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Vertex-Magic Total Labeling on G-sun Graphs

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Abstract

Graph labeling is an immense area of research in mathematics, specifically graph theory. There are many types of graph labelings such as harmonious, magic, and lucky labelings. This paper will focus on magic labelings. Graph theorists are particularly interested in magic labelings because of a simple problem regarding tree graphs introduced in the 1990's. The problem is still unsolved after almost thirty years. Researchers have studied magic labelings on other graphs in addition to tree graphs. In this paper we will consider vertex-magic labelings on G-sun graphs. We will give vertex-magic total labelings for ladder sun graphs and complete bipartite sun graphs. We will also show when there is no vertex-magic total labeling for other types of G-sun graphs.

1 Introduction

Graph theory dates back to a conundrum regarding bridges in the City of Königsberg. The problem asked if it was possible to walk over all seven bridges exactly once. Leonhard Euler, a famous mathematician, represented the town and bridges using a structure we refer to as a graph today. He was able to use the graph to show it was impossible to walk over all seven bridges just once [1; 5]. Figure 1 shows the layout of the seven bridges and the graph representation.

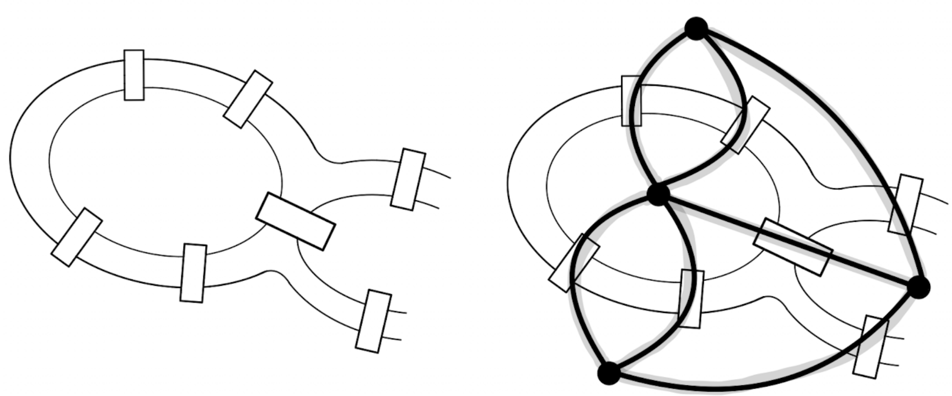

Figure 1: The 7 Bridges of The City of Königsberg
and The Graph Model

We can see the many uses and applications of graphs in [5]. They can be used to model a variety of problems. They are used in science specifically in graph enumeration techniques, showing covalent bonds between atoms. They are also used in computer science where graph algorithms solve problems modeled by graphs. Graphs are used in operations research, where they may model transport networks or activity networks.

A **graph**, G, consists of a finite nonempty set of **vertices**, or points, and a set of edges, or lines, between the vertices [1]. Two vertices are **adjacent** if there is an edge between them. The number of vertices in G is called the **order** of G, denoted by v. The number of edges in G is called the **size** of G, denoted by e. For example in Figure 2 the middle graph is a path graph with 5 vertices and 4 edges. Thus the size of this path graph is 4, and the order is 5.

A vertex is **incident** to an edge, or an edge is incident to a vertex, if the vertex is an endpoint of that edge. In Figure 2, the star graph on the right has a vertex that is incident to all the edges of the graph. We say the **degree** of a vertex is the number of incident edges connecting to the vertex. In the cycle graph on the left, the degree of every vertex is 2.

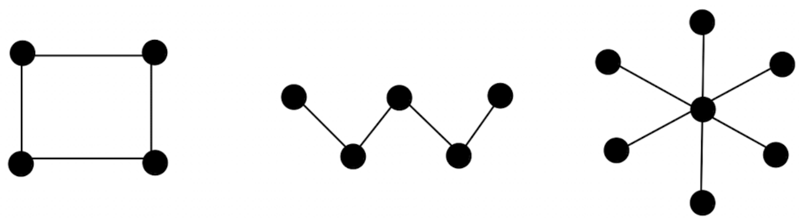

Figure 2: Cycle, Path and Star Graph

One interesting area of research in graph theory is graph labeling. A graph labeling is an assignment of integers to the vertices and/or edges of a graph. Labelings were first introduced in the 1960's. There are over 200 types of labelings including harmonious, graceful, and magic labelings [3].

2 Magic Graphs

Magic graphs are the focus of our research. Magic Graphs are graphs with a **vertex-magic total labeling**. Label the vertices and edges of a graph with integers 1, 2, ..., $e + v$. If the sum of the labels of a vertex and its incident edges is the same for all vertices then the labeling is a vertex-magic total labeling, VMTL. When a VMTL exists, the graph is called a magic graph. On a graph with a VMTL, the sum of a vertex label and its incident edges' labels is called the **magic number**, denoted by k.

Example 1. Figure 3 gives a VMTL for a cycle graph with 3 vertices and 3 edges. We label the graph with integers from 1, 2, ..., 6. In Figure 3 notice that the graph on the far left all the vertices and their incident edges sum to the same number, the magic number, 9.

Not all graphs can have a VMTL. Some graphs can have multiple VMTL's with the same magic number or with different magic numbers, k. In Figure 3, the cycle graph C_3 is shown with VMTL's having different magic numbers. On the graph on the left, $k = 9$, the graph in the middle, $k = 10$, and the graph on the right, $k = 12$.

If a graph has VMTL then the following lemma from [2], gives us a formula for the magic number.

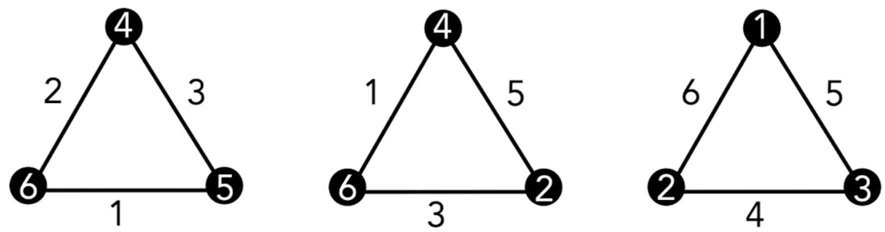

Figure 3: Cycle graph with VMTL

Lemma 1. *[2] If G is a vertex-magic graph with v vertices and e edges, then*

$$k = \frac{(v+e)(v+e+1)}{2v} + \frac{e_{sum}}{v}.$$

The proof of this lemma can be found in [2]. We use the result of this lemma to find a formula for the sum of the edges, or e_{sum}, that will be useful when we are finding labelings.

$$e_{sum} = vk - \frac{(v+e+1)(v+e)}{2} \tag{1}$$

Using this lemma we are able to find the **magic spectrum**, which is the range of possible k values. Not all integers are able to be the magic number. In Figure 3, we saw the cycle graph C_3 can have a VMTL with multiple magic numbers. The following theorem from [2] provides us with bounds on the k value.

Theorem 1. *[2] Let G be a graph with v vertices and e edges. If G is a vertex-magic graph, then the magic number, k, is bounded such that*

$$\frac{e(e+1)+(e+v+1)(e+v)}{2v} \leq k \leq e + \frac{e(e+1)+(e+v+1)(e+v)}{2v}.$$

A proof of Theorem 1 is found in [2]. Using the magic spectrum for k from this theorem makes labeling our graph G more efficient.

Example 2. We find a VMTL for the path graph, P_5. We start with finding the magic spectrum using Theorem 1. Since $e = 4$ and $v = 5$, by Theorem 1

$$\frac{4(4+1)+(4+5+1)(4+5)}{2(5)} \leq k \leq 4 + \frac{4(4+1)+(4+5+1)(4+5)}{2(5)}$$
$$11 \leq k \leq 15$$

The magic number for P_5 will have a lower bound of 11 and an upper bound of 15. This means the magic number is between 11 and 15. Using Lemma 1, we find the corresponding

e_{sum} for the k value we are using to label P_5. We'll choose $k = 11$. We use the formula for e_{sum} that follows lemma 1.

Using equation (1)

$$e_{sum} = (5)(11) - \frac{(4+5+1)(4+5)}{2}$$
$$e_{sum} = 10$$

The possible labels for P_5 are 1, 2, ..., 9. We now determine labelings by first making partitions of the labels that sum $e_{sum} = 10$. We see there is only one partition of the labels that sum to 10. We label our edges and vertices so that the labels of the vertices and incident edges sum to $k = 11$. Figure 4 shows a VMTL with magic number 11.

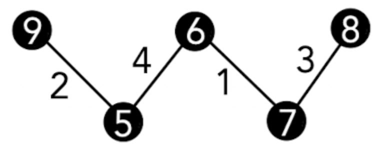

Figure 4: P_5 with a VMTL

3 G-sun Graphs

Let's consider VMTL on G-sun graphs.

Given a graph G of order v and size e a G-sun is a graph with order $2v$ formed by adjoining v new vertices of degree one to the vertices of G [4]. The new graph attaches a vertex of degree one to each existing vertex of graph G. To transform a graph into a G-sun graph, an edge and vertex are added to each existing vertex. We will refer to the outer, degree one vertices and edges as ray vertices and ray edges.

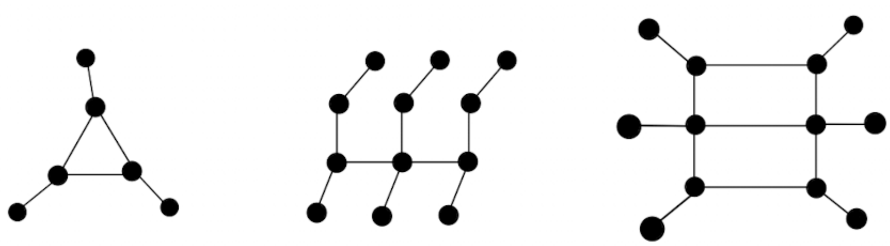

Figure 5: From left to right, cycle-sun, centipede-sun, and ladder-sun

There are several results that will be useful as we investigate G-sun graphs. The following theorem is from MacDougall [4].

Theorem 2. *[4] Let G be any graph of order v. If G has e edges, then a G-sun graph, G^*, has no labeling whenever*

$$e > \frac{-1 + \sqrt{1 + 8v^2}}{2}. \tag{2}$$

A proof of theorem 2 is found in [4].

Theorem 2 uses the size and order from the base graph G. If the inequality is satisfied there is no VMTL for the G-sun graph. If the inequality is not satisfied we have a possibility for a VMTL on the G-sun graph. The inequality eliminates graphs that will not have VMTL on the G-sun graph. We set $\frac{-1+\sqrt{1+8v^2}}{2}$ equal to z. If z is greater than or equal to e, from the base graph G, then there is a possibility for VMTL on the G-sun graph.

3.1 Ladder Sun Graphs

In this section we consider ladder sun graphs.

A **ladder graph**, L_n, is a graph with $2n$ vertices, consisting of two copies of P_n with an additional edge between corresponding vertices of the two paths. We will call these additional edges the rungs of the ladder.

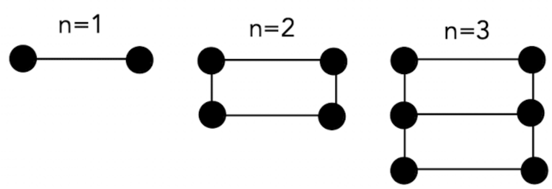

Figure 6: Ladder graphs L_1, L_2, and L_3

To find the size of L_n, we look at how L_n is formed. Since there are two path graphs and each path graph has $n-1$ edges, then we have $2n-2$ edges on the sides of the ladder. Then with the addition of n rungs we have a total of $e = 3n - 2$ edges. The number of vertices is $v = 2n$ by the definition of a ladder graph.

Example 3. We provide an example of labeling the ladder sun graph, L_3-sun. We begin by determining if there is a potential for labeling the L_3-sun graph using Theorem 2. We use the e and v values from L_3. If z is greater than or equal to $e = 7$ then there is a possibility that L_3-sun will have a VMTL.

$$\begin{aligned} z &= \frac{-1 + \sqrt{1 + 8v^2}}{2} \\ &= \frac{-1 + \sqrt{1 + 8(6)^2}}{2} \\ &= \frac{-1 + \sqrt{289}}{2} \\ &= 8 \end{aligned}$$

Since $7 \not> 8$ the inequality is not satisfied, this indicates that there is a potential for a vertex-magic total labeling of the ladder sun graph, L_3-sun.
Next, we find the magic spectrum using Theorem 1 with the e and v values from the graph of L_3-sun.

$$\frac{13(13+1)+(13+12+1)(13+12)}{2(12)} \leq k \leq 13+\frac{13(13+1)+(13+12+1)(13+12)}{2(12)}$$
$$34.667 \leq k \leq 47.667$$

We round up our lower bound for k to 35 and round down our upper bound for k to 47 because our labels are integers and therefore our magic number will also be an integer. Now we know our k value will be between 35 and 47. We can give better bounds for k. The label of each ray vertex and its incident edge must add to k. Since there are six ray vertices, then we need six pairs of labels that add to k. We label our graph with integers between 1 and $e+v$, so we are using labels 1 to 25. When $k=47$, the only pairs of labels that add to 47 are 25 and 22, and 24 and 23. We would not be able to label all the ray vertices and edges. The largest k with at least 6 pairs of labels that sum to k is $k=39$.

We find a VMTL for the largest k in the magic spectrum. We first begin by labeling all the ray vertices and edges. We label the ray vertices and edges with the six pairs. The pairs of labels that add to $k=39$ are

$$\{20,19\},\{21,18\},\{22,17\},\{23,16\},\{24,15\}, \text{ and } \{25,14\}$$

Label the vertices with the larger of the 2 numbers on the ray vertices. Using equation (1), we find the corresponding e_{sum}.

$$e_{sum} = (12)(39) - \frac{(12+13+1)(12+13)}{2}$$
$$e_{sum} = 143$$

Now we can use the fact that our edges will all sum to 143. We already have 6 edges labeled. When we deduct the value of those labels from 143 we are left with 44. Next, we determine partitions with the remaining labels that sum to 44. A few partitions that sum to 44 are as follows,

$$\{13,12,9,4,3,2,1\}$$
$$\{12,11,10,5,3,2,1\}$$
$$\{12,11,9,6,3,2,1\}$$

There are more partitions that add to 44. Using the first partition we can find the VMTL with magic number 39 shown in Figure 7.

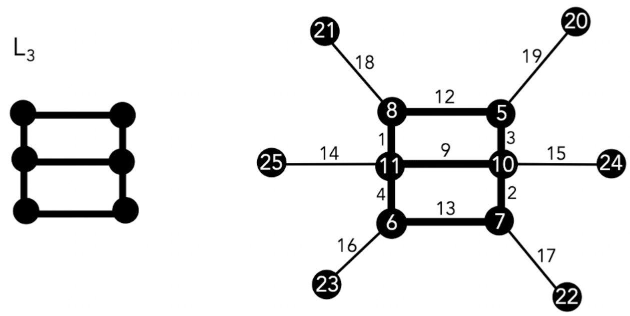

Figure 7: L_3 Ladder Sun graph with magic number 39

Theorem 3. *There is no vertex-magic total labeling of a L_n-sun when $n > 8$.*

Proof. Let L_n be a ladder sun graph. We show that there is no VMTL when $n > 8$. There are $e = 3n - 2$ edges. There are $v = 2n$ vertices. To determine when $e > z$, we first see when $e = z$.

$$\begin{aligned} 3n - 2 &= \frac{-1 + \sqrt{1 + 8(2n)^2}}{2} \\ 6n - 4 &= -1 + \sqrt{1 + 32n^2} \\ 4n^2 - 36n + 8 = 0 \end{aligned}$$

When we solve the equation $4n^2 - 36n + 8 = 0$ the roots are 0.228 and 8.772. We can determine that $e - z < 0$ on the interval $(0.228, 8.772)$ and positive everywhere else. This means there is no VMTL when $n \geq 9$ because of Theorem 2. □

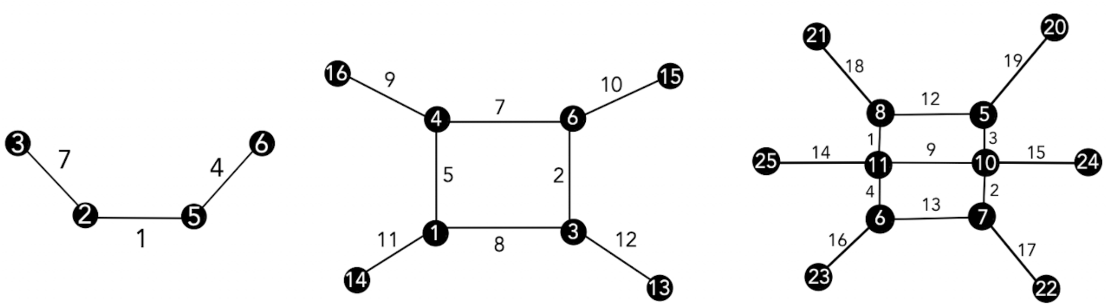

Figure 8: Ladder Sun graphs with a VMTL

In Figure 8 we see VMTL's for the L_1-sun, L_2-sun, and L_3-sun.

3.2 Complete Bipartite Graphs

We now consider complete bipartite sun graphs. Complete bipartite graphs, $K_{m,n}$, have two sets of vertices, set A and set B where the size of set $A = m$ and the size of set $B = n$. All vertices in set A are adjacent to all of the vertices in set B and the only edges in $K_{m,n}$ are edges joining a vertex in A to a vertex in B. In Figure 9 we can see complete bipartite graphs with different set A and set B sizes.

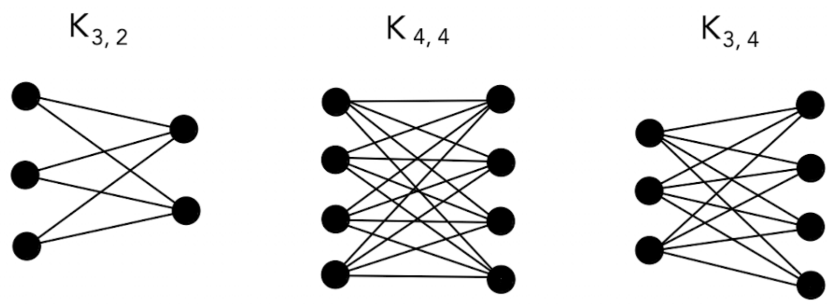

Figure 9: Complete Bipartite Graphs

3.2.1 Complete Bipartite Graphs $K_{x,1}$

In this section we consider complete bipartite $K_{x,1}$-sun graphs. In Figure 10 we see an example of the graph of $K_{x,1}$ and $K_{x,1}$-sun.

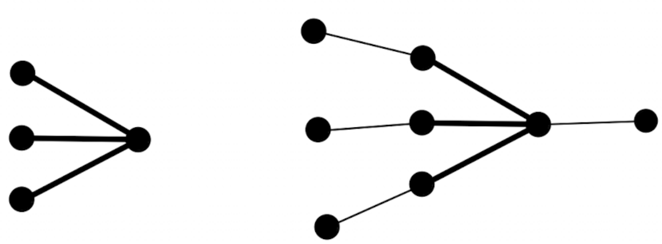

Figure 10: Complete Bipartite Graph $K_{3,1}$ and $K_{3,1}$-sun

Theorem 4. *There is always a potential for a VMTL of a $K_{x,1}$-sun when $x \geq 1$.*

Proof. Let $K_{x,1}$ be a complete bipartite graph where $x \geq 1$. In $K_{x,1}$ there are $e = x$ edges. There are $v = x + 1$ vertices. We want to show complete bipartite graphs $K_{x,1}$ have a potential for vertex-magic total labeling in the $K_{x,1}$-sun. We determine first where $e = z$.

$$
\begin{aligned}
x &= \frac{-1+\sqrt{1+8(x+1)^2}}{2} \\
-4x^2-12x-8 &= 0
\end{aligned}
$$

The equation $-4x^2 - 12x - 8 = 0$ is positive between -2 and -1. We can determine that $e < z$ for all positive values of x. This means there is always a possible VMTL because of Theorem 2. □

Example 4. We look at the labeling process for the $K_{3,1}$ graph. By Theorem 4 we know that there is always a possibility for a VMTL on $K_{x,1}$-sun graphs when $x \geq 1$. Now we can determine the magic spectrum using Theorem 1. Using the order and size from the $K_{3,1}$-sun graph we have $v = 8$ and $e = 7$.

$$
\begin{aligned}
\frac{7(7+1)+(7+8+1)(7+8)}{2(8)} \leq & k \leq 7+\frac{7(7+1)+(7+8+1)(7+8)}{2(8)} \\
18.5 \leq & k \leq 25.5
\end{aligned}
$$

The k value can only be an integer so we change that inequality to $19 \leq k \leq 25$. Next, we eliminate even more k values because we need four 2 number combinations to be able to label the ray edges and vertices. Using this information we reduce our magic spectrum to $19 \leq k \leq 23$. Now we choose our largest k value and find the corresponding e_{sum} using equation (1).

$$
\begin{aligned}
e_{sum} &= (8)(23) - \frac{(16)(15)}{2} \\
e_{sum} &= 64
\end{aligned}
$$

We label the ray edges and vertices with the four 2 number combinations that sum to $k = 23$. The ray vertex adjacent to the vertex of maximum degree is labeled with the larger number of the pair. The vertex of maximum degree is labeled with 1 because it has the highest degree. The remaining ray vertices are labeled with the smaller number in the pair.

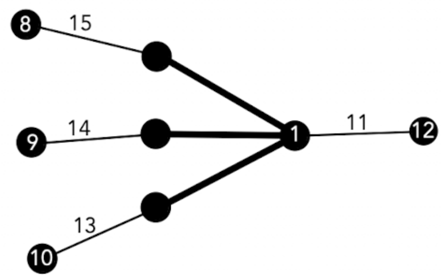

Figure 11: $K_{3,1}$-sun with ray vertices and edges labeled

Using the remaining numbers we form partitions of potential edge values that sum to our calculated e_{sum} of 11 after subtracting the edge values that are already labeled. The complete bipartite graph $K_{3,1}$-sun, with a VMTL is shown in Figure 12.

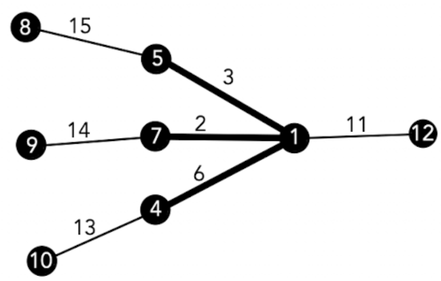

Figure 12: $K_{3,1}$-sun Vertex-Magic Total Labeling

In general, when we find a VMTL for $K_{x,1}$-sun graphs we follow the same process. We determine the magic spectrum using Theorem 1. Then we reduce the spectrum by getting rid of larger k values for which there are not enough 2-number combinations that sum to k. With our reduced k spectrum we pick the largest remaining k as the magic number and find the corresponding e_{sum} using equation 1. Now we label the ray vertices and edges and label the vertex of maximum degree with 1. Next we determine partitions of the remaining labels that sum to e_{sum} along with the edge labels of the ray edges. Then we test the partitions to find a VMTL for the $K_{x,1}$-sun graph.

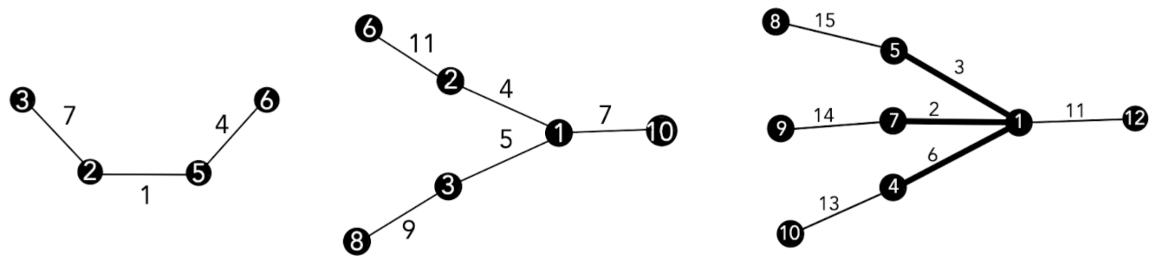

Figure 13: Complete Bipartite Graphs, $K_{1,1}$-sun, $K_{2,1}$-sun, and $K_{3,1}$-sun with VMTL

In Figure 13 we can see additional $K_{x,1}$ graph with VMTL's, $K_{1,1}$-sun, $K_{2,1}$-sun, and $K_{3,1}$-sun.

3.2.2 Complete Bipartite Graphs $K_{x,2}$

In this section we consider complete bipartite $K_{x,2}$-sun graphs.

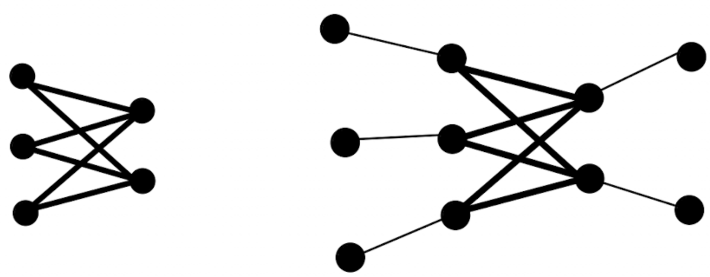

Figure 14: $K_{x,2}$ Complete Bipartite Graph and Sun Graph

Theorem 5. *There is no vertex-magic total labeling for a $K_{x,2}$-sun when $x \geq 4$.*

Proof. Let $K_{x,2}$ be a complete bipartite graph. There are $e = 2x$ edges and $v = x + 2$ vertices. We want to show $K_{x,2}$-sun can not have a VMTL when $x \geq 4$. To determine when $e > z$, we will first see when $e = z$.

$$2x = \frac{-1 + \sqrt{1 + 8(x+2)^2}}{2}$$
$$8x^2 - 24x - 32 = 0$$

When we solve the equation $8x^2 - 24x - 32 = 0$ the roots are -1 and 4. $e - z > 0$ when $x > 4$. When $e > z$ there is no labeling. There is no VMTL on $K_{x,2}$-sun when $x \geq 4$ because of Theorem 2. □

3.2.3 Complete Bipartite Graphs $K_{x,x}$

In this section we will consider complete bipartite $K_{x,x}$-sun graphs.

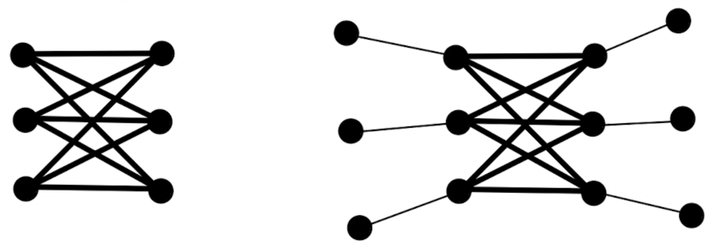

Figure 15: $K_{x,x}$ and $K_{x,x}$-sun

Theorem 6. *Let $K_{x,x}$ be a complete bipartite graph and let $x \geq 3$. Then there is no VMTL on $K_{x,x}$-sun.*

Proof. Let $K_{x,x}$ be a complete bipartite graph where x is the number of vertices in one set. There are $e = x^2$ edges and $v = 2x$ vertices. We want to show a $K_{x,x}$-sun can not have vertex-magic total labeling when $x \geq 3$. To determine when $e > z$, we will first see when $e = z$.

$$\begin{aligned} x^2 &= \frac{-1 + \sqrt{1 + 8(2x)^2}}{2} \\ 4x^4 - 28x^2 &= 0 \end{aligned}$$

The roots of the equation, $4x^4 - 28x^2 = 0$, are -2.646, 0, and 2.646. $e - z > 0$ when $x \geq 3$. When $e > z$ there is no labeling. There is no VMTL on $K_{x,x}$-sun when $x \geq 3$ because of Theorem 2.

□

3.3 Prism Graphs

In this section we will consider prism-sun graphs. A **prism graph**, Y_n, is a graph with $2n$ vertices that consists of two cycle graphs, an exterior and interior cycle graph, with an additional edge between corresponding vertices of the two cycle graphs.

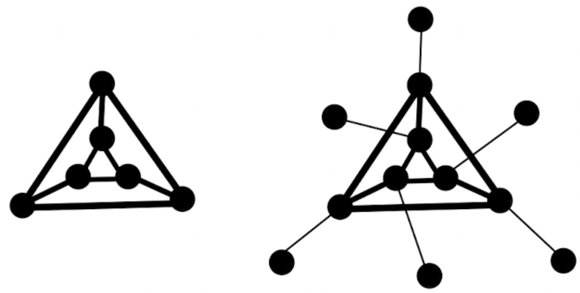

Figure 16: Prism Graphs Y_3 and Y_3-sun

Theorem 7. *There is no vertex-magic total labeling for a Y_n-sun when $n \geq 1$.*

Proof. Let Y_n be a prism graph where n is the number of exterior vertices. There are $e = 3n$ edges. There are $v = 2n$ vertices. We want to show there is no VMTL on Y_n-sun. To determine when $e > z$, we will first see when $e = z$.

$$\begin{aligned} 3n &= \frac{-1 + \sqrt{1 + 8(2n)^2}}{2} \\ 4n^2 + 12n &= 0 \end{aligned}$$

The roots of the equation, $4n^2 + 12n = 0$, are -3 and 0. $e - z > 0$ when $n > 0$. When $e > z$ there is no labeling. There is no VMTL on Y_n-sun when $n \geq 1$ because of Theorem 2. $\square$

Theorem 7 shows that a prism-sun graph does not have a VMTL.

3.4 Wheel Graphs

In this section we will consider wheel-sun graphs.

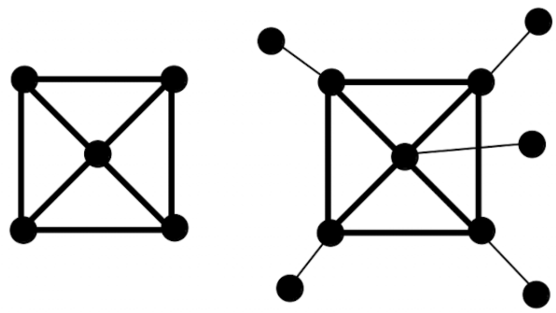

Figure 17: Wheel Graph W_4 and W_4-sun

A **Wheel graph**, W_n, is a cycle graph, C_n with an additional vertex adjacent to all the other vertices.

Theorem 8. *There is no vertex-magic total labeling for a W_n-sun when $n \geq 2$.*

Proof. Let W_n be a wheel graph where n is the number of vertices in the cycle graph. There are $e = 2n$ edges and $v = n + 1$ vertices. We want to show where wheel-sun graphs can not have a vertex-magic total labeling.

$$\begin{aligned} 2n &= \frac{-1 + \sqrt{1 + 8(n+1)^2}}{2} \\ 8n^2 - 8n - 8 = 0 & \end{aligned}$$

$e - z > 0$ when $n \geq 2$. The roots of the equation, $8n^2 - 8n - 8 = 0$, are -0.618 and 1.618. When $e > z$ there is no VMTL. There is no VMTL on W_n-sun when $n \geq 2$ because of Theorem 2. $\square$

4 Summary

The table gives a summary of when each graph has a possible VMTL or no VMTL.

Graph	**Possible VMTL**	**No VMTL**
L_n-sun	$1 \leq n \leq 8$	n$\geq$ 9
$K_{x,1}$-sun	$x > 0$	Not proven
$K_{x,2}$-sun	$1 \leq x \leq 3$	$x \geq 4$
$K_{x,x}$-sun	$1 \leq x \leq 2$	$x \geq 3$
Y_n-sun	Never	$x \geq 1$
W_n-sun	$n = 1$	$n \geq 2$

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