the hammock localization of a symmetric monoidal category to be DK-equivalent to a category with a symmetric monoidal structure.

### 1.1 Organization of the Dissertation

In Chapter 2 we review the main types of objects we will use in this dissertation: simplicial sets, symmetric monoidal categories, enriched categories, and operads, and provide several examples. Chapter 3 explores the connection between operads and symmetric monoidal categories. We construct, for every operad $\mathcal{O}$ in an appropriate $\mathcal{V}$, a symmetric monoidal category enriched in $\mathcal{V}$, called its symmetric monoidal envelope in Definition 3.1.2. We use the properties of this construction as inspiration for defining the category $\mathsf{SM}_\mathsf{O}$ in Theorem 3.2.3, and prove our main theorem of the equivalence between $\mathsf{SM}_\mathsf{O}$ and the category of operads in $\mathcal{V}$ in Theorem 3.3.3.

We then turn our attention to localizations. In Chapter 4 we define the hammock localization and tree hammock localization. We then carefully describe the functor $R$, including how to manage the symmetric action, in Chapter 5. We also interpret some of the properties of the hammock localization in the language of operads. In Chapter 6, we review a first attempt to prove that $R$ is a DK-equivalence by containing it in a diagram of other DK-equivalences. Finally in Chapter 7, we introduce quasicategories and marked simplicial sets and we describe an alternative approach to the proof which involves viewing the tree hammock localization as an $\infty$-localization.

# CHAPTER 2

## Categories, Simplicial Sets, and Operads

We will work primarily with two types of objects: symmetric monoidal categories and operads. We also introduce here simplicial sets and enriched categories, with a particular focus on simplicially enriched categories.

### 2.1 Symmetric Monoidal Categories

Recall that a category $\mathcal{C}$ can be defined as follows:

**Definition 2.1.1.** A *category* $\mathcal{C}$ consists of the following data:

- a collection of *objects* of $\mathcal{C}$. We often write $A \in \mathcal{C}$ to denote an object $A$ in $\mathcal{C}$, even though the objects may not necessarily form a set,
- for every pair of objects $A, B \in \mathcal{C}$, a set of *morphisms* from $A$ to $B$, $\mathcal{C}(A, B)$, sometimes denoted as $\mathrm{Hom}_{\mathcal{C}}(A, B)$,
- for every object $A \in \mathcal{C}$, an *identity morphism* $\mathrm{Id}_A : A \to A$,
- for every triple of objects $A, B, C \in \mathcal{C}$, a *composition map*

$$\mathcal{C}(B, C) \times \mathcal{C}(A, B) \to \mathcal{C}(A, C),$$

such that

– the composition is associative; that is for composable maps $h, g, f$,

$$h \circ (g \circ f) = (h \circ g) \circ f,$$

– the composition is unital; that is for any map $f : A \to B$,

$$f \circ \mathrm{Id}_A = f = \mathrm{Id}_B \circ f.$$

*Example* 2.1.2. There are several familiar examples of categories:

- $\mathsf{Set}$ is the category with objects sets and morphisms functions of sets.
- $\mathsf{Grp}$ is the category with objects groups and morphisms group homomorphisms.
- $\mathsf{Top}$ is the category with objects topological spaces and morphisms continuous maps.

*Example* 2.1.3. Suppose we have a category $\mathcal{C}$. Then the category $\mathcal{C}^{\mathrm{op}}$ is defined to be the category with the same objects as $\mathcal{C}$ and with morphisms $\mathcal{C}^{\mathrm{op}}(A, B) := \mathcal{C}(B, A)$.

**Definition 2.1.4.** A *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ between two categories is a map which assigns

- for every object $A \in \mathcal{C}$, an object $F(A) \in \mathcal{D}$,
- for every morphism $f : A \to B$ in $\mathcal{C}$, a morphism $F(f) : F(A) \to F(B)$ in $\mathcal{D}$, such that
  – for every $A \in \mathcal{C}$, $F(\mathrm{Id}_A) = \mathrm{Id}_{F(A)}$,
  – for every pair $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, $F(g \circ f) = F(g) \circ F(f)$.

*Example* 2.1.5. We collect here several familiar functors.

- $U : \mathsf{Grp} \to \mathsf{Set}$ is the forgetful functor, where $U(G)$ is the underlying set of $G$ and $U(f)$ is the underlying function of $f$.

- $F : \mathsf{Set} \to \mathsf{Grp}$ is the free functor, which assigns to every set the free group generated by that set.
- $\pi_1 : \mathsf{Top} \to \mathsf{Grp}$ is the functor assigning to each topological space its fundamental group.

All of the categories in Example 2.1.2 have objects with underlying sets, and morphisms with underlying functions of sets, though this is not true of all categories. Such categories are called concrete categories:

**Definition 2.1.6.** A pair $(\mathcal{C}, U)$ consisting of a category $\mathcal{C}$ and a faithful functor $U : \mathcal{C} \to \mathsf{Set}$, meaning that $U$ induces injections on the hom-sets, is called a *concrete category*. We will often refer to $\mathcal{C}$ as a concrete category without mention of $U$.

*Example* 2.1.7 (Simpicial Sets). We define a category $\Delta$ whose objects are finite, non-empty, ordered sets $[n] = \{0, 1, \dots, n\}$ for nonnegative integers $n$, and whose morphisms are order-preserving functions. A *simplicial set* is a functor $X : \Delta^{\text{op}} \to \mathsf{Set}$, which we will often refer to as $X_\bullet$, or when the context is clear, $X$. We call the elements of the set $X_n := X([n])$ the $n$-simplices.

$\Delta$ has a generating set of morphisms consisting of coface maps $d^i : [n-1] \to [n]$ for $0 \le i \le n$ which skip the element $i$ in the image, and codegeneracy maps $s^i : [n+1] \to [n]$ for $0 \le i \le n$ which repeats the element $i$ in the image. In other words

$$d^i(t) = \begin{cases} t & t < i \\ t+1 & t \ge i \end{cases}, \qquad s^i(t) = \begin{cases} t & t \le i \\ t-1 & t > i \end{cases}.$$

We call the collection $d_i := X(d^i) : X_n \to X_{n-1}$ the *face maps* of $X$ and $s_i := X(s^i) : X_n \to X_{n+1}$ the *degeneracy maps* of $X$. To describe a simplicial set $X_\bullet$, it is enough to give the sets $X_n$ and the set of face and degeneracy maps (which are required to follow certain relations).

A map of simplicial sets $f : X_\bullet \to Y_\bullet$ is a set of functions $f_i : X_i \to Y_i$ for $i \geq 0$ which commute with the face and degeneracy maps. We then have a category $\mathsf{sSet}$ with objects simplicial sets and morphisms maps of simplicial sets.

*Example* 2.1.8. $C : \mathsf{Set} \to \mathsf{sSet}$ takes a set $X$ to the *constant simplicial set* $X_\bullet$, where $X_n = X$ for all $n \geq 0$, and the face and degeneracy maps are identities. We call any simplicial set that is isomorphic to a constant simplicial set a *discrete simplicial set*.

*Example* 2.1.9. There is a functor $|-| : \mathsf{sSet} \to \mathsf{Top}$ which associates to every simplicial set its *geometric realization*, see [Rie11, 4.5].

**Definition 2.1.10.** A *natural transformation* $\alpha$ from functor $F : \mathcal{C} \to \mathcal{D}$ to functor $G : \mathcal{C} \to \mathcal{D}$, denoted $\alpha : F \Rightarrow G$, is a collection of morphisms in $\mathcal{D}$ consisting of one morphism for each object $X \in \mathcal{C}$, $\alpha_X : F(X) \to G(X)$, called the *component of* $\alpha$ *at* $X$, such that for every $f : X \to Y$ in $\mathcal{C}$, $\alpha_Y \circ F(f) = G(f) \circ \alpha_X$.

**Definition 2.1.11.** A pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ gives an *equivalence of categories* if there exists natural isomorphisms $\alpha : GF \Rightarrow \mathrm{Id}_{\mathcal{C}}$ and $\beta : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$.

**Definition 2.1.12.** Let $\mathcal{K}$ be a category. We call $\mathcal{K}$ a *monoidal category* if it has the following additional structure:

- a bifunctor $\otimes : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$, called the *monoidal product*,
- an object $1_{\mathcal{K}}$, called the *unit object*,
- an isomorphism $\alpha$, natural in each of $A, B, C$ with components $\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$, called the *associator*,
- natural isomorphisms $\lambda$, $\rho$ with components $\lambda_A : 1_{\mathcal{K}} \otimes A \to A$ and $\rho_A : A \otimes 1_{\mathcal{K}} \to A$, called the *left* and *right unitors*, respectively.

such that the following coherence diagrams hold for all $A, B, C, D \in \mathcal{K}$:

$$\begin{array}{ccccc} A\otimes(B\otimes(C\otimes D)) & \xrightarrow{\alpha_{A,B,C\otimes D}} & (A\otimes B)\otimes(C\otimes D) & \xrightarrow{\alpha_{A\otimes B,C,D}} & ((A\otimes B)\otimes C)\otimes D \\ {\scriptstyle \mathrm{Id}_A\otimes\alpha_{B,C,D}}\downarrow & & & & \uparrow{\scriptstyle \alpha_{A,B,C}\otimes \mathrm{Id}_D} \\ A\otimes((B\otimes C)\otimes D) & & \xrightarrow[\alpha_{A,B\otimes C,D}]{} & & (A\otimes(B\otimes C))\otimes D \end{array}$$

$$\begin{array}{ccc} A\otimes(1_{\mathcal{K}}\otimes B) & \xrightarrow{\alpha_{A,1_{\mathcal{K}},B}} & (A\otimes 1_{\mathcal{K}})\otimes B \\ & {\scriptstyle \mathrm{Id}_A\otimes\lambda_B}\searrow \quad \swarrow{\scriptstyle \rho_A\otimes\mathrm{Id}_B} & \\ & A\otimes B & \end{array}$$

The first diagram is known as the *pentagon axiom*.

**Definition 2.1.13.** We call a monoidal category $\mathcal{K}$ a *strict monoidal category* if $\alpha$, $\lambda$, and $\rho$ are all identities.

**Definition 2.1.14.** We call a monoidal category $\mathcal{K}$ a *symmetric monoidal category* if it is additionally equipped with a *symmetry isomorphism*, also called a *symmetry map*, which is a natural isomorphism $s$ with components $s_{A,B} : A\otimes B \to B\otimes A$ such that the following coherence diagrams, called the unit, associativity, and inverse diagrams respectively, hold for all $A, B, C \in \mathcal{K}$:

$$\begin{array}{ccc} 1_{\mathcal{K}}\otimes A & \xrightarrow{s_{1_{\mathcal{K}},A}} & A\otimes 1_{\mathcal{K}} \\ & {\scriptstyle \lambda_A}\searrow \quad \swarrow{\scriptstyle \rho_A} & \\ & A & \end{array}$$

$$\begin{array}{ccccc} A\otimes(B\otimes C) & \xrightarrow{\alpha_{A,B,C}} & (A\otimes B)\otimes C & \xrightarrow{s_{A\otimes B,C}} & C\otimes(A\otimes B) \\ {\scriptstyle \mathrm{Id}_A\otimes s_{B,C}}\downarrow & & & & \downarrow{\scriptstyle \alpha_{C,A,B}} \\ A\otimes(C\otimes B) & \xrightarrow[\alpha_{A,C,B}]{} & (A\otimes C)\otimes B & \xrightarrow[s_{A,C}\otimes\mathrm{Id}_B]{} & (C\otimes A)\otimes B \end{array}$$

$$\begin{array}{ccc} & B\otimes A & \\ & {\scriptstyle s_{A,B}}\nearrow \quad \searrow{\scriptstyle s_{B,A}} & \\ A\otimes B & \xrightarrow[\mathrm{Id}_{A\otimes B}]{} & A\otimes B \end{array}$$

*Example* 2.1.15. $\mathsf{Set}$ is a symmetric monoidal category. The monoidal product is the regular Cartesian product, the unit object is the singleton set, the associator is trivial, and the symmetry map sends a coordinate pair $(a, b) \in A \times B$ to $(b, a) \in B \times A$.

Similarly, $\mathsf{sSet}$ is a symmetric monoidal category, where for $X, Y \in \mathsf{sSet}$, $(X \otimes Y)_n = X_n \times Y_n$, and the unit object is the terminal simplicial set $*$ (also denoted $\Delta^0$).

*Example* 2.1.16. Let $M$ be a *monoid*; that is a set equipped with an associative binary operation and a unit. Then we can construct a category $\mathcal{M}$ which has a single object $*$ and automorphisms of $*$ given by the objects in $M$, with composition given by the product in $M$. Then $\mathcal{M}$ is a strict monoidal category.

*Example* 2.1.17. Let $G$ be a finite group, $A$ an abelian group, and $\alpha : G \times G \times G \to A$ a 3-cocycle, meaning that for all $g, h, k, l \in G$,

$$\alpha(h, k, l)\alpha(g, hk, l)\alpha(g, h, k) = \alpha(gh, k, l)\alpha(g, h, kl)$$

We then can construct a monoidal category $\mathcal{G}$, where

- the objects of $\mathcal{G}$ are $g \in G$,
- the morphisms are given by

$$\mathrm{Hom}_{\mathcal{G}}(g, h) = \begin{cases} A & g = h \\ \emptyset & g \neq h \end{cases},$$

  with composition given by multiplication in $A$,
- the monoidal product is given by multiplication in $G$, meaning $g \otimes h := gh$,
- the unit object is the identity of $G$,
- the associator $\alpha_{g,h,k} : g \otimes (h \otimes k) \to (g \otimes h) \otimes k$ is given by $\alpha(g, h, k) \in A = \mathrm{Hom}_{\mathcal{G}}(ghk, ghk)$.

Note that this gives several examples of a monoidal category with non-trivial associator. Furthermore, unless $G$ is abelian, $\mathcal{G}$ cannot be symmetric monoidal, since whenever $gh \neq hg$, there are no morphisms $g \otimes h \to h \otimes g$. $\mathcal{G}$ is an example of a 2-group.

For several more examples see [EGNO10, Section 2.3]

**Definition 2.1.18.** A *monoidal functor* $F : \mathcal{K} \to \mathcal{L}$ between monoidal categories is a functor $F$ equipped with a natural transformation $\phi$ with components $\phi_{A,B} : F(A) \otimes_{\mathcal{L}} F(B) \to F(A \otimes_{\mathcal{K}} B)$ and a morphism $U : 1_{\mathcal{L}} \to F(1_{\mathcal{K}})$, such that for every triple $A, B, C \in \mathcal{K}$, the following diagrams commute in $\mathcal{D}$:

$$
\begin{array}{ccc}
(F(A) \otimes_{\mathcal{L}} F(B)) \otimes_{\mathcal{L}} F(C) & \xleftarrow{\alpha_{\mathcal{L}}} & F(A) \otimes_{\mathcal{L}} (F(B) \otimes_{\mathcal{L}} F(C)) \\
{\scriptstyle \phi_{A,B} \otimes_{\mathcal{L}} \mathrm{Id}_{F(C)}} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{Id}_{F(A)} \otimes_{\mathcal{L}} \phi_{B,C}} \\
F(A \otimes_{\mathcal{K}} B) \otimes_{\mathcal{L}} F(C) & & F(A) \otimes_{\mathcal{L}} F(B \otimes_{\mathcal{K}} C) \\
{\scriptstyle \phi_{A \otimes_{\mathcal{K}} B, C}} \big\downarrow & & \big\downarrow {\scriptstyle \phi_{A, B \otimes_{\mathcal{K}} C}} \\
F((A \otimes_{\mathcal{K}} B) \otimes_{\mathcal{K}} C) & \xleftarrow[F(\alpha_{\mathcal{K}})]{} & F(A \otimes_{\mathcal{K}} (B \otimes_{\mathcal{K}} C))
\end{array}
$$

$$
\begin{array}{ccc}
F(A) \otimes_{\mathcal{L}} 1_{\mathcal{L}} & \xrightarrow{\mathrm{Id}_{F(A)} \otimes_{\mathcal{L}} U} & F(A) \otimes_{\mathcal{L}} F(1_{\mathcal{K}}) \\
{\scriptstyle \rho_{\mathcal{L}}} \big\downarrow & & \big\downarrow {\scriptstyle \phi_{A, 1_{\mathcal{K}}}} \\
F(A) & \xleftarrow[F(\rho_{\mathcal{K}})]{} & F(A \otimes_{\mathcal{K}} 1_{\mathcal{K}})
\end{array}
\qquad
\begin{array}{ccc}
1_{\mathcal{L}} \otimes_{\mathcal{L}} F(B) & \xrightarrow{U \otimes_{\mathcal{L}} \mathrm{Id}_{F(B)}} & F(1_{\mathcal{K}}) \otimes_{\mathcal{L}} F(B) \\
{\scriptstyle \lambda_{\mathcal{L}}} \big\downarrow & & \big\downarrow {\scriptstyle \phi_{1_{\mathcal{K}}, B}} \\
F(B) & \xleftarrow[F(\lambda_{\mathcal{K}})]{} & F(1_{\mathcal{K}} \otimes_{\mathcal{K}} B)
\end{array}
$$

The maps $\phi_{A,B}$ and $U$ are called the *coherence maps*.

*Remark* 2.1.19. In some texts, this is called a *lax monoidal functor*.

**Definition 2.1.20.** A *symmetric monoidal functor* $F : \mathcal{K} \to \mathcal{L}$ between symmetric monoidal categories is a monoidal functor such that the following diagram commutes:

$$
\begin{array}{ccc}
F(A) \otimes_{\mathcal{L}} F(B) & \xrightarrow{s^{\mathcal{L}}_{F(A),F(B)}} & F(B) \otimes_{\mathcal{L}} F(A) \\
{\scriptstyle \phi_{A,B}} \big\downarrow & & \big\downarrow {\scriptstyle \phi_{B,A}} \\
F(A \otimes_{\mathcal{K}} B) & \xrightarrow[F(s^{\mathcal{K}}_{A,B})]{} & F(B \otimes_{\mathcal{K}} A)
\end{array}
$$

In our work, we will primarily use strict symmetric monoidal functors, which are symmetric monoidal functors where the coherence maps are identities. Or more explicitly,

**Definition 2.1.21.** A *strict symmetric monoidal functor* $F : \mathcal{K} \to \mathcal{L}$ between symmetric monoidal categories (not necessarily strict) is a functor $F$ such that for all $A, B \in \mathcal{K}$,

- $F(A) \otimes_{\mathcal{L}} F(B) = F(A \otimes_{\mathcal{K}} B)$,
- $F(1_{\mathcal{K}}) = 1_{\mathcal{L}}$,
- $F(\alpha_{\mathcal{K}}) = \alpha_{\mathcal{L}}$,
- $F(\rho_{\mathcal{K}}) = \rho_{\mathcal{L}}$ and $F(\lambda_{\mathcal{K}}) = \lambda_{\mathcal{L}}$,
- $F(s_{\mathcal{K}}) = s_{\mathcal{L}}$.

*Example* 2.1.22. The functor $C : \mathsf{Set} \to \mathsf{sSet}$ in Example 2.1.8 is a strict symmetric monoidal functor.

*Example* 2.1.23. Let $M$, $M'$ be monoids and $F : M \to M'$ a morphism of monoids. The induced functor $F : \mathcal{M} \to \mathcal{M}'$ is a strict monoidal functor.

*Example* 2.1.24. Let $G$ be a finite group and $A$ an abelian group, and let $\alpha, \alpha' : G \times G \times G \to A$ be 3-cocycles. Let $\mathcal{G}$ denote the 2-group as in Example 2.1.17 with associator $\alpha$, and $\mathcal{G}'$ denote the one with assocaiator $\alpha'$. Let $F : \mathcal{G} \to \mathcal{G}'$ be the functor which is the identity on objects and morphisms.

In order to have $F$ be a monoidal functor, we need $\phi_{g,h} : F(g) \otimes F(h) \to F(g \otimes h) \in \mathrm{Hom}_{\mathcal{G}}(gh, gh) = A$ such that

$$\alpha(\phi_{g,hk})(\phi_{h,k}) = (\phi_{gh,k})(\phi_{g,h})\alpha'$$

This is equivalent to saying that $\alpha, \alpha'$ are cohomologous. In fact, $F$ is a monoidal functor if and only if $\alpha, \alpha'$ are cohomologous [Lur22, Example 00E5]. Note that if $\alpha \neq \alpha'$, $F$ cannot be strict.

**Definition 2.1.25.** We define the category $\mathsf{SymmMod}$ to be the category with objects symmetric monoidal categories and morphisms symmetric monoidal functors.

## 2.2 Enriched Categories

We introduce enriched categories following the description in [Rie14, Chapter 3].

In this section, let $(\mathcal{V}, \otimes, 1_{\mathcal{V}})$ be a symmetric monoidal category which is complete and cocomplete, that is, a category in which all small limits and colimits exist.

**Definition 2.2.1.** A *category enriched in* $\mathcal{V}$, $\mathcal{C}$, also called a $\mathcal{V}$-category, consists of:

- a collection of objects in $\mathcal{C}$,
- for each pair of objects $A, B \in \mathcal{C}$, a *hom-object* $\mathcal{C}(A, B) \in \mathcal{V}$,
- for each $A \in \mathcal{C}$, a morphism $\mathrm{Id}_A : 1_{\mathcal{V}} \to \mathcal{C}(A, A)$ in $\mathcal{V}$,
- for each triple $A, B, C$, a morphism $\circ : \mathcal{C}(B, C) \otimes \mathcal{C}(A, B) \to \mathcal{C}(A, C)$ in $\mathcal{V}$,

such that the following diagrams commute for all $A, B, C, D \in \mathcal{C}$:

$$\begin{array}{ccc} \mathcal{C}(C,D)\otimes\mathcal{C}(B,C)\otimes\mathcal{C}(A,B) & \xrightarrow{1\otimes\circ} & \mathcal{C}(C,D)\otimes\mathcal{C}(A,C) \\ {\scriptstyle \circ\otimes 1}\downarrow & & \downarrow{\scriptstyle\circ} \\ \mathcal{C}(B,D)\otimes\mathcal{C}(A,B) & \xrightarrow[\circ]{} & \mathcal{C}(A,D)\end{array}$$

$$\begin{array}{ccc} \mathcal{C}(A,B)\otimes 1_{\mathcal{V}} & \xrightarrow{1\otimes \mathrm{Id}_A} & \mathcal{C}(A,B)\otimes\mathcal{C}(A,A) \\ & {\scriptstyle \rho_{\mathcal{C}(A,B)}}\searrow & \downarrow{\scriptstyle\circ} \\ & & \mathcal{C}(A,B)\end{array} \qquad \begin{array}{ccc} \mathcal{C}(B,B)\otimes\mathcal{C}(A,B) & \xleftarrow{\mathrm{Id}_B\otimes 1} & 1_{\mathcal{V}}\otimes\mathcal{C}(A,B) \\ {\scriptstyle\circ}\downarrow & \swarrow{\scriptstyle \lambda_{\mathcal{C}(A,B)}} & \\ \mathcal{C}(A,B) & & \end{array}$$

*Example* 2.2.2. We will be particularly interested in categories enriched in $\mathsf{sSet}$, which we will call *simplicial categories*. A simplicial category $\mathcal{C}$ is a category where the hom-objects are simplicial sets. For $A \in \mathcal{C}$, the identity morphism $\mathrm{Id}_A$ will be a 0-simplex in $\mathcal{C}(A, A)$, and composition will be given by a simplicial map.

*Remark* 2.2.3. In general, the term simplicial category is ambiguous. Depending on the context, it could mean a simplicial object in categories; i.e., a functor $\mathcal{C}_\bullet : \Delta^{\mathrm{op}} \to \mathsf{Cat}$, or it could be used as above. A category enriched in $\mathsf{sSet}$ is actually an example of a simplicial object in categories where the face and degeneracy maps are the identity on objects. Throughout this dissertation, we will use simplicial categories to exclusively refer to categories enriched in $\mathsf{sSet}$.

*Example* 2.2.4. An ordinary category $\mathcal{C}$ can be seen as a simplicial category with the same objects as $\mathcal{C}$ and hom-objects the discrete simplicial sets $\mathcal{C}(A, B)$ (as in Example 2.1.8).

**Definition 2.2.5.** A $\mathcal{V}$-*functor* $F : \mathcal{C} \to \mathcal{D}$ between $\mathcal{V}$-categories consists of

- an object map which sends $A \in \mathcal{C}$ to $F(A) \in \mathcal{D}$,
- for each $A, B \in \mathcal{C}$, morphisms $F_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(F(A), F(B))$ in $\mathcal{V}$ such that the following diagrams commute for all $A, B, C \in \mathcal{C}$:

$$\begin{array}{ccc} \mathcal{C}(B,C) \otimes \mathcal{C}(A,B) & \xrightarrow{\ \circ\ } & \mathcal{C}(A,C) \\ {\scriptstyle F_{B,C}\otimes F_{A,B}}\downarrow & & \downarrow{\scriptstyle F_{A,C}} \\ \mathcal{D}(F(B),F(C)) \otimes \mathcal{D}(F(A),F(B)) & \xrightarrow[\circ]{} & \mathcal{D}(F(A),F(C)) \end{array} \qquad \begin{array}{ccc} 1_{\mathcal{V}} & \xrightarrow{\mathrm{Id}_A} & \mathcal{C}(A,A) \\ & {\scriptstyle \mathrm{Id}_{F(A)}} \searrow & \downarrow{\scriptstyle F_{A,A}} \\ & & \mathcal{D}(F(A),F(A)) \end{array}$$

*Example* 2.2.6. A functor $F : \mathcal{C} \to \mathcal{D}$ between simplicial categories will consist of an object map and simplicial maps $F_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(F(A), F(B))$.

**Definition 2.2.7.** $\mathsf{sCat}$ is the category with objects simplicial categories and morphisms functors of simplicial categories.

Note that $\mathcal{V}$ is a symmetric monoidal ordinary category; in other words, the hom-objects of $\mathcal{V}$ are sets. Furthermore, the functor $\mathcal{V}(1_{\mathcal{V}}, -) : \mathcal{V} \to \mathsf{Set}$ is a monoidal functor. In particular, the coherence maps of $\mathcal{V}(1_{\mathcal{V}}, -)$ give, for any $U, V \in \mathcal{V}$, a function of sets:

$$\mathcal{V}(1_{\mathcal{V}}, U) \times \mathcal{V}(1_{\mathcal{V}}, V) \to \mathcal{V}(1_{\mathcal{V}}, U \otimes V) \tag{2.2.1}$$

**Definition 2.2.8.** Let $\mathcal{C}$ be a $\mathcal{V}$-category. Then the *underlying category* $\mathcal{C}_0$ *of* $\mathcal{C}$ is the ordinary category with the same objects as $\mathcal{C}$ and hom-objects the sets $\mathcal{C}_0(A, B) := \mathcal{V}(1_{\mathcal{V}}, \mathcal{C}(A, B))$.

The identities $\mathrm{Id}_A \in \mathcal{C}_0(A, A)$ are the specified morphisms $\mathrm{Id}_A \in \mathcal{V}(1_{\mathcal{V}}, \mathcal{C}(A, A))$. Composition is given by

$$\begin{array}{ccccc} \mathcal{C}_0(B,C) \times \mathcal{C}_0(A,B) & \dashrightarrow & & & \mathcal{C}_0(A,C) \\ \| & & & & \| \\ \mathcal{V}(1_{\mathcal{V}}, \mathcal{C}(B,C)) \times \mathcal{V}(1_{\mathcal{V}}, \mathcal{C}(A,B)) & \longrightarrow & \mathcal{V}(1_{\mathcal{V}}, \mathcal{C}(B,C) \otimes \mathcal{C}(A,B)) & \longrightarrow & \mathcal{V}(1_{\mathcal{V}}, \mathcal{C}(A,C)) \end{array}$$

where the first arrow is Equation 2.2.1 and the second is the functor $\mathcal{V}(1_{\mathcal{V}}, -)$ applied to the composition morphism from $\mathcal{C}$.

*Example* 2.2.9. The underlying category $\mathcal{C}_0$ of a simplicial category $\mathcal{C}$ is the category with the same objects as $\mathcal{C}$ and hom-sets $\mathcal{C}_0(A, B) = \mathcal{C}(A, B)_0$, the 0-simplices of the simplicial set $\mathcal{C}(A, B)$.

In later sections, we will need the notion of an equivalence of simplicial categories. We first need some definitions.

**Definition 2.2.10.** Let $X_\bullet$ be a simplicial set. Then the *set of path components of* $X$ is the coequalizer of the parallel functions

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0$$

More explicitly, the set of path components of $X$ is given by $X_0/\approx$ where $\approx$ is the equivalence relation generated by the relation $\sim$ on $X_0$, defined by $a \sim b$ if there exists a 1-simplex $\alpha \in X_1$ such that $d_0\alpha = a$ and $d_1\alpha = b$. We let $[a]$ denote the equivalence class of $a \in X_0$ in $X_0/\approx$.

There is a bijection between the set of path components of $X$ and the set of connected components of $X$. Given $a \in X_0$, we can roughly define the connected component of $X$ to be the set of those $n$-simplices which only have vertices in $[a]$. We will interchangeably use $\pi_0 X$ to refer to both the set of connected components and of path components of $X$, and will make the meaning clear in context.

**Definition 2.2.11.** A map of simplicial sets $f : X_\bullet \to Y_\bullet$ is a *weak equivalence* if the induced functor of geometric realizations $|f| : |X| \to |Y|$ is a weak homotopy equivalence of topological spaces.

**Definition 2.2.12.** [Ber07a] Let $\mathcal{C} \in \mathsf{sCat}$. The *category of components* of $\mathcal{C}$, denoted $\pi_0\mathcal{C}$ is the regular category with the same objects as in $\mathcal{C}$, and hom-sets given by $(\pi_0\mathcal{C})(A, B) := \pi_0(\mathcal{C}(A, B))$, the path components of the simplicial set $\mathcal{C}(A, B)$. Composition in $\pi_0\mathcal{C}$ is defined by $[g] \circ [f] = [gf]$, which one can show is well defined.

If a morphism $f \in \mathcal{C}(A, B)_0$ is sent to an isomorphism in $\pi_0\mathcal{C}$, then we call $f$ a *homotopy equivalence*. If $F : \mathcal{C} \to \mathcal{D}$ is a functor of simplicial categories, then $\pi_0F : \pi_0\mathcal{C} \to \pi_0\mathcal{D}$ denotes the induced map on the categories of components of $\mathcal{C}$ and $\mathcal{D}$.

**Definition 2.2.13.** [Ber07a] A *DK-equivalence* between simplicial categories $\mathcal{C}$, $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ satisfying the following two conditions:

- for any objects $A, B \in \mathcal{C}$, the map $\mathcal{C}(A, B) \to \mathcal{D}(F(A), F(B))$ is a weak equivalence of simplicial sets;
- the induced functor $\pi_0F$ is an equivalence of categories.

These are the weak equivalances of a cofibrantly generated model category structure on $\mathsf{sCat}$ described in [Ber07a, Theorem 1.1].

These equivalences are usually called DK-equivalences after Dwyer and Kan who first defined a version of them in [DK80b]. Note that if $F$ is the identity on objects, as it was in the original description, the second condition follows from the first.

## 2.3 Nerves

We can associate to every category a simplicial set:

**Definition 2.3.1.** Let $\mathcal{C}$ be a category. The *nerve of* $\mathcal{C}$, denoted $N(\mathcal{C})$, is a simplicial set with $N(\mathcal{C})_0$ the objects of $\mathcal{C}$, $N(\mathcal{C})_1$ the morphisms of $\mathcal{C}$, and $N(\mathcal{C})_n$ strings of $n$-composable arrows in $\mathcal{C}$ for $n \geq 2$. The face maps $d_i : N(\mathcal{C})_n \to N(\mathcal{C})_{n-1}$ are given by composing the $i$th and $i+1$-th maps for $0 < i < n$, and by removing the first and last map for $i = 0$ and $n$ respectively. The degeneracy maps $s_i : N(\mathcal{C})_n \to N(\mathcal{C})_{n+1}$ are given by inserting an identity in the $i$th position (thinking of the first map as the 0th position). For example,

$$A \xrightarrow{f} B \xrightarrow{g} C \in N(\mathcal{C})_2$$

$$d_0(A \xrightarrow{f} B \xrightarrow{g} C) = B \xrightarrow{g} C, \quad d_1(A \xrightarrow{f} B \xrightarrow{g} C) = A \xrightarrow{gf} C, \quad d_2(A \xrightarrow{f} B \xrightarrow{g} C) = A \xrightarrow{f} B,$$

$$s_0(A \xrightarrow{f} B \xrightarrow{g} C) = A \xrightarrow{\mathrm{Id}_A} A \xrightarrow{f} B \xrightarrow{g} C, \quad s_1(A \xrightarrow{f} B \xrightarrow{g} C) = A \xrightarrow{f} B \xrightarrow{\mathrm{Id}_B} B \xrightarrow{g} C,$$

$$s_2(A \xrightarrow{f} B \xrightarrow{g} C) = A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\mathrm{Id}_C} C$$

A functor $F : \mathcal{C} \to \mathcal{D}$ induces a map of simplicial sets $N(F) : N(\mathcal{C}) \to N(\mathcal{D})$.

In order to capture the higher dimensional data of a simplicial category, we need a different construction.

**Definition 2.3.2.** Let $\mathcal{C} \in \mathsf{sCat}$. We define $N_{coh}(\mathcal{C})$, the *homotopy coherent nerve* of $\mathcal{C}$ to be the simplicial set with

- $N_{coh}(\mathcal{C})_0$ the objects of $\mathcal{C}$;
- $N_{coh}(\mathcal{C})_1$ the morphisms of the underlying category $\mathcal{C}_0$, i.e., the 0-simplices of the hom-objects of $\mathcal{C}$;
- for $n > 1$, the $n$-simplices of $N_{coh}(\mathcal{C})$ consist of a string of $n$ composable morphisms of the underlying category $\mathcal{C}_0$ along with coherence data.

For a full definition of $N_{coh}$, see [Lur22, Subsection 00KM], [Rie10].

**Lemma 2.3.3.** *[Lur09, 1.2.3.1] When $\mathcal{C}$ is an ordinary category, viewed as a discrete simplicial category, $N_{coh}(\mathcal{C}) = N(\mathcal{C})$.*

## 2.4 Operads

Let $(\mathcal{V}, \otimes, 1_{\mathcal{V}})$ be a closed symmetric monoidal category tensored over sets which has finite colimits. We will most commonly work in the cases where $\mathcal{V}$ is $\mathsf{Set}$ or $\mathsf{sSet}$. Throughout this dissertation, we will assume $0 \in \mathbb{N}$. An operad $\mathcal{O}$ in $\mathcal{V}$ is, roughly, a collection of operations with $n$ inputs and a single output, along with a composition rule that 'behaves nicely'. More precisely,

**Definition 2.4.1.** An *operad* $\mathcal{O}$ in $\mathcal{V}$ is a collection of objects $\mathcal{O}(n) \in \mathcal{V}$ for $n \in \mathbb{N}$, called the *$n$-ary operations* along with

- a map $1_{\mathcal{O}} : 1_{\mathcal{V}} \to \mathcal{O}(1)$, giving the *unit of the operad*,
- a right action of the symmetric group $\Sigma_n$ on $\mathcal{O}(n)$ for all $n \geq 0$,
- *structure maps*

$$\gamma : \mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \ldots \otimes \mathcal{O}(j_k) \to \mathcal{O}(j)$$

  for all $k \geq 1$, $j_i \geq 0$, and $\sum j_i = j$,

such that the following diagrams commute for all $i_t$, $j_s$, and $k$, where $j = \sum j_s$:

- $\gamma$ is associative:

$$\begin{array}{ccc}
\begin{array}{lll} \mathcal{O}(k) & \otimes & \mathcal{O}(j_1) \otimes \ldots \otimes \mathcal{O}(j_k) \\ & \otimes & \mathcal{O}(i_1) \otimes \ldots \otimes \mathcal{O}(i_{j_1}) \\ & \vdots & \\ & \otimes & \mathcal{O}(i_{j-j_k+1}) \otimes \ldots \otimes \mathcal{O}(i_j) \end{array} & \longrightarrow & \begin{array}{ll} & \mathcal{O}(j_1 + \ldots + j_k) \\ \otimes & \mathcal{O}(i_1) \otimes \ldots \otimes \mathcal{O}(i_{j_1}) \\ \vdots & \\ \otimes & \mathcal{O}(i_{j-j_k+1}) \otimes \ldots \otimes \mathcal{O}(i_j) \end{array} \\
\downarrow & & \downarrow \\
\begin{array}{lll} \mathcal{O}(k) & \otimes & \mathcal{O}(i_1 + \ldots + i_{j_1}) \\ & \vdots & \\ & \otimes & \mathcal{O}(i_{j-j_k+1} + \ldots + i_j) \end{array} & \longrightarrow & \mathcal{O}(i_1 + \ldots + i_j)
\end{array}$$

- $1_{\mathcal{O}}$ is a unit for $\gamma$:

$$\begin{array}{ccc}
1_{\mathcal{V}} \otimes \mathcal{O}(k) & & \\
{\scriptstyle 1_{\mathcal{O}} \otimes \mathrm{Id}} \downarrow & \overset{\cong}{\searrow} & \\
\mathcal{O}(1) \otimes \mathcal{O}(k) & \xrightarrow[\gamma]{} & \mathcal{O}(k)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{O}(k) \otimes 1_{\mathcal{V}} \otimes \ldots \otimes 1_{\mathcal{V}} & & \\
{\scriptstyle \mathrm{Id} \otimes 1_{\mathcal{O}} \otimes \ldots \otimes 1_{\mathcal{O}}} \downarrow & \overset{\cong}{\searrow} & \\
\mathcal{O}(k) \otimes \mathcal{O}(1) \otimes \ldots \otimes \mathcal{O}(1) & \xrightarrow[\gamma]{} & \mathcal{O}(k)
\end{array}$$

- the structure maps are equivariant with respect to the $\Sigma_n$ actions in the following ways for $\sigma \in \Sigma_k$, $\tau_s \in \Sigma_{j_s}$:

$$\begin{array}{ccc}
\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \ldots \otimes \mathcal{O}(j_k) & \xrightarrow{\sigma \otimes \mathrm{Id} \otimes \ldots \otimes \mathrm{Id}} & \mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \ldots \otimes \mathcal{O}(j_k) \\
{\scriptstyle \mathrm{Id} \otimes \sigma^*} \downarrow & & \\
\mathcal{O}(k) \otimes \mathcal{O}(j_{\sigma^{-1}(1)}) \otimes \ldots \otimes \mathcal{O}(j_{\sigma^{-1}(k)}) & & \downarrow {\scriptstyle \gamma} \\
{\scriptstyle \gamma} \downarrow & & \\
\mathcal{O}(j_{\sigma^{-1}(1)} + \ldots + j_{\sigma^{-1}(k)}) & \xrightarrow[\sigma(j_1, \ldots, j_k)]{} & \mathcal{O}(j_1 + \ldots + j_k)
\end{array}$$

where $\sigma(j_1, \ldots, j_k)$ is the permutation which results from partitioning $j$ letters into $k$ blocks as given and then applying $\sigma$ to the $k$ blocks.

$$\begin{array}{ccc}
\mathcal{O}(k)\otimes\mathcal{O}(j_1)\otimes\ldots\otimes\mathcal{O}(j_k) & \xrightarrow{\mathrm{Id}\otimes\tau_1\otimes\ldots\otimes\tau_k} & \mathcal{O}(k)\otimes\mathcal{O}(j_1)\otimes\ldots\otimes\mathcal{O}(j_k) \\
\downarrow\gamma & & \downarrow\gamma \\
\mathcal{O}(j_1+\ldots+j_k) & \xrightarrow[\tau_1\oplus\ldots\oplus\tau_k]{} & \mathcal{O}(j_1+\ldots+j_k)
\end{array}$$

where $\tau_{j_1}\oplus\ldots\oplus\tau_{j_k}$ is the image of $(\tau_{j_1},\ldots,\tau_{j_k})$ under the canonical inclusion $\Sigma_{j_1}\times\ldots\times\Sigma_{j_k}\to\Sigma_j$. As needed, we will let $e_j\in\Sigma_j$ denote the identity permutation.

*Remark* 2.4.2. Note that we do not require our operads to be reduced; in other words we allow for operads with $\mathcal{O}(0)\neq *$, the terminal object of $\mathcal{V}$.

We can also define the *ith composition*, $\circ_i:\mathcal{O}(k)\otimes\mathcal{O}(j)\to\mathcal{O}(k+j-1)$ for $1\leq i\leq k$ as follows:

$$\begin{array}{ccc}
\mathcal{O}(k)\otimes U\otimes\ldots\otimes U\otimes\mathcal{O}(j)\otimes U\otimes\ldots\otimes U & \xrightarrow{\cong} & \mathcal{O}(k)\otimes\mathcal{O}(j) \\
\downarrow \mathrm{Id}\otimes 1_{\mathcal{O}}\otimes\ldots\otimes 1_{\mathcal{O}}\otimes\mathrm{Id}\otimes 1_{\mathcal{O}}\otimes\ldots\otimes 1_{\mathcal{O}} & & \downarrow\circ_i \\
\mathcal{O}(k)\otimes\mathcal{O}(1)\otimes\ldots\otimes\mathcal{O}(1)\otimes\mathcal{O}(j)\otimes\mathcal{O}(1)\otimes\ldots\otimes\mathcal{O}(1) & \xrightarrow[\gamma]{} & \mathcal{O}(j+k-1)
\end{array}$$

where $\mathcal{O}(j)$ is in the $(i+1)$-th spot of the tensor product.

*Remark* 2.4.3. We will primarily study operads in concrete categories, in which case we can think of the $n$-ary operations as elements of the underlying set of $\mathcal{O}(n)$. In particular, we will think of the unit $1_{\mathcal{O}}$ as an element of $\mathcal{O}(1)$. This allows us to use equations to interpret the commutative diagrams above, which are a bit easier to digest.

It is usually helpful to think about working with operads by depicting the operations as trees, where an $n$-ary operation is depicted as a tree with $n$ branches (see Figure 2.1). We also provide graphical depictions of the properties, which hold for all $k$, $j_s$, and $i_t$:

- (Figure 2.2) $\gamma$ is associative: for all $f\in\mathcal{O}(k)$, $g_s\in\mathcal{O}(j_s)$, $j=\sum j_s$, and $h_t\in\mathcal{O}(i_t)$,

$$\gamma(\gamma(f;g_1,\ldots g_k);h_1,\ldots,h_{j_1},\ldots,h_{j-j_k+1},\ldots,h_j)=$$

(a) $\alpha \in \mathcal{O}(3)$ (b) $\beta \in \mathcal{O}(2)$

Figure 2.1

$$\gamma(f; \gamma(g_1; h_1, \ldots, h_{j_1}), \ldots, \gamma(g_k; h_{j-j_k+1}, \ldots, h_j)$$

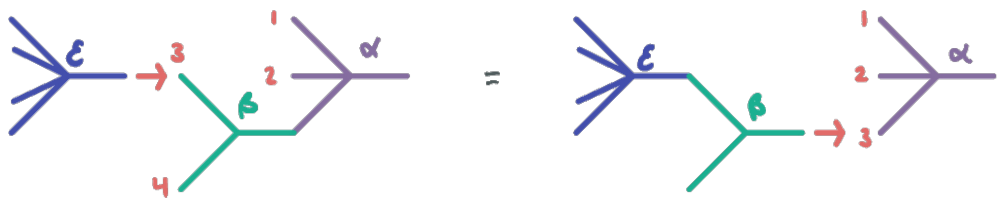



Figure 2.2: Example of associativity of $\gamma$: $(\alpha \circ_3 \beta) \circ_3 \epsilon = \alpha \circ_3 (\beta \circ_1 \epsilon)$.

- (Figure 2.3) $1_\mathcal{O}$ is a unit for $\gamma$: for all $f \in \mathcal{O}(n)$,

$$\gamma(f; \underbrace{1_\mathcal{O}, \ldots, 1_\mathcal{O}}_{n\text{-times}}) = f = \gamma(1_\mathcal{O}; f)$$

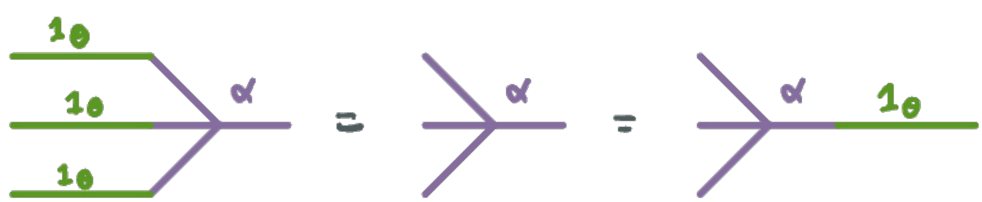



Figure 2.3: Example of $1_\mathcal{O}$ as a unit with $\alpha \in \mathcal{O}(3)$: $\gamma(\alpha; 1_\mathcal{O}, 1_\mathcal{O}, 1_\mathcal{O}) = \alpha = \gamma(1_\mathcal{O}; \alpha)$.

- the structure maps are equivariant with respect to the $\Sigma_n$ actions (Figure 2.4):
  - (Figure 2.5a) for all $f \in \mathcal{O}(k)$, $g_s \in \mathcal{O}(j_s)$, and $\sigma \in \Sigma_k$,

$$\gamma(f \cdot \sigma; g_1, \ldots, g_k) = \gamma(f; g_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(k)}) \cdot \sigma(j_1, \ldots, j_k)$$

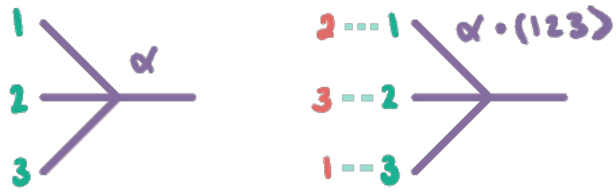
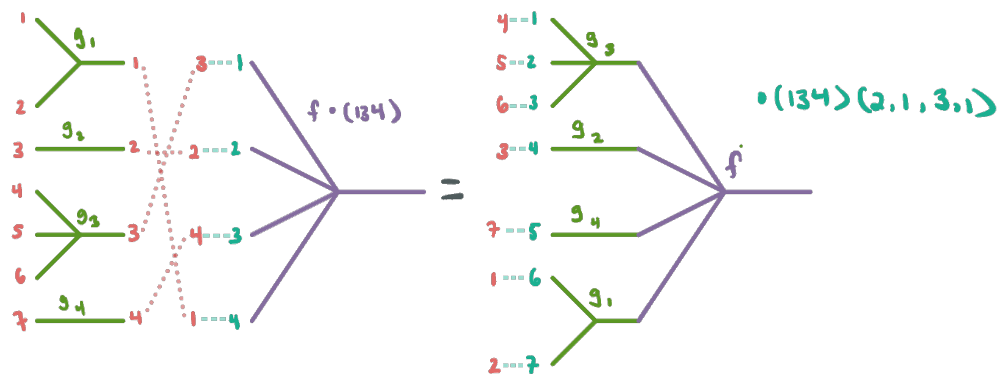


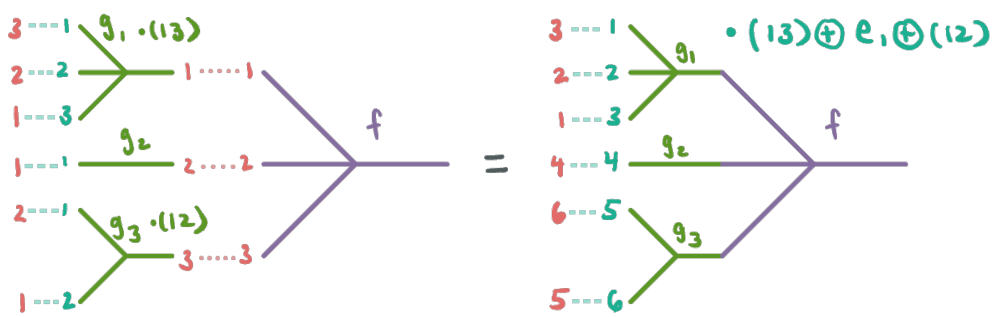

Figure 2.4: We can depict the symmetric action by permuting the "input labels" on an operation. Here, for $\alpha \in \mathcal{O}(3)$, the unpermuted labels are green and the permuted ones are red, so the figure on the right depicts $\alpha \cdot (123)$.

– (Figure 2.5b) for all $f \in \mathcal{O}(k)$, $g_s \in \mathcal{O}(j_s)$ and $\tau_s \in \Sigma_s$,

$$\gamma(f; g_1 \cdot \tau_1, \ldots, g_k \cdot \tau_k) = \gamma(f; g_1, \ldots g_k) \cdot (\tau_1 \oplus \ldots \oplus \tau_k)$$




(a)




(b)

Figure 2.5: (a) This depicts the equality $\gamma(f \cdot (134); g_1, g_2, g_3, g_4) = \gamma(f; g_3, g_2, g_4, g_1) \cdot ((134)(2,1,3,1))$. We can think of this as permuting the labels on $f$ first, then reordering the $g_i$ to match that permutation before applying $\gamma$. The $g_i$ carry with them the original labels on their "inputs" which gives the permutation applied to the composition. (b) This depicts the equality $\gamma(f; g_1 \cdot (13), g_2, g_3 \cdot (12)) = \gamma(f; g_1, g_2, g_3) \cdot ((13) \oplus e_1 \oplus (12))$.

*Example* 2.4.4 (Endomorphism Operad). Let $X \in \mathcal{V}$, let $\mathrm{End}_X(n) = \mathrm{Hom}(X^n, X)$, where $X^n$ is shorthand for $X^{\otimes n}$. The $i$th composition is $f \circ_i g$ is given by using the output of $g$ as the $i$th input of $f$, and the symmetric action permutes the inputs.

*Example* 2.4.5 (Associative Operad). Let $\mathrm{Assoc}(n) = \Sigma_n$. The $i$th composition is given by block permutations, and the symmetric action is given by multiplication in $\Sigma_n$.

*Example* 2.4.6 (Commutative Operad). Let $\mathrm{Comm}(n) = *$. The structure maps and symmetric action should be clear.

*Example* 2.4.7. Given a monoid $M$, we can define an operad $\mathcal{M}$ where $\mathcal{M}(1) = M$ and $\mathcal{M}(n) = \emptyset$ for $n \neq 1$, and the structure maps are given by the multiplication in $M$.

*Example* 2.4.8 (Little $k$-cubes operad). Let $C_k(n)$ consist of the spaces of linear embeddings of $n$ little $k$-cubes into the unit $k$-dimensional cube such that the interiors of the little $k$ cubes are disjoint. The $i$th composition is given by scaling down a unit cube and putting it in the place of the $i$th little $k$-cube, and the symmetric action is given by shuffling the labels on the little $k$-cubes. See Figure 2.6.

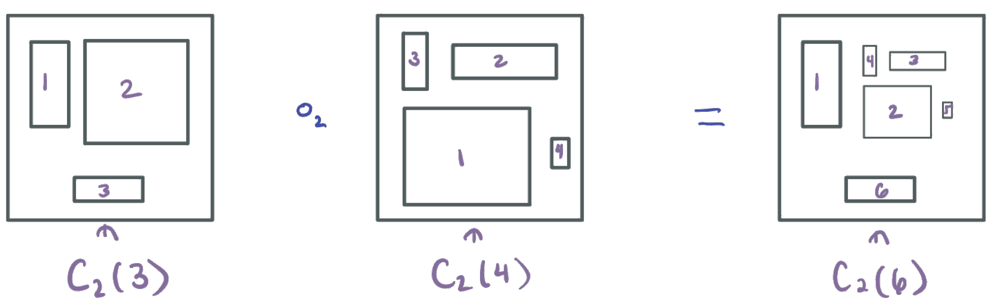



Figure 2.6: An example of $i$th composition of a 3-ary and a 4-ary operation in the little 2-cubes operad.

*Example* 2.4.9 (Surface operad). Let $\mathcal{S}_{g,n}$ denote the moduli space of Riemann surfaces of genus $g$ with $n$ labeled and ordered boundary components. By replacing these spaces by their total singular complexes, we can define an operad $\mathcal{S}$ where

$$\mathcal{S}(n) := \coprod_{g \geq 0} \mathrm{Sing}_*(\mathcal{S}_{g,n+1}).$$

Roughly, we can think of an operation in $\mathcal{S}(n)$ as a surface of genus $g$ with $n+1$ labeled and ordered boundary components, with the first $n$ components the "inputs" of the operation and the last one the "output". For example, the torus with two boundary components is in $\mathcal{S}(1)$, with one component labeled as an input and the other as the output. Then the $i$th composition is given by gluing the output component of one operation to the $i$th input component of another, and the symmetric action is given by permuting the labels on the input components.

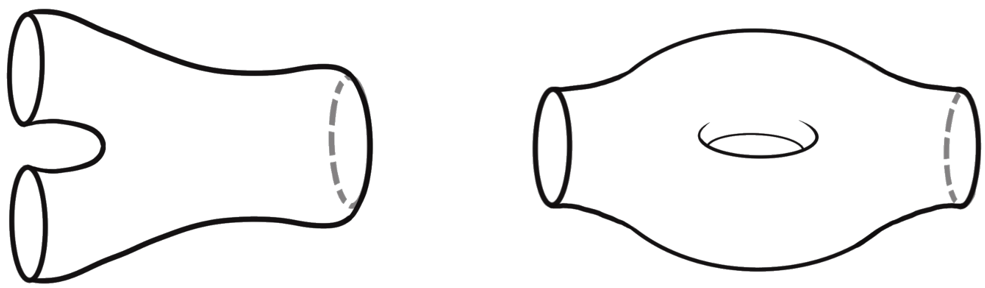


Figure 2.7: An example of a 2-ary operation in the surface operad and a 1-ary operation, called the torus operation.

There is a symmetric monoidal category where the objects are disjoint finite unions of circles and morphisms are disjoint unions of $\mathcal{S}_{g,n}$ with the boundary components divided into inputs and outputs, where unlike above we allow for multiple output components. Segal [Seg04] then defines a conformal field theory as a symmetric monoidal functor from this category to an appropriate linear category. To understand when this theory behaves stably, we want the action of the torus operation to be invertible. This is a motivation for constructing a localization of an operad in [BBP$^+$18].

**Definition 2.4.10.** A *map of operads* $F : \mathcal{O} \to \mathcal{P}$ is a collection of $\Sigma_n$-equvariant maps $F_n : \mathcal{O}(n) \to \mathcal{P}(n)$ such that $F_1(1_{\mathcal{O}}) = 1_{\mathcal{P}}$ and $F$ commutes with the structure maps, meaning that for all $f_i \in \mathcal{O}(i)$ and $\sum j_i = j$,

$$F_j(\gamma_{\mathcal{O}}(f_k; f_{j_1}, \ldots, f_{j_k})) = \gamma_{\mathcal{P}}(F_k(f_k); F_{j_1}(f_{j_1}), \ldots, F_{j_k}(f_{j_k})).$$

When the context is clear, we will drop the subscript $n$ on the individual maps of $F$.

*Example* 2.4.11. A map of operads $F : \mathcal{O} \to \mathrm{End}_X$ gives $X$ the structure of an *algebra over the operad* $\mathcal{O}$. We can thus view the abstract operations of $\mathcal{O}$ as actual operations on an object $X$. The power of operads is that we can study a general abstract structure that we then realize as actual structure on an object.

For example, an algebra over the commutative operad is a commutative algebra. To see this, consider $\mathrm{Comm}(2) = *$ . A map of operads $F : \mathrm{Comm} \to \mathrm{End}_X$ will pick out a product on $X$, or in other words, an operation $f : X^2 \to X$. The structure of Comm determines the properties of this product. In particular, since $F$ is $\Sigma_n$-equivariant,

$$f \cdot (12) = F(*) \cdot (12) = F(* \cdot (12)) = F(*) = f.$$

# CHAPTER 3

## An Equivalence of Operads and Symmetric Monoidal Categories

Let $(\mathcal{V}, \otimes, 1_{\mathcal{V}})$ be a closed symmetric monoidal concrete category tensored over sets which has finite colimits. Primarily, we will think of the cases when $\mathcal{V}$ is $\mathsf{Set}$ or $\mathsf{sSet}$. Let $\mathsf{Operad}$ denote the category of operads in $\mathcal{V}$, and $\mathsf{SymmMod}$ denote the category of symmetric monoidal categories enriched in $\mathcal{V}$.

In this chapter, we associate to every operad $\mathcal{O}$ in $\mathsf{Operad}$ a symmetric monoidal category $\widetilde{\mathcal{O}}$, following [BBP+18] and [MZZ20]. We then define a subcategory $\mathsf{SM}_{\mathsf{O}}$ of $\mathsf{SymmMod}$ which is more restrictive than in the existing literature, and show that this assignment gives rise to a functor $\mathfrak{F} : \mathsf{Operad} \to \mathsf{SM}_{\mathsf{O}}$. We also define a functor $\mathfrak{G} : \mathsf{SM}_{\mathsf{O}} \to \mathsf{Operad}$ and prove that together these two functors give an equivalence of categories.

Note that we have reversed many of the conventions in [BBP+18]. For example, they give $\widetilde{\mathcal{O}}(1, n) := \mathcal{O}(n)$, and we give $\widetilde{\mathcal{O}}(n, 1) := \mathcal{O}(n)$.

### 3.1 Symmetric Monoidal Envelope

**Lemma 3.1.1.** *Every operad $\mathcal{O} \in \mathsf{Operad}$ gives rise to a symmetric strict monoidal category enriched in $\mathcal{V}$.*

*Proof.* Let $\mathcal{O} \in \mathsf{Operad}$. We will construct a symmetric strict monoidal category $\widetilde{\mathcal{O}}$. The objects of $\widetilde{\mathcal{O}}$ are $n \in \mathbb{N}$. The morphisms are given by:

$$\widetilde{\mathcal{O}}(n, 1) := \mathcal{O}(n)$$

$$\widetilde{\mathcal{O}}(m,n) := \coprod_{\sum m_i = m} \left(\widetilde{\mathcal{O}}(m_1,1)\otimes \ldots \otimes \widetilde{\mathcal{O}}(m_n,1)\right) \otimes_{\Sigma_{m_1}\times\ldots\times\Sigma_{m_n}} \Sigma_m.$$

The terms of the coproduct are coequalizers, where $\Sigma_{m_i}$ acts on $\widetilde{\mathcal{O}}(m_i,1) = \mathcal{O}(m_i)$ on the right by the symmetric action on $\mathcal{O}$, and $\Sigma_{m_1}\times\ldots\times\Sigma_{m_n}$ acts on $\Sigma_m$ on the left by multiplication by the image of the natural inclusion of $\Sigma_{m_1}\times\ldots\times\Sigma_{m_n}$ into $\Sigma_m$.

For $g \in \widetilde{\mathcal{O}}(m,n)$, we will write

$$g = [(g_1,\ldots,g_n,\sigma),(m_1,\ldots,m_n)] \tag{3.1.1}$$

where $\sum m_i = m$, each $g_i \in \widetilde{\mathcal{O}}(m_i,1)$, and $\sigma \in \Sigma_m$. See Figure 3.1 for examples. For $f \in \mathcal{O}(n)$, we view $f$ as a morphism in $\widetilde{\mathcal{O}}(n,1)$ by writing $f = [(f,e_n),(n)]$, where $e_n \in \Sigma_n$ is the identity permutation. We will have

$$[(g_1,\ldots,g_n,\sigma),(m_1,\ldots,m_n)] = [(g_1',\ldots,g_n',\sigma'),(m_1',\ldots,m_n')]$$

if each $m_i = m_i'$ and if there exists $(\alpha_1,\ldots,\alpha_n) \in \Sigma_{m_1}\times\ldots\times\Sigma_{m_n}$ such that $g_i\cdot\alpha_i = g_i'$ for all $i$ and $(\alpha_1\oplus\ldots\oplus\alpha_n)\sigma' = \sigma$.

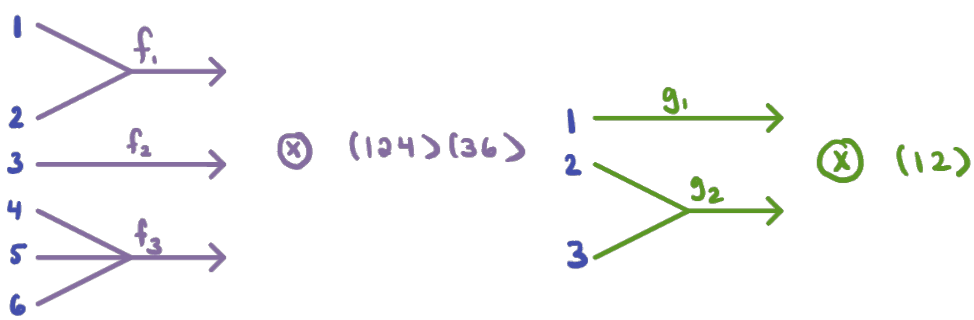



Figure 3.1: Depictions of example morphisms $f = [(f_1,f_2,f_3,(124)(36)),(2,1,3)] \in \widetilde{\mathcal{O}}(6,3)$ and $g = [(g_1,g_2,(12)),(1,2)] \in \widetilde{\mathcal{O}}(3,2)$.

Note that the inclusion of the permutation from $\Sigma_m$ encodes the symmetric action of $\Sigma_m$ on $\mathcal{O}(m)$. For $f \in \mathcal{O}(m)$, $\sigma \in \Sigma_m$, we can equivalently express $f\cdot\sigma \in \widetilde{\mathcal{O}}(m,1)$ as either of the following:

$$[(f,\sigma),(m)] = [(f\cdot\sigma, e_m),(m)].$$

We also define $\widetilde{\mathcal{O}}(0,0) := 1_{\mathcal{V}}$ and $\widetilde{\mathcal{O}}(b,0) := \emptyset$ for $b>0$. Note that $\widetilde{\mathcal{O}}(0,a) = \mathcal{O}(0)^{\otimes a}$ for $a>0$.

The identity morphisms $\mathrm{Id}_n \in \widetilde{\mathcal{O}}(n,n)$ are given by

$$\mathrm{Id}_n = [(\underbrace{1_{\mathcal{O}},\ldots,1_{\mathcal{O}}}_{n\text{-times}}, e_n),(\underbrace{1,\ldots,1}_{n\text{-times}})]$$

Composition is given by maps $\circ : \widetilde{\mathcal{O}}(n,p)\otimes\widetilde{\mathcal{O}}(m,n)\to\widetilde{\mathcal{O}}(m,p)$. For

$$g\otimes f = [(g_1,\ldots,g_p,\sigma),(n_1,\ldots,n_p)]\otimes[(f_1,\ldots,f_n,\tau),(m_1,\ldots,m_n)] \in \widetilde{\mathcal{O}}(n,p)\otimes\widetilde{\mathcal{O}}(m,n),$$

we define

$$\begin{aligned} g\circ f = [(&\gamma(g_1; f_{\sigma^{-1}(1)},\ldots,f_{\sigma^{-1}(n_1)}),\ldots,\gamma(g_p; f_{\sigma^{-1}(n-n_p+1)},\ldots,f_{\sigma^{-1}(n)}),\\ &\sigma(m_1,\ldots,m_n)\tau),\left(\sum_{i=1}^{n_1} m_{\sigma^{-1}(i)},\ldots,\sum_{i=n-n_p+1}^{n} m_{\sigma^{-1}(i)}\right)\Bigg] \in \widetilde{\mathcal{O}}(m,p). \end{aligned} \tag{3.1.2}$$

Note that $g_1\in\mathcal{O}(n_1)$, so we can calculate $\gamma(g_1; f_{\sigma^{-1}(1)},\ldots,f_{\sigma^{-1}(n_1)})$ as it has the correct number of terms. Furthermore, the image of this calculation will lie in $\mathcal{O}\left(\sum_{i=1}^{n_1} m_{\sigma^{-1}(i)}\right)$, as required by the first term of the partition of $m$. The same applies for the other $\gamma$ terms of $g\circ f$. Also, we can verify that we do in fact give a partition of $m$ since each $m_i$ term appears exactly once.

Finally we describe $\sigma(m_1,\ldots,m_n)\tau\in\Sigma_m$. This is the permutation resulting from first applying $\tau$, separating the resulting permutation into $n$ blocks with the $i$th block of length $m_i$, and then applying $\sigma$ to the blocks. For example, given $f,g$ as in Figure 3.1, we have

$\sigma = (12) \in \Sigma_3$, $\tau = (124)(36) \in \Sigma_6$, and $(m_1, m_2, m_3) = (2, 1, 3)$. We follow the above steps:

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ [2 & 4] & [6] & [1 & 5 & 3] \\ 6 & 2 & 4 & 1 & 5 & 3 \end{array}$$

resulting in the permutation (1634). In Figure 3.2, we depict the composition $g \circ f$ of the morphisms from Figure 3.1.

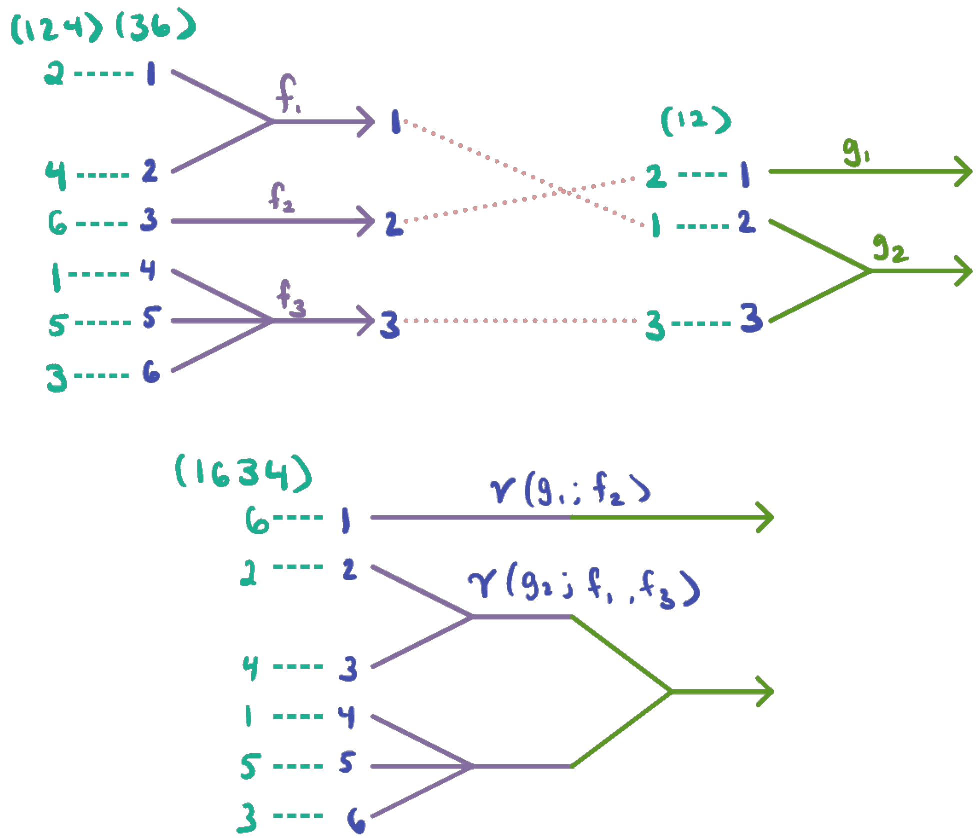



Figure 3.2: The composition $g \circ f = [(\gamma(g_1; f_2), \gamma(g_2; f_1, f_3), (1634)), (1, 5)]$

The composition is well defined because the structure maps are suitably equivariant with respect to the $\Sigma_n$ actions. We can verify that composition on either side of $f$ with the appropriate identity results in $f$, and that the composition is associative, and thus $\widetilde{\mathcal{O}}$ is a category.

The monoidal product is given by a functor $\boxtimes : \widetilde{\mathcal{O}} \times \widetilde{\mathcal{O}} \to \widetilde{\mathcal{O}}$. On objects, $m \boxtimes n := m + n$.

On morphisms, for

$$f \otimes g = [(f_1, \ldots, f_p, \sigma), (m_1, \ldots, m_p)] \otimes [(g_1, \ldots, g_q, \tau), (n_1, \ldots, n_q)] \in \widetilde{\mathcal{O}}(m, p) \otimes \widetilde{\mathcal{O}}(n, q),$$

we define

$$f \boxtimes g = [(f_1, \ldots, f_p, g_1, \ldots, g_q, \sigma \oplus \tau), (m_1, \ldots, m_p, n_1, \ldots, n_q)] \in \widetilde{\mathcal{O}}(m + n, p + q)$$

where $\sigma \oplus \tau$ is the image of $(\sigma, \tau)$ in the canonical inclusion $\Sigma_m \times \Sigma_n \to \Sigma_{m+n}$. In other words, the permutation $\sigma \oplus \tau$ applies $\sigma$ to the first $m$ letters and $\tau$ to the last $n$ letters.

It is straightforward to check that $\boxtimes$ preserves identities and compositions in $\widetilde{\mathcal{O}} \times \widetilde{\mathcal{O}}$, and thus is a bifunctor. The identity object of $\widetilde{\mathcal{O}}$ is 0, and we let the associator and unit maps be the identity. Thus $\widetilde{\mathcal{O}}$ is a strict monoidal category.

We then define the symmetry isomorphisms $s_{m,n} : m \boxtimes n \to n \boxtimes m$ for each pair $m, n \in \widetilde{\mathcal{O}}$ by

$$s_{m,n} = [(\underbrace{1_{\mathcal{O}}, \ldots, 1_{\mathcal{O}}}_{m+n\text{-times}}, (m, n)), (\underbrace{1, \ldots, 1}_{m+n\text{-times}})]$$

where $(m, n)$ is the permutation sending the first $m$ letters to the end. Note that for a letter $i$ in $\{1, \ldots m + n\}$ and $\sigma \in \Sigma_m$, $\tau \in \Sigma_n$,

$$(m, n)(\sigma \oplus \tau)(i) = \begin{cases} m + \sigma(i) & i \leq m \\ \tau(i - m) & i > m \end{cases} = (\tau \oplus \sigma)(m, n)(i) \tag{3.1.3}$$

Thus for $f \in \widetilde{\mathcal{O}}(m, p)$, $g \in \widetilde{\mathcal{O}}(n, q)$, we can show the following diagram commutes:

$$\begin{array}{ccc} m \boxtimes n & \xrightarrow{s_{m,n}} & n \boxtimes m \\ {\scriptstyle f \boxtimes g}\downarrow & & \downarrow{\scriptstyle g \boxtimes f} \\ p \boxtimes q & \xrightarrow[s_{p,q}]{} & q \boxtimes p \end{array}$$

Thus $s_{m,n}$ is natural in both $m$ and $n$. We can then show that

$$s_{m,0} = \mathrm{Id}_m,$$

$$(s_{m,p} \boxtimes \mathrm{Id}_n) \circ (\mathrm{Id}_m \boxtimes s_{n,p}) = s_{m+n,p},$$

$$s_{m,n} \circ s_{n,m} = \mathrm{Id}_{m+n}.$$

Hence $\widetilde{\mathcal{O}}$ is a symmetric strict monoidal category. □

**Definition 3.1.2.** Let $\mathcal{O} \in \mathsf{Operad}$. Then we call $\widetilde{\mathcal{O}} \in \mathsf{SymmMod}$ its *symmetric monoidal envelope* [MZZ20, 10.9].

*Remark* 3.1.3. For every $n$, we can define a map of monoids $i_n : \Sigma_n \to \widetilde{\mathcal{O}}(n,n)$ as the following composition:

$$\Sigma_n \cong 1_{\mathcal{V}}^{\otimes n} \otimes \Sigma_n \xrightarrow{1_{\mathcal{O}}^{\otimes n} \otimes \mathrm{Id}_{\Sigma_n}} \mathcal{O}(1)^{\otimes n} \otimes \Sigma_n = (\widetilde{\mathcal{O}}(1,1))^{\otimes n} \otimes \Sigma_n \hookrightarrow \widetilde{\mathcal{O}}(n,n)$$

For $\sigma \in \Sigma_n$,

$$i_n(\sigma) = [(\underbrace{1_{\mathcal{O}}, \ldots, 1_{\mathcal{O}}}_{n\text{-times}}, \sigma), (\underbrace{1, \ldots, 1}_{n\text{-times}})].$$

Then for every $m \geq 0$, $n \geq 1$, $\widetilde{\mathcal{O}}(m,n)$ has a right $\Sigma_m$-action defined by pre-compostion with $i_m$, and a left $\Sigma_n$-action defined by post-composition with $i_n$. Thus for $f \in \widetilde{\mathcal{O}}(m,n)$, $\tau \in \Sigma_m$, and $\epsilon \in \Sigma_n$,

$$f \cdot \tau = f \circ i_m(\tau) = [(f_1, \ldots, f_n, \sigma\tau), (m_1, \ldots, m_n)],$$

$$\epsilon \cdot f = i_n(\epsilon) \circ f = [f_{\sigma^{-1}(1)}, \ldots, f_{\sigma^{-1}(n)}, \epsilon(m_1, \ldots, m_n)\sigma), (m_{\sigma^{-1}(1)}, \ldots, m_{\sigma^{-1}(n)})].$$

In particular, since these actions are defined by pre- and post-composition, they commute with one another. In other words, $\epsilon \cdot (f \cdot \tau) = (\epsilon \cdot f) \cdot \tau$. Also, this gives us that $s_{m,n} = i_{m+n}((m,n))$.

## 3.2 The Category $\mathsf{SM_O}$

In this section, we draw inspiration from the properties of $\widetilde{\mathcal{O}}$ to develop a description of the objects in $\mathsf{SM_O}$. We define the category $\mathsf{SM_O}$ and describe its relationship to PROPs.

**Lemma 3.2.1.** *Let $\mathcal{K}$ be a symmetric strict monoidal category enriched in $\mathcal{V}$ with object monoid $(\mathbb{N}, 0, +)$ and a map of monoids $i_n : \Sigma_n \to \mathcal{K}(n,n)$ for all $n$ such that*

$$i_m(\sigma) \boxtimes i_n(\tau) = i_{m+n}(\sigma \oplus \tau)$$

*for all $\sigma \in \Sigma_m$, $\tau \in \Sigma_n$. Then the map below is well defined for all $m \geq 0$ and $n \geq 1$.*

$$\begin{array}{c} \coprod_{\sum m_i = m} (\mathcal{K}(m_1,1) \otimes \ldots \otimes \mathcal{K}(m_n,1)) \otimes_{\Sigma_{m_1} \times \ldots \times \Sigma_{m_n}} \Sigma_m \\ \Big\downarrow {\scriptstyle \boxtimes \times i_m} \\ \mathcal{K}(m,n) \otimes \mathcal{K}(m,m) \\ \Big\downarrow {\scriptstyle \circ} \\ \mathcal{K}(m,n) \end{array} \tag{3.2.1}$$

*Proof.* In the domain of 3.2.1, the action of $\Sigma_{m_i}$ on $\mathcal{K}(m_i, 1)$ is given by pre-composition with the image of $i_{m_i}$, and the action of $\Sigma_{m_1} \times \ldots \times \Sigma_{m_n}$ is given by left multiplication by the image of the inclusion into $\Sigma_m$. We will use notation 3.1.1 to refer to objects in the domain.

Suppose $(f_1, \ldots, f_n, \sigma) \sim (g_1, \ldots, g_n, \tau)$ in the domain of map 3.2.1. Then there exists $(\alpha_1, \ldots, \alpha_n) \in \Sigma_{m_1} \times \ldots \times \Sigma_{m_n}$ such that $g_i = f_i \cdot \alpha_i = f_i \circ i_{m_i}(\alpha_i)$ and

$$\sigma = (\alpha_1 \oplus \ldots \oplus \alpha_n)\tau.$$

Then applying map 3.2.1, we get

$$\begin{aligned}(f_1 \boxtimes \ldots \boxtimes f_n) \circ i_m(\sigma) &= (f_1 \boxtimes \ldots \boxtimes f_n) \circ i_m((\alpha_1 \oplus \ldots \oplus \alpha_n)\tau) \\ &= (f_1 \boxtimes \ldots \boxtimes f_n) \circ ((i_{m_1}(\alpha_1) \boxtimes \ldots \boxtimes i_{m_n}(\alpha_n)) \circ i_m(\tau)) \\ &= ((f_1 \circ i_{m_1}(\alpha_1)) \boxtimes \ldots \boxtimes (f_n \circ i_{m_n}(\alpha_n))) \circ i_m(\tau) \\ &= (g_1 \boxtimes \ldots \boxtimes g_n) \circ i_m(\tau).\end{aligned}$$

Thus map 3.2.1 is well defined. □

For a fixed $m, n$, we call map 3.2.1 $\eta_{m,n}$. Note that if $\eta_{m,n}$ is an isomorphism, then for all $f \in \mathcal{K}(m,n)$,

$$f = \eta_{m,n}(\eta_{m,n}^{-1}(f)) = (f_1 \boxtimes \ldots \boxtimes f_n) \circ i_m(\sigma)$$

though this decomposition description may not be unique as shown above.

$\mathcal{K}(m,n)$ has a right-$\Sigma_m$ and a left-$\Sigma_n$ action defined by pre- and post-composition with $i_m$ and $i_n$ respectively. The domain of $\eta_{m,n}$ also has a right-$\Sigma_m$ action defined by right multiplication on the $\Sigma_m$ factor.

**Lemma 3.2.2.** *We can define a left $\Sigma_n$ action on*

$$\coprod_{\sum m_i = m} (\mathcal{K}(m_1, 1) \otimes \ldots \otimes \mathcal{K}(m_n, 1)) \otimes_{\Sigma_{m_1} \times \ldots \times \Sigma_{m_n}} \Sigma_m$$

*as follows for $\tau \in \Sigma_n$:*

$$\begin{aligned}\tau \cdot [(f_1, \ldots, f_n, \sigma), (m_1, \ldots, m_n)] &= [(f_{\tau^{-1}(1)}, \ldots, f_{\tau^{-1}(n)}, \tau(m_1, \ldots m_n)\sigma), \\ &\quad (m_{\tau^{-1}(1)}, \ldots m_{\tau^{-1}(n)})].\end{aligned}$$

*Proof.* We have

$$e_n \cdot [(f_1, \ldots, f_n, \sigma), (m_1, \ldots, m_n)] = [(f_1, \ldots, f_n, \sigma), (m_1, \ldots, m_n)].$$

Let $\tau, \epsilon \in \Sigma_n$, and let $(m_1, \ldots, m_n)$ be a partition of $m$. Then note that

$$(\epsilon\tau)(m_1, \ldots, m_n) = \epsilon(m_{\tau^{-1}(1)}, \ldots, m_{\tau^{-1}(n)})\tau(m_1, \ldots, m_n).$$

Thus

$$\begin{aligned}
\epsilon \cdot (\tau \cdot [(f_1, \ldots, f_n, \sigma), (m_1, \ldots, m_n)]) &= \\
\epsilon \cdot [(f_{\tau^{-1}(1)}, \ldots, f_{\tau^{-1}(n)}, \tau(m_1, \ldots m_n)\sigma), (m_{\tau^{-1}(1)}, \ldots, m_{\tau^{-1}(n)})] &= \\
\big[f_{\tau^{-1}(\epsilon^{-1}(1))}, \ldots f_{\tau^{-1}(\epsilon^{-1}(n))}, \epsilon(m_{\tau^{-1}(1)}, \ldots, m_{\tau^{-1}(n)})\tau(m_1, \ldots, m_n)\sigma) & \\
(m_{\tau^{-1}(\epsilon^{-1}(1))}, \ldots, m_{\tau^{-1}(\epsilon^{-1}(n))}\big] &= \\
\big[[(f_{(\epsilon\tau)^{-1}(1)}, \ldots, f_{(\epsilon\tau)^{-1}(n)}, (\epsilon\tau)(m_1, \ldots m_n)\sigma), (m_{(\epsilon\tau)^{-1}(1)}, \ldots, m_{(\epsilon\tau)^{-1}(n)})]\big] &= \\
(\epsilon\tau) \cdot [(f_1, \ldots, f_n, \sigma), (m_1, \ldots, m_n)] &
\end{aligned}$$

Hence we have a left $\Sigma_n$ action. □

**Theorem 3.2.3.** *There exists a subcategory of* SymmMod*, called* $\mathsf{SM_O}$ *with*

- *objects* $\mathcal{K}$ *symmetric strict monoidal categories enriched in* $\mathcal{V}$ *with*
  - *object monoid* $(\mathbb{N}, 0, +)$*;*
  - *for all* $n$*, a map of monoids* $i_n : \Sigma_n \to \mathcal{K}(n, n)$ *such that* $i_m(\sigma) \boxtimes i_n(\tau) = i_{m+n}(\sigma \oplus \tau)$ *for all* $\sigma \in \Sigma_n$*,* $\tau \in \Sigma_m$*;*
  - $s_{m,n} = i_{m+n}((m, n))$ *for all* $m, n$*,*

*and where for all* $m \geq 0$ *and* $n \geq 1$*, the map*

$$\begin{array}{c} \coprod_{\sum m_i = m} (\mathcal{K}(m_1,1)\otimes \ldots \otimes \mathcal{K}(m_n,1)) \otimes_{\Sigma_{m_1}\times\ldots\times\Sigma_{m_n}} \Sigma_m \\ \Big\downarrow{\scriptstyle \boxtimes \times i_m} \\ \mathcal{K}(m,n)\otimes\mathcal{K}(m,m) \\ \Big\downarrow{\scriptstyle \circ} \\ \mathcal{K}(m,n) \end{array} \tag{3.2.2}$$

*is a left-$\Sigma_n$ and right-$\Sigma_m$ equivariant isomorphism.*

- *morphisms $F : \mathcal{K} \to \mathcal{L}$ strict symmetric monoidal functors such that $F$ is the identity on objects and $F \circ i_n^{\mathcal{K}} = i_n^{\mathcal{L}}$.*

*Proof.* We first check that the chosen $s_{m,n}$ fulfill the necessary properties. We claim that $s_{p,q} \circ (f \boxtimes g) = (g \boxtimes f) \circ s_{m,n}$ for $f \in \mathcal{K}(m,p)$, $g \in \mathcal{K}(n,q)$:

$$\begin{aligned} i_{p+q}((p,q)) \circ (f \boxtimes g) &= i_{p+q}((p,q)) \circ (((f_1 \boxtimes \ldots \boxtimes f_p) \circ i_m(\sigma)) \boxtimes ((g_1 \boxtimes \ldots \boxtimes g_q) \circ i_n(\tau))) \\ &= i_{p+q}((p,q)) \circ ((f_1 \boxtimes \ldots \boxtimes f_p \boxtimes g_1 \boxtimes \ldots \boxtimes g_q) \circ (i_m(\sigma) \boxtimes i_n(\tau)) \\ &= (g_1 \boxtimes \ldots \boxtimes g_q \boxtimes f_1 \boxtimes \ldots \boxtimes f_p) \circ \\ &\quad\; i_{m+n}((p,q)(m_1,\ldots,m_p,n_1,\ldots,n_q)(\sigma \oplus \tau)) \\ &= (g_1 \boxtimes \ldots \boxtimes g_q \boxtimes f_1 \boxtimes \ldots \boxtimes f_p) \circ i_{m+n}((\tau \oplus \sigma)(m,n)) \\ &= ((g_1 \boxtimes \ldots \boxtimes g_q \boxtimes f_1 \boxtimes \ldots \boxtimes f_p) \circ (i_n(\tau) \boxtimes i_m(\sigma))) \circ i_{m+n}((m,n)) \\ &= (g \boxtimes f) \circ i_{m+n}((m,n)) \end{aligned}$$

Where the third equality is given by left-$\Sigma_n$ equivariance of 3.2.2 and the monoidal property on the $i_n$, and the fourth equality is given by 3.1.3, since $(p,q)(m_1,\ldots,m_p,n_1,\ldots,n_q) = (m,n)$.

Then since $(m,0) = e_m$,

$$i_{m+0}((m,0)) = \mathrm{Id}_m .$$

Also, $((m,p) \oplus e_n)(e_m \oplus (n,p)) = (m+n,p)$ implies

$$(i_{m+p}((m,p)) \boxtimes \mathrm{Id}_n) \circ (\mathrm{Id}_m \boxtimes i_{n+p}((n,p))) = i_{(m+n)+p}((m+n,p)).$$

Finally, $(m,n)(n,m) = e_{n+m}$ gives

$$i_{m+n}((m,n)) \circ i_{n+m}((n,m)) = \mathrm{Id}_{m+n}\,.$$

Thus setting $s_{m,n} = i_{m,n}((m,n))$ will fulfil the properties given in Defintion 2.1.14.

Next we check that $\mathsf{SM_O}$ is a subcategory of $\mathsf{SymmMod}$. In other words, we verify that $\mathsf{SM_O}$ contains identities for all objects in $\mathsf{SM_O}$ and is closed under composition. Identities are in particular the identity on objects and on the image of the $i_n$, and thus for any $\mathcal{K} \in \mathsf{SM_O}$, $\mathrm{Id}_{\mathcal{K}} \in \mathsf{SM_O}$.

Suppose $F : \mathcal{K} \to \mathcal{L}$, $G : \mathcal{L} \to \mathcal{M}$ are maps in $\mathsf{SM_O}$. Note that the composition of two strict symmetric monoidal functors is a strict symmetric monoidal functor. Furthermore, $GF$ is the identity on objects, and for all $\sigma \in \Sigma_n$, $G(F(i_n^{\mathcal{K}}(\sigma))) = G(i_n^{\mathcal{L}}(\sigma)) = i_n^{\mathcal{M}}(\sigma)$. Thus $GF$ is in $\mathsf{SM_O}$, and $\mathsf{SM_O}$ is therefore a subcategory of $\mathsf{SymmMod}$. □

Note that for any operad $\mathcal{O}$, $\widetilde{\mathcal{O}}$ is of course in $\mathsf{SM_O}$.

**Corollary 3.2.4.** *For $\mathcal{K} \in \mathsf{SM_O}$, $f = (f_1 \boxtimes \ldots \boxtimes f_n) \circ i_m(\sigma) \in \mathcal{K}(m,n)$, $\epsilon \in \Sigma_m$, $\tau \in \Sigma_n$, we can describe pre- and post-composition with $i_m$ and $i_n$ respectively as*

$$f \circ i_m(\epsilon) = (f_1 \boxtimes \ldots \boxtimes f_n) \circ i_m(\sigma\epsilon)$$

$$i_n(\tau) \circ f = (f_{\tau^{-1}(1)} \boxtimes \ldots \boxtimes f_{\tau^{-1}(n)}) \circ i_m(\tau(m_1, \ldots, m_n)\sigma)$$

*Proof.* This follows immediately from the equivariance of 3.2.2. □

*Remark* 3.2.5. Versions of $\mathsf{SM_O}$ were described in [BBP+18] and [MZZ20] but in order to give the equivalence in Theorem 3.3.3, a more restrictive definition is needed.

*Remark* 3.2.6. $\mathsf{SM_O}$ is in fact a subcategory of $\mathsf{Perm}$, the category of permutative categories, which consists of symmetric strict monoidal categories and strict symmetric monoidal functors.

*Remark* 3.2.7. The objects in $\mathsf{SM_O}$ fit the definition of a PROP as given by Mac Lane [Lan65, 24]. Let $\mathcal{K} \in \mathsf{SM_O}$. As in Mac Lane's definition, we have a category with objects the natural numbers, a monoidal product given by addition on the objects, and we can see $\Sigma_n$ as a subgroup of the invertible morphisms in $\mathcal{K}(n,n)$ through the image of $i_n$, where $i_n(e_n) = \mathrm{Id}_n$. $\mathcal{K}$ also satisfies the three additional axioms given by Mac Lane:

- The monoidal product is strictly associative since $\mathcal{K}$ is a symmetric strict monoidal category.
- $i_m(\sigma) \boxtimes i_n(\tau) = i_{m+n}(\sigma \oplus \tau)$.
- For $f \in \mathcal{K}(m,p)$, $g \in \mathcal{K}(n,q)$, $i_{p+q}((p,q)) \circ (f \boxtimes g) = (g \boxtimes f) \circ i_{m+n}((m,n))$.

The second and third axioms are the additional properties that we require of the $i_n$ maps for objects in $\mathsf{SM_O}$ which were not included in [BBP+18] and [MZZ20]. In Theorem 3.3.3, we will prove that $\mathsf{SM_O}$ is equivalent to $\mathsf{Operad}$. Since operads were initially defined as a special type of PROP [Vog98], it is reasonable to assume that the above axioms would be the minimal additional structure we would need.

Another way to view Theorem 3.3.3 is that the decomposition of $\mathcal{K}(m,n)$ along with the $\Sigma_n$ and $\Sigma_m$ equivariance is a way to identify PROPs which are in fact operads; or equivalently ones that can be expressed "in standard form" as in [Vog98].

### 3.3 An Equivalence Between Operad and $\mathsf{SM_O}$

In this section we will prove that $\mathsf{Operad} \simeq \mathsf{SM_O}$. We will first construct a functor $\mathfrak{F} : \mathsf{Operad} \to \mathsf{SM_O}$ (Lemma 3.3.1), then a functor $\mathfrak{G} : \mathsf{SM_O} \to \mathsf{Operad}$ (Lemma 3.3.2), and finally show that $\mathfrak{F}$ and $\mathfrak{G}$ give an equivalence of categories (Theorem 3.3.3).

**Lemma 3.3.1.** *There exists a functor $\mathfrak{F} : \mathsf{Operad} \to \mathsf{SM_O}$ which takes $\mathcal{O} \in \mathsf{Operad}$ to its symmetric monoidal envelope, $\widetilde{\mathcal{O}}$.*

*Proof.* We will verify that the assignment on objects $\mathfrak{F}(\mathcal{O}) = \widetilde{\mathcal{O}}$ extends to a functor $\mathfrak{F} : \mathsf{Operad} \to \mathsf{SM_O}$. For $F : \mathcal{O} \to \mathcal{P}$, define $\widetilde{F} := \mathfrak{F}(F)$ as follows:

- On objects, $\widetilde{F}$ is the identity.
- For $g \in \widetilde{\mathcal{O}}(n,1) = \mathcal{O}(n)$, $\widetilde{F}(g) = F_n(g) \in \mathcal{P}(n) = \widetilde{\mathcal{P}}(n,1)$.
- For $g \in \widetilde{\mathcal{O}}(m,n)$, we apply $F$ to each component of $g$:

$$g = [(g_1, \ldots, g_n, \sigma), (m_1, \ldots, m_n)] \in \coprod_{\sum m_i = m} \left(\widetilde{\mathcal{O}}(m_1,1) \otimes \ldots \otimes \widetilde{\mathcal{O}}(m_n,1)\right) \otimes_{\Sigma_{m_1} \times \ldots \times \Sigma_{m_n}} \Sigma_m$$

  maps to

$$\widetilde{F}(g) = [(F_{m_1}(g_1), \ldots, F_{m_n}(g_n), \sigma), (m_1, \ldots, m_n)]$$

$$\in \coprod_{\sum m_i = m} \left(\widetilde{\mathcal{P}}(m_1,1) \otimes \ldots \otimes \widetilde{\mathcal{P}}(m_n,1)\right) \otimes_{\Sigma_{m_1} \times \ldots \times \Sigma_{m_n}} \Sigma_m$$

We claim that $\widetilde{F}$ gives a functor $\widetilde{O} \to \widetilde{P}$ in $\mathsf{SM_O}$.

$\widetilde{F}$ **is well defined:** Suppose

$$[(g_1, \ldots, g_n, \sigma), (m_1, \ldots, m_n)] = [(h_1, \ldots, h_n, \tau), (m_1, \ldots, m_n)],$$

so that there exists $(\alpha_1, \ldots, \alpha_n) \in \Sigma_{m_1} \times \ldots \times \Sigma_{m_n}$ such that $g_i \cdot \alpha_i = h_i$ for $1 \leq i \leq n$ and $(\alpha_1 \oplus \ldots \oplus \alpha_n)\tau = \sigma$. Then $F_{m_i}(g_i) \cdot \alpha_i = F_{m_i}(g_i \cdot \alpha_i) = F_{m_i}(h_i)$, since $F$ is a map of operads. Thus $\widetilde{F}$ is well defined.

$\widetilde{F}$ **is a functor in $\mathsf{SM_O}$:** We must verify that $\widetilde{F}$ is a functor; meaning that it preserves identities and composition, that it is in particular a strict symmetric monoidal functor, that it is the identity on objects, and that $\widetilde{F} \circ i_n^{\widetilde{\mathcal{O}}} = i_n^{\widetilde{\mathcal{P}}}$.

We claim that $\widetilde{F}$ preserves identities:

$$\widetilde{F}(\mathrm{Id}_n^{\widetilde{\mathcal{O}}}) = \widetilde{F}\left([(1_{\mathcal{O}}, \ldots, 1_{\mathcal{O}}, e_n), (1, \ldots, 1)]\right) = [(1_{\mathcal{P}}, \ldots, 1_{\mathcal{P}}, e_n), (1, \ldots, 1)] = \mathrm{Id}_n^{\widetilde{\mathcal{P}}}$$

We claim that $\widetilde{F}$ preserves composition. Let $g \otimes h \in \mathcal{O}(n,p) \otimes \mathcal{O}(m,n)$, so that

$$\begin{aligned} g &= [(g_1, \ldots, g_p, \sigma), (n_1, \ldots, n_p)] \\ h &= [(h_1, \ldots, h_n, \tau), (m_1, \ldots, m_n)] \end{aligned}$$

and hence

$$gh = \Big[(\gamma(g_1; h_{\sigma^{-1}(1)}, \ldots, h_{\sigma^{-1}(n_1)}), \ldots, \gamma(g_p; h_{\sigma^{-1}(n-n_p+1)}, \ldots, h_{\sigma^{-1}(n)}),$$

$$\sigma(m_1, \ldots, m_n)\tau), \left(\sum_{i=1}^{n_1} m_{\sigma^{-1}(i)}, \ldots, \sum_{i=n-n_p+1}^{n} m_{\sigma^{-1}(i)}\right)\Big]$$

Note that as a map of operads, $F$ commutes with composition. Hence

$$\begin{aligned} \widetilde{F}(gh) &= \Big[(F(\gamma(g_1; h_{\sigma^{-1}(1)}, \ldots, h_{\sigma^{-1}(n_1)})), \ldots, F(\gamma(g_p; h_{\sigma^{-1}(n-n_p+1)}, \ldots, h_{\sigma^{-1}(n)})), \\ &\quad \sigma(m_1, \ldots, m_n)\tau), \left(\sum_{i=1}^{n_1} m_{\sigma^{-1}(i)}, \ldots, \sum_{i=n-n_p+1}^{n} m_{\sigma^{-1}(i)}\right)\Big] \\ &= \Big[(\gamma(F(g_1); F(h_{\sigma^{-1}(1)}), \ldots, F(h_{\sigma^{-1}(n_1)})), \ldots, \\ &\quad \gamma(F(g_p); F(h_{\sigma^{-1}(n-n_p+1)}), \ldots, F(h_{\sigma^{-1}(n)})), \\ &\quad \sigma(m_1, \ldots, m_n)\tau), \left(\sum_{i=1}^{n_1} m_{\sigma^{-1}(i)}, \ldots, \sum_{i=n-n_p+1}^{n} m_{\sigma^{-1}(i)}\right)\Big] \\ &= \widetilde{F}(g)\widetilde{F}(h) \end{aligned}$$

We next claim that $\widetilde{F}$ is a strict symmetric monoidal functor. Since $\widetilde{F}$ is the identity on objects, it is immediate that $\widetilde{F}$ preserves the monoidal unit and product on objects. We have that $\widetilde{F}(s^{\mathcal{O}}_{m,n}) = s^{\mathcal{P}}_{m,n}$, so $\widetilde{F}$ respects the symmetry. $\widetilde{F}$ also preserves the monoidal product on morphisms. Let $g \in \mathcal{O}(m,p)$, $h \in \mathcal{O}(n,q)$:

$$\begin{aligned}
\widetilde{F}(g \boxtimes h) &= \widetilde{F}\Big(\big[(g_1,\dots,g_p,\sigma),(m_1,\dots,m_p)\big] \boxtimes \big[(h_1,\dots,h_q,\tau),(n_1,\dots,n_q)\big]\Big) \\
&= \widetilde{F}\Big(\big[(g_1,\dots,g_p,h_1,\dots,h_q,\sigma\times\tau),(m_1,\dots,m_p,n_1,\dots,n_q)\big]\Big) \\
&= \big[(F_{m_1}(g_1),\dots,F_{m_p}(g_p),F_{n_1}(h_1),\dots,F_{n_q}(h_q),\sigma\times\tau), \\
&\quad (m_1,\dots,m_p,n_1,\dots,n_q)\big] \\
&= \big[(F_{m_1}(g_1),\dots,F_{m_p}(g_p),\sigma),(m_1,\dots,m_p)\big] \boxtimes \\
&\quad \big[(F_{n_1}(h_1),\dots,F_{n_q}(h_q),\tau),(n_1,\dots,n_q)\big] \\
&= \widetilde{F}(g) \boxtimes \widetilde{F}(h)
\end{aligned}$$

Finally we verify that $\widetilde{F} \circ i_n^{\widetilde{\mathcal{O}}} = i_n^{\widetilde{\mathcal{P}}}$ for all $n$. Let $\sigma \in \Sigma_n$:

$$\begin{aligned}
\widetilde{F} \circ i_n^{\widetilde{\mathcal{O}}}(\sigma) &= \widetilde{F}([1_{\mathcal{O}},\dots,1_{\mathcal{O}},\sigma),(1,\dots,1)]) \\
&= [(F_1(1_{\mathcal{O}}),\dots,F_1(1_{\mathcal{O}}),\sigma),(1,\dots,1)] \\
&= [1_{\mathcal{P}},\dots,1_{\mathcal{P}},\sigma),(1,\dots,1)] \\
&= i_n^{\widetilde{\mathcal{P}}}(\sigma)
\end{aligned}$$

Thus $\widetilde{F} : \widetilde{\mathcal{O}} \to \widetilde{\mathcal{P}}$ is indeed a morphism in $\mathsf{SM_O}$.

$\mathfrak{F} : \mathsf{Operad} \to \mathsf{SM_O}$ **is a functor:** Let $\mathcal{O} \in \mathsf{Operad}$. Then $\mathfrak{F}(\mathrm{Id}_{\mathcal{O}}) = \widetilde{\mathrm{Id}_{\mathcal{O}}}$. We claim this is equal to $\mathrm{Id}_{\widetilde{\mathcal{O}}}$. $\widetilde{\mathrm{Id}_{\mathcal{O}}}$ is the identity on objects, so it only remains to check that it is the identity on morphisms:

$$\begin{aligned}
\widetilde{\mathrm{Id}_{\mathcal{O}}}([(g_1,\dots,g_n,\sigma),(m_1,\dots,m_n)]) &= [(\mathrm{Id}_{\mathcal{O}}(g_1),\dots,\mathrm{Id}_{\mathcal{O}}(g_n),\sigma),(m_1,\dots,m_n)] \\
&= [(g_1,\dots,g_n,\sigma),(m_1,\dots,m_n)]
\end{aligned}$$

Hence $\mathfrak{F}$ preserves identities. Let $F : \mathcal{O} \to \mathcal{P}$ and $G : \mathcal{P} \to \mathcal{Q}$ be maps of operads. Since $\widetilde{F}$ and $\widetilde{G}$ are both the identity on objects, it is clear that on objects, $\widetilde{G} \circ \widetilde{F} = \widetilde{G \circ F}$. We verify that this holds on morphisms as well:

$$\begin{aligned}
\widetilde{G} \circ \widetilde{F}([(g_1, \dots, g_n, \sigma), (m_1, \dots, m_n)]) &= \widetilde{G}([(F(g_1), \dots, F(g_n), \sigma), (m_1, \dots, m_n)]) \\
&= [((G \circ F)(g_1), \dots, (G \circ F)(g_n), \sigma), (m_1, \dots, m_n)] \\
&= \widetilde{G \circ F}([(g_1, \dots, g_n, \sigma), (m_1, \dots, m_n)])
\end{aligned}$$

Thus $\mathfrak{F}$ preserves composition and is a functor.

□

**Lemma 3.3.2.** *There exists a functor* $\mathfrak{G} : \mathsf{SM_O} \to \mathsf{Operad}$.

*Proof.* For $\mathcal{K} \in \mathsf{SM_O}$, define $\mathcal{K}^{\mathsf{O}} := \mathfrak{G}(\mathcal{K}) \in \mathsf{Operad}$ where $\mathcal{K}^{\mathsf{O}}$ is the operad with

- $\mathcal{K}^{\mathsf{O}}(n) := \mathcal{K}(n, 1)$,
- For $\sigma \in \Sigma_n$, $f \in \mathcal{K}^{\mathsf{O}}(n)$, $f \cdot \sigma := f \circ (i_n(\sigma))$,
- $1_{\mathcal{K}^{\mathsf{O}}} := \mathrm{Id}_1 \in \mathcal{K}(1, 1)$,
- the composition $\gamma_{\mathcal{K}^{\mathsf{O}}}$ is given by the composites

$$\begin{array}{c}
\mathcal{K}^{\mathsf{O}}(k) \otimes \mathcal{K}^{\mathsf{O}}(j_1) \otimes ... \otimes \mathcal{K}^{\mathsf{O}}(j_k) = \mathcal{K}(k,1) \otimes \mathcal{K}(j_1,1) \otimes ... \otimes \mathcal{K}(j_k,1) \\
\big\downarrow {\scriptstyle \mathrm{Id} \otimes \boxtimes} \\
\mathcal{K}(k,1) \otimes \mathcal{K}(\Sigma j_i, k) \\
\big\downarrow {\scriptstyle \circ} \\
\mathcal{K}(\Sigma j_i, 1) = \mathcal{K}^{\mathsf{O}}(\Sigma j_i)
\end{array}$$

We verify that $\mathcal{K}^{\mathsf{O}}$ is an operad:

- $\gamma_{\mathcal{K}^{\mathsf{O}}}$ is associative: Let $f \in \mathcal{K}^{\mathsf{O}}(k)$, $g_i \in \mathcal{K}^{\mathsf{O}}(j_i)$, $\sum j_i = j$, $h_s \in \mathcal{K}^{\mathsf{O}}(t_s)$. Then

$$\begin{aligned}
\gamma(\gamma(f; g_1, \dots, g_k); h_1, \dots, h_{j_1}, \dots, h_{j-j_k+1}, \dots, h_j) &= \\
(f \circ (g_1 \boxtimes \dots \boxtimes g_k)) \circ (h_1 \boxtimes \dots \boxtimes h_{j_1} \boxtimes \dots \boxtimes h_{j-j_k+1} \boxtimes \dots \boxtimes h_j) &= \\
f \circ [(g_1 \circ (h_1 \boxtimes \dots \boxtimes h_{j_1})) \boxtimes \dots \boxtimes (g_k \circ (h_{j-j_k+1} \boxtimes \dots \boxtimes h_j))] &= \\
\gamma(f; \gamma(g_1; h_1, \dots, h_{j_1}), \dots, \gamma(g_k; h_{j-j_k+1}, \dots h_j)).
\end{aligned}$$

- $1_{\mathcal{K}^{\mathsf{O}}}$ is a unit for $\gamma_{\mathcal{K}^{\mathsf{O}}}$: Let $f \in \mathcal{K}^{\mathsf{O}}(n)$. Then

$$\gamma(1_{\mathcal{K}^{\mathsf{O}}}; f) = \mathrm{Id}_1 \circ f = f,$$

$$\gamma(f; \underbrace{1_{\mathcal{K}^{\mathsf{O}}}, \dots, 1_{\mathcal{K}^{\mathsf{O}}}}_{n\text{-times}}) = f \circ (\mathrm{Id}_1 \boxtimes \dots \boxtimes \mathrm{Id}_1) = f \circ \mathrm{Id}_n = f.$$

- $\gamma_{\mathcal{K}^{\mathsf{O}}}$ is equivariant with respect to the action of the symmetric groups: Let $f \in \mathcal{K}^{\mathsf{O}}(k)$, $g_i \in \mathcal{K}^{\mathsf{O}}(j_i)$, $j = \sum j_i$, $\sigma \in \Sigma_k$, $\tau_i \in \Sigma_{j_i}$. Then

$$\begin{aligned}
\gamma(f \cdot \sigma; g_1, \dots, g_k) &= (f \circ i_k(\sigma)) \circ (g_1 \boxtimes \dots \boxtimes g_k) \\
&= f \circ (g_{\sigma^{-1}(1)} \boxtimes \dots \boxtimes g_{\sigma^{-1}(k)}) \circ i_j(\sigma(j_1, \dots, j_k)) \\
&= \gamma(f; g_{\sigma^{-1}(1)}, \dots, g_{\sigma^{-1}(k)}) \cdot (\sigma(j_1, \dots, j_k))
\end{aligned}$$

where the second equality comes from the left $\Sigma_k$ equivariance of $\eta_{j,k}$. Also,

$$\begin{aligned}
\gamma(f; g_1 \cdot \tau_1, \dots g_k \cdot \tau_k) &= f \circ ((g_1 \circ i_{j_1}(\tau_1)) \boxtimes \dots \boxtimes (g_k \circ i_{j_k}(\tau_k)) \\
&= f \circ (g_1 \boxtimes \dots \boxtimes g_k) \circ (i_{j_1}(\tau_1) \boxtimes \dots \boxtimes i_{j_k}(\tau_k)) \\
&= f \circ (g_1 \boxtimes \dots \boxtimes g_k) \circ i_j(\tau_1 \oplus \dots \oplus \tau_k) \\
&= \gamma(f; g_1, \dots, g_k) \cdot (\tau_1 \oplus \dots \oplus \tau_k)
\end{aligned}$$

For $F : \mathcal{K} \to \mathcal{L}$, we define $F^{\mathsf{O}} := \mathfrak{G}(F)$ by $F^{\mathsf{O}}_n(f) = F(f)$.

We verify that $F^{\mathsf{O}}$ is a map of operads:

- $F^{\mathsf{O}}$ preserves the unit: $F^{\mathsf{O}}_1(1_{\mathcal{K}^{\mathsf{O}}}) = F(\mathrm{Id}^{\mathcal{K}}_1) = \mathrm{Id}^{\mathcal{L}}_1 = 1_{\mathcal{L}^{\mathsf{O}}}$.
- $F^{\mathsf{O}}$ commutes with the action of the symmetric groups: let $f \in \mathcal{K}^{\mathsf{O}}(n)$, $\sigma \in \Sigma_n$. Then

$$F^{\mathsf{O}}_n(f \cdot \sigma) = F(f \circ i^{\mathcal{K}}_n(\sigma)) = F(f) \circ F(i^{\mathcal{K}}_n(\sigma)) = F(f) \circ i^{\mathcal{L}}_n(\sigma) = F^{\mathsf{O}}_n(f) \cdot \sigma.$$

- $F^{\mathsf{O}}$ commutes with the composition $\gamma$: let $(f_k, f_{j_1}, \dots, f_{j_n}) \in \mathcal{K}^{\mathsf{O}}(k) \otimes \mathcal{K}^{\mathsf{O}}(j_1) \otimes \dots \otimes \mathcal{K}^{\mathsf{O}}(j_n)$. Then

$$\begin{aligned}
\gamma_{\mathcal{L}^{\mathsf{O}}}(F^{\mathsf{O}}_k(f_k); F^{\mathsf{O}}_{j_1}(f_{j_1}), \dots, F^{\mathsf{O}}_{j_n}(f_{j_n})) &= F(f_k) \circ (F(f_{j_1}) \boxtimes \dots \boxtimes F(f_{j_n})) \\
&= F(f_k) \circ F(f_{j_1} \boxtimes \dots \boxtimes f_{j_n}) \\
&= F(f_k \circ (f_{j_1} \boxtimes \dots \boxtimes f_{j_n})) \\
&= F^{\mathsf{O}}_{\Sigma j_i}(\gamma_{\mathcal{K}^{\mathsf{O}}}(f_k; f_{j_1}, \dots, f_{j_n})).
\end{aligned}$$

Let $\mathcal{K} \in \mathsf{SM}_{\mathsf{O}}$ and $f \in \mathcal{K}^{\mathsf{O}}(n)$. Then $(\mathrm{Id}_{\mathcal{K}})^{\mathsf{O}}_n(f) = \mathrm{Id}_{\mathcal{K}}(f) = f = \mathrm{Id}_{\mathcal{K}^{\mathsf{O}}n}(f)$, so $\mathfrak{G}$ preserves identity. Let $F : \mathcal{K} \to \mathcal{L}$ and $G : \mathcal{L} \to \mathcal{M}$ be maps in $\mathsf{SM}_{\mathsf{O}}$. Then

$$(G^{\mathsf{O}} \circ F^{\mathsf{O}})_n(f) = G^{\mathsf{O}}_n(F(f)) = G(F(f)) = (G \circ F)^{\mathsf{O}}_n(f)$$

so $\mathfrak{G}$ preserves composition. Therefore $\mathfrak{G} : \mathsf{SM}_{\mathsf{O}} \to \mathsf{Operad}$ is a functor. □

**Theorem 3.3.3.** *$\mathfrak{F}$ and $\mathfrak{G}$ give an equivalence between* $\mathsf{Operad}$ *and* $\mathsf{SM}_{\mathsf{O}}$.

*Proof.* We will show that $\mathfrak{G}\mathfrak{F} = \mathrm{Id}_{\mathsf{Operad}}$ and then that $\mathfrak{F}\mathfrak{G} \cong \mathrm{Id}_{\mathsf{SM}_{\mathsf{O}}}$ by constructing a natural isomorphism $\eta : \mathfrak{F}\mathfrak{G} \Rightarrow \mathrm{Id}_{\mathsf{SM}_{\mathsf{O}}}$.

We first claim that $\mathfrak{G}\mathfrak{F} = \mathrm{Id}_{\mathsf{Operad}}$. Let $\mathcal{O} \in \mathsf{Operad}$. Then $\mathfrak{G}\mathfrak{F}(\mathcal{O}) = \widetilde{\mathcal{O}}^{\mathsf{O}}$ is an operad where

- $\widetilde{\mathcal{O}}^{\mathsf{O}}(n) = \widetilde{\mathcal{O}}(n,1) = \mathcal{O}(n)$,

- $1_{\widetilde{\mathcal{O}}^{\mathsf{O}}} = \mathrm{Id}_1 = 1_{\mathcal{O}}$,

- for $\sigma \in \Sigma_n$, $f \in \widetilde{\mathcal{O}}^{\mathsf{O}}(n)$,

$$\begin{aligned} f \cdot_{\widetilde{\mathcal{O}}^{\mathsf{O}}} \sigma &= f \circ (i_n^{\widetilde{\mathcal{O}}}(\sigma)) \\ &= [(f, e_n), (n)] \circ [(1_{\mathcal{O}}, \dots, 1_{\mathcal{O}}, \sigma), (1, \dots, 1)] \\ &= [(\gamma_{\mathcal{O}}(f; 1, \dots, 1), e_n(1, \dots, 1)\sigma, (n)] \\ &= [(f, \sigma), (n)] \\ &= [(f \cdot_{\mathcal{O}} \sigma, e_n), (n)] = f \cdot_{\mathcal{O}} \sigma. \end{aligned}$$

- for $(f, g_1, \dots, g_k) \in \widetilde{\mathcal{O}}^{\mathsf{O}}(k) \otimes \widetilde{\mathcal{O}}^{\mathsf{O}}(j_1) \otimes \dots \otimes \widetilde{\mathcal{O}}^{\mathsf{O}}(j_k)$, we have

$$\begin{aligned} \gamma_{\widetilde{\mathcal{O}}^{\mathsf{O}}}(f; g_1, \dots, g_k) &= f \circ (g_1 \boxtimes \dots \boxtimes g_k) \\ &= [(f, e_k), (k)] \circ [(g_1, \dots, g_k, (e_{j_1} \times \dots \times e_{j_k})), (j_1, \dots, j_k)] \\ &= [(\gamma_{\mathcal{O}}(f; g_1, \dots, g_k), e_k(j_1, \dots, j_k)(e_{j_1} \times \dots \times e_{j_k})), (\Sigma j_i)] \\ &= [\gamma_{\mathcal{O}}(f; g_1, \dots, g_k), e_j), (j)] \\ &= \gamma_{\mathcal{O}}(f; g_1, \dots, g_k) \end{aligned}$$

Thus $\mathfrak{G}\mathfrak{F}(\mathcal{O}) = \mathcal{O}$. Let $F : \mathcal{O} \to \mathcal{P}$ be a map of operads. Then $\mathfrak{G}\mathfrak{F}(F) = \widetilde{F}^{\mathsf{O}}$ is a map of operads where for $f \in \widetilde{\mathcal{O}}^{\mathsf{O}}(n) = \mathcal{O}(n)$,

$$\widetilde{F}_n^{\mathsf{O}}(f) = \widetilde{F}(f) = F_n(f)$$

Therefore $\mathfrak{G}\mathfrak{F} = \mathrm{Id}_{\mathsf{Operad}}$.

Next, we claim that $\mathfrak{F}\mathfrak{G} \cong \mathrm{Id}_{\mathsf{SM}_{\mathsf{O}}}$. Let $\mathcal{K} \in \mathsf{SM}_{\mathsf{O}}$. Then $\mathfrak{F}\mathfrak{G}(\mathcal{K}) = \widetilde{\mathcal{K}^{\mathsf{O}}}$ is a symmetric strict monoidal category in $\mathsf{SM}_{\mathsf{O}}$ with

- monoid of objects $\mathbb{N}$,
- $\widetilde{\mathcal{K}^{\mathsf{O}}}(n,1) = \mathcal{K}^{\mathsf{O}}(n) = \mathcal{K}(n,1)$,
- $\widetilde{\mathcal{K}^{\mathsf{O}}}(m,n) = \coprod_{\sum m_i = m} (\mathcal{K}(m_1,1) \otimes \ldots \otimes \mathcal{K}(m_n,1)) \otimes_{\Sigma_{m_1} \times \ldots \times \Sigma_{m_n}} \Sigma_m \cong \mathcal{K}(m,n)$,
- $i_m^{\widetilde{\mathcal{K}^{\mathsf{O}}}} : \Sigma_m \to \widetilde{\mathcal{K}^{\mathsf{O}}}(m,m)$ is defined as follows: for $\sigma \in \Sigma_m$,

$$\begin{aligned} i_m^{\widetilde{\mathcal{K}^{\mathsf{O}}}}(\sigma) &= [(1_{\mathcal{K}^{\mathsf{O}}}, \ldots, 1_{\mathcal{K}^{\mathsf{O}}}, \sigma), (1, \ldots, 1)] \\ &= [(\mathrm{Id}_1, \ldots, \mathrm{Id}_1, \sigma), (1, \ldots, 1)] \end{aligned}$$

- The monoidal product on objects remains addition. On morphisms, it is $\widetilde{\mathcal{K}^{\mathsf{O}}}(m,p) \otimes \widetilde{\mathcal{K}^{\mathsf{O}}}(n,q) \to \widetilde{\mathcal{K}^{\mathsf{O}}}(m+n, p+q)$ is given by

$$[(f_1, \ldots, f_p, \sigma), (m_1, \ldots, m_p)] \boxtimes [(g_1, \ldots, g_q, \tau), (n_1, \ldots, n_q)] =$$

$$[(f_1, \ldots, f_p, g_1, \ldots, g_q, \sigma \times \tau), (m_1, \ldots, m_p, n_1, \ldots, n_q)]$$

- Composition $\widetilde{\mathcal{K}^{\mathsf{O}}}(n,p) \otimes \widetilde{\mathcal{K}^{\mathsf{O}}}(m,n) \to \widetilde{\mathcal{K}^{\mathsf{O}}}(m,p)$ is given by

$$\begin{aligned} &[(g_1, \ldots, g_p, \sigma), (n_1, \ldots, n_p)] \circ [(h_1, \ldots, h_n, \tau), (m_1, \ldots, m_n)] = \\ &[(\gamma_{\mathcal{K}^{\mathsf{O}}}(g_1; h_{\sigma^{-1}(1)}, \ldots, h_{\sigma^{-1}(n_1)}), \ldots, \gamma_{\mathcal{K}^{\mathsf{O}}}(g_p; h_{\sigma^{-1}(n-n_p+1)}, \ldots, h_{\sigma^{-1}(n)}), \\ &\qquad \sigma(m_1, \ldots, m_n)\tau), \left( \sum_{i=1}^{n_1} m_{\sigma^{-1}(i)}, \ldots, \sum_{i=n-n_p+1}^{n} m_{\sigma^{-1}(i)} \right) \Bigg] = \\ &[(g_1 \circ (h_{\sigma^{-1}(1)} \boxtimes \ldots \boxtimes h_{\sigma^{-1}(n_1)}), \ldots, g_p \circ (h_{\sigma^{-1}(n-n_p+1)} \boxtimes \ldots \boxtimes h_{\sigma^{-1}(n)}), \\ &\qquad \sigma(m_1, \ldots, m_n)\tau), \left( \sum_{i=1}^{n_1} m_{\sigma^{-1}(i)}, \ldots, \sum_{i=n-n_p+1}^{n} m_{\sigma^{-1}(i)} \right) \Bigg] \end{aligned}$$

- Symmetry is given by $s_{m,n} : m \boxtimes n \to n \boxtimes m$ where

$$s_{m,n} = [(\underbrace{\mathrm{Id}_1, \ldots, \mathrm{Id}_1}_{m+n\text{-times}}, (m,n)), (\underbrace{1, \ldots, 1}_{m+n\text{-times}})]$$

Let $F : \mathcal{K} \to \mathcal{L}$ be a morphism in $\mathsf{SM}_{\mathsf{O}}$. Then we have $\mathfrak{F}\mathfrak{G}(F) = \widetilde{F^{\mathsf{O}}} : \widetilde{\mathcal{K}^{\mathsf{O}}} \to \widetilde{\mathcal{L}^{\mathsf{O}}}$ where

- $\widetilde{F^{\mathsf{O}}}$ is the identity on objects,
- for $g \in \widetilde{\mathcal{K}^{\mathsf{O}}}(n,1)$, $\widetilde{F^{\mathsf{O}}}(g) = F^{\mathsf{O}}_n(g) = F(g)$,
- for $g = [(g_1, \ldots, g_n, \sigma), (m_1, \ldots, m_n)] \in \widetilde{\mathcal{K}^{\mathsf{O}}}(m,n)$,

$$\begin{aligned} \widetilde{F^{\mathsf{O}}}(g) &= [(F^{\mathsf{O}}_{m_1}(g_1), \ldots, F^{\mathsf{O}}_{m_n}(g_n), \sigma), (m_1, \ldots, m_n)] \\ &= [(F(g_1), \ldots, F(g_n), \sigma), (m_1, \ldots, m_n)] \end{aligned}$$

We now construct a natural isomorphism $\eta : \mathfrak{F}\mathfrak{G} \Rightarrow \mathrm{Id}_{\mathsf{SM}_{\mathsf{O}}}$. For each $\mathcal{K} \in \mathsf{SM}_{\mathsf{O}}$, define $\eta_{\mathcal{K}} : \widetilde{\mathcal{K}^{\mathsf{O}}} \to \mathcal{K}$ by

- $\eta_{\mathcal{K}}(n) = n$,
- for $g = [(g_1, \ldots, g_n, \sigma), (m_1, \ldots, m_n)] \in \widetilde{\mathcal{K}^{\mathsf{O}}}(m,n)$,

$$\eta_{\mathcal{K}}(g) = (g_1 \boxtimes \ldots \boxtimes g_n) \circ i^{\mathcal{K}}_m(\sigma) \in \mathcal{K}(m,n).$$

Note that on morphisms, for a fixed pair $m, n$, $\eta_{\mathcal{K}}$ is the isomorphism $\eta_{m,n}$

$$\widetilde{\mathcal{K}^{\mathsf{O}}}(m,n) = \coprod_{\sum m_i = m} (\mathcal{K}(m_1,1) \otimes \ldots \otimes \mathcal{K}(m_n,1)) \otimes_{\Sigma_{m_1} \times \ldots \times \Sigma_{m_n}} \Sigma_m \cong \mathcal{K}(m,n)$$

described in Diagram 3.2.2.

$\eta_{\mathcal{K}}$ **is a map in $\mathsf{SM_O}$:** We first verify that $\eta_{\mathcal{K}}$ is a functor. We have

$$\eta_{\mathcal{K}}(\mathrm{Id}_n^{\widetilde{\mathcal{K}^{\mathsf{O}}}}) = \eta_{\mathcal{K}}([(\mathrm{Id}_1^{\mathcal{K}}, \ldots, \mathrm{Id}_1^{\mathcal{K}}, e_n), (1, \ldots, 1)]) = (\mathrm{Id}_n^{\mathcal{K}}) \circ i_n^{\mathcal{K}}(e_n) = \mathrm{Id}_n^{\mathcal{K}}.$$

Thus $\eta_{\mathcal{K}}$ preserves identities. We must now verify that for $g \otimes f \in \widetilde{\mathcal{K}^{\mathsf{O}}}(n,p) \otimes \widetilde{\mathcal{K}^{\mathsf{O}}}(m,n)$ $\eta_{\mathcal{K}}(g) \circ \eta_{\mathcal{K}}(f) = \eta_{\mathcal{K}}(gf)$:

$$\begin{aligned}
\eta_{\mathcal{K}}(g) \circ \eta_{\mathcal{K}}(f) &= ((g_1 \boxtimes \ldots \boxtimes g_p) \circ i_n(\tau)) \circ ((f_1 \boxtimes \ldots \boxtimes f_n) \circ i_m(\sigma)) \\
&= (g_1 \boxtimes \ldots \boxtimes g_k) \circ (f_{\tau^{-1}(1)} \boxtimes \ldots \boxtimes f_{\tau^{-1}(n)}) \circ i_m(\tau(m_1, \ldots, m_n)\sigma) \\
&= [(g_1 \circ (f_{\tau^{-1}(1)} \boxtimes \ldots \boxtimes f_{\tau^{-1}(n_1)})) \boxtimes \ldots \boxtimes \\
&\qquad (g_n \circ (f_{\tau^{-1}(n-n_p+1)} \boxtimes \ldots \boxtimes f_{\tau^{-1}(n)}))] \circ i_m(\tau(m_1, \ldots, m_n)\sigma) \\
&= \eta_{\mathcal{K}}(gf)
\end{aligned}$$

Thus $\eta_{\mathcal{K}}$ preserves composition and is therefore a functor.

Note that since $\eta_{\mathcal{K}}$ is the identity on objects, in particular it preserves both the monoidal unit and the monoidal product on objects. We check that it also preserves the monoidal product on morphisms; i.e., for $f \otimes g \in \widetilde{\mathcal{K}^{\mathsf{O}}}(m,p) \otimes \widetilde{\mathcal{K}^{\mathsf{O}}}(n,q)$, $\eta_{\mathcal{K}}(f \boxtimes g) = \eta_{\mathcal{K}}(f) \boxtimes \eta_{\mathcal{K}}(g)$:

$$\begin{aligned}
\eta_{\mathcal{K}}(f) \boxtimes \eta_{\mathcal{K}}(g) &= ((f_1 \boxtimes \ldots \boxtimes f_p) \circ i_m(\sigma)) \boxtimes ((g_1 \boxtimes \ldots \boxtimes g_q) \circ i_n(\tau)) \\
&= ((f_1 \boxtimes \ldots \boxtimes f_p) \boxtimes (g_1 \boxtimes \ldots \boxtimes g_q)) \circ (i_m(\sigma) \boxtimes i_n(\tau)) \\
&= ((f_1 \boxtimes \ldots \boxtimes f_p) \boxtimes (g_1 \boxtimes \ldots \boxtimes g_q)) \circ (i_{m+n}(\sigma \oplus \tau)) \\
&= \eta_{\mathcal{K}}(f \boxtimes g)
\end{aligned}$$

Thus $\eta_{\mathcal{K}}$ is a monoidal functor.

We also have for all $m, n$,

$$\begin{aligned}
\eta_{\mathcal{K}}(s_{m,n}^{\widetilde{\mathcal{K}^{\mathsf{O}}}}) &= \eta_{\mathcal{K}}([(\underbrace{\mathrm{Id}_1^{\mathcal{K}}, \ldots, \mathrm{Id}_1^{\mathcal{K}}}_{m+n\text{-times}}, (m,n)), (\underbrace{1, \ldots, 1}_{m+n\text{-times}})]) \\
&= i_{m+n}^{\mathcal{K}}((m,n)) \\
&= s_{m,n}^{\mathcal{K}}.
\end{aligned}$$

Therefore $\eta_{\mathcal{K}}$ is a strict symmetric monoidal functor.

For $\sigma \in \Sigma_n$,

$$\eta_{\mathcal{K}}(i_n^{\widetilde{\mathcal{K}^{\mathsf{O}}}}(\sigma)) = \eta_{\mathcal{K}}([(\mathrm{Id}_1, \ldots, \mathrm{Id}_1, \sigma), (1, \ldots, 1)]) = i_n^{\mathcal{K}}(\sigma).$$

Since $\eta_{\mathcal{K}}$ is also the identity on objects, $\eta_{\mathcal{K}}$ is a map in $\mathsf{SM}_{\mathsf{O}}$.

**$\eta$ is a natural isomorphism with components $\eta_{\mathcal{K}}$.** Let $F : \mathcal{K} \to \mathcal{L}$ in $\mathsf{SM}_{\mathsf{O}}$. We must verify that the following diagram commutes:

$$\begin{array}{ccc}
\mathfrak{F}\mathfrak{G}(\mathcal{K}) = \widetilde{\mathcal{K}^{\mathsf{O}}} & \xrightarrow{\eta_{\mathcal{K}}} & \mathcal{K} \\
{\scriptstyle \mathfrak{F}\mathfrak{G}(F) = \widetilde{F^{\mathsf{O}}}} \downarrow & & \downarrow {\scriptstyle F} \\
\mathfrak{F}\mathfrak{G}(\mathcal{L}) = \widetilde{\mathcal{L}^{\mathsf{O}}} & \xrightarrow[\eta_{\mathcal{L}}]{} & \mathcal{L}
\end{array}$$

On objects this diagrams commutes trivially. Suppose we have $f \in \widetilde{\mathcal{K}^{\mathsf{O}}}(m, n)$. Then

$$\begin{aligned}
F(\eta_{\mathcal{K}}(f)) &= F((f_1 \boxtimes \ldots \boxtimes f_n) \circ i_m^{\mathcal{K}}(\sigma)) \\
&= (F(g_1) \boxtimes \ldots \boxtimes F(g_n)) \circ i_m^{\mathcal{L}}(\sigma) \\
&= \eta_{\mathcal{L}}([(F(g_1), \ldots, F(g_n), \sigma), (m_1, \ldots, m_n)]) \\
&= \eta_{\mathcal{L}}(\widetilde{F^{\mathsf{O}}}(f))
\end{aligned}$$

Thus $\eta$ is a natural transformation. Since each $\eta_{\mathcal{K}}$ is the identity on objects and an isomorphism on hom-objects, each $\eta_{\mathcal{K}}$ is an isomorphism and thus $\eta$ is an isomorphism.

Therefore we have that $\mathfrak{G}\mathfrak{F} = \mathrm{Id}_{\mathsf{Operad}}$ and $\mathfrak{F}\mathfrak{G} \cong \mathrm{Id}_{\mathsf{SM_O}}$, so $\mathfrak{F}$ and $\mathfrak{G}$ give an equivalence of categories between $\mathsf{Operad}$ and $\mathsf{SM_O}$. □

*Remark* 3.3.4. For $X \in \mathcal{V}$, since $\mathfrak{G}\mathfrak{F} = \mathrm{Id}_{\mathsf{Operad}}$, a map $F : \mathcal{K} \to \widetilde{\mathrm{End}_X}$ in $\mathsf{SM_O}$ will give $X$ the structure of an algebra over the operad $\mathcal{K}^{\mathsf{O}}$.

*Remark* 3.3.5. There exists a generalization of operads, called *colored operads* or *symmetric multicategories*. These can be thought of as categories with objects called colors that allow morphisms from a finite set of colors to a single color, along with composition maps, units, and a symmetric action. Classical operads then can be defined as colored operads with just one color. Unlike in the classical case, every symmetric monoidal category gives rise to a colored operad [Lur07, Example 1.1.5]. One future direction for this work is to understand how the above equivalence compares with this correspondence.

# CHAPTER 4

## Hammock and Tree Hammock Localizations

Often, within a category we have a designated collection of morphisms, called weak equivalences, that are almost like isomorphisms. The existence of a weak equivalence between two objects might indicate some sort of sameness of the objects; for example a homotopy equivalence between two topological spaces or a DK-equivalence between simplicial categories. Note that while in applications the weak equivalences will almost always encode a "sameness", the weak equivalences of a category can be any collection of morphisms in the category.

Weak equivalences, however, are not necessarily invertible. In fact, the existence of a weak equivalence from $X$ to $Y$ does not even imply the existence of a weak equivalence from $Y$ to $X$. Instead, for a category $\mathcal{C}$ with a collection of weak equivalences $\mathcal{W}$, referred to as $(\mathcal{C}, \mathcal{W})$, we can construct an associated category in which the weak equivalences are in some sense "invertible".

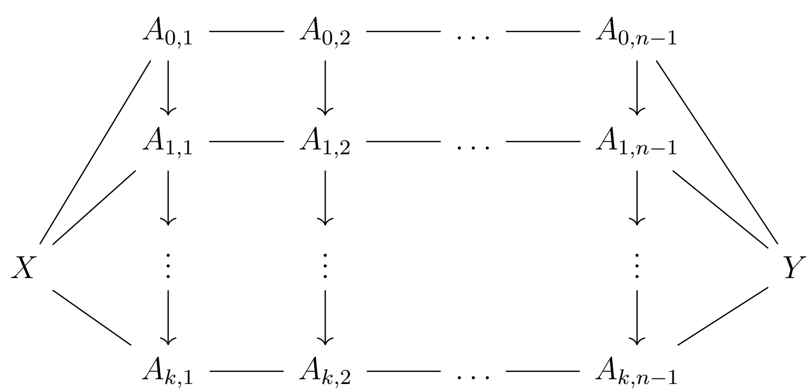
The simplest such construction is the Gabriel–Zisman category of fractions (see [Rie19, Section 2.1]), which produces a functor to an ordinary category $\mathcal{C}[\mathcal{W}^{-1}]$ where the morphisms in $\mathcal{W}$ are sent to isomorphisms. While this construction is relatively easy to describe, it has certain disadvantages. For example, if $\mathcal{C}$ is locally small, there is no guarantee that $\mathcal{C}[\mathcal{W}^{-1}]$ is locally small as well. Furthermore, in general, $N(\mathcal{C})$ and $N(\mathcal{C}[\mathcal{W}^{-1}])$ do not have the same homotopy type.

Instead, in this chapter we review a simplicial localization construction for $(\mathcal{C}, \mathcal{W})$, called the hammock localization [DK80a], as well as a localization construction for operads, called the tree hammock localization [BBP+18].

### 4.1 Hammock Localizations

**Definition 4.1.1.** The category $\mathsf{RelCat}$ of *relative categories* consists of objects $(\mathcal{C}, \mathcal{W})$ where $\mathcal{C}$ is a category and $\mathcal{W}$ is a *wide subcategory*; meaning a subcategory that contains all the objects of $\mathcal{C}$. The morphisms $F : (\mathcal{C}, \mathcal{W}) \to (\mathcal{C}', \mathcal{W}')$ of $\mathsf{RelCat}$ are functors $F : \mathcal{C} \to \mathcal{C}'$ such that the restriction to $\mathcal{W}$ induces a functor $F : \mathcal{W} \to \mathcal{W}'$. The category $\mathsf{RelsCat}$ of *relative simplicial categories* is defined similarly, where $\mathcal{C}$ is a simplicial category.

**Definition 4.1.2.** Let $(\mathcal{C}, \mathcal{W}) \in \mathsf{RelCat}$. Let $X, Y \in \mathcal{C}$. A *reduced hammock from $X$ to $Y$ of height $k$ and length $n$* is a commutative diagram in $\mathcal{C}$ of the type:




such that

- all vertical maps are in $\mathcal{W}$;
- in each column of horizontal maps, all maps point in the same direction; and if they point left they are all in $\mathcal{W}$;
- the maps in adjacent columns point in different directions;
- no column of maps contains only identity maps.

For a fixed $X$ and $Y$, we can form a simplicial set called $L^H_{\mathcal{W}}\mathcal{C}(X, Y)$ where the $k$-simplices are the reduced hammocks from $X$ to $Y$ of height $k$ and arbitrary length. The $i$th face map is given on a $k$-simplex by deleting the $i$th row of the hammock and reducing as necessary. The $i$th degeneracy map is given by repeating the $i$th row.

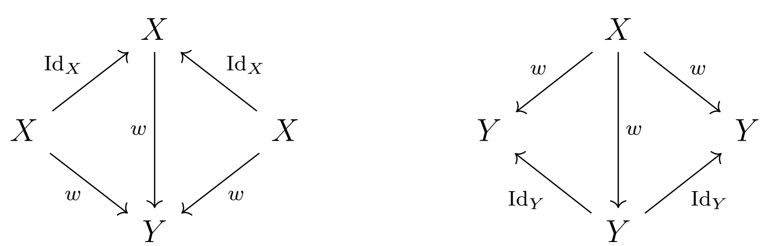
**Definition 4.1.3.** [DK80a, 2.1] Let $(\mathcal{C}, \mathcal{W}) \in \mathsf{RelCat}$. We define the *hammock localization of* $\mathcal{C}$ *with respect to* $\mathcal{W}$, $L^H_{\mathcal{W}}\mathcal{C}$, to be the simplicial category with the same objects as $\mathcal{C}$, with hom-objects $L^H_{\mathcal{W}}\mathcal{C}(X, Y)$, and with composition of a reduced hammock from $X$ to $Y$ with one from $Y$ to $Z$ given by gluing them at $Y$ and reducing if needed.

There is a canonical functor $\mathcal{C} \to L^H_{\mathcal{W}}\mathcal{C}$, which is the identity on objects and which takes a morphism $f : X \to Y$ to the height 0, width 1 hammock from $X$ to $Y$ consisting of a single forward facing arrow labeled by $f$.

Suppose $w : X \to Y$ is a map in $\mathcal{W}$. Then its image in $L^H_{\mathcal{W}}\mathcal{C}$ is *not* an isomorphism (unless $w$ is already an isomorphism in $\mathcal{C}$) since the height 0 hammock

$$X \xrightarrow{w} Y \xleftarrow{w} X$$

is not equal to the identity on $X$. However, we do have the following height 1 hammocks




which give that $w$ is sent to an isomorphism in $\pi_0 L^H_{\mathcal{W}}\mathcal{C}$. In other words, we have the following lemma:

**Lemma 4.1.4** (Weak Invertibility)**.** *[DK80a, 3.3] Let $w : X \to Y \in \mathcal{W}$. The $w$ induces, for every object $V \in \mathcal{C}$, weak homotopy equivalences of simplicial sets*

$$w_* : L^H_{\mathcal{W}}\mathcal{C}(V, X) \to L^H_{\mathcal{W}}\mathcal{C}(V, Y), \qquad w^* : L^H_{\mathcal{W}}\mathcal{C}(Y, V) \to L^H_{\mathcal{W}}\mathcal{C}(X, V).$$

*Remark* 4.1.5. If $(\mathcal{C}, \mathcal{W}) \in \mathsf{RelsCat}$, we can define the hammock localization of $\mathcal{C}$ with respect to $\mathcal{W}$ as the diagonal of $L^H_{\mathcal{W}}\mathcal{C}$. In other words, the $k$-simplices of the localization will be height $k$ hammocks built out of $k$-simplices from $\mathcal{C}$ and $\mathcal{W}$.

We collect here some useful properties of the hammock localization.

**Lemma 4.1.6** (Functoriality)**.** *[DK80a, 3.1] Hammock localization defines a functor $L^H$ : $\mathsf{RelCat} \to \mathsf{sCat}$. In other words, a functor $F : (\mathcal{C}, \mathcal{W}) \to (\mathcal{C}', \mathcal{W}')$ in $\mathsf{RelCat}$ induces a functor $L^H F : L^H_{\mathcal{W}}\mathcal{C} \to L^H_{\mathcal{W}'}\mathcal{C}'$.*

**Lemma 4.1.7** (Homotopy Lemma)**.** *[DK80a, 2.4] Let $F : (\mathcal{C}, \mathcal{W}) \to (\mathcal{C}', \mathcal{W}')$ be a functor in $\mathsf{RelsCat}$. If $F$ and its restriction $F : \mathcal{W} \to \mathcal{W}'$ are both DK-equivalences, then so is the induced function $\operatorname{diag} L^H_{\mathcal{W}}\mathcal{C} \to \operatorname{diag} L^H_{\mathcal{W}'}\mathcal{C}'$. In other words, DK-equivalent simplicial categories induce DK-equivalent localizations.*

**Definition 4.1.8.** Let $(\mathcal{C}, \mathcal{W}) \in \mathsf{RelCat}$. Then we say $(\mathcal{C}, \mathcal{W})$ *admits a calculus of left fractions* if

- For each diagram $X' \xrightarrow{u} X \xrightarrow{f} Y \in \mathcal{C}$ with $u \in \mathcal{W}$, there exists a commutative diagram with $v \in \mathcal{W}$:

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[v]{} & Y \end{array}$$

- If $g, f : X \to Y \in \mathcal{C}$ and $u : X' \to X \in \mathcal{W}$ such that $fu = gu$, then there exists $v \in \mathcal{W}$ such that $vf = vg$.

If, in addition, we have the property that

- for any two composable maps $f, g \in \mathcal{C}$, if two of the three of $f$, $g$, and $gf$ are in $\mathcal{W}$, then so is the third,

then $(\mathcal{W}, \mathcal{W})$ also admits a calculus of left fractions.

Recall that we can view $\pi_0 L^H_{\mathcal{W}}\mathcal{C}$ as a simplicial category as in Example 2.2.4.

**Lemma 4.1.9.** *[DK80a, 7.3] If $(\mathcal{C}, \mathcal{W})$ and $(\mathcal{W}, \mathcal{W})$ admit a calculus of left fractions, then the natural map of simplicial categories $L^H_{\mathcal{W}}\mathcal{C} \to \pi_0 L^H_{\mathcal{W}}\mathcal{C}$ is a DK-equivalence. In other words, for every pair of objects $X, Y \in \mathcal{C}$, the components of $L^H_{\mathcal{W}}\mathcal{C}(X, Y)$ are contractible.*

## 4.2 Tree Hammock Localization

We now define a localization construction for operads, following [BBP$^+$18].

Let $\mathcal{O}$ be an operad in $\mathsf{Set}$. Note that $\mathcal{O}(1)$ is a monoid with unit $1_{\mathcal{O}}$. Thus we can consider submonoids of $\mathcal{O}(1)$, that is collections of 1-ary operations which are closed under $\gamma$ and contain $1_{\mathcal{O}}$. Let $\mathcal{W}$ be such a submonoid. We will use the notation $(\mathcal{O}, \mathcal{W})$ to refer to an operad and a submonoid $\mathcal{W} \subset \mathcal{O}(1)$. We can consider $\mathcal{W}$ as an operad where $\mathcal{W}(1) = \mathcal{W}$, $\mathcal{W}(n) = \emptyset$ for all $n \neq 1$, and the operad structure maps are inherited from $\mathcal{O}$, as in Example 2.4.7.

We would like to construct an operad where the operations in $\mathcal{W}$ are in some sense "invertible". The obvious approach would be to consider the relative category $(\widetilde{\mathcal{O}}, \widetilde{\mathcal{W}})$ and take the hammock localization. However, there is no reason to expect that $L^H_{\widetilde{\mathcal{W}}}\widetilde{\mathcal{O}}$ is symmetric monoidal, let alone in $\mathsf{SM}_{\mathsf{O}}$. Thus it may not correspond to an operad. Instead, we work with a new construction.

We will build directed planar trees out of atomic directed pieces. We have two types of *atomic pieces*: $O_n : n \to 1$ for $n \geq 0$ which has a single root node and $n$ leaf nodes, and $W_1 : 1 \leftarrow 1$ which has a single root node and a single leaf node. See Figure 4.1.

(a) $O_1$ (b) $O_3$ (c) $W_1$

Figure 4.1: In each atomic piece, the red node is the root node and the green node(s) the leaf node(s).

By gluing together atomic pieces, identifying the root node of one atomic piece with the leaf node of another, with the resulting node called an *internal node*, we can construct a tree

$\tau \in \mathbb{T}(n)$ for $n \geq 0$, the set of directed planar trees with one root and $n$ leaves labeled 1 through $n$, with possibly some unlabeled leaves.

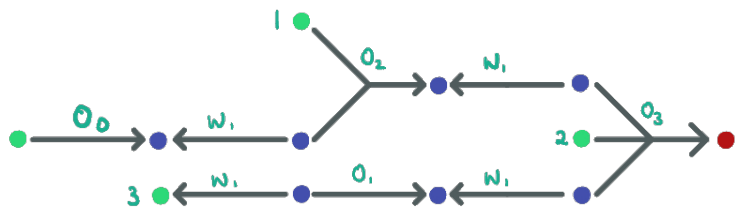



Figure 4.2: An example $\tau \in \mathbb{T}(3)$. Here the blue nodes are the interior nodes, the red one is the root node, and the green ones are the leaf nodes. Note one leaf node is unlabeled, corresponding to the atomic piece $O_0$.

**Definition 4.2.1.** Let $(\mathcal{O}, \mathcal{W})$ be an operad in $\mathsf{Set}$ and a submonoid of $\mathcal{O}(1)$. A reduced tree hammock of height $k$ and type $\tau \in \mathbb{T}(n)$ is a three dimensional diagram consisting of $k$ copies of $\tau$ arranged in parallel horizontal planes, connected by vertical downward arrows between corresponding interior, root, and leaf nodes. Each atomic piece and vertical edge is labeled by an operation in $\mathcal{O}$ so that the diagram commutes.

We also require that

- atomic pieces $O_n$ are labeled by operations in $\mathcal{O}(n)$;
- atomic pieces $W_1$ and vertical arrows are labeled by operations in $\mathcal{W}$;
- atomic pieces in adjacent columns point in different directions;
- no column of atomic pieces contains arrows all labeled by $1_{\mathcal{O}}$. Note by a column, we mean the corresponding atomic pieces in each copy of $\tau$. In Figure 4.3, $t_1$ and $t_2$ are in the same column, but $s_1$ and $t_1$ are not.

An example reduced tree hammock is given in Figure 4.3.

Let $n \geq 0$. We can form a simplicial set called $L^{TH}_{\mathcal{W}}\mathcal{O}(n)$ where the $k$-simplices are the reduced tree hammocks of height $k$ and type $\tau \in \mathbb{T}(n)$. The $i$th face map is given on a $k$-simplex by deleting the $i$th row of the tree hammock and reducing as necessary. The $i$th degeneracy map is given by repeating the $i$th row.

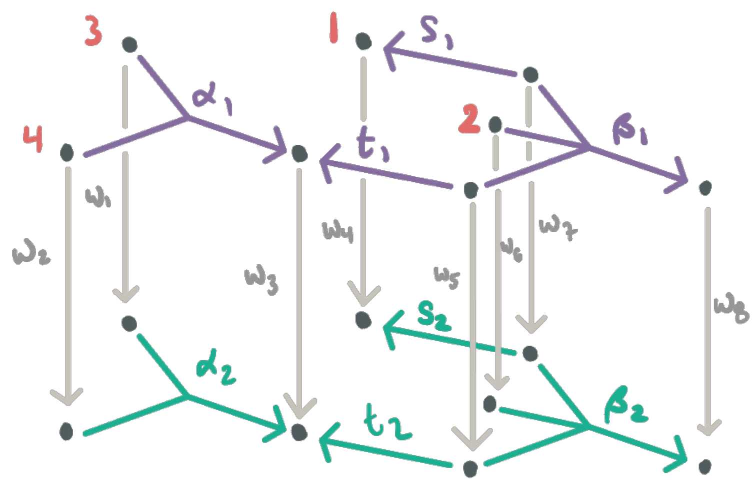



Figure 4.3: An example of a height 1 reduced tree hammock; in other words a 1-simplex in $L^{TH}_{\mathcal{W}}\mathcal{O}(4)$. If, for example, $t_1 = t_2 = 1_{\mathcal{O}}$, then this would be unreduced.

**Definition 4.2.2.** Let $(\mathcal{O}, \mathcal{W})$ be an operad in $\mathsf{Set}$ and a submonoid of $\mathcal{O}(1)$. We define the *tree hammock localization* of $\mathcal{O}$ with respect to $\mathcal{W}$, $L^{TH}_{\mathcal{W}}\mathcal{O}$, to be the simplicial operad with the $n$-ary operations given by $L^{TH}_{\mathcal{W}}\mathcal{O}(n)$, and $i$th composition given by grafting the root node of one operation with the $i$th leaf node of the other, and reducing as necessary. The symmetric action is given by shuffling the labels on the leaf nodes.

*Remark* 4.2.3. If $\mathcal{O}$ is a simplicial operad, we can repeat a similar process to construct an operad $L^{TH}_{\mathcal{W}}\mathcal{O}$ enriched in bisimpicial sets.

There is a canonical map of operads $\mathcal{O} \to L^{TH}_{\mathcal{W}}\mathcal{O}$, which takes $\alpha \in \mathcal{O}(n)$ to the height 0 tree hammock in $L^{TH}_{\mathcal{W}}\mathcal{O}(n)$ consisting of a single atomic piece $O_n$ labled by $\alpha$.

We collect here some useful properties of the tree hammock localization.

Since $\mathcal{O}(1)$ is a monoid, we can consider it as an operad as in Example 2.4.7, and we can do the same with submonoid $\mathcal{W}$. Note that a reduced tree hammock built entirely out of atomic pieces $O_1$ and $W_1$ is just a reduced hammock without the initial and final maps of the hammock. Since every monoid is a category with one object, we have the following lemma:

**Lemma 4.2.4.** *[BBP+18, 5.3] There is an isomorphism of simplicial monoids*

$$L^H_{\mathcal{W}}(\mathcal{O}(1)) \cong L^{TH}_{\mathcal{W}}(\mathcal{O}(1))(1).$$

Compare the following lemmas to Lemmas 4.1.6 and 4.1.4.

**Lemma 4.2.5** (Functoriality)**.** *[BBP+18, 5.1] Let $(\mathcal{O}, \mathcal{W})$, $(\mathcal{O}', \mathcal{W}')$ be a pair of operads with submonoids of their respective 1-ary operations. Let $\phi : \mathcal{O} \to \mathcal{O}'$ be a map of operads with $\phi(\mathcal{W}) \subset \mathcal{W}'$. Then $\phi$ induces a map on localizations $L^{TH}_{\mathcal{W}}\mathcal{O} \to L^{TH}_{\mathcal{W}'}\mathcal{O}'$.*

**Lemma 4.2.6** (Weak Invertibility)**.** *[BBP+18, 5.2] Let $w \in \mathcal{W}$ be a 1-ary operation. Then $w$ induces weak homotopy equivalences*

$$w \circ - : L^{TH}_{\mathcal{W}}\mathcal{O}(n) \to L^{TH}_{\mathcal{W}}\mathcal{O}(n) \qquad - \circ_i w : L^{TH}_{\mathcal{W}}\mathcal{O}(n) \to L^{TH}_{\mathcal{W}}\mathcal{O}(n).$$

# CHAPTER 5

## A Functor from $L^H_{\widetilde{\mathcal{W}}}\widetilde{\mathcal{O}}$ to $\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}$

Unless otherwise noted, from now on we assume all operads are in $\mathsf{Set}$ or $\mathsf{sSet}$. Fix an operad $\mathcal{O}$ and a submonoid $\mathcal{W} \subset \mathcal{O}(1)$. We then have two constructions we might consider for creating a category associated to $\mathcal{O}$ where the maps from $\mathcal{W}$ are in some sense invertible. The first is to take the tree hammock localization of $\mathcal{O}$ with respect to $\mathcal{W}$, and then take its symmetric monoidal envelope, $\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}$. The second option is to first take the symmetric monoidal envelopes of $\mathcal{O}$ and $\mathcal{W}$, and then take the hammock localization of $\widetilde{\mathcal{O}}$ with respect to $\widetilde{\mathcal{W}}$, $L^H_{\widetilde{\mathcal{W}}}\widetilde{\mathcal{O}}$. Ideally, these two constructions should be DK-equivalent.

Consider a reduced hammock in $L^H_{\widetilde{\mathcal{W}}}\widetilde{\mathcal{O}}(n, 1)$ for $n \geq 0$. This will be a hammock where an arrow $a \to b$ will be labeled by an ordered list of $b$ operations from $\mathcal{O}$, with a total of $a$ "inputs", along with a permutation from $\Sigma_a$ (see below for an example), where the vertical and backwards-pointing arrows are labeled by operations from $\mathcal{W}$.

We can view this as a (potentially unreduced) tree hammock with $n$ labeled leaf nodes, where the paths from the labeled leaf nodes to the root node are all the same length. To construct this tree hammock, we consider each commutative triangle or square in the hammock to create diagrams of atomic pieces that we can then glue together, recalling that the permutations may shuffle the order of the $b$ operations for composition. This is easier to understand after seeing an example. Consider $K \in L^H_{\widetilde{\mathcal{W}}}\widetilde{\mathcal{O}}(4, 1)$ as in Figure 5.1.

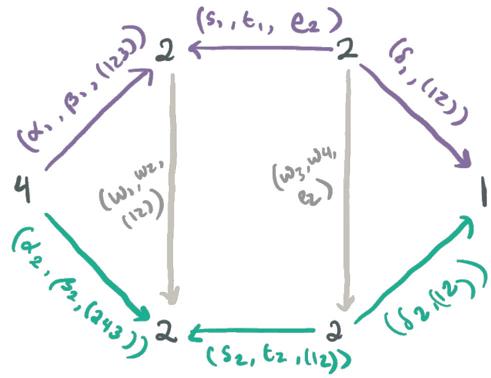



Figure 5.1: A reduced hammock $K \in L^H_{\widetilde{W}}\widetilde{\mathcal{O}}(4,1)$. To keep the diagram from crowding, we've not included the partition of $m$ given in the usual description of morphisms in $\widetilde{\mathcal{O}}$ as in 3.1.1.

For this example, all operations labeled by Roman letters are 1-ary operations, and those labeled by Greek letters are 2-ary operations. We can start by looking at the right triangle. Recall the composition formula 3.1.2 for $\widetilde{\mathcal{O}}$, which will reorder components of the morphisms before composing. The commutativity of the triangle gives us

$$\begin{aligned}(\delta_2, (12)) \circ (w_3, w_4, e_2) &= (\delta_1, (12)) \\ (\gamma(\delta_2; w_4, w_3), (12)(1,1)e_2) &= (\delta_1, (12)) \\ (\gamma(\delta_2; w_4, w_3), (12)) &= (\delta_1, (12))\end{aligned}$$

That the permutations agree simply verifies that the commutativity of the right triangle makes sense. We learn that $\gamma(\delta_2; w_4, w_3) = \delta_1 = \gamma(1_{\mathcal{O}}; \delta_1)$, which we can express in the diagram of atomic pieces given in Figure 5.2.

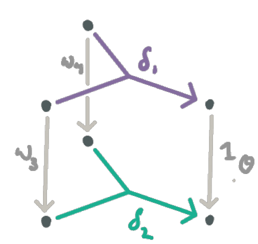



Figure 5.2: A tree hammock demonstrating $\gamma(\delta_2; w_4, w_3) = \delta_1 = \gamma(1_{\mathcal{O}}; \delta_1)$.

Next, we consider the middle commuting square. We have the following:

$$
\begin{aligned}
(w_1, w_2, (12)) \circ (s_1, t_1, e_2) &= (s_2, t_2, (12)) \circ (w_3, w_4, e_2) \\
(\gamma(w_1; t_1), \gamma(w_2; s_1), (12)(1,1)e_2) &= (\gamma(s_2; w_4), \gamma(t_2; w_3), (12)(1,1)e_2)
\end{aligned}
$$

Matching up the components in the last line we get the two diagrams of atomic pieces in Figure 5.3.

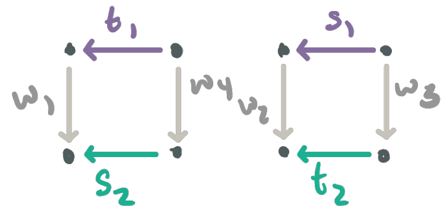



Figure 5.3: Two tree hammock demonstrating $\gamma(w_1; t_1) = \gamma(s_2; w_4)$ and $\gamma(w_2; s_1) = \gamma(t_2; w_3)$.

We then repeat the process for the left commuting triangle:

$$
\begin{aligned}
(w_1, w_2, (12)) \circ (\alpha_1, \beta_1, (123)) &= (\alpha_2, \beta_2, (243)) \\
(\gamma(w_1; \beta_1), \gamma(w_2; \alpha_1), (12)(2,2)(123)) &= (\alpha_2, \beta_2, (243)) \\
(\gamma(w_1; \beta_1), \gamma(w_2; \alpha_1), (243)) &= (\alpha_2, \beta_2, (243)) \qquad (5.0.1)
\end{aligned}
$$

which gives us two more diagrams of atomic pieces in Figure 5.4.

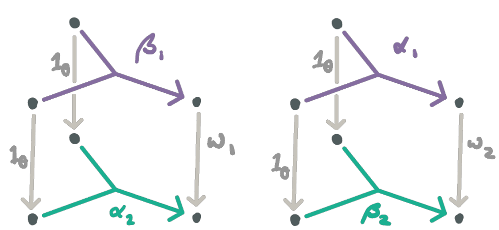


Figure 5.4: Two tree hammock demonstrating $\gamma(w_1; \beta_1) = \alpha_2 = \gamma(\alpha_2; 1_{\mathcal{O}}, 1_{\mathcal{O}})$ and $\gamma(w_2; \alpha_1) = \beta_2 = \gamma(\beta_2; 1_{\mathcal{O}}, 1_{\mathcal{O}})$.

Finally, we can glue the diagrams together along their vertical maps to create a tree hammock with 4 labeled leaf nodes. We use the permutation from the composition of the leftmost diagram to label the leaf nodes - here, this means using (243) as in Figure 5.5.

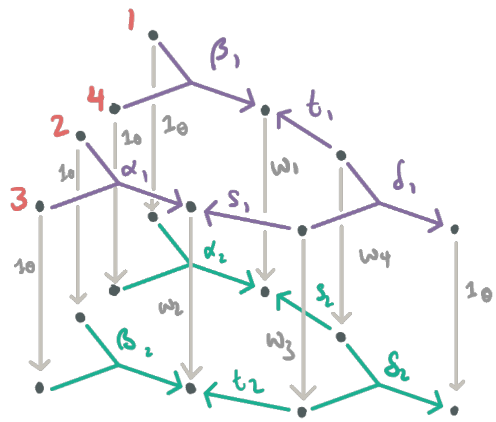



Figure 5.5: The tree hammock resulting from stitching Figures 5.2, 5.3, and 5.4 together. The leaves are labeled according to the permutation in Equation 5.0.1.

It is possible that this could be unreduced, even if the original hammock was reduced. In this example, if $t_1 = s_2 = 1_\mathcal{O}$, and thus $w_1 = w_4$, we would have a column of identities, even though the original hammock $K$ had no column of identities. In this case, we would reduce our tree hammock to finally have an operation in $L_\mathcal{W}^{TH}\mathcal{O}(4)$ as in Figure 5.6.

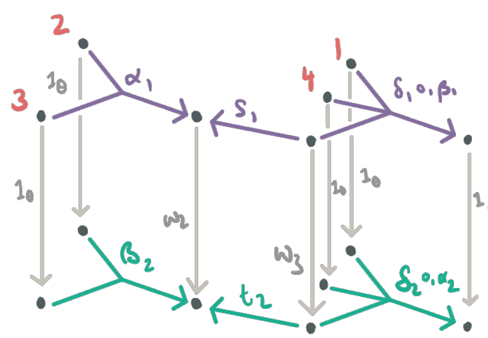



Figure 5.6: If, in Figure 5.5, $t_1 = s_2 = 1_\mathcal{O}$, then we would need to reduce Figure 5.5 to this tree hammock.

This of course gives a morphism in $\widetilde{L_\mathcal{W}^{TH}\mathcal{O}}(4,1) = L_\mathcal{W}^{TH}\mathcal{O}(4)$, and we can label this morphism as in Figure 5.7.

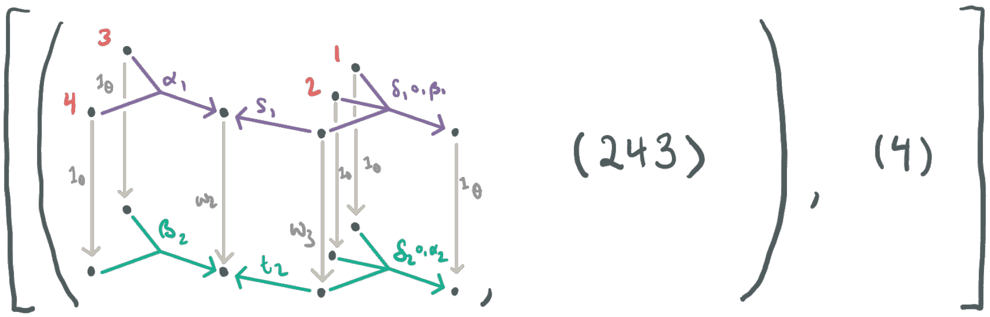



Figure 5.7: This is the image $R(K) \in \widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}(4,1)$, given in the form of morphisms in the symmetric monoidal envelope, as in 3.1.1.

where the permutation tells us how to permute the labelings on the leaf nodes before composing with another tree hammock.

Following the above steps, for any $K \in L^H_{\widetilde{\mathcal{W}}}\widetilde{\mathcal{O}}(m,n)$, we can construct $R(K) \in \widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}(m,n)$, where the steps will give $n$ tree hammocks with a total of $m$ labeled leaf nodes, along with a permutation in $\Sigma_m$. Since reduction preserves the simplicial structure and $R$ commutes with composition (of hammocks and tree hammocks), this assembles into a (bi)simplicial functor $R$ which is the identity on objects.

**Theorem 5.0.1.** *[BBP+18, 5.4] Hammock reduction defines a full functor of (bi)simplicially enriched categories*

$$R : L^H_{\widetilde{\mathcal{W}}}\widetilde{\mathcal{O}} \to \widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}.$$

**Corollary 5.0.2.** *[BBP+18, 5.3] Let $M$ be a monoid. Consider the operad $\mathcal{M}$ as in Example 2.4.7, and let $W$ be a submonoid of $M = \mathcal{M}(1)$. Then $R$ induces an isomorphism on morphism spaces.*

In general, we expect that $R$ should give $\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}$ as a suitably equivalent construction to $L^H_{\widetilde{\mathcal{W}}}\widetilde{\mathcal{O}}$.

**Conjecture 5.0.3.** *$R$ induces a homotopy equivalence on morphism spaces.*

Conjecture 5.0.3 would have several nice consequences. In general, the hammock localization of a symmetric monoidal category is not necessarily symmetric monoidal. This theorem would give sufficient conditions on $(\mathcal{C}, \mathcal{W})$ so that its hammock localization is equivalent to a symmetric monoidal category.

Furthermore, Conjecture 5.0.3 would also imply the following corollaries giving properties of the tree hammock localization inherited from Lemmas 4.1.7 and 4.1.9 about the hammock localization:

**Conjecture 5.0.4** (Homotopy Lemma)**.** *Let $\mathcal{O}, \mathcal{P}$ be operads in* sSet *with $\mathcal{W}, \mathcal{V}$ submonoids of their respective 1-ary operations. Let $f : \mathcal{O} \to \mathcal{P}$ be a map of operads which sends all of $\mathcal{W}$ into $\mathcal{V}$. If $f$ and its restriction $f : \mathcal{W} \to \mathcal{V}$ are both weak equivalences, meaning that $f$ induces a weak equivalence of simplicial sets for each $n$ [BM03, 3.3.1] then so is the induced map $L^{TH}_{\mathcal{W}}\mathcal{O} \to L^{TH}_{\mathcal{V}}\mathcal{P}$.*

**Lemma 5.0.5.** *Let $(\mathcal{O}, \mathcal{W})$ be an operad and a submonoid of $\mathcal{O}(1)$. Suppose that*

1. *for all $f \in \mathcal{O}(n)$, $u \in \mathcal{W}$, and for each $0 \le i \le n$ there exists $f' \in \mathcal{O}(n)$, $v \in \mathcal{W}$ such that $f \circ_i u = v \circ_1 f'$ and*

2. *for all $f, g \in \mathcal{O}(n)$, $u \in \mathcal{W}$ such that $f \circ_i u = g \circ_i u$, there exists $v \in \mathcal{W}$ such that $v \circ_1 f = v \circ_1 g$.*

*Then $(\widetilde{\mathcal{O}}, \widetilde{\mathcal{W}})$ admits a calculus of left fractions (4.1.8).*

*Suppose further that*

3. *for any $f, g \in \mathcal{O}(1)$, if two-out-of-three of $f$, $g$, and $g \circ_1 f$ are in $\mathcal{W}$, so is the third.*

*Then $(\widetilde{\mathcal{W}}, \widetilde{\mathcal{W}})$ admits a calculus of left fractions.*

*Proof.* First note that (1) can be equivalently expressed as the property

- For all $f \in \mathcal{O}(n)$, $u_1, \ldots, u_n \in \mathcal{W}$, there exists $f' \in \mathcal{O}(n)$, $v \in \mathcal{W}$ such that $\gamma(f; u_1, \ldots, u_n) = \gamma(v; f')$

by iterating property (1) $n$ times for $1 \le i \le n$. We can derive property (1) from the above by letting $u_j = 1_{\mathcal{O}}$ for $j \ne i$.

Similarly (2) can be equivalently expressed as

- for all $f, g \in \mathcal{O}(n)$, $u_1, \ldots, u_n \in \mathcal{W}$ such that $\gamma(f; u_1, \ldots, u_n) = \gamma(g; u_1, \ldots, u_n)$, there exists $v \in \mathcal{W}$ such that $\gamma(v; f) = \gamma(v; g)$.

Suppose now that $(\mathcal{O}, \mathcal{W})$ has properties (1) and (2). Let

$$f = [(f_1, \ldots, f_n, \sigma), (m_1, \ldots, m_n)] \in \widetilde{\mathcal{O}}(m, n),$$

$$u = [(u_1, \ldots, u_m, \tau), (1, \ldots, 1)] \in \widetilde{\mathcal{W}}(m, m).$$

Since $u \in \widetilde{\mathcal{W}}$, all of the $u_i$ are in $\mathcal{W}$. Then

$$fu = [(\gamma(f_1; u_{\sigma^{-1}(1)}, \ldots, u_{\sigma^{-1}(m_1)}), \ldots, \gamma(f_n; u_{\sigma^{-1}(m-m_n+1)}, \ldots, u_{\sigma^{-1}(m)}), \sigma\tau), (m_1, \ldots, m_n)].$$

By property (1), there exists $f_1' \in \mathcal{O}(m_1)$, $v_1 \in \mathcal{W}$ such that

$$\gamma(f_1; u_{\sigma^{-1}(1)}, \ldots, u_{\sigma^{-1}(m_1)}) = \gamma(v_1; f_1').$$

We can repeat this process to find $f_j' \in \mathcal{O}(m_j)$, $v_j \in \mathcal{W}$ for $1 \leq j \leq n$. Then let

$$f' = [(f_1', \ldots, f_n', \sigma\tau), (m_1, \ldots, m_n)],$$

$$v = [(v_1, \ldots, v_n, e_n), (1, \ldots, 1)].$$

We can verify that $fu = vf'$.

Suppose that there exists $f, g \in \widetilde{\mathcal{O}}(m, n)$, $u \in \widetilde{\mathcal{W}}(m, m)$ where $f, u$ are as above and

$$g = [(g_1, \ldots, g_n, \epsilon), (m_1', \ldots, m_n')]$$

such that $gu = fu$; i.e.

$$[(\gamma(f_1; u_{\sigma^{-1}(1)}, \ldots, u_{\sigma^{-1}(m_1)}), \ldots, \gamma(f_n; u_{\sigma^{-1}(m-m_n+1)}, \ldots, u_{\sigma^{-1}(m)}), \sigma\tau), (m_1, \ldots, m_n)] =$$

$$[(\gamma(g_1; u_{\epsilon^{-1}(1)}, \ldots, u_{\epsilon^{-1}(m_1')}), \ldots, \gamma(g_n; u_{\epsilon^{-1}(m-m_n'+1)}, \ldots, u_{\epsilon^{-1}(m)}), \epsilon\tau), (m_1', \ldots, m_n')].$$

This equality first implies that $m_j = m_j'$ for all $j$ since the partitions of $m$ in the compositions need to match.

We then have that there exists

$$\alpha_1 \times \ldots \times \alpha_n \in \Sigma_1 \times \ldots \times \Sigma_n$$

such that $(\alpha_1 \oplus \ldots \oplus \alpha_n)\epsilon = \sigma$ and

$$\begin{aligned}
\gamma(f_1; u_{\sigma^{-1}(1)}, \ldots, u_{\sigma^{-1}(m_1)}) \cdot \alpha_1 &= \gamma(g_1; u_{\epsilon^{-1}(1)}, \ldots, u_{\epsilon^{-1}(m_1)}) \\
&\vdots \\
\gamma(f_n; u_{\sigma^{-1}(m-m_n+1)}, \ldots, u_{\sigma^{-1}(m)}) \cdot \alpha_n &= \gamma(g_n; u_{\epsilon^{-1}(m-m_n+1)}, \ldots, u_{\epsilon^{-1}(m)}).
\end{aligned}$$

Note that for $1 \leq i \leq m_1$,

$$\sigma^{-1}(i) = \epsilon^{-1}((\alpha_1 \times \ldots \times \alpha_n)^{-1}(i)) = \epsilon^{-1}(\alpha_1^{-1}(i))$$

since $\alpha_1$ permutes only the first $m_1$ letters. Then we have

$$\begin{aligned}
\gamma(f_1; u_{\sigma^{-1}(1)}, \ldots, u_{\sigma^{-1}(m_1)}) \cdot \alpha_1 &= \gamma(f_1; u_{\epsilon^{-1}(\alpha_1^{-1}(1))}, \ldots, u_{\epsilon^{-1}(\alpha_1^{-1}(m_1))}) \cdot \alpha_1 \\
&= \gamma(f_1 \cdot \alpha_1; u_{\epsilon^{-1}(1)}, \ldots, u_{\epsilon^{-1}(m_1)}) \\
&= \gamma(g_1; u_{\epsilon^{-1}(1)}, \ldots, u_{\epsilon^{-1}(m_1)})
\end{aligned}$$

by the $\Sigma_{m_1}$-equivariance of $\gamma$. Thus by property (2), there exists $v_1 \in \mathcal{W}$ such that $\gamma(v_1; f_1 \cdot \alpha_1) = \gamma(v_1; g_1)$. Similarly, we can find $v_j \in \mathcal{W}$ such that $\gamma(v_j; f_j \cdot \alpha_j) = \gamma(v_j; g_j)$. Let

$v = [(v_1, \ldots, v_n, e_n), (1, \ldots, 1)] \in \widetilde{\mathcal{W}}(n,n)$. Then

$$\begin{aligned} vf &= [(\gamma(v_1; f_1), \ldots, \gamma(v_n; f_n), \sigma), (m_1, \ldots, m_n)] \\ &= [(\gamma(v_1; f_1) \cdot \alpha_1, \ldots, \gamma(v_n; f_n) \cdot \alpha_n, (\alpha_1 \oplus \cdot \oplus \alpha_n)\sigma), (m_1, \ldots, m_n)] \\ &= [(\gamma(v_1; f_1 \cdot \alpha_1), \ldots, \gamma(v_n; f_n \cdot \alpha_n), (\alpha_1 \oplus \cdot \oplus \alpha_n)\sigma), (m_1, \ldots, m_n)] \\ &= [(\gamma(v_1; g_1), \ldots, \gamma(v_n; g_n), \epsilon), (m_1, \ldots, m_n)] = vg. \end{aligned}$$

Let $f \in \widetilde{\mathcal{O}}(m,n)$, $g \in \widetilde{\mathcal{O}}(n,p)$ such that two-out-of-three of $f$, $g$, and $gf$ are in $\widetilde{\mathcal{W}}$. Since the only maps in $\widetilde{\mathcal{W}}$ are endomorphisms, this implies $m = n = p$. Then we have

$$g = [(g_1, \ldots, g_m, \epsilon), (1, \ldots, 1)],$$

$$f = [(f_1, \ldots, f_m, \sigma), (1, \ldots, 1)],$$

$$gf = [(\gamma(g_1; f_{\epsilon^{-1}(1)}), \ldots, \gamma(g_m; f_{\epsilon^{-1}(m)}), \epsilon\sigma), (1, \ldots, 1)]$$

For whichever maps are in $\widetilde{\mathcal{W}}$, this implies that each component of the maps is in $\mathcal{W}$. Thus two-out-of-three of $g_1$, $f_{\epsilon^{-1}(1)}$, or $\gamma(g_1; f_{\epsilon^{-1}(1)})$ are in $\mathcal{W}$, so thus the third is as well. The same holds for each triple $g_j$, $f_{\epsilon^{-1}(j)}$, or $\gamma(g_j; f_{\epsilon^{-1}(j)})$. Therefore the third of $f$, $g$, and $gf$ is in $\widetilde{\mathcal{W}}$. Then $(\widetilde{\mathcal{W}}, \widetilde{\mathcal{W}})$ admits a calculus of left fractions as well. □

When (1) and (2) of Lemma 5.0.5 hold, we say that $(\mathcal{O}, \mathcal{W})$ *admits a calculus of left fractions*, and if (3) holds as well we say that $(\mathcal{W}, \mathcal{W})$ *admits a calculus of left fractions.*

**Conjecture 5.0.6.** *If $(\mathcal{O}, \mathcal{W})$ and $(\mathcal{W}, \mathcal{W})$ admit a calculus of left fractions, then for all $n$, the components of $L^{TH}_{\mathcal{W}}\mathcal{O}$ are contractible.*

A natural first attempt to proving Conjecture 5.0.3 would be to construct a homotopy inverse to $R$. If we start with a tree hammock where the path from each labeled leaf node to the root node is the same length as in Figure 5.8a, then we can view it as a hammock by condensing each horizontal "column" of atomic pieces in each row to a single arrow as in Figure 5.8b. However, due to reduction, this does not assemble into a map of simplicial categories, as described in Figure 5.8.

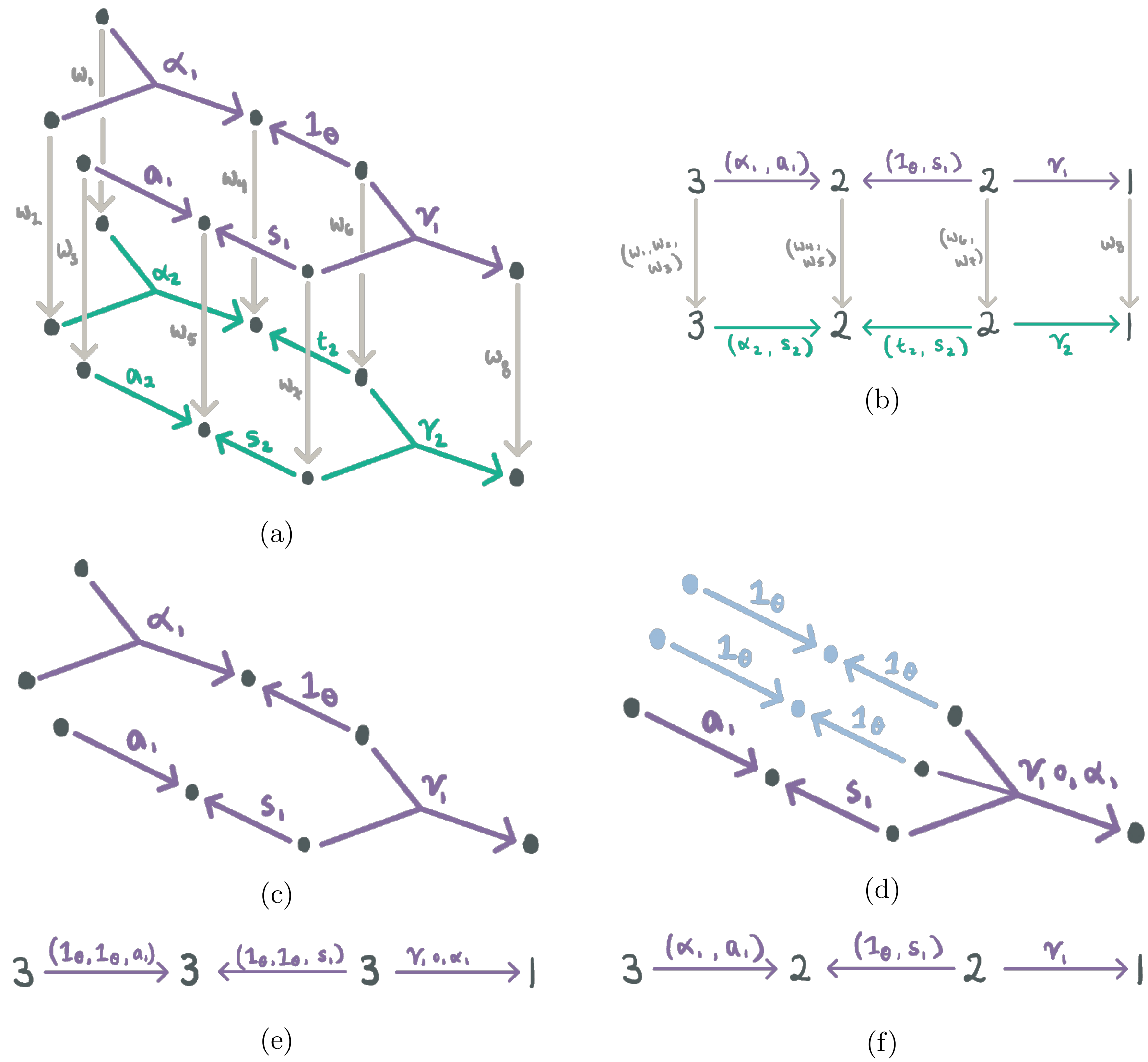



Figure 5.8: (a) A map $F$ in $\widetilde{L_{\mathcal{W}}^{TH}\mathcal{O}}(3,1)$. (b) The image of $F$ under our guess of $R^{-1}$, where we have left off the anchoring intial and final maps for clarity. Additionally, each map here is paired with the identity permutation in the appropriate $\Sigma_n$. (c) The first step of finding $d_1(F)$, which deletes the bottom row of $F$. (d) $d_1(F)$ is Figure 5.8c after reduction, since the previous image had a column of identities. For the next step, we will think of "exetending" this map by atomic pieces labeled by $1_{\mathcal{O}}$ so that the paths from the root node to the leaf nodes are all the same length, but these atomic pieces are not actually part of $d_1(F)$. (e) The image of $d_1(F)$ under our guess of $R^{-1}$. (f) $d_1$ of Figure 5.8b. Note this is not the same as Figure 5.8e, showing that our guess for $R^{-1}$ is not a map of simplicial categories.

# CHAPTER 6

## A First Attempt at Proving Conjecture 5.0.3

Conjecture 5.0.3 remains unproved, but in the remainder of this dissertation we collect a few promising approaches to the proof. As a test case, we only consider operads in Set, but we believe it should be straightforward to expand a successful proof to operads in sSet. In this case, Conjecture 5.0.3 can be rewritten:

**Conjecture 6.0.1** (5.0.3 for operads in Set)**.** *$R : L^H_{\widetilde{\mathcal{W}}}\widetilde{\mathcal{O}} \to \widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}$ is a DK-equivalence of simplicial categories.*

### 6.1 A Hammock Localization of the Tree Hammock Localization

We can view $\widetilde{\mathcal{W}}$ as a simplicial subcategory of $\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}$ by considering each hom-object as a discrete simplicial category, meaning that there are no non-degenerate $k$-simplices for $k > 0$. For example, for each $w \in \widetilde{\mathcal{W}}(n,n)_0$, there is an $k$-simplex in $\widetilde{\mathcal{W}}(n,n)_k$ which is a width 1 height $k$ tree hammock, of type a single forward pointing atomic piece, labeled in every row by $w$ and with vertical arrows all identities.

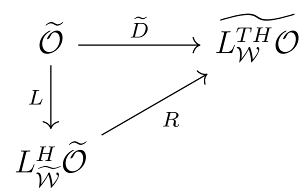



Figure 6.1

**Lemma 6.1.1.** *For a fixed operad $\mathcal{O}$ and a submonoid $\mathcal{W} \subset \mathcal{O}$, the diagram in Figure 6.1 commutes, where $D$ is the canonical map $\mathcal{O} \to L^{TH}_{\mathcal{W}}\mathcal{O}$, and $L$ is the canonical map $\widetilde{\mathcal{O}} \to L^H_{\widetilde{\mathcal{W}}}\widetilde{\mathcal{O}}$.*

*Proof.* Since all of the functors in Figure 6.1 are the identity on objects, it only remains to check that the diagram commutes on morphisms. Let $f = [(f_1, \ldots, f_n, \sigma), (m_1, \ldots, m_n)] \in$

$\widetilde{\mathcal{O}}(m,n)$. Then $L(f)$ is a hammock consisting of a single forward pointing arrow $m \to n$ labeled by $f$, and $R(L(f)) = [(f_1, \ldots, f_n, \sigma), (m_1, \ldots, m_n)]$ where here each $f_i$ is a tree hammock with $m_i$ labeled leaf nodes consisting of a single forward atomic piece $O_{m_i}$ labeled by $f_i$. We also have

$$\begin{aligned} \widetilde{D}(f) &= [(D(f_1), \ldots, D(f_n), \sigma), (m_1, \ldots, m_n)] \\ &= [(f_1, \ldots, f_n, \sigma), (m_1, \ldots, m_n)] \end{aligned}$$

where each $f_i$ is again a tree hammock with a single forward facing atomic piece labeled by $f_i$. Thus the diagram commutes. □

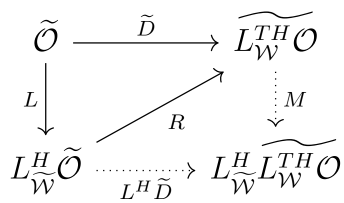


Figure 6.2

We can extend this diagram by taking the hammock localization of $\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}$ with respect to $\widetilde{\mathcal{W}}$, resulting in Figure 6.2. This is a diagram in the category of bisimplicial categories, where the hom-objects of $\widetilde{\mathcal{O}}$ are constant in two directions, and the hom-objects of $\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}$ and $L^H_{\widetilde{\mathcal{W}}}\widetilde{\mathcal{O}}$ are each constant in one direction and in the opposite direction of each other. Here, $M$ is the canonical map that comes with the hammock localization, while $L^H\widetilde{D}$ is the image of $\widetilde{D}$ under the functor $L^H : \mathsf{RelCat} \to \mathsf{sCat}$.

Three of the corners in this diagram have a similar flavor and it is easy to get confused about how they differ. All three categories have the natural numbers as their objects. In $L^H_{\widetilde{\mathcal{W}}}\widetilde{\mathcal{O}}$, the morphisms are hammocks whose components are labeled by morphisms in $\widetilde{\mathcal{O}}$. In $\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}$, the morphisms are products of tree hammocks labeled by operations in $\mathcal{O}$. Finally, in $L^H_{\widetilde{\mathcal{W}}}\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}$, the morphisms are hammocks labeled by morphisms in $\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}$; in other words they are hammocks built out of tree hammocks.

By a similar argument to Lemma 6.1.1, we can show that the outer square commutes. The bottom triangle, however, does not. For example, if we start with a height 1 hammock in $L^H_{\widetilde{\mathcal{W}}}\widetilde{\mathcal{O}}$, this will map under $L^H\widetilde{D}$ to a height 1 hammock in $L^H_{\widetilde{\mathcal{W}}}\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}$ built of height 0 tree hammocks. But when we look at its image under $M \circ R$, we get a height 0 hammock consisting of a single forward facing arrow labeled by a height 1 tree hammock. Clearly these images will not be equal. However, the bottom triangle does commute up to homotopy:

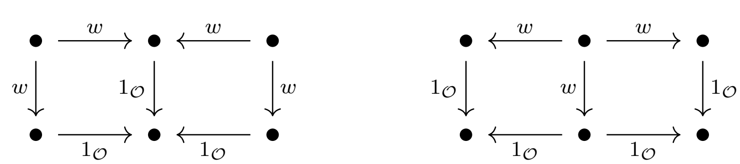
**Lemma 6.1.2.** *For an operad $\mathcal{O}$ and a submonoid $\mathcal{W} \subset \mathcal{O}(1)$, $M \circ R : L^H_{\widetilde{\mathcal{W}}}\widetilde{\mathcal{O}} \to L^H_{\widetilde{\mathcal{W}}}\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}$ is homotopic to $L^H\widetilde{D}$.*

*Proof.* Since the outer square of Figure 6.2 commutes, we have that $[M \circ \widetilde{D}] = [L^H\widetilde{D} \circ L]$ in Ho(sCat), the category obtained from sCat by inverting all of the DK-equivalences. Since the upper triangle also commutes, this gives us $[M \circ (R \circ L)] = [L^H\widetilde{D} \circ L]$. From the first part of [TV05, Remark 2.2.1], we can conclude that the induced map $[L]$ in Ho(sCat) is injective. Thus $[M \circ R] = [L^H\widetilde{D}]$. ☐

**Lemma 6.1.3.** *$M : \widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}} \to L^H_{\widetilde{\mathcal{W}}}\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}$ is a DK-equivalence.*

*Proof.* We first claim that $\widetilde{\mathcal{W}}$ is *neglectable* [BK12, 2.3] in $\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}$, meaning that every map in $\widetilde{\mathcal{W}}$ goes to an isomorphism in $\pi_0(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})$. This is proved using a similar argument to [BBP+18, 5.2].

Let $w \in \widetilde{\mathcal{W}}(1,1)_0$, meaning we have a height 0 tree hammock $\cdot \xrightarrow{w} \cdot$. In $\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}(1,1)_0$, we have the following height 0 tree hammock which we will call $\overline{w}$: $\cdot \xleftarrow{w} \cdot$. We can construct the following tree hammocks:



The first diagram gives that $\overline{w}$ is a right inverse for $w$ in $\pi_0(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})$, and the second diagram gives that it is a left inverse, since taking $\pi_0$ identifies the top and bottom rows of a height 1 tree hammock. Thus $w$ maps to an isomorphism in $\pi_0(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})$. We can repeat a similar construction for any $w \in \widetilde{\mathcal{W}}(n,n)_0$. Therefore by [BK12, 2.4.ii], $M$ is a DK-equivalence. ☐

Since $M \circ R$ and $L^H\widetilde{D}$ have the same image in Ho(sCat), if one is a DK-equivalence, so is the other. Furthermore, since DK-equivalences are the weak equivalences in the model category structure on sCat, they have the 2-out-of-3 property. Thus we can reduce the proof of Conjecture 6.0.1 to proving that $L^H\widetilde{D}$ is a DK-equivalence, since then $M \circ R$ will be a DK-equivalence, and combined with Lemma 6.1.3, this would imply that $R$ is a DK-equivalence.

In attempting to prove that $L^H\widetilde{D}$ is a DK-equivalence, it became clear that the key was to study $\widetilde{D}$ itself.

# CHAPTER 7

## $\widetilde{D}$ as an $\infty$-localization

Another approach to proving Conjecture 6.0.1 is to show that the tree hammock localization is an $\infty$-localization as defined in [Lur17].

In this chapter, we will introduce $\infty$-categories, using the model of quasicategories, and marked simplicial sets. We will define a localization for $\infty$-categories and review a proof that the traditional hammock localization fits this definition. Finally, we will discuss the first steps of a proof that the tree hammock localization fits this definition, which will ultimately imply that $R$ is a DK-equivalence.

### 7.1 Quasicategories and Marked Simplicial Sets

For this approach, we want to study $\infty$-categories, which is a type of higher category theory. We start by reviewing the notion of a *2-category*. Rather than restricting ourselves to a category with just objects and morphisms, in a 2-category, we have morphisms between morphisms, which we call *2-morphisms*. If we require that the composition of these 2-morphisms is strictly associative and unital, than we have a *strict 2-category*, and if we instead only require an isomorphism witnessing these properties, we get a *weak 2-category*, or bicategory. It turns out that every weak 2-category is equivalent to a strict 2-category. We could then include 3-morphisms between 2-morphisms, to get a 3-category, and so on. However, it is not the case that every weak 3-category is equivalent to a strict 3-category.

In general, for $1 \leq n \leq \infty$, a *n-category* will have $i$-morphisms for all $i \leq n$. Finding the correct notion of an $n$-category can quickly get out of hand for $n > 3$, and certainly for

$n = \infty$. Instead, we will work with the following:

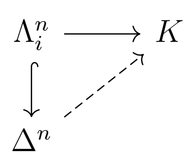
**Definition 7.1.1.** An $(\infty, 1)$-*category* is an $\infty$-category where all $k$-morphisms are invertible for $k > 1$.

Since we only discuss $(\infty, 1)$-categories here, we will use $\infty$-categories to refer to $(\infty, 1)$-categories. There are several models for $\infty$-categories, including simplicial categories, Segal categories, and complete Segal spaces (see [Ber07b] for an overview). We will primarily use quasicategories, also known as weak Kan complexes, which have been studied by Joyal [Joy05], Boardman-Vogt [BV73], and Lurie [Lur09]. More recently, the information about quasicategories has been collected by Rezk [Rez22].

### 7.1.1 Quasicategories

**Definition 7.1.2.** The *ith horn* of the simplex $\Delta^n$, $\Lambda^n_i \subset \Delta^n$, is obtained from $\Delta^n$ by deleting the interior and the face opposite the $i$th vertex. If $i = 0$ or $n$ we call $\Lambda^n_i$ an *outer horn*, and if $0 < i < n$, we call $\Lambda^n_i$ an *inner horn*.

**Definition 7.1.3.** Let $K$ be a simplicial set. We say that $K$ is a Kan complex if for any $n$, any $0 \leq i \leq n$, and any map $\Lambda^n_i \to K$, there exists an extension $\Delta^n \to K$ making the diagram commute.

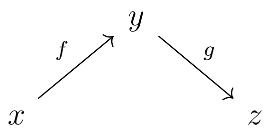


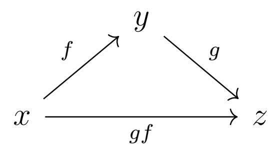
Colloquially, we call the extension a *filler* of the horn and say that the horn can be *filled*.

**Definition 7.1.4.** Let $K$ be a simplicial set. We say that $K$ is a *quasicategory* (or a *weak Kan complex*) if every inner horn can be filled.

*Example* 7.1.5. Every Kan complex is a quasicategory.

*Example* 7.1.6. Let $\mathcal{C}$ be a category. Then its nerve $N(\mathcal{C})$ is a quasicategory. For example, here is an inner horn of a 2-simplex in $\mathcal{C}$ and its filler.





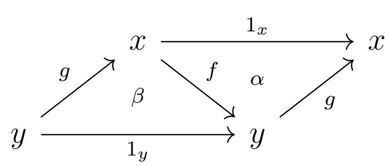

In particular, the fillers in $N(\mathcal{C})$ are unique, which is not true in a general quasicategory.

Drawing inspiration from the previous example, we often call the 0-simplices of a quasicategory $\mathcal{C}$ *objects* and the 1-simplices *morphisms*. For vertices $x, y \in \mathcal{C}_0$, let $\hom_{\mathcal{C}}(x, y)$ denote the collection of morphisms in $\mathcal{C}_1$ where $d_1(f) = x$ and $d_0(f) = y$, and $1_x$ denote the morphism $s_0 x \in \hom_{\mathcal{C}}(x, x)$.

*Example* 7.1.7. If $\mathcal{C}$ is a simplicial category such that for all objects $X, Y \in \mathcal{C}$, $\mathcal{C}(X, Y)$ is a Kan complex, then $N_{coh}(\mathcal{C})$ is a quasicategory.

**Definition 7.1.8.** A morphism of quasicategories $F : \mathcal{C} \to \mathcal{D}$ is a map of simplicial sets between two quasicategories. No additional structure is required. We will call these *functors of quasicategories.*

**Definition 7.1.9.** A morphism $f \in \hom_{\mathcal{C}}(x, y)$ is an *isomorphism* if there exists $g \in \mathcal{C}_1$ and $\alpha, \beta \in \mathcal{C}_2$ such that we can produce the following diagram:




In other words,

$$d_2\alpha = f = d_0\beta, \quad d_0\alpha = g = d_2\beta, \quad d_1\alpha = 1_x, \quad d_1\beta = 1_y.$$

### 7.1.2 Marked simplicial sets

**Definition 7.1.10.** A *marked simplicial set* is a pair $(X, E)$ where $X$ is a simplicial set and $E$ is a set of edges of $X$ containing all degenerate edges. An edge of $X$ is *marked* if it belongs to $E$. A *map of marked simplicial sets* $f : (X, E) \to (Y, F)$ is a map of simplicial sets $f$ such that $f(E) \subset F$. These assemble into the category of marked simplicial sets, $\mathsf{sSet}_+$. For a marked simplicial set $(X, E)$, we call $X$ its *underlying simplicial set.*

*Example* 7.1.11. There are three important types of marked simplicial sets:

- $X^\sharp$ is the marked simplicial set with every edge marked, called the *maximal marking.*
- $X^\flat$ is the marked simplicial set with only degenerate edges marked, called the *minimal marking.*
- For a quasicategory $\mathcal{C}$, $\mathcal{C}^\natural$ denotes the marked simplicial set with the isomorphisms marked.

*Remark* 7.1.12. In more recent texts such as [RV22], marked simplicial sets can include marked simplices in all dimensions, and all degenerate simplices are required to be included in $E$. We primarily use the definition above from [Lur09, 3.1], which is just a special case of the newer definition where no nondegenerate $k$-simplices are marked for $k > 1$.

Our goal in section 7.3 will be to construct a particular weak equivalence of marked simplicial sets. To describe what that means, we first introduce some definitions, combining [RV22] and [Lur09]. Lurie defines most of these structures for the category $\mathsf{sSet}_{+/S}$, or the category of marked simplicial sets equipped with a map to $S^\sharp$ for a simplicial set $S$. We are interested in the case $\mathsf{sSet}_+ \cong \mathsf{sSet}_{+/*}$. This simplifies many of the constructions greatly, so we only describe them for the case $S = *$. In particular, several constructions depend on Cartesian fibrations over $S$ (see [Lur09, 2.4.2]), and a simplicial map $X \to *$ is a Cartesian fibration if and only if $X$ is a quasicategory.

**Lemma 7.1.13.** *[RV22, D.3.3]* $\mathsf{sSet}_+$ *is Cartesian closed, where*

- *the* Cartesian product of marked simplicial sets $(X, E)$ *and* $(Y, F)$ *is the Cartesian product of the underlying simplicial sets where a simplex is marked if and only if it is the product of marked simplices; i.e.,* $(X, E) \times (Y, F) = (X \times Y, E \times F)$*;*

- *the* internal hom $(Y, F)^{(X,E)}$ *has as its $n$-simplices maps of marked simplicial sets* $X \times (\Delta^n)^\flat \to Y$*, where* $\sigma : X \times \Delta^1 \to Y$ *is marked exactly when the restriction* $\sigma : E \times (\Delta^1)_1 \to Y_1$ *is entirely contained in* $F$*.*

**Definition 7.1.14.** Let $X = (X, E)$ and $Y = (Y, F)$ denote marked simplical sets. We define $\mathrm{Map}^\flat(X, Y)$ to be the underlying simplicial set of $Y^X$; i.e. the simplicial set with $n$-simplices maps of marked simplicial sets $X \times (\Delta^n)^\flat \to Y$. The simplicial structure here is induced by the geometric maps $D_i$ and $S_i$ on the standard $n$-simplices [Fri08, 3].

We define $\mathrm{Map}^\sharp(X, Y) \subset \mathrm{Map}^\flat(X, Y)$ to be the simplicial subset of $\mathrm{Map}^\flat(X, Y)$ consisting of simplices $\sigma$ such that every edge of $\sigma$ is marked in $Y^X$.

*Remark* 7.1.15. [Lur09, 3.1.3.1] If $Y$ is a quasicategory and $X = (X, E)$ a marked simplicial set, then $\mathrm{Map}^\flat(X, Y^\natural)$ is a quasicategory and $\mathrm{Map}^\sharp(X, Y^\natural)$ is the largest Kan complex contained in $\mathrm{Map}^\flat(X, Y^\natural)$.

**Lemma 7.1.16.** *[Lur09, 3.1.3.3] Let* $f : (X, E) \to (Y, F)$*. The following are equivalent:*

1. *For every quasicategory $Z$, the induced map*

$$\mathrm{Map}^\flat(Y, Z^\natural) \to \mathrm{Map}^\flat(X, Z^\natural)$$

*is an equivalence of quasicategories.*

2. *For every quasicategory $Z$, the induced map*

$$\mathrm{Map}^\sharp(Y, Z^\natural) \to \mathrm{Map}^\sharp(X, Z^\natural)$$

*is a homotopy equivalence of Kan complexes.*

**Definition 7.1.17.** A map $f : (X, E) \to (Y, F)$ is a *weak equivalence of marked simplicial sets* if it satisfies either condition of Lemma 7.1.16.

### 7.1.3 Fibrant Objects

We will need the notion of fibrant objects in several related categories.

**Lemma 7.1.18.** *$X \in \mathsf{sSet}$ is fibrant if and only if $X$ is a Kan complex.*

*Remark* 7.1.19. Here we are using the classical model structure on $\mathsf{sSet}$. In the Joyal model structure, the fibrant objects are exactly the quasicategories.

**Lemma 7.1.20.** *[Lur09, 3.1.4.1] $(X, E) \in \mathsf{sSet}_+$ is fibrant if and only if $(X, E) \cong Y^\natural$ where $Y$ is an $\infty$-category.*

**Lemma 7.1.21.** *$\mathcal{C} \in \mathsf{sCat}$ is fibrant if and only if $\mathcal{C}(X, Y)$ is a Kan complex for all $X, Y \in \mathcal{C}$.*

There exists a fibrant replacement functor $\mathrm{Ex}^\infty : \mathsf{sSet} \to \mathsf{sSet}$ which gives, for every simplicial set $X$, a weakly equivalent Kan complex $\mathrm{Ex}^\infty(X)$. We review the construction here.

**Definition 7.1.22.** For the standard $n$-simplex $\Delta^n$, the nondegenerate $m$-simplices correspond to an ordered list of $m+1$ vertices of $\Delta^n$; i.e. a subset of $\{0, 1, \ldots, n\}$ of cardinality $m + 1$. These nondegenerate simplices form a poset ordered by inclusion, which we call $\mathrm{nd}\,\Delta^n$. Then the *subdivision of* $\Delta^n$ is defined to be

$$\mathrm{sd}\,\Delta^n := N(\mathrm{nd}\,\Delta^n).$$

We depict $\mathrm{sd}\,\Delta^1$ and $\mathrm{sd}\,\Delta^2$ in Figure 7.1.

**Definition 7.1.23.** For $X \in \mathsf{sSet}$, we define $\mathrm{Ex}(X)$ to be the simplicial set with

$$\mathrm{Ex}(X)_n := \mathrm{Hom}_{\mathsf{sSet}}(\mathrm{sd}\,\Delta^n, X).$$

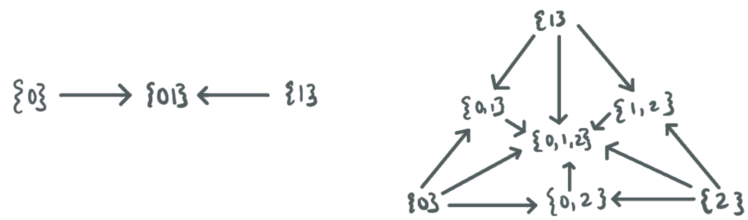


Figure 7.1: Depictions of the subdivisions of $\Delta^1$ and $\Delta^2$.

A $k$-simplex in $\mathrm{sd}\,\Delta^n$ is an ordered list $[a_0, \dots, a_k]$ where each $a_i$ is a subset of $\{0, 1, \dots, n\}$ such that $a_i \subset a_{i+1}$ for $0 \le i \le k-1$. If this is a non-degenerate $k$-simplex, then each of the $a_i$ is distinct.

In other words, each $a_i$ is a list of vertices of $\Delta^n$. For each $a_i = \{v_0^i, \dots v_s^i\} \subset \{0, \dots, n\}$, where $v_j^i < v_{j+1}^i$ for $0 \le j \le s-1$, we let $v^i := v_s^i$ denote the *last vertex* of $a_i$. Note for the $k$-simplex $[a_0, \dots, a_k]$, $v^i \le v^{i+1}$ for $0 \le i \le k-1$.

**Lemma 7.1.24.** *[GJ91, III.4] The* last vertex map $lv : \mathrm{sd}\,\Delta^n \to \Delta^n$*, which is defined by*

$$lv([a_0, \dots, a_k]) = [v^0, \dots v^k]$$

*is a simplicial weak equivalence.*

**Definition 7.1.25.** For $X \in \mathsf{sSet}$, we define the map $j_X : X \to \mathrm{Ex}(X)$ to be the simplicial map induced by precomposition with $lv$.

*Example* 7.1.26. $\mathrm{sd}\,\Delta^1$ has two non-degenerate 1-simplices, $\{0\} \to \{0,1\}$ and $\{1\} \to \{0,1\}$. We have

$$lv(\{0\} \to \{0,1\}) = 0 \to 1, \qquad lv(\{1\} \to \{0,1\}) = 1 \xrightarrow{s_0 1} 1.$$

Then for a 1-simplex $x \xrightarrow{f} y$ in $X_1$, we have

$$j_X(x \xrightarrow{f} y) = x \xrightarrow{f} y \xleftarrow{s_0 y} y \in \mathrm{Ex}(X)_1.$$

**Definition 7.1.27.** For $X \in \mathsf{sSet}$, define $\mathrm{Ex}^\infty(X)$ to be the colimit of the directed system

$$X \xrightarrow{j_X} \mathrm{Ex}(X) \xrightarrow{j_{\mathrm{Ex}(X)}} \mathrm{Ex}^2(X) \xrightarrow{j_{\mathrm{Ex}^2(X)}} \dots$$

which comes with a map $j_X^\infty : X \to \mathrm{Ex}^\infty(X)$. By construction, this assembles into a functor $\mathrm{Ex}^\infty : \mathsf{sSet} \to \mathsf{sSet}$.

**Lemma 7.1.28.** *[Gui, Section 4] We collect a few important properties of* $\mathrm{Ex}^\infty$*:*

- $\mathrm{Ex}^\infty(X)$ *is a Kan complex for any* $X \in \mathsf{sSet}$.
- $j_X^\infty$ *is a weak equivalence for any* $X$ *(and a cofibration).*
- $\mathrm{Ex}^\infty$ *preserves 0-simplices.*

*The first two bullets give that* $\mathrm{Ex}^\infty$ *is a fibrant replacement functor.*

The definition of $\mathrm{Ex}^\infty$ gives

$$\mathrm{Ex}^\infty(X)_k = \left(\coprod_n \mathrm{Ex}^n(X)_k\right) / \approx$$

where the equivalence relation $\approx$ is generated by the relation $\sim$ where $\alpha \sim \beta$ if $\beta = j_{\mathrm{Ex}^t(X)}(\alpha)$ for some $t$. For example, we have

$$x \xrightarrow{\alpha} y \sim x \xrightarrow{\alpha} y \xleftarrow{s_0 y} y \sim x \xrightarrow{\alpha} y \xleftarrow{s_0 y} y \xrightarrow{s_0 y} y \xleftarrow{s_0 y} y \sim \dots$$

Essentially, we can think of the representatives of these equivalence classes of $k$-simplices as maps $\mathrm{sd}^i \Delta^k \to X$ for some $i$.

**Lemma 7.1.29.** *There exists a fibrant replacement functor* $\mathrm{Ex}^\infty : \mathsf{sCat} \to \mathsf{sCat}$ *where for* $\mathcal{C} \in \mathsf{sCat}$, $\mathrm{Ex}^\infty(\mathcal{C})$ *has the same objects as* $\mathcal{C}$ *and*

$$\mathrm{Ex}^\infty(\mathcal{C})(X, Y) := \mathrm{Ex}^\infty(\mathcal{C}(X, Y))$$

*Remark* 7.1.30. Using the model category structure on $\mathsf{sCat}$, we can construct the *right derived homotopy coherent nerve functor*,

$$\boldsymbol{R}N_{coh} : \mathsf{sCat} \to \mathsf{sSet}.$$

Explicitly, up to weak equivalence we can define $\boldsymbol{R}N_{coh}(\mathcal{C}) := N_{coh}(\mathcal{C}^f)$, where $\mathcal{C}^f$ is a fibrant replacement for $\mathcal{C}$ in $\mathsf{sCat}$. When we refer to $\boldsymbol{R}N_{coh}(\mathcal{C})$, we will always mean $N_{coh}(\mathrm{Ex}^\infty(\mathcal{C}))$.

### 7.2 $\infty$-localization

**Definition 7.2.1.** [Lur17, 1.3.4.1] Let $\mathcal{C}$ be an ordinary category and $\mathcal{W}$ be a collection of morphisms of $\mathcal{C}$, and $\mathcal{D}$ an $\infty$-category. We say that $f : N(\mathcal{C}) \to \mathcal{D}$ *exhibits* $\mathcal{D}$ *as the* $\infty$-*category obtained from* $N(\mathcal{C})$ *by inverting the set of morphisms* $\mathcal{W}$ if, for every $\infty$-category $\mathcal{E}$, composition with $f$ induces a fully faithful embedding $\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(N(\mathcal{C}), \mathcal{E})$ whose essential image is the collection of functors $F : N(\mathcal{C}) \to \mathcal{E}$ which carry each morphism in $\mathcal{W}$ to an equivalence in $\mathcal{E}$.

In this case, the $\infty$-category $\mathcal{D}$ is determined up to equivalence by $\mathcal{C}$ and $\mathcal{W}$, and will be denoted by $\mathcal{C}[\mathcal{W}^{-1}]$. We will often call $\mathcal{C}[\mathcal{W}^{-1}]$ the $\infty$-*localization of* $\mathcal{C}$ *with respect to* $\mathcal{W}$.

*Remark* 7.2.2. [Lur17, 1.3.4.2] If we assume without loss of generality that $\mathcal{W}$ contains all identity morphisms, then $\mathcal{C}[\mathcal{W}^{-1}]$ can be identified with the underlying simplicial set of a fibrant replacement of $(N(\mathcal{C}), \mathcal{W})$ in $\mathsf{sSet}_+$.

**Lemma 7.2.3.** *Let* $\mathcal{C}$ *be a category, and* $\mathcal{W}$ *a collection of morphisms in* $\mathcal{C}$ *which includes all identity morphisms. Then the hammock localization of* $\mathcal{C}$ *with respect to* $\mathcal{W}$ *is an* $\infty$-*localization as in Definition 7.2.1 in the following sense: there exists a weak equivalence*

$$(N(\mathcal{C}), \mathcal{W}) \to \boldsymbol{R}N_{coh}(L^H_{\mathcal{W}}\mathcal{C})^\natural.$$

*Proof.* [1] The map of marked simplicial sets $(N(\mathcal{C}), \mathcal{W}) \to \boldsymbol{R}N_{coh}(L^H_{\mathcal{W}}\mathcal{C})^\natural$ is given as follows: we begin with the canonical map $\mathcal{C} \hookrightarrow L^H_{\mathcal{W}}\mathcal{C}$, and take the fibrant replacement of $L^H_{\mathcal{W}}\mathcal{C}$ in $\mathsf{sCat}$ by applying the functor $Ex^\infty$ to each hom-object. We can consider $\mathcal{C}$ as a simplicial category where each hom-object is constant, and apply the homotopy coherent nerve to obtain a map of simplicial sets $N_{coh}(\mathcal{C}) \to \boldsymbol{R}N_{coh}(L^H_{\mathcal{W}}\mathcal{C})$. Since $\mathcal{C}$ is an ordinary category, $N_{coh}(\mathcal{C}) = N(\mathcal{C})$ [Lur09, 1.2.3.1]. The marked edges of $\boldsymbol{R}N_{coh}(L^H_{\mathcal{W}}\mathcal{C})^\natural$ are exactly the isomorphisms, and following the argument proceeding Lemma 4.1.4, the morphisms in $\mathcal{W}$ are isomorphisms. Thus we have a map of marked simplicial sets $(N(\mathcal{C}), \mathcal{W}) \to \boldsymbol{R}N_{coh}(L^H_{\mathcal{W}}\mathcal{C})^\natural$ as desired.

Constant simplicial sets are Kan complexes, so therefore $\mathcal{C}$ considered as a simplicial category is a fibrant simplicial category. Thus we can apply [Hin13, Proposition 1.2.1] to conclude that the map is a weak equivalence in $\mathsf{sSet}_+$.

□

Drawing inspiration from the above lemma, instead of comparing the tree hammock localization to the hammock localization directly, we claim that the tree hammock localization is an $\infty$-localization as in Definition 7.2.1. Using a similar method to the above proof, we can construct a map $(N(\widetilde{\mathcal{O}}), \widetilde{\mathcal{W}}) \to \boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^\natural$. Note that since $\mathcal{W}$ is a submonoid of $\mathcal{O}(1)$, it contains $1_{\mathcal{O}}$ and thus $\widetilde{\mathcal{W}}$ contains all identity morphisms in $\widetilde{\mathcal{O}}$.

$$\begin{array}{ccc} N(\widetilde{\mathcal{O}}) & \longrightarrow & \boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^\natural \\ \downarrow & \nearrow & \\ \boldsymbol{R}N_{coh}(L^H_{\widetilde{\mathcal{W}}}\widetilde{\mathcal{O}})^\natural & & \end{array}$$

Figure 7.2

**Conjecture 7.2.4.** *Let $\mathcal{O}$ be an operad and $\mathcal{W}$ a submonoid of $\mathcal{O}(1)$. $H : (N(\widetilde{\mathcal{O}}), \widetilde{\mathcal{W}}) \to \boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^\natural$ is a weak equivalence of marked simplicial sets.*

If the above conjecture is true, we will have the commutative diagram in Figure 7.2 where the top and left maps are $\infty$-localization functors. Then by [Lur22, Remark 01MX], the diagonal map is a categorical equivalence of the underlying simplicial sets.

[1]This proof is adapted from a MathOverflow discussion [Mar21].

## 7.3 First Steps of Proof of Conjecture 7.2.4

In this section, we give a rough overview of the work completed so far towards proving the conjecture. We first need to understand $\boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^\natural$, and then will describe the map $H : (N(\widetilde{\mathcal{O}}), \widetilde{\mathcal{W}}) \to \boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^\natural$. We will review a technique for proving a map is a homotopy equivalence of Kan complexes, and use (2) in Lemma 7.1.16 to begin a proof of Conjecture 7.2.4.

We start by examining $\mathrm{Ex}^\infty(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})$. This is a simplicial category with the same objects as $\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}$, and with hom-objects $\mathrm{Ex}^\infty(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}(m,n))$. We collect some notes on $X := \mathrm{Ex}^\infty(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}(m,n))$ for a fixed $m, n$:

- The 0-simplices of $X$ are the 0-simplices of $\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}(m,n)$ since $\mathrm{Ex}^\infty$ preserves 0-simplices; i.e. they are products of height 0 tree hammocks.

- The 1-simplices of $X$ are zig-zags of 1-simplices of $\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}(m,n)$, i.e. they are zig-zags of (products of) height 1 tree hammocks from $m$ to $n$. For example, a simple zig-zag would be a pair of height 1 tree hammocks $A$ and $B$ such that $d_0A = d_0B$, so their bottom rows match after any necessary reduction. Note that $A$ and $B$ may not be of the same type because of this reduction (see Figure 7.3).

- The $k$-simplices of $X$ are essentially the maps $\mathrm{sd}^i\,\Delta^k \to \widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}(m,n)$. The data of one of these maps can be quite restrictive. For example, if we just think about the bottom left diamond of $\mathrm{sd}\,\Delta^2$, the image of that in $\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}(m,n)$ would be two height 2 tree hammocks from $m$ to $n$, $C$ and $D$, such that they share the same top and bottom rows and so that $d_1C = d_1D$.

Then $\boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}) := N_{coh}(\mathrm{Ex}^\infty(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}))$. This is a simplicial set where:

- The 0-simplices are $\mathbb{N}$, the objects of $\mathrm{Ex}^\infty(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}(m,n))$.

- A 1-simplex consists of $m, n \in \mathbb{N}$ and a map $f \in \mathrm{Ex}^\infty(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}(m,n))_0$, i.e. a product of height 0 tree hammocks from $m$ to $n$. We then have $d_1 f = m$, $d_0 f = n$.

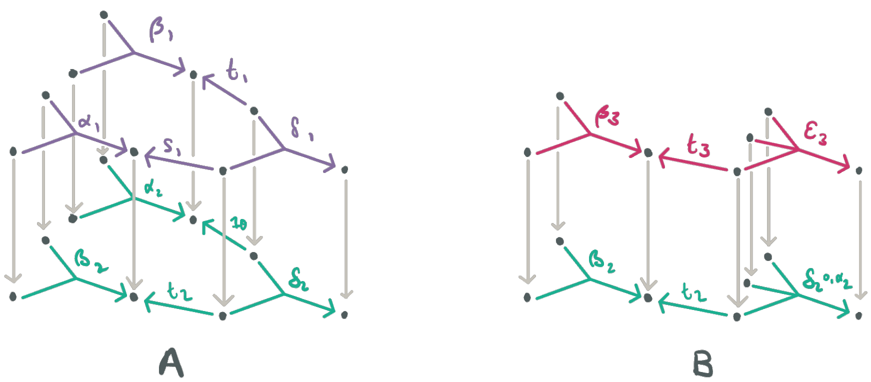



Figure 7.3: Here are height 1 tree hammocks $A$ and $B$ such that $d_0A = d_0B$. Recall that applying the simplicial structure maps also includes reducing as necessary, and the bottom row of $A$ includes an identity arrow and is thus unreduced.

- A 2-simplex consists of $m, n, p \in \mathbb{N}$, $f \in \mathrm{Ex}^\infty(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}(m,n))_0$, $g \in \mathrm{Ex}^\infty(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}(n,p))_0$, $h \in \mathrm{Ex}^\infty(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}(m,p))_0$, and a 1-simplex $A$ in $\mathrm{Ex}^\infty(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}(m,p))_1$ such that $d_1A = h$ and $d_0A = gf$. In other words, $A$ is a zig-zag of (a product of) height 1 tree hammocks from $m$ to $p$ connecting the height 0 tree hammock $h$ to the height 0 tree hammock $gf$.
- In general, a $k$-simplex consists of a string of $k$-composable height 0 tree hammocks along with additional structure.

As in Example 7.1.7, $\boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})$ is a quasicategory, so we can consider the marked simplicial set $\boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^\natural$. The marked edges here are the isomorphisms. This means $f \in \mathrm{Ex}^\infty(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}(m,n))_0$ is an isomorphism if and only if there exists $g \in \mathrm{Ex}^\infty(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}(n,m))_0$ such that there are zigzags of height 1 tree hammocks connecting $gf$ and $\mathrm{Id}_m$ and connecting $fg$ and $\mathrm{Id}_n$. We believe that this can only occur if $f$ is built entirely out of $O_1$ and $W_1$ atomic pieces, with the $O_1$ pieces labeled by operations in $\mathcal{W}$ or by operations which were already invertible in $\mathcal{O}$. For example, the following $f$ and $g$ would be isomorphisms, where $w, a, v \in \mathcal{W}$:

$$f = \cdot \xrightarrow{w} \cdot \xleftarrow{a} \cdot \xrightarrow{v} \cdot \qquad g = \cdot \xleftarrow{v} \cdot \xrightarrow{a} \cdot \xleftarrow{w} \cdot$$

The map $H : (N(\widetilde{\mathcal{O}}), \widetilde{\mathcal{W}}) \to \boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^\natural$ is induced similarly to the map in Lemma

7.2.3: we apply $N_{coh}$ to the composition

$$\widetilde{\mathcal{O}} \xrightarrow{\widetilde{D}} \widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}} \to \mathrm{Ex}^{\infty}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}),$$

remembering that $N_{coh}(\widetilde{\mathcal{O}}) = N(\widetilde{\mathcal{O}})$ since $\widetilde{\mathcal{O}}$ is a regular category. It will be useful, however, to have an idea of where $H$ sends $k$-simplices:

- The functors above are the identity on objects, and $N_{coh}$ sends objects to 0-simplices, so $H$ is the identity on 0-simplices.

- The 1-simplices of $N(\widetilde{\mathcal{O}})$ are the morphisms of $\widetilde{\mathcal{O}}$. $\widetilde{D}$ takes these morphisms to height 0 tree hammocks which are the 0-simplices of the hom-objects of $\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}}$. Since $\mathrm{Ex}^{\infty}$ is the identity on 0-simplices of the hom-objects, $H$ is also the identity on the 1-simplices.

- A 2-simplex in $N(\widetilde{\mathcal{O}})$ is a pair of composable arrows $(g, f)$. This can be seen as a 2-simplex in $N_{coh}(\widetilde{\mathcal{O}})$ as in the following diagram:

$$\begin{array}{ccccc} m & & \xrightarrow{\quad gf \quad} & & p \\ {\scriptstyle \mathrm{Id}_m}\downarrow & & & & \downarrow{\scriptstyle \mathrm{Id}_p} \\ m & \xrightarrow[f]{} & n & \xrightarrow[g]{} & p \end{array}$$

  If we consider $f$, $g$, and $gf$ as height 0 tree hammocks, the above diagram looks like a 2-simplex in $\boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^{\natural}$, where $h = gf$ and our "zig-zag" of connecting height 1-tree hammocks is just a single height 1 tree hammock, giving the image of $(g, f)$ under $H$.

- Similarly, for a $k$-simplex in $N(\widetilde{\mathcal{O}})$, we can view a string of $k$ composable morphisms from $\widetilde{\mathcal{O}}$ as a $k$-simplex in $\boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^{\natural}$ where the additional structure is trivial, giving the image of the $k$-simplex under $H$.

From the description of the marked simplices in $\boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^{\natural}$ it should be clear that $H$ sends the morphisms in $\widetilde{\mathcal{W}} \subset \widetilde{\mathcal{O}}$ to marked simplices, so $H$ is map of marked simplicial sets.

Using Lemma 7.1.16, to prove Conjecture 7.2.4, it is enough to show that $H$ induces a homotopy equivalence of Kan complexes for all quasicategories $Z$:

$$H^* : \mathrm{Map}^\sharp(\boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^\natural, Z^\natural) \to \mathrm{Map}^\sharp((N(\widetilde{\mathcal{O}}), \widetilde{\mathcal{W}}), Z^\natural).$$

We will make use of the following theorem:

**Theorem 7.3.1** (Simplicial Whitehead)**.** *A simplicial map $f : X \to Y$ between Kan complexes is a homotopy equivalence if and only if for any commuting square as below with $n > 0$, there exists a lift $d$ such that the upper triangle commutes and the lower triangle commutes up to homotopy.*

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{a} & X \\ {\scriptstyle i}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle f} \\ \Delta^n & \xrightarrow[b]{} & Y \end{array}$$

To explore how we might use this theorem, we start with the case $n = 1$. Suppose we have a commuting diagram

$$\begin{array}{ccc} \{0,1\} & \xrightarrow{a} & \mathrm{Map}^\sharp(\boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^\natural, Z^\natural) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle H^*} \\ \Delta^1 & \xrightarrow[b]{} & \mathrm{Map}^\sharp((N(\widetilde{\mathcal{O}}), \widetilde{\mathcal{W}}), Z^\natural) \end{array}$$

$a$ picks out two 0-simplices in $\mathrm{Map}^\sharp(\boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^\natural, Z^\natural)$; i.e. two maps of marked simplicial sets

$$a_0, a_1 : \boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^\natural \cong \boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^\natural \times (\Delta^0)^\flat \to Z^\natural.$$

$b$ picks out a 1-simplex in $\mathrm{Map}^\sharp((N(\widetilde{\mathcal{O}}), \widetilde{\mathcal{W}}), Z^\natural)$. Since $Z$ is a quasicategory, $\mathrm{Map}^\sharp((N(\widetilde{\mathcal{O}}), \widetilde{\mathcal{W}}), Z^\natural)$ is the largest Kan complex contained in $\mathrm{Map}^\flat((N(\widetilde{\mathcal{O}}), \widetilde{\mathcal{W}}), Z^\natural)$. Using [Jar19, 17.2], we can conclude that $b$ picks out a 1-simplex in $\mathrm{Map}^\flat((N(\widetilde{\mathcal{O}}), \widetilde{\mathcal{W}}), Z^\natural)$ which is an isomorphism. Also, since the square commutes, we have

$$d_1 b = a_0 \circ H, \qquad d_0 b = a_1 \circ H.$$

For example, we have that for $(m, 0) \in N(\widetilde{\mathcal{O}}) \times \Delta^0$,

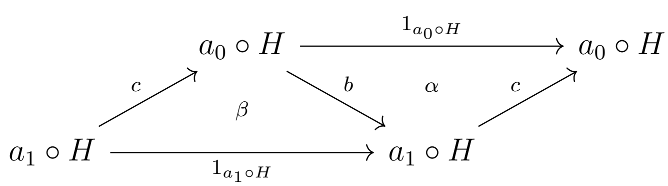
$$\begin{aligned} d_0 b(m, 0) :&= (b \circ (\mathrm{Id}_{N(\widetilde{\mathcal{O}})} \times D_0))(m, 0) = b(m, 1) \\ a_1 \circ H(m, 0) &= a_1(H(m), 0) = a_1(m, 0) \end{aligned} \tag{7.3.1}$$

since $H$ is the identity on objects. Thus $b(m, 1) = a_1(m, 0)$, and similarly $b(m, 1) = a_0(m, 0)$.

Together, this gives us that $b$ is a map of marked simplicial sets

$$(N(\widetilde{\mathcal{O}}), \widetilde{\mathcal{W}}) \times (\Delta^1)^\flat \to Z^\natural$$

such that there exists $c \in \mathrm{Map}^\flat((N(\widetilde{\mathcal{O}}), \widetilde{\mathcal{W}}), Z^\natural)_1$, $\alpha, \beta \in \mathrm{Map}^\flat((N(\widetilde{\mathcal{O}}), \widetilde{\mathcal{W}}), Z^\natural)_2$ making the diagram commute:




Our goal then is to construct a simplicial map

$$d : \Delta^1 \to \mathrm{Map}^\sharp(\boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^\natural, Z^\natural),$$

i.e., to pick out an isomorphism $d$ in $\mathrm{Map}^\flat(\boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^\natural, Z^\natural)_1$ such that $d_1 d = a_0$, $d_0 d = a_1$, and $d \circ (H \times \mathrm{Id}_{\Delta^1})$ is homotopic to $b$. We will focus on satisfying the first two properties.

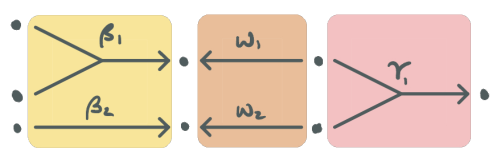
We want to understand where $d$ sends $(\epsilon, t) \in \boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^\natural \times \Delta^1$. In order to have $d_1 d = a_0 : \boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^\natural \times \Delta^0 \to Z^\natural$ and $d_0 d = a_1$ we must have for $m \in \boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^\natural_0$

and $\epsilon \in \boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^{\natural}_n$ for $n > 0$:

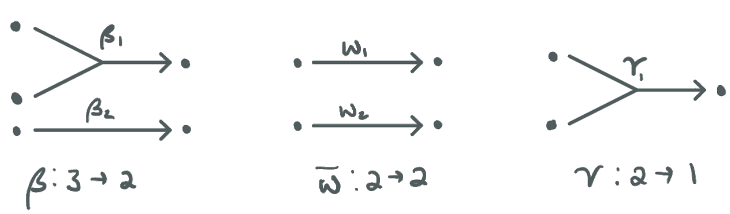
$$\begin{aligned}
a_0(m,0) &= (d_1 d)(m,0) = d((\mathrm{Id}_{\boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^{\natural}} \times D_1)(m,0)) = & d(m,0); \\
a_0(\epsilon, \underbrace{s_0 \dots s_0 0}_{n\text{-times}}) &= (d_1 d)(\epsilon, s_0 \dots s_0 0) = d((\mathrm{Id} \times D_1)(\epsilon, s_0 \dots s_0 0)) = & d(\epsilon, s_0 \dots s_0 0) \\
a_1(m,0) &= (d_0 d)(m,0) = d((\mathrm{Id} \times D_0)(m,0)) = & d(m,1); \\
a_1(\epsilon, \underbrace{s_0 \dots s_0 0}_{n\text{-times}}) &= (d_0 d)(\epsilon, s_0 \dots s_0 0) = d((\mathrm{Id} \times D_0)(\epsilon, s_0 \dots s_0 0)) = & d(\epsilon, s_0 \dots s_0 1)
\end{aligned}$$

where the $D_i$ are the geometric maps induced on the standard geometric simplices by the coface maps $d^i$. Thus $d$ has been entirely determined on 0-simplices.

For a 1-simplex $\epsilon$ in $\boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^{\natural}$, there are three corresponding 1-simplices in $\boldsymbol{R}N_{coh}(\widetilde{L^{TH}_{\mathcal{W}}\mathcal{O}})^{\natural} \times (\Delta^1)^{\flat}$: $(\epsilon, s_0 1)$, $(\epsilon, s_0 0)$, and $(\epsilon, 0 \to 1)$. By the above equations, $d$ has been determined on the the first two 1-simplices. We will work through an example to illustrate a conjecture for the construction of $d(\epsilon, 0 \to 1)$. Let $\epsilon$ be a height 0 tree hammock from 3 to 1 as depicted below.




Note that we can think of $\epsilon$ as the composition of three width 1 tree hammocks, corresponding to the highlighted sections. Each of these width 1 tree hammocks corresponds to a 1-simplex in $N(\widetilde{\mathcal{O}})$, where we let $\bar{w}$ be the forward pointing arrows labeled by the labels in $w$.



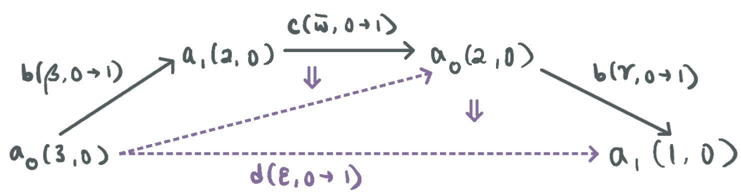
Since these are now 1-simplices in $N(\widetilde{\mathcal{O}})$, we can apply $b$ and $c$. From the properties given above for $b, c$, we have the following:

$$d_1(b(\beta, 0 \to 1)) = a_0(3,0) \qquad d_0(b(\beta, 0 \to 1)) = a_1(2,0)$$
$$d_1(c(\bar{w}, 0 \to 1)) = a_1(2,0) \qquad d_0(c(\bar{w}, 0 \to 1)) = a_0(2,0)$$
$$d_1(b(\gamma, 0 \to 1)) = a_0(2,0) \qquad d_0(b(\gamma, 0 \to 1)) = a_1(1,0)$$

For example, recalling Equation 7.3.1, we have

$$d_0(b(\beta, 0 \to 1)) = b(d_0(\beta, 0 \to 1)) = b(2,1) = a_1(2,0)$$

We can then construct the following diagram in $Z^\natural$, where because $Z$ is a quasicategory, we can fill the inner horns as depicted.




We would like to call the resulting 1-simplex from $a_0(3,0)$ to $a_1(1,0)$ $d(\epsilon, 0 \to 1)$. Note that this would respect $d_0 d = a_1$ and $d_1 d = a_0$.

The general idea for constructing $d(\epsilon, 0 \to 1)$ for $\epsilon : m \to n$ would then be as follows:

- Divide $\epsilon$ into width 1 tree hammocks.
- For every backwards facing piece $w$, consider $\bar{w}$ as above.
- If $\epsilon$ has an even width, extend it to odd width by adding a final extra piece, $\mathrm{Id}_n : n \to n$.
- For the first, third, ... pieces, apply $b(-, 0 \to 1)$.
- For the second, fourth, ... pieces, apply $c(-, 0 \to 1)$.

- Fill the resulting sequence of inner horns, from left to right, to construct

$$a_0(m,0) \xrightarrow{d(\epsilon,0\to 1)} a_1(n,0).$$

The immediate problem with this approach is that the inner horn fillers in a quasicategory are not unique, though they are equal in the homotopy category of the quasicategory. This means that we would need to take careful steps to insure that the above construction of $d$ is well-defined. Additionally, we still need to define $d$ on $n$-simplices for $n > 1$, and check that this gives that the bottom triangle commutes up to homotopy. By the construction of $d$, it should be a simplicial map.

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