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Optimal Size and Location of Warehouses under Risk of Failure

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Optimal Size and Location of Warehouses under Risk of Failure

by

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Optimal Size and Location of

Warehouses under Risk of Failure

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Declaration of Co-Authorship / Previous Publications

I. Co-Authorship

I hereby declare that the key ideas, primary contributions, experimental designs, data analysis and interpretation, in the papers mentioned in the table below, were performed by the author, and supervised by Dr. Richard J. Caron.

I am aware of the University of Windsor Senate Policy on Authorship and I certify that I have properly acknowledged the contribution of other researchers to my dissertation, and have obtained written permission from each of the co-authors to include the published, accepted and submitted papers in my thesis.

I certify that, with the above qualification, this dissertation, and the research to which it refers, is the product of my own work.

II. Previous Publication

This dissertation includes original papers that have been previously published/accepted for publication in peer reviewed journals and proceedings of refereed conferences as well as 1 submitted paper.

Thesis Chapter	Publication title/ /full citation	Publication status
Chapter 2	R. J. Caron, **T. Oshan**, Optimal Warehouse Location and Size In Practice **Int. J. of Operational Research**	In Press
Chapter 3	**T. Oshan**, R. J. Caron, Optimal Warehouse Location and Size Under Risk of Failure	Submitted to PLOS ONE
Chapter 4	**T. Oshan**, Designing a Supply Chain Network with Capacitated Warehouses Fortification Under Risk of Failure **2021 8th International Conference on Frontiers of Industrial Engineering (ICFIE 2021) Singapore, September 16-18, 2021**	In Press

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Abstract

Nowadays, every company faces challenges that seem to be loaded with a contradiction: how to reduce operations and transportation costs while increasing customer satisfaction levels. Designing a supply chain network is an effective solution to such an issue. Supply chain network design involves making decisions about the number, sizes, and locations of the facilities in a supply chain. The focus of this study is how to choose appropriate warehouse locations and sizes in supply chain network design. The study is divided into two parts. In the second part, the risk of warehouse failure is considered while in the first part, it is not.

Three sets of mathematical optimization models for warehouse location and branch assignment were developed. The first set of mathematical optimization models covered the case of warehouse location without risk. Two sets of decision variables were introduced to determine the locations for new warehouses and assign warehouses to branches. The second set of mathematical optimization models covered the warehouse location problem under the risk of warehouse failure. Again, two sets of decision variables were introduced. The first set of decision variables helped in determining the locations for new warehouses, and the second set helped in assigning a primary and a backup warehouse to each branch. The backup warehouse to be used in case of failure of the primary warehouse. The third set of mathematical optimization models covered the case in which

some warehouses can be fortified to become totally risk-free. Each branch was either assigned to a primary fortified warehouse only or to a primary warehouse that was not fortified and a secondary fortified warehouse. Fortification model required an additional variables indicating which warehouses to be fortified.

Warehouses with multiple capacity levels and multiple part category types were considered, which is a contribution to the topic of warehouse disruption risk. Specialized warehouses were also considered in this dissertation, which is another contribution of this dissertation.

Some linearization and relaxation methods were used to help in solving the three models. Further, a solution methodology was presented based on the solution to scenario subproblems that are more easily, i.e., more quickly, solved. This requires an algorithm to determine the scenarios. Each scenario represents the number and sizes of warehouses needed to be built. The scenarios are novel in that they do not specify a subset of warehouses to be opened, but rather they specify the number of warehouses of each size to be opened.

The results showed the effectiveness of the proposed solution methodology by application to an example based on a case study of a Canadian company; and a created example based on European cities.

Dedication

I present this dissertation to my family.

Acknowledgments

I extend my most sincere thanks to Dr. Richard J. Caron for his support, leadership and encouragement. His guidance made the creation of this document a most interesting and enjoyable experience for me. I also offer my thanks to my committee members, Dr. Saman Hassanzadeh Amin, Dr. Yash Aneja, Dr. Waguih El Maraghy and Dr. Guoqing Zhang for their helpful comments in improving my dissertation. My deepest gratitude is extended to my family for their encouragement and support. My rewards and accomplishments would not have been achieved without their support.

Table of Contents

List of Tables

List of Figures

Table of Notation

Notation	Description
B, b, m	B is the set of branch indices b and has cardinality m.
W, w, n	W is the set of warehouse location indices w and has cardinality n.
S, s, q	S is the set of warehouse size indices s and has cardinality q.
J, j, g	J is the set of product category indices j and has cardinality g.
E	The set of (w, s) indices indicating warehouses currently built or that must be built.
$\mathscr{R}$	The set of (w, j) indicating warehouse unable to process category j parts.
A^s	The footprint in square feet, of a warehouse with size index s.
x_w^s	A binary variable equal to one if and only if a warehouse of size s is built at location w.
y_{wbj}	A binary variable equal to one if and only if a branch b is supplied by a warehouse at location w with its demand of items from category j.
r	Warehouse branch assignment level, primary (r=1) and backup (r=2).
y_{wbj}^r	A binary variable equal to one if and only if a branch b is supplied with its demand of items from category j by a warehouse at location w as a primary (r=1) or backup (r=2).
K	A common volume unit to measure demand.
d_{bj}	The demand, expressed in K, of items from category j, from branch b.
f_w	The cost per square foot, in dollars, during the planning horizon, where the number of square feet is given by A^s, for a warehouse built at location w.
ℓ_w	The cost per square foot, in dollars, during the planning horizon, where the number of square feet is given by A^s, for industrial land at location w.

Table 1: Table of Notation

Continue: Table of Notation

Notation	Description
$\boldsymbol{V}^s$	The volume, expressed in K, of storage space available in a warehouse with size index s.
ν_j^s	The operational cost of handling one K of items from category j at a warehouse with size index s.
τ_{wbj}	The cost to ship one K of items from category j from the warehouse at location w to branch b.
p_w	The probability that a warehouse w fails over the planning horizon.
U	The predetermined limit, or upper bound, on the number of warehouses to be built.
P^*	$\max\limits_w p_w$.
P_*	$\min\limits_w p_w$.
TD	Total demand.
ED	Expected demand.
$\underline{\mathrm{ED}}$	The product of $(1 + P_*)$ and TD.
$\overline{\mathrm{ED}}$	The product of $(1 + P^*)$ and TD.
z_{wbj}^s	The product of x_w^s and y_{wbj}.
z_{wbj}^{rs}	The product of x_w^s and y_{wbj}^r.
Q_{wbj}^s	The product of z_{wbj}^{2s} and $\sum\limits_{w' \neq w} (p_{w'} y_{w'bj}^1)$.
O_{wbj}	The product of y_{wbj}^2 and $\sum\limits_{w' \neq w} (p_{w'} y_{w'bj}^1)$.
Γ_w^s	The product of x_w^s and γ_w.
ζ_{wbj}^s	The product of z_{wbj}^{1s} and $(1 - \gamma_w)$.
η_{wbj}	The product of y_{wbj}^1 and $(1 - \gamma_w)$.
ϕ_{wbj}	The product of y_{wbj}^1 and γ_w.
ψ_{wbj}	The product of y_{wbj}^2 and γ_w.
$\mathfrak{S}$	Set of all possible scenarios.
$\mathfrak{s}_i$	The ith scenario.
ρ_w	The percentage of fixed cost that represents the fortification cost for warehouse w.
γ_w	A binary variable equal to one if and only if a warehouse w is fortified.

Chapter 1

Introduction

1.1 What is Supply Chain Management?

Global competition, items with short life cycles, and customer expectations have forced companies to invest in their supply chain networks. According to Chopra et al. [25], and using Figure 1.1 [7], we see that a supply chain consists of all parties involved, directly or indirectly, in fulfilling a customer demand. Thus, the supply chain includes not only the manufacturers and suppliers but also transporters, warehouses, retailers, and even customers themselves. In other words, supply chain management means integrating suppliers, producers, warehouses, and stores to ensure the goods are produced and distributed at the right volume, to the right locations, and at the right time, so that the total system cost is minimized and the required service is satisfied. Throughout this work, facilities, warehouses, and distribution centers are considered to be the same.

In the supply chain network, raw materials are used by factories to produce items that are shipped to warehouses for intermediate storage before being shipped to retailers or directly to customers. So, in order to reduce cost and enhance the service, the interaction between the various levels in the supply chain network must be taken into consideration.

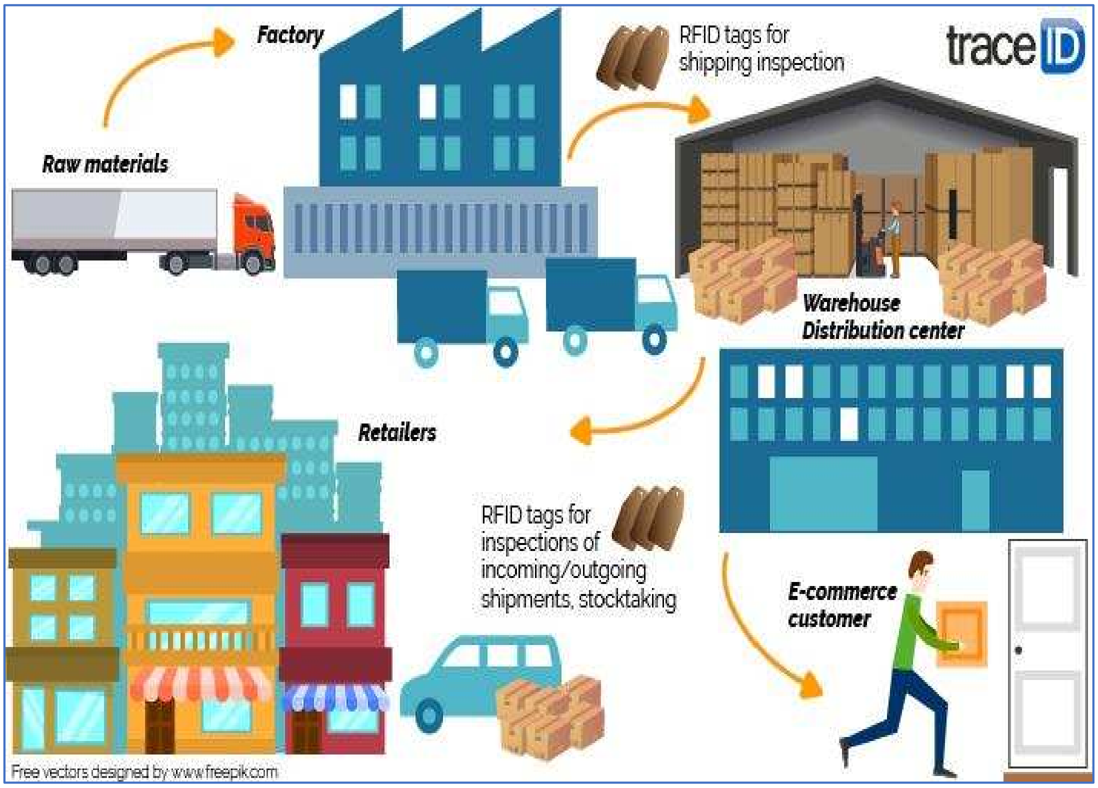

Figure 1.1: The supply chain network [7]

Supply chain management decisions are not easy to be determined for many reasons, such as: first, supply chain strategies are directly affected by another chain called the development chain that includes the set of activities associated with new product introduction. Second, it is not easy to design and operate a supply chain with an objective of minimizing the costs and all service levels are maintained. Finally, uncertainty and risk are inherited in every supply chain; customer demand can be forecasted but this forecasting is not exact and accurate. Also, a facility faces the risk of failure which affect the topology of the supply chain network.

1.2 Supply Chain Network Design

Supply chain *network design decisions* include the assignment of facility role; location of manufacturing, storage, or transportation-related facilities; and the allocation of capacity and more markets to each facility [25]. The main goal of such decisions is to maximize the company's overall profit and satisfy customer demands in the shortest possible time with the minimal possible cost.

Any supply chain network is affected by many factors. Figure 1.2 shows some factors that influence decisions with design of supply chains networks. The factors are strategic, technological, Macroeconomic, political, infrastructure, competitive, and operational factors. The strategy of any firm is the main factor in supply chain network designing. For example, some firms focus on cost leadership and tend to save on the cost of location and manufacturing systems. As for the firms that focus on the response rate, they tend to place their facilities in the locations which react quickly to any change in the market needs. Technology factors are represented by the availability of product technologies that play an important role in the network design. It can help in performing and enhancing economies of scale. The Macroeconomic factors include tariffs and tax incentives, Exchange-rate, demand risk, freight and fuel costs. In terms of political factors, firms prefer to locate their facilities in politically stable countries which helps in providing a clear rules and regulations of trade and ownership. The availability of good infrastructure is a very important factor for all firms before locating their facilities as it will reduce the needed time to construct a new facility. Poor infrastructure in a certain place means an extra cost for any firm that wants to establish their supply chain network in that area. As for the competitive factors, companies must consider the strategy, size, and location of competitors when designing their supply chain network. Finally, operational factors include the daily needed operations to run the network such as the used technology, employees, electricity bills, etc. They are important as they have direct impact on the overall network design cost.

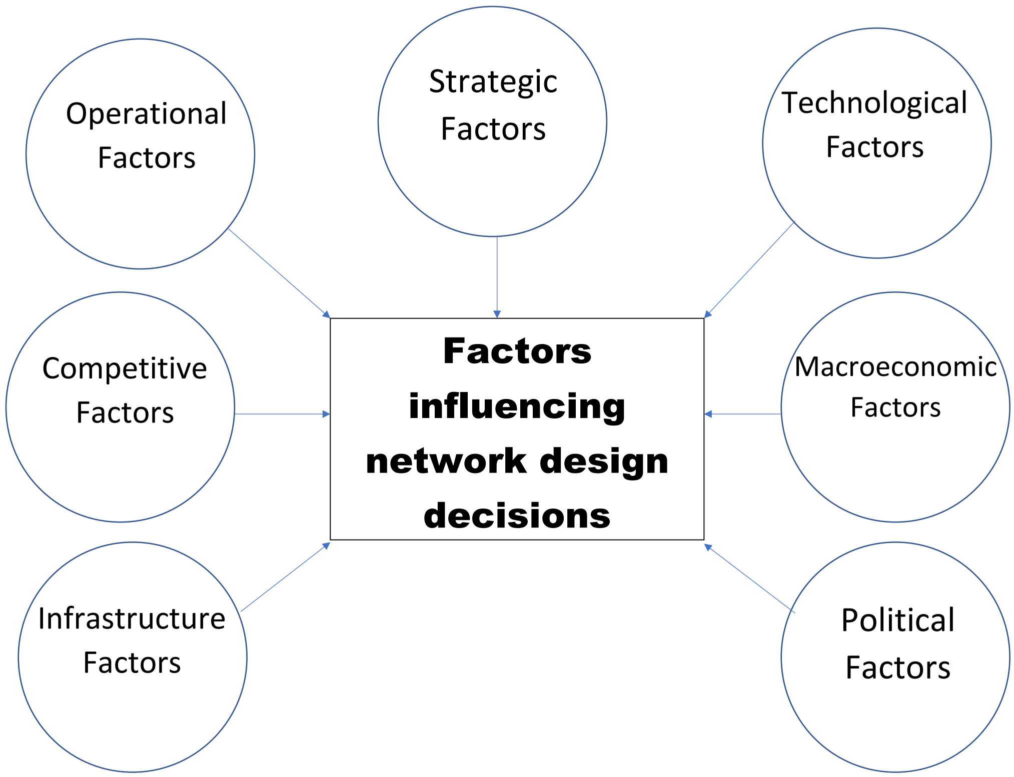

Figure 1.2: Factors influencing network design decisions

1.3 Supply Chain Risk

There are many ways to define a supply chain risk. Zsidisin [106] defined the supply chain risk as the probability that is associated with the inability of a certain supplier to meet customer demand or cause threats to customer life and safety. Ellis et al. [39], defined the supply chain risk as an individual's perception of the incurred loss as a result of the disruption in the supply of a certain purchased item from a particular supplier. Juttner et al. [54] defined supply chain risk as any risk that prevents the final product to be delivered.

Some authors identify supply chain risk types without classification. For example Chopra et. al. [25] presented some risk types such as disruption (natural disaster, war, terrorism, and labor disputes), delays, system risk, forecast risk (due to long lead times, and seasonality), intellectual property risk, procurement risk (number of customer, financial strength of customers), inventory risk (inventory hold cost, demand and supply uncertainty), and capacity risk (cost of capacity and capacity flexibility). On the other hand, some authors classified the risk types into two categories, internal and external. Figure 1.3 illustrates the classification of supply chain risk as described by Wu et al.[100]. The figure shows that the internal and external risks can be classified into controllable, partially controllable, and uncontrollable factors.

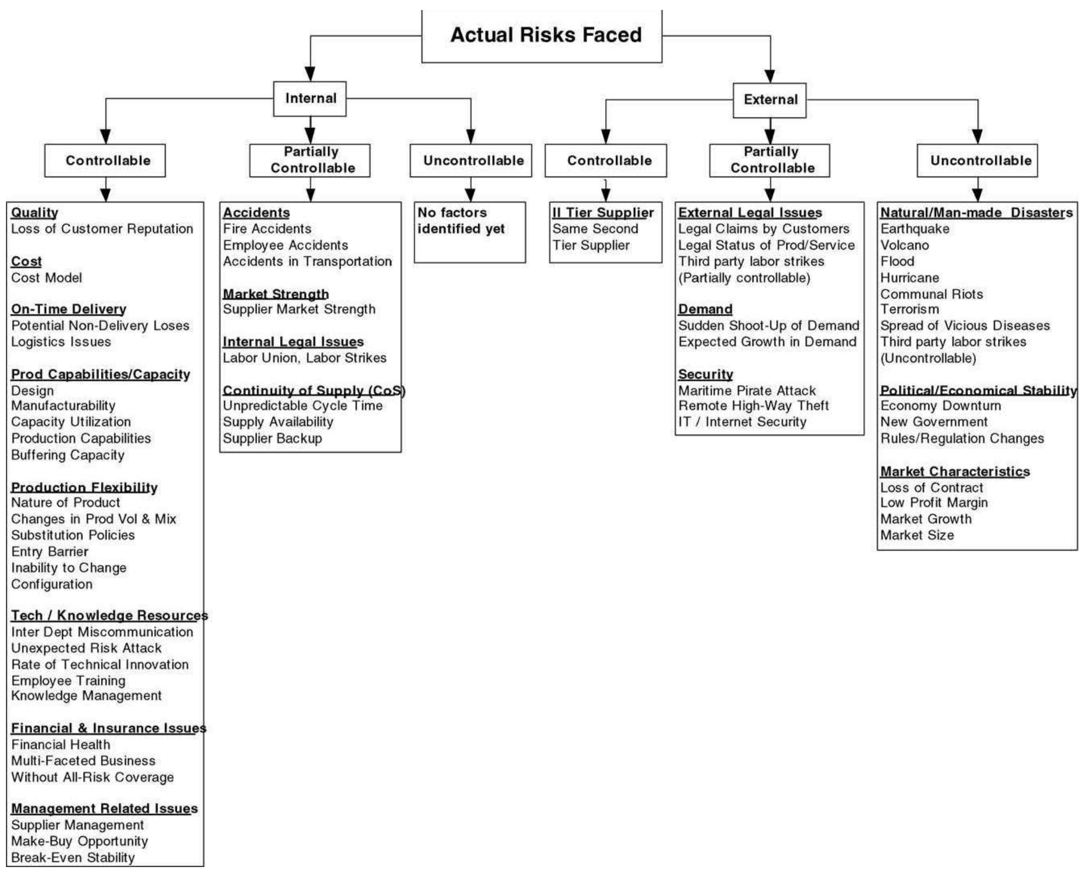

Figure 1.3: Supply Chain Risk Classification [100]

1.4 Facility Location Problem

The Facility Location Problem (FLP), sometimes known as location analysis, is a branch of Operations Research and Computational Geometry. FLP is an optimization problem where we determine the sites for factories and warehouses. The problem consists of selecting the best site among potential sites, taking into account that demands at several points must be satisfied by those facilities. The objective of the FLP is to select the sites so that total cost is minimized.

The FLP has many applications in different areas, such as in Computer Vision [91], Data Summarization and Clustering [66, 95], and Network Design [38, 83].

The FLP can be divided, based on the facility capacity, into the Uncapacitated Facility Location Problem (UFLP) and the Capacitated Facility Location Problem (CFLP). Uncapacitated facility location problem (UFLP) assumes that each facility can produce and ship unlimited quantities of a certain commodity under consideration. The first models of UFLP were introduced in the 60's of the last century, when the Simple Plant Location Problem (SPLP) [59, 19] and the P-Median Problem [46, 47] were introduced. On the other hand, CFLP assumes that each facility has a limited capacity to produce and ship quantities. CFLP is an important extension of UFLP in which capacity values are considered with the goal of maximizing the demand that can be satisfied by each potential warehouse. According to [96], the UFLP is the simplest version, concerning getting a solution, of the FLP. However, the FLP is an NP-hard problem, even if it is UFLP [31].

1.5 Literature Review

This section presents the literature related to the Supply Chain Network Design while focusing on facility location and facility location with the risk of facility failure.

1.5.1 Facility Location Problem

Dasci and Verter [33] presented a formal definition of the plant location and technology acquisition problem and provided a mathematical model for this problem in a multi product environment that selects the facility location while minimizing the total cost. The authors assumed that there were no limitations on the availability of technology and the capacity to be built-in at the potential sites. So their problem is called Uncapacitated Plant Location and Technology Acquisition problem (UPL & TAP) which is a single-level, deterministic and static problem. UPL & TAP reduces to the UFLP in case of having a single product and linear technology costs. But Krarup and Pruzan [57] showed that UFLP is NP-Complete, so UPL & TAP is NP-Complete as well. A solution algorithm, based on the Progressive Piecewise Linear Underestimation, was presented which gives upper and lower bounds on the optimal solution with an objective function of minimizing the total cost.

Holmberg et al. [50] presented a heuristic solution for the capacitated facility location problem where each customer is served by a single facility. Their solution method was based on Lagrangian heuristic with a branch -and- bound framework. The computational results showed that the proposed method was efficient in getting a solution to the FLP.

In [41], Etemadnia et al. addressed the wholesale hub location problem in food supply chains. Hubs are used to connect a set of origin and destination nodes so that we get the maximum utilization of facilities and minimize transportation costs. They are responsible for redirecting the aggregated inbound and disaggregated outbound flows. The main purpose of the authors' work was to design a supply chain network that includes optimal hub locations to serve food consumption markets. A mixed-integer programming model (MIP) has been introduced with the objective of minimizing the total network costs that include the transportation of goods in addition to hub construction costs. The authors applied the proposed mathematical model to a meat supply chain in the Northeast United States. The network consists of 13 federal states and 433 counties. Different cases have been studied such as unlimited average traveled distance from the production sites to the hub locations and from the hub locations to the consumption markets. The paper also presented the effect of road conditions on hub locations in the network. Road conditions include road service, road accessibility and the capacity of the roads in the network.

Daskin and Jones [35] suggested a new approach to solve capacitated facility location problems (CFLP). Their new approach was introduced to solve facility location and single sourcing customer allocation problems. The authors observed from real decision problems that in most location problems only one or two additional sites for the facilities are selected from a small list of candidate alternatives. This makes the total enumeration is possible and make it better for decision makers to look at the full range of possible options. Also, the authors found that the violation incurred by relaxing the single sourcing constraint is so small. This is because of fewer of facilities compared to the customer in most of the cases.

In [45], Hajiaghayi et al. introduced a generalized version of the facility location problem where the facility cost is a function of the number of clients assigned to the facility. The authors focused on a concave facility cost function and found that this problem can be reduced to the uncapacitated facility location problem. Also, the authors improved a greedy algorithm to solve the problem. The greedy algorithm that has been used was found to be helpful in solving their problem.

Melkote and Daskin [73] presented a mixed integer programming formulation of the Capacitated Facility Location Problem

(CFLP). The authors used LP relaxation with this problem and derived some valid inequalities to strengthen the LP relaxation. Test problems showed that more than a third of them were solved in under 5 minutes within 5% optimality using the proposed model. Further, the authors did some sensitivity analysis to check the model's behaviours. It has been shown that, compared with the uncapacitated model, when capacity constraints are imposed, contrary to what the authors were expecting, transportation costs decreased. On the other hand, as the authors expected, the network became more expensive.

Lin et al. [67] formulated and analyzed a strategic design model of a distribution system with four echelons including plants, consolidation centers, distribution centers and retailers. The authors assumed significant economies of scale in the transportation cost which leaded them to have a concave cost function. Also, the authors assumed that plant locations, capacities, and capabilities are assumed to be known and fixed. The authors proposed a greedy heuristic method that can efficiently find near optimal solutions. The solutions that they got using their heuristic methods where within 1% of the optimal solution.

In [40], Etemadnia et al. examined wholesale facility (hub) locations in food supply chain systems on a national scale to facilitate the efficient transfer of food from production regions to consumption locations. The mathematical formulation that has been used is a mixed integer linear programming (MILP) model that minimizes the transportation cost and fixed cost such as building but not the operational cost. The authors considered an upper and lower bound on the capacity for each facility. In order to reduce the computational size of the problem, the binary variable that represents building or not building a facility, has been relaxed. In the model with the relaxed binary variable, all hubs with zero assignment values were removed and a small sized model called the Intermediate Model has been built with binary variables representing to build or not build the facility. The introduced Intermediate Model solutions were very efficient as they were close to the model without relaxing the binary variables.

Ashtab et al. [16] presented a Binary Quadratic Optimization mathematical model for multi capacitated, three level supply chain design including suppliers, distribution centres (DCs), and customer zones. The authors applied the proposed mathematical model, with three model simplifications that allow for the solution to the model, on a real case study with 47 suppliers, 13 distribution centres and 2,976 customer zones. The first model simplification was to cluster the customer's zones according to the postal code to reduce the problem size. The second model simplification was to relax the binary variable that assigns customer zones to a warehouse. The third model simplification was to linearize the quadratic function that represent the variables cost by assigning to each DC a variable cost based on capacity rather than using the multiplication of the location and size variables. The outputs that the authors obtained were very close to the optimal solution if such simplifications were not used.

In [48], Holmberg studied the capacitated facility location problem with staircase costs and fixed costs that appear at several levels of productions. The problem was modeled with discrete different sizes. The authors proposed a solution method based on convex piecewise linear relaxation and Benders decomposition. The study showed that convex linearization technique is a promising approach for large stair case cost problems. In [49], Holmberg and Ling proposed a Lagrangian relaxation heuristic and used the convex piecewise linearization on a staircase shaped cost function of a capacitated facility location problem with different discrete sizes. The authors found that the Lagrangian relaxation heuristic is quite promising compared to the ADD heuristic. The ADD heuristic is used to obtain an initial feasible solution by finding the location that provides the largest reduction in the objective function, for the capacitated plant location.

In [99], Wu et al. presented a CFLP with multiple facilities in one site. The authors considered multiple types of facilities to serve types of commodities. Their problem was modeled as a non-linear integer programming. A heuristic algorithm based on Lagrangian heuristic was used to find an approximate solution to the problem.

Montoya et al. [76] introduced the mulit-product capacitated facility location problem with production and building cost (MP-

CFLPGC) that can be linear, concave, or convex which allowed to have economies of scale and congestion. MP-CFLPGC was formulated as a MILP. The authors evaluated the proposed formulation of the problem by analyzing the results of commercial optimizer using 288 randomly generated test instances for the supply chain of a Colombian cement industry. The MP-CFLPGC allowed to answer to important question in multi-product supply chain: (1) Which one is better to have: few big facilities or many small facilities? and (2) Which one is better to have: specialized facility for each product or non-specialized for multi-product facilities? To solve the problem, the authors proposed a randomized mathematical programming model based heuristic. On average, the authors found that the optimality gap was 3.7%. Further, the authors found that in 5.5% of the instances, after 60 minutes of running time, the optimality gap drops to less than 1%.

In [53], Jayaraman presented a mixed integer programming model for a multi-product warehouses logistics problem. The author considered a limit on each demand that can be supplied by a particular warehouse. He applied Lagrangian relaxation and a heuristic solution procedure to solve the problem. The author found that the heuristic procedure performs well in terms of both approximations to optimality and solution times.

EL Amrani and Benadada [14] worked on the multi-capacitated location problem (MCLP) with budget constraints. MCLP consists of locating facilities on a network with the objective of minimizing the total cost of assigning each customer to the nearest facility. Each facility is assigned one capacity level ffrom a pre-determined set of capacity levels. The authors tested several approaches to solve the MCLP problem. They used a data set that consisted of five instance classes with five levels of difficulty (easy, medium, difficult, very difficult, and complex). The difficulty depends on the number of customers but not the number of facilities and their capacity levels. The authors found that the Lagrangian relaxation method worked very well with easy instances. However, with difficult instances, Lagrangian relaxation violated demand constraints and generated bad solutions.

Amiri [13] worked on locating production plants and distribution warehouses to determine the best strategy to distribute the product from plants to warehouses and from warehouses to customers. His objective was to select the optimum numbers, location and capacities for plants and distribution warehouses so that all customer demand is satisfied with minimum cost. The author allowed for multiple capacity levels to the plants and distribution warehouses. He developed a mixed-integer programming model and provided a Lagrangian heuristic solution procedure. The computational tests showed that his solution procedure is efficient and effective for getting the optimal location and capacity to each built plant and distribution center.

In [42], Fischetti et al. did a computational study for CFLP problems. The authors implemented Benders' decomposition on their CFLP so that to reduce the computations needed to minimize the sum of facility opening costs, and customer allocation costs. The authors found that their implementation was simpler, saved a lot of computational time, and more efficient than other heuristic approaches for CFLP.

Basker [21] stated that great circle distance (GCD) is preferable to Euclidean distance (ED) for optimal facility location problems. The author presented mathematical models that combine GCD, demand points, and demand weights (weights are given to the demand points based on their importance). The models covered the case of single and multi-facility location problems. The author used the Weiszfeld ([98]) algorithm to solve for the optimal facility location.

1.5.2 Facility Failure Risk Problem

Constructing facilities is usually very costly, so it is not easy to modify the location of a built facility. Thus, facility location is so crucial in the supply chain network design. Risk has always been part of the supply chain. It is a reality inside and outside any organization. So, any facility faces the risk of disruption for many reasons caused by man-made and natural disasters. If a facility fails, its customer will either be assigned to another non disrupted facility or give up service. In both cases, the supply chain network will face some losses. This issue forces supply chain network designers to consider such risks in designing supply chain networks.

In [104], Zhang et al. worked on scenarios of facility failing that causes a supply disruption. The authors formulated a joint location-inventory model with an objective to minimize the expected total cost across all possible facility failure scenarios. The model determines the optimal number of facilities and their locations. Two methods have been used to solve this problem. The first method is special ordered sets of type two (SOS2) and the second one is a heuristic method based on Lagrangian relaxation. The output that the authors got was: although it is expected because of the economies of scale, inventory costs tend to reduce the optimal number of open facilities, but in reality because of the risk diversification effect, supply disruptions tend to result in more open facilities to reduce expected transportation cost.

In [87], Simchi-Levi et al. studied the low-probability high-impact risk such as shutting of a supplier's factory or flood at a distribution center. The authors developed a mathematical model that can help companies in quantifying the financial and operational impact that can arise because of a critical supplier's facility were out of commission for a period of time. Also, the mathematical model can help companies in reducing their exposure to supply chain risk. One of the main features of the mathematical model is its ability to determine time to recovery (TTR) which is the time that it would take for a particular node (such as a supplier facility, a distribution center, or a transportation hub) to be restored to full functionality after disruption.

In three year research engagement with Ford Motor Company, Simchi-Levi et al. [88] developed a novel risk-exposure model that assesses the impact of a disruption originating anywhere in a firm's supply chain. The proposed model reduces the need to estimate the likelihood of low probability of high impacts events as it focuses on understanding the impact of any disruption rather than knowing its source.

Church et al [29] introduced two new optimization models called the r-interdiction median problem and the r-interdiction covering problem. Both models helped in identifying the most critical facility assets in a service / supply system that if lost, a big impact can happen to the service delivery . Such a model can be used in identifying the worse-case of loss. In [27], Church and Scaparra presented a family of models that identify possible effects that are caused by loosing some facilities in a supply chain network as a result of natural disasters or international strike. The models can identify the worst-case and best-case expected loss.

In [90], Snyder and Daskin introduced the reliability fixed charge location problem (RFLP) and the reliability p-median problem (RPMP). Their mathematical model assigns each customer, whose original facility was disrupted, to the nearest non-disrupted facility. They assumed that all facilities, except those that cannot be disrupted, have the same disruption probabilities. Also, in their problem the facilities were assumed to be uncapacitated. The authors considered the trade-off between operating costs and expected failure cost over all all facilities. They showed that substantial increases in reliability are possible with minimal increases in operating costs. The authors proposed a Lagrangian Relaxation to solve the proposed problem.

Cui et al. [32] proposed a mixed integer program (MIP) formulation and a continuous approximation (CA) model to study the reliable uncapacitated fixed charge location problem (RUFL) with an objective to minimize total costs during normal and failure scenarios. The authors randomly generated the facility disruption probability from a uniform distribution between 0 and 0.2. The authors used a Lagrangian Relaxation (LR) algorithm and found that it is efficient in mid-sized RUFL problems. They also found that for large-scale problems, the CA method is better than the LR method because it provides a fast heuristic with which near-optimum solutions. Their solution balanced the trade-off between normal and emergency operating costs.

Shen et al. [86] studied an uncapacitated reliable facility location problem. Their problem was formulated using a mathematical model such that if a facility fails, then its customers are reassigned to other (operational) facilities. Unlike the work done by Snyder and Daskin [90], their work assumes that all facilities have independent and differing disruption probabilities. To solve the problem, the authors proposed two approaches. The first approach is the scenarios method, by which the disruption scenarios are entailed and the problem is formulated as a stochastic programming model. The second approach involve the use of nonlinear terms to calculate the probability that each customer is served by its closest rth facility.

In [82], Peng et al. came up with a mixed-integer programming model that minimizes the nominal cost (the cost of the scenario in which no disruptions occur) while reducing the disruption costs using the p-robustness criterion that bounds the cost in disruption scenarios. The authors proposed a metaheuristic algorithm that produces very close to optimal results given a fraction of time required by CPLEX.

In [63] Li et al. presented a reliable p-median problem (RPMP) and a reliable uncapacitated fixed-charge location problem (RUFL). Both models consider heterogeneous facility failure probabilities, one layer of supplier backup and facility fortification with limited budget. RPMP and RUFL are non linear integer programming models that are proved to be NP-hard. The authors developed a Lagrangian Relaxation-based (LR) solution algorithms and showed its efficiency. The limitation for both models was in their consideration of suppliers unlimited capacities.

Azad et al. [18] proposed a capacitated supply chain network design (SCND) model under random disruptions in both facility and transportation links. The authors considered partial DCs disruption which means that DCs can serve from their non disrupted part. The problem has been solved using a modified version of Benders' Decomposition (BD).

In [93] Tang et al. proposed an integer programming model for the reliable facility location problem with facility protection, that allows for site specific failure probability to protect the supply chain network against random facility disruption. The authors assumed that their problem is of uncapacitated facilities. Two solution approaches were proposed; Lagrangian Relaxation and Local Search.

Hoseinpour and Javid [51] considered the design of an immobile service system in which each facility is exposed to the risk of interruptions. The objective of their work, which involve location-capacity decisions and allocations, is to maximize the difference between the service providers' profit and the sum of transportation and waiting costs. The authors formulated the problem as a mixed-integer nonlinear program and solved the problem using an algorithm based on Lagrangian Relaxation that can solve a large-scale problem with 500 customers and 50 service facilities in a few seconds.

In [102], Yun et al. proposed a nonlinear integer programming mathematical model that can help in balancing the initial facility investments and expected long-term operational cost by finding the optimal facility locations. Each facility is assumed to be uncapacitated and have a disruption probability that is site-dependent. The authors applied a linearization technique to reduce the difficulty of the problem. The results revealed the benefit of having backup facilities and system robustness against variations of the loss-of-service penalty.

Aryanezhad et al. [15] proposed a non-linear integer programming for supply chain network design with distribution centers are subject to random disruption. The objective of the model is to minimize the expected costs that include location, inventory, transportation and lost sales costs. The authors studied the impact of distribution centers on facility location and inventory decision. The model suggested to assign some multiple backup distribution centers to cope with the disruption issue. The authors developed a solution method based on genetic algorithm to solve the problem.

In [97], Wang et al., extended the work done by Snyder and Daskin [90] on uncapacitated facilities by considering heterogeneous facility failure probabilities and assuming the presence of two types of facilities: reliable and non-reliable. The authors proposed a mixed integer programming model that minimizes the sum of initial facility construction costs and expected transportation costs in both the regular and failure scenarios. Also, the authors presented a Lagrangian Relaxation algorithm to solve the problem.

Aguila and ElMaraghy [9] developed a mixed integer linear programming model to design the supply chain and product architecture.

The objective function for this model was to minimize the risk of natural disasters in the supply chain. The model can be used to evaluate the risk of any proposed configuration for the supply chain network.

In [52], Hosseini et al offered an efficient solution to the problem of resilient supplier network and order allocation under disruption risks. Their model can accommodate the case of having a large number of disruptive events with no computational burden as a result of using the Noisy-OR technique [81, 56, 105].

Yu et al. [101] developed a risk-averse optimization formulation to the RUFL problem to compute resilient location and customer assignment solution in the case of independent and correlated disruption. Facilities were assigned random disruption probabilities. Their problem was MINP in the case of independent disruption probabilities and MILP in the case of dependent disruption probabilities. For the MINP, the authors developed a branch-and-cut algorithm combined with augmented Lagrangian decomposition to get the solution. The computation results showed that the risk-averse models did better than the classical-neutral models in improving the reliability.

In [69], Lu and Cheng studied the disruption of capacitated facilities that does not only affect the facilities capacities but also the demand of customer manner. The authors presented a three two-stage robust optimization formulations with different objectives and performance bounds are built to model the problem. They used column-and-constraint generation algorithm and Benders decomposition method to solve the models. The study found that facility disruption correlated demand, which is the demand affected by facility disruption, has an affect on the network design

Momayezi et al. [75] studied the capacitated hub location problem under the risks of hub disruption. The authors assumed that if any hub fails, then its customers assigned to another operational hub. They modeled the problem as a two-stage stochastic program and used a metaheuristic algorithm to solve it.

In [60], Kungwalsong et al. considered a four echelon supply chain network designing problem with the facility disruption risk. The authors proposed a two stage programming to model the problem and developed a simulated annealing (SA) algorithm [17] to determine the optimal facilities location and their capacity.

1.5.3 Facility Fortification

This section presents the literature related to the facility fortification. A fortified facility means that it becomes a non-disrupted facility or risk free facility.

In [78], Namdar et al. examined a reliable capacitated facilities under partial and complete disruption. The authors applied multiple mitigation strategies such as: DCs fortification, transshipment between DCs, facility location to overcome disruption. They found that transshipment strategy is more effective than the other two strategies.

Church et al. [29] introduced two new optimization models called the r-interdiction median problem (RIM) and the r-interdiction covering problem (RIC) to identify the most important facilities in a supply system. Both models help in identifying for, a given supply system, the set of facilities that have the most effect on the service delivered if lost.

In [28], Church et al. presented an integer-linear programming model that optimally allocate fortification resources, assumed to be limited, so that the impact of facility interdiction is minimized. Facilities are assumed to be uncapacitated. The authors tested the presented models on two different geographical data sets and used the Implicit Enumeration method to solve the problem. The authors found that a solution to the fortification problem contains at least one cite in the solution to the RIM model.

Aksen et al. [11] studied the added budget constraint on the r-interdiction median problem with fortification (RIMF). Their objective was to find the optimal allocation of protection resources to an existing sytem of P uncapacitated facilities. The authors used binary enumeration tree at each node to solve the problem. They found that the number of facilities to be fortified and the objective function values do not depend only on the allocated fortification budget, but also depend on the cost of protecting each facility individually.

In [65], Liberatore et al. studied the problem of optimally protecting a capacitated median system with limited protecting resources. They presented a mathematical model that finds that best protection plans against disruptions that affect regions rather than single elements in the system such as earthquakes, storms, hurricanes, spread of a disease...etc. The authors created a correlation matrix Q_{kj} that shows how the lose in the capacity of facility j when facility k is disrupted. They found that ignoring the correlation effects may lead to have an unnecessary increase in the overall cost to the network when any disruption happen. Finally, the authors proposed an exact solution algorithm called the tree-search procedure to find which facilities to protect.

Lasada et al. [68] presented a bi-level mixed integer linear program for protecting uncapacitated facilities and reducing the impact of worse-case facilities disruption with the consideration of the recovery time role on the system and the possibility to have multiple disrupted facilities over time. The authors used Bender's decomposition and Super Valid Inequalities to solve instances of the problem.

It was mentioned in Section 1.5.2 that Li et al. [63] presented a reliable p-median problem (RPMP) and a reliable uncapacitated fixed-charge location problem (RUFL). Further, the authors studied the impact of facility fortification on the improvement of network reliability. The proposed models enabled for periodic fortification upgrades when needed and depend on the availability of fortification resources. The authors found that the selected facilities to be fortified are those located in the areas with the highest demands.

In [72], Medal et al. studied the minimax uncapacitated facility location and hardening problem (MFLHP). MFLHP minimize the maximum distance from a demand point to the closest facility after disruption. The problem was formulated as MIP and decomposed into subproblems and solved using binary search algorithm. The authors found that if the cost of hardening the built facilities is low, then it is possible to reduce the post-disruption radius by hardening more facilities that does not affect the pre-disruption radius.

Mahmoodjanloo et al. [70] presented a tri-level defense capacitated facility location model for full coverage in the r-interdiction problem. The model makes a decision on the number and the location of the defense facilities (defense facilities are facilities that defend service facilities against attacks). After any attack, the model has the option to out-source part of the demand that cannot be satisfied by the available facilities. Further, the author studied the probability of a fortification facility to defend a service facility with respect to the distance between them. The authors proposed a hybrid meta-heuristic method to solve the problem. The authors found that an increase in the shipping cost, in the outsourcing cost to cover the demand in the interdiction budget, and a decrease in the defender's system success probability upper bound will lead to have an increase in the overall network cost.

Akbari-Jafarabadi et al. [10] proposed a conceptual framework for the capacitated facility location problem to minimize the total cost before and after interdiction. The problem was an integer program and the authors used explicitly enumeration method and meta heuristic algorithms to solve it. The work proved the importance of planning and designing defense system so that the vulnerability of the system is reduced.

In [64], Li et al. developed an agent-based simulation model over 10 years to study the effects of facility disruption and fortification on the total cost of the supply chain network. The decisions on fortification were done based on anticipating disruptions to occur on the most important facilities that are the facilities that cover the largest demand.

Khanduzi et al. [55] presented a partial interdiction / fortification problem for capacitated facilities and budget constraint. The defender in their problem is looking for allocating the available resources to protect the whole system so that the total system losses are minimized. On the other hand, the attacker is looking to interdict and maximize the system losses. The authors used a metaheuristic algorithm called PSO and a population-based algorithm called TLBO to solve the NP-hard problem. They found that the objective function values increase with an increase in the interdiction level and a decrease in the fortification level.

In [36], Dey and Jenamani presented the problem of a fortification plan for capacitated facilities with maximum limit on the traveled distance and budget limit. The work suggested robust fortification plans under facility disruption. The maximum limit on the traveled distance is needed for some items such as emergency and perishable items and it was helpful in the computational complexity. An implicit enumeration algorithm was used to solve the problem.

In [77], Monzon et al. used a real case study for the 2018 storm that hit Mozambique to test their presented model for a pre disaster Humanitarian logistic model (Anysia and Kopczak defined the Humanitarian logistic in [94]). Their model captures the fortification of element of the distribution network, the location of emergency inventories, and the definition of their capacity.

Cheng et al. [24] studied a robust fixed charge location problem under facility disruption and demand uncertainty. The authors studied the possibility of fortifying the existing facilities to protect them from disruption. They proposed a mathematical model that allows to determine an optimal and robust facility location so that the overall network can face all types of uncertainties. The authors implement the C&CG method proposed by Zeng and Zhao [103] to solve the problem. They also developed a C&CG algorithm and compare it with the other one in the literature.

In [12], Alikhani et al. studied the problem of designing / re-designing a resilient supply chain network under uncertainty using multiple resilience strategies, including facility fortification, reserved capacity, inventory repositioning and network design quality. The authors used two stage stochastic programming (TSSP) to model their problem. According to [44] and [80], TSSP has been recommended to solve optimization problem of network design under disruptions. They found that using a mixture of such resilience strategies increases the network's resilient and decreases the post disruption costs.

1.5.4 Summary and Research Gap

This section summarizes the studies covered in Sections 1.5.1 through 1.5.3 and presents them in tables. The multiple capacity levels column in Table 1.1 shows that there is a scarcity of research on the facility location problem with multiple capacity levels. This dissertation focused on this topic by developing a mathematical model to solve the facility location problem with multiple capacity levels and applying it to a Canadian case study and a created European example.

Concerning the facility failure risk problem, Table 1.2 shows many gaps in the capacitated / uncapacitated column and the multiple capacity levels column. Literature review showed that there is a scarcity of research in studies on the disruption of facilities with multiple capacity levels. Further, most of the literature involves the demand for a single item, this dissertation deals with the demand in multiple part category types in the presented examples. As a contribution, a scenario model was used to decrease the solution time required by CPLEX to design the network. This model is different from those in the related literature in that it does not specify a set

Table 1.1: Facility Location Literature Review Summary

Author(s)	**Year**	**Capacitated / Uncapacitated**	**Multiple Capacity Levels**	**Solution Method**
Daskin and Jones [35]	1993	Capacitated	No	Relaxing single source variable
Holmberg and Ling [49]	1997	Capacitated	Yes	Lagrangean relaxation heuristic
Jayaraman [53]	1998	Capacitated	No	Lagrangian Relaxation Heuristic solution
Holmberg et al. [50]	1999	Capacitated	No	Lagrangian heuristic with a branch and bound framework
Dasci and Verter [33]	2001	Uncapacitated	N.A.	Progressive Piecewise Linear Underestimation
Melkote and Daskin [73]	2001	Capacitated	No	Valid inequalities
Hajiaghayi et al. [45]	2003	Uncapacitated	N.A.	Greedy Algorithm
Lin et al. [67]	2006	Capacitated	No	Greedy Algorithm
Amiri [13]	2006	Capacitated	Yes	Lagrangian Relaxation
Ashtab et al. [16]	2014	Capacitated	Yes	Three model simplifications
Etemadnia et al. [40]	2015	Capacitated	No	Relaxing build or not to build variable
Fischetti et al. [42]	2016	Capacitated	No	Bender's decomposition
Montoya et al. [76]	2016	Capacitated	Yes	Heursitc
EL Amrani and Benadada [14]	2018	Capacitated	Yes	Lagrangian Relaxation
Basker [21]	2021	Uncapacitated	N.A.	Weiszfeld's Algorithm
Omar and Morales [22]	2021	Capacitated	No	Gaussian mixture models (GMMs) and dispersion reductions

of failed warehouses. Rather, it specifies the number and sizes of warehouses to be built. So, it generates solutions much easier than those in the literature.

Table 1.2: Facility Failure Risk Literature Review Summary

Author(s)	**Year**	**Capacitated / Uncapacitated**	**Multiple Capacity Levels**	**Part Categories**	**Solution Method**
Snyder and Daskin [90]	2005	Uncapacitated	N.A.	No	Same disruption probabilities Lagrangian Relaxation
Cui et al. [32]	2010	Uncapacitated	N.A.	No	Randomly generated facility disruption probabilities Lagrangian Relaxation
Shen et al.[86]	2011	Uncapacitated	N.A.	No	Different disruption probabilities Scenarios method
Aryanezhad et al. [15]	2010	Uncapacitated	N.A.	No	Solution method based on Genetic Algorithm
Peng et al. [82]	2011	Capacitated	No	No	Metaheuristic Algorithm based on genetic algorithm
Li et al. [63]	2013	Uncapacitated	N.A.	No	Different disruption probabilities Lagrangian Relaxation
Azad et al. [18]	2013	Capacitated	No	No	Partial DC disruption Benders' decomposition
Simchi-Levi et al. [87]	2014	N.A.	N.A.	No	DC Time to Recovery
Zhang et al. [104]	2016	Uncapacitated	N.A.	No	Lagrangian Relaxation
Tang et al. [93]	2016	Uncapacitated	N.A.	No	Lagrangian Relaxation Local search
Hoseinpour and Javid [51]	2016	Capacitated	No	No	Location-Capacity decisions and allocations Lagrangian Relaxation
Yun et al. [102]	2017	Uncapacitated	N.A.	No	NIP Linearization techniques
Yu et al.[101]	2017	Uncapacitated	N.A.	No	Lagrangian Relaxation Branch and Cut
Wang et al. [97]	2018	Uncapacitated	N.A.	No	Different disruption probabilities Lagrangian Relaxation
Lu and Cheng [69]	2021	Capacitated	No	No	Column-and-constraint generation algorithm and Benders decomposition
Momayezi et al. [75]	2021	Capacitated	No	No	Metaheuristic Algorithm

From Table 1.3, one can see that there is a scarcity of research on the topic of fortification plans for facilities with multiple capacity levels that deal with multiple part category types. Thus, the contribution of the dissertation is the presentation of this topic and the development of a mathematical model to solve the presented problem related to this topic. A version of the scenarios model that was developed for the risk model was introduced to help in reducing the time required to solve the problem.

Table 1.3: Facility Fortification Literature Review Summary

Author(s)	**Year**	**Capacitated / Uncapacitated**	**Multiple Capacity Levels**	**Part Categories**	**Solution Method**
Church et al. [29]	2004	Uncapacitated	N.A.	No	Implicit Enumeration.
Church et al. [28]	2007	Uncapacitated	N.A.	No	ILP Solver
Aksen et al. [11]	2010	Uncapacitated	N.A.	No	Binary enumeration.
liberatore et al. [65]	2012	Capacitated	No	No	Tree-search procedure
Lasada et al. [68]	2012	Uncapacitated	N.A.	No	Benders decomposition, Super Valid Inequalities.
Li et al. [63]	2013	Uncapacitated	N.A.	No	Lagrangian Relaxation
Medal et al. [72]	2014	Uncapacitated	N.A.	No	Binary search
Namdar et al. [78]	2016	Capacitated	No	No	Solver
Mahmoodjanloo et al. [70]	2016	Capacitated	No	No	Hubrid Metaheuristic.
Akbari-Jafarabadi et al. [10]	2017	Capacitated	No	No	Explicitly enumeration, Metaheuristic Algorithm.
khanduzi et al. [55]	2018	Capacitated	No	No	PSO TLBO
Dey and Jenamani [36]	2019	Capacitated	No	No	Implicitly Enumeration Algorithm
Alikhani et al. [12]	2021	Capacitated	No	No	Two stage stochastic programming (TSSP)
Cheng et al. [24]	2021	Uncapacitated	N.A.	No	C&CG method

The use of specialized warehouses is another contribution of this research. Specialized warehouses are those that can serve some (but not all) product categories. This topic was covered in cases with and without facility disruptions.

1.6 Problem Statement

This study covered the topic of the facility location problem under the risk of failure with the consideration of multiple capacity levels, different product categories and specialized warehouses. The study also considered the assignment of branches to warehouses which make the problem a two-echelon supply chain problem. Each branch is assigned to one warehouse to supply its demand for a certain product category. The case of specialized warehouses is considered in this study. By specialized warehouse we mean that there are some warehouses that cannot supply all types of product categories. When risk is considered, each branch was assigned to a primary and a secondary warehouse. The secondary warehouse supply the expected demand of the branch in case of the failure of the primary warehouse. Finally, facility fortification was considered in this dissertation. Fortification costs were calculated as a percentage of the fixed costs.

The dissertation presented three sets of mathematical models for facility location that can help in determining the locations and sizes of new facilities and how to assign branches to those facilities. In this dissertation, a scenarios model was introduced that helped in solving the presented problems. The scenarios model was unique because it is based on the solution to scenario subproblems that are more easily, i.e. more quickly, solved.

1.7 Research Objective

The objective of this dissertation was to develop a supply chain network design model for cases with and without supply chain risk, represented by warehouse failure. Some methods were presented and used to get solutions to the mathematical models. After all, the developed mathematical models were applied to a real case study for a Canadian company and a created European example. This work helped the Canadian company determine where to locate its warehouses and what branches to assign to them. In the second part of this dissertation, the potential warehouses were assumed to have some disruption probabilities, and the problem was solved by assigning two warehouses to each branch, a primary and a backup warehouse. Finally, warehouses were assumed to be possibly fortified, and then the case study and the created example were solved under this assumption.

1.8 Research Scope

The main question that was studied in this research: Are there mathematical models that can help us to answer the following:

1. What are the optimal warehouse locations?

2. What are the optimal warehouse sizes?

3. What branches are served by each warehouse?

4. What are the answers of 1-3 under the risk that a warehouse will fail?

5. What are the answers of 1-4 under the warehouse fortification process?

1.9 Methodology and Tools

Mixed integer non linear programming (MINLP) models were developed to model the CFLP with and without the risk of warehouse failure. The developed mathematical models are based on a realistic scenario which is a Canadian case study and a created example that were introduced later in this dissertation. The problems created out of such models are NP-hard problems. By NP-hard problems we mean a class of problems that are at least as hard as the hardest problems in NP, where NP problem, stands for non-deterministic polynomial problem, is the class of problems which a given yes-solution can be verified as a solution in a polynomial time. A software package, CPLEX, was used to solve the problems. In some cases of the inability to solve the presented problems, some linearization techniques, relaxation methods, and a scenario algorithm were developed and applied.

1.10 IDEF0

IDEF0 is a function modeling method used to model the decision, actions, and activities of an organization or a system. It helps in organizing the analysis of a system and establishing a good communication between the analyst and customer. The IDEF0 for our problems can be found in Figure 1.4. From Figure 1.4, the input for our problems is the set of all parameters that were presented in the case study mathematical model section. The main objective of these problems is to minimize the fixed and variable costs in addition to the transportation costs. The objective function along with the model constraints are the control of the problem. CPLEX software package was used to solve these problems that represent the mechanism in the IDEF0. Finally, the outputs are the sizes and locations of the selected warehouses by the software, the assignment of warehouses to branches, and warehouses selected to be fortified.

1.11 Document Organization

This Dissertation is organized as follows: Chapter 1 has the introduction that contains some basic concepts needed in this dissertation followed by the literature review. In Chapter 2, the Capacitated Facility Location Problem (CFLP) without the existence of warehouses failure risk was studied. Chapter 3 covered the problem of CFLP with the risk of facility failure and presented the needed

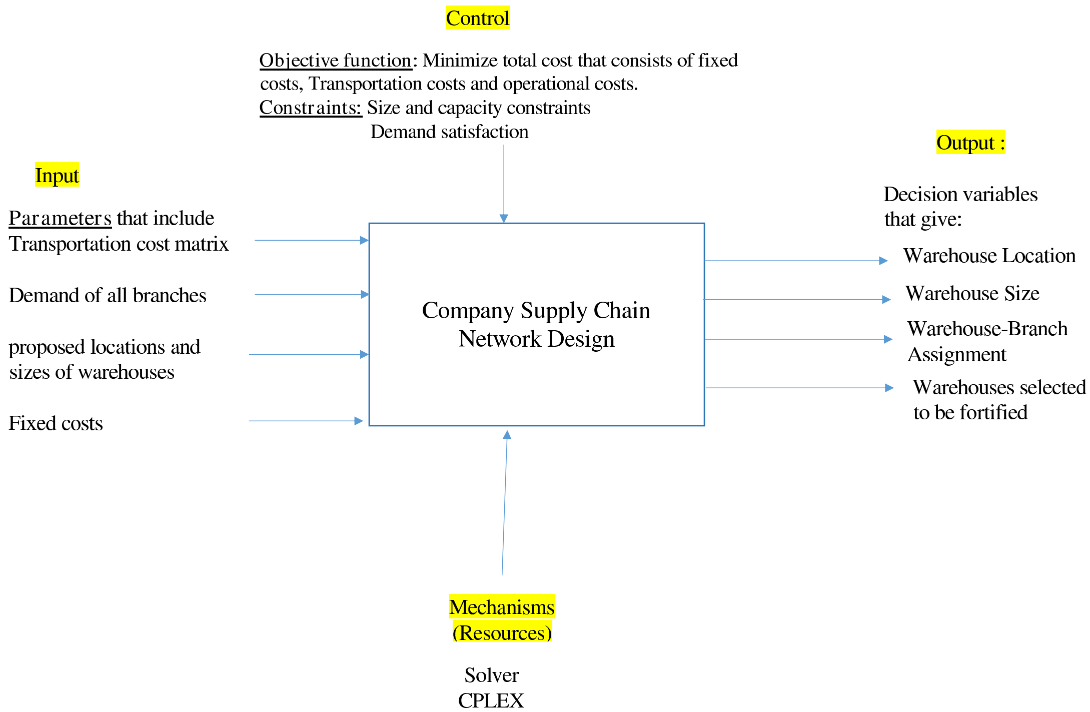

Figure 1.4: IDEF0

mathematical model to address it. In chapter 4, a mathematical model, for the case of fortifying warehouses so that they become non disrupted, was created. Chapter 5 presents the final conclusion and future work. In chapters 2 - 4, some linearization techniques, relaxation methods, and a solution algorithm were developed to help in solving the presented problems. Further, a Canadian case study and a created European example were used to the test the presented mathematical models and the solution methods.

Chapter 2

Capacitated Facility Location Problem without Risk

2.1 Introduction

This chapter is about supply chain network design. It presents a quadratic binary variable mathematical model for a Capacitated Facility Location Problem (CFLP). The model allows for multiple product categories, for pre-selection of warehouses to be built, and for warehouse specialization. The pre-selection allows us to accommodate existing warehouses. The specialization constraint allows us to accommodate the situation in which some warehouses may not be able to handle all of the product categories. A linearization method and a relaxation method were used to reduce the needed computational time for the presented mathematical model. The mathematical model with the linearization and relaxation techniques were applied to a case study based on the network of a Canadian company and a created example of some European cities. CPLEX solver was used to solve both examples.

2.2 Optimization Model

A Mixed Integer Quadratic Optimization model for the network design of a two echelon supply chain that consists of warehouses and branches is presented in this section. The solution of the optimization model determines the locations and sizes of the warehouses to be built. It will also help in assigning built warehouses to branches.

The set of branches is indexed by $b \in B = \{1, 2, ..., m-1, m\}$ and the set of potential warehouse locations is indexed by $w \in W = \{1, 2, ..., n-1, n\}$. The set of warehouse sizes is indexed by $s \in S = \{1, 2, ..., q-1, q\}$, and a warehouse with size index s has a footprint of A^s square feet. Our model allows for a large variety of items. Typically, the items can be categorized by product categories. Product categories are used to group products with similar features, such as weight, size, and usage. The set of product categories is indexed by $j \in J = \{1, 2, ..., g-1, g\}$. When we say product j we are referring to all products in category j, and when there is a single product category, we drop the index j.

Sometimes, for simplicity, we refer to branch b, rather than saying the branch indexed by b. Likewise, when we talk about warehouse w, size s, and category j, we mean the warehouse at the location indexed by w, with size indexed by s, and the product category indexed by j, respectively.

We define x_w^s as the binary variable equal to 1 if and only if a warehouse of size s is built at location w. For ease of presentation, we will often use (w, s) to denote warehouse w with size s. y_{wbj} is a binary variable equal to 1 if and only if a branch b is supplied with its demand of items from product category j by the warehouse at location w.

To ensure that only a single size is selected for each built warehouse, we add the constraints

$$\sum_s x_w^s \leq 1, \ \forall w \in W. \tag{2.2.1}$$

To ensure that all units from product category j demanded at branch b are supplied by a single warehouse, we add the constraints

$$\sum_w y_{wbj} = 1, \ \forall b \in B, \ \forall j \in J. \tag{2.2.2}$$

Other constraints, which might be termed management constraints, are

$$\sum_{s,w} x_w^s \leq U, \tag{2.2.3}$$

to ensure that no more than U warehouses are built. (We use the convention that the summation is over all values of all variables under the $\sum$.)

To account for existing, built warehouses, and to account for warehouses preselected to be built at a specific size, we set

$$x_w^s = 1, \forall (w, s) \in E, \tag{2.2.4}$$

where $E \subset W \times S$ is the set of existing warehouses. For example, if there was an existing warehouse at location 2 with size index 3, we would have $x_2^3 = 1$ and $(2, 3) \in E$. If there are no such warehouses, then $E = \emptyset$ and constraint (2.2.4) is removed. Also, if there is no limit on the number of warehouses to be built, constraint (2.2.3) is removed.

The thesis considers specialized warehouses. For example, suppose that warehouse 2 cannot handle product type 3. Then, we would want $y_{2b3} = 0$ for all values of b. The set of restricted allocation is enforced by the constraints

$$y_{wbj} = 0, \forall\, b \in B \quad \text{and} \quad \forall (w, j) \in \mathscr{R}, \tag{2.2.5}$$

where $\mathscr{R} \subset W \times J$ is the set of restricted assignments. The set $\mathscr{R}$ has a potential influence on the the upper limit U because the warehouses selected must be able to cover demand for all product types. For example, if the product categories $j = 1, 2, 3, 4,$ and 5 can only be handled by warehouses $w = 1, 2, 3, 4,$ and 5, respectively, then we would need to have $U \geq 5$. Let $\mathbb{R}$ be the $|J| \times |W|$ binary matrix with the (j, w)-th entry equal to zero if and only if $(w, j) \in \mathscr{R}$. Let u^* be the optimal objective function value of the set-covering problem

$$\min\{\, e^\top u \,|\, \mathbb{R}u \geq e \,\},$$

where e is a column vector of ones of length $|J|$ and $e^\top$ is a row vector of ones of length $|W|$. We must have

$$U \geq e^\top u^*.$$

Note that if there is a warehouse that can handle all product categories, then $e^\top u^* = 1$.

Constraints (2.2.5) are removed in case of not having specialized warehouses.

In this dissertation, the case study and the created example involve many and varied items to be handled. So, let K be a common volume unit used to measure the demand. Let d_{bj} be the demand in K from product category j at branch b. $\boldsymbol{V}^s$ is defined as the volume of storage space, expressed in K, available in a warehouse with a size index s. To ensure that the storage space, expressed in K, required to store the demanded items from all product categories from all branches supplied by warehouse w is less than or equal to the available storage space of that warehouse, we have the constraints

$$\sum_{j,b} d_{bj} y_{wbj} \leq \sum_{s} \boldsymbol{V}^s x_w^s, \ \forall w \in W. \tag{2.2.6}$$

Let f_w be the cost per square foot, in dollars, during the planning horizon, where the number of square feet is given by A^s, for a warehouse of size s built at location w; and let ℓ_w be the cost per square foot, in dollars, during the planning horizon, for industrial land at location w. The fixed warehouse cost in dollars, during the planning horizon, is

$$C_F(x) = \sum_w (f_w + l_w) \sum_s A^s x^s_w. \tag{2.2.7}$$

The operational cost such as labor, used machines and equipment, and utility costs, are related to the warehouse's activity level. Let ν^s_j represents the operational cost required to handle one K of items from product category j at a warehouse with a size index s. As ν^s_j depends on s, it can capture economies of scale and the technology level. One can assume that large warehouses use more advanced equipment and machines to handle items, so they tend to have a lower operational cost compared to the small warehouses. The total operational warehouse cost in dollars, during the planning horizon, is

$$C_O(x, y) = \sum_{j,s} \nu^s_j \left(\sum_b d_{bj} \sum_w x^s_w \, y_{wbj} \right). \tag{2.2.8}$$

Let τ_{wbj} be the dollar cost of shipping one K of items from product category j from the warehouse at location w to the branch b. Thus, the total transportation cost in dollars, during the planning horizon, is

$$C_T(y) = \sum_{j,b} d_{bj} \sum_w \tau_{wbj} \, y_{wbj}. \tag{2.2.9}$$

The complete cost function to be minimized is

$$C(x, y) = C_F(x) + C_O(x, y) + C_T(y).$$

Putting everything together, the Mixed Integer Quadratic Optimization problem is to

$$\begin{aligned}
\boldsymbol{M}\textbf{:}\ \text{Minimize}\ \ & C(x, y) = C_F(x) + C_O(x, y) + C_T(y) \\
\text{Subject to}\quad & (2.2.1) - (2.2.6), \\
& x_w^s \in \{0, 1\},\ \forall s \in S, \forall w \in W,\ \text{and} \\
& y_{wbj} \in \{0, 1\},\ \forall w \in W, \forall b \in B, \forall j \in J.
\end{aligned}$$

To eliminate the binary quadratic terms $x_w^s\ y_{wbj}$ in (2.2.8), one can use the standard substitution [43]

$$z_{wbj}^s = x_w^s\ y_{wbj}, \tag{2.2.10}$$

insisting that $\forall s \in S,\ \forall w \in W,\ \forall b \in B,$ and $\forall j \in J$,

$$z_{wbj}^s \leq x_w^s, \tag{2.2.11}$$

$$z_{wbj}^s \leq y_{wbj}, \tag{2.2.12}$$

$$z_{wbj}^s \geq x_w^s + y_{wbj} - 1,\ \text{and} \tag{2.2.13}$$

$$z_{wbj}^s \geq 0. \tag{2.2.14}$$

Hence, z_{wbj}^s is a continuous variable that is, because of (2.2.11)-(2.2.14), equal to 1 if and only if a warehouse of size s is built at location w and supplies the demand from branch b of items from category j and is zero otherwise. So, the total operational cost is

$$C_O(z) = \sum_{j,s} \nu_j^s \left(\sum_b d_{bj} \sum_w z_{wbj}^s \right). \qquad (2.2.15)$$

Putting everything together, the Mixed Integer Linear Optimization problem is to

$$\begin{aligned} \boldsymbol{LM}\text{:}\ \text{Minimize}\quad & C(x,y,z) = C_F(x) + C_O(z) + C_T(y) \\ \text{Subject to}\quad & (2.2.1)-(2.2.6), (2.2.11)-(2.2.14) \\ & x_w^s \in \{0,1\},\ \forall s \in S, \forall w \in W,\ \text{and} \\ & y_{wbj} \in \{0,1\},\ \forall w \in W, \forall b \in B, \forall j \in J. \end{aligned}$$

If there is a single product category, the subscripts j are removed from models ***M*** and ***LM***. Models ***RM*** and ***RLM***, derived from models ***M*** and ***LM***, respectively, are derived by relaxing the y_{wbj} variables, that is, replacing $y_{wbj} \in \{0,1\}$ with $0 \leq y_{wbj} \leq 1$.

Lemma 2.2.1, shows that it is enough to impose the constraint $y_{wbj} \geq 0$ $\forall w \in W$, $b \in B$, and $j \in J$ in models ***RM*** and ***RLM*** to get $y_{wbj} \leq 1$. Lemma 2.2.2 states that imposing the constraints $y_{wbj} \geq 0$ does not affect the substitution $z_{wbj}^s = y_{wbj}\ x_w^s$.

Lemma 2.2.1. In models ***RM*** and ***RLM***, if $y_{wbj} \in \{0,1\}$ is replaced with $y_{wbj} \geq 0$, then $y_{wbj} \leq 1$ is implicit.

Proof. Let y_{wbj} be such that $y_{wbj} \geq 0$. Using constraints (2.2.2), which indicate that the summation for the y_{wbj} variables over all warehouses should be equal to 1, it follows that $y_{wbj} \leq 1$. $\square$

Lemma 2.2.2. The substitution $z^s_{wbj} = x^s_w \; y_{wbj}$ is valid for model $\boldsymbol{RLM}$.

Proof. In model $\boldsymbol{RLM}$, we have that $y_{wbj} \geq 0$. From lemma 2.2.1, it follows that $0 \leq y_{wbj} \leq 1$. If $x^s_w = 0$, then (2.2.11)-(2.2.14) imply that $z^s_{wbj} \leq 0$, $z^s_{wbj} \leq y_{wbj}$, $z^s_{wbj} \geq y_{wbj} - 1$, and $z^s_{wbj} \geq 0$. Thus, $z^s_{wbj} = 0$. In the same manner, if $x^s_w = 1$, then $z^s_{wbj} \leq 1$, $z^s_{wbj} \leq y_{wbj}$, $z^s_{wbj} \geq y_{wbj}$, and $z^s_{wbj} \geq 0$. Thus, $z^s_{wbj} = y_{wbj}$. □

Theorem 2.2.1 explains the consequences of relaxing the y_{wbj} variables in model $\boldsymbol{LM}$. The same theorem can be used for model $\boldsymbol{M}$. Figure 2.1 explains the theorem with the assumption, for simplicity, that we only have one product category. Assume that, warehouse w_1 is the warehouse with the minimum operating and transportation costs to satisfy the demand of branch b. However, w_1 is unable to satisfy the whole demand of branch b as a result of its limited capacity. Then warehouse w_2 can take part of the remaining demand but again it is unable to satisfy the remaining whole demand. So, w_3 can serve branch b with its remaining demand. Thus, the fractional assignment appears if a certain warehouse is unable to satisfy the whole demand of the branch that is supposed to serve. So it serves part of the demand and the rest will be satisfied by one or more warehouses. Theorem 2.2.1 states that we obtain fractional assignments in the case that a warehouse w^* assigned to a branch b^* and cannot accommodate its full demand. The rest of the demand of b^* will be satisfied by one or more warehouses that are not at full ca-

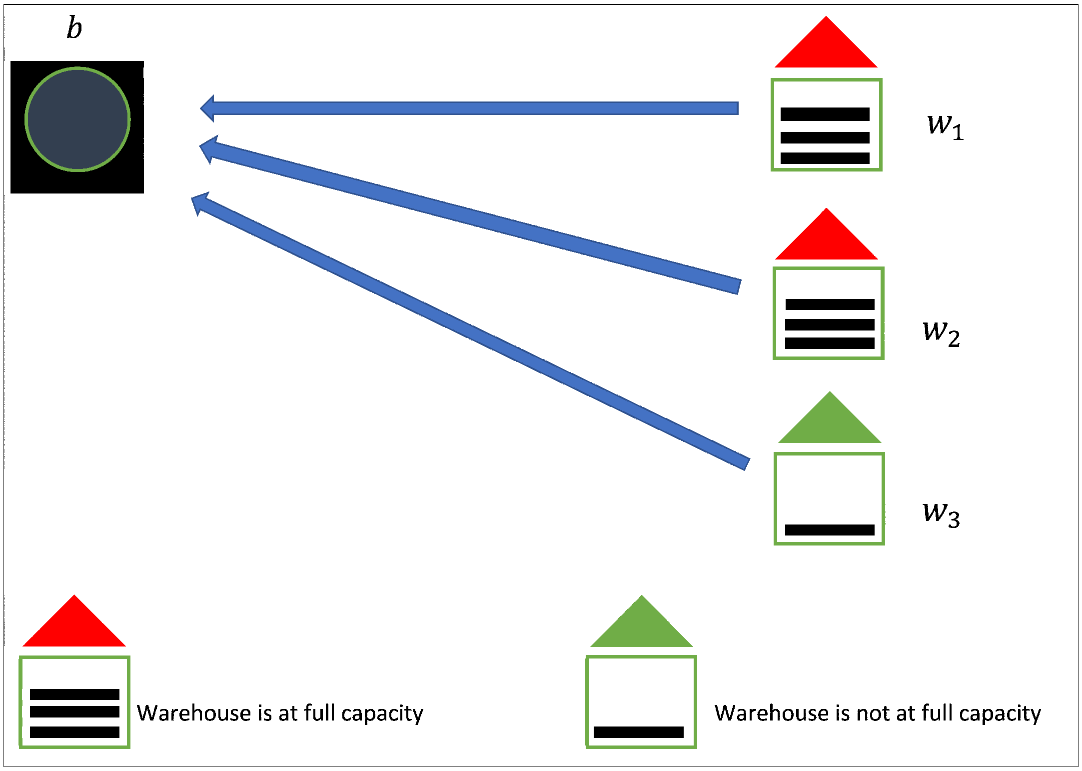

Figure 2.1: Illustration for Theorem 2.2.1

pacity before serving the demand, or part of it, for branch b^*, where at most one of which will not be operating at full capacity after such assignment. It is assumed that no two warehouses are identical in transportation and operational costs when serving the same amount of demand from branch b^*. This assumption is called as $no-ties$ Assumption 1. We will see, after stating the proof of Theorem 2.2.1, the reason to have Assumption 1.

From lemma 2.2.1, implicit in $\boldsymbol{RLM}$ are the constraints $y_{wbj} \leq 1$. So, the optimal solution to $\boldsymbol{RLM}$, for all $w \in W$, $b \in B$, and $j \in J$, either $y_{wbj} = 0$, $y_{wbj} = 1$, or $0 < y_{wbj} < 1$, that is, y_{wbj} is fractional. Theorem 2.2.1 considers the case when y_{wbj} is fractional.

Assumption 1. That for any branch b and product j the sum of the per unit transportation cost to a warehouse and the per unit operational cost at that warehouse is unique. That is, for warehouses w_1 and w_2, with sizes s_1 and s_2, respectively,

$$\nu_j^{s_1} + \tau_{w_1bj} \neq \nu_j^{s_2} + \tau_{w_2bj}.$$

Theorem 2.2.1. *Let (x^*, y^*, z^*) be an optimal solution to **RLM** and suppose that the no-ties assumption, Assumption 1, is satisfied. If there exist ω, β, and ϕ such that $0 < (y_{\omega\beta\phi})^* < 1$ and if*

$$W^* = \{\, w \,|\, 0 < (y_{w\beta\phi})^* < 1\} \tag{2.2.16}$$

then there exists at most one warehouse $w \in W^*$ that is not running at full capacity, that is,

$$\sum_{b,j} d_{bj}(y_{wbj})^* < \boldsymbol{V}^s, \tag{2.2.17}$$

where s is the size of built warehouse w.

Proof. If, for all $w \in W^*$, we have

$$\sum_{b,j} d_{bj}\,(y_{wbj})^* = \boldsymbol{V}^s,$$

we are done. Suppose that there exist built warehouses $w_1 \in W^*$

and $w_2 \in W^*$ with sizes s_1 and s_2, respectively, such that

$$\sum_{b,j} d_{bj}\,(y_{w_1bj})^* < V^{s_1} \quad \text{and} \quad \sum_{b,j} d_{bj}\,(y_{w_2bj})^* < V^{s_2}.$$

We will show that this contradicts optimality. With Assumption 1 we can assume, without loss of generality, that

$$\nu_\phi^{s_1} + \tau_{w_1\beta\phi} < \nu_\phi^{s_2} + \tau_{w_2\beta\phi}. \tag{2.2.18}$$

Let δ be such that

$$\sum_{b\neq\beta, j\neq\phi} d_{bj}\,(y_{w_1bj})^* + d_{\beta\phi}((y_{w_1\beta\phi})^* + \delta) \leq V^{s_1},$$

$$\sum_{b\neq\beta, j\neq\phi} d_{bj}\,(y_{w_2bj})^* + d_{\beta\phi}((y_{w_2\beta\phi})^* - \delta) \leq V^{s_2},$$

$$0 \leq (y_{w_1\beta\phi})^* + \delta \leq 1, \quad \text{and}$$

$$0 \leq (y_{w_2\beta\phi})^* - \delta \leq 1.$$

Thus, the solution given by (x^*, y^*, z^*) with $(y_{w_1\beta\phi})^*$ and $(y_{w_2\beta\phi})^*$ replaced with $(y_{w_1\beta\phi})^* + \delta$ and $(y_{w_2\beta\phi})^* - \delta$, respectively, is feasible. Denote this solution by $\overline{(x^*, y^*, z^*)}$. In this solution, we transferred $\delta d_{\beta\phi}$ of the demand for product ϕ from branch β from warehouse w_2 to w_1. From (2.2.18) it follows that

$$C\overline{(x^*, y^*, z^*)} < C(x^*, y^*, z^*),$$

which contradicts optimality. $\square$

In Theorem 2.2.1, we show the importance of Assumption 1. In Figure 2.2, we have branch b with demand $d_{bj} = 100\ K$ from product category j. Let w_1, w_2, and w_3 be the only available warehouses that can satisfy partial demand of branch b with their remaining capacities of 40 K, 60 K, and 50 K, respectively. Also, consider the following costs $\nu_j^{s_1} + \tau_{w_1bj} = \$8$, $\nu_j^{s_2} + \tau_{w_2bj} = \nu_j^{s_3} + \tau_{w_3bj} = \$10$, which means that Assumption 1 is not satisfied. Because w_1 has the minimum operational and transportation costs, it will serve branch b with 40 K of its demand. The remaining demand of branch b is 60 K, and there are many possibilities for satisfying it, one of which is that 30 K will be served by both w_2 and w_3. In such a case, we have two warehouses instead of one warehouse that are not operating at their full capacities.

2.3 Canadian Case Study

2.3.1 Introduction

This section presents a case study of a Canadian company with an existing network of 158 branches and 2 warehouses that exist across all Canadian provinces with a concentration in Quebec, Ontario, and Alberta. The warehouses handle close to 19,000 different products differing in size, shape, weight and density. The company wanted to build up to three new warehouses to position themselves for success 15 years into the future. The company's suppliers are

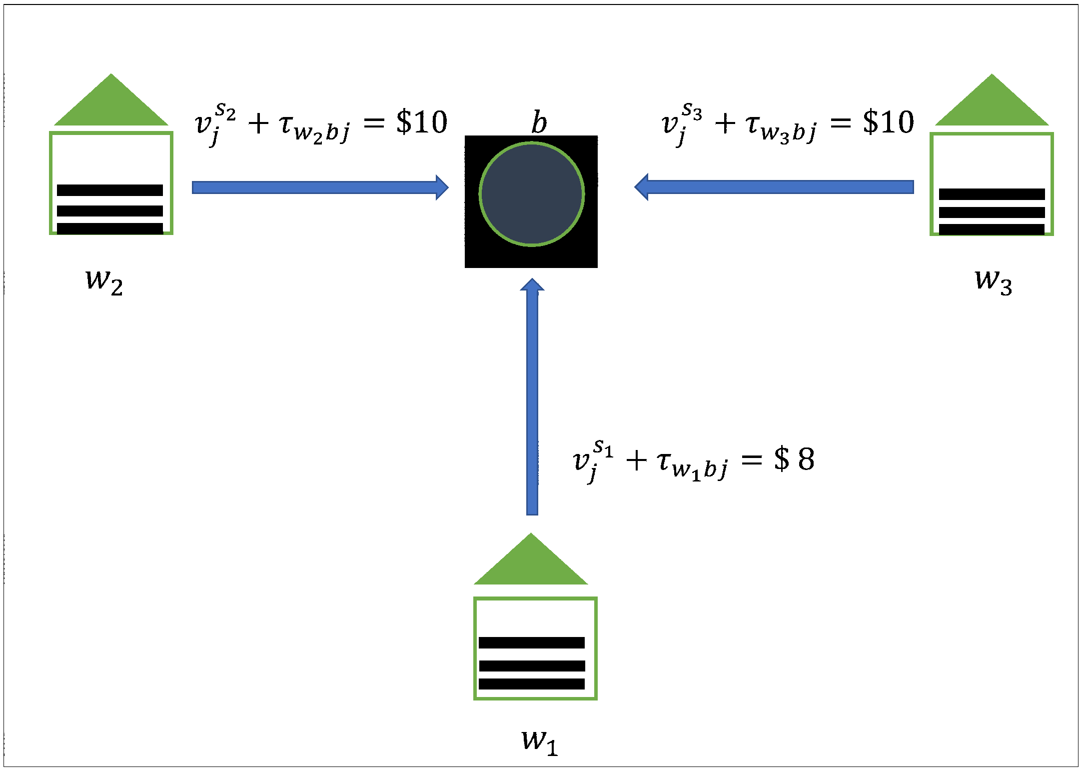

Figure 2.2: The case of dropping Assumption 1

from all over the world. Suppliers ship to the warehouses, and in some cases, directly to the branches and job sites. As the company suggested that we do not consider suppliers, this makes our case study a two-echelon supply chain network that includes warehouses and branches. A total of 32 cities with the largest populations and/or significant geographical locations in Canada were selected to be potential warehouse locations. The demand for each branch was determined from historical data over 45 day periods; the average time an item remained in inventory. Consequently, we adopted a 45 day planning horizon.

In the following sections, the parameters that are used in the mathematical model to solve this case study are presented. Then, the solution steps to the mathematical model are presented, followed by some scenarios and sensitivity analysis.

2.3.2 Parameters for model $\boldsymbol{M}$

As the company has a large product variety it was a challenge to determine a common demand unit. The variety in the parts is handled by adopting a common demand unit for product categories. The common volume unit K equals 1,000 cubic inches in our case study and the company converted all product demand into units of K, effectively reducing the number of parts to one, so the subscript j is removed from the models $\boldsymbol{M}$ and $\boldsymbol{LM}$.

Potential and current warehouses were labeled from 1 to 34. The company suggested three possible sizes for their potential warehouses that are: large (A^3 =250,000 square feet), medium (A^2 =150,000 square feet), and small (A^1 =70,000 square feet). The current warehouses are large and have indices 1 and 2.

We define $\boldsymbol{V}^s$ as the volume of storage space in K available in a warehouse of A^s square feet where one square foot of space can store $3,868.08$ cubic inches (this figure was given by the company). Thus, the three possible warehouse sizes given by the company were converted into the following capacities: small, $\boldsymbol{V}^1 = 270,765.60\ K$; medium, $\boldsymbol{V}^2 = 580,212.00\ K$; and large, $\boldsymbol{V}^3 = 967,020.00\ K$.

Since the company has 158 branches across the Canadian provinces, so the set of branches is indexed by $b \in B = \{1, 2, ..., 158\}$ and the set of current and potential warehouse locations is indexed by $w \in W = \{1, 2, ..., 34\}$.

The upper bound U on the number of warehouses to be built is 5. Also, to force the existence of the warehouses 1 and 2 with their large sizes, we need $x_{w^*}^{s^*} = 1$, where $(w^*, s^*) \in E$ and $E = \{(1, 3), (2, 3)\}$.

At the time of the study, warehouse fabrication costs across Canada were between \$145 and \$165 per square foot. We adopted the midpoint of \$155 as the cost per square foot, regardless of location, the assumption being that there was little variation across the country. We assumed that the cost would be amortized over 15 years, with 3% interest and payments made every 45 days to match the planning horizon period. So, the fabrication cost is $f = \$1.61$ per square foot every 45 days. Unlike fabrication costs, land costs have significant variation with dependence on location and city size. We surveyed seven of the potential warehouse locations to get an estimate of land costs. We then used linear regression (price vs population) to estimate land costs at the remaining warehouse locations. We use l_w to denote the cost in dollars per square foot, amortized at 3% over 15 years payable every 45 days, to purchase industrial land at location w. We found that l_w for all potential warehouses ranges between \$0.13 to \$0.24 per square foot every 45 days.

Moving to the operational costs, let ν^s represents the operational cost required to handle one K at a warehouse with a size index of s. The operational costs given by the company are are $\nu^1 = \$0.33$, $\nu^2 = \$0.27$, and $\nu^3 = \$0.19$.

The transportation cost parameter is τ_{wb}, the cost of shipping one K of product from the warehouse at the location indexed by w to branch b. The cost parameters were determined first by a calculation of the road distances between every warehouse location w and the location of branch b, and then from a company estimate of the cost of shipping one K of product one kilometer. There were three K-kilometer cost constants depending on whether the distance was short, medium, or long. The parameters used in this study are 0.00228 per K-km for short haul distances less than or equal to 300 km, 0.001163 per K-km for medium haul distances between 300 and 900 km, and 0.000697 per K-km for long haul distances of 900 KM or greater.

2.3.3 Solution to models *M* and *LM*

Figure 2.3 shows that the solution to the mathematical models ***M*** and ***LM*** determines the locations and sizes of new warehouses to be built out of the potential warehouses and the assignment of branches to the built warehouses.

Model ***M*** was solved using CPLEX Optimization Studio 12.10.0 running on an Intel $i7$ Asus laptop with 16 GB of RAM and 2.80 GHz processor with four cores. The parameters were stored in

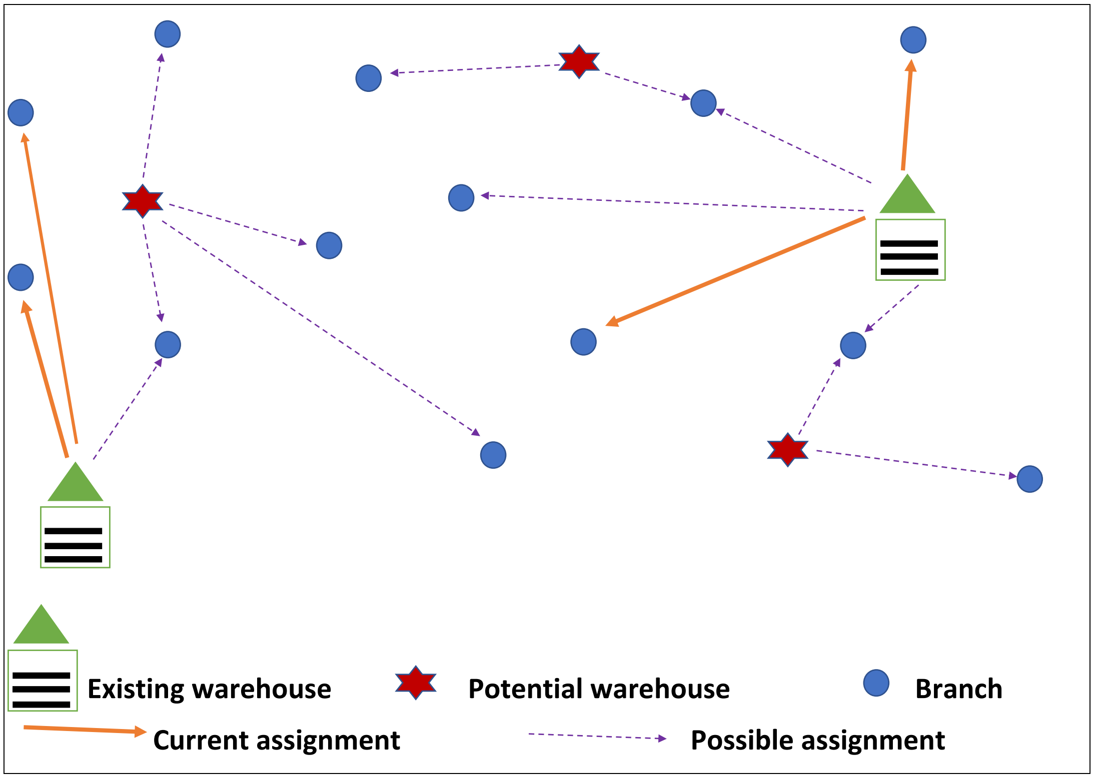

Figure 2.3: Two Echelon Supply Chain Example

an Excel sheet.

Table 2.1 shows that model $\boldsymbol{M}$ has $5,472$ variables where all of them are binary and 229 constraints. CPLEX returned an optimal solution to model $\boldsymbol{M}$ in under two minutes with $176,129$ iterations. After relaxing the y_{wb} variables, CPLEX returned an optimal solution to model $\boldsymbol{RM}$ in less than one minute. Both models gave the same built warehouses.

The solutions to models $\boldsymbol{M}$ and $\boldsymbol{RM}$ show the need to have three new warehouses; two were medium sized, and the other was large. The medium warehouses are in Quebec and British Columbia, and the large warehouse is in Alberta. Not surprisingly, the warehouses are located near the population centers. The exact locations are withheld because of a non-disclosure agreement.

Table 2.1: Numerical results for the Canadian case study using CPLEX (Quadratic Models)

Model	$\boldsymbol{M}$	$\boldsymbol{RM}$
Total Variables	5,472	5,472
Binary Variables	5,472	100
Constraints	229	5,601
Iterations	176,129	24,140
Time (Hr:Min:Sec)	00:01:52	00:00:43
Best Objective $\times 10^6$	3.9138	3.9057
Objective Bound $\times 10^6$	3.9138	3.89
Built Warehouses (w, s)	(1, 3), (2, 3) (4, 2), (28, 3) (33, 2)	(1, 3), (2, 3) (4, 2), (28, 3) (33, 2)
Status	Optimal	Optimal

Now it is the time to solve the linearized models. Model $\boldsymbol{LM}$ was solved using CPLEX on the same laptop mentioned above. There are $21,588$ total variables where $5,472$ of them are binary. Further, there are $64,693$ constraints. Table 2.2 shows that CPLEX took almost two and half minutes with $184,000$ iterations to solve model $\boldsymbol{LM}$. Then the y_{wb} variables were relaxed and we still have the same number of total variables which is $21,588$, but the number of binary variables dropped to 100 variables. CPLEX took one minute and 12 seconds to solve model $\boldsymbol{RLM}$ with around $100,000$ iterations.

The objective function has three components and, at optimality of models $\boldsymbol{M}$ and $\boldsymbol{LM}$, the costs components are

$$C_F = 1.8518 \times 10^6, \quad C_O = 0.8161 \times 10^6, \quad \text{and } C_T = 1.2459 \times 10^6.$$

The breakdown of C_F into its components is 1.6905×10^6 for building

Table 2.2: Numerical results for the Canadian case study using CPLEX (Linear Models)

Model	***LM***	***RLM***
Total Variables	21,588	21,588
Binary Variables	5,472	100
Constraints	64,493	69,865
Iterations	184,097	100,156
Time (Hr:Min:Sec)	00:02:24	00:01:12
Best Objective $\times 10^6$	3.9138	3.9057
Objective Bound $\times 10^6$	3.9127	3.8699
Built Warehouses (w, s)	(1, 3), (2, 3) (4, 2), (28, 3) (33, 2)	(1, 3), (2, 3) (4, 2), (28, 3) (33, 2)
Status	Optimal	Optimal

costs and 0.1613×10^6 for land costs. The building costs dominate, followed by transportation costs, then operational costs. As all costs are estimates, the sensitivity of the final solutions on the costs estimates is explored in the next section.

2.3.4 Analysis

In this section, some sensitivity analysis were studied and conducted for the Canadian case study. Only CPLEX solver was used in this section.

2.3.4.1 Limit on the Number of Warehouses

Management constraint (2.2.3) in model $\boldsymbol{M}$ limits the number of warehouses to five. Since the total demand from the system is about 3,881,722 K, and since the largest warehouse has a capacity of 967,020 K, it takes at least five warehouses (four large and one small) to satisfy the whole demand. The optimal solution has 3 large and 2 medium warehouses. Table 2.3 and Figure 2.4 show the impact on the solutions as U increases. As U increases, the optimal number of warehouses also increases until $U = 7$. For $U \geq 7$, the optimal solution has two large warehouses at locations 1 and 2, the existing warehouses, medium warehouses at locations 4 and 28, and small warehouses at locations 22, 30, and 33. For $U \geq 7$, the savings in transportation costs, resulting from more warehouses, is smaller than the cost of building and operating the new warehouses.

For the coming sections, we set $U = +\infty$ to better understand the sensitivity of the optimal solution to changes in some key parameters. In all cases, the two existing warehouses are included.

Table 2.3: Impact of U on Total Cost ($M) and the Location and Size of Warehouses.

U/Number Built	$C(x,y)$	Built Warehouses (w,s)
5/5	3.9138	(1,3),(2,3),(4,2), (28,3), (33,2)
6/6	3.7905	(1,3),(2,3),(4,2),(22,1), (28,2), (33,2)
7/7	3.6769	(1,3),(2,3),(4,2),(22,1), (28,2), (30,1),(33,1)
8/7	3.6769	(1,3),(2,3),(4,2),(22,1), (28,2), (30,1),(33,1)

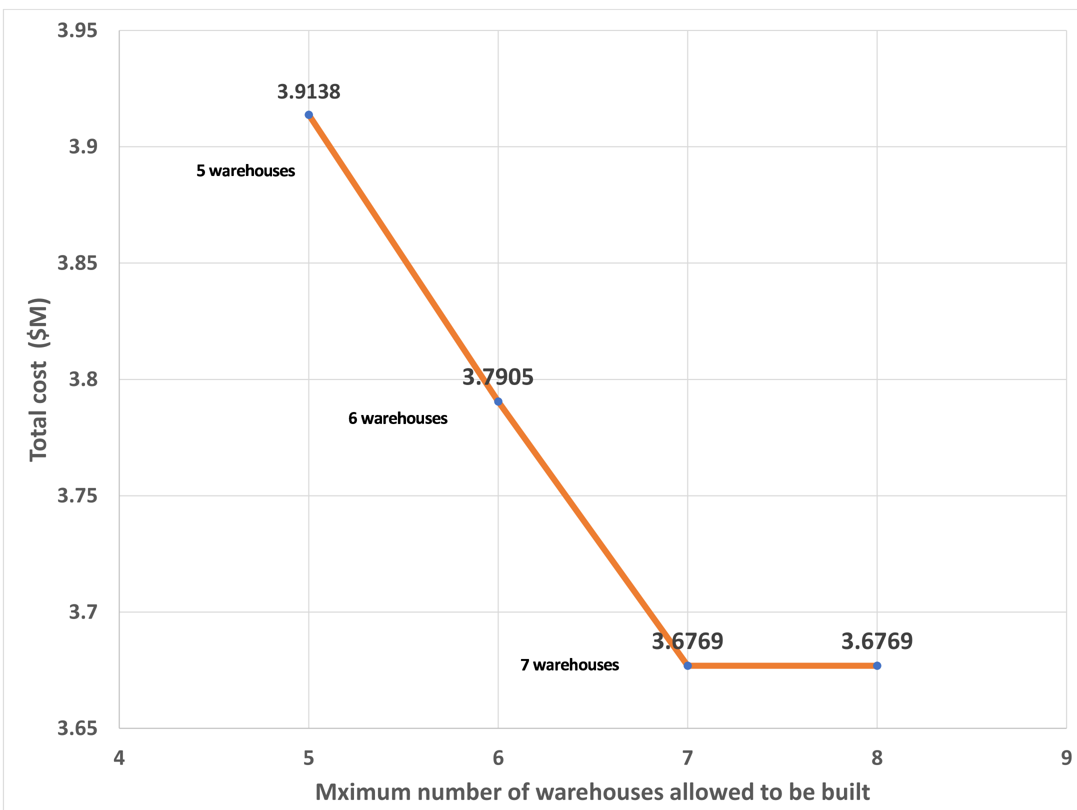

Figure 2.4: Impact of U on Total Cost ($M).

2.3.4.2 Transportation Costs

The results in Table 2.4 and Figure 2.5 demonstrate the sensitivity of the optimal solution as a function of the transportation cost factors τ_{wb}. The first column of the table gives the value of α, and model $\boldsymbol{M}$ is solved with cost factors $\alpha\tau_{wb}$. As expected, an increase to the transportation cost multiplier α increases the total costs. Further, Table 2.4 and Figure 2.6 show the impact of transportation cost on the number of built warehouses. In all cases, the only large warehouses are the two existing warehouses. Table 2.4 shows as α increases the number of medium warehouses decrease and the number of small warehouses increase. In Figure 2.6, we see that the solution can be very sensitive on the cost multipliers. For example, when we move from 1 to 1.1, the optimal number of warehouses went from 7 to 9. So, we suggest that managers use this model to do some sensitivity around price changes. In a country like Canada, it is not surprising that transportation costs would dominate building costs leading to more, and smaller, warehouses. In fact, the client company has branches that act as inventory hubs, that is, like very small warehouses. This might change if supplier costs were included as the cost of transportation from the supplier to many small warehouses would likely impact the number of warehouses built.

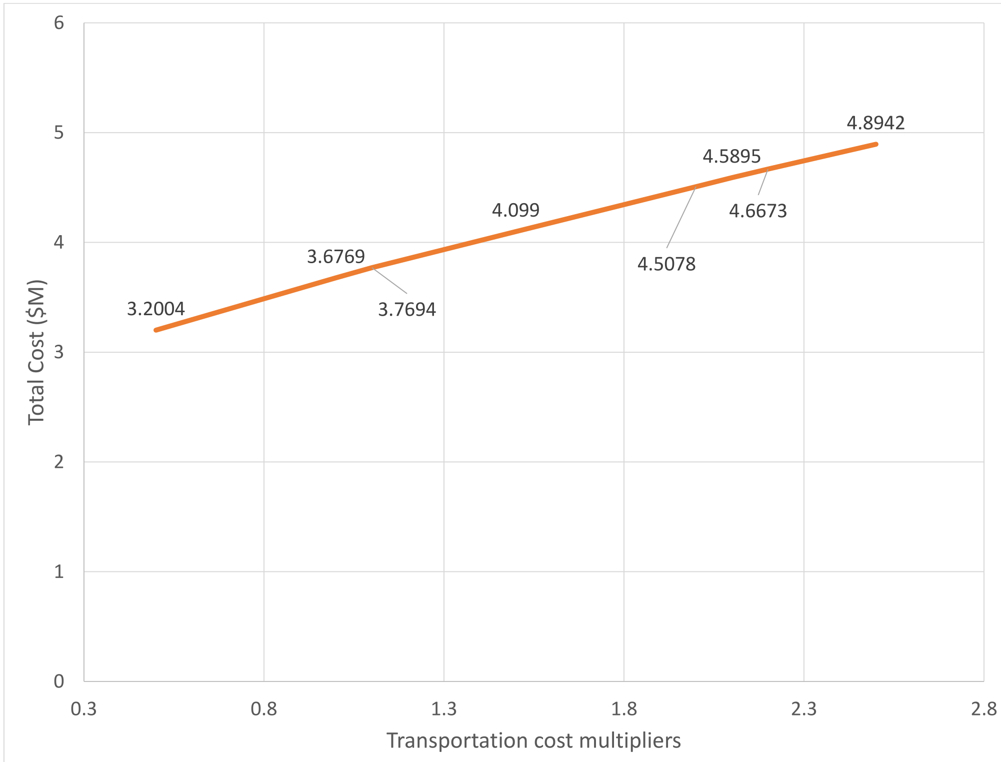

Figure 2.5: Impact of Transportation cost on Total cost ($M)

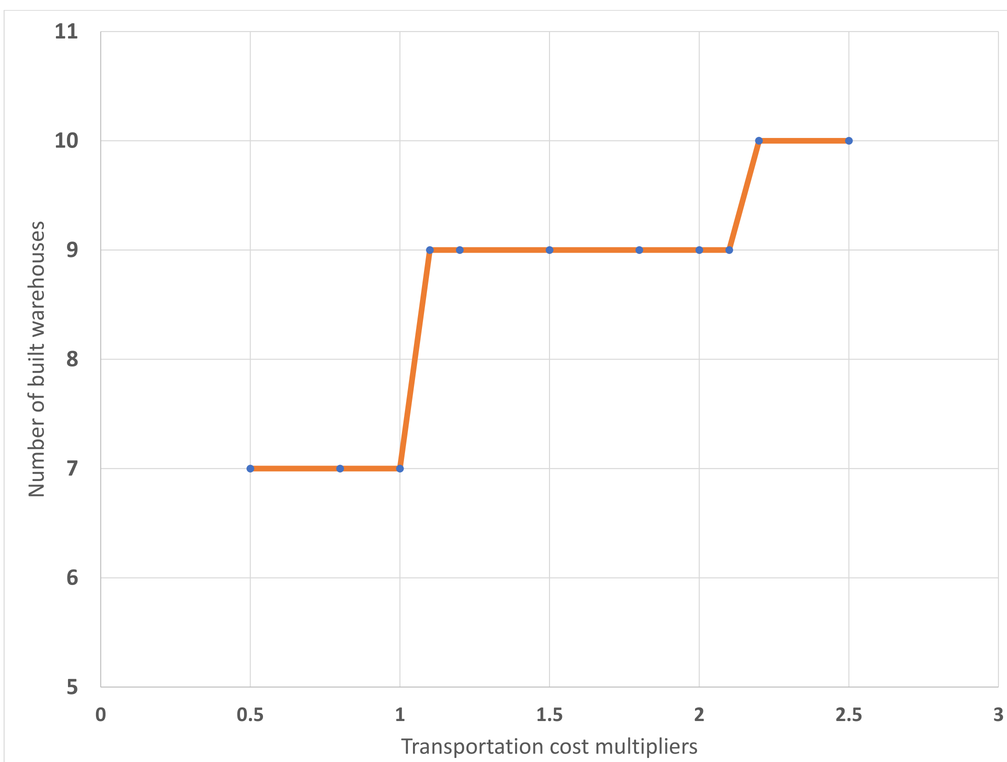

Figure 2.6: Impact of Transportation cost on the number of built warehouses

Table 2.4: Impact of Transportation Cost on cost ($M) and the Number of Warehouses.

α	$C(x,y)$	Large	Medium	Small	Total
0.5	3.2004	2	2	3	7
1.0	3.6769	2	2	3	7
1.1	3.7694	2	1	6	9
1.5	4.0990	2	1	6	9
2.0	4.5078	2	1	6	9
2.1	4.5895	2	1	6	9
2.2	4.6673	2	1	7	10
2.5	4.8942	2	1	7	10

2.3.4.3 Demand

This section explores the sensitivity of the cost and of the number and size of warehouses with changes to demand. At each branch, the existing demand was multiplied by β. The first column of Table 2.5 gives the values of β considered. The remaining columns give the optimal value of total cost, the number of large, medium and small warehouses to be built, and the total number of warehouses to be built. As expected, and can be found in Table 2.5 and Figures 2.7 and 2.8, cost increases with demand; but, perhaps unexpected, is the sensitivity of the number and size of warehouses built on changes in demand. This is, perhaps a consequence of the fact that demand influences both transportation and operational costs as well as the number of warehouses required.

Table 2.5: Impact of Demand on Total Cost ($M) and the Number of Warehouses.

β	$C(x, y)$	Large	Medium	Small	Total
0.9	3.3497	2	0	6	8
1.0	3.6769	2	2	3	7
1.01	3.7129	2	1	6	9
1.05	3.8482	2	1	6	9
1.06	3.8805	2	1	7	10
1.1	3.9803	2	1	7	10

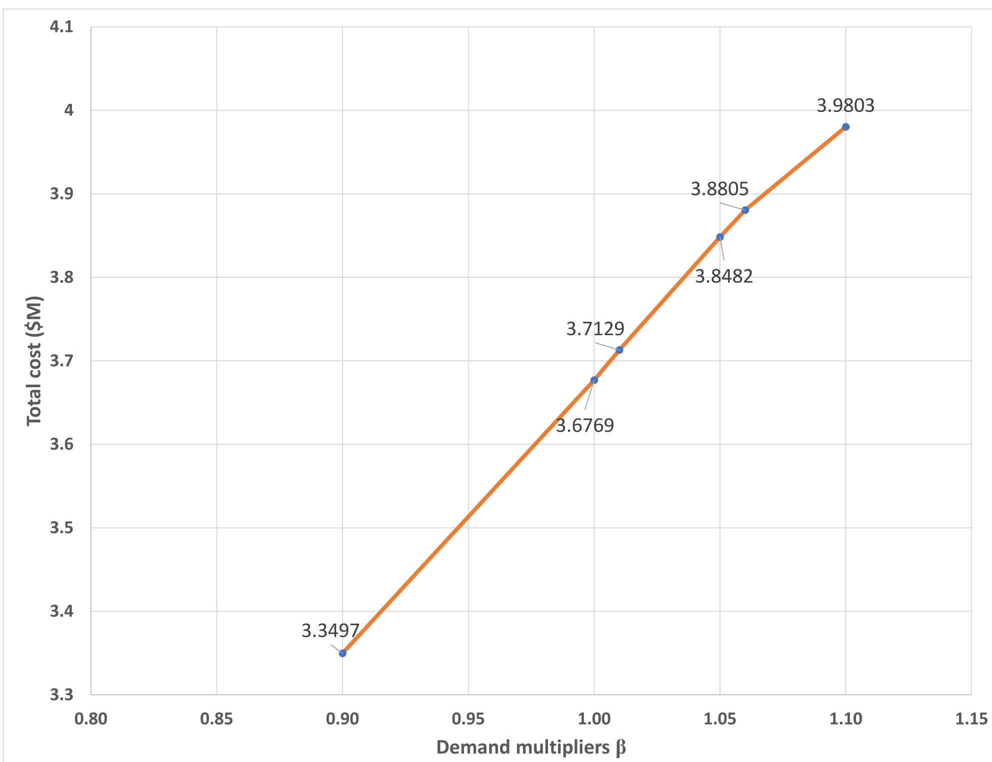

Figure 2.7: Impact of Demand on the Total Cost ($M).

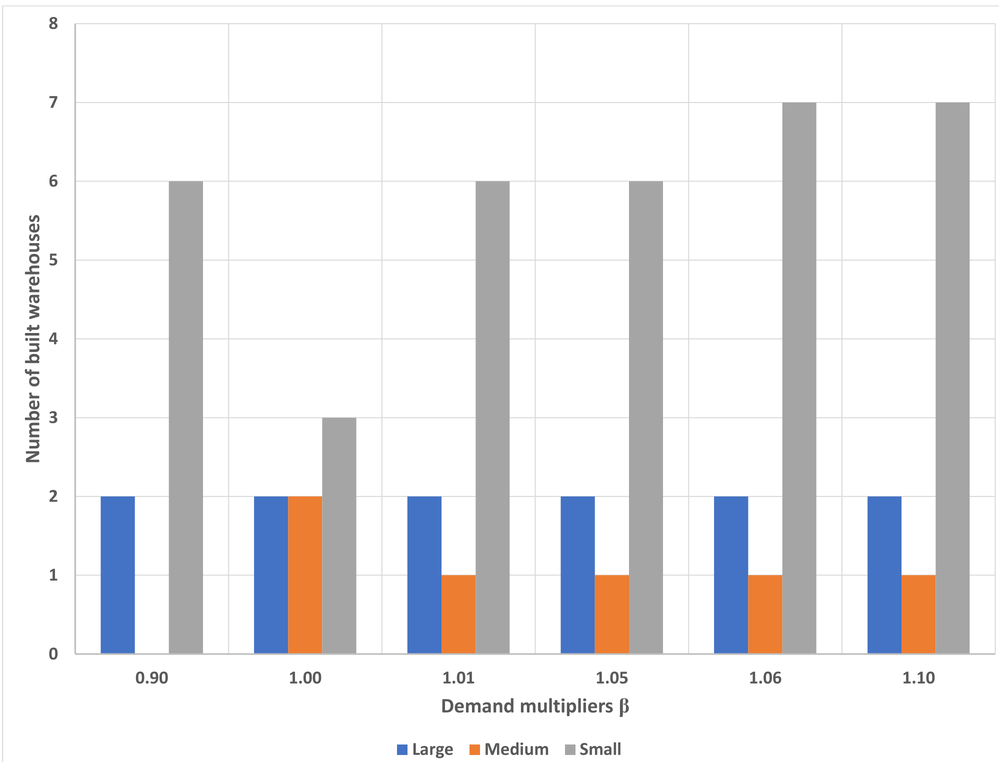

Figure 2.8: Impact of Demand on the Number of Warehouses.

Table 2.6 and Figure 2.9 show the value of the total cost and the cost components; as well as the percentage change in costs compared to the base case of $\beta = 1$. For example, when β decreases from 1.0 to 0.9, i.e., total demand decreases by 10%, the two medium warehouses are replaced with three additional small warehouses. The transportation costs decrease by 11%, the fixed costs decrease by 9% and the operating costs by 6%. For $\beta = 1.06$, the total demand increases by 6%, the total cost increase by 6%, but, by replacing a medium warehouse with 4 small warehouses, the transportation cost decrease of 15%, is offset by increases in the fixed costs by 13% and in the operational cost by 12%.

Table 2.6: Impact of Demand on the Total Cost Components ($M) with %-age change.

β	$C(x,y)$	$C_F(x)$	$C_O(x,y)$	$C_T(y)$
0.9 (-10)	3.3497 (-9)	1.6199 (-9)	0.8832 (-6)	0.8466 (-11)
1.0 (0)	3.6769 (0)	1.7823 (0)	0.9416 (0)	0.9530 (0)
1.01 (1)	3.7129 (6)	1.8853 (6)	0.9889 (5)	0.8387 (-12)
1.05 (5)	3.8482 (1)	1.8853 (6)	1.0400 (10)	0.9229 (-3)
1.06 (6)	3.8805 (6)	2.0080 (13)	1.0582 (12)	0.8143 (-15)
1.1 (10)	3.9803 (8)	2.0080 (13)	1.1064 (18)	0.8659 (-9)

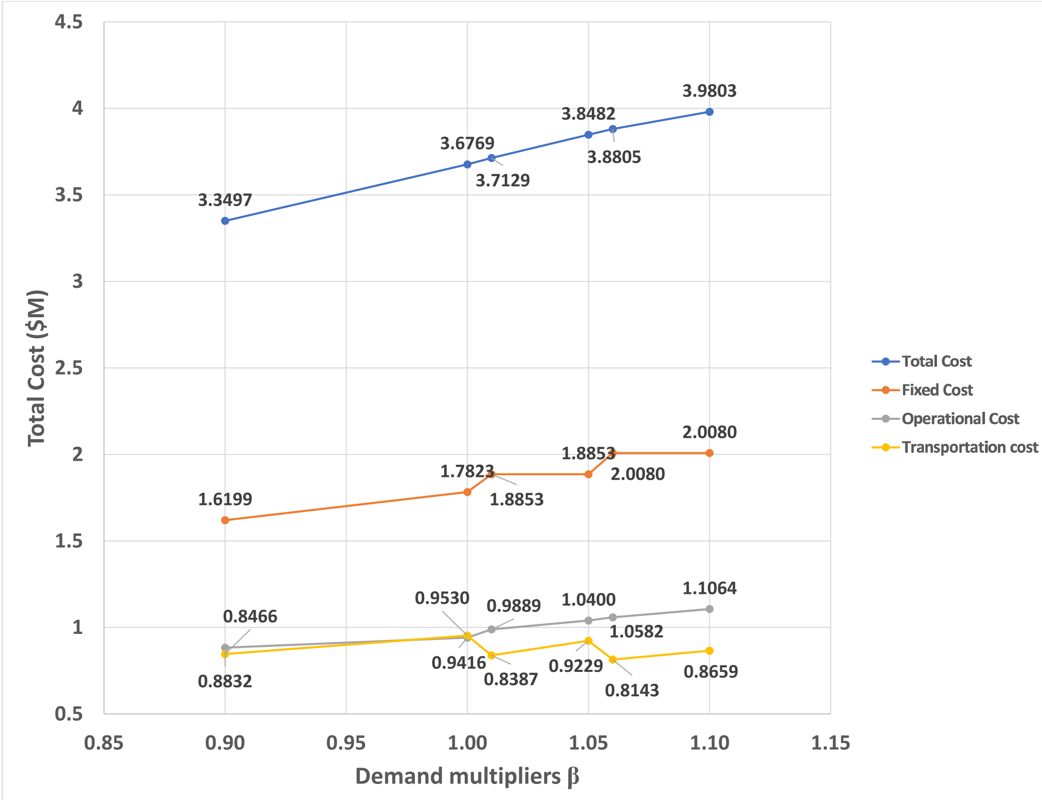

Figure 2.9: Impact of Demand on the Total Cost Components.

Table 2.7 and Figure 2.10 explore the sensitivity with an analysis of capacity. The first column shows the value of β, the second column shows total demand, the third column excess capacity in the built warehouses after the demand is met, and the last column shows the increase in demand from one value of β to the next. Recall that the capacity of a small warehouse is $V^1 = 270,765.6$, of a medium warehouse is $V^2 = 580,212.0$, and of a large warehouse is $V^3 = 967,020.0$. When β is increased from 0.9 to 1.0 the increase in demand is 388,172 but the excess capacity in the system is only 65,084. Another small warehouse will not accommodate the increase, so at least one new medium size (or a large) must be built. In fact, to reach optimality with the new demand three small warehouses are replaced by 2 medium warehouses. When β is increased from 1 to 1.01, the increase in demand is 38,817, but the excess capacity when $\beta = 1$ is 25,039. So additional warehouses must be built. To reach optimality, a medium warehouse is replaced by three small warehouses. When β is increased from 1.01 to 1.05 the increased demand of 155,269 can be met with the excess capacity of 218,306 so no new warehouses need to be built. When β increases to 1.06 another small warehouse is needed, and when β increases to 1.1, excess capacity can meet the additional demand.

In summary, increases to demand together with fixed warehouse size, have a significant influence on the optimal solution.

Table 2.7: Impact of Demand on Capacity.

β	Total Demand	Excess Capacity	Increase in Demand
0.9	3,493,550	65,084	0
1.0	3,881,722	25,039	388,172
1.01	3,920,539	218,306	38,817
1.05	4,075,808	63,037	155,269
1.06	4,114,625	294,986	38,817
1.1	4,269,894	139,717	155,269

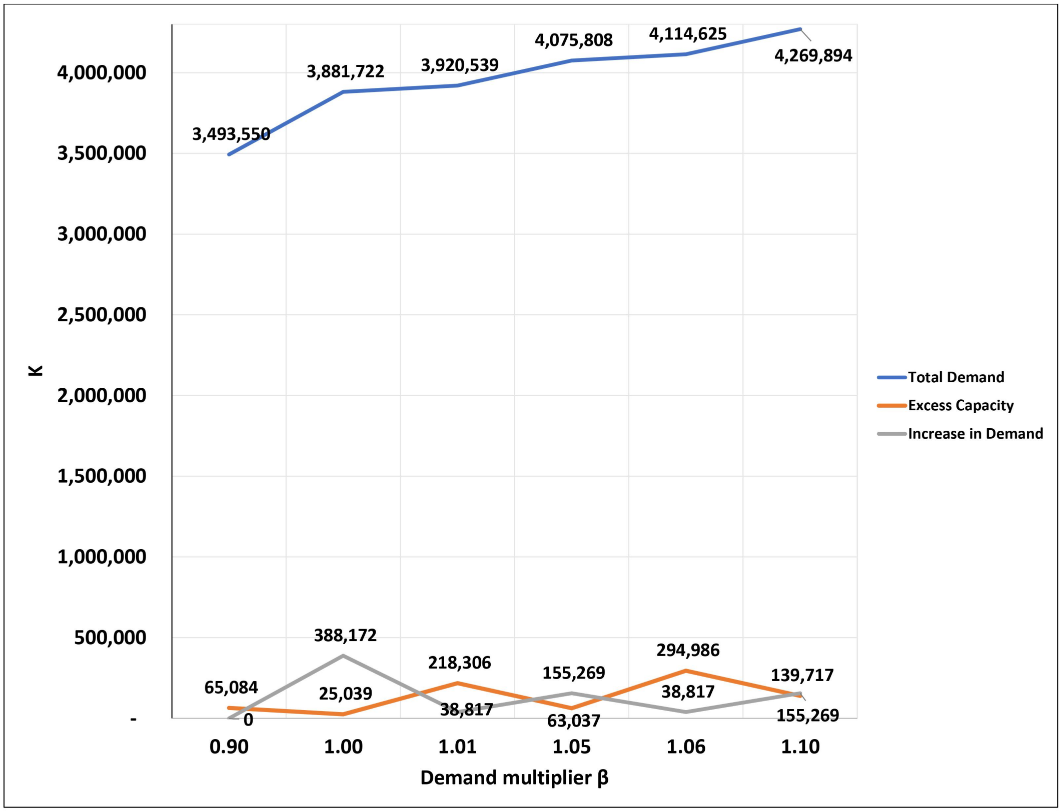

Figure 2.10: Impact of Demand on Capacity (K).

2.3.4.4 Operational Cost

The results in Table 2.8 show the impact of changes in the operation cost ν^3 for large warehouses. With the idea that larger warehouses can be more efficient, the cost factor ν^3 was replaced with $\gamma\nu^3$ with decreasing values of γ as given in the first column of the table. In Section 2.3.2 it was noted that the operational cost was the smallest of the three cost components. Table 2.8 shows that the results are insensitive to dramatic changes to the operational cost factor for the large warehouses.

Table 2.8: Impact of Operational Cost on Total cost ($M) and the Number of Warehouses.

γ	$C(x,y)$	Large	Medium	Small	Total
0.9	3.6769	2	2	3	7
0.8	3.6769	2	2	3	7
0.7	3.6769	2	2	3	7
0.6	3.6769	2	2	3	7
0.5	3.6769	2	2	3	7
0.4	3.6769	2	2	3	7
0.1	3.6769	2	2	3	7

2.4 European Example

2.4.1 Introduction

Models $\boldsymbol{M}$ and $\boldsymbol{LM}$ were tested using an example was created having a 37 European cities: Amsterdam Antwerp, Athens, Barcelona, Berlin, Bern, Brussels, Calais, Cologne, Copenhagen, Edinburgh, Frankfurt, Geneva, Genoa, Hamburg, Le Havre, Lisbon, London, Luxembourg, Lyon, Madrid, Marseille, Milan, Munich, Naples, Nice, Paris, Prague, Rome, Rotterdam, Strasbourg, Stuttgart, The Hague, Turin, Venice, Vienna, and Zurich.

From [2], the distances between the cities were found. All of the 37 cities are both branches and potential warehouses. Five product categories were considered in this example and we did not, as we did in the previous case study, unify them into one product category. The five product categories demand data were made up and two cases were considered with regard to the built warehouses. With the first one, all warehouses can serve the five product categories and in the second case, all warehouses are specialized with some product categories to serve. As in the Canadian case study, the demand for all branches from each product category was specified in K that is 1000 cubic inches. A matrix of the transportation cost, with a size of 185×37, was created. The matrix contains the cost of transporting one K of each product category between all of the 37 cities. The costs per KM are 0.0012 for the first product category, 0.0006 for the second product category, 0.00072 for the third product category, 0.00084 for the fourth product category, and 0.00096 for the fifth product category. The difference in the shipping costs is a result of the differences in the weight of the product categories. The same sizes and capacities of the warehouses in the Canadian case study were adopted in this example. Building costs, per square foot, were randomly generated from $U \sim [1, 2]$. Also, land costs, per square foot, were randomly generated from $U \sim [0.15, 0.25]$. The operational costs per K were randomly generated from $U \sim [0.05, 0.4]$ and depends on the warehouse size and the product category as Table 2.9 shows. Finally, set $E = \emptyset$ and $U = 5$.

Table 2.9: Operational cost per K

Category	Small Warehouse	Medium Warehouse	Large Warehouse
$j = 1$	0.33	0.27	0.19
$j = 2$	0.25	0.21	0.14
$j = 3$	0.40	0.27	0.22
$j = 4$	0.20	0.15	0.05
$j = 5$	0.37	0.30	0.26

2.4.2 Solution to *M* and *LM*

2.4.2.1 Case 1: Flexible Warehouses

By flexible warehouse, we mean that all warehouses can handle all products. The created example was solved using CPLEX solver on the Asus laptop mentioned above. A maximum of five warehouses is allowed to be built.

Using CPLEX, Table 2.10 shows that model ***M*** was solved in almost five minutes with around 513 thousand iterations. When the y_{wbj} variables were relaxed then CPLEX took close to six minutes to get an optimal solution to model $\boldsymbol{RM}$ and the number of iterations increased to almost 552 thousand. Selected warehouses are the same in both solutions. The objective function value of model $\boldsymbol{M}$ is 3.2321×10^6, whereas it is 3.2296×10^6 in model $\boldsymbol{RM}$. The gap between the two solutions is only 0.07%.

Table 2.10: Numerical results for the European Example with flexible warehouses using CPLEX (Quadratic Models).

Model	$\boldsymbol{M}$	$\boldsymbol{RM}$
Total Variables	6,956	6,956
Binary Variables	6,956	111
Constraints	260	7,105
Iterations	512,638	551,656
Time (Hr:Min:Sec)	00:04:59	00:05:48
Best Objective $\times 10^6$	3.2321	3.2296
Best Bound $\times 10^6$	3.1820	3.1919
Built warehouses (w, s)	(7, 3), (12, 3) (20,3),(21, 1) (35, 3)	(7, 3), (12, 3), (20,3),(21, 1), (35, 3)
Status	Optimal	Optimal

Moving to model $\boldsymbol{LM}$. Table 2.11 shows that CPLEX took less than 16 minutes with around one million iterations to get an optimal solution to model $\boldsymbol{LM}$. Objective function value and selected warehouses were exactly the same as in the solutions to model $\boldsymbol{M}$ in Table 2.10.

After relaxing the y_{wbj} variables, Table 2.11 shows that CPLEX saved around 50% of the time in solving model $\boldsymbol{RLM}$. Same built warehouses were found as in model $\boldsymbol{LM}$.

Table 2.11: Numerical results for the European Example with flexible warehouses using CPLEX (Linear Models)

Model	***LM***	***RLM***
Total Variables	27,491	27,491
Binary Variables	6,956	111
Constraints	82,400	89,245
Iterations	1,081,492	339,647
Time (Hr:Min:Sec)	00:15:26	00:07:44
Objective $\times 10^6$	3.2321	3.2296
Bound $\times 10^6$	3.2012	3.1919
Built warehouses $(w,\ s)$	(7, 3), (12, 3), (20,3), (21, 1), (35, 3)	(7, 3), (12, 3), (20,3), (21, 1), (35, 3)
Status	Optimal	Optimal

2.4.2.2 Case 2: Specialized Warehouses

It is common for warehouses to be specialized in specific product categories. In the created example with 37 European cities, we let warehouses 1 through 10 could not serve product categories 1 and 2 and can only serve product categories 3, 4, and 5. Also, warehouses 11 through 20 could not serve product categories 3 and 4, warehouses 21 through 30 could not serve product categories 1 and 3, and finally, warehouses 31 through 37 could not serve product category 4. In constraint (2.2.5) we have

$$\begin{aligned}\mathscr{R} = \{1,\dots,10\} \times \{1,2\} \cup \{11,\dots,20\} \times \{3,4\} \cup \{21,\dots,30\} \\ \times \{1,3\} \cup \{31,\dots,37\} \times \{4\}.\end{aligned} \tag{2.4.1}$$

Although the inclusion of the specialized warehouses would lead to a higher total costs, it would match the problem with many realistic situations.

Table 2.12 shows that CPLEX spent 22 seconds to get an optimal solution to model $\boldsymbol{M}$ with the case of specialized warehouses. The objective function value is 3.7061×10^6 which is 14.76% higher than the value in model $\boldsymbol{M}$ with the flexible warehouses (Table 2.10). On the other hand, CPLEX spent 19 seconds to get an optimal solution of 3.7013×10^6 to model $\boldsymbol{RM}$. Built warehouses are identical in the solutions to models $\boldsymbol{M}$ and $\boldsymbol{RM}$. Note that in Table 2.12, we got the same sizes of built warehouses as in Table 2.10. Warehouse 12 is common in both tables. Further, warehouses 7, 20, and 35 that are of large size, in the solution to the flexible warehouses, were replaced with warehouses 6, 9, and 14 in the solution to the specialized warehouses. Also, warehouse 21 of small size was replaced with warehouse 17 of the same size.

Table 2.12: Numerical results for the European Example with specialized warehouses using CPLEX (Quadratic Models)

Model	$\boldsymbol{M}$	$\boldsymbol{RM}$
Total Variables	6,956	6,956
Binary Variables	6,956	111
Constraints	2,739	9,584
Iterations	84,104	50,803
Time (Hr:Min:Sec)	00:00:22	00:00:19
Objective $\times 10^6$	3.7061	3.7013
Bound $\times 10^6$	3.6740	3.6891
Built Warehouses (w, s)	(6, 3), (9, 3) (12,3),(14, 3) (17, 1)	(6, 3), (9, 3) (12,3),(14, 3) (17, 1)
Status	Optimal	Optimal

For the linearized models, Table 2.13 shows that CPLEX spent

Table 2.13: Numerical results for the European Example with specialized warehouses using CPLEX (Linear Models)

Model	***LM***	***RLM***
Total Variables	27,491	27,491
Binary Variables	6,956	111
Constraints	84,879	91,724
Iterations	645,018	107,525
Time (Hr:Min:Sec)	00:02:09	00:00:39
Objective $\times 10^6$	3.7061	3.7013
Bound $\times 10^6$	3.7049	3.6824
Built Warehouses (w, s)	(6, 3), (9, 3) (12,3),(14, 3) (17, 1)	(6, 3), (9, 3) (12,3),(14, 3) (17, 1)
Status	Optimal	Optimal

2 minutes and 9 seconds to solve model ***LM*** and 39 seconds to solve model ***RLM***. Objective function values and built warehouses are exactly the same to what we got in Table 2.12.

2.5 Conclusion

A mixed integer, non-linear, multiple capacity levels, single source facility location mathematical model was applied to a large Canadian company and to a created problem of 37 European cities. Model modifications included the addition of management constraints and the inclusion of land and building costs, operational costs and transportation costs. Management constraints included a limit on the number of warehouses to be built and the existing warehouses would be maintained. The cost functions included separate costs for land and building; operational costs that depended on warehouse size and finally transportation costs. Based on the solution of the Canadian company, the company started to build a medium warehouse at location 33.

The analysis section of the Canadian company showed that the management constraint on the number of warehouses built was active and that consideration should be given to building more, but smaller warehouses. It was also shown in Table 2.4 that the solution was sensitive to transportation cost so that organizations using this model should give these costs careful attention. Tables 2.5 to 2.7 show that changes in demand have a great impact on the number and size of warehouses built. This is because demand influence transportation cost, operational cost, and the number of warehouses, because of their chosen capacities.

Relaxing the y_{wbj} variables showed a great saving in the solution time and the number of iterations, when solving model $\boldsymbol{RLM}$, needed to get an optimal solution. Whenever we got an optimal solution to models $\boldsymbol{LM}$ and $\boldsymbol{RLM}$, it was noticed that both models have the same built warehouses along with their sizes.

Finally, it is clear that there is an increase in the objective value function as the specialized warehouses were introduced in the created example of the 37 European cities. In this problem, an increase of 14.76% in the objective function was found as a result of the introduction of the specialized warehouses.

Chapter 3

Capacitated Facility Location Problem under Risk of Warehouse Failure

3.1 Facility Failure Risk

Early studies on the facility location of the supply chain network design assume that once the facilities are built, they will remain functioning all the time. Recent studies show an increased recognition of the fact that constructed facilities may be disrupted at any time. In fact, many factors, such as natural disasters, power outages, water floods, labor strikes, machine break downs, and transportation damages, can lead to having a facility disruption. For example, because of the electricity cut-off in China in 2008, many companies such as Intel, Isuzu Motors, and Suzuki stopped their production as they were unable to get their demand from China warehouses [92]. In 2001, Ericsson lost a substantial portion of its market to Nokia because of a disruption at a Philips Semiconductor plant that caused a shortage of cell phone parts that were to be provided to Ericsson [61]. The initial outbreak of the Corona virus in China disrupted global supply chains. In the USA, 3.28 million workers applied for unemployment benefits in the week ending March 21, 2020 [4]. On February 17, 2020, Apple said that it is expected to have a decrease in its quarterly earnings [1]. Two reasons behind these expectations. First, the constrained global supply of iPhones. Second, the significant decrease in the demand of the Chinese markets. Hence, several studies have been done to obtain an optimal facility location with the consideration of disruption. To get the optimal facility location design with the consideration of possible facility failure, a number of reliable facility location models have been proposed [102].

Our approach is to assign a nonzero probability of failure to all warehouses, and to assign each branch a primary warehouse and a secondary warehouse. All demand by the branch for a product category will be satisfied by its primary warehouse, unless it fails. If its primary warehouse fails, its secondary warehouse will provide some of its demand, the expected demand calculated from the probability of failure of the primary warehouse and the total demand.

A cubic binary variable optimization model is presented, the solution of which will determine network design under risk. It assumes that no warehouses will work as backups for themselves. It covers the failure of all built warehouses with the assumption, as in

[63], that for any branch, if the primary warehouse fails, then the backup warehouse will be available. It also assumes that warehouses fail independently with site-specific failure probabilities.

3.2 Mathematical Model

In this section, a mathematical model that accounts for the risk of facility failure is presented. To design for the risk of failure of warehouses, for product j required for branch b, we assign both a primary warehouse and a secondary warehouse; the secondary warehouse to take over the delivery only in the event of failure of the primary warehouse. $r = 1$ is used to denote a primary warehouse and $r = 2$ to denote a secondary warehouse. This is called the warehouse level. Let y^r_{wbj} be the binary variable equal to one if and only if the demand for product j at branch b is fulfilled by warehouse w at level r

To include the risk of warehouse failure in our operational and transportation costs, we first need to model that risk. Let $0 < p_w < 1$ be the probability that the warehouse w fails. The dependence on w is important as risk of failure depends on location. A warehouse in a coastal city with frequent tropical storms, or in a city on or near a fault line has a greater probability of failure than those more isolated from large natural events. Likewise, warehouses in politically unstable cities, or in cities without a stable supply of electricity and water, have higher risk that those in more stable environments.

If we do not account for the risk of failure, the primary warehouses will supply their assigned branches and the operational cost is

$$\sum_{w,s,j} v_j^s \sum_b (d_{bj}\ x_w^s\ y_{wbj}^1). \tag{3.2.1}$$

Since the probability that warehouse w does not fail is $(1 - p_w)$, the expected operational cost at the primary warehouses is

$$\sum_{w,s,j} v_j^s \sum_b \Big(d_{bj}\ x_w^s\ y_{wbj}^1\ (1 - p_w)\Big). \tag{3.2.2}$$

Now, think of w as the secondary warehouse. The probability that this warehouse supplies its assigned branches is the probability that its corresponding primary warehouse fails, which is

$$\Big(\sum_{w' \neq w} p_{w'}\ y_{w'bj}^1\Big). \tag{3.2.3}$$

Thus, the expected operational cost associated with the secondary warehouse is

$$\sum_{w,s,j} v_j^s \sum_b \Big(d_{bj}\ x_w^s\ y_{wbj}^2\ \Big(\sum_{w' \neq w} p_{w'}\ y_{w'bj}^1\Big)\Big). \tag{3.2.4}$$

Putting equations (3.2.2) and (3.2.4) together gives the total expected operational cost function

$$\hat{C}_O(x,y) = \sum_{w,s,j} \left[\nu_j^s \sum_b d_{bj} \Big(x_w^s \; y_{wbj}^1 \; (1-p_w) + x_w^s \; y_{wbj}^2 \sum_{w' \neq w} p_{w'} \; y_{w'bj}^1 \Big) \right] \tag{3.2.5}$$

which is cubic in the binary variables; and the cost is in dollars.

To illustrate the expected operational cost, consider Figure 3.1. Let b be a branch, j be a product category, and w and w' be two warehouses. Then for warehouse w, we have one of the following cases

(i) w does not serve b, with product category j, either as a primary or a backup warehouse, so $y_{wbj}^r = 0$ for $r = 1, 2$. Thus, the expected operational cost of serving branch b by warehouse w is 0,

(ii) w is a primary warehouse for branch b to serve its demand from j. So, the expected operational cost of serving branch b by warehouse w, as a primary warehouse, is $\nu_j^s \; d_{bj} \; (1-p_w) = 11(100)(0.96) = 1,056$,

(iii) w is a backup warehouse for branch b to serve its demand from j. This will happen only if the primary warehouse, say w', of b is failed. So the expected operational cost of serving branch b by warehouse w, as a backup warehouse, is $\nu_j^s \; d_{bj}(p_{w'}) = 11(100)(0.05) = 55$.

So, for this example, the expected operational cost of warehouse w

to serve branch b with its demand from product category j, is either zero in case w is neither a primary nor a backup warehouse for b, or it is $1,056$ in case w is the primary warehouse for b, or 55 in case w is the backup warehouse for b when its primary warehouse w' fails.

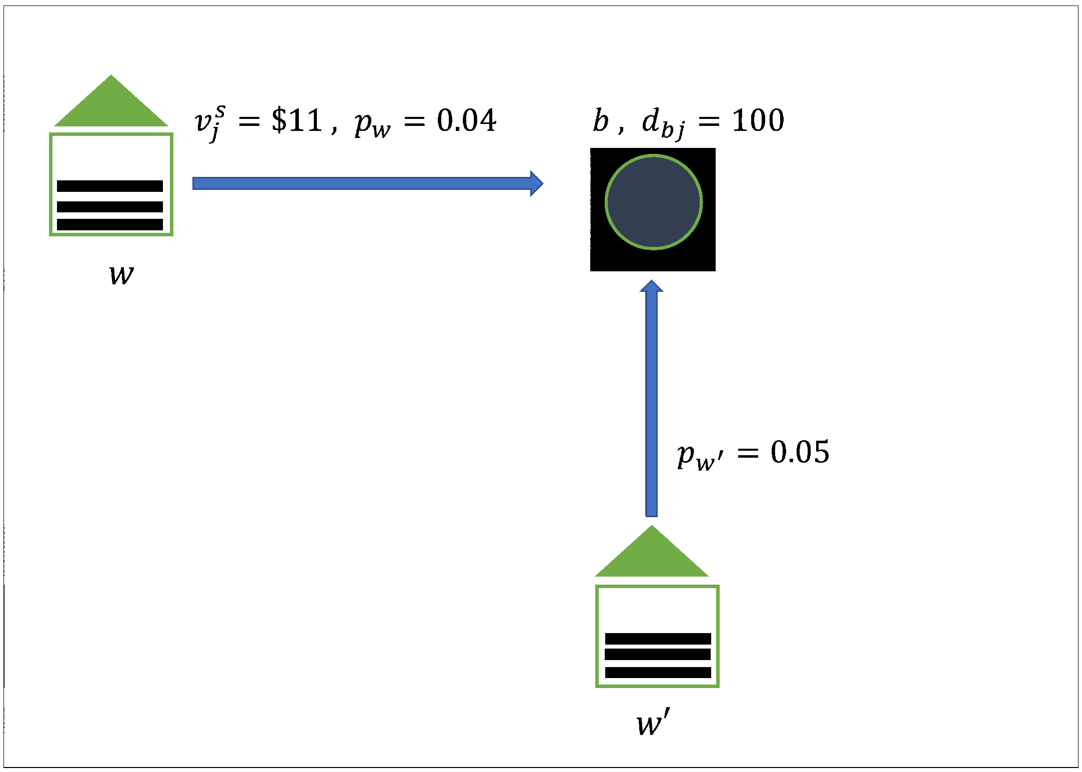

Figure 3.1: Expected Operational Cost

Using the same development as for operational cost, the total expected transportation cost, in dollars, is

$$\hat{C}_T(y) = \sum_{w,b,j} \left[d_{bj}\tau_{wbj}\left(y^1_{wbj}\ (1-p_w) + y^2_{wbj} \sum_{w' \neq w} p_{w'}\ y^1_{w'bj} \right) \right]. \tag{3.2.6}$$

The complete cost function to be minimized is

$$\hat{C}(x, y) = C_F(x) + \hat{C}_O(x, y) + \hat{C}_T(y)$$

.

We now develop the constraints. We have

$$\sum_w y^r_{wbj} = 1, \ \forall b \in B, \ \forall j \in J, \ \forall r \in R, \tag{3.2.7}$$

to ensure that each branch b is assigned to a single primary warehouse and a single secondary warehouse to meet its demand of items from product category j; and

$$\sum_r y^r_{wbj} \leq 1, \ \forall w \in W, \ \forall b \in B, \ \forall j \in J, \tag{3.2.8}$$

to prevent a warehouse from being both the primary and secondary warehouse for product j at branch b.

To ensure that the total storage space, in K, available in a warehouse w, to meet the expected demand in K of items from all product categories for all branches supplied by the warehouse w, whether w was a primary or a secondary warehouse, is less than or equal to the volume of w in K, we have

$$\sum_{b,j} d_{bj} \Big(y^1_{wbj} + y^2_{wbj} \sum_{w' \neq w} (p_{w'} \ y^1_{w'bj}) \Big) \leq \sum_s \boldsymbol{V}^s x^s_w, \ \forall w \in W. \tag{3.2.9}$$

Note that constraints (3.2.9) are quadratic. The term

$d_{bj}\ y^2_{wbj} \sum_{w' \neq w} (p_{w'}\ y^1_{w'bj})$ is the expected demand, of branch b, supplied by warehouse w as a secondary warehouse in case the primary warehouse for branch b fails. Note that if we did not use the probability in (3.2.9), we will end up building double the space of the warehouse w.

Putting everything together, the cubic binary optimization problem is to

$$\begin{aligned}
\boldsymbol{M_r}\textbf{:}\ \text{Minimize} \quad \hat{C}(x,y) \quad &= C_F(x) + \hat{C}_O(x,y) + \hat{C}_T(y) \\
\text{Subject to:} \quad &(2.2.1), (2.2.3) - (2.2.5), \\
&(3.2.7) - (3.2.9), \\
x^s_w \quad &\in \{0,1\},\ \forall s \in S, \forall w \in W,\ \text{and} \\
y^r_{wbj} \quad &\in \{0,1\},\ \forall w \in W, \forall b \in B, \forall j \in J, \\
&\qquad \forall r \in R.
\end{aligned}$$

Model $\boldsymbol{M_r}$ is a binary, cubic optimization problem. If there is a single product, or single product category, the subscripts j are removed.

In the case of the failure of a certain warehouse, it is possible that some of its served branches are not served by the backup warehouse due to the shortage of the availability of the demanded products. The reason for this is because of the limited available capacity as a result of the consideration of the failure probabilities. The penalty cost of not serving a certain demand for any branch, in case of the failure of its primary warehouse, is not considered in this study and will be recommended as a future work.

In Corollary 3.2.2, given below, we see that the model $\boldsymbol{M_r}$ is consistent. That is, if a warehouse is unbuilt, then it has no assigned branches either as a primary or a secondary warehouse. Consequently, no need to have explicit constraints to ensure consistency.

Lemma 3.2.1. *If $\hat{x}$ and $\hat{y} \geq 0$ satisfy (3.2.9), and if $\hat{w}$ is such that $\hat{x}^s_{\hat{w}} = 0$ for all s, then $\hat{y}^r_{\hat{w}bj} = 0$ for all $b \in B$, $j \in J$, and $r \in R$.*

Proof. For $w = \hat{w}$, $(x, y) = (\hat{x}, \hat{y})$ and $\hat{x}^s_{\hat{w}} = 0$ for all s, (3.2.9) gives

$$\sum_{b,j} d_{bj} \left(\hat{y}^1_{\hat{w}bj} + \hat{y}^2_{\hat{w}bj} \sum_{w' \neq \hat{w}} (p_{w'} \hat{y}^1_{w'bj}) \right) \leq 0$$

and, since all quantities are nonnegative, for all $b \in B$ and $j \in J$, we have

$$\hat{y}^1_{\hat{w}bj} + \hat{y}^2_{\hat{w}bj} \sum_{w' \neq \hat{w}} (p_{w'} \hat{y}^1_{w'bj}) \leq 0.$$

Again, since all quantities are nonnegative it follows that $\hat{y}^1_{\hat{w}bj} = 0$. Since branch b has to be assigned a primary warehouse for product j, this means that

$$\sum_{w' \neq \hat{w}} (p_{w'} \hat{y}^1_{w'bj}) > 0,$$

which implies that $\hat{y}^2_{\hat{w}bj} = 0$. □

Corollary 3.2.2. *Let $(\hat{x}, \hat{y})$ be a feasible solution to model $\boldsymbol{M_r}$ and let $\hat{w}$ be such that $\hat{x}^s_{\hat{w}} = 0$ for all $s \in S$. Then $y^r_{\hat{w}bj} = 0$ for all $b \in B$, $j \in J$, and $r \in R$.*

Proof. Feasibility implies that $\hat{x}$ and $\hat{y} \geq 0$ satisfy (3.2.9) so that the result follows from Lemma 3.2.1. $\square$

Proposition 3.2.3 says that at the optimality of model $\boldsymbol{M_r}$, if for each branch b and each product category j, the sizes of the primary and backup warehouses that satisfy the demand of product j at branch b are equal to the size of the warehouse that serves branch b of items from category j at the optimality of model $\boldsymbol{M}$, then the operational cost in models $\boldsymbol{M}$ and $\boldsymbol{M_r}$ are equal. (i.e. the operational costs of risk and non risk models are equal).

Proposition 3.2.3. *Let $(\tilde{x}, \tilde{y})$ be an optimal solution to model $\boldsymbol{M_r}$. Let $b \in B$, and $j \in J$ be arbitrary but fixed. Let $\overline{w}$, and $\overline{\overline{w}} \in W$ be such that $\tilde{x}_{\overline{w}}^{\overline{s}} = \tilde{x}_{\overline{\overline{w}}}^{\overline{\overline{s}}} = 1$ and $\tilde{y}_{\overline{w}bj}^{1} = \tilde{y}_{\overline{\overline{w}}bj}^{2} = 1$ where $\overline{s}$, $\overline{\overline{s}}$ are the sizes of $\overline{w}$, $\overline{\overline{w}}$, respectively. Also, let $(\hat{x}, \hat{y})$ be the optimal solution to model $\boldsymbol{M}$. Suppose that for the same b and j, there exists $\hat{w} \in W$ with size $\hat{s}$ such that $\hat{x}_{\hat{w}}^{\hat{s}} = 1$ and $\hat{y}_{\hat{w}bj} = 1$. If $\overline{s} = \overline{\overline{s}} = \hat{s}$, then $\hat{C}_O(x, y) = C_O(x, y)$, where $\hat{C}_O(x, y)$ is as in (3.2.5) and $C_O(x, y)$ is as in (2.2.15).*

Proof. In model $\boldsymbol{M_r}$, at optimality, the operational cost of satisfying the demand of product j at branch b, by warehouses $\overline{w}$ and $\overline{\overline{w}}$, is

$$\begin{aligned} \hat{C}_O(x, y)\Big|_{b,j} = d_{bj}\Big(& \nu_j^{\overline{s}}\ \tilde{x}_{\overline{w}}^{\overline{s}}\ \tilde{y}_{\overline{w}bj}^{1}\ (1 - p_{\overline{w}}) \\ & + \nu_j^{\overline{\overline{s}}}\ \tilde{x}_{\overline{\overline{w}}}^{\overline{\overline{s}}}\ \tilde{y}_{\overline{\overline{w}}bj}^{2}\ \ p_{\overline{w}}\ \ \tilde{y}_{\overline{w}bj}^{1}\Big). \end{aligned} \tag{3.2.10}$$

Since we know that $\tilde{x}_{\overline{w}}^{\overline{s}} = \tilde{y}_{\overline{w}bj}^{1} = 1$, and $\tilde{x}_{\overline{\overline{w}}}^{\overline{\overline{s}}} = \tilde{y}_{\overline{\overline{w}}bj}^{2} = 1$, so (3.2.10) can be written as

$$\hat{C}_O(x,y)\Big|_{b,j} = d_{bj}\left(\nu_j^{\overline{s}}\ (1 - p_{\overline{w}})\ +\ \nu_j^{\overline{\overline{s}}}\ p_{\overline{w}}\right). \tag{3.2.11}$$

Since we know that $\overline{s} = \overline{\overline{s}}$, it follows that

$$\hat{C}_O(x,y)\Big|_{b,j} = \nu_j^{\overline{s}}\ d_{bj}. \tag{3.2.12}$$

On the other hand, at optimality, the operational cost of satisfying the demand for product j at branch b in model $\boldsymbol{M}$, by warehouse w, is

$$C_O(x,y)\Big|_{b,j} = d_{bj}\left(\nu_j^{\hat{s}}\ \hat{x}_{\hat{w}}^{\hat{s}}\ \hat{y}_{\hat{w}bj}\right). \tag{3.2.13}$$

As $\hat{x}_{\hat{w}}^{\hat{s}} = \hat{y}_{\hat{w}bj} = 1$, and $\overline{s} = \overline{\overline{s}} = \hat{s}$, it follows that (3.2.12) is equal to (3.2.13). So the proposition follows. $\square$

3.2.1 Linearization and Relaxation to Model $\boldsymbol{M_r}$

Model $\boldsymbol{M_r}$ has non linear terms because of the multiplication between x_w^s, y_{wbj}^2, and $y_{w'bj}^1$ in (3.2.5), and between y_{wbj}^2 and $y_{w'bj}^1$ in (3.2.6) and (3.2.9). Two methods were used to remove the nonlinearity.

3.2.1.1 Model $\boldsymbol{L_1M_r}$

The multiplication of x_w^s and y_{wbj}^r in the operational cost (3.2.5), can be linearized by the standard linearization using the substitution $z^{rs}_{wbj} = x^s_w \;\; y^r_{wbj}$, with the constraints, $\forall s \in S,\ \forall w \in W,\ \forall b \in B,\ \forall j \in J,\ and\ \forall r \in R$,

$$z^{rs}_{wbj} \leq x^s_w, \tag{3.2.14}$$

$$z^{rs}_{wbj} \leq y^r_{wbj}, \tag{3.2.15}$$

$$z^{rs}_{wbj} \geq x^s_w + y^r_{wbj} - 1,\ and \tag{3.2.16}$$

$$z^{rs}_{wbj} \geq 0\ . \tag{3.2.17}$$

z^{rs}_{wbj} is a continuous variable and because of (3.2.14) - (3.2.17) it is equal to 1 if and only if a warehouse of size s is built at location w and supplies branch b at level r with its demand from product category j. The reformulated expected operational cost is now

$$\begin{aligned}\hat{C}_O(z,y) = \sum_{w,s,j} \Bigg[\nu^s_j \sum_b d_{bj} \Big(z^{1s}_{wbj} \;\; (1-p_w) \\ + z^{2s}_{wbj} \;\; \sum_{w' \neq w} (p_{w'} \;\; y^1_{w'bj}) \Big) \Bigg].\end{aligned} \tag{3.2.18}$$

The next step is to linearize the remaining quadratic terms in (3.2.6), (3.2.9) and (3.2.18). We set

$$Q^s_{wbj} = z^{2s}_{wbj} \sum_{w' \neq w} (p_{w'} y^1_{w'bj}) \quad \text{and} \quad O_{wbj} = y^2_{wbj} \sum_{w' \neq w} (p_{w'} \, y^1_{w'bj}).$$

We add constraints analogous to those in (3.2.14) and (3.2.17) and we introduce cuts to the equivalent of (3.2.15) and (3.2.16), using P^*, where

$$P^* = \max_w \, p_w \quad \text{and} \quad P_* = \min_w \, p_w, \tag{3.2.19}$$

and where P_* is introduced at this point, for convenience. The constraints are

$$0 \leq Q^s_{wbj} \leq P^* \ z^{2s}_{wbj}, \quad \forall w, b, s, j, \tag{3.2.20}$$

$$\sum_{w' \neq w} (p_{w'} \ y^1_{w'bj}) - P^* \ (1 - z^{2s}_{wbj}) \leq Q^s_{wbj} \leq \sum_{w' \neq w} (p_{w'} \ y^1_{w'bj}), \quad \forall w, b, s, j. \tag{3.2.21}$$

$$0 \leq O_{wbj} \leq P^* \ y^2_{wbj}, \quad \forall w, b, j, \tag{3.2.22}$$

and

$$\sum_{w' \neq w} (p_{w'} \, y^1_{w'bj}) - P^*(1 - y^2_{wbj}) \leq O_{wbj} \leq \sum_{w' \neq w} (p_{w'} \ y^1_{w'bj}) \quad \forall w, b, j. \tag{3.2.23}$$

It is straightforward to see that, since $P^* \leq 1$, the cut constraints are valid. For example, the right-hand inequality in (3.2.20) implies $Q^s_{wbj} \leq z^{2s}_{wbj}$.

We use these linearizations to reformulate the operational and transportation cost functions. We have

$$\hat{C}_O(z, Q) = \sum_{w,s,j} \left[\nu^s_j \sum_b d_{bj} \left(z^{1s}_{wbj} \ \ (1 - p_w) \ + Q^s_{wbj} \right) \right] \tag{3.2.24}$$

and

$$\hat{C}_T(y, O) = \sum_{w,b,j} \left[d_{bj} \, \tau_{wbj} \left(y^1_{wbj} \ \ (1 - p_w) \ + O_{wbj} \right) \right], \tag{3.2.25}$$

and the reformulated capacity constraints (3.2.9) are

$$\sum_{b,j} d_{bj}\ (y^1_{wbj} + O_{wbj}) \leq \sum_s \boldsymbol{V}^s x^s_w,\ \forall\ w, \tag{3.2.26}$$

The linearization of model $\boldsymbol{M_r}$ is to

$$\begin{aligned}
\boldsymbol{L_1 M_r}\textbf{:}\ \text{Minimize}\ \hat{C}(x,y,z,O,Q) &= C_F(x) + \hat{C}_O(z,Q) + \hat{C}_T(y,O) \\
\text{Subject to:} \qquad & (2.2.1), (2.2.3) - (2.2.5), (3.2.7), \\
& (3.2.8), (3.2.14) - (3.2.17), \\
& (3.2.20) - (3.2.23),\ \text{and}\ (3.2.26), \\
x^s_w \qquad & \in \{0,1\},\ \forall w \in W, s \in S, \\
y^r_{wbj} \qquad & \in \{0,1\},\ \forall w \in W, b \in B, j \in J, \\
& \qquad\qquad r \in R.
\end{aligned}$$

The following lemma shows that model $\boldsymbol{L_1 M_r}$, like model $\boldsymbol{M_r}$, is consistent in that no branches are assigned to unbuilt warehouses.

Lemma 3.2.4. *If* $(\hat{x}, \hat{y}, \hat{z}, \hat{Q}, \hat{O})$ *is feasible for* $\boldsymbol{L_1 M_r}$*, then, for any* $\hat{w}$ *with* $\hat{x}^s_{\hat{w}} = 0$ *for all* s*,* $\hat{y}^r_{\hat{w}bj} = 0$ *for all* $r \in R$*,* $b \in B$*, and* $j \in J$*.*

Proof. For $w = \hat{w}$, since $\hat{x}^s_{\hat{w}} = 0$ for all s, the right-hand-side of constraint (3.2.26) is zero. Thus,

$$\sum_{b,j} d_{bj}\ (\hat{y}^1_{\hat{w}bj} + \hat{O}_{\hat{w}bj}) \leq 0.$$

Since d_{bj}, $\hat{y}^1_{\hat{w}bj}$, and $\hat{O}_{\hat{w}bj}$ are nonnegative, it follows that $\hat{y}^1_{\hat{w}bj} = 0$ and $\hat{O}_{\hat{w}bj} = 0$ for all b and j. It remains to show that $\hat{y}^2_{\hat{w}bj} = 0$. With

$\hat{O}_{\hat{w}bj} = 0$, the left-most inequality on (3.2.23) gives

$$\sum_{w' \neq \hat{w}} (p_{w'}\, \hat{y}^1_{w'bj}) \leq P^*(1 - \hat{y}^2_{\hat{w}bj}). \tag{3.2.27}$$

From constraint (3.2.7) we have, for $r = 1$,

$$\sum_{w \neq \hat{w}} \hat{y}^1_{wbj} + \hat{y}^1_{\hat{w}bj} = 1, \ \forall\ b \in B, j \in J.$$

Since $\hat{y}^1_{\hat{w}bj} = 0$ and since the y variables are binary, there is an index $w' \neq \hat{w}$ with $\hat{y}^1_{w'bj} = 1$ so that the left-hand side of (3.2.27) is strictly greater than zero, which means the right-hand side is strictly greater than zero which implies that $\hat{y}^2_{\hat{w}bj} = 0$. □

3.2.1.2 Model $\boldsymbol{RL_1M_r}$

In this section, model $\boldsymbol{L_1M_r}$ is relaxed by replacing $y^1_{wbj} \in \{0,1\}$ with $y^1_{wbj} \geq 0$ to get

$\boldsymbol{RL_1M_r}$**:** Minimize $\hat{C}(x, y, z, O, Q) = C_F(x) + \hat{C}_O(z, Q) + \hat{C}_T(y, O)$

$$\begin{aligned}
\text{Subject to:} \quad & (2.2.1), (2.2.3) - (2.2.5), (3.2.7), \\
& (3.2.8), (3.2.14) - (3.2.17), \\
& (3.2.20) - (3.2.23), \text{ and } (3.2.26), \\
x^s_w \quad & \in \{0,1\}, \ \forall w \in w, s \in S, \\
y^1_{wbj} \quad & \geq 0, \ \forall\ w \in W, b \in B, j \in J, \\
y^2_{wbj} \quad & \in \{0,1\}, \ \forall w \in W, b \in B, j \in J.
\end{aligned}$$

Lemma 3.2.5 shows that $y^1_{wbj} \leq 1$ is implicit in $\boldsymbol{RL_1M_r}$, so that it need not be stated explicitly in the problem statement. Lemmas 3.2.6 to 3.2.8 show that the substitutions remain valid in $\boldsymbol{RL_1M_r}$.

Lemma 3.2.5. *If* $(x_w^s\ ,\ y_{wbj}^r\ ,\ z_{wbj}^{rs}\ ,\ Q_{wbj}^s\ ,\ O_{wbj})$ *is a feasible solution for* $\boldsymbol{RL_1M_r}$*, then* $y_{wbj}^1 \leq 1$ *for all* $w \in W$*,* $b \in B$*, and* $j \in J$*.*

Proof. Feasibility gives $y_{wbj}^1 \geq 0$. This, together with (3.2.7) and $r = 1$, implies that $y_{wbj}^1 \leq 1$. □

Lemma 3.2.6. *The substitution* $z_{wbj}^{rs} = x_w^s\ y_{wbj}^r$ *is valid in model* $\boldsymbol{RL_1M_r}$*.*

Proof. If $r = 2$, then z_{wbj}^{rs} is unchanged. Suppose that $r = 1$. If $x_w^s = 0$, then (3.2.14) and (3.2.17) imply that $z_{wbj}^{rs} = 0$. If $x_w^s = 1$, then (3.2.15) and (3.2.16) imply that $z_{wbj}^{rs} = y_{wbj}^r$. If $z_{wbj}^{rs} = 0$, then we have two cases. If $r = 1$, then (3.2.14) to (3.2.16), together with Lemma 3.2.5 and the fact that x_w^s is binary, gives us that either $x_w^s = 0$ and $0 \leq y_{wbj}^1 \leq 1$ or that $x_w^s = 1$ and $y_{wbj}^1 = 0$. □

Lemma 3.2.7. *The substitution* $z_{wbj}^{2s} \sum_{w' \neq w} (p_{w'} y_{w'bj}^1) = Q_{wbj}^s$ *is valid in model* $\boldsymbol{RL_1M_r}$*.*

Proof. Since we only relaxed the variables y_{wbj}^1, it follows that the variables z_{wbj}^{2s} are still binary valued. If $z_{wbj}^{2s} = 0$, then $Q_{wbj}^s = 0$, from (3.2.20). If $z_{wbj}^{2s} = 1$, $Q_{wbj}^s = \sum_{w' \neq w} (p_{w'} y_{w'bj}^1)$ from (3.2.21). □

Lemma 3.2.8. *The substitution* $y_{wbj}^2 \sum_{w' \neq w} (p_{w'} y_{w'bj}^1) = O_{wbj}$ *is valid in model* $\boldsymbol{RL_1M_r}$*.*

Proof. Analogous to the proof of Lemma 3.2.7. □

Table 3.1 gives the relative sizes of our three models. Models $\boldsymbol{M_r}$ and $\boldsymbol{L_1M_r}$ have the same number of binary variable, but $\boldsymbol{L_1M_r}$ has continuous variables because of the linearization. In $\boldsymbol{RL_1M_r}$, the relaxation of the y^1 variables decreased the number of binary variables, again increasing the number of continuous variables. Both linearization and relaxation increased the number of constraints.

Table 3.1: Comparison of Problem Size.

Model	$\boldsymbol{M_r}$	$\boldsymbol{L_1M_r}$	$\boldsymbol{RL_1M_r}$
Binary Variables	$n(q+2mg)-\lvert E\rvert-\lvert\mathscr{R}\rvert$	$n(q+2mg)-\lvert E\rvert-\lvert\mathscr{R}\rvert$	$n(q+mg)-\lvert E\rvert-\lvert\mathscr{R}\rvert$
Continuous Variables	0	$nmg(1+3q)$	$nmg(2+3q)$
Constraints	$2(n+mg)$ $+nmg+\lvert E\rvert+\lvert\mathscr{R}\rvert+1$	$mg(2+5n+12nq)$ $+2n+\lvert E\rvert+\lvert\mathscr{R}\rvert+1$	$mg(2+6n+12nq)$ $+2n+\lvert E\rvert+\lvert\mathscr{R}\rvert+1$

Theorem 3.2.9 is an extension of Theorem 2.2.1. It specifies the cases where we get fractional assignments after relaxing the y^1_{wbj} variables to create model $\boldsymbol{RL_1M_r}$.

Theorem 3.2.9. *Let $(x^*, y^*, z^*, Q^*, O^*)$ be an optimal solution to $\boldsymbol{RL_1M_r}$ and suppose that the no-ties assumption, Assumption 1, is satisfied. If there exist ω, β, and ϕ such that $0 < (y^1_{\omega\beta\phi})^* < 1$ and if*

$$W^* = \{\, w \mid 0 < (y^1_{w\beta\phi})^* < 1 \,\} \tag{3.2.28}$$

then there exists at most one warehouse $w \in W^*$ that is not running *at full capacity, that is,*

$$\sum_{b,j} d_{bj} \left((y^1_{wbj})^* + (O_{wbj})^* \right) < \boldsymbol{V}^s,$$

where s is the size of built warehouse w.

Proof. If, for all $w \in W^*$, we have

$$\sum_{b,j} d_{bj} \left((y^1_{wbj})^* + (O_{wbj})^* \right) = \boldsymbol{V}^s,$$

we are done. Suppose that there exists built warehouses $w_1 \in W^*$ and $w_2 \in W^*$ with sizes s_1 and s_2, respectively, such that

$$\sum_{b,j} d_{bj} \left((y^1_{w_1bj})^* + (O_{w_1bj})^* \right) < V^{s_1} \quad \text{and}$$

$$\sum_{b,j} d_{bj} \left((y^1_{w_2bj})^* + (O_{w_2bj})^* \right) < V^{s_2}.$$

We will show that this contradicts optimality. With Assumption 1 we can assume, without loss of generality, that

$$\nu^{s_1}_{\phi} + \tau_{w_1\beta\phi} < \nu^{s_2}_{\phi} + \tau_{w_2\beta\phi}. \tag{3.2.29}$$

Let δ be such that

$$\sum_{b\neq\beta, j\neq\phi} d_{bj} \left((y^1_{w_1bj})^* + (O_{w_1bj})^* \right) + \left(d_{\beta\phi}((y^1_{w_1\beta\phi})^* + \delta) \right) \leq V^{s_1},$$

$$\sum_{b\neq\beta, j\neq\phi} d_{bj} \left((y^1_{w_2bj})^* + (O_{w_2bj})^* \right) + \left(d_{\beta\phi}((y^1_{w_2\beta\phi})^* - \delta) \right) \leq V^{s_2},$$

$$0 \leq (y^1_{w_1\beta\phi})^* + \delta \leq 1, \quad \text{and}$$

$$0 \leq (y^1_{w_2\beta\phi})^* - \delta \leq 1.$$

Thus, the solution given by $(x^*, y^*, z^*, Q^*, O^*)$ with $(y^1_{w_1\beta\phi})^*$ and $(y^1_{w_2\beta\phi})^*$ replaced with $(y^1_{w_1\beta\phi})^* + \delta$ and $(y^1_{w_2\beta\phi})^* - \delta$, respectively, is feasible. Denote this solution by $\overline{(x^*, y^*, z^*, Q^*, O^*)}$. In this solution, we transferred $\delta d_{\beta\phi}$ of the demand for product ϕ from branch β from warehouse w_2 to w_1. From (3.2.29) it follows that

$$C\overline{(x^*, y^*, z^*, Q^*, O^*)} < C(x^*, y^*, z^*, Q^*, O^*),$$

which contradicts optimality. $\square$

Corollary 3.2.10 shows that there can be alternate optimal solutions.

Corollary 3.2.10. *Let $(x^*, y^*, z^*, Q^*, O^*)$ be an optimal solution to $\boldsymbol{RL_1M_r}$. If there exists ω, β, and ϕ such that $0 < (y^1_{\omega\beta\phi})^* < 1$ and if*

$$\nu^{s_1}_\phi + \tau_{w_1\beta\phi} = \nu^{s_2}_\phi + \tau_{w_2\beta\phi}$$

for $w_1, w_2 \in W^*$, where W^* is as given in (3.2.36) then there are alternate optimal solutions to $\boldsymbol{RL_1M_r}$.

Proof. We can shift demand for product ϕ from branch β between warehouse w_2 to w_1 without affecting the values of the objective function, giving alternate solutions to $\boldsymbol{RL_1M_r}$. $\square$

As constraints (3.2.20)-(3.2.23) rely on z^{2s}_{wbj} being binary valued, we did not relax the y^2_{wbj} variables. Further, if the y^2_{wbj} variables are relaxed, we could assign to a branch an unbuilt warehouse as a back-up. Suppose that we have $\hat{w}$ such that $x^s_{\hat{w}} = 0$ for all s. That is, no warehouse is built at location $\hat{w}$. Let $\hat{b}$ and $\hat{j}$ be arbitrary but fixed. From (3.2.26), it follows that $y^1_{\hat{w}\hat{b}\hat{j}} = 0$ and $O_{\hat{w}\hat{b}\hat{j}} = 0$. Then, using (3.2.23), we get $y^2_{\hat{w}\hat{b}\hat{j}} < 1$. If $y^2_{\hat{w}\hat{b}\hat{j}}$ is relaxed, then it can take a fractional value thereby making an assignment of a branch to an unbuilt warehouse.

3.2.1.3 Risk and Expected Demand

This section explores the dependence of expected demand (ED) on the failure probabilities. Let the total risk-free demand be

$$TD = \sum_{b,j} d_{bj}.$$

Lemma 3.2.11. *If* $\underline{\mathrm{ED}} = (1 + P_*)\,\mathrm{TD}$ *and* $\overline{\mathrm{ED}} = (1 + P^*)\,\mathrm{TD}$, *then*

$$\underline{\mathrm{ED}} \leq \mathrm{ED} \leq \overline{\mathrm{ED}}. \tag{3.2.30}$$

Proof. Expected demand ED is given by the left-hand side of (3.2.9), from which a rearrangement of the terms gives

$$\mathrm{ED} = \sum_{b,j} d_{b,j} \left(1 + \sum_{w} y^2_{wbj} \sum_{w' \neq w} (p_{w'} y^1_{w'bj}) \right).$$

Using (3.2.7), (3.2.8) and the definition of TD yields

$$(1 + P_*)\,\mathrm{TD} \leq \mathrm{ED} \leq (1 + P^*)\,\mathrm{TD} \Longleftrightarrow \underline{\mathrm{ED}} \leq \mathrm{ED} \leq \overline{\mathrm{ED}}.$$

□

Corollary 3.2.12. *The additional warehouse capacity required to plan for risk is no more that* $P^*(\mathrm{TD})$*.*

If $P^* = 0$, then $P_* = 0$ and the problem is risk free. In this case, ED=TD. If $P_* = 1$, then $P^* = 1$ giving ED=2TD. This would mean that all warehouses would need a duplicate.

3.2.1.4 Scenario based Solution Algorithm

The numerical results to be presented in sections 3.3 and 3.4 will show that instances of model $\boldsymbol{RL_1M_r}$ may require long solution times, i.e., more than 24 hours. This section presents a solution strategy that solves a finite set of much simpler, i.e., they take less time to solve, models each based on a specific scenario. Unlike the scenarios in, for example, [82, 86], in this paper a scenario is a fixed number of warehouses of each size such that the total capacity of the warehouses is greater than or equal to $\underline{\mathrm{ED}}$, and, with the possible exception of one scenario, has total capacity less than $\overline{\mathrm{ED}}$.

Denote the set of scenarios by $\mathfrak{S}$. Elements of the set are vectors $\mathfrak{s}_i$ where i is the scenario index. Each component $\mathfrak{s}_i^s$ of $\mathfrak{s}_i$ gives the number of warehouses of size s to be built including those in the set E that have already been built. For each size s, define

$$q^s = \sum_{(w,s)\in E} x_w^s.$$

If, for a particular s, there is no w with $(w, s) \in E$, then set $q^s = 0$. This will be used in the third step of Algorithm 1.

Finally, the total number of warehouses needed for a scenario cannot exceed the upper limit on the total number of warehouses, that is, $\sum_s \mathfrak{s}^s \leq U$. Before we state Algorithm 1, which is used to determine $\mathfrak{S}$, we give a formal definition of a scenario.

Definition 3.2.13. The vector $\mathfrak{s}_i$ is a scenario if there is a corresponding x with $\sum_w x_w^s = \mathfrak{s}_i^s$ and an assignment vector y that satisfies (2.2.1), (2.2.3)- (2.2.5), (3.2.7), (3.2.8), and the revised capacity constraint

$$\sum_{b,j} d_{bj} \left(y_{wbj}^1 + y_{wbj}^2 P_* \right) \leq \sum_s V^s \; x_w^s, \; \forall \; w. \tag{3.2.31}$$

Algorithm 1: Determination of $\mathfrak{S}$.

1. Set $\mathfrak{S} = \emptyset$, $i = 1$, and $f(x_0) = \underline{\mathrm{ED}} - 1$.

2. Let $(x_w^s)^*$ be the optimal solution to

$$\begin{array}{lrcl} \text{Minimize} & f(x) & = & \sum_{w,s} \boldsymbol{V}^s \ x_w^s \\ \text{Subject to} & & & (2.2.1),\ (2.2.3)- \\ & & & (2.2.5),\ (3.2.7), \\ & & & (3.2.8),\ (3.2.31), \\ & f(x) & \geq & f(x_{i-1}) + 1, \\ & y_{wbj}^1 & \geq & 0,\ \forall w \in W,\ b \in B,\ j \in J, \\ & y_{wbj}^2 & \in & \{0,1\},\ \forall w \in W,\ b \in B,\ j \in J, \\ & x_w^s & \in & \{0,1\},\ \forall w \in W,\ s \in S. \end{array}$$

3. Calculate $\mathfrak{s}_i^s = \sum_w (x_w^s)^* + q^s$, $\forall s$, and $f(x_i) = \sum_{w,s} \boldsymbol{V}^s \ (x_w^s)^*$. Set $\mathfrak{S} = \mathfrak{S} \cup \{\mathfrak{s}_i\}$.

4. While $f(x_i) < \overline{\mathrm{ED}}$, replace i with $i+1$ and return to step 2.

Notice that the total capacity of the last scenario may well be larger than $\overline{\mathrm{ED}}$, while all other scenarios have a total capacity in the interval $[\underline{\mathrm{ED}}, \overline{\mathrm{ED}}]$. In Step 2 of Algorithm 1, the constraint $f(x) \geq f(x_{i-1}) + 1$ ensures that no two scenarios produced by the algorithm have the same capacity. A consequence is that the algorithm may not produce the complete set $\mathfrak{S}$. For example, if the possible warehouse sizes are 100 and 200, then $(2,0)$ and $(0,1)$ are two scenarios that give the same total warehouse capacity.

Assumption 2. That the complete scenario set contains no two scenarios with the same total warehouse capacity.

Theorem 3.2.14. *Under Assumption 2, Algorithm 1 produces the complete set of scenarios* $\mathfrak{S}$.

Proof. Let $\hat{\mathfrak{s}} \notin \mathfrak{S}$ be a scenario with corresponding function value $f(\hat{x})$. There exists a index i with $\mathfrak{s}_i \in \mathfrak{S}$ and $\mathfrak{s}_{i+1} \in \mathfrak{S}$ and $f(x_i) + 1 \leq f(\hat{x}) \leq f(x_{i+1})$. Since x_{i+1} gives the optimal objective value in iteration $(i+1)$ of Algorithm 1 we must have $f(\hat{x}) = f(x_{i+1})$ which violates Assumption 2. $\square$

For each $\mathfrak{s}_i \in \mathfrak{S}$, we formulate

$$
\begin{aligned}
\boldsymbol{RL_1M_{r\mathfrak{s}_i}}\text{: Minimize } \quad \hat{C}(x_i, y_i, z_i, Q_i, O_i) &= C_F(x) + \hat{C}_O(z, Q) \\
&\quad +\hat{C}_T(y, O) \\
\text{Subject to:} \quad & (2.2.1), (2.2.3) - (2.2.5), \\
& (3.2.7), (3.2.8), \\
& (3.2.14) - (3.2.17), \\
& (3.2.20) - (3.2.23), \\
& (3.2.26), \\
\sum_w x_w^s &= \mathfrak{s}_i^s, \ \forall\ s \in S, \\
x_w^s &\in \{0, 1\}, \ \forall\ s \in S, w \in W, \\
y_{wbj}^1 &\geq 0, \ \forall\ w \in W, b \in B, j \in J, \\
y_{wbj}^2 &\in \{0, 1\}, \ \forall\ w \in W, b \in B, \\
& \qquad j \in J.
\end{aligned}
$$

That is, $\boldsymbol{RL_1M_{r\mathfrak{s}_i}}$ is $\boldsymbol{RL_1M_r}$ with the additional constraint set

$$\sum_w x_w^s = \mathfrak{s}_i^s, \ \forall\ s \in S, \tag{3.2.32}$$

that ensures that the number of warehouses of each size that are built is determined by the scenario $\mathfrak{s}_i$. Theorem 3.2.15 shows that the solution to $\boldsymbol{RL_1M_r}$ can be obtained from the solutions $(x_i^*, y_i^*, z_i^*, Q_i^*, O_i^*)$ to the $\boldsymbol{RL_1M_{r\mathfrak{s}_i}}$.

Theorem 3.2.15. *Under Assumption 2, the optimal solution to model* $\boldsymbol{RL_1M_r}$ *is* $(x_k^*, y_k^*, z_k^*, Q_k^*, O_k^*)$ *where*

$$C(x_k^*, y_k^*, z_k^*, Q_k^*, O_k^*) = \min_{\mathfrak{s}_i \in \mathfrak{S}} C(x_i^*, y_i^*, z_i^*, Q_i^*, O_i^*).$$

Proof. Let $(\hat{x}, \hat{y}, \hat{z}, \hat{Q}, \hat{O})$ be an optimal solution to model $\boldsymbol{RL_1M_r}$ with corresponding objective function $\hat{C}$. Thus, $\hat{C} \leq C^*$. Let $\hat{\mathfrak{s}}$ be the scenario determined by $\hat{x}$, that is,

$$\hat{\mathfrak{s}}^s = \sum_w \hat{x}^s + q^s, \quad \forall s.$$

Theorem 3.2.14 implies that $\hat{\mathfrak{s}}$ would have been determined by algorithm 1. Thus, $C^* \leq \hat{C}$, and $(x_k^*, y_k^*, z_k^*, Q_k^*, O_k^*)$ is an optimal solution to $\boldsymbol{RL_1M_r}$. $\square$

3.2.1.5 Model L_2M_r

The idea for the second linearization method is to use the average failure probabilities. If we Look into the term $z^{2s}_{wbj} \sum_{w' \neq w} p_{w'}\ y^1_{w'bj}$ in (3.2.18), it is clear that, because of (3.2.7), we will end up having only one w', say $\hat{w'}$, in $\sum_{w' \neq w} p_{w'}\ y^1_{w'bj}$, such that $y^1_{\hat{w'}bj} = 1$. This leads us to have $z^{2s}_{wbj} \sum_{w' \neq w} p_{w'}\ y^1_{w'bj} = z^{2s}_{wbj}\ p_{\hat{w'}}$. So, in order to get rid of the nonlinearity, we can approximate the term $\sum_{w' \neq w} p_{w'}\ y^1_{w'bj}$ by $\sum_{w' \neq w} \frac{p_{w'}}{n-1}$, where n is the total number of warehouses and $n-1$ is the total number of warehouses w' such that $w' \neq w$. In other words, we take the average failure probability of warehouses w' such that $w' \neq w$. So, using this technique, (3.2.18) becomes

$$\begin{aligned} \hat{\hat{C}}_O(z) = \sum_{j,s,w} \Bigg[\nu^s_j \sum_b \ \ d_{bj} \Bigg(& z^{1s}_{wbj} \ \ (1-p_w) \\ & + z^{2s}_{wbj} \ \ \sum_{w' \neq w} \frac{p_{w'}}{n-1} \Bigg) \Bigg]. \end{aligned} \tag{3.2.33}$$

Also, (3.2.6) becomes

$$\begin{aligned} \hat{\hat{C}}_T(y) = \sum_{j,b,w} \Bigg[\ \ d_{bj} \tau_{wbj} \Bigg(& y^1_{wbj} \ \ (1-p_w) \\ & + y^2_{wbj} \sum_{w' \neq w} \frac{p_{w'}}{n-1} \Bigg) \Bigg]. \end{aligned} \tag{3.2.34}$$

Finally, (3.2.9) can be replaced by

$$\sum_{b,j} d_{bj} \left(y^1_{wbj} + y^2_{wbj} \sum_{w' \neq w} \frac{p_{w'}}{n-1} \right) \leq \sum_s \boldsymbol{V}^s x^s_w, \ \forall w \in W. \tag{3.2.35}$$

Thus, model $\boldsymbol{M_r}$ will be replaced by model $\boldsymbol{L_2M_r}$ which is

$$
\begin{aligned}
\boldsymbol{L_2M_r}\text{: Minimize} \quad \hat{\hat{C}}(x,y,z) \quad & = C_F(x) + \hat{\hat{C}}_O(z) + \hat{\hat{C}}_T(y) \\
\text{Subject to:} \quad & (2.2.1), (2.2.3) - (2.2.5), \\
& (3.2.7), (3.2.8), \\
& (3.2.14) - (3.2.17), \text{ and} \\
& (3.2.35), \\
x_w^s \quad & \in \{0,1\}, \ \forall s \in S, \forall w \in W, \text{ and} \\
y_{wbj}^r \quad & \in \{0,1\}, \ \forall w \in W, \forall b \in B, \forall j \in J, \\
& \qquad \forall r \in R.
\end{aligned}
$$

The variables y_{wbj}^r can be relaxed so that $y_{wbj}^r \geq 0$. Theorem 3.2.16 specified the case where we get fractional assignment after relaxing y_{wbj}^r variables and create model $\boldsymbol{RL_2M_r}$.

Theorem 3.2.16. *Let* $(x^*, y^*, z^*, Q^*, O^*)$ *be an optimal solution to* $\boldsymbol{RL_2M_r}$ *and suppose that the no-ties assumption, Assumption 1, is satisfied. If there exists* λ, ω, β, *and* ϕ *such that* $0 < (y_{\omega\beta\phi}^{\lambda})^* < 1$ *and if*

$$
W^* = \{\, w \,|\, 0 < (y_{w\beta\phi}^{\lambda})^* < 1 \} \tag{3.2.36}
$$

then there exists at most one warehouse in W^* *that is not running at full capacity, that is,*

$$
\sum_{b,j} d_{bj} \left(y_{wbj}^1 + y_{wbj}^2 \sum_{w' \neq w} \frac{p_{w'}}{n-1} \right) < \sum_s \boldsymbol{V}^s x_w^s,
$$

where s is the size of built warehouse w.

Proof. Analogous to the proof of Theorems 2.2.1 and 3.2.9. $\square$

Note that, in Theorem 3.2.16, both of y^1_{wbj} and y^2_{wbj} variables were relaxed.

It is not necessary that model $\boldsymbol{RL_2M_r}$ to be easier to be solved, i.e. faster, compared with the models $\boldsymbol{RL_1M_{r\mathfrak{s}_i}}$. However, model $\boldsymbol{RL_2M_r}$ will be so helpful and faster when we have high number of scenarios in models $\boldsymbol{RL_1M_{r\mathfrak{s}_i}}$.

3.2.2 The Solution Methodology

For a particular instance, the goal is to have a solution to $\boldsymbol{M_r}$. We do this by first attempting to solve $\boldsymbol{L_1M_r}$ and $\boldsymbol{L_2M_r}$. Starting with model $\boldsymbol{L_1M_r}$, if the software package, e.g., CPLEX, finds a solution in *reasonable time*, then we are done. Unfortunately, as we will see in Sections 3.3 and 3.4, we can expect that CPLEX will not find a solution to model $\boldsymbol{L_1M_r}$ within, say, 24 hours. Then, we try to solve model $\boldsymbol{RL_1M_r}$, and again in some cases,CPLEX is expected to be unable to get an optimal solution within 24 hours. We then use Algorithm 1 to find a set of scenarios $\mathfrak{S}$. For each scenario $\mathfrak{s}_i$ in $\mathfrak{S}$, we solve $\boldsymbol{RL_1M_{r\mathfrak{s}_i}}$ and determine the index k as in the statement of Theorem 3.2.15, which gives us $(x^*_k, y^*_k, z^*_k, Q^*_k, O^*_k)$. Set $(x^*, y^*, z^*, Q^*, O^*) = (x^*_k, y^*_k, z^*_k, Q^*_k, O^*_k)$, which is a solution to $\boldsymbol{RL_1M_r}$. If y^*_k is binary, then x^*_k and y^*_k give a solution to $\boldsymbol{L_1M_r}$ and $\boldsymbol{M_r}$. If y^*_k is not binary, update the set of built warehouses E to include all built warehouses indicated by x^* and solve model $\boldsymbol{L_1M_r}$. Use this as the best solution to model $\boldsymbol{M_r}$. Since models $\boldsymbol{RL_1M_r}$ and $\boldsymbol{RL_1M_{rs_i}}$ are relaxed versions of model $\boldsymbol{L_1M_r}$, when solving instances of those problems, we setup the optimality gap to 1%.

As for models $\boldsymbol{L_2M_r}$ and $\boldsymbol{RL_2M_r}$, we start by solving model $\boldsymbol{L_2M_r}$ and in case of not getting an optimal solution within 24 hours, we solve model $\boldsymbol{RL_2M_r}$.

3.3 Canadian Case Study-Risk

3.3.1 Introduction

The Canadian case study of Section 2.3 was extended to include risk. Each built warehouse can fail with the assumption, as mentioned above, that no primary and backup warehouses fail for the same branch. As in [36] and [63] the warehouses failure probabilities p_w were randomly generated from $U \sim [0, 0.05]$. Figure 3.2 shows that the solutions to model $\boldsymbol{L_1M_r}$ and model $\boldsymbol{L_2M_r}$ will select new warehouses and assign to them along with the built warehouses, if any, branches as primary and backup warehouses

3.3.2 Solution to Model $\boldsymbol{L_1M_r}$

Model $\boldsymbol{L_1M_r}$ was solved using CPLEX Optimization Studio 12.10.0 on an Acer Intel $i7$ laptop with 16 GB of RAM and 3.30 GHz processor with four cores. Model $\boldsymbol{L_1M_r}$ has a total of $64,564$ variables, $10,844$ of which are binary variables.

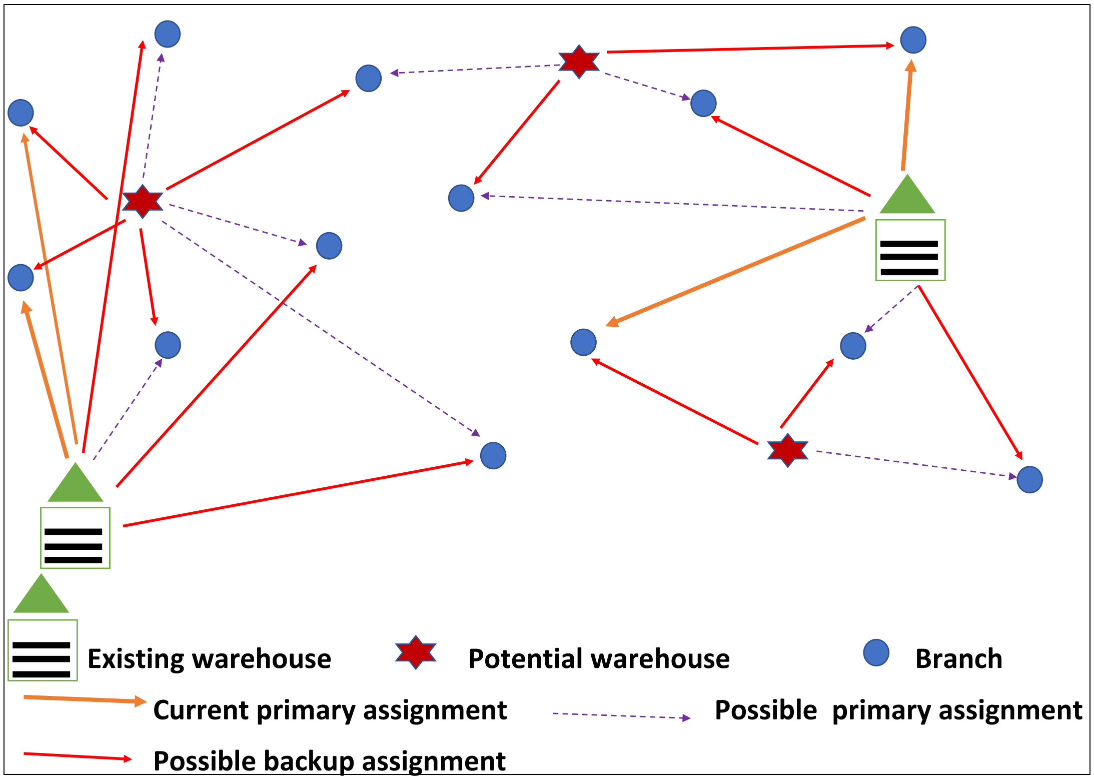

Figure 3.2: Primary and backup assignments Example

Table 3.2 shows that CPLEX returned a feasible, but not optimal, solution to model $\boldsymbol{L_1M_r}$. The objective function value was 3.9309×10^6. In addition to the existing warehouses, two medium and one large warehouses were built. Then we solved model $\boldsymbol{RL_1M_r}$ and found that in less than three hours, CPLEX returned an optimal solution of 3.9288×10^6. Built warehouses in the solution to models $\boldsymbol{L_1M_r}$ and $\boldsymbol{RL_1M_r}$ are identical.

We now use our scenario algorithm. We start with Algorithm 1. We calculate $\underline{\text{ED}} = 3,883,824.86$ and $\overline{\text{ED}} = 4,064,868.04$. The first iteration produced the scenario $\mathfrak{s}_1 = (0,2,3)$ with $f(x_1) = 4,061,484$ and the second iteration produced $\mathfrak{s}_2 = (1,0,4)$ with $f(x_2) = 4,138,845.6$ in the second iteration. Note that $f(x_2)$ is higher than $\overline{\text{ED}}$. Thus, the set of scenarios is $\mathfrak{S} = \{\mathfrak{s}_1, \mathfrak{s}_2\}$. CPLEX took less than 30 minutes to get an optimal solution of 3.9285×10^6 to model $\boldsymbol{RL_1M_{r\mathfrak{s}_1}}$. For model $\boldsymbol{RL_1M_{r\mathfrak{s}_2}}$ CPLEX took about an hour to get an optimal solution of 4.0062×10^6. Thus, the optimal solution to model $\boldsymbol{RL_1M_r}$ is when we use the first scenario $\mathfrak{s}_1$. Some values of the y^1_{wb} variables in the solution to model $\boldsymbol{RL_1M_r}$, are not binary, so we set up

$$E = \{\,(1,3),\ (2,3),\ (4,2), (28,3), (33,2)\}. \tag{3.3.1}$$

The last column of Table 3.2 shows that CPLEX solved the modified $\boldsymbol{L_1M_r}$ in 2 minutes. The optimal objective function is 3.9309×10^6. From this example, we conjecture that the the solution methodology should omit the first two steps, that is, the solution of $\boldsymbol{L_1M_r}$ and $\boldsymbol{RL_1M_r}$.

Table 3.2: Numerical Results for the Canadian case study using CPLEX.

	$\boldsymbol{L_1M_r}$	$\boldsymbol{RL_1M_r}$	$\boldsymbol{RL_1M_{r\mathfrak{s}_1}}$	$\boldsymbol{L_1M_r}$ with (3.3.1)
Total Variables	64,564	64,564	64,564	64,561
Binary Variables	10,844	5,472	5,472	10,841
Constraints	220,639	226,011	226,013	220,642
Iterations	19,432,300	13,319,054	1,520,545	770,206
Time (Hr:Min:Sec)	24:00:00	02:51:03	00:29:05	00:02:01
Best Objective $\times 10^6$	3.9309	3.9288	3.9285	3.9309
Objective Bound $\times 10^6$	3.8966	3.8915	3.8895	3.9305
Warehouses (w, s)	(1,3), (2,3) (4,2), (28,3) (33,2)	(1,3), (2,3) (4,2), (28,3) (33,2)	(1,3), (2,3) (4,2), (28,3) (33,2)	(1,3), (2,3) (4,2), (28,3) (33,2)
Status	Feasible	Optimal	Optimal	Optimal

In Table 3.3, we show the number of branches assigned to the built warehouses as the primary warehouse and as the secondary warehouse. The Total demand is given by TD = 3,881,722.2 and this is the total capacity needed to satisfy demand from the primary branches. The total excess capacity in the warehouses is 179,761.8 which is enough to satisfy the expected demand from the secondary warehouses. The total available capacity is just below $\overline{\mathrm{ED}}$. As P^* increases, $\overline{\mathrm{ED}}$ increases, and more total capacity would have to be built.

Table 3.3: Comparison of Warehouse Allocation and Capacity - Canada.

Warehouse	#Primary Branches	#Secondary Branches	Available Capacity	Primary Allocation	Excess Capacity	Secondary Allocation
1 (Large)	24	19	967,020	966,670.1	349.9	349.8
2 (Large)	49	10	967,020	964,490.3	2,529.7	2,529.7
4 (Medium)	22	9	580,212	569,168.9	11,043.1	11,042.3
28 (Large)	47	74	967,020	933,328.6	33,691.4	18,398.6
33 (Medium)	16	46	580,212	448,064.3	132,147.7	4,182.1
Totals	158	158	4,061,484	3,881,722.2	179,761.8	36,502.5

3.3.3 Solution to Model $\boldsymbol{L_2M_r}$

Models $\boldsymbol{L_2M_r}$ and $\boldsymbol{RL_2M_r}$ were solved using CPLEX on the Acer laptop described in Section 3.3.2.
The second column of Table 3.4 shows that it took CPLEX almost one hour to get an optimal solution to model $\boldsymbol{L_2M_r}$. The third column of Table 3.4 shows that CPLEX spent less than 17 minutes with around 700 thousand iterations to solve model $\boldsymbol{RL_2M_r}$.

Update the set E to be

$$E = \{\,(1,3),\ (2,3),\ (4,2), (28,2), (30,3)\}. \tag{3.3.2}$$

CPLEX took 16 seconds with only 461 iterations to solve model $\boldsymbol{L_2M_r}$ with (3.3.2). Table 3.4 shows that CPLEX got an objective function of 3.9977×10^6 to model $\boldsymbol{L_2M_r}$ and 3.9956×10^6 to model $\boldsymbol{RL_2M_r}$.

Table 3.4: Numerical results for the Canadian case study using CPLEX.

Model	$\boldsymbol{L_2M_r}$	$\boldsymbol{RL_2M_r}$	$\boldsymbol{L_2M_r}$ with (3.3.2)
Total Variables	43,076	43,076	43,073
Binary Variables	10,844	100	10,841
Constraints	134,687	134,687	134,687
Iterations	2,357,452	697,601	461
Time (Hr:Min:Sec)	00:59:13	00:16:49	00:00:16
Best Objective $\times 10^6$	3.9977	3.9956	3.9977
Objective Bound $\times 10^6$	3.9803	3.9297	3.9975
Built warehouses (w, s)	(1, 3), (2, 3) (4, 2), (28, 2) (30, 3)	(1, 3), (2, 3) (4, 2),(28, 2) (30, 3)	(1, 3), (2, 3) (4, 2),(28, 2) (30, 3)
Status	Optimal	Optimal	Optimal

Further, built warehouses in models $\boldsymbol{L_2M_r}$ and $\boldsymbol{RL_2M_r}$ were of the same sizes as those in model $\boldsymbol{RL_1M_{r\mathfrak{s}_1}}$ in Table 3.2. The only difference in the selected warehouses is that the size of the warehouse 28 became medium instead of large size and the warehouse 30 with large size replaced the warehouse 33 with medium size.

To summarize, the total built capacities of the selected warehouses for model $\boldsymbol{RL_1M_{r\mathfrak{s}_1}}$ and model $\boldsymbol{RL_2M_r}$ were exactly the same. Further, the warehouses selected using model $\boldsymbol{RL_1M_{r\mathfrak{s}_1}}$ were exactly the same as those selected using model $\boldsymbol{LM}$. We also found that there were 3.53% and 2.38% unused capacities in the solutions to models $\boldsymbol{RL_1M_{r\mathfrak{s}_1}}$ and $\boldsymbol{L_2M_r}$, respectively, compared to 4.44% unused capacity in the solution to model $\boldsymbol{LM}$. Further, the total cost in model $\boldsymbol{L_1M_r}$ is 3.9309×10^6, and it is 3.9977×10^6 in model $\boldsymbol{L_2M_r}$ with. Hence, we got a 1.7% higher cost in model $\boldsymbol{L_2M_r}$ than in model $\boldsymbol{L_1M_r}$. Finally, the cost in models $\boldsymbol{L_1M_r}$and $\boldsymbol{L_2M_r}$ are higher by 0.44% and 2.14%, respectively, than the cost in the solution of model $\boldsymbol{LM}$.

Selected warehouses in models $\boldsymbol{L_2M_r}$ and $\boldsymbol{RL_2M_r}$ were forced to exist in model $\boldsymbol{L_1M_r}$. Table 3.5summarizes the results using CPLEX. CPLEX gave an objective function of 3.9393×10^6 which is higher by only 0.21% than the objective function value in the solution to model $\boldsymbol{L_1M_r}$ with (3.3.1) in Table 3.2. So, one can recommend getting the needed warehouses using model $\boldsymbol{RL_2M_r}$ and force them to exist in model $\boldsymbol{L_1M_r}$. This way can save some time in case of having a lot of solution scenarios in models $\boldsymbol{RL_1M_{r\mathfrak{s}_i}}$.

Table 3.5: Numerical results for the Canadian case study using CPLEX.

Model	$\boldsymbol{L_1 M_r}$ with (3.3.2)
Total Variables	64,561
Binary Variables	10,841
Constraints	220,642
Iterations	411,700
Time (Hr:Min:Sec)	00:00:58
Best Objective $\times 10^6$	3.9393
Objective Bound $\times 10^6$	3.9389
Built Warehouses (w, s)	(1, 3), (2, 3), (4, 2) (28, 2),(30, 3)
Status	Optimal

3.3.4 Sensitivity Analysis on Failure Probabilities

In the previous sections, risk probabilities were set between $0-5\%$. In this section, with the same limit of five warehouses are allowed to be built, probabilities were increased to cover the cases of $5-10\%$, $10-15\%$, $15-20\%$, and $20-25\%$. When the probabilities $25-30\%$ are used, more than five warehouses are needed to be built.

Table 3.6: Impact of Failure probability on cost ($M) and the Sizes of Warehouses.

p	$C(x,y)$	Large	Medium	Small
5-10%	4.1224	4	1	0
10-15%	4.1968	4	1	0
15-20%	4.4089	5	0	0
20-25%	4.5975	5	0	0

From Table 3.6, one can find that when the probabilities are between $5-10\%$, four large and one medium warehouses are needed to cover the total demand with an objective of 4.1224×10^6. So compared to the $0-5\%$, one medium warehouse became large and the rest are the same. The same capacities of the warehouses are built when the probabilities are between $10-15\%$ and the objective function increased by 1.8% and became 4.1968×10^6. When the probabilities are increased to be $15-20\%$ and $20-25\%$, all five warehouses are built of large sizes with objective functions of 4.4089×10^6 and 4.5975×10^6, respectively.

3.4 European Example - Risk

The created example in Section 2.4 of 37 European cities was used here with the addition of failure probabilities of warehouses of $0-5\%$. The same limit on the number of built warehouses that is five was applied here.

3.4.1 Solution to model $\boldsymbol{L_1M_r}$

3.4.1.1 Flexible Warehouses

Table 3.7 shows that after 24 hours CPLEX returned a feasible, but not optimal, solution to model $\boldsymbol{L_1M_r}$ with an objective function of 3.2680×10^6 and selected four large and one small size warehouses to be built. Then CPLEX was applied on $\boldsymbol{RL_1M_r}$. CPLEX returned a feasible, but not optimal, solution of 3.2661×10^6 in 24 hours. The warehouses to be built were the same as in the solution to $\boldsymbol{L_1M_r}$, except the fourth large warehouse is built at location 35 rather than 23. We now use the scenario Algorithm 1 with $\underline{\mathrm{ED}} = 3,897,202.62$ and $\overline{\mathrm{ED}} = 4,082,005.04$. We found that we will need to have two scenarios. The first scenario is $\mathfrak{s}_1 = (0,2,3)$ with $f(x_1) = 4,061,484$ and the second scenario is $\mathfrak{s}_2 = (1,0,4)$ with $f(x_2) = 4,138,845.60$. Thus, the set of scenarios is $\mathfrak{S} = \{\mathfrak{s}_1, \mathfrak{s}_2\}$. After around 3.5 hours CPLEX returned an optimal solution of 3.3510×10^6 to model $\boldsymbol{RL_1M_{r\mathfrak{s}_1}}$. On the other hand, Table 3.7 shows that CPLEX took almost 2.5 hours to get an optimal solution of 3.2655×10^6 to model $\boldsymbol{RL_1M_{r\mathfrak{s}_2}}$. Using

Theorem 3.2.15, the optimal solution to model $\boldsymbol{RL_1M_r}$ is 3.2655×10^6. We found that the solution to model $\boldsymbol{RL_1M_r}$ had fractional y^1_{wbj} values, so we replaced $E = \emptyset$ with

$$E = \{\,(7,3),(12,3),(20,3),(21,1),(35,3)\,\} \tag{3.4.1}$$

and solved the modified model $\boldsymbol{L_1M_r}$. CPLEX found on optimal solution with an objective value of 3.2673×10^6 in thirty-seven seconds. This example supports the conjecture that the solution methodology should omit the first two steps and begin with the scenario algorithm.

Table 3.7: Numerical results for the European example with flexible warehouses using CPLEX.

	$\boldsymbol{L_1M_r}$	$\boldsymbol{RL_1M_r}$	$\boldsymbol{RL_1M_{rs_2}}$	$\boldsymbol{L_1M_r}$ with (3.4.1)
Total Variables	82,251	82,251	82,251	82,246
Binary Variables	13,801	6,956	6,956	13,796
Constraints	281,090	287,935	287,937	281,095
Iterations	85,999,261	64,843,202	6,886,512	13,042
Time (Hr:Min:Sec)	24:00:00	24:00:00	02:26:48	00:00:37
Best Objective $\times 10^6$	3.2680	3.2661	3.2655	3.2673
Objective Bound $\times 10^6$	3.0037	3.0646	3.2246	3.2669
Built warehouses (w,s)	(7,3),(12,3) (20,3),(21,1) (23,3)	(7,3), (12,3) (20,3), (21,1) (35,3)	(7,3), (12,3) (20,3), (21,1) (35,3)	(7,3), (12,3) (20,3), (21,1) (35,3)
Status	Feasible	Feasible	Optimal	Optimal

Table 3.8 is analogous to Table 3.3. It shows the number of branches assigned to the built warehouses as the primary warehouse and as the secondary warehouse. The Total capacity needed to satisfy demand from the primary branches is 3,889,658.82 with a total excess capacity in the warehouses is 249,186.78 which is enough to satisfy the expected demand from the secondary warehouses of 91,402.51. The total available capacity is just above $\overline{\text{ED}}$. As P^* increases, $\overline{\text{ED}}$

increases, and more total capacity would have to be built.

Table 3.8: Comparison of Warehouse Allocation and Capacity - Flexible Warehouses.

Warehouse	#Primary Branches	#Secondary Branches	Available Capacity	Primary Allocation	Excess Capacity	Secondary Allocation
7 (Large)	60	31	967,020	923,216.52	43,803.48	33,383.38
12 (Large)	37	77	967,020	953,372.20	13,647.8	13,508.84
20 (Large)	41	49	967,020	803,924.17	163,095.83	29,968.89
21 (Small)	9	6	270,765.60	267,507.11	3,258.49	1,235.80
35 (Large)	38	22	967,020	941,638.82	25,381.18	13,305.60
Totals	185	185	4,138,845.6	3,889,658.82	249,186.78	91,402.51

3.4.1.2 Specialized warehouses

As in Section 2.4.2.2, some warehouses will be specialized in some categories. The same assumptions were applied, in this section, on the primary and backup levels. For example, in Section 2.4.2.2 it was assumed that warehouses 1 to 10 do not serve product categories 1 and 2, and in this section same warehouses will not serve product categories 1 and 2 whether as primary or backup warehouses and so on with the other assumptions. We started by solving model $\boldsymbol{L_1 M_r}$. Table 3.9 shows that after 24 hours, CPLEX returned a feasible, but not optimal, solution to model $\boldsymbol{L_1 M_r}$. The objective function is 3.8086×10^6. A small warehouse was built at location 17 and large warehouses were built at locations 6, 9, 12, and 14. On the other hand, CPLEX took 7 hours and 47 minutes to return an optimal solution to model $\boldsymbol{RL_1 M_r}$. The objective function value is 3.8085×10^6. The warehouses to be built were the same as for model $\boldsymbol{L_1 M_r}$. Using Algorithm 1, as in the case of flexible warehouses, we found that we need to have two scenarios to the case of specialized warehouses. The first scenario is $\mathfrak{s}_1 = (0, 2, 3)$ and the second scenario is $\mathfrak{s}_2 = (1, 0, 4)$. CPLEX took 1.5 hours to get an optimal solution of 3.8219×10^6 to model $\boldsymbol{RL_1M_{r\mathfrak{s}_1}}$. On the other hand, Table 3.9 shows that CPLEX took around one hour and 18 minutes to solve model $\boldsymbol{RL_1M_{r\mathfrak{s}_2}}$ and gave an objective function value of 3.8087×10^6 which is the solution to model $\boldsymbol{RL_1M_r}$.

We found that the solution to $\boldsymbol{RL_1M_r}$ had fractional y^1_{wbj} values, so we set

$$E = \{ (6, 3), (9, 3), (12, 3), (14, 3), (17, 1) \}. \tag{3.4.2}$$

The last column in Table 3.9 shows that it took CPLEX 18 seconds to get an optimal solution to model $\boldsymbol{L_1M_r}$ with (3.4.2) with an objective function of 3.8088×10^6 which is higher by 16.6% than the case of flexible warehouses.

Table 3.9: Numerical Results of the European Example with specialized warehouses using CPLEX.

Model	$\boldsymbol{L_1M_r}$	$\boldsymbol{RL_1M_r}$	$\boldsymbol{RL_1M_{r\mathfrak{s}_2}}$	$\boldsymbol{L_1M_r}$ with (3.4.2)
Total Variables	82,251	82,251	82,251	82,246
Binary Variables	13,801	6,956	6,956	13,796
Constraints	286,048	292,891	292,893	286,053
Iterations	67,186,435	19,873,951	4,612,400	7,497
Time (Hr:Min:Sec)	24:00:00	07:47:31	01:18:33	00:00:18
Best Objective $\times 10^6$	3.8086	3.8085	3.8087	3.8088
Objective Bound $\times 10^6$	3.7971	3.7721	3.7817	3.8074
Built warehouses (w, s)	(6, 3), (9, 3) (12, 3), (14,3) (17, 1)	(6, 3), (9, 3) (12, 3), (14,3) (17, 1)	(6, 3), (9, 3) (12, 3), (14,3) (17, 1)	(6, 3), (9, 3) (12, 3), (14,3) (17, 1)
Status	Feasible	Optimal	Optimal	Optimal

Table 3.10 gives information to analyze the allocation of branches to primary and secondary warehouses.

Table 3.10: Comparison of Warehouse Allocation and Capacity - Specialized Warehouses.

Warehouse	#Primary Branches	#Secondary Branches	Available Capacity	Primary Allocation	Excess Capacity	Secondary Allocation
6 (Large)	58	39	967,020	865,737.56	101,282.44	25,047.14
9 (Large)	50	42	967,020	936,177.81	30,842.19	28,967.27
12 (Large)	51	40	967,020	916,995.20	50,024.8	49,999.82
14 (Large)	19	61	967,020	912,910.98	54,109.02	54,100.83
17 (Small)	7	3	270,765.60	257,837.27	12,928.33	952.88
Totals	185	185	4,138,845.6	3,889,658.82	249,186.78	159,067.94

3.4.2 Solution to model $\boldsymbol{L_2M_r}$

3.4.2.1 Flexible Warehouses

Table 3.11 shows that it took CPLEX around four hours to get an optimal solution to model $\boldsymbol{L_2M_r}$ and around 48 minutes to get an optimal solution to model $\boldsymbol{RL_2M_r}$. Built warehouses are identical in the solution to both models and have the same size of what we got from the solutions to models $\boldsymbol{L_1M_r}$ and $\boldsymbol{RL_1M_{rs_2}}$ in Table 3.7. Warehouses 7, 12, and 21 are common in the solutions to models $\boldsymbol{L_1M_r}$ and $\boldsymbol{L_2M_r}$.

The built warehouse in Table 3.11 were forced into model $\boldsymbol{L_1M_r}$ by replacing $E = \emptyset$ with

$$E = \{\, (7,3), (12,3), (13,3), (14,3), (21,1) \,\}. \tag{3.4.3}$$

Table 3.12 shows that CPLEX found on optimal solution to $\boldsymbol{L_1M_r}$ with (3.4.3) in 14 seconds with an objective function of 3.3061×10^6 which is higher by only 1.19% than what we got from the solution to $\boldsymbol{L_1M_r}$ with (3.4.1) in Table 3.7.

Table 3.11: Numerical Results of the European Example with flexible warehouses using CPLEX.

Model	L_2M_r	RL_2M_r
Total Variables	54,871	54,871
Binary Variables	13,801	111
Constraints	171,570	185,260
Iterations	6,558,957	869,700
Time (Hr:Min:Sec)	04:08:02	00:48:21
Best Objective $\times 10^6$	3.2682	3.2660
Objective Bound $\times 10^6$	3.2651	3.2650
Built warehouses (w, s)	(7,3),(12,3) (13,3),(14,3) (21,1)	(7,3),(12,3) (13,3),(14,3) (21,1)
Status	Optimal	Optimal

Table 3.12: Numerical Results of the European Example with flexible warehouses using CPLEX.

Model	L_1M_r with (3.4.3)
Total Variables	82,246
Binary Variables	13,796
Constraints	286,053
Iterations	50,499
Time (Hr:Min:Sec)	00:00:14
Best Objective $\times 10^6$	3.3061
Objective Bound $\times 10^6$	3.3057
Built warehouses (w, s)	(7,3),(12,3),(13,3) (14,3), (21,1)
Status	Optimal

3.4.2.2 Specialized warehouses

With the case of specialized warehouses, Table 3.13 shows that it took CPLEX around 13 minutes to get an optimal solution to model L_2M_r and around 10 minutes to get an optimal solution to model RL_2M_r. Built warehouses are identical to what we got in the solution to model L_1M_r with specialized warehouses in Table 3.9. Thus, forcing the built warehouses in Table 3.13 into model L_1M_r

will give the same objective function that we got for model $\boldsymbol{L_1M_r}$ with (3.4.2) in Table 3.9.

Table 3.13: Numerical Results of the European Example with specialized warehouses using CPLEX.

Model	$\boldsymbol{L_2M_r}$	$\boldsymbol{RL_2M_r}$
Total Variables	54,871	54,871
Binary Variables	13,801	111
Constraints	176,528	190,218
Iterations	350,516	269,859
Time (Hr:Min:Sec)	00:13:44	00:10:23
Best Objective $\times 10^6$	3.7349	3.7316
Objective Bound $\times 10^6$	3.7334	3.7303
Built warehouses (w, s) (w, s)	(6,3),(9,3),(12,3) (14,3),(17,1)	(6,3),(9,3),(12,3) (14,3),(17,1)
Status	Optimal	Optimal

3.5 Conclusion

This chapter presented a cubic risk mathematical model for the optimal selection of warehouses and the assignment of branches to warehouses under risk of warehouse failure. Two methods were used to linearize the model. Then a scenario-based algorithm with which to solve the model was presented, and for the presented examples, the number of scenarios is quite small. A theorem showed that scenarios subproblems produces an optimal solution to the relaxed problem. The Canadian case study and the created European example showed the scenarios subproblems are quickly solved. The solution to the Canadian case study showed that for warehouses failure with probabilities between 0% and 5%, the total built capacity is exactly the same as those for which no risk exists. The built capacity increases as we increase the failure probability until 25%, and then the problem requires more than five warehouses to be built. In the European example, again with the same warehouse failure probabilities between 0% and 5%, we got the same built capacity as in the no risk case. The introduction of specialized warehouses increased the total cost function by 16.6%.

Future work includes consideration objective functions that penalize unmet demand, models allowing branches to be supplied from more than one warehouse, and a sensitivity analysis to examine the trade-off between risk avoidance and transportation costs. Finally, future work should address the assumption that no two scenarios have the same total capacity including its connection to the existence of multiple non-negative solutions to the Diophantine $\sum_s V^s \mathfrak{s}^s = f(\mathfrak{s}_i^*)$.

Chapter 4

Facility Fortification under Risk of Failure

4.1 Facility Fortification

This chapter presents the problem of how to chose optimal warehouse locations and sizes and which warehouses to fortify against disruption. The term "fortification" was used to represent the action taken against any expected disruption to the warehouses, whether natural or man made. According to [28], facility fortification can be done by using approaches such as (1) enhancing security systems in the facility; (2) gathering intelligence about potentially disruptive events; and (3) stockpiling critical components so that a facility can be returned to service as soon as possible after a disruption. Warehouses can become completely reliable once they are fortified [78]. A non zero probability of failure was assigned to all warehouses, which means that all warehouses face some risk of failure. A mixed integer non linear mathematical model was developed in this chapter. The model will help in selecting the location and sizes of warehouses, as well as which warehouses to be fortified. As in [63], each branch will be assigned to either a fortified warehouse as its primary warehouse or a non-fortified warehouse as its primary warehouse and a fortified warehouse as its backup warehouse. Thus, each branch should be assigned to a fortified warehouse, whether it is a primary or backup warehouse. The demand of the branch will be satisfied by its primary warehouse, unless that warehouse fails in case that it is not fortified. When a primary warehouse fails, the backup warehouse will satisfy some of its demand, the expected demand calculated from the probability of failure of the primary warehouse and the total demand. Figure 4.1 shows the idea of warehouse fortification. It shows a network of existing and potential warehouses, both fortified and non-fortified. It also shows how the branches are assigned to these warehouses.

According to [36], there is a scarcity of published papers in the area of capacitated facilities, interdiction, and fortification budget limit tri-level formulation. The tri-level means defender-attacker-defender. The first defender determines which facilities to be fortified to protect the system. The attacker identifies the unfortified facilities to impose the maximal harm to the system. The second defender tries to minimize the overall harm to the system. The authors mentioned that this type of problems is hard to solve. As a contribution, this dissertation considers the fortification of warehouses with mul-

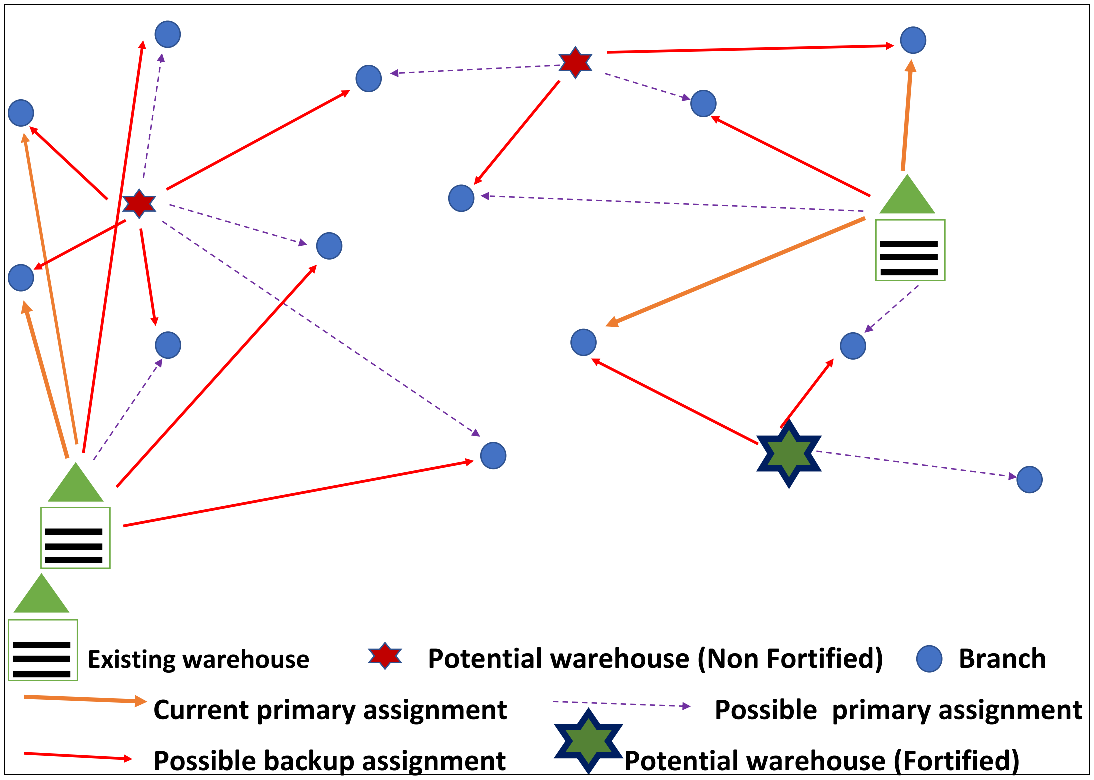

Figure 4.1: Fortification: Primary and backup assignments example

tiple capacity levels while considering multiple part category types. It also considers the specialized warehouses. As in [63], fortification costs will be calculated as a percentage of the fixed cost. The Canadian case study and the European example, introduced in the previous chapters, were extended in this chapter to cover the case of facility fortification.

4.2 Mathematical Model

This section presents a developed mathematical model for determining the location and size of built warehouses and determine which warehouses need to be fortified against disruption. Let ρ_w be the percentage of fixed cost that represents the fortification cost for warehouse w. So, $\rho_w(f_w + \ell_w)$ is the cost per square foot, in dollars, during the planning horizon, for a warehouse at location w to be fortified . Let γ_w equal 1 if and only if a warehouse w is fortified. The fixed and fortification cost in dollars, during the planning horizon, is

$$C_{F_f}(x,\gamma) = \sum_w \Big(\big(f_w + \ell_w\big)\big(1 + \rho_w\ \gamma_w\big)\Big) \sum_s A^s x_w^s. \tag{4.2.1}$$

The expected operational cost, in dollars, during the planning horizon is

$$\begin{aligned} C_{O_f}(z,y,\gamma) = \sum_{j,s,w} \Bigg[\ \nu_j^s \sum_b d_{bj} \Big(z_{wbj}^{1s} \ \ \big(1 - p_w\big(1 - \gamma_w\big)\big) \\ + z_{wbj}^{2s} \ \ \sum_{w' \neq w} p_{w'}\ y_{w'bj}^1 \Big)\Bigg]. \end{aligned} \tag{4.2.2}$$

$C_{O_f}(z,y,\gamma)$ can be explained in the same way that (3.2.18) was explained with the consideration that the term $\Big(1 - p_w(1-\gamma_w)\Big)$ is 1 if the warehouses w is fortified and $1 - p_w$ if it was not fortified.

The expected transportation cost, in dollars, during the planning horizon is

$$\begin{aligned} C_{T_f}(y,\gamma) = \sum_{j,b,w} \Bigg[\ d_{bj}\tau_{wbj} \Big(y_{wbj}^1 \ \ \big(1 - p_w(1 - \gamma_w)\big) \\ + y_{wbj}^2 \sum_{w' \neq w} p_{w'} \ \ y_{w'bj1} \Big)\Bigg]. \end{aligned} \tag{4.2.3}$$

$C_{T_f}(y,\gamma)$ can be explained in the same way that the term $\Big(1 - p_w(1 -$

$\gamma_w)\Big)$ and (3.2.18) were explained.

The complete cost function to be minimized is

$$C_f(x, y, z, \gamma) = C_{F_f}(x, \gamma) + C_{O_f}(z, y, \gamma) + C_{T_f}(y, \gamma)$$

.

To ensure that each branch b is assigned to a single primary warehouse to meet its demand of items from category j, we add the constraints

$$\sum_w y^1_{wbj} = 1, \ \forall b \in B, \ \forall j \in J. \tag{4.2.4}$$

To ensure that a branch b is assigned to a single backup warehouse, in case that its primary warehouse is not fortified, to meet its demand of items from category j, we add the constraints

$$\sum_w y^2_{wbj} = 1 - \sum_w y^1_{wbj} \ \gamma_w, \ \forall b \in B, \ \forall j \in J. \tag{4.2.5}$$

To ensure that a fortified warehouse is a built warehouse, we add the constraints

$$\gamma_w \leq \sum_s x^s_w, \ \forall w \in W. \tag{4.2.6}$$

To ensure the existence of at least one fortified warehouse in the network, we add the constraint

$$\sum_w \gamma_w \geq 1. \tag{4.2.7}$$

To ensure that for each branch b, either its primary or backup warehouse is fortified, we add the constraints

$$\sum_{w} \left(y^1_{wbj} + y^2_{wbj}\right)\gamma_w = 1,\ \forall b \in B,\ \forall j \in J. \tag{4.2.8}$$

Putting everything together, the binary optimization problem is to

$$\begin{aligned}
\boldsymbol{M_{rf}}\textbf{:}\ \text{Minimize}\ C_f(x,y,z,\gamma) &= C_{F_f}(x,\gamma) + C_{O_f}(z,y,\gamma) + C_{T_f}(y,\gamma) \\
\text{Subject to:}\quad & (2.2.1), (2.2.3) - (2.2.5), (3.2.8), (3.2.9), \\
& (3.2.14) - (3.2.17), (4.2.4) - (4.2.8), \\
x^s_w \quad & \in \{0,1\},\ \forall s \in S, \forall w \in W, \\
y^r_{wbj} \quad & \in \{0,1\},\ \forall w \in W, \forall b \in B, \forall j \in J, \\
& \qquad\quad \forall r \in R, \\
\gamma_w \quad & \in \{0,1\},\ \forall w \in W.
\end{aligned}$$

4.3 Linearization and Relaxation to Model $\boldsymbol{M_{rf}}$

Model $\boldsymbol{M_{rf}}$ has non-linear terms in $C_f(x,y,z,\gamma)$ and (3.2.9). The same techniques that were used to linearize models $\boldsymbol{M}$ and $\boldsymbol{M_r}$ were used to linearize model $\boldsymbol{M_{rf}}$.

4.3.1 Model $\boldsymbol{L_1 M_{rf}}$

Define Q and O exactly as they were defined in Section 3.2.1.1. Further, the linearization technique used in model $\boldsymbol{M}$ to linearize the multiplication of the variables x^s_w and y_{wbj} will be used in this section. For the fixed cost (4.2.1), let $\Gamma^s_w = x^s_w\, \gamma_w$, with the following constraints, $\forall s \in S,\ \forall w \in W$,

$$\Gamma_w^s \leq x_w^s, \tag{4.3.1}$$

$$\Gamma_w^s \leq \gamma_w, \tag{4.3.2}$$

$$\Gamma_w^s \geq x_w^s + \gamma_w - 1, \text{ and} \tag{4.3.3}$$

$$\Gamma_w^s \geq 0. \tag{4.3.4}$$

So, the linearization of (4.2.1) is

$$C_{F_f}(x,\Gamma) = \sum_w \Bigg(\big(f_w + \ell_w\big)\big(\sum_s A^s x_w^s\big) + \rho_w\big(f_w + \ell_w\big)\big(\sum_s A^s \Gamma_w^s\big)\Bigg). \tag{4.3.5}$$

In the expected operational cost (4.2.2), let $\zeta_{wbj}^s = z_{wbj}^{1s}\ (1-\gamma_w)$, with the following constraints $\forall s \in S,\ \forall w \in W,\ \forall b \in B,\ \forall j \in J$,

$$\zeta_{wbj}^s \leq z_{wbj}^{1s}, \tag{4.3.6}$$

$$\zeta_{wbj}^s \leq (1 - \gamma_w), \tag{4.3.7}$$

$$\zeta_{wbj}^s \geq z_{wbj}^{1s} - \gamma_w, \text{ and} \tag{4.3.8}$$

$$\zeta_{wbj}^s \geq 0. \tag{4.3.9}$$

So, the linearization of (4.2.2) is

$$C_{O_f}(z,\zeta,Q) = \sum_{j,s,w} \Bigg[\nu_j^s \sum_b d_{bj} \Bigg(z_{wbj}^{1s} \ - p_w\ \zeta_{wbj}^s + Q_{wbj}^s \Bigg)\Bigg]. \tag{4.3.10}$$

For the transportation cost (4.2.3), let $\eta_{wbj} = y_{wbj}^1\ (1-\gamma_w)$, with the following constraints, $\forall w \in W,\ \forall b \in B,\ \forall j \in J,$

$$\eta_{wbj} \leq y^1_{wbj}, \tag{4.3.11}$$

$$\eta_{wbj} \leq (1 - \gamma_w), \tag{4.3.12}$$

$$\eta_{wbj} \geq y^1_{wbj} - \gamma_w,\ \text{and} \tag{4.3.13}$$

$$\eta_{wbj} \geq 0. \tag{4.3.14}$$

So, the linearization of (4.2.3) is

$$C_{T_f}(y, \eta, O) = \sum_{j,b,w} \left[d_{bj} \tau_{wbj} \left(y^1_{wbj} \ - p_w \ \eta_{wbj} + O_{wbj} \right) \right]. \tag{4.3.15}$$

Let $\phi_{wbj} = y^1_{wbj}\ \gamma_w$ and let $\psi_{wbj} = y^2_{wbj}\ \gamma_w$ with the following two sets of constraints $\forall w \in W,\ \forall b \in B,\ \forall j \in J,$

$$\phi_{wbj} \leq y^1_{wbj}, \tag{4.3.16}$$

$$\phi_{wbj} \leq \gamma_w, \tag{4.3.17}$$

$$\phi_{wbj} \geq y^1_{wbj} + \gamma_w - 1, \tag{4.3.18}$$

$$\phi_{wbj} \geq 0, \tag{4.3.19}$$

and,

$$\psi_{wbj} \leq y^2_{wbj}, \tag{4.3.20}$$

$$\psi_{wbj} \leq \gamma_w, \tag{4.3.21}$$

$$\psi_{wbj} \geq y^2_{wbj} + \gamma_w - 1, \tag{4.3.22}$$

$$\psi_{wbj} \geq 0. \tag{4.3.23}$$

So, (4.2.5) and (4.2.8), respectively, become

$$\sum_w y_{wbj}^2 = 1 - \sum_w \phi_{wbj},\ \forall b \in B,\ \forall j \in J, and \tag{4.3.24}$$

$$\sum_w \left(\phi_{wbj} + \psi_{wbj}\right) = 1,\ \forall b \in B,\ \forall j \in J. \tag{4.3.25}$$

Thus, the reformulated $\boldsymbol{M_{rf}}$ is

$$\begin{aligned}
\boldsymbol{L_1M_{rf}}\text{:}\ \text{Minimize } C_f(x,y,z,\Gamma,\zeta,Q,O,\eta) &= C_{F_f}(x,\Gamma) + C_{O_f}(z,\zeta,Q) \\
&\quad + C_{T_f}(y,\eta,O) \\
\text{Subject to:} \qquad & (2.2.1),(2.2.3)-(2.2.5), \\
& (3.2.8),(3.2.14)-(3.2.17), \\
& (3.2.20)-(3.2.23),(3.2.26), \\
& (4.2.4),(4.2.6),(4.2.7), \\
& (4.3.1)-(4.3.4), \\
& (4.3.6)-(4.3.9), \\
& (4.3.11)-(4.3.14), \\
& (4.3.16)-(4.3.25), \\
x_w^s \qquad & \in \{0,1\},\ \forall s \in S, \forall w \in W, \\
y_{wbj}^r \qquad & \in \{0,1\},\ \forall w \in W, \forall b \in B, \\
& \qquad \forall j \in J, \forall r \in R \text{ and}, \\
\gamma_w \qquad & \in \{0,1\},\ \forall w \in W.
\end{aligned}$$

As in section (3.2.1.2), variables y_{wbj}^1 were relaxed and we call the new model $\boldsymbol{RL_1M_{rf}}$. Also Algorithm 1 was modified so that we get the scenarios needed to solve model $\boldsymbol{RL_1M_{rf}}$. The modification is done by letting $f(x_0) = TD - 1$ in step 1 and removing $y^2_{wbj}P_*$ from the constraints $\sum_{b,j} d_{bj} \left(y^1_{wbj} + y^2_{wbj}P_* \right) \leq \sum_s \boldsymbol{V}^s \ x^s_w, \ \forall \ w$. So the new scenario capacity constraints are

$$\sum_{b,j} d_{bj} \left(y^1_{wbj} \right) \leq \sum_s \boldsymbol{V}^s \ x^s_w, \ \forall \ w \tag{4.3.26}$$

The reason for this modification is that all built warehouses can be fortified, so we will only need primary assignments. The modified algorithm will be called Algorithm 2.

Algorithm 2: Determination of $\mathfrak{S}$ when facilities can be fortified.

1. Set $\mathfrak{S} = \emptyset$, $i = 1$, and $f(x_0) = TD - 1$.

2. Let $(x_w^s)^*$ be the optimal solution to

$$\begin{aligned}
\text{Minimize} \quad f(x) &= \sum_{w,s} \boldsymbol{V}^s \ x_w^s \\
\text{Subject to} \quad & \quad (2.2.1),\ (2.2.3)- \\
& \quad (2.2.5),\ (3.2.7), \\
& \quad (3.2.8),\ (4.3.26) \\
f(x) &\geq f(x_{i-1}) + 1, \\
y_{wbj}^1 &\geq 0, \forall w \in W,\ b \in B,\ j \in J, \\
y_{wbj}^2 &\in \{0,1\}, \forall w \in W,\ b \in B,\ j \in J, \\
x_w^s &\in \{0,1\}, \forall w \in W,\ s \in S.
\end{aligned}$$

3. Calculate $\mathfrak{s}_i^s = \sum_w (x_w^s)^* + q^s,\ \forall s$, and $f(x_i) = \sum_{w,s} \boldsymbol{V}^s \ (x_w^s)^*$. Set $\mathfrak{S} = \mathfrak{S} \cup \{\mathfrak{s}_i\}$.

4. While $f(x_i) < \overline{\text{ED}}$, replace i with $i+1$ and return to step 2.

Constraints (3.2.32) are added to model $\boldsymbol{RL_1M_{rf}}$ and call the new model $\boldsymbol{RL_1M_{rf\mathfrak{s}_i}}$.

4.3.2 Model $\boldsymbol{L_2M_{rf}}$

As in Section 3.2.1.5, the average probabilities were used to linearize the operational and the transportation costs as well as the capacity constraint. Thus, (4.3.10) can be written as

$$\hat{C}_{O_f}(z,\zeta) = \sum_{j,s,w} \Bigg[\ \nu_j^s \sum_b d_{bj} \Bigg(z_{wbj}^{1s} \ - p_w \ \zeta_{wbj}^s \\ + z_{wbj}^{2s} \ \sum_{w' \neq w} \frac{p_{w'}}{n-1} \Bigg) \Bigg], \tag{4.3.27}$$

further, (4.3.15) can be written as

$$\hat{C}_{T_f}(y,\eta) = \sum_{j,b,w} \Bigg[\ d_{bj}\tau_{wbj} \Bigg(y_{wbj}^1 \ - p_w \ \eta_{wbj} \\ + y_{wbj}^2 \sum_{w' \neq w} \frac{p_{w'}}{n-1} \Bigg) \Bigg]. \tag{4.3.28}$$

Finally, capacity constraints (3.2.9) was replaced by (3.2.35).

Thus, the reformulated $\boldsymbol{M_{rf}}$ is

$$\begin{aligned} \boldsymbol{L_2 M_{rf}}\textbf{:}\ \text{Minimize } \hat{C}_f(x,y,z,\Gamma,\zeta,\eta) &= C_{F_f}(x,\Gamma) + \hat{C}_{O_f}(z,\zeta) \\ &\quad + \hat{C}_{T_f}(y,\eta) \\ \text{Subject to:} \qquad & (2.2.1), (2.2.3) - (2.2.5), \\ & (3.2.8), (3.2.14) - (3.2.17), \\ & (3.2.35), (4.2.4), (4.2.6), \\ & (4.2.7), (4.3.1) - (4.3.4), \\ & (4.3.6) - (4.3.9), \\ & (4.3.11) - (4.3.14), \\ & (4.3.16) - (4.3.25), \end{aligned}$$

$$
\begin{aligned}
x_w^s \quad & \in \{0,1\}, \ \forall s \in S, \forall w \in W, \\
y_{wbj}^r \quad & \in \{0,1\}, \ \forall w \in W, \forall b \in B, \\
& \qquad\qquad \forall j \in J, \forall r \in R \text{ and}, \\
\gamma_w \quad & \in \{0,1\}, \ \forall w \in W.
\end{aligned}
$$

As in model $\boldsymbol{L_2M_r}$, the variables y_{wbj}^r, in model $\boldsymbol{L_2M_{rf}}$, can be relaxed to create model $\boldsymbol{RL_2M_{rf}}$.

4.3.3 The Solution Methodology

The solution methodology to the case of facility fortification is the same as what we had in Section 3.2.2. We start, using CPLEX, by trying to have a solution to model $\boldsymbol{L_1M_{rf}}$ then model $\boldsymbol{RL_1M_{rf}}$. In case of not getting an optimal solution we use Algorithm 2 to get a set of scenarios $\mathfrak{S}$. Theorem 3.2.15 can be used here to specify the optimal solution to $\boldsymbol{L_1M_{rf}}$. Then we check if the values for the y_{wbj}^r variables are binary or not and follow the same procedure of Section 3.2.2 to get an optimal solution to model $\boldsymbol{L_1M_{rf}}$. Further, optimality gap were set to 1% when we solved models $\boldsymbol{RL_1M_{rf}}$ and $\boldsymbol{RL_1M_{rfs_i}}$. Finally, we do the same thing, as in solving model $\boldsymbol{RL_2M_r}$ in Section 3.2.2, when we need to solve models $\boldsymbol{L_2M_{rf}}$ and $\boldsymbol{RL_2M_{rf}}$.

4.4 Canadian Case Study - Fortification

4.4.1 Introduction

The case study in Section 3.3 was extended to include facility fortification. Same warehouse probability of failures p_w used in the previous chapter were used here. Thus non fortified warehouses can fail with probabilities p_w that were randomly generated from $U \sim [0, 0.05]$. In [63], the fortification costs are randomly generated from a uniform distribution so that they represent between 2% to 12% of the facility fabrication cost. In this dissertation, fortification costs were randomly generated from $U \sim [0.05, 0.1]$ of the facility fixed costs. Then, as we will see later, some sensitivity analysis was performed on the fortification costs of $10-15\%$, $15-20\%$, $20-25\%$, and $25-30\%$ of the facility fixed costs.

4.4.2 Solution to Model $\boldsymbol{L_1M_{rf}}$

Model $\boldsymbol{L_1M_{rf}}$ was solved using CPLEX on the Acer laptop as described in Section 3.3.2. Table 4.1 shows that, after 24 hours with around 49.4 million iterations, CPLEX returned a feasible, but not optimal, solution to the problem with an objective function value of 3.9764×10^6 and an optimality gap of 5.53%. The additional built warehouses that were selected were two medium warehouses and one large warehouse. Fortified warehouses were one medium and two large warehouses.

Table 4.1: Numerical results for the Canadian Case study using CPLEX.

Model	$\boldsymbol{L_1M_{rf}}$	$\boldsymbol{RL_1M_{rf}}$	$\boldsymbol{RL_1M_{rf\mathfrak{s}_1}}$	$\boldsymbol{L_1M_{rf}}$ with (4.4.1)
Total Variables	96,932	96,932	96,932	96,929
Binary Variables	10,878	5,506	5,506	10,875
Constraints	350,168	355,540	355,542	350,171
Iterations	49,448,207	28,424,649	3,084,540	17,143
Time (Hr:Min:Sec)	24:00:00	07:57:04	01:14:00	00:00:19
Best Objective $\times 10^6$	3.9764	3.9498	3.9475	3.9492
Objective Bound $\times 10^6$	3.7564	3.9123	3.9266	3.9469
Built warehouses (w, s)	(1, 3), (2, 3) (4, 2), (28, 2) (30, 3)	(1, 3), (2, 3) (4, 2), (28, 3) (33, 2)	(1, 3), (2, 3) (4, 2), (28, 3) (33, 2)	(1, 3), (2, 3) (4, 2), (28, 3) (33, 2)
Fortified warehouses (w)	1, 4, 30	1, 33	1, 33	1, 33
Status	Feasible	Optimal	Optimal	Optimal

Using the second step in the solution methodology, the y^1_{wb} variables were relaxed. CPLEX spent almost eight hours hours with around 28.5 million iterations to get an optimal solution to model $\boldsymbol{RL_1M_{rf}}$ with an objective function value of 3.9498×10^6. Selected warehouses were of the same sizes as those in the solution to model $\boldsymbol{L_1M_{rf}}$ with the difference that warehouse 33 replaced warehouse 30. One large and one medium warehouses were selected to be fortified.

Using Algorithm 2, we found that we need to have two scenarios that are exactly of what we had in model $\boldsymbol{RL_1M_r}$. So, $\mathfrak{s}_1 = (0, 2, 3)$ and $\mathfrak{s}_2 = (1, 0, 4)$. The optimal solution that we got to model $\boldsymbol{RL_1M_{rf\mathfrak{s}_1}}$ is 3.9475×10^6. CPLEX was able to get this optimal solution in one hour and 14 minutes with about 3 million iterations. Built warehouses with their sizes were identical to what we had in the solution to model $\boldsymbol{RL_1M_{rf}}$. Further, those warehouses are exactly as what we had in Table 2.1 (non-risk model $\boldsymbol{M}$) and Table 3.2 (risk model $\boldsymbol{L_1M_r}$). Fortified warehouses were warehouse 1 of

large size and warehouse 33 of medium size. When we solved model $\boldsymbol{RL_1M_{rfs_2}}$, we got an optimal solution in almost two hours with an objective function value of 4.0231×10^6. Thus, the solution to model $\boldsymbol{RL_1M_{rfs_1}}$ in Table 4.1 is the optimal solution to model $\boldsymbol{RL_1M_{rf}}$.
We got some fractional values to the variables y^1_{wb}, in the solution to model $\boldsymbol{L_1M_{rf}}$, thus built warehouses in the solution to model $\boldsymbol{RL_1M_{rf}}$ were forced to exist in model $\boldsymbol{L_1M_{rf}}$, and update the set E to be

$$E = \{\, (1,3),\ (2,3),\ (4,2), (28,3), (33,2)\}. \tag{4.4.1}$$

The last column of Table 4.1 shows that CPLEX took 19 seconds to get an optimal solution with objective function value of 3.9492×10^6 to model $\boldsymbol{L_1M_{rf}}$ with (4.4.1). This value is higher by 0.47% than the value of the risk model $\boldsymbol{L_1M_r}$ without considering warehouses fortification (Table 3.2). As in the solution to models $\boldsymbol{RL_1M_{rf}}$ and $\boldsymbol{RL_1M_{rfs_1}}$, warehouses 1 and 33 were selected to be fortified.
The solution to the Canadian case study showed the effectiveness of Algorithm 2 in solving such type of problems. The total time for both scenarios of Algorithm 2 was less than what model $\boldsymbol{RL_1M_r}$ took to get a solution to the case study.

Table 4.2 shows that non fortified warehouses 2, 4, and 28 did not provide any secondary allocations. This is expected since each warehouse of those warehouses cannot be, according to model $\boldsymbol{L_1M_{rf}}$, either a secondary warehouse to themselves, or to the other warehouses that are fortified.

Table 4.2: Comparison of Warehouse Allocation and Capacity - Canada.

Warehouse	#Primary Branches	#Secondary Branches	Available Capacity	Primary Allocation	Excess Capacity	Secondary Allocation
1 (Large)	22	67	967,020	960,823.25	6,196.75	2,534.11
2 (Large)	50	0	967,020	965,622.02	1,397.98	0
4 (Medium)	24	0	580,212	575,015.77	5,196.23	0
28 (Large)	41	0	967,020	966,551.56	468.44	0
33 (Medium)	21	48	580,212	413,709.50	166,502.5	3,804.16
Totals	158	115	4,061,484	3,881,722.1	179,761.9	6,338.27

4.4.3 Solution to Model $\boldsymbol{L_2M_{rf}}$

Table 4.3 shows that it took CPLEX one hour and 25 minutes to get an optimal solution of 3.9887×10^6 to model $\boldsymbol{L_2M_{rf}}$. Built warehouses have the same sizes as what we got in model $\boldsymbol{L_1M_{rf}}$ in Table 4.1. Three warehouses were selected to be fortified that two of them are large and one medium.

Table 4.3: Numerical results for the Canadian Case study using CPLEX.

Model	$\boldsymbol{L_2M_{rf}}$	$\boldsymbol{RL_2M_{rf}}$
Total Variables	75,444	75,444
Binary Variables	10,878	134
Constraints	264,214	274,958
Iterations	5,693,466	1,812,850
Time (Hr:Min:Sec)	01:25:38	00:36:05
Best Objective $\times 10^6$	3.9887	3.9858
Objective Bound $\times 10^6$	3.9893	3.9543
Built warehouses (w, s)	(1, 3), (2, 3) (4, 2), (28, 2) (30, 3)	(1, 3), (2, 3) (4, 2), (28, 2) (30, 3)
Fortified warehouses (w)	1, 2, 28	1, 2, 4, 28
status	Optimal	Optimal

On the other hand, CPLEX was able to get an optimal solution of 3.9858×10^6 to model $\boldsymbol{RL_2M_{rf}}$ in 36 minutes. Same built warehouses, to model $\boldsymbol{L_2M_{rf}}$, were found with an extra fortified warehouse, that is warehouse 4 of medium size. Built warehouses in Table 4.3 are forced into model $\boldsymbol{L_1M_{rf}}$. So the set E is updated as

$$E = \{\, (1,3), (2,3), (4,2), (28,2), (30,3) \,\}. \tag{4.4.2}$$

Then model $\boldsymbol{L_1M_{rf}}$ with (4.4.2) was solved using CPLEX. An optimal solution of 3.9607×10^6 was found in 29 seconds as can be seen in Table 4.4.

Table 4.4: Numerical results for the Canadian Case study using CPLEX.

Model	$\boldsymbol{L_1 M_{rf}}$ with (4.4.2)
Total Variables	96,929
Binary Variables	10,875
Constraints	350,171
Iterations	37,698
Time (Hr:Min:Sec)	00:00:29
Best Objective $\times 10^6$	3.9607
Objective Bound $\times 10^6$	3.9586
Built warehouses (w, s)	(1, 3), (2, 3) (4, 2), (28, 2) (30, 3)
Fortified warehouses (w)	1, 30
Status	Optimal

The optimal solution in Table 4.4 is only higher by 0.3% from the optimal solution of $\boldsymbol{L_1 M_{rf}}$ with (4.4.1) in Table 4.1. Two large warehouses were selected to be fortified.

4.4.4 Model Validation and Sensitivity Analysis on Failure Probabilities and Fortification Costs

In the previous section, failure probabilities and fortified costs were set between $0-5\%$ and $5-10\%$, respectively. In this section, with the same limit of five warehouses are allowed to be built, failure probabilities were increased to cover the cases of $5-10\%$, $10-15\%$, $15-20\%$, $20-25\%$, and $25-30\%$. Also warehouses fortification cost percentages cover the cases of $5-10\%$, $10-15\%$, $15-20\%$, $20-25\%$, and $25-30\%$.

From Table 4.5, with fortification of $5-10\%$ of the fixed cost, one can see that across all probabilities, warehouses were three large and two medium. With all probability ranges except the case of $5-10\%$, all of the built warehouses were fortified and the objective function is the same.

Proposition 4.4.1 validates model $\boldsymbol{M_{rf}}$. It proves that once the number of fortified warehouses is equal to the number of built warehouses, i.e. all built warehouses are fortified, then increasing the failure probability of warehouses does not change the objective function as long as the problem is feasible.

Proposition 4.4.1. *Let* $(\bar{x}, \bar{y}, \bar{z}, \bar{\gamma})$ *be the optimal solution to model* $\boldsymbol{M_{rf}}$ *when* $\bar{P}$ *is the set of probabilities of warehouses failure. Also, let* $(\hat{x}, \hat{y}, \hat{z}, \hat{\gamma})$ *be the optimal solution to model* $\boldsymbol{M_{rf}}$ *when* $\hat{P}$ *is the set of probabilities of warehouses failure. If* $\bar{x} = \hat{x}$ *and* $\forall w \in W$ *such*

Table 4.5: Impact of Failure probability on cost (\$M) and the Number of built and fortified Warehouses. Fortification is $5-10\%$ of the fixed cost.

p	$C(x,y)$	Large	Medium	Small	Fortified
5-10%	4.0163	3	2	0	4
10-15%	4.0232	3	2	0	5
15-20%	4.0232	3	2	0	5
20-25%	4.0232	3	2	0	5
25-30%	4.0232	3	2	0	5

that $\sum_s \bar{x}_w^s = 1$, *and* $\sum_s \hat{x}_w^s = 1$, *we have* $\bar{\gamma}_w = 1$ *and* $\hat{\gamma}_w = 1$, *then* $C_f(\bar{x}, \bar{y}, \bar{z}, \bar{\gamma}) = C_f(\hat{x}, \hat{y}, \hat{z}, \hat{\gamma})$.

Proof. Since $\forall w \in W$ such that $\sum_s \bar{x}_w^s = \sum_s \hat{x}_w^s = 1$ we have $\bar{x} = \hat{x}$ with $\bar{\gamma}_w = \hat{\gamma}_w = 1$, it follows that the fixed and fortification costs are equal i.e. $C_{F_f}(\bar{x}, \bar{\gamma}) = C_{F_f}(\hat{x}, \hat{\gamma})$. As for the expected operational cost,

$$C_{O_f}(\bar{z}, \bar{y}, \bar{\gamma}) = \sum_{j,s,w} \left[\nu_j^s \sum_b d_{bj} \left(\bar{z}_{wbj}^{1s} \; \Big(1 - \bar{p}_w \big(1 - \bar{\gamma}_w\big)\Big) \right.\right. \\ \left.\left. + \; \bar{z}_{wbj}^{2s} \sum_{w' \neq w} \bar{p}_{w'} \; \bar{y}_{w'bj}^{1} \right) \right]. \tag{4.4.3}$$

Since we know that $\forall w$ such that $\sum_s \bar{x}_w^s = 1$, we have $\bar{\gamma}_w = 1$, so using (4.2.5), it follows that $\bar{z}_{wbj}^{2s} = 0 \; \forall w \in W, \; b \in B, \; s \in S, \; j \in J$. Thus, $C_{O_f}(\bar{z}, \bar{y}, \bar{\gamma}) = \sum_{j,s,w} \nu_j^s \sum_b d_{bj} \; \bar{z}_{wbj}^{1s}$.

Since $\bar{z}_{wbj}^{1s}$ and $\hat{z}_{wbj}^{1s}$ are optimal and $\bar{x} = \hat{x}$, it follows that $\bar{z}_{wbj}^{1s} = \hat{z}_{wbj}^{1s}$

$\forall w \in W,\ b \in B,\ s \in S,\ j \in J$. Thus,

$$\begin{aligned} C_{O_f}(\bar{z}, \bar{y}, \bar{\gamma}) &= \sum_{j,s,w} \nu_j^s \sum_b d_{bj}\ \hat{z}_{wbj}^{1s} \\ &= \hat{C}_{O_f}(\hat{z}, \hat{y}, \hat{\gamma}) \end{aligned}$$

In the same way, one can show that $C_{T_f}(\bar{y}, \bar{\gamma}) = C_{T_f}(\hat{y}, \hat{\gamma})$ and the proposition follows. $\square$

Tables 4.6 - 4.9 validate model $\boldsymbol{M_{rf}}$. They show that when we fixed the probability of warehouse failure and increased the fortification cost, the number of fortified warehouses is decreased. With fortification cost of $25 - 30\%$ of the fixed cost, we did not reach the case of having five fortified warehouses as Table 4.9 shows.

Table 4.6: Impact of Failure probability on cost (\$M) and the Number of built and fortified Warehouses. Fortification is $10 - 15\%$ of the fixed cost.

p	$C(x, y)$	Large	Medium	Small	Fortified
5-10%	4.0864	3	2	0	4
10-15%	4.0864	3	2	0	4
15-20%	4.1159	3	2	0	5
20-25%	4.1159	3	2	0	5
25-30%	4.1159	3	2	0	5

Table 4.7: Impact of Failure probability on cost (\$M) and the Number of built and fortified Warehouses. Fortification is $15 - 20\%$ of the fixed cost.

p	$C(x, y)$	Large	Medium	Small	Fortified
5-10%	4.1445	3	2	0	3
10-15%	4.1823	3	2	0	4
15-20%	4.2020	3	2	0	4
20-25%	4.2084	3	2	0	5
25-30%	4.2084	3	2	0	5

Table 4.8: Impact of Failure probability on cost ($M) and the Number of built and fortified Warehouses. Fortification is $20-25\%$ of the fixed cost.

p	$C(x,y)$	Large	Medium	Small	Fortified
5-10%	4.2013	3	2	0	3
10-15%	4.2525	3	2	0	4
15-20%	4.2782	3	2	0	4
20-25%	4.2916	3	2	0	4
25-30%	4.3009	3	2	0	5

Table 4.9: Impact of Failure probability on cost ($M) and the Number of built and fortified Warehouses. Fortification is $25-30\%$ of the fixed cost.

p	$C(x,y)$	Large	Medium	Small	Fortified
5-10%	4.2393	3	2	0	2
10-15%	4.3227	3	2	0	4
15-20%	4.3484	3	2	0	4
20-25%	4.3710	3	2	0	4
25-30%	4.3828	3	2	0	4

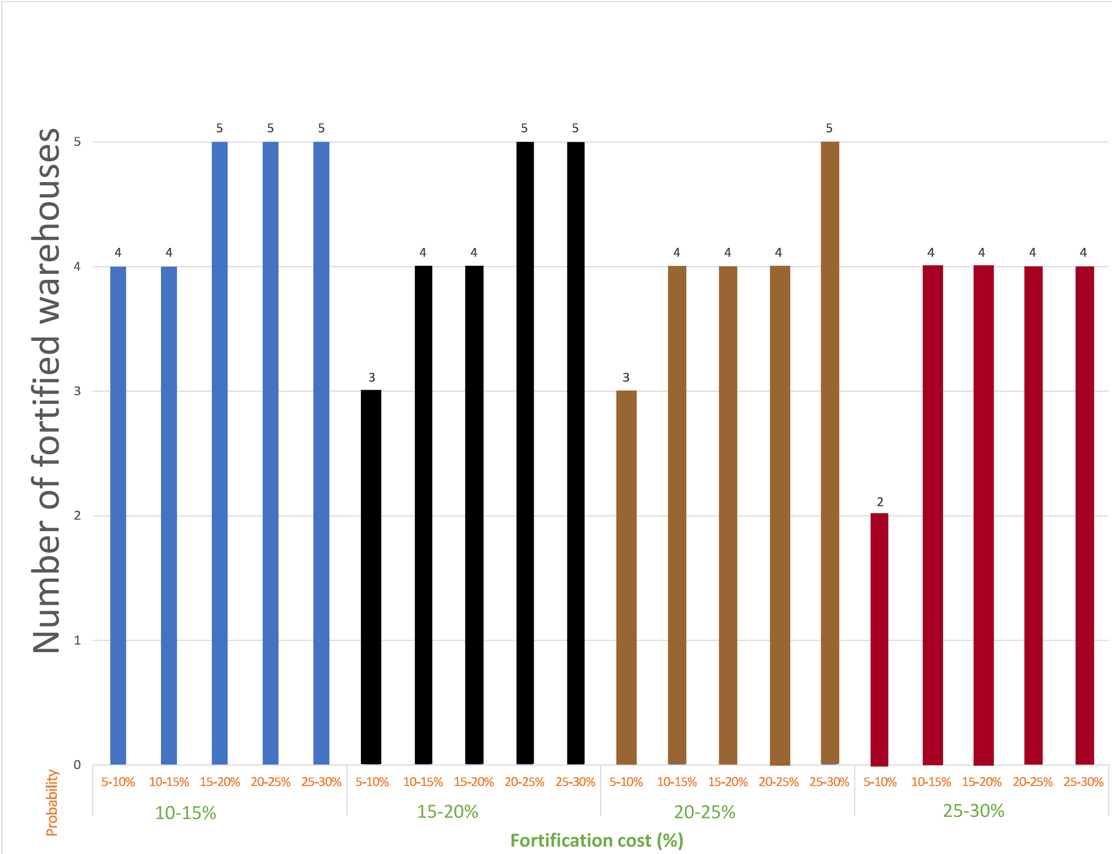

Figure 4.2: Impact of warehouses fortification cost (%) and probability on the number of built warehouses

Figure 4.2 summarizes Tables 4.5 - 4.9.

4.5 European Example - Warehouses Fortification

4.5.1 Introduction

The European example in Section 3.4 was extended to include warehouses fortification. The 37 warehouses will have the same failure probabilities p_w of Section 3.4 that were randomly generated from $U \sim [0, 0.05]$.

4.5.2 Solution to model $\boldsymbol{L_1M_{rf}}$

4.5.2.1 Flexible warehouses

Model $\boldsymbol{L_1M_{rf}}$ was solved using CPLEX on the Acer laptop as described in Section 3.3.2. As can be seen in Table 4.10, after 24 hours with more than 55 million iterations, CPLEX returned a feasible, but not optimal, solution to the problem with an objective function value of 3.2759×10^6 and an optimality gap of 7.2%. Four large warehouses that are 7, 12, 20, and 23 and one small warehouse, warehouse 21, were selected to be built. Warehouse 12 of large size, and warehouse 21 of small size were selected to be fortified. On the other hand, CPLEX spent 24 hours to get a feasible, but not optimal, solution of $3.2982\times^6$ to model $\boldsymbol{RL_1M_{rf}}$ with an optimality gap of 5.16%. Warehouse 13 replaced warehouse 20 in the built and fortified warehouses list.

Using Algorithm 2, we found that, as in Section 3.4.2.1, we need to

Table 4.10: Numerical results for the European Example with flexible warehouses using CPLEX.

Model	$\boldsymbol{L_1M_{rf}}$	$\boldsymbol{RL_1M_{rf}}$	$\boldsymbol{RL_1M_{rfs_2}}$	$\boldsymbol{L_1M_{rf}}$ with (4.5.1)
Total Variables	123,469	123,469	123,469	123,464
Binary Variables	13,838	6,993	6,993	13,833
Constraints	446,037	459,726	459,728	446,042
Iterations	55,110,851	22,593,030	3,858,870	43,083
Time (Hr:Min:Sec)	24:00:00	24:00:00	02:39:35	00:00:35
Best Objective $\times 10^6$	3.2759	3.2982	3.2761	3.2751
Objective Bound $\times 10^6$	3.0389	3.1281	3.2427	3.2749
Built warehouses (w, s)	(7, 3), (12, 3) (20, 3), (21, 1) (23, 3)	(7, 3), (12, 3) (13, 3), (21, 1) (23,3)	(7, 3), (12, 3) (20, 3), (21, 1) (23, 3)	(7, 3), (12, 3) (20, 3), (21, 1) (23, 3)
Fortified warehouses (w)	12, 21	13, 21	12, 21	12, 21
Status	Feasible	Feasible	Optimal	Optimal

have two scenarios that are $\mathfrak{s}_1 = (0, 2, 3)$ and $\mathfrak{s}_2 = (1, 0, 4)$. In about three hours, CPLEX found an optimal solution of 3.3339×10^6 to model $\boldsymbol{RL_1M_{rfs_1}}$. On the other hand, Table 4.10 shows that after two hours and 39 minutes, CPLEX found an optimal solution to model $\boldsymbol{RL_1M_{rfs_2}}$ with an objective function value of 3.2761×10^6. Built and fortified warehouses were exactly as in the solution to model $\boldsymbol{L_1M_{rf}}$. Thus, the scenario with four large and one small warehouses gave the optimal solution to model $\boldsymbol{RL_1M_{rf}}$. The solution to model $\boldsymbol{RL_1M_{rfs_2}}$ have some fractional values to the y^1_{wbj} variables. So, model $\boldsymbol{L_1M_{rf}}$ was solved again by forcing the built warehouses in the solution to model $\boldsymbol{RL_1M_{rfs_2}}$. The set E was updated as

$$E = \{\, (7, 3), (12, 3), (20, 3), (21, 1), (23, 3) \,\}. \tag{4.5.1}$$

The last column of Table 4.10 shows that it took CPLEX 35 seconds to solve model $\boldsymbol{L_1M_{rf}}$ with (4.5.1) and get an objective function value of 3.2751×10^6 which is higher by 0.76% than the solution to the risk model without fortification, model $\boldsymbol{RL_1M_r}$, in Table 3.7. As in the solution to $\boldsymbol{L_1M_{rf}}$ and $\boldsymbol{RL_1M_{rfs_2}}$, the warehouses 12 of large size and 21 of small size were selected to be fortified.

Using the scenarios of Algorithm 2, the European example was solved in total time less than 6 hours. This shows how the proposed scenarios algorithm method was helpful in reducing the needed time to get an optimal solution to this problem.

Table 4.11 shows that, as in Table 4.2, non fortified warehouses have no secondary allocations.

Table 4.11: Comparison of Warehouse Allocation and Capacity - Flexible Warehouses.

Warehouse	#Primary Branches	#Secondary Branches	Available Capacity	Primary Allocation	Excess Capacity	Secondary Allocation
7 (Large)	62	0	967,020	965,657.25	1,362.75	0
12 (Large)	34	127	967,020	950,671.64	16,348.36	16,347.04
20 (Large)	41	0	967,020	739,152.51	227,867.49	0
21 (Small)	9	15	270,765.60	267,507.11	3,258.49	1,547.02
35 (Large)	39	0	967,020	966,670.30	349.7	0
Totals	185	142	4,138,845.6	3,889,658.81	249,186.79	17,894.06

4.5.2.2 Specialized warehouses

As in Section 3.4.1.2, some warehouses were specialized in some categories. The same assumptions are used in this section. Table 4.12 shows that after 24 hours CPLEX got a feasible, but not optimal, solution to model $\boldsymbol{L_1M_{rf}}$ with an objective function of 3.7852×10^6 and an optimality gap of 6.3%. Built warehouses were $6, 9, 12$, and 14 with large sizes and warehouse 17 with small size. Fortified warehouses were $9, 12, 14$ and 17. With model $\boldsymbol{RL_1M_{rf}}$, CPLEX spent 24 hours to get a feasible, but not optimal, solution of 3.7844×10^6 with an optimality gap of 1.97%. Built warehouses are identical to the warehouses in the solution to model $\boldsymbol{L_1M_{rf}}$. However, warehouse 6 in the fortified warehouses list in the solution to model $\boldsymbol{RL_1M_{rf}}$ replaced warehouse 9 in the solution to model $\boldsymbol{L_1M_{rf}}$.

Table 4.12: Numerical results for the European Example with specialized warehouses using CPLEX.

Model	$\boldsymbol{L_1M_{rf}}$	$\boldsymbol{RL_1M_{rf}}$	$\boldsymbol{RL_1M_{rfs_2}}$	$\boldsymbol{L_1M_{rf}}$ with (4.5.2)
Total Variables	123,469	123,469	123,469	123,464
Binary Variables	13,838	6,993	6,993	13,834
Constraints	450,996	457,841	457,843	451,001
Iterations	30,270,743	29,581,484	2,079,478	6,695
Time (Hr:Min:Sec)	24:00:00	24:00:00	01:08:11	00:00:22
Best Objective $\times 10^6$	3.7852	3.7844	3.7799	3.7836
Objective Bound $\times 10^6$	3.5486	3.7099	3.7467	3.7831
Built warehouses (w, s)	(6, 3), (9, 3) (12, 3),(14, 3) (17, 1)	(6, 3), (9, 3) (12, 3),(14, 3) (17, 1)	(6, 3), (9, 3) (12, 3),(14, 3) (17, 1)	(6, 3), (9, 3) (12, 3),(14, 3) (17, 1)
Fortified warehouses (w)	9, 12, 14, 17	6, 12, 14, 17	9, 12, 14, 17	9, 12, 14, 17
Status	Feasible	Feasible	Optimal	Optimal

Using Algorithm 2, same as the case of flexible warehouses, we got $\mathfrak{s}_1 = (0, 2, 3)$ and $\mathfrak{s}_2 = (1, 0, 4)$. CPLEX found, in 2.5 hours, an optimal solution of 3.8360×10^6 to model $\boldsymbol{RL_1M_{rfs_1}}$. On the other hand, Table 4.12 shows that CPLEX took one hour and 8 minutes to get an optimal solution to model $\boldsymbol{RL_1M_{rfs_2}}$, with an objective function of 3.7799×10^6. Built and fortified warehouses were exactly as in the solution to model $\boldsymbol{L_1M_{rf}}$. The solution to model $\boldsymbol{RL_1M_{rfs_2}}$ had fractional values to the variables y^1_{wbj}. Thus, model $\boldsymbol{L_1M_{rf}}$ was solved again by using the built warehouses in the solution to model $\boldsymbol{RL_1M_{rfs_2}}$. So the set E was updated as

$$E = \{\,(6,3),(9,3),(12,3),(14,3),(17,1)\,\}. \tag{4.5.2}$$

The last column of Table 4.12 shows that CPLEX took 22 seconds to get an optimal solution to model $\boldsymbol{L_1M_{rf}}$ with (4.5.2). The objective function value is 3.7836×10^6 which is less by 0.66% than the case of risk without warehouses fortification, model $\boldsymbol{L_1M_r}$, in Table 3.9. Fortified warehouses are as in the solution to model $\boldsymbol{L_1M_{rf}}$ and to model $\boldsymbol{RL_1M_{rfs_2}}$.

The solution to the case of specialized warehouses confirms the effectiveness of the proposed solution method using the scenarios algorithm, Algorithm 2, in solving the problem of warehouse fortification.

In Table 4.13, one can see that, as in Tables 4.2 and 4.11, the non fortified warehouse 6 has no secondary allocation. Also, warehouse 17 of small size has no secondary allocation although it is a fortified warehouse. This is because the only non fortified warehouse is warehouse 6. So warehouse 6 is the only warehouse that needs backup warehouses to satisfy the demand of branches that it serves as

Table 4.13: Comparison of Warehouse Allocation and Capacity - Specialized Warehouses.

Warehouse	#Primary Branches	#Secondary Branches	Available Capacity	Primary Allocation	Excess Capacity	Secondary Allocation
6 (Large)	50	0	967,020	852,823.75	114,196.25	0
9 (Large)	50	45	967,020	933,563.59	33,456.41	29,241.87
12 (Large)	51	1	967,020	967,016.01	3.99	3.79
14 (Large)	28	4	967,020	966,779.86	240.14	223.30
17 (Small)	6	0	270,765.60	169,475.61	101,289.99	0
Totals	185	50	4,138,845.6	3,889,658.82	249,186.78	29,468.96

a primary warehouse. One can see that warehouse 6 got its secondary assignments from warehouses 9, 12, and 14 as they were able to serve its demand, as secondary warehouses, with lower transportation and operational costs than warehouse 17.

4.5.3 Solution to Model $\boldsymbol{L_2M_{rf}}$

4.5.3.1 Flexible warehouses

Model $\boldsymbol{L_2M_{rf}}$ was solved using CPLEX using the Acer laptop mentioned above. Table 4.14 shows that after 24 hours CPLEX returned a feasible but not optimal solution to model $\boldsymbol{L_2M_{rf}}$. After relaxing the y^r_{wbj} variables, CPLEX found an optimal solution to model $\boldsymbol{RL_2M_{rf}}$ in 9 hours and 40 minutes. Same built sizes were given by both solutions with the difference that warehouse 14, of large size, in the solution to model $\boldsymbol{RL_2M_{rf}}$ replaced warehouse 23, of large size, in the solution to model $\boldsymbol{L_2M_{rf}}$. In the solution to both models, warehouse 13, of large size, was selected to be fortified.

Table 4.14: Numerical results for the European Example with flexible warehouses using CPLEX.

Model	$\boldsymbol{L_2M_{rf}}$	$\boldsymbol{RL_2M_{rf}}$	$\boldsymbol{L_2M_{rf}}$ with (4.5.3)
Total Variables	96,089	96,089	96,084
Binary Variables	13,838	148	13,833
Constraints	336,517	365,371	336,522
Iterations	20,301,625	5,435,908	4,603
Time (Hr:Min:Sec)	24:00:00	9:40:57	00:00:08
Best Objective $\times 10^6$	3.3085	3.2914	3.3210
Objective Bound $\times 10^6$	3.0619	3.2889	3.2939
Built warehouses (w, s)	(7,3), (12,3) (13,3),(21,1) (23,3)	(7,3), (12,3) (13,3), (14,3) (21,1)	(7,3), (12,3) (13,3), (14,3) (21,1)
Fortified warehouses (w)	13	13	13
status	Feasible	Optimal	Optimal

Built warehouses in the solution of model $\boldsymbol{RL_2M_{rf}}$ were used to update the set E in model $\boldsymbol{L_2M_{rf}}$ as

$$E = \{\, (7,3), (12,3), (13,3), (14,3), (21,1)\, \}. \tag{4.5.3}$$

CPLEX spent 8 seconds to get an optimal solution to $\boldsymbol{L_2M_{rf}}$ with (4.5.3) and warehouse 13 was selected to be fortified.

Now, let us force the warehouses in the solution to model $\boldsymbol{RL_2M_{rf}}$ into model $\boldsymbol{L_1M_{rf}}$. So, add (4.5.3) to model $\boldsymbol{L_1M_{rf}}$. Table 4.15 shows that, it took CPLEX one minute and 11 seconds to get an optimal solution to model $\boldsymbol{L_1M_{rf}}$ with (4.5.3). Fortified warehouses were two of large size that are warehouses 12 and 14 and one small warehouse that is warehouse 21. If we compare the solution to $\boldsymbol{L_1M_{rf}}$ with (4.5.3) in Table 4.15 to the solution to $\boldsymbol{L_1M_{rf}}$ with (4.5.1) in Table 4.10, we will find that there is an increase of 1.03% in the objective function and one extra large warehouse, warehouse 14, was added to the fortified warehouse list.

Table 4.15: Numerical results for the European Example with flexible warehouses using CPLEX.

Model	$\boldsymbol{L_1M_{rf}}$ with (4.5.3)
Total Variables	123,464
Binary Variables	13,833
Constraints	446,042
Iterations	89,123
Time (Hr:Min:Sec)	00:01:11
Best Objective $\times 10^6$	3.3089
Objective Bound $\times 10^6$	3.3088
Built warehouses (w, s)	(7, 3), (12, 3), (13, 3) (14, 3), (21, 1)
Fortified warehouses	12, 14, 21
Status	Optimal

4.5.3.2 Specialized warehouses

With the case of specialized warehouses, Table 4.16 shows that, it took CPLEX three hours and 47 minutes to get an optimal solution to model $\boldsymbol{L_2M_{rf}}$ and one hour and 42 minutes to get an optimal solution to model $\boldsymbol{RL_2M_{rf}}$. Built and fortified warehouses are identical in both solutions.

Table 4.16: Numerical results for the European Example with specialized warehouses using CPLEX.

Model	$\boldsymbol{L_2M_{rf}}$	$\boldsymbol{RL_2M_{rf}}$
Total Variables	96,089	96,089
Binary Variables	13,838	148
Constraints	341,475	355,165
Iterations $\times 10^6$	3.1633	1.2089
Time (Hr:Min:Sec)	03:47:38	01:42:55
Best Objective $\times 10^6$	3.7614	3.7587
Objective Bound $\times 10^6$	3.7609	3.7461
Built warehouses (w, s)	(6,3), (9,3) (12,3),(14,3) (17,1)	(6,3), (9,3) (12,3),(14,3) (17,1)
Fortified warehouses (w)	9, 14	9, 14
status	Optimal	Optimal

Note that built warehouses in Table 4.16 are exactly what we got for model $\boldsymbol{L_1M_{rf}}$ in Table 4.12.

4.6 Conclusion

This chapter presented a mixed-integer non-linear mathematical model $\boldsymbol{M_{rf}}$ for solving the problem of how to choose optimal warehouse locations and sizes in a supply chain network design and to select some warehouses to fortify so that they become completely reliable. The standard linearization method and the average probabilities method were used to linearize the model $\boldsymbol{M_{rf}}$. A modified version of Algorithm 1, Algorithm 2, was also introduced to help in solving the problem.

The Canadian case study and the created European example showed the effectiveness of the proposed solution methodology. The sensitivity analysis validated model $\boldsymbol{M_{rf}}$ by showing that as the fortification cost increased while the warehouse failure probabilities were fixed, the number of fortified warehouses required decreased. Further, once all built warehouses are fortified, increasing the failure probability of warehouses has no effect on the value of the objective function as long as the problem is feasible. The study showed that with 5% to 10% of fortification percentage cost out of the fixed cost, it is not necessary to reduce the total cost compared to the case of warehouses under failure risk without fortification. The results showed that there was a 0.47% increase in total costs in the Canadian case study, and a 0.78% increase in total costs in the European example with flexible warehouses and a reduction of 0.66%

in total costs in the European example with specialized warehouses. A limited fortification budget was not considered in this study. Fortification budgeting in the case of warehouses with multiple capacity levels should be explored in future work.

Chapter 5

Conclusion and Future Work

5.1 Conclusion

In section 1.8, five questions were listed to be answered in this thesis. The questions were answered for non risk model (Chapter 2), a risk model (Chapter 3), and a risk model with fortification (Chapter 4).

Three sets of mathematical models were presented for a two-echelon capacitated facility location problem (CFLP). The first set of mathematical models was developed to locate warehouses and assign them to branches to minimize costs, including fixed, variable, and transportation costs. In the second set of mathematical models, built warehouses are assumed to fail. Therefore, two warehouses are assigned to each branch. The first warehouse is the primary warehouse, and the second one is the backup warehouse that is used in the case of the failure of the primary warehouse. The case of the failure of the primary and backup warehouses at the same time was ignored. Also, no penalty was applied if the whole demand was not satisfied in the case of the failure of the primary warehouse. In the third set of mathematical models, the case of fortifying warehouses to become risk-free was considered. Each branch was assigned to either a primary fortified warehouse or a non-fortified primary warehouse and a fortified backup warehouse.

In the three presented mathematical models, the binary variables that assign warehouses to branches were relaxed. The results showed that relaxing these binary variables yielded fractional assignments only in the case in which a built warehouse with the minimum variable and transportation costs cannot accommodate the whole demand for its assigned branch. A linearization method, an approximation method, and a scenario solution algorithm were developed to solve the second and third problems. The Canadian case study and the created European example demonstrated the effectiveness of those methods.

The comparison between the solutions to the three presented models (i.e., the non risk model, the risk model with $0-5\%$ failure probabilities, and the warehouse fortification model with fortification costs of $5-10\%$ of the fixed costs) showed that the sizes of the built warehouses are exactly the same. As expected, there was an increase in the objective function when we moved from the non risk model to the risk model. In chapter 4, we saw that allowing fortification could decrease the overall cost.

In the three presented problems, the introduction of the specialized warehouses showed that there was an increase in the overall total cost.

The sensitivity analysis of the first problem showed that changes in demand have a great impact on the number and sizes of built warehouses. The second problem showed that built capacity increased as the failure probability increased. In the third problem, the sensitivity analysis showed that once all built warehouses are fortified, increasing the failure probability of warehouses had no effect on the value of the objective function as long as the problem was feasible. Further analysis showed that increasing the fortification costs while fixing the warehouse failure probability decreases the number of warehouses required to be fortified. Finally, increasing the warehouse failure probabilities increases the number of warehouses required to be fortified.

In practice, when managers are involved in decisions about the planning out of new warehouse locations, we suggest they begin with a solution to the scenarios algorithm followed by solution of the scenario $\mathfrak{s}_k$ problems and then use that as the basis for decisions.

The risk of failure is not the only source of uncertainty. Many researches have considered the uncertainty of demand. So, it is good to create the presetned solution methodologies on uncertainties on demand.

5.2 Future Work

The topic of variables, but not pre-determined, capacities in the case of warehouse failure can be explored in future work. Also, in the case of warehouse failure, one can consider studying the penalty cost of not serving the whole demand for a certain branch in case its primary warehouse failed. Also, one can consider relaxing the assumption that the primary and backup warehouses cannot fail together. The topic of limited fortification budgets in the case of warehouses with multiple capacity levels can be explored. Further, future work should address the assumption, in Algorithm 1 and Algorithm 2, that no two scenarios have the same total capacity including its connection to the Diophantine equation

$$\sum_{s} \boldsymbol{V}^s \mathfrak{s}^s = f(\mathfrak{s}_i^*)$$

and the existence of alternate non-negative solutions.

Another topic that can be explored for future work is that the presented three models can be formulated in consideration of suppliers. In this case, there will be three-echelon supply chain network design problems, instead of two-echelon ones. Finally, the partial disruption of specialized warehouses with multiple capacity levels can be considered in future work.

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