Article

The Worst Case GARCH-Copula CVaR Approach for Portfolio Optimisation: Evidence from Financial Markets

Tahani S. Alotaibi [1,2,]*, Luciana Dalla Valle [1,]* and Matthew J. Craven [1]

[1] School of Engineering, Computing and Mathematics, University of Plymouth, Drake Circus, Plymouth PL4 8AA, UK

[2] Department of Mathematics, Faculty of Science and Humanities, Shaqra University, Al Duwadimi Road, Shaqra 11911, Saudi Arabia

* Correspondence: tahani.alotaibi@students.plymouth.ac.uk or taldajani@su.edu.sa (T.S.A.); luciana.dallavalle@plymouth.ac.uk (L.D.V.)

Abstract: Portfolio optimisation aims to efficiently find optimal proportions of portfolio assets, given certain constraints, and has been well-studied. While portfolio optimisation ascertains asset combinations most suited to investor requirements, numerous real-world problems impact its simplicity, e.g., investor preferences. Trading restrictions are also commonly faced and must be met. However, in adding constraints to Markowitz's basic mean-variance model, problem complexity increases, causing difficulties for exact optimisation approaches to find large problem solutions inside reasonable timeframes. This paper addresses portfolio optimisation complexities by applying the Worst Case GARCH-Copula Conditional Value at Risk (CVaR) approach. In particular, the GARCH-copula methodology is used to model the portfolio dependence structure, and the Worst Case CVaR (WCVaR) is considered as an alternative risk measure that is able to provide a more accurate evaluation of financial risk compared to traditional approaches. Copulas model the marginal of each asset separately (which may be any distribution) and also the interdependencies between assets This allows an accurate risk to investment assessment to be applied in order to compare it with traditional methods. In this paper, we present two case studies to evaluate the performance of the WCVaR and compare it against the VaR measure. The first case study focuses on the time series of the closing prices of six major market indexes, while the second case study considers a large dataset of share prices of the Gulf Cooperation Council's (GCC) oil-based companies. Results show that the values of WCVaR are always higher than those of VaR, demonstrating that the WCVaR approach provides a more accurate assessment of financial risk.

Keywords: copula; VaR; WCVaR; GARCH; portfolio optimisation

1. Introduction

A major component in portfolio optimisation problems is risk, and the most common risk measure used by practitioners and researchers is the variance (Graham and Craven 2021; Markowitz 1952). A model known as mean-variance (M-V) depends on the strict assumption that asset returns are multivariate and normally distributed. However, this assumption is not supported in practice (Jin et al. 2016), and various studies have therefore made proposals for alternative risk measures that seek to move beyond the M-V model's limitations (Hoe et al. 2010; Konno and Yamazaki 1991; Young 1998).

Calculating the risk can be performed in various other ways, such as using the Value at Risk (VaR) and the Conditional Value at Risk (CVaR), as well as the Worst Case Conditional Value at Risk (WCVaR). The latter considers asset dependence structures using copulas. VaR is a standard recommended by the Basel committee, but it has been recently criticised (Kakouris and Rustem 2014). The first reason is VaR's lack of satisfactory sub-additivity, making it an incoherent risk measure. It can also have multiple local minima, and it is not convex (Salahi et al. 2013). The second is that the percentile of distribution tail loss is

inadequate, according to Kakouris and Rustem (2014). This criticism is also used by Szegö (2005) to argue that "VaR does not measure risk" and the author advises alternatives such as CVaR, since this is the distribution expectation above VaR and is impacted by distribution tail fatness, providing an improved picture of distribution tail loss. Marimoutou et al. (2009) used conditional and unconditional Extreme Value Theory models to estimate VaR in both short and long oil market trading positions. They compared their model with Historical Simulation and Filtered Historical Simulation methods and GARCH, demonstrating that significant improvements are achieved with conditional Extreme Value Theory and Filtered Historical Simulation in contrast to conventional approaches. They also note that the GARCH (1,1)-t model has the potential to yield similarly improved results, particularly for the left tail. Their results corroborate that a filtering process is an important factor in success when using standard methods such as Historical Simulation and Extreme Value Theory.

To achieve a linear or convex problem for risk management and portfolio optimisation, Rockafellar and Uryasev (2000; 2002) suggest a minimisation formulation. Some assumptions are required for the CVaR calculation in their formula, concerning the assets' underlying distribution. Zhu and Fukushima (2009) describe this as a hypercube, ellipsoidal set or other uncertainty domain, or a multivariate distribution.

Other approaches include the use of the wavelet technique for assessing the co-movement in both time-frequency spaces via a decomposition of time series into their time scale component (Aloui and Hkiri 2014; Ben Mabrouk 2020). Mensi et al. (2018) studied the multifractality and the dynamic weak-form efficiency of stock markets using a Multifractal Detrended Fluctuation Analysis (MF-DFA) approach. Ghahtarani (2021) employed a fuzzy neural network to predict the market value of assets in different economic conditions as input data of the developed model.

The most common multivariate distribution is the Gaussian, since calibration is simple and efficient simulation algorithms are available. Elliptical distributions also have these advantages to a point, and Credit Risk often uses the Student's t distribution (Chan and Kroese 2010). However, the Gaussian distribution also has disadvantages, such as its symmetry, indicating that losses and gains have equal probability (Patton 2004). Furthermore, stronger comovements of assets are seen in crisis than in prosperity with respect to financial markets (Hu 2006). Another drawback is that dependence measurements use linear correlation, which may not adequately measure dependence in non-linear asymmetric comovements (Kakouris and Rustem 2014). Mixture distributions can mitigate elliptical distributions' underlying symmetry and their limitations (Kakouris and Rustem 2014) and have been used by Hu (2006) and Smillie (2008) in the bivariate case.

Copulas are defined as multivariate distribution functions with one-dimensional margins distributed equally on the [0,1] closed interval (Cherubini et al. 2004; Nelsen 2007). Univariate cumulative distributions of random variables can replace uniform margins (Cherubini et al. 2004; Nelsen 2007). Thus, consideration is not on dependent between random variables but on their marginal distributions, making them more flexible in comparison to standard distributions due to the separation of the multivariate dependency and univariate distribution selection (Kakouris and Rustem 2014). Di Clemente and Romano (2021) advocate the use of copulas for financial applications to determine the dependence structure of the financial asset returns in the portfolio. The suitability of copulas for modelling financial data has also been demonstrated by Alexander (2001) and Bouyé et al. (2000). For an overview about copula functions, see, for example, Dalla Valle (2017b); for a review of copula applications in finance, see Dalla Valle (2017a). Following the risk management research stream, Dalla Valle et al. (2016) illustrate the use of copulas for credit risk, while Dalla Valle and Giudici (2008); Dalla Valle (2009); Fantazzini et al. (2008) focus on operational risk measurement. Kakouris and Rustem (2014) are motivated by Hu (2006) and Zhu and Fukushima (2009) to bring in mixture copulas to extend CVaR to WCVaR in a worst-case scenario. Their Archimedean copulas each characterise a different type of dependency. Mixing them allows a greater spectrum of dependencies, from which they can derive CVaR and WCVaR for copulas. Mixing copulas can result in a solution to convex optimisation

problems in the WCVaR framework. Copulas also give distribution selection more flexibility in the CVaR case.

For robust results in portfolio optimisation and risk measurement, dependency should be considered (Kakouris and Rustem 2014). The authors propose copulas and mixture copulas as one way to achieve this, and compare them against Gaussian CVaR and Worst Case Markowitz. Empirical analysis reveals that WCVaR outperforms them in crisis periods.

Robust portfolio optimisation has also been combined with copula models by Sabino da Silva and Ziegelman (2017) in a WCVaR framework. They made comparisons with a Gaussian Copula-CVaR portfolio (GCCVaR), alongside the S&P 500 index and an equally weighted portfolio ($1/N$). Improved hedges were found with the copula basis than the $1/N$ portfolio against losses. Improved downside risk statistics were found in portfolios with the WCVaR approach than the GCCVaR in rebalancing periods, as well as more profitability when daily or weekly rebalancing. Chukwudum (2018) examined the shaping of the extremal dependence structure with a combination of copulas and Generalised Pareto distribution, demonstrating the effect that tail dependence has on risk measures, reinsurance net premium, and risk allocation. Applied to the insurance sector in Nigeria, it was shown that correlating losses increases risk measure values and impacts risk allocation. It was also shown that the use of the Clayton copula generally lowered reinsurance premiums, whereas with the Gumbel copula, these were generally increased, compared with premiums hypothesised independently.

The objective of this paper is to test the accuracy of the WCVaR approach, based on copulas with GARCH marginals, compared to traditional approaches. The performance of the WCVaR approach is assessed in two different case studies. The first case study focuses on the time series of the closing prices of six major market indexes; the second case study considers a large dataset of the share prices of the Gulf Cooperation Council's (GCC) oil-based companies.

This paper is organised as follows. First, the methodology is illustrated, introducing copulas, particularly the Student's t-test, the Kendall's τ dependence measure, and relevant inference strategy. Then, the GARCH time series model is discussed, and the copula-GARCH approach and WCVaR using the copula-GARCH model are explained in detail. Further, the WCVaR and VaR for the two financial datasets are computed, and then the results obtained are compared. Concluding remarks are provided in the Discussion section.

2. Research Methods

In this section we illustrate in detail the Worst Case GARCH-Copula VaR approach for portfolio optimisation. We start by introducing the definition of copula and the Kendall's τ dependence measure. We focus on the Student's t copula, since, due to its ability to capture tail dependence, it is the most appropriate copula family for financial data. Hence, we discuss the Maximum Likelihood Estimation (MLE) and the Inference Function for Margins (IFM) methods. The GARCH time series approach to model the copula marginals is then described. Finally, the WCVaR using the copula-GARCH model is explained in detail.

2.1. Copulas

Copulas are tools that permit distributions of individual assets to be separated from their dependence structure. They are also used to model nonlinear dependence. The benefits of this are that any distribution can be assumed when considering marginal distributions of assets, with copulas being used to obtain the joint multivariate distribution, taking marginals dependence structure into account. Moreover, in general, if we consider the loss distribution of financial data, the left and the right tail represent profit and loss, respectively, and they are often not the same (i.e., one tail may be heavier than the other). Therefore, a symmetric distribution, such as the normal distribution, may not be appropriate. Using a copula in this situation is a more flexible choice because, depending on the type of copula and on the parameter values, we can have different shapes of distributions. Thus, a more accurate representation of the joint distribution of returns and losses is ob-

tained. If we consider several assets, a multivariate distribution describes simultaneously what happens with all assets, and in a copula setting, the marginal of each asset may be different (Joe 1997; Nelsen 2007).

Definition 1 (Copula). *(Mai and Scherer 2017) A function $C : [0,1]^d \rightarrow [0,1]$ is called a (d-dimensional) copula if there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a random vector $(U_1, \ldots, U_d)$ such that $U_k \sim U[0,1]$ for all $k = 1, \ldots, d$ and*

$$C(u_1, \ldots, u_d) = \mathbb{P}(U_1 \leq u_1, \ldots, U_d \leq u_d),$$

where $u_1, \ldots, u_d \in [0,1]$.

Theorem 1 (Sklar's Theorem). *(Daníelsson 2011, p. 27) Let F be the distribution of X, let G be the distribution of Y, and let H be the joint distribution of (X, Y). Assume that F and G are continuous. Then, there exists a unique copula C such that the following is the case.*

$$H(X, Y) = C(F(X), G(Y)). \tag{1}$$

Sklar's Theorem describes how to use copulas to derive the joint multivariate distribution and embeds the dependence structure between the marginals.

2.1.1. Kendall's τ Coefficient

Kendall's τ is a measure of rank correlation. Assume that (X_1, Y_1) and (X_2, Y_2) are two independent pairs of random variables drawn from random vector (X, Y). Kendall's τ is defined by the following.

$$\tau(X, Y) = Pr[(X_1 - X_2)((Y_1 - Y_2) > 0] - Pr[(X_1 - X_2)((Y_1 - Y_2) < 0]. \tag{2}$$

The term $Pr[(X_1 - X_2)(Y_1 - Y_2) > 0]$ refers to Pr[concordance], while the term $Pr[(X_1 - X_2)(Y_1 - Y_2) < 0]$ refers to Pr[discordance]; thus, $\tau(X, Y)$ is a measure of the relative difference between the probability of concordance and the probability of discordance.

2.1.2. Student's t-Copula

Definition 2 (Student's t-Copula). *(Demarta and McNeil 2005) The Student's t-copula is given by the following:*

$$C_{\nu,P}^t(U, V) = \int_{-\infty}^{t_\nu^{-1}(u)} \int_{-\infty}^{t_\nu^{-1}(v)} \frac{\Gamma(\frac{\nu+2}{2})}{\Gamma(\frac{\nu}{2})\sqrt{(\pi\nu)^2|P|}} \left(1 + \frac{X'P^{-1}X}{\nu}\right)^{-\frac{\nu+2}{2}} dX, \tag{3}$$

where $X = (X_1, X_2)'$ is the two-dimensional random vector that has a (non-singular) multivariate t distribution with degrees of freedom ν, P is the correlation matrix, t_ν is the density function, and t_ν^{-1} is the quantile function of a standard univariate t_ν distribution.

The Student's t-copula includes all Fréchet lower and upper bounds. The Student's t-copula is flexible in that it can describe both negative and positive dependence and is radially symmetric and has dependence levels equal in the upper and lower tails.

2.1.3. Copula Inference with the Maximum Likelihood Method

Suppose $\{(X_{i1}, \ldots, X_{ip})^T, i = 1, \ldots, n\}$ are n independent realisations from a multivariate distribution. Let F_i be the cumulative distribution functions (cdfs) and f_i be the probability density functions (pdfs) of the p margins with $j = 1, \ldots, p$ and let c be a copula density. Let $\theta = (\beta^T, \alpha^T)$ be the parameter vector to be estimated, β be the vector of marginal parameters and α be the vector of copula parameters. The loglikelihood function can be defined as follows:

$$l(\theta) = \sum_{i=1}^{n} \log c\{F_1(X_{i1}; \beta), \ldots, F_p(X_{ip}; \beta); \alpha\} + \sum_{i=1}^{n} \sum_{j=1}^{p} \log\{f_i(X_{ij}; \beta)\} \qquad (4)$$

The ML estimator of θ is

$$\hat{\theta}_{ML} = \text{argmax}_{\theta \in \Delta} l(\theta), \qquad (5)$$

where Δ is the parameter space (Yan 2007).

2.1.4. Inference Functions for Margins (IFM)

The number of parameters increases when the dimension p becomes large. Thus, the optimisation problem becomes more difficult. Joe and Xu (1996) presented the inference functions for the margins (IFM) method. This is a two-step method that estimates marginal parameter β as follows:

$$\hat{\beta}_{IFM} = \text{argmax}_\beta \sum_{i=1}^{n} \sum_{j=1}^{p} \log f_i(X_{ij}; \beta). \qquad (6)$$

After that the association parameters α is estimated as follows.

$$\hat{\alpha}_{IFM} = \text{argmax}_\alpha \sum_{i=1}^{n} \log c\big((F_1(X_{i1}; \hat{\beta}_{IFM}), \ldots, F_p(X_{ip}; \hat{\beta}_{IFM}); \alpha\big). \qquad (7)$$

As a result there is small number of parameters for each maximisation task. This leads to diminished computational difficulty (Yan 2007).

2.2. The GARCH (p, q) Model

The GARCH model is a time series model that is probably the most widely used in finance. It is used to evaluate the volatility of returns in financial markets. It is known that stock returns exhibit heavy-tailed probability distributions. This might be because the conditional variance is variable. Using a GARCH model can capture fat tails and volatility clustering within financial time series. Bollerslev (1986) developed the Generalized AutoRegressive Conditional Heteroscedasticity (GARCH) model as a generalization of the ARCH model proposed by Engle (1982).

A GARCH(p,q) model takes the following form:

$$\begin{aligned} \epsilon_t &= W_t \sqrt{h_t} \\ h_t &= \alpha_0 + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-i}, \end{aligned} \qquad (8)$$

where ϵ_t is the return at time t, W_t is the residual, and $h_t = VaR[\epsilon_t \mid \mathcal{F}_{t-1}]$ where $\mathcal{F}_{t-1}$ is the information up to time $t - 1$; $\alpha_0 > 0$, α_i and β_j are model parameters.

The GARCH (1,1) Case

The GARCH (1,1) model is the most popular model used in empirical finance. It can be easily derived from Equation (8) and can be written as follows.

$$\begin{aligned} \epsilon_t &= W_t \sqrt{h_t} \\ h_t &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 h_{t-1} \end{aligned} \qquad (9)$$

2.3. The Copula-GARCH Model

The copula-GARCH model is used to measure financial risks such as portfolio risk and volatility over time. This model aims to describe asymmetric dependence and complex non-linear relations between assets. Copulas have been used to estimate risk measures in

financial markets such as (VaR) which is the risk measure extensively used by financial analysts (Messaoud and Aloui 2015). There are several studies that have used copulas for determining VaR. For example, Huang et al. (2009) used the copula-GARCH model to estimate VaR of portfolios. The authors found that the copula-GARCH model is flexible in separating marginal distributions from the dependence structure, which is more effective in describing high volatility. This is contrary to traditional methods that computed VaR using the variance–covariance and the Monte Carlo approaches. Embrechts et al. (2005) introduced a methodology that is based on the theory of copulas to calculate VaR in the worst case scenarios. Cherubini and Luciano (2001) used copula functions to estimate VaR and allocate capital. They applied their approach to the data of five years of daily returns on two stock market indices: FTSE100 and S&P100. They found that copulas separate marginal distributions from the dependence between returns results in joint probabilities of extreme losses. Jondeau and Rockinger (2006) proposed a new measure of conditional dependence using copula functions with GARCH processes. Fortin and Kuzmics (2002) used convex linear combinations of copula with the GARCH-model to estimate VaR on a set of European stock indices: FTSE and DAX. Messaoud and Aloui (2015) used the GJR-GARCH based on a Student's t-copula to estimate the VaR and CVaR of a portfolio and described asymmetric dependence. The authors consider daily returns of market indices from four countries: Egypt, Malaysia, South Africa, and Turkey.

2.4. The Worst Case Conditional Value at Risk

There are many measures to calculate risk. A common used risk measure is VaR, but it has been criticised because it is not a coherent measure of risk as well as being a percentile of loss distribution which does not describe possible losses on the tails of distribution. That leads Szegö (2005) to propose the CVaR as an alternative measure. CVaR is a coherent measure, and it provides some improved properties with respect to losses on the tail of distribution than VaR (Bertsimas et al. 2004; Quaranta and Zaffaroni 2008; Rockafellar and Uryasev 2002). Kakouris and Rustem (2014) presented the copula formulation of the CVaR of a portfolio. The authors extended the CVaR to WCVaR using mixture copulas.

The definitions of VaR, CVaR, and WCVaR will be presented in the following.

Definition 3 (VaR). *Let $w \in \mathbb{W} \subset \mathbb{R}^m$ be a decision vector, let $u \in \mathbb{I}^n$ be a random vector, let $\tilde{g}(w, u)$ be the cost function, and let $F(x) = (F_1(x_1), \ldots, F_n(x_n))^T$ be a set of marginal distributions where $u = F(x)$. Moreover, assume that u follows a continuous distribution with copula function $C(\cdot)$. Then, for a confidence level β, VaR_β is defined as follows.*

$$VaR_\beta(w) = \min\{\alpha \in \mathbb{R} : C(u \mid \tilde{g}(w, u) \leq \alpha) \geq \beta\}. \tag{10}$$

Definition 4 (CVaR). *Given w, u, $F(x)$, and $\tilde{g}(w, u)$ as in Definition 3, $CVaR_\beta$ for a confidence level β is defined as follows.*

$$CVaR_\beta(w) = \frac{1}{1-\beta} \int\limits_{\tilde{g}(w,u) \geq VaR_\beta(w)} \tilde{g}(w, u)c(u)du. \tag{11}$$

Definition 5 (WCVaR). *The WCVaR for fixed $w \in \mathbb{W}$ and confidence level β with respect to C is defined as follows.*

$$WCVaR_\beta(w) = sup_{c(\cdot) \in C} CVaR_\beta(w). \tag{12}$$

2.5. Worst Case GARCH-Copula CVaR Portfolio Optimisation

In order to apply the Worst Case Copula-CVaR Portfolio Optimisation (Sabino da Silva and Ziegelman 2017), we need firstly to model marginals with the GARCH model assuming skew t-distributed innovations. Then, we extract the residuals and transform them into pseudo-observations. Next, we estimate a t-copula to model dependencies between marginals. Then, we generate a high number of scenarios (e.g., simulations) using

the dependence structure that we obtained from estimating the copula. After that, we transform these scenarios into quantiles of the t-distribution. Then, we derive the simulated daily asset log-returns using standard deviations that have been estimated by the GARCH model. Finally, to optimise the portfolio's weights, we apply the WCVaR technique to find the optimal portfolio.

3. Application of the Worst Case GARCH-Copula CVaR to Financial Datasets

In this section, the Worst Case GARCH-Copula CVaR technique is applied to two financial time series datasets. The first dataset is a relatively small set that includes six major market indexes considered between 2009 and 2013. The second dataset is a large collection of stock prices of the Gulf Cooperation Council's (GCC) oil-based companies collected between 2008 and 2019. The GCC data were previously analysed, for example, by Mensi et al. (2018); Aloui and Hkiri (2014) and Ben Mabrouk (2020).

3.1. The Financial Market Indexes Dataset

This Section demonstrates the computation of the WCVaR for a portfolio using multivariate copula simulation with GARCH marginals. We downloaded data from Yahoo Finance, and imported the closing prices of six market indexes that we modelled with GARCH processes. The indexes are SPY: Large Cap US (S&P 500); EEM: Emerging Markets Equity; TLT: 20+ Year Treas Bond (iShares Barclays); COY: US High-Yield Bond; GSP: Commodities broad (iPath S&P GSCI Total Return Index); and RWR: Real estate (REIT Index). We use daily data over the period from 1 October 2009 to 24 June 2013. We then applied the methodology that we described above in Section 2.5 on these data. Figure 1 shows the relative price movements of each index. The initial level of each index has been normalised to unity to facilitate the comparison of relative performance over the historical record (Pfaff 2016).
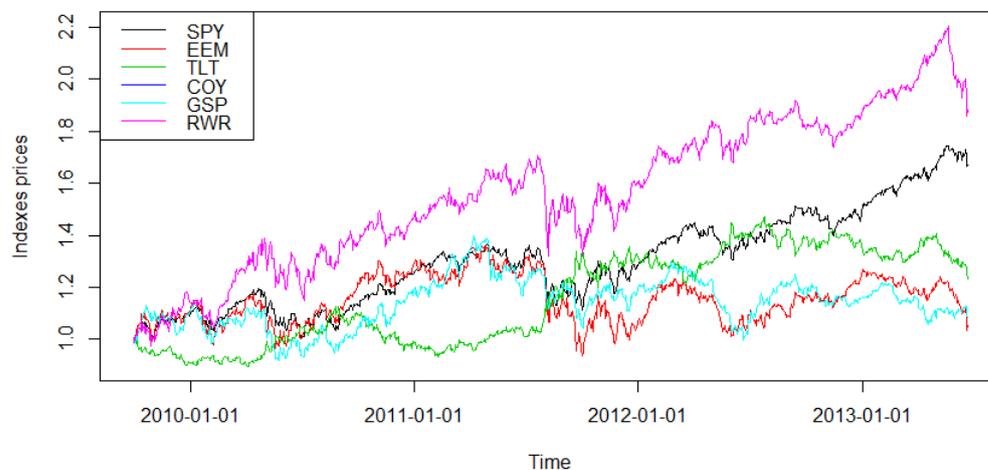
**Figure 1.** Performance of price movements of each index over the historical record.

We characterise individually the distribution of returns of each index. Since copulas allow us to capture dependencies between marginals, we investigated the existence of correlations between returns using scatterplots and correlation coefficients.

Figure 2 illustrates scatterplots between each pair of index returns in the lower triangular panels, and histograms of each marginal in the diagonal and Pearson's correlation coefficients in the upper triangular panels. For instance, it is evident that there is a positive strong correlation between SPY and EEM as well as between SPY and RWR and between EEM and RWR, whereas we notice a negative correlation between SPY and TLT. However, we can see that there is a weak correlation, for example, between COY and GSP.
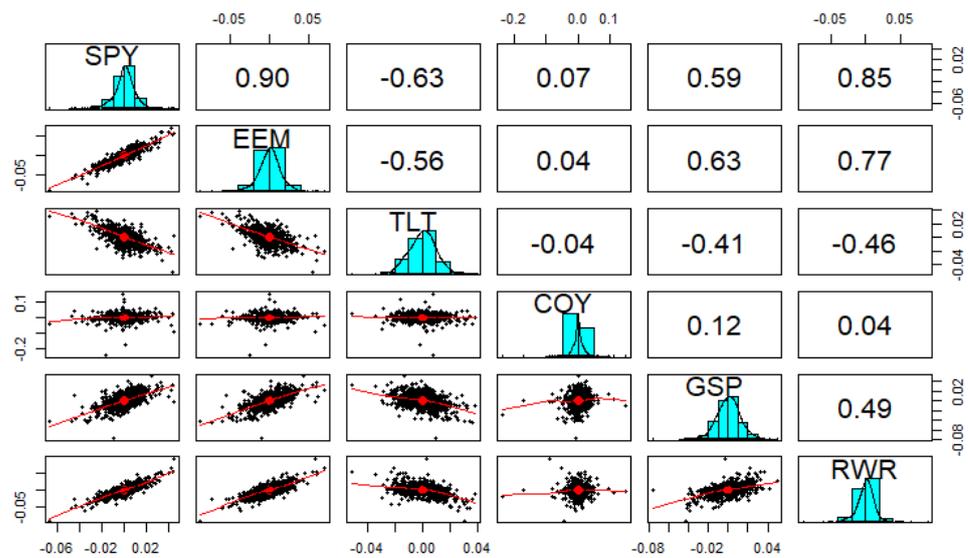
Figure 2. Scatterplots between each pair of index returns in the lower triangular panels, histograms of each marginal in the diagonal, and Pearson's correlation coefficients in the upper triangular panels.

Then, we specify and estimate GARCH(1,1) models for each marginal with t-distributed innovation processes and we extracted the standardised residuals. Figure 3 shows the Student's t-quantile functions of the residuals, allowing us to check that the assumptions of the GARCH models are met.
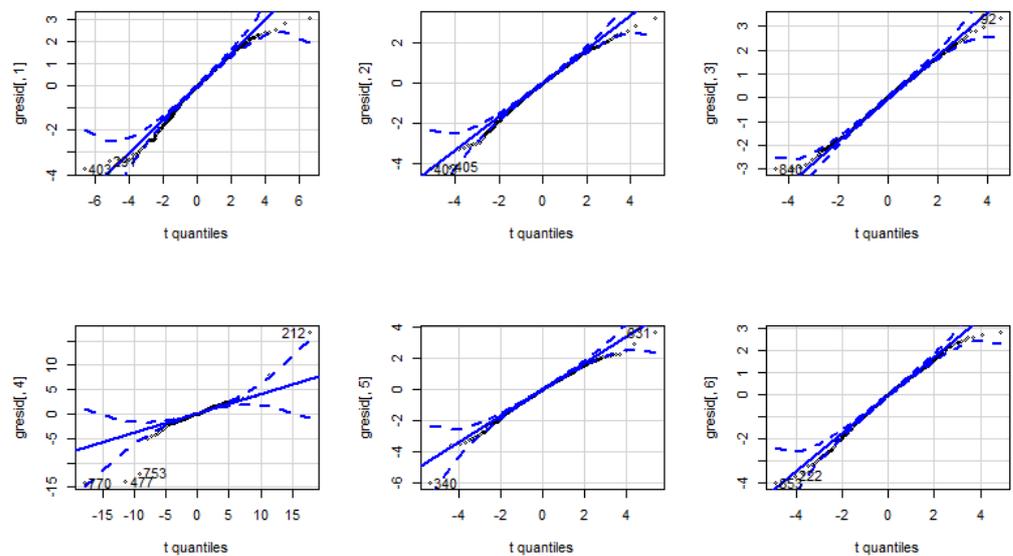
Figure 3. The Student's t-quantile functions of the residuals of GARCH(1,1) for each marginal.

In Figure 4, the plots demonstrate the autocorrelation functions (ACF) of the residual values. The ACF calculates a single time series' similarity against a delayed version (with a delayed copy) of itself (Welsh 1999). It is clear from the plots that there is no residual serial correlation in the residuals and thus the GARCH model is appropriate.
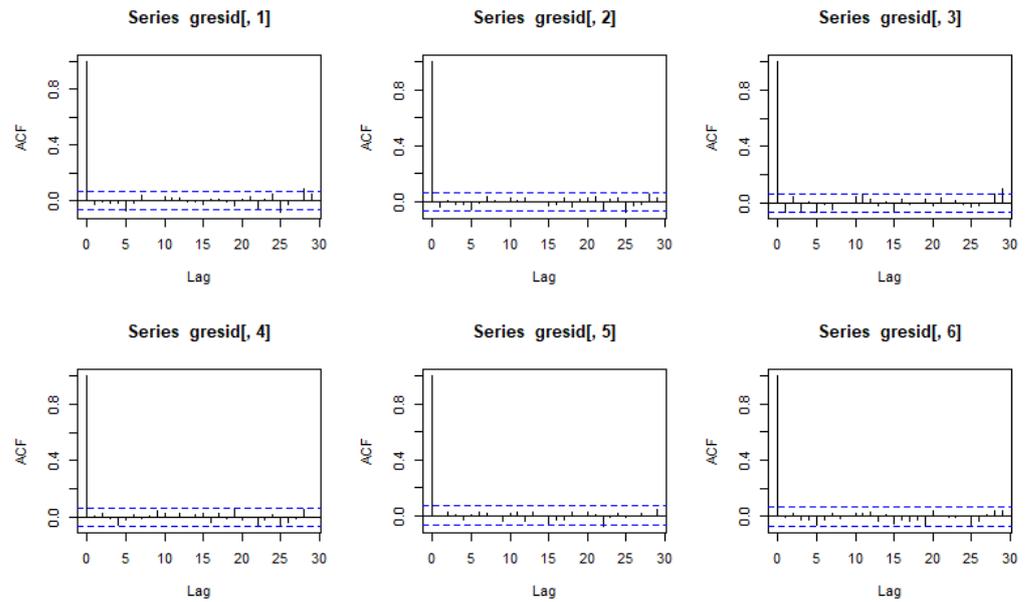
Figure 4. Plots of autocorrelation functions (ACF).

Figure 5 was produced after transforming the standardised residuals into pseudo-observations, which are constrained in order to belong to set $[0,1]$. Figure 5 illustrates scatterplots between pseudo-observations of each pair of indexes in the lower triangular panels, histograms of each marginal in the diagonal and Pearson's correlation coefficients in the upper triangular panels. Comparing Figure 5 with Figure 2, it can be seen that there is a similar behaviour in terms of dependence between marginals.
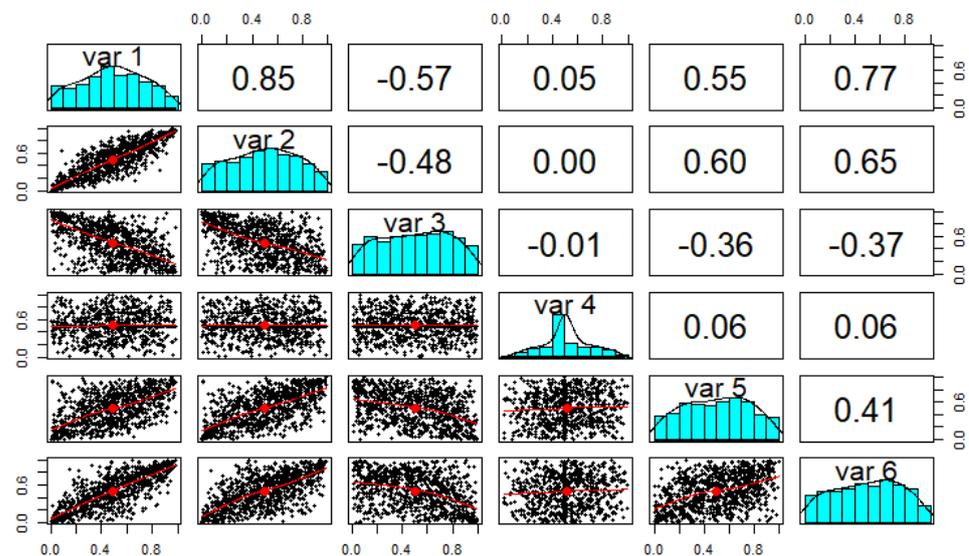

Figure 5. Scatterplots between pseudo-observations of each pair of indexes in the lower triangular panels, histograms of each marginal in the diagonal, and Pearson's correlation coefficients in the upper triangular panels.

The dependence structure between marginals is estimated using a t-copula. Then, the dependence determined by the estimated t-copula was used for generating $N = 10,000$ random variates for the pseudo-uniformly distributed variables. After that, we use these quantiles in conjunction with standard deviations to calculate $N = 10,000$ portfolio return scenarios.

Next, we transformed the uniform variates to daily centered returns via the inverse CDF of each index using Student's t distributions for each marginal. The multivariate simulations from the copula model can be used to compute the VaR and CVaR of a sample portfolio. We could also find optimal portfolio weights that give us a minimum risk for a certain level of return. We can perform this using the WCVaR technique, which allows us to find optimal portfolio weights that give us a minimum risk for a certain level of return. Using the WCVaR method, we obtained the following weights: 0.37916193 for SPY, 0 for EEM, 0.52727024 for TLT, 0.03467091 for COY, 0.05889691 for GSP, and 0 for RWR. We found that VaR and CVaR were 1.340576 and 1.675549, respectively. We also calculated VaR and CVaR for a multivariate normal as a benchmark traditional method, and we obtained optimal weights of 1.087546 and 1.267103, respectively.

In order to compare the WCVaR technique with the multivariate normal, we also calculated the VaR and CVaR by adopting randomly chosen weights (0.1 for SPY, 0.2 for EEM, 0.3 for TLT, 0.2 for COY, 0.1 for GSP, and 0.1 for RWR). The VaR is 1.641517 and CVaR is 2.122687. Finally, we compared the results obtained above with data generated from a multivariate normal distribution, obtaining a VaR and CVaR of 1.619638 and 1.836164, respectively.

Table 1 shows the comparison between the traditional multivariate normal method and the WCVaR technique. These results reveal that the traditional multivariate normal method produces lower VaR and CVaR and underestimates risks. While the WCVaR method takes into proper account the strong dependence in the tails using a Student's t-copula, the traditional multivariate normal fails to do that, yielding lower risk measures.

Table 1. Comparison between the traditional multivariate normal method and WCVaR technique.

	With Optimal Weights		With Random Weights	
	WCVaR	**Multivariate Normal**	**WCVaR**	**Multivariate Normal**
VaR	1.340576	1.087546	1.641517	1.619638
CVaR	1.675549	1.267103	2.122687	1.836164

Figure 6 shows the comparison between the WCVaR and minimum-variance portfolio values. It is clear that the WCVaR and the minimum-variance are very similar in the period 1 October 2009 to 1 September 2011, while the minimum variance measure exceeds the WCVaR from 2 September 2011 to 1 October 2012. From 2 October 2012 to 1 June 2013, there is a sharp increase in the value of WCVaR, which outperforms the minimum-variance.
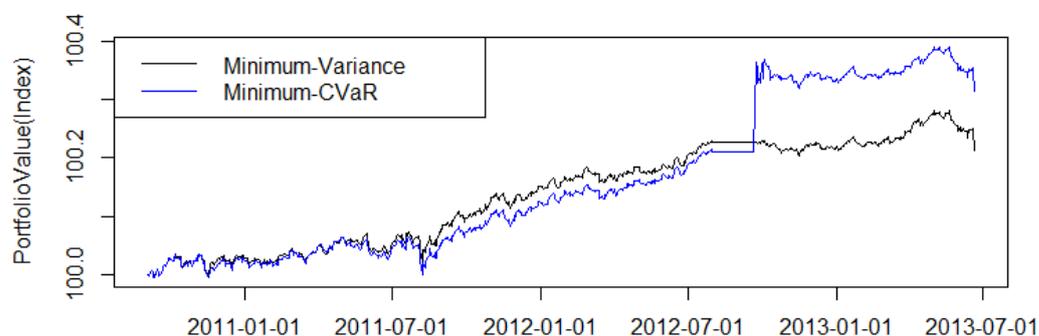
Figure 6. Trajectory of WCVaR and minimum-variance portfolio values.

3.2. The Gulf Cooperation Council (GCC) Dataset

In this section, we apply the WCVaR method to the GCC dataset, which includes stock markets from Saudi Arabia, Qatar, the United Arab Emirates, Oman, Bahrain, and Kuwait. In addition, oil prices and the prices of three precious metals (gold, silver, and copper) are included. The daily returns of the GCC stock indexes are considered and taken from the

Thomson Datastream service and the logarithmic return was used to calculate the rate of
return on assets. The sample period of the index data runs from 1 January 2008 to 31 January
2019 and comprises a total of 2894 observations. The arithmetic mean of returns is used,
and it is identical to what is used in the standard Markowitz model. It should be mentioned
that stocks with missing values at the beginning of the period of the study were dropped
from the dataset. As above, the data cover the period from January 2008 to January 2019, an
eventful period beginning in the year of the global financial crisis, and covering the oil crisis
when prices saw a significant drop. However, the price of gold was rising at that time, and
this provided a balance to the negative impact of the crisis, particularly in GCC countries
where economic growth was actually significant (Maghyereh et al. 2017). The study period
continues through gold's price peak in 2010 and the fall that followed, and onwards to
2014, which is the beginning of the oil crisis. These two periods impacted GCC oil exporters
negatively, along with central bank reserves, adding to the already turbulent political climate
of the Middle East and creating added insecurity for investments (Maghyereh et al. 2017).
As a result, the subject at hand is an interesting case to investigate.

     We use this data because the GCC markets are developing and have a significant
growth economy. This may help investors to locate profitable investment opportunities.
In addition, GCC countries are oil producers which affects on their economies. Moreover,
there is a lack of literature in constructing optimal portfolios in GCC markets.

     In this study, we divided the sample into 13 classes: 11 classes comprising stocks of
firms, 1 class comprising solely oil, and 1 final class with three precious metals. The number
and type of assets in each class is provided in Table 2. The grouping was determined by the
Global Industry Classification Standard (GICS), which is a standardised classification sys-
tem for equities that used by investors and established indices (or investment community).
The GICS was developed by Standard & Poor's and Morgan Stanley Capital International in
1999 to provide a global standard for classifying firms into sectors and industries. The GICS
structure contains 11 sectors, 24 industry groups, 68 industries, and 157 sub-industries. All
companies are categorised regarding their principal business activity (similar operating
characteristics). The sectors are: Energy, Materials, Industrials, Consumer Discretionary,
Consumer Staples, Health Care, Financials, Information Technology, Communication Ser-
vices, Utilities, and Real Estate. We use the Global Industry Classification Standard because
some sectors are pro-cyclical (e.g., the financial sector), while others are counter-cyclical or
do not depend on the business cycle (e.g., pharmaceuticals) (Basu 1996). Pro-cyclical stocks
refer to a positive correlation between any stock's price and the overall state of the economy
(in other words, any economic quantity that tends to move in the same direction as the
economy, increasing in expansion and declining in a recession). In contrast, any stock's
price that is negatively correlated with the overall state of the economy is said to constitute
counter-cyclical stocks.

Table 2. Generating VaR and WCVaR using WCVaR technique for the GCC dataset.

Class	Type of Asset	Number of Assets	VaR	WCVaR
1	Energy	20	0.4749	0.6532
2	Materials	68	0.1702	0.2218
3	Industrials	52	0.3766	0.5712
6	Healthcare	9	1.1005	1.4873
8	Information Technology	5	1.0342	1.8063
9	Communication Services	13	1.3899	2.2619
10	Utilities	7	1.6091	2.1049
11	Real Estate	47	0.0128	0.0173

     Since the dataset used in this section is very large, the High Performance Computer
(HPC) cluster at the University of Plymouth was used to process the experiment. Table 2
shows the results of the VaR and WCVaR measures for the GCC dataset. Since the number
of considered assets is large (496 assets), it was partitioned by class constraints, and the

methodology was applied to each class individually; the number of assets assigned to each class is shown in Table 2. We focused on the following eight classes: Energy (20 assets), Materials (68 assets), Industrials (52 assets), Healthcare (9 assets), Information Technology (5 assets), Communication Services (13 assets), Utilities (7 assets), and Real Estate (47 assets).

In Table 2, classes 4, 5, 7, 12, and 13 are not included. In particular, classes 4, 5, and 7 are affected by multicollinearity, which arises when two or more variables are linearly related, and there is a lack of orthogonality between them (for more information the reader is referred to Alin 2010). We noticed that the stock market prices of classes 4, 5, and 7 of this dataset remained stable for parts of the considered time horizon. In addition, class 4 has several missing data, which prevents the application of the methodology. Moreover, class 7 has over 200 assets, making the application of the methodology computationally unfeasible. Moreover, classes 12 and 13 consist of only 3 and 1 assets, respectively, which are datasets that are too small for the application of the methodology. The WCVaR and VaR are calculated at the 10% confidence level. It can be seen that the values of WCVaR are higher than those of VaR for all classes.

When we compared between various classes, we can see that Communication Services has the highest WCVaR, which means that it is the most risky class, while Real Estate has the lowest WCVaR, meaning that it can be interpreted as less risky than the other classes. Moreover, it was observed that classes 2, 3, and 11, which have the largest number of assets, had the smallest values of WCVaR. Therefore, the value of WCVaR decreases when the number of assets increases.

Applying WCVaR on a large dataset is very computationally demanding and requires an HPC. However, the WCVaR provides us with a more accurate estimation method than the traditional VaR to measure risks.

4. Conclusions and Future Work

In this paper, an advanced risk measure (WCVaR) was used to calculate the risk of portfolio in an accurate manner. The WCVaR technique involves two main components, namely: copulas that allow us to calculate the dependence structure of assets, and time series analysis such as GARCH models to analyse the marginals. The approach was applied firstly on a small dataset, including prices of six indices collected over the period between 2009 and 2013 and then on the large GCC dataset, including the prices of 500 assets collected over the period between 2008 and 2019. Then, we evaluated the performance of the WCVaR and compared it against the VaR measure. The results showed that the WCVaR outperformed VaR and measured risks more accurately. Moreover, it was found that including more assets in the portfolio leads to a decrease in the WCVaR value. On the other hand, the use of the WCVaR approach is, in general, computationally demanding, making it challenging to work with large datasets and requiring powerful computers. The main advantage of the WCVaR method is that it measures risk more accurately compared to other traditional methods. Hence, this approach can be very beneficial to decision-makers, investors, financial institutions, and banks in their decision-making regarding minimising the overall risk of their portfolio investments.

In this paper, we considered only the t-copula, and it would be interesting to extend this study using other copula families or a mixture of copulas (Hu 2006; Kakouris and Rustem 2014). A mixture of copulas is a combination of a set of different types of copulas. They are a flexible method in the case in which data contains many asymmetries and they consider various patterns of dependence in order to determine an appropriate measure of dependence to achieve robust results (Kakouris and Rustem 2014). Then, it would be interesting to compare the results of this method to that which were presented in this paper. These new methods can also be applied to other datasets, such as cryptocurrency datasets (Bitcoin, for example) or data in other fields (for example, environmental risk data such as floods or natural hazards). Additional suggestions include using different types of models for the marginals such as Generalized Additive Models (GAM) instead of GARCH models (Coussement et al. 2010; Hastie and Tibshirani 1990).

References

Alexander, Carol. 2001. *Market Models: A Guide to Financial Data Analysis*. New York: John Wiley.

Alin, Aylin. 2010. Multicollinearity. *Wiley Interdisciplinary Reviews: Computational Statistics* 2: 370–74. [CrossRef]

Aloui, Chaker, and Besma Hkiri. 2014. Co-movements of GCC emerging stock markets: New evidence from wavelet coherence analysis. *Economic Modelling* 36: 421–31. [CrossRef]

Basu, Susanto. 1996. Procyclical productivity: Increasing returns or cyclical utilization? *The Quarterly Journal of Economics* 111: 719–51. [CrossRef]

Ben Mabrouk, Anouar. 2020. Wavelet-based systematic risk estimation: Application on GCC stock markets: The Saudi Arabia case. *Quantitative Finance and Economics* 4: 542–95.

Bertsimas, Dimitris, Geoffrey J. Lauprete, and Alexander Samarov. 2004. Shortfall as a risk measure: Properties, optimization and applications. *Journal of Economic Dynamics and Control* 28: 1353–81. [CrossRef]

Bollerslev, Tim. 1986. Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics* 31: 307–27. [CrossRef]

Bouyé, Eric, Valdo Durrleman, Ashkan Nikeghbali, Gaël Riboulet, and Thierry Roncalli. 2000. Copulas for Finance—A Reading Guide and Some Applications. Available online: https://ssrn.com/abstract=1032533 (accessed on 1 October 2021).

Chan, Joshua C. C., and Dirk P. Kroese. 2010. Efficient estimation of large portfolio loss probabilities in t-copula models. *European Journal of Operational Research* 205: 361–67. [CrossRef]

Cherubini, Umberto, and Elisa Luciano. 2001. Value-at-risk trade-off and capital allocation with copulas. *Economic Notes* 30: 235–56. [CrossRef]

Cherubini, Umberto, Elisa Luciano, and Walter Vecchiato. 2004. *Copula Methods in Finance*. Hoboken: John Wiley & Sons.

Chukwudum, Queensley. 2018. Extreme Value Theory and Copulas: Reinsurance in the Presence of Dependent Risks. Available online: https://hal.archives-ouvertes.fr/hal-01855971/file/EVT%20copulas%20and%20reinsurance.pdf (accessed on 1 September 2021).

Coussement, Kristof, Dries F. Benoit, and Dirk Van den Poel. 2010. Improved marketing decision making in a customer churn prediction context using generalized additive models. *Expert Systems with Applications* 37: 2132–43. [CrossRef]

Dalla Valle, Luciana, and Paolo Giudici. 2008. A bayesian approach to estimate the marginal loss distributions in operational risk management. *Computational Statistics & Data Analysis* 52: 3107–27.

Dalla Valle, Luciana, Maria Elena De Giuli, Claudia Tarantola, and Claudio Manelli. 2016. Default probability estimation via pair copula constructions. *European Journal of Operational Research* 249: 298–311. [CrossRef]

Dalla Valle, Luciana. 2009. Bayesian copulae distributions, with application to operational risk management. *Methodology and Computing in Applied Probability* 11: 95–115. [CrossRef]

Dalla Valle, Luciana. 2017a. Copula and vine modeling for finance. In *Wiley StatsRef: Statistics Reference Online*; Hoboken: John Wiley & Sons, Ltd.

Dalla Valle, Luciana. 2017b. Copulas and vines. In *Wiley StatsRef: Statistics Reference Online*; Hoboken: John Wiley & Sons, Ltd.

Daníelsson, Jón. 2011. *Financial Risk Forecasting: The Theory and Practice of Forecasting Market Risk with Implementation in R and Matlab*. Hoboken: John Wiley & Sons.

Demarta, Stefano, and Alexander J. McNeil. 2005. The t copula and related copulas. *International Statistical Review* 73: 111–29. [CrossRef]

Di Clemente, Annalisa, and Claudio Romano. 2021. Calibrating and simulating copula functions in financial applications. *Frontiers in Applied Mathematics and Statistics* 7: 642210. [CrossRef]

Embrechts, Paul, Rudiger Frey, and Alexander McNeil. 2005. Quantitative risk management. *Princeton Series in Finance* 10, 116–237.

Engle, Robert F. 1982. Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica: Journal of the Econometric Society* 50: 987–1007. [CrossRef]

Fantazzini, Dean, Luciana Dalla Valle, and Paolo Giudici. 2008. Copulae and operational risks. *International Journal of Risk Assessment and Management* 9: 238–57. [CrossRef]

Fortin, Ines, and Christoph Kuzmics. 2002. Tail-dependence in stock-return pairs. *Intelligent Systems in Accounting, Finance & Management* 11: 89–107.

Ghahtarani, Alireza. 2021. A new portfolio selection problem in bubble condition under uncertainty: Application of z-number theory and fuzzy neural network. *Expert Systems with Applications* 177: 114944. [CrossRef]

Graham, David I., and Matthew J. Craven. 2021. An exact algorithm for small-cardinality constrained portfolio optimisation. *Journal of the Operational Research Society* 72: 1415–31. [CrossRef]

Hastie, Trevor, and Robert Tibshirani. 1990. *Generalized Additive Models, Volume 43 of Monographs on Statistics and Applied Probability*. Washington: Chapman & Hall, CRC Press.

Hoe, Lam Weng, Jaaman Saiful Hafizah, and Isa Zaidi. 2010. An empirical comparison of different risk measures in portfolio optimization. *Business and Economic Horizons* 1: 39–45.

Hu, Ling. 2006. Dependence patterns across financial markets: A mixed copula approach. *Applied Financial Economics* 16: 717–29. [CrossRef]

Huang, Jen-Jsung, Kuo-Jung Lee, Hueimei Liang, and Wei-Fu Lin. 2009. Estimating value at risk of portfolio by conditional copula-GARCH method. *Insurance: Mathematics and Economics* 45: 315–24. [CrossRef]

Jin, Yan, Rong Qu, and Jason Atkin. 2016. Constrained portfolio optimisation: The state-of-the-art markowitz models. In *Proceedings of the 5th International Conference on Operations Research and Enterprise Systems—Volume 1: ICORES, Rome, Italy, February 23–25*. Cham: Springer, pp. 388–95.

Joe, Harry, and James Jianmeng Xu. 1996. The Estimation Method of Inference Functions for Margins for Multivariate Models. Available online: https://open.library.ubc.ca/soa/cIRcle/collections/facultyresearchandpublications/52383/items/1.0225985 (accessed on 1 May 2021).

Joe, Harry. 1997. *Multivariate Models and Multivariate Dependence Concepts*. Washington, DC: CRC Press.

Jondeau, Eric, and Michael Rockinger. 2006. The copula-GARCH model of conditional dependencies: An international stock market application. *Journal of International Money and Finance* 25: 827–53. [CrossRef]

Kakouris, Iakovos, and Berç Rustem. 2014. Robust portfolio optimization with copulas. *European Journal of Operational Research* 235: 28–37. [CrossRef]

Konno, Hiroshi, and Hiroaki Yamazaki. 1991. Mean-absolute deviation portfolio optimization model and its applications to tokyo stock market. *Management Science* 37: 519–31. [CrossRef]

Maghyereh, Aktham I., Basel Awartani, and Panagiotis Tziogkidis. 2017. Volatility spillovers and cross-hedging between gold, oil and equities: Evidence from the gulf cooperation council countries. *Energy Economics* 68: 440–53. [CrossRef]

Mai, Jan-Frederik, and Matthias Scherer. 2017. *Simulating Copulas: Stochastic Models, Sampling Algorithms, and Applications, Volume 6 of Series in Quantitative Finance*. Singapore: World Scientific.

Marimoutou, Velayoudoum, Bechir Raggad, and Abdelwahed Trabelsi. 2009. Extreme value theory and value at risk: Application to oil market. *Energy Economics* 31: 519–30. [CrossRef]

Markowitz, Harry. 1952. Portfolio selection. *The Journal of Finance* 7: 77–91.

Mensi, Walid, Atef Hamdi, and Seong-Min Yoon. 2018. Modelling multifractality and efficiency of gcc stock markets using the mf-dfa approach: A comparative analysis of global, regional and islamic markets. *Physica A: Statistical Mechanics and Its Applications* 503: 1107–16. [CrossRef]

Messaoud, Samia Ben, and Chaker Aloui. 2015. Measuring risk of portfolio: GARCH-copula model. *Journal of Economic Integration* 30: 172–205. [CrossRef]

Nelsen, Roger B. 2007. *An Introduction to Copulas*. New York: Springer Science & Business Media.

Patton, Andrew J. 2004. On the out-of-sample importance of skewness and asymmetric dependence for asset allocation. *Journal of Financial Econometrics* 2: 130–68. [CrossRef]

Pfaff, Bernhard. 2016. *Financial Risk Modelling and Portfolio Optimization with R*. New York: John Wiley & Sons.

Quaranta, Anna Grazia, and Alberto Zaffaroni. 2008. Robust optimization of conditional value at risk and portfolio selection. *Journal of Banking & Finance* 32: 2046–56.

Rockafellar, R. Tyrrell, and Stanislav Uryasev. 2000. Optimization of conditional value-at-risk. *Journal of Risk* 2: 21–42. [CrossRef]

Rockafellar, R. Tyrrell, and Stanislav Uryasev. 2002. Conditional value-at-risk for general loss distributions. *Journal of Banking & Finance* 26: 1443–71.

Sabino da Silva, Fernando B., and Flávio Ziegelman. 2017. Robust Portfolio Optimization with Multivariate Copulas: A Worst-case CVaR Approach. Available online: http://dx.doi.org/10.2139/ssrn.3076283 (accessed on 1 April 2021).

Salahi, Maziar, Farshid Mehrdoust, and Farzaneh Piri. 2013. CVaR robust mean-CVaR portfolio optimization. *International Scholarly Research Notices* 2013: 570950. [CrossRef]

Smillie, Alan. 2008. New Copula Models in Quantitative Finance. Ph.D. thesis, Imperial College London, London, UK.

Szegö, Giorgio. 2005. Measures of risk. *European Journal of Operational Research* 163: 5–19. [CrossRef]

Welsh, William F. 1999. On the reliability of cross-correlation function lag determinations in active galactic nuclei. *Publications of the Astronomical Society of the Pacific* 111: 1347. [CrossRef]

Yan, Jun. 2007. Enjoy the joy of copulas: With a package copula. *Journal of Statistical Software* 21: 1–21. [CrossRef]

Young, Martin R. 1998. A minimax portfolio selection rule with linear programming solution. *Management Science* 44: 673–83. [CrossRef]

Zhu, Shushang, and Masao Fukushima. 2009. Worst-case conditional value-at-risk with application to robust portfolio management. *Operations Research* 57: 1155–68. [CrossRef]