

Solvency II - An important case in Applied VaR

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Abstract:

Value-at-Risk (VaR) is an extremely popular risk measure and many financial companies have successfully used it to manage their risks. Recent developments towards a general single European financial regulation, lead to a great increase in the use of VaR. At least, for European Bank and Insurance industry, VaR is no longer an optional risk management tool, but it became mandatory.

In this chapter we focus on the Insurance business and discuss the use of VaR as it has been proposed in the context of the Solvency II (undergoing) negotiations. Our goals are, on the one hand, to present the underlying assumptions of the models that have been proposed in the Quantitative Impact Studies (QIS) and, on the other hand, to suggest alternative VaR implementations, based upon estimation methods and firm specific characteristics. Our suggestions may be used to develop internal models as suggested in Solvency II context. Finally, we analyze the case a of Portuguese insurer operating in the motor branch and compare QIS and internal model VaR implementations. In our concrete application, (one year horizon) capital requirements are similar under the two alternatives, allowing us to conclude for the robustness of the models proposed in QIS.

Keywords: Value-at-Risk, Financial risk regulation, insurance regulation, Solvency II

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1. Introduction

Insurance is a risky and a risk business and insurance companies' role, in the economic activity as an institutional investors, is increasingly important.

On the one hand and given the purpose of its business, an insurer is a *risk taker* by selling insurance. So, to be able to survive in their industry, insurers needs to make sure they are able to satisfy the responsibilities assumed towards their clients. To fulfil that purpose an insurer needs not only, to appropriately evaluate the risks at stake charging appropriate insurance premiums, but also to calculate the amount of capital that should be kept in order to face other losses that, in some situations, can be quite adverse. Naturally, such capital requirements are based on risk taking and depend upon the concrete nature and amount of risks a specific insurer covers. Risks that can be associated with the insurance firm core activity are many times called *insurance risks*. Examples of such risks are the following: possible wrong assessment of the risks insured (premiums charged may not be enough to cover the assumed responsibilities), defaults from their clients (in paying agreed future premiums); or counterparty risk from re-insurers (other entities whom they sold their risks to and may default on their contracts leaving the insurance company with the original obligations they thought they no longer had as they had been resold).

On the other hand, an insurer is an *investor* and a natural player on financial or capital markets. As any investor, an insurer's portfolio typically consists of various assets of different types with different risk profiles. The assets portfolio of an insurer include all types of products, from simple deposits, to stocks, bonds, investment funds units, real estate investments and various types of derivative products on a variety of possible underlying. These risks should also be assessed, both in nature and amount, and taken

into account when computing capital requirements. These are risks that do not result directly from the insurer's core activity but from the fact that cash-flows resulting from that activity are invested in risky financial products. They are, therefore, globally called *market risk*.

The insurance sector carries significant importance in Europe. On voluntary, as well as on statutory basis, it provides cover against various risks facing the citizens, corporations and other organizations. In addition, collecting long-term savings of millions of Europeans, it is the largest institutional investor on EU stock exchanges. An appropriate prudential framework for insurance is therefore of extreme importance. Recent catastrophic events – whether natural or man-made – have highlighted the significance of having a stable and solvent insurance sector.

The previous EU solvency system, introduced in the 1970s, was conceived in a period when the general economic features, as well as insurance practices, were different. Capital requirements of insurance undertakings were determined based on simple ratios that were calculated as percentages of risk exposure measures (e.g. technical provisions, premiums or claims). Nowadays, insurance companies are faced with a different business situation with increasing competition, convergence between financial sectors as well as international dependence. At the same time insurance, asset, and risk management methods and techniques have been significantly refined. The recent economic downturn and volatile financial markets have also put the insurance sector solvency and risk management under significant pressure.

The inadequacy of the old solvency system started to be discussed in 1998, when the European Commission acknowledge the need of “enhancing consumer confidence by

promoting full financial market integration while ensuring high levels of consumer protection” [European Commission (1998)] and initiated several changes to the existing directives, leading the European control authorities (of the insurance business) into the development of a new solvency project. For further details, on the background and international context of the new project, we refer to Linder and Ronkainen (2004).

The new *European Solvency Project* was organized in two phases: In the first phase, known as Solvency I, fundamental arrangements were specified, a general framework was defined, and several studies were ordered by the European Commission. The most well-known studies are the KPMG (2002) study and what became known as the Sharma (2002) Report. For a survey on all studies ordered by the European Commission and their recommendations see Eling, Schmeiser and Schmit (2007). This first phase ended during 2003 and its recommendations became effective as of January 2004.⁴ In a second phase, called Solvency II, these fundamentals are being developed into specific rules and guidelines. This project follows in spirit the so called Basel II agreement, established and already in application in the European banking industry. The solvency working group is, thus, moving towards convergence of insurance and banking regulatory systems. Still, a significant difference between the two is that Solvency II focuses more heavily on a holistic risk management approach rather than on management of single risks independently. The Solvency II project is not yet established, as is still under scrutiny by different supervisory authorities in each EU country and changes may still occur. In fact, this input capacity was specifically sought by the EU as part of the regulatory development process. See European Commission (2003). Value-at-Risk (VaR) has

⁴ Solvency I addressed many of the coordination issues across regulatory bodies and provided an initial rules-based set of capital requirements (see EU Directive 2002/13/EC for non-life insurers and EU Directive 2002/83/EC for life insurers).

emerged as a key instrument and will continue to be so. This study discusses the use of VaR, as it has been proposed in the context of the Solvency II and it is organized as follows. In the next section, Section 2, we make a summary presentation of the Solvency II project and discuss the possibility insurance companies have of developing their own internal models when computing capital requirements. In Section 3 we do a critical presentation the standard method for capital requirement calculation suggested in Solvency II and put into practice by the *Third Quantitative Impact Study (QIS3)*⁵. We highlight the standard method assumptions and propose alternatives that could be useful in building internal models. In Section 4, we consider the case of an insurer operating in the Portuguese motor branch. For this concrete insurer we compute (one year) capital requirement both directly applying the QIS3 rules and thought an internal model that takes into account our suggestions. In the last section we finish the work by addressing final remarks and criticisms on the models used and results obtained.

2. Solvency II and QIS3

Traditional European solvency systems, monitored by control authorities in each country, were based on solvency margins and technical reserves for each branch. Even in the current system – Solvency I – the solvency level depends only on the amount of premiums or claims, and there is no relation with capital requirements and risk taken.

Solvency II aims to be a major improvement over the previous schemes. For the first time a solvency scheme acknowledge the important role insurers play in the world financial markets, and has the ambition of taking into account the financial market risk associated

⁵ Several *quantitative impact studies* have been put in place by the CEA-*Comité Européen des Assurances* by request of CEIOPS-Committee of European Insurance and Occupational Pension Supervision. Their purpose is to study the impact of the introduction of Solvency II rules.

with insurers' investments – the *market risk*. In addition, Solvency II also acknowledges the business of risk is risky in itself, classifying and taking into account various insurance risks: *underwriting risk*, *counterparty risk*, *operational risk*. Similarly to its banking industry counterpart, the Basel II project, Solvency II has been structured along three main objectives – called the three pillars. *Pillar I* defines the financial resources that a company needs to hold in order to be considered solvent. Two thresholds are defined, the first called the Solvency Capital Requirement (SCR), and another lower threshold called the Minimum Capital Requirement (MCR). The SCR level is a first action level, that is, supervisory action will be triggered if resources fall below its level. The MCR is a severe action level by the control authority, which can include company closure to new business. While Pillar I focus on quantitative requirements *Pillar II* defines more qualitative requirements and supplements the first. For instance, it defines the framework of intervention of the supervisory authority . Finally, *Pillar III* addresses issues such as risk disclosure requirements, transparency or free access to information.

In this chapter our main focus is on the quantitative aspects of Solvency II and thus, on Pillar I and, in particular, in evaluating the SCR. For the calculation of the SCR, Solvency II proposes a standard (simple) model but encourages the development of internal models that would provide a better adequacy to the kind, spread and amount of risks taken by each insurer. This is underlined by Ronkainen *et al.* (2007). Still, as mentioned in Liebwein (2006), any internal model alternative would have to accomplish legal requirements, provide greater added value to shareholders when risk management processes are included, and be subject to approval by the control authorities.

On what concerns technical reserves, i.e. those directly connected with *insurance risks*, the premium reserves and the reserves for pending claims must be computed separately. In the absence of an active market, the technical reserves must be estimated and a risk margin must be added. For the estimating procedure a probability distribution of the cash-flows must be estimated, the mean value of this distribution will then be used as an estimate. This is known as the *Best Estimate*. The risk margin works as loading and intends to cover the volatility of the risk factors, the uncertainty of the best estimate, the risk associated to the insurance portfolio of the company and natural estimation errors, from the model and its parameters. It is also used to compute a market value for the insurer or branch. The Best Estimate will be used as an approximation to the future costs that the insurer is expected to cover, and this will be the basis of the computation of its capital needs. For such calculation the insurer must have reliable and organized loss past data to allow the use of proper statistical methods. For instance, claim payments of accidents occurring in one year develop along one or more future years. Typically, we need to fill out an empty triangle with properly estimated values.⁶ The discounted sum of the estimated claim payments of each occurrence year will be our estimate for the loss reserves for that year. There are several methods to compute the future payments; the most popular is the so called *Chain Ladder*. For more details on this and other methods please see Taylor (2000). Further discussion on this subject can also be found in Mack (1994) and Verral (1994).

⁶ An example is shown in Table 3 with the data of our application. There, we can see, for instance that losses occurred in 2001 can develop until 2006 or even more, that needs to be estimated as well. Along each line we have payments for losses occurred in each starting year, here the values are incremental payments.

Market risk can, in a more straightforward way, be computed from market quotes. These risks are also not related with each insurance firm, instead they are related to specificities of particular classes of assets. So, one can think of general rules that may apply to such classes of assets and that should be implemented by all firms.

Whether implied from market quotes or estimated, future uncertainty should be evaluated using a *risk measure*. Although the European working parties suggest the use of both the *VaR measure* and the *Conditional Tail Expectation measure* – also known as Tail VaR – at the moment only the VaR measure is considered as a standard ⁷.

Definition 1: Let X be a random variable representing the values of an asset or a return rate for instance. The **Value-at-Risk**, denoted as VaR_{α} , is defined as

$$\text{VaR}_{\alpha \times 100\%} = \inf \left\{ x \in \mathbb{R} : \Pr(X > x) \leq \alpha \right\} \quad (1)$$

In the following we critically present the standard model proposed by the Committee of European Insurance and Occupational Pension Supervision (CEIOPS), as it stands in the latest Quantitative Impact Study (QIS3) and discuss possible alternatives for implementing VaR, whenever we feel it is appropriate.

One first point we would like to make is that the VaR measure makes no assumption concerning the distribution of the random variable we are interested in computing the quantile. Thus, various approaches exist, some called *parametric* and based upon

⁷ In any situation, we will stick to the use of VaR as it is the only risk measure considered as standard in the context of Solvency II. We notice, however, that VaR as a risk measurement has been quite criticized, in the financial literature. A subtle technical problem is that VaR is not sub-additive. That is, it is possible to construct two portfolios, A and B, in such a way that $\text{VaR}(A+B) > \text{VaR}(A) + \text{VaR}(B)$, which contradicts the idea of diversification. A theory of *coherent risk measures* exists outlining the properties we would want any measure of risk to possess. As opposed to VaR, Tail VaR is a coherent risk measure. For further details on this subject we refer to the canonical paper on the subject: Artzner, et al (1999).

assumptions about the underlying distributions – the most commonly used is the Gaussian distribution, others based upon empirical distributions. This last approach requires collecting (usually historical) past information on the random variable we are interested in and observing the appropriate quantile and is, thus, known as *historical* approach to VaR. Both parametric and historical approaches to implementing VaR have their own advantages and disadvantages. On the one hand, any parametric model requires parameter estimation which require choosing the appropriate estimation method and evaluating the ability of the model to explain the data. Furthermore, many of the variables we are interested in are far from the most well known parametric distributions (and clearly far from Gaussian), indicating a historical approach could be best. The historical approach to determining VaR is a nonparametric approach that requires no direct estimation. On the other hand, however, and especially because we are concerned with extreme events the amount of historical information one may need to collect in order to capture well the tail behavior of a distribution can be substantial. Also, we may want to consider the possibility that some extreme events that have not yet occurred in the past, may occur in the future. This advocates in favor of parametric methods. Probably the sensible attitude is to use either both methods – choosing the worst case scenario – or a mixture of them. For a detailed discussion on various VaR implementations we refer to Linsmeier and Pearson (1996) and Jorion (2001).

As we will see, the standard model, as proposed in QIS3, relies considerably on a parametric Gaussian implementation of VaR. We propose, instead that, provided there is enough information on the relevant variables – which is typically the case at least on what concerns with financial data – an historical approach will be more accurate evaluating

VaR. The QIS3 computations rely considerably on the following VaR result for normally distributed random variables.

Result 2: *Suppose you have N random variables, X_n for $n = 1, 2, \dots, N$, all normally distributed with mean zero and variance σ_n . Let $Y = \sum_{n=1}^N w_n X_n$ define a linear combination of the random variables X_n , where the w_n 's are constants. We denote shortly by $\text{VaR}(X_n)$ the VaR of order α of each random variable. Then the VaR of Y ,*

$$\text{VaR}(Y) = \sqrt{\sum_{i,j=1}^N \rho_{ij} w_i w_j \text{VaR}(X_i) \text{Var}(X_j)}, \quad (2)$$

where all VaR are computed for the same probability α .

For the remaining of this section we discuss in detail the implementation of both alternatives. The various risks were divided according to their nature into: *market risk, counterparty risk, underwriting risk* and *operational risk*. The final assessment of the *capital requirement* is obtained by a risk aggregation, considering all the above mentioned risks.

We aim at applying the methods here discussed to an insurer operating in the motor branch. We, therefore focus, as far as insurance risks are concerned, on risks specific of that branch. Also, we do not include in our study catastrophe risk as appropriate data would be hard to find and its complexity is beyond the scope of our application. As far as market risks are concerned, we refrain from discussing the impact of two important classes of assets in capital requirements: risk management assets (such as derivatives) and real estate assets. Both these classes of assets were absent from our concrete insurer

portfolio, but deserve special attention in QIS3. For details on how to include risks or these types of assets not mentioned in this study we refer to CEIOPS (2007a,b,c).

2.1 Market Risk

When evaluating market risk we are mainly concerned with measuring the impact that changes in the market value of various assets have in the overall investments that insurers do when applying the cash-flows of its activity, in risky financial markets. The standard approach suggested in QIS3 relies on two key assumptions: (i) that financial returns are normally distributed and (ii) that such a distribution is static, i.e., is does not evolve over time. Both these assumptions have been widely studied in the financial literature and it has been found enough evidence allowing us to clearly reject any of them. See, for instance, Cont (2001). Moreover, capital requirements are computed assuming a static portfolio, i.e. that a portfolio does not change over time. Good risk management practice, however, requires rebalancing the portfolio periodically. There is a clear inconsistency between accessing risk (via capital requirements) assuming a static portfolio when good risk management lead to a dynamic portfolio. The best we can suggest to overcome this difficulty is to periodically re-evaluate the capital requirements estimation; with a periodicity similar to the periodicity each firm changes their portfolio. We now look deeper into each asset class, present the QIS3 standard model suggestion and propose our alternative. For exact formulas concerning the QIS3 approach we refer to CEIOPS (2007a,b,c).

As far as **equity risk** is concerned, QIS3 divides it into specific and systematic risk. As the first type of risk can be eliminated in a well diversified portfolio, QIS3 considers it as concentration risk (and not equity risk). Having to deal only with the systematic part of

equity risk, QIS3 relies on two indices: a “Global index” that includes all stocks from European countries and other global markets, and a second index called “Others index” where the remaining countries should be included as well as any non quoted stocks (irrespective of their country). Based upon collected historical information and assuming a Gaussian distribution with mean zero for each index returns, QIS3 estimated each index $\text{VaR}_{99.5\%}$ obtaining -32% and -45%, for the Global and Others indices, respectively. Also, QIS 3 suggests the use of a correlation of 0.75 between the two indices and equation (2) to take into account the diversification effect across indices.⁸

An insurer is exposed to **interest rate risk** via all assets and liabilities whose value is sensitive to changes in the term structure of interest rates. Example of such assets and liabilities are fixed-income investments, insurance liabilities, loans, etc. The QIS3 standard model assumes that default-free interest rates (also known as zero-rates) maturing each year from 1y to 20y are independent variables. For each of these variables, QIS3 estimates VaR both with $\alpha = 0.005$ and $\alpha = 0.995$, assuming that changes in rates of any maturity are normally distributed with mean zero. Even though the independence across maturities and the Gaussian assumption are questionable, the proposed standard method allows considering the risk of various movements of the term structure taking into account that rates with different maturities have different volatilities. Also, computing two quantiles for each variable makes sense as an insurer may be more exposed to the risk of interest rates increasing, or decreasing. Finally, when implementing the standard method each insurer must decompose the payoffs of each interest rate

⁸ To the best of our knowledge, only the “Global Index” volatility was estimated based upon quarterly data on the MSCI Developed Markets index, from 1970 to 2005. The “Others index” volatility and correlation between both indices are *add-hoc* values suggested in QIS3.

sensitive asset or liability to match the yearly maturities up to 20 years⁹. Table 1 presents QIS3 VaR estimates for changes in zero rates of different maturities¹⁰.

VaR \ T	1	2	3	4	5	6	7	8	9	10-15	16	17	18	19	20+
$\alpha = 0.005$	0.94	0.77	0.69	0.62	0.56	0.52	0.49	0.46	0.44	0.42	0.41	0.40	0.39	0.38	0.37
$\alpha = 0.995$	-0.51	-0.47	-0.44	-0.42	-0.40	-0.38	-0.37	-0.35	-0.34	-0.34	-0.33	-0.33	-0.32	-0.31	-0.31

Table 1 :VaR for changes in zero rates of several maturities (QIS3 estimates).

Currency risk arises from the level of volatility of currency exchange rates. An insurer is exposed to that risk whenever she directly invests in foreign currencies or when some of her assets or liabilities are denominated in a currency other than its home currency. Once again, the standard model in QIS3 assumes that changes in exchange rates follow a Normal distribution with zero mean. Supposing quotes are in a (Foreign/Home) form, whenever a foreign currency depreciates the mentioned ratio increases. So, provided we subtract the liabilities to the assets denominated in foreign currency (i.e. we consider only the net asset value), adverse movements are measured by VaR with $\alpha = 0.005$. QIS3, considered only the variation of the euro relative to a index of foreign currencies and estimates, therefore, only one VaR which turned out to be of 20%.¹¹

Spread or credit risk is part of the risk an insurer is exposed to via its investments in financial markets. It is associated with the credit worthiness of some financial products issued by corporations. The typical example are corporate bonds whose value is lower than a (otherwise equivalent) government bond because it may happen that the corporation fails to pay some of the coupons or capital. Credit risk can be measured by

⁹ The QIS3 recommends for maturities longer than 20 years, the use of the 20 year VaR.

¹⁰ These estimates we obtained using German monthly zero-coupon rates, from 1y to 10y maturities since 1972 and also extracted from daily data European Swaps with 1y,5y,10y,15y,20y, 25y and 30y, since 1997.

¹¹ This estimation was performed using monthly exchange rate quotes towards the euro from 1958 to 2006, excluding the Bretton-Woods period (1992-2001). The foreign currencies considered were the US dollar, the British pound, the Japanese yen, the Swedish crone, the Swiss franc and the Australian dollar.

the difference in yields between corporate and government bonds. Such a difference is called the *credit spread*. Clearly, in the same way interest rates vary across maturities also credit spreads will do so, originating what is known as a *credit curve* or credit spread terms structure. Moreover, naturally, credit spreads vary across ratings: products with less credit risk (higher rating) will have lower spreads than those with lower credit worthiness (lower rating). The QIS3 standard approach takes both these facts into account and analyse various pairs of (duration¹², credit rating class). Nonetheless, credit spreads of each pair are taken as independent random variables. For an insurer the risk is that their investments will be worth less due to a decrease in credibility, which is the same as an increase in credit spreads. Thus, the appropriate figure we are looking for when trying to access this risk is a VaR with $\alpha = 0.995$. Table 2 presents such a VaR for some combinations of duration/credit class¹³.

VaR _{99.5%}	2	4	6	8	10	12	14
AAA or AA	0.5%	1%	1.5%	2%	2.5%	3%	3.5%
A	2.06%	4.12%	6.18%	8.24%	10.30%	12.36%	14.42%
BBB	2.25%	4.50%	6.75%	9%	11.25%	13.05%	15.30%
BB	6.78%	13.56%	20.34%	27.12%	27.12%	27.12%	27.12%
B	11.2%	22.4%	33.6%	33.6%	33.6%	33.6%	33.6%
CCC	22.4%	44.8%	44.8%	44.8%	44.8%	44.8%	44.8%
Unrated	4%	8%	12%	16%	20%	24%	28%

Table 2: VaR for assets with different durations and credit spreads. (Based upon QIS3 calibrations).

The last risk we must refer to before aggregating all market risks is **concentration risk**.

In QIS3, the definition of concentration risk is restricted to the risk regarding the accumulation of exposures with the same counterparty also called *name*. It does not

¹² Here we refer to the *Fisher-Weil Duration measure* that can be interpreted as an *average* payment time of a cash-flow stream paid at several points in time. For further detail we refer to Weil (1973).

¹³ QIS are not presented as in Table 2. Instead, they present two separate functions, a function dependent upon the rating class and an independent function that depends upon duration alone. The final VaR measure is then obtained multiplying the two functions. We believe the above table allows a better understanding of the risks at stake and refer to CEIOPS (2007c) for further details.

include other types of concentrations such as geographical concentration, industry sector concentration, etc. All entities belonging to the same group are interpreted as the same name. An *add-hoc* concentration threshold is defined based upon the credit rating of each product: for products with A or higher rating the threshold is 5% for the insurer total assets value while for products with lower rating is 3%. QIS proposes an excess expose measure based upon those thresholds and VaR corrections that should be performed when the insurer's assets are not well diversified enough. Most insurers do have well diversified portfolios, so it is quite common to obtain zero concentration risk. This is the case in our case study, we refer to CEIOPS (2007a) for calibration details and actual VaR corrections. Finally, to compute the **overall market risk VaR**, QIS3 simply suggests the use of equation (2) with an *add-hoc* correlation matrix¹⁴ and take into account each asset class weight in terms of the total assets considered in the market risk analysis.

We now briefly discuss an alternative, to the QIS3 standard VaR, for measuring all market risks. In the case study we will use this alternative as our **Internal Model**. Our suggestion relies on two main ideas: (i) to consider the historical approach to VaR when evaluating market risks and (ii) to use information each insurer has easy access to its own current portfolio composition. The historical approach to VaR is considered by many authors as “the simplest and most transparent method of calculation” [Dowd (2005)]. It involves running the current portfolio across a set of historical asset values and to build a historical return distribution to obtain the relevant percentile (the VaR). The key aspect here is that we do not assume a normal distribution of asset returns. Moreover, using the insurer specific portfolio composition, we focus on the exact risks the insurer take and not

¹⁴ The exact correlation matrix suggested in QIS3 is presented in Table 4 where it is compared with an alternative matrix estimated considering our case study Insurer specific portfolio.

all the risk it could take. The main drawbacks that could be pointed out are the requirement for a large market database and the computationally intensive calculations. The first is hardly a problem in the concrete case of market risks as long financial time series are typically available. The second drawback depends upon the complexity of the insurer portfolio and can be overcome with a good computer program. Our suggestion, for computing market risk takes into account the current issuer portfolio and, for each risk class, can be summarized in the following **5 steps procedure**:

1. Collect as much historical information as possible on the market value of any product subject to that risk. Denote the collection of all such products by $W_p = (w_1, \dots, w_N)$ where w_n for all $n = 1, \dots, N$ are the weights of a specific asset or liability.¹⁵

2. Use those weights to build a time series of the collection value. Let $V_n(t)$ be the market price of a specific asset or liability at some past date t . Then, current collection value at the same date, $V_p(t)$, is given by $V_p(t) = \sum_{n=1}^N w_n V_n(t)$.

3. From that time series deduce the time series of collection value returns, R_p . For data with high frequency the actual returns may be well approximated by log returns,

$$R_p(t) = \frac{V_p(t+1) - V_p(t)}{V_p(t)} \approx \ln \left(\frac{V_p(t+1)}{V_p(t)} \right) \quad (3)$$

4. Assume the annual returns are given by $R_p^a(t) = \Delta^{-1} \times R_p^\Delta(t)$ where Δ denotes the periodicity of the data expressed in years and built a time series of annual returns.

5. Using the series of annual returns determine the appropriate VaR for the risk under analysis observing the appropriate quantile from the historical distribution.

¹⁵ We take assets to have a positive weight and liabilities to have a negative one.

To aggregate the total market risk, estimate the correlations between each type of risks from the various time series determined in 5. Then, using the estimated correlations and the previously obtained VaR for each risk, we apply equation (2) and obtain the SCR for market risk using.

We note that our Internal Model approach to estimate market risk is similar, at least in spirit, to the ideas underlying the QIS3 proposal. To be more concrete, when dealing with *interest rate risk*, we recommend considering several points of the interest rate term structure and computing the VaR for each of them. The difference is that we believe the relevant points should be chosen taking into account the durations of the assets and liabilities exposed to interest rate risk in the insurer current portfolio. For *credit spread risk*, we also suggest grouping the products according to their rating and duration. However, taking a historical VaR approach would mean estimating a VaR for each pair (duration, rating) based upon past yield changes. Finally, we notice that using this approach it makes no sense to define a concentration risk as we have not considered only the systematic risk of each product but its actual total risk¹⁶ We now go on analyzing *insurance risks*, i.e. risks that are more related to an insurer's core activity.

2.2 Counterparty Default Risk

This is the risk of default of a counterparty to risk mitigation contracts like reinsurance and financial OTC¹⁷ derivatives. In terms of the amounts at stake, reinsurance counterparty risk tends to be the most important. In our case-study it is also the only one.

Here we analyze only reinsurance counterparty risk. If case of default from a reinsurance

¹⁶ Due to space restrictions we do not describe in detail our approach to each of the market risks. Instead, we present the main steps and refer to Vicente (2008) for further details.

¹⁷ Over-the-Counter (OTC) derivatives are bilateral contracts traded outside exchanges and thus subject to counterparty risk.

counterparty, the insurer will incur in replacement costs (RC), which can be evaluated as the sum of the technical reserves of the ceded reinsurance plus the extra (previously paid) premium minus any recoveries. QIS3 assumes zero recovery in case of default and estimates probabilities of default (PD) for different reinsurance rating classes. The rating classes are the same as those in Table 2, except that the first two, AAA and AA, come separated and the last two come merged. The figures (in %) in a decreasing rating scale come, 0.002%, 0.01%, 0.05%, 0.24%, 1.2%, 6.04% and 30.41%, respectively. The VaR for this risk is computed in three steps: (1) calculation of the so called Herfindahl index¹⁸, (2) calculation of capital requirement per counterparty and (3) aggregation. The final formula for the VaR depends on an implicit correlation of counterparty default. In our case-study the correlation is 1 and the formula is give by $VaR = RC \times \min\{100 \times PD; 1\}$. We refer to CEIOPS (2007a) for details on this method or on how QIS3 handles derivative related counterparty risk. Given the complexity of counterparty risk modeling we suggest no **Internal Model** to the QIS3 approach.

2.3 Underwriting Risk

This sort of risk arises directly from the nature of the insurance activity. In the case of non-life insurance, it basically includes premium risk and the reserve risk.¹⁹ On what concerns premium and reserve risk, QIS3 standard approach rely on two measures: a premium volume measure (PVM) and a reserve volume measure (RVM) and in evaluating the variations of such measures to compute their volatilities. For the **premium risk** the QIS3 suggestion are either to directly observe the PMV from the insurer's

¹⁸ The Herfindahl index is a measure of the size of firms in relationship to the industry and an indicator of the amount of competition among them.

¹⁹ Catastrophe risk (*Cat* risk) is also considered as underwriting risk. However, as previously mentioned, we do not analyse this risk in the context of this study.

estimate of the premiums for next year, net of reinsurance or to consider an increase of 5% on the actual net premiums. Even though 5% is an *add-hoc*, lack of information makes this second option quite used in practice. This will also be the option we will choose for our case-study. QIS3, then suggests to obtain the PVM volatility, σ^{pr} , by the well known Bühlmann and Straub's credibility formula: $\sigma^{pr} = \sqrt{c\sigma_I^2 + (1-c)\sigma_M^2}$ where σ_I is the insurer's sigma, σ_M is the market sigma and c is a constant dependent upon the sample size n ($c = n / (n + 4)$ if $n \geq 7$ and $c = 0$ otherwise).²⁰ QIS 3 uses a fixed value of 10% for σ_M , and suggest σ_I to be computed by (historical) estimation, calculating a (weighted) sample standard deviation of the loss ratio, annual claims cost over premiums net of reinsurance. For the **reserve risk**, their suggestion is to use the figure for the loss reserves from the insurer's last year balance sheet as RVM. This is an amount supervised by the control authorities. QIS3 computes the RVM volatility, σ^{res} , first computing the volatility of the line of business *third party liabilities* and of the *other classes* and then aggregating it. QIS3 recommends the use of 12.5% and 7.5%, respectively, for the two mentioned volatilities. Finally, the **overall underwriting VaR** is obtained using the sum of the volume measures $VM = PVM + RVM$ and to the following formula,

$$\text{VaR} = VMf(\sigma) \quad \text{with} \quad f(\sigma) = \frac{e^{\left\{N_{99.5\%} \sqrt{\ln(\sigma^2 + 1)}\right\}}}{\sqrt{\sigma^2 + 1}} - 1 \quad (4)$$

where $N_{99.5\%}$ stands for the 99.5 percentile of a standard normal distribution and σ is an overall volatility given by $\sigma^2 = (PVM \times RVM \times \sigma^{pr} \times \sigma^{res}) / VM^2$. The function

²⁰ See Bühlmann and Gisler (2005) for more details on the credibility formula.

$f(\sigma)$ implicitly assumes that the underwriting risk follows a lognormal distribution and is such that, for a small and medium size σ , gives roughly 3σ .

Next, we discuss alternative procedures, that may be used an **Internal Model**, to evaluate the underwriting risk. For **premium risk** we consider a sum of loss and expense ratios, called combined ratio CR , $CR = LR + ER$. The loss ratio (incurred losses and loss-adjustment expenses divided by net earned premium) is added to the expense ratio (underwriting expenses divided by net premium written) to determine the company's combined ratio. The combined ratio is a reflection of the company's overall premium profitability. A combined ratio of less than 100 percent indicates premium profitability, while anything over 100 indicates an underwriting loss. The expense ratio, ER , tends to be relatively constant over time, so most of the premium risk is, in fact associated with the loss ratio that should be properly evaluated. Our suggestion is to follow El-Bassiouni's (1991) model where LR is assumed to follow a lognormal distribution. The proposed model can be understood a linear regression model on the logarithm of the loss ratio, $Y_{it} = \ln(LR_{it}) = \lambda_i + \beta_t + \varpi_{it}$, with i and t the insurer and year indices. The expected value Y_{it} is given by λ_i , while $\beta_t \sim N(0; \theta_2)$ and $\varpi_{it} \sim N(0; \theta_1 / RP_{it})$ with RP_{it} the received premiums. Note that, to estimate an insurer loss ratio all the loss ratios of other insurers in the same branch are used. The model has $(n+2)$ parameters that need to be estimated. Its (stepwise) estimation is a mix of data estimation and simulation. With these estimates and the distribution assumed we simulated 5000 values for Y_{it} and built the empirical distribution. The $\text{VaR}_{99.5\%}(LR)$ is then used to compute the VaR for overall risk $\text{VaR}^{pr} = (\text{VaR}(LR) + ER - 100)RP$. As far as the **reserve risk** we follow

Hersterber *et al.* (2003). In some sense the computations are related to those of the best estimate for the loss reserving but they go further computing an (estimated) distribution for the future liabilities, since we should estimate a maximum value for reserves under that distribution. For that we use the Bootstrap re-sampling technique to simulate the residues obtained from the Chain-Ladder method. Finally, we produce an estimate distribution of reserves. The VaR for this risk will be the difference between that 99.5% percentile of the estimated distribution and its mean value.²¹ The VaR that evaluates the **overall underwriting risk** will be the result of the aggregation of the premium risk and the reserve risk using the formula in equation (2).

2.4 Operational Risk

Operational risk arises with potential losses from a group of misconducts of internal systems, procedures, human resources, external events, legal risk and others. For this risk we follow fully the QIS3 lines, i.e. we consider the same approach in our internal model. By its nature it is not easy to make an allocation of this risk among all possible different branches. QIS3 considers information along major branches like life, non-life and health insurance. For our application, we only need information on the non-life branch. Based upon studies on operational misconducts in non-life insurance QIS3 suggests a calculation formula for this risk, underlining however that it is not definite as it needs further developments. QIS3 computes the solvency operational capital requirement, $SCR^{operational}$, as be the minimum between two risk figures: (i) 30% of they call the SCR^{basic} and (ii) 2% of either gross technical reserves or gross premiums, whichever is

²¹ For more details on the procedures concerning the underwriting risk, we refer to Vicente (2008) where all details can be found.

bigger. The basic SCR results from the aggregation of the market, counterparty and underwriting VaRs using equation (2) and assuming correlations of 0.25, 0.25 and 0.5 between the pairs (market, counterparty), (market, underwriting) and (counterparty, underwriting), respectively. Operational risk assessment is complex and reliable information hard to obtain, so we propose no Internal models concerning this risk.

2.5 Final Capital Requirements and Risk Margin

The final Solvency Capital Requirement, SCR^{final} , for the period under analysis is the sum of the SCR^{basic} and the $SCR^{operational}$. Under the *Swiss Solvency Test*, the solvency of other periods than the one we are interested in – *run-off* periods - should also be considered in building up a prudential *risk margin*. Given our main goal – discuss the use of VaR in the context of Solvency II – we refrain from explaining the exact risk margin computations that are non VaR related.

3. Case Study

In this section we illustrate the ideas previously presented and compute the Solvency Capital Requirements for the case of a Portuguese Insurer operating in the motor branch. From now on we refer to this insurer, shortly, as the “Insurer”. We present in parallel the results from QIS3 standard model and the Internal Model (IM) proposed by the authors. Our analysis is a static one (thus, limited) and based upon the Insurer information at the end of 2006. We note that the data from the automobile insurance includes some other additional, but smaller, covers. We have decided not to work the models for the different lines of business as were not quite sure on the reliability of the separation. Figure 1 presents the composition of the Insurer’s assets and liabilities.

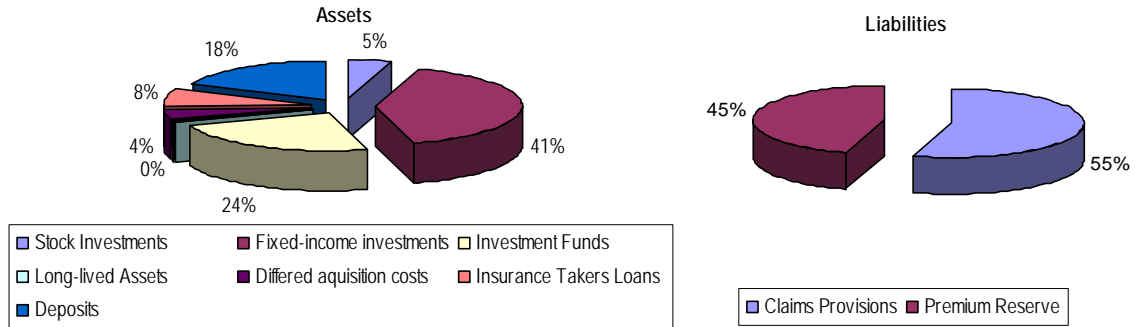


Figure 1: Assets and Liabilities of the Insurer

3.1 Market Risks

The Insurer “portfolio”, i.e. the amount of assets or liabilities that is exposed to market risk, represented 70.5 % of its total assets and 54.6% of its total liabilities. We note that the same asset, for instance corporate bonds in foreign currencies, exposes the Insurer to several market risks at the same time: interest rate risk, credit spread risk and currency risk. This, together with the fact that risk (and risk measures) are non additive (due to the diversification effects), makes far from trivial to present the proportion of each risk class on the total “portfolio” market risk. Table 3 compares the QIS3 and the IM VaR results. For the IM estimates we have used daily market data, from the previous five years²² on all possible portfolio products. The analysis can be easily extended to a longer data series.²³ All information – stock quotes, bonds technical information and quotes, interest rate quotes and exchange rates – was extracted from the *Bloomberg, L.P. information system*. The deviations column in Table 3 shows the difference between the QIS3 and Internal Model VaR estimates both in absolute terms and measured in percentage of the QIS3 results. We start by noticing that the internal model VaR results can be considerably higher or lower. For instance the IM currency risk VaR is 42.1% lower and the IM

²² For a few stocks which were not listed during the entire period the information was somewhat shorter.

²³ It is far from clear whether going further into the past would be more accurate in estimating future risks, which is really what we are trying to access.

interest risk VaR is 32.5% higher than the QIS3 VaRs. These deviations, however, partially compensate each other and the final SCR^{market} in the IM is only 9.4% higher than according to QIS3. The Insurer exposure to **equity risk** was mainly (95%) due to investments in stocks belonging to QIS3 “Global Index”. However, 17.5% of those investments are in home stocks, which clearly lead to a non perfect correlation of the portfolio with the “Global Index” and, consequently, to a higher risk. This is clearly a concentration risk not captured by QIS 3 because it does not consider geographical concentration. The IM captures this higher risk, producing a higher equity VaR estimate. As far as **interest rate risk** is concerned the Insurer risk was that of an increase in interest rates. The differences are a direct result of the difference in the way VaR is estimated under QIS3 under our IM. Figure 2 illustrates such differences and we note that for maturities between 2 and 14 years IM model predicted higher VaRs. Most of the interest rate sensitive products had durations within that interval, which justifies the increase of 32.5% of the QIS3 VaR.

	QIS 3				Internal Model (IM)				Deviations (IM-QIS3)		
		€	% Asset Class	% Overall Portfolio		€	% Asset Class	% Overall Portfolio	€	% of QIS 3 VaR	
Equity	Global (95%)	2.108.575									
	Other (5%)	144.472									
	VaR	2.113.519	31%	2,8%	VaR	2.308.447	33,4%	3,0%	VaR Dev	194.928	9,2%
Interest Rate	Assets	3.012.414			Assets	3.682.230			Assets		
	Liabilities	-658.332			Liabilities	-562.716			Liabilities		
	VaR	2.354.082	15%	3,2%	VaR	3.119.514	20,0%	4,2%		765.432	32,5%
Currency	VaR	848.494	20%	1,1%	VaR	491.162	11,6%	0,6%		-357.332	-42,1%
CreditSpread	VaR	544.989	3,4%	0,70%	VaR	544.989	3,4%	0,70%	VaR	0	0%
Concentration	VaR	0	0%	0%	-----	-----	-----	-----	-----	-----	-----
SCR market		3.563.370		4,80%		3.897.404		5,20%		334.034	9,4%

Table 3: Market Risk QIS3 versus IM comparison.

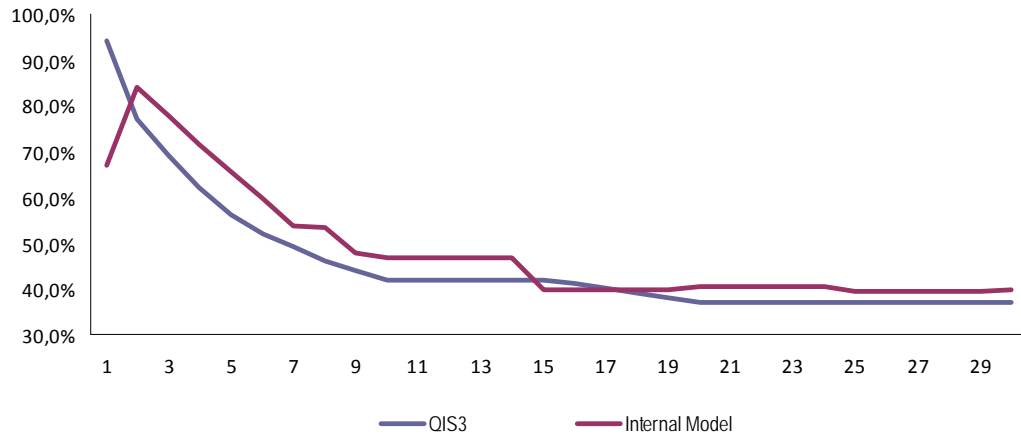


Figure 2: Comparison of QIS3 and IM VaR for interest rate changes

The Insurer was exposed to **currency risk** both because of direct investments in foreign currency and because it had investments in products denominated in foreign currencies. The exposure was, however, only in the following currencies: GBP (38%), CHF (32%), SEK (17%) and NOK (13%). The lower VaR produced by the IM can easily be understood as all these currencies are strong currencies with low variation relative to the euro. Despite that the QIS3 approach fixed a 20% depreciation to compute the VaR. The absence of **credit spread risk** difference results from having adopted, for the IM, the standard approach proposed in QIS3. In our Insurer case 65.2% of credit sensitive investment were rated AAA or AA, 30.3% A and the remaining BBB. So, to some extent, this exposure was too small to justifying the development of an IM alternative.

QIS 3	Equity	Interest Rate	Currency	Credit Spread	IM	Equity	Interest Rate	Currency	Credit Spread
Equity	1				Equity	1			
Interest Rate	0	1			Interest Rate	-0,033	1		
Currency	0,25	0,25	1		Currency	-0,064	0,045	1	
Credit Spread	0,25	0,25	0,25	1	Credit Spread	0,079	-0,154	-0,007	1

Table 4: QIS3 versus Internal Model estimated correlations between the various market risks.

Table 4 compares the correlations used by the QIS3 and IM for aggregating all market risk. We note that the QIS3 numbers are not based upon estimations while the IM are. The QIS3 suggested matrix gives higher correlations than what seems to realistic.

3.2 Insurance Risks

Table 6, below, presents the results concerning the **underwriting risk**. To estimate the premium risk in the IM we have used premium and loss data on 20 Portuguese insurers during 20 years. Starting values for the parameters were $\theta_1^{(0)} = 0.0539$ and $\theta_2^{(0)} = 0.0125$. After 4 additional steps we reached the final values of $\theta_1^{(4)} = 1.019.543\text{€}$ and $\theta_2^{(4)} = 0\text{€}$. The final 20 values for the λ_i 's can be seen in Vicente (2008), $\lambda_{14} = 4.2969$ is the value for the Insurer. From there we got the values for the mean and variance of $\widehat{E}[Y_{2006}] = 4.297$ and $\widehat{V}[Y_{2006}] = 0.189$. Concerning reserve risk there was available a triangle of data from 2001 to 2006 with the occurrence year and the corresponding claim payment developments, see Table 5.

	1	2	3	4	5	6
2001	7.792.345	2.262.154	166.760	393.132	206.338	86.268
2002	9.545.760	4.324.696	768.550	613.266	240.614	
2003	11.879.234	5.522.645	646.284	571.696		
2004	13.804.560	4.993.800	1.230.524			
2005	15.336.473	7.390.653				
2006	18.170.211					

Table 5: Matrix of the incremental paid claims

Because we have insufficient data for the estimation of the correlation coefficients we use as an estimate the average of the coefficients of the Portuguese insurers in the motor branch. The standard deviations are calculated with the data of received premiums and loss reserves. The final results for the premium risk and the reserve risk are shown in Table 6. More details on the all procedure can be found in Vicente (2008). The main difference between the QIS3 and IM underwriting risk evaluation is due to the fact that the IM uses basically information of the Portuguese instead of the European market as the

QIS3 does. This application concerns with the motor insurance and is known that Portuguese drivers have a loss rate greater than the European counterpart, giving support to using Portuguese data only.

	QIS3		IM		Deviation	
Premium Risk	PMV	58.610.179	RP	51,549,852		
	σ_I	49,8%	LR	120.03%		
	σ_M	10,0%	ER	25,46%		
	C	0,667	VaR	23,448,828		
	σ^{pr}	41%				
Reserve Risk	RMV	28.972.444	Mean	14,615,425		
	σ^{res}	12,5%	RES	27,483,984		
			SE	5,339,722		
			VaR	12,868,559		
Underwriting Risk	VM	87.582.623				
	σ	10,70%	$\rho_{pr,res}$	0.3789		
	$f(\sigma)$	0,3133				
SCR ^{underwriting}	27.395.091		28.805.711		1.410.620	5,15%

Table 6: Underwriting Risk, QIS3 versus IM comparison

On what concerns **counterparty risk** and **operational risk** we followed fully the QIS3 directives. Our Insurer had only one reinsurance contract and no financial derivatives. As the reinsurer had rated “A” by Standard&Poors its estimated probability of default was considered to be of 0.05%. The replacement cost was of 6,079,915€ and, thus the final $SCR^{counterparty} = 303,996€$. For operational risk, we got the following values for the pairs (Gross Premiums, Gross Technical Reserves) as (55,660,560, 51,549,852) and (39,026,522, 51,549,852) respectively for the QIS3 and the Internal Model.

Across all insurance risks the main difference, between the two models are in the way they handle the underwriting risk. Not surprisingly it is here that we find bigger SCR discrepancies.

3.3 Conclusion

	QIS 3	IM	Deviations (%)
SCR ^{market}	3.563.370	3.897.404	9,40%
SCR ^{counterparty}	3.039.96	303.996	0%
SCR ^{underwriting}	27.395.091	28.805.711	5,10%
SCR ^{basic}	28.144.636	29.627.565	5,30%
SCR ^{operational}	1.113.211	1.030.997	-7,40%
SCR Final	29.257.847	30.658.562	4,80%

Table 7: SCR, QIS3 vs Internal Model

Table 7 summarizes the quantities for the two approaches we worked out for the different risks. The structures of the two methods are similar allowing for a risk to risk comparison. On a quick look we find that the capital requirements are quite similar although the Internal Model gives (slightly) higher requirements. Given a company value, at the end of 2006, of about 50 million €, the Insurer remain solvent under any of the proposed approaches. However, we must say that the models used have obvious limitations. The analysis is not dynamic, for instance it assumes that the business structure remains the unchanged, no new insurance contracts and the asset structure is the same. Also, the IM is quite simplified, to ease the calculations and adapt to the information available, which quite often is limited. Finally, the QIS3 approach is not yet settled and it is expected soon that a new project, called QIS4, is launched by CEIOPS.

Besides that, there are risks not developed like cat risk, by its complexity (e.g. see the model by Lescourret and Robert (2006)). Also, as we said earlier, operational risk needs a better development. The possibility of a derivatives market was not considered here and could serve as risk mitigation. Also we made the (usual) assumptions concerning probability distributions fit of some random factors, like the lognormal for the loss ratio, without enough supporting data.

As a final remark we must say that a great step has been taken concerning risk assessment for insurers, and that the official model proposed by the European control bodies can be the model built by the insurer herself, although subject to approval. This is in fact encouraged.

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