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How many claims does it take to get ruined and recovered?☆

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Abstract

We consider in the classical surplus process the number of claims occurring up to ruin, by a different method presented by Stanford and Stroiński [Astin Bulletin 24 (2) (1994) 235]. We consider the computation of Laplace transforms (LTs) which can allow the computation of the probability function. Formulae presented are general.

The method uses the computation of the probability function of the number of claims during a negative excursion of the surplus process, in case it gets ruined. When initial surplus is zero this probability function allows us to completely define the recursion for the transform above. This uses the fact that in this particular case, conditional time to ruin has the same distribution as the time to recovery, given that ruin occurs.

We consider also the computation of moments of the number of claims during recovery time, which with initial surplus zero allows us to compute the moments of the number of claims up to ruin. We end this work by giving some insight on the shapes of the two types of probability functions involved.

Keywords: Probability of ruin; Claim number up to ruin; Claim number up to recovery; Time to ruin; Duration of negative surplus; Severity of ruin; Recursive methods

1. Introduction

In this work we consider the classical risk process, where claims occur as a Poisson process. Much has been studied and said in the actuarial literature over the classical model about ruin probabilities, either finite or infinite time. We know that if ruin is to occur it does at the instant of a claim. We can thus think of ruin not by directly addressing the *waiting time* of the event 'ruin' (finite time ruin probability), but think of the waiting time in terms of number of claims that occur until the process gets ruined, if it does. This is not a new approach, for instance, Stanford and Stroiński (1994) dealt with this problem in the classical model for phase-type distributed claim sizes. Recently, Stanford et al. (2000) extended the same approach to some non-Poisson claim processes. Both papers deal with the problem by studying the increment (positive or negative) on the risk reserve between two consecutive claims as the difference between the revenue earned and the claim amount. Their approach involves a recursive evaluation of Laplace–Stieltjes transforms allowing the calculation of the probability of ruin on the nth claim occurrence

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$(n = 1, 2, \dots)$, by evaluating the transform at the origin. The transform is based on the joint probability of non-ruin up to the nth claim and the reserve remaining after the nth claim occurrence less than an appropriate level, say y. The authors considered phase-type distributed claim sizes, in particular exponential, mixtures of exponentials and Erlang.

We restrict here to the study of the same problem in the classical model, the evaluation of the probability of ruin occurring at the nth claim $(n = 1, 2, \dots)$ in the classical model in a more general way, using a completely different approach, more direct, say *classical*, by enhancing the relationship between time to ruin and duration of a negative surplus, once ruin has occurred, with an initial surplus zero. This has been explained by Egídio dos Reis (1993). We extend the study to the number of claims occurring during a first period of negative surplus. This study is fundamental in our approach to our main problem and was not considered by Stanford and Stroiński (1994).

Let $\{U(t), t \geq 0\}$ be the classical continuous time surplus process so that

$$U(t) = u + ct - S(t), \quad t \geq 0,$$

where u (≥ 0) is the insurer's initial surplus, c the constant insurer's rate of premium income per unit time, $S(t) = \sum_{j=1}^{N(t)} X_j$ the aggregate claim amount up to time t, $N(t)$ the number of claims in the same time interval having a Poisson (λt) distribution, $S(t) = 0$ if $N(t) = 0$, and $\{X_j\}_{j=1}^{\infty}$ a sequence of i.i.d. random variables representing the individual claim amounts. $\{X_j\}$ is independent of $\{N(t)\}$. We denote by $B(x)$ and $b(x)$ the common distribution and density function of X_j, respectively, with $B(0) = 0$, so that all claim amounts are positive. We also assume that the mean of X_j, which we denote by b_1, is finite and that $c > \lambda b_1$. For simplicity we write $a = \lambda/c$. We will further assume in some parts of this paper the existence of the moment generating function of X_j for some strictly positive argument, which we denote by $m(s) = E[e^{sX_j}]$, and we state that clearly where appropriate.

Define the time until ruin, denoted T, by

$$T = \begin{cases} \inf\{t : U(t) < 0\}, & \\ \infty, & \text{if } U(t) \geq 0\ \forall t. \end{cases}$$

We denote by $T_c = T|T < \infty$ the conditional random variable time to ruin, given that ruin occurs. The probability of ultimate ruin from initial surplus u for this risk process is defined as

$$\psi(u) = \Pr\{U(t) < 0 \text{ for some positive } t | U(0) = u\} = \Pr\{T < \infty | U(0) = u\}$$

and let $\delta(u) = 1 - \psi(u)$ denote the survival probability. It is well known that $\psi(0) = ab_1$. If the moment generating function of X_j exists for some strictly positive argument, then the adjustment coefficient for this risk process is the unique positive number R such that

$$\lambda + cR = \lambda m(R). \tag{1}$$

Let $G(u; x)$ and $g(u; x)$ be the (defective) distribution and density function of the probability and severity of ruin, respectively

$$G(u; x) = \Pr\{T < \infty \quad \text{and} \quad U(T) > -x | U(0) = u\} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x} G(u; x) = g(u; x).$$

It is well known that $g(0; x) = a[1 - B(x)]$, see for instance Bowers et al. (1986). We denote by Y and Y_c the defective random variable of the severity of ruin and the conditional severity of ruin, given that ruin occurs, respectively.

Let $P(u; n)$ be the probability that ruin occurs before or at the nth claim $(n = 1, 2, \dots)$ from initial surplus $u \geq 0$ and $p(u; n)$ be the respective probability function. Denote the associated random variable by M. Obviously, we have that

$$p(u; 1) = P(u; 1), \qquad p(u; n + 1) = P(u; n + 1) - P(u; n), \quad n \geq 1$$

and $\psi(u) = \lim_{n \to \infty} P(u; n)$.

Consider the surplus process ongoing even if ruin occurs at some instant. Once ruin has occurred, the process will be passing through negative values temporarily, as it will recover back to positive surplus values with probability 1 [please see Egídio dos Reis (1993)]. Let $\tilde{T}_c$ be the duration of the surplus excursion through negative surplus values up to recovery or time to recovery, conditional on $T < \infty$. Denote by $q(u; n)$ the (conditional) probability of having n claims before the surplus process recovers to non-negative values and after the process has been ruined, given that ruin has occurred from initial surplus u. The support of the r.v., say K, is the set $\{0, 1, 2, \ldots\}$.

In the next section we consider the calculation of the Laplace transform (LT) for the probability $p(u; n)$, $n = 1, 2, \ldots$. Section 3 deals with the computation of the probability function $q(u; n)$, $n = 0, 1, 2, \ldots$. In Section 4 we consider the particular case $u = 0$ and establish the relation between the probability functions $p(0; n + 1)$ and $q(0; n)$ which will allow us to completely define the transform derived in Section 1. Section 5 deals with the calculation of the moments of the number of claims during the time to recovery, through the calculation of the moment generating function, in case it exists. In the last section we compute a couple of examples and plot the probability functions previously studied.

2. On the number of claims before ruin

Considering the first claim occurrence we have that

$$P(u; 1) = \int_0^\infty \lambda \exp\{-\lambda t\} \int_{u+ct}^\infty b(x)\,\mathrm{d}x\,\mathrm{d}t = \int_0^\infty \lambda \exp\{-\lambda t\}[1 - B(u + ct)]\,\mathrm{d}t,$$

$$P(u; n + 1) = P(u; 1) + \int_0^\infty \lambda \exp\{-\lambda t\} \int_0^{u+ct} b(x) P(u + ct - x; n)\,\mathrm{d}x\,\mathrm{d}t, \quad n \geq 1$$

and

$$p(u; n + 1) = \int_0^\infty \lambda \exp\{-\lambda t\} \int_0^{u+ct} b(x) p(u + ct - x; n)\,\mathrm{d}x\,\mathrm{d}t, \quad n \geq 1 \tag{2}$$

$$p(u; n + 1) = c^{-1} \int_{r=u}^\infty \lambda \exp\left\{-\lambda \left(\frac{r - u}{c}\right)\right\} \int_{x=0}^r b(x) p(r - x; n)\,\mathrm{d}x\,\mathrm{d}r = a \int_{r=u}^\infty \mathrm{e}^{-a(r-u)} b * p(r; n)\,\mathrm{d}r \tag{3}$$

setting $r = u + ct$, $a = \lambda/c$ and the convolution $b * p(r; n) = \int_{x=0}^r b(x) p(r - x; n)\,\mathrm{d}x$. Similarly, we have that

$$p(u; 1) = a\,\mathrm{e}^{au} \int_u^\infty \mathrm{e}^{-ax}[1 - B(x)]\,\mathrm{d}x. \tag{4}$$

Let $\bar{p}(s; n + 1) = \int_0^\infty \mathrm{e}^{-su} p(u; n + 1)\,\mathrm{d}u$ be the LT of $p(u; n + 1)$. The LT of (3) comes, for $n = 1, 2, \ldots$

$$\begin{aligned}\bar{p}(s; n + 1) &= a \int_0^\infty \mathrm{e}^{-su} \int_{r=u}^\infty \mathrm{e}^{-a(r-u)} b * p(r; n)\,\mathrm{d}r\,\mathrm{d}u = a \int_{r=0}^\infty \mathrm{e}^{-ar} b * p(r; n) \int_{u=0}^r \mathrm{e}^{-(s-a)u}\,\mathrm{d}u\,\mathrm{d}r \\ &= \frac{a}{s - a} \int_0^\infty \mathrm{e}^{-ar} b * p(r; n)[1 - \mathrm{e}^{-(s-a)r}]\,\mathrm{d}r = \frac{a}{s - a}[\bar{p}(a; n)\bar{b}(a) - \bar{p}(s; n)\bar{b}(s)], \quad s \neq a,\end{aligned} \tag{5}$$

taking into account of the fact that the LT of a convolution is the product of LT's, and where $\bar{b}(s)$ is the LT of $b(x)$.

Similarly, for $p(u;1)$ we get its LT using (4)

$$\bar{p}(s;1)=\int_0^\infty a\,\mathrm{e}^{-(s-a)u}\int_u^\infty \mathrm{e}^{-ax}[1-B(x)]\,\mathrm{d}x\,\mathrm{d}u=\int_0^\infty a\,\mathrm{e}^{-ax}[1-B(x)]\int_0^x \mathrm{e}^{-(s-a)u}\,\mathrm{d}u\,\mathrm{d}x$$
$$=(s-a)^{-1}\left(\int_0^\infty a\,\mathrm{e}^{-ax}[1-B(x)]\,\mathrm{d}x-\int_0^\infty a\,\mathrm{e}^{-sx}[1-B(x)]\,\mathrm{d}x\right)$$
$$=(s-a)^{-1}(\bar{g}(0;a)-\bar{g}(0;s)),\quad s\neq a, \tag{6}$$

where $\bar{g}(0;s)$ is the LT of the density $g(0,x)$. Note that using l'Hôpital's rule, we have that

$$\bar{p}(a;1)=\lim_{s\to a}\bar{p}(s;1)=-\lim_{s\to a}\frac{\mathrm{d}}{\mathrm{d}s}\bar{g}(0;s)=\int_0^\infty x\,\mathrm{e}^{-ax}g(0,x)\,\mathrm{d}x=-\bar{g}'(0,a),$$

where $\bar{g}'(0,a)=(\mathrm{d}/\mathrm{d}s)\bar{g}'(0,s)|_{s=a}$.

Back to the transform $\bar{p}(s;n+1)$, $n=1,2,\dots$, from (2) setting $u=0$ we get

$$p(0;n+1)=\int_{s=0}^\infty a\,\mathrm{e}^{-as}\int_{x=0}^s b(x)p(s-x;n)\,\mathrm{d}x\,\mathrm{d}s$$
$$=\int_{s=0}^\infty a\,\mathrm{e}^{-as}b*p(s;n)\,\mathrm{d}s=a\bar{p}(a;n)\bar{b}(a),\quad n=1,2,\dots \tag{7}$$

so that we can write

$$\bar{p}(s;n+1)=\frac{p(0;n+1)-a\bar{b}(s)\bar{p}(s;n)}{s-a},\quad n\geq 1\,(s\neq a). \tag{8}$$

Formula (8) is a recursive formula for the transform $\bar{p}(s;n+1)$ $(n\geq 1)$. To evaluate the recursion we need to compute $p(0;n+1)$, which is the LT of the convolution $ab*p(s;n)$ evaluated at $a=\lambda/c$. Recall that this is a positive constant. For $n=1$ (and $u=0$) we have immediately from (4)

$$p(0;1)=\bar{g}(0;a). \tag{9}$$

If we take (5) and compute the limit as $s\to a$, we see that

$$\lim_{s\to a}\bar{p}(s;n+1)=-\lim_{s\to a}a\left[\bar{p}(s;n)\frac{\mathrm{d}}{\mathrm{d}s}\bar{b}(s)+\bar{b}(s)\frac{\mathrm{d}}{\mathrm{d}s}\bar{p}(s;n)\right],\quad n=1,2,\dots$$

using l'Hôpital's rule. We will see in subsequent sections how to better compute $p(0;n+1)$, $n=1,2,\dots$.

By successive substitution in (8) and using (6) we get

$$\bar{p}(s;n)=(-1)^n\frac{(a\bar{b}(s))^{n-1}}{(s-a)^n}\bar{g}(0;s)+\sum_{i=0}^{n-1}(-1)^i\frac{(a\bar{b}(s))^i}{(s-a)^{i+1}}p(0;n-i),$$
$$\bar{p}(s;1)=(s-a)^{-1}(\bar{g}(0;a)-\bar{g}(0;s)),\quad s\neq a,$$

where $\bar{g}(0;s)=a[1-\bar{b}(s)]/s$ and $\bar{g}(0;a)=1-\bar{b}(a)$. $\bar{p}(s;n)$ depends upon the transforms $\bar{b}(s)$ and $\bar{g}(0;s)$, which in turn depends also on $\bar{b}(s)$. Inverting $\bar{p}(s;n)$ we get the probability distribution $p(u;n)$, which depend on the probabilities $p(0;i)$, $i=1,2,\dots,n$. We will see in subsequent sections that we are able to compute these.

3. On the number of claims before recovery

Let us now consider the calculation of $q(u;n)$, the probability of having n claims before the surplus process recovers to non-negative values, given that ruin has occurred, or the (conditional) probability of having n claims during a negative surplus excursion. The support of the r.v. K is the set $\{0, 1, 2, \dots\}$, we note that we do not include the claim that has (just) caused ruin. This one is included on the number of "claims to get ruined". We can take different approaches for the computation of $q(u; n)$, as follows.

First, we follow the paper by Gerber (1990) and consider the particular case, when the surplus process starts with $u = 0$. Let T_x be the time of the first passage of the surplus process through a fixed positive level x starting from initial surplus zero. From Dickson and Gray (1984) we know that the process gets to positive level x without having occurred a single claim is equivalent to be for the first time at this level at time $T_x = c/x$, and the probability is

$$\Pr\left[T_x = \frac{x}{c}\right] = \mathrm{e}^{-\lambda x/c} = \mathrm{e}^{-ax}. \tag{10}$$

Let us go now back to the general surplus process with $u \geq 0$, and consider that ruin has occurred at some instant (T), with given deficit y. Given $Y_\mathrm{c} = y$, the time that the process gets back to the zero level for the first time without any claim occurrence has probability $\exp\{-ay\}$. The proper distribution for the deficit at the time of ruin is $G(u, y)/\psi(u)$. Then to get the probability of having zero claims until the process recovers to level zero is got by *averaging* $\exp\{-ay\}$ over the distribution of Y_c. That is,

$$q(u; 0)\psi(u) = \int_0^\infty \mathrm{e}^{-ax} g(u; x)\,\mathrm{d}x = \bar{g}(u; a), \tag{11}$$

giving the LT of $g(u; x)$ evaluated at $s = a$. For a positive integer n we need to establish an equation for $q(u; n)$.

Consider first the calculation of $q(u; 1)$. Suppose that ruin occurs at time T with $-U(T) = y$ and restart the process from $-y$ and let the process upcross the level zero, or start from zero and cross the positive fixed level y. If no claim occurs, the process will recover at time $T_y = y/c$. Consider that one claim occurs at instant t before recovery. The instant must lie in the interval $(0, y/c)$ with density $\lambda\exp\{-\lambda t\}$. The amount of the claim is x with density $b(x)$. Just after this event the surplus will be $-y + ct - x$. No claim occurs until the process recovers from here. This has probability $\exp\{-a(x + y - ct)\}$ (see above). If we then average with the distribution of the conditional severity of ruin, we get that

$$\begin{aligned}
\psi(u)q(u; 1) &= \int_{y=0}^\infty \int_{t=0}^{y/c} \lambda\,\mathrm{e}^{-\lambda t} \int_{x=0}^\infty \mathrm{e}^{-a(x+y-ct)} b(x)\,\mathrm{d}x\,\mathrm{d}t\; g(u; y)\,\mathrm{d}y \\
&= \int_{y=0}^\infty \mathrm{e}^{-ay} g(u; y) \int_{t=0}^{y/c} \lambda\,\mathrm{e}^{-\lambda t}\,\mathrm{e}^{\lambda t} \int_{x=0}^\infty \mathrm{e}^{-ax} b(x)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \\
&= \bar{b}(a) \int_{y=0}^\infty ay\,\mathrm{e}^{-ay} g(u; y)\,\mathrm{d}y = -a\bar{b}(a)\bar{g}'(u; a),
\end{aligned}$$

where $\bar{g}'(u; a)$ is the derivative of the LT of $g(u; x)$ evaluated at $s = a$. For $n = 2, 3, \dots$, we can proceed recursively.

Let $q(n|y)$ be the conditional probability of having n claims before the process recovers to non-negative values, for a given severity $Y_\mathrm{c} = y$, or the conditional probability of having exactly n claims before the process upcrosses the positive level y. Note that $q(n|y)$ is independent of u. Following the reasoning above we have that

$$\psi(u)q(u; n) = \int_{y=0}^\infty \int_{t=0}^{y/c} \lambda\,\mathrm{e}^{-\lambda t} \int_{x=0}^\infty q(n-1|x+y-ct) b(x)\,\mathrm{d}x\,\mathrm{d}t\; g(u; y)\,\mathrm{d}y, \quad n = 1, 2, \dots. \tag{12}$$

Recall that one claim must occur at time, say t, before time y/c with amount x. At this instant the surplus will be at the (negative) value $-(x + y - ct)$, and then has to upcross level 0 after a further $(n - 1)$ claims. Note that we can have the starting expression for $q(0|\cdot)$ from (10). To better evaluate (12) we can use direct results from Gerber (1990) just like what follows.

For a given positive value x and $S_k = X_1 + X_2 + \cdots + X_k$ and $S_0 \equiv 0$, for $k = 0, 1, 2, \ldots$, Gerber (1990) shows that, our notation is $q(k|x, S_k)$,

$$q(k|x, S_k) = x\frac{a^k (S_k + x)^{k-1}}{k!} \mathrm{e}^{-a(S_k+x)}$$

is the "conditional probability that the process $\{U(t)|u = 0\}$ will pass *for the first time* through the level x between the kth and the $(k + 1)$th jump", in his terminology. Hence,

$$q(n-1|x+y-ct) = \int_0^\infty (x+y-ct)\frac{a^{n-1}(z+x+y-ct)^{n-2}}{(n-1)!}\mathrm{e}^{-a(z+x+y-ct)} b^{*(n-1)}(z)\,\mathrm{d}z,$$

where $b^{*(n-1)}(\cdot)$ is the $(n-1)$th convolution of $b(\cdot)$. However, to compute (12) we do not need to evaluate the inner double integral on the right-hand side. We can compute directly the probability

$$q(k|x) = \int_0^\infty x\frac{a^k (z+x)^{k-1}}{k!}\mathrm{e}^{-a(z+x)} b^{*k}(z)\,\mathrm{d}z, \quad k = 1, 2, \ldots .$$

We have that

$$(z+x)^{k-1} = \sum_{n=0}^{k-1}\binom{k-1}{n} z^n x^{k-1-n} \quad (k = 1, 2, \ldots).$$

Then

$$\begin{aligned}
q(k|x) &= \frac{xa^k}{k!}\mathrm{e}^{-ax}\int_0^\infty \sum_{n=0}^{k-1}\binom{k-1}{n} z^n x^{k-1-n}\,\mathrm{e}^{-az} b^{*k}(z)\,\mathrm{d}z \\
&= \frac{xa^k}{k!}\mathrm{e}^{-ax}\sum_{n=0}^{k-1}\binom{k-1}{n} x^{k-1-n}\int_0^\infty z^n\,\mathrm{e}^{-az} b^{*k}(z)\,\mathrm{d}z \\
&= \frac{xa^k}{k!}\mathrm{e}^{-ax}\left(\sum_{n=1}^{k-1}\binom{k-1}{n} x^{k-1-n}(-1)^n\,\frac{\mathrm{d}^n}{\mathrm{d}s^n}\int_0^\infty \mathrm{e}^{-sz} b^{*k}(z)\,\mathrm{d}z\Big|_{s=a} + x^{k-1}\bar{b}(a)^k\right) \\
&= \frac{a^k}{k!}\mathrm{e}^{-ax}\left(\sum_{n=1}^{k-1}\binom{k-1}{n} x^{k-n}(-1)^n\,\frac{\mathrm{d}^n}{\mathrm{d}s^n}\bar{b}(s)^k\Big|_{s=a} + x^k\bar{b}(a)^k\right).
\end{aligned}$$

Now,

$$\begin{aligned}
\psi(u)q(u;k) &= \int_0^\infty q(k|x)g(u,x)\,\mathrm{d}x = \frac{a^k}{k!}\left(\sum_{n=1}^{k-1}\binom{k-1}{n}(-1)^n\left(\frac{\mathrm{d}^n}{\mathrm{d}s^n}\bar{b}(s)^k\Big|_{s=a}\right)\right. \\
&\quad \left.\times\int_0^\infty x^{k-n}\,\mathrm{e}^{-ax} g(u,x)\,\mathrm{d}x + \bar{b}(a)^k\int_0^\infty x^k\,\mathrm{e}^{-ax} g(u,x)\,\mathrm{d}x\right) \\
&= \frac{a^k}{k!}\left(\sum_{n=1}^{k-1}\binom{k-1}{n}(-1)^n\left(\frac{\mathrm{d}^n}{\mathrm{d}s^n}\bar{b}(s)^k\Big|_{s=a}\right)(-1)^{k-n}\left(\frac{\mathrm{d}^{k-n}}{\mathrm{d}s^{k-n}}\bar{g}(u;s)\Big|_{s=a}\right)\right. \\
&\qquad \left. + \bar{b}(a)^k(-1)^k\left(\frac{\mathrm{d}^k}{\mathrm{d}s^k}\bar{g}(u;s)\Big|_{s=a}\right)\right)
\end{aligned}$$

$$= \frac{(-a)^k}{k!}\left(\sum_{n=1}^{k-1}\binom{k-1}{n}\left(\frac{\mathrm{d}^n}{\mathrm{d}s^n}\bar{b}(s)^k\bigg|_{s=a}\right)\left(\frac{\mathrm{d}^{k-n}}{\mathrm{d}s^{k-n}}\bar{g}(u;s)\bigg|_{s=a}\right)+\bar{b}(a)^k\left(\frac{\mathrm{d}^k}{\mathrm{d}s^k}\bar{g}(u;s)\bigg|_{s=a}\right)\right)$$

$$= \frac{(-a)^k}{k!}\left(\sum_{n=0}^{k-1}\binom{k-1}{n}\left(\frac{\mathrm{d}^n}{\mathrm{d}s^n}\bar{b}(s)^k\bigg|_{s=a}\right)\left(\frac{\mathrm{d}^{k-n}}{\mathrm{d}s^{k-n}}\bar{g}(u;s)\bigg|_{s=a}\right)\right),$$

where the symbol $((\mathrm{d}^0/\mathrm{d}s^0)\bar{b}(s)^k)|_{s=a} = \bar{b}(a)^k$.

If we apply Leibnitz's rule for derivatives of products we get

$$\psi(u)q(u;k) = \frac{(-a)^k}{k!}\left(\frac{\mathrm{d}^{k-1}}{\mathrm{d}s^{k-1}}(\bar{b}(s)^k\bar{g}'(u;s))\bigg|_{s=a}\right).$$

In summary we have

$$q(u;0) = \frac{\bar{g}(u;a)}{\psi(u)}, \qquad q(u;1) = -\frac{a\bar{b}(a)\bar{g}'(u;a)}{\psi(u)},$$
$$q(u;n) = \frac{(-a)^n}{\psi(u)n!}\left(\frac{\mathrm{d}^{n-1}}{\mathrm{d}s^{n-1}}(\bar{b}(s)^n\bar{g}'(u;s))\bigg|_{s=a}\right), \quad n=2,3,\dots. \tag{13}$$

The calculation of the probabilities of the (conditional) number of claims during a negative excursion of the surplus process involves the computation of the LTs of the claim amount distribution and the distribution of the probability and severity of ruin, and, of course, the ultimate ruin probability. The LTs exist at least for $s \geq 0$, see Gerber (1979), and we should be able to compute them, at least numerically. In the subsequent section we will consider the particular case $u = 0$. Different authors have considered the computation of the distribution $G(u, x)$. A reference paper is Gerber et al. (1987). For other references please see Lin and Willmot (1999). See also Willmot (2000).

4. The case with zero initial surplus, $u = 0$

We consider here, in particular, the computation of $q(0;n)$, $n = 0, 1, 2\dots$, which will allow us to compute $p(0;n)$, $n = 1, 2, \dots$, as we will establish a relationship between these two probability functions.

For the density of the (defective) severity of ruin with $u = 0$ we know that $g(0;x) = a[1-B(x)]$ giving $\bar{g}(0;s) = a[1-\bar{b}(s)]/s$ and $\bar{g}'(0;s) = -[\bar{g}(0;s)+a\bar{b}'(s)]/s$, so that $\bar{g}(0;a) = 1-\bar{b}(a)$ and $\bar{g}'(0;a) = -\bar{g}(0;a)/a-\bar{b}'(a)$. Knowing that $\psi(0) = ab_1$, we have that

$$q(0;0) = \frac{\bar{g}(0;a)}{\psi(0)} = \frac{1-\bar{b}(a)}{ab_1}, \qquad q(0;1) = -\frac{a\bar{b}(a)\bar{g}'(0;a)}{\psi(0)} = \bar{b}(a)\frac{\bar{g}(0;a)/a+\bar{b}'(a)}{b_1},$$
$$q(0;n) = -\frac{(-a)^{n-1}}{b_1 n!}\left(\frac{\mathrm{d}^{n-1}}{\mathrm{d}s^{n-1}}(\bar{b}(s)^n\bar{g}'(0;s))\bigg|_{s=a}\right), \quad n=2,3,\dots. \tag{14}$$

We can establish easily a direct relation between the claims arriving during a negative surplus excursion and claim number until ruin when $u = 0$. Consider both the conditional random variables of the time to ruin and the time to recovery, given that $T < \infty$, T_c and $\tilde{T}_c$ with initial surplus $u = 0$. As explained by Egídio dos Reis (1993), these two have the same distribution in this particular case. Thus, there is an obvious relation between the conditional r.v.'s $M|T < \infty$ and K. They do not have exactly the same distribution, the support set is different. Once ruin has occurred we may have zero claim occurrence until the process recovers. That is, the claim that causes ruin is not counted as a claim in the negative excursion until recovery. On the other hand, once the process starts with initial surplus zero it will never get ruined without any claim. So we need at least one claim, the claim that causes ruin, to have a negative deficit at T.

We recall that the support of the r.v. K is the set $\{0, 1, 2, \dots\}$. Given the above reasons we can conclude that

$$\psi(0)q(0; n) = p(0; n+1), \quad n = 0, 1, 2, \dots . \tag{15}$$

We see from (9) and (14) that $p(0; 1) = \psi(0)q(0; 0) = \bar{g}(0; a)$. We could show an analytical proof by induction, using (3) and (7), and formulae (14). Thus, we can evaluate completely recursion (8) and then by inversion compute the probability function $\{p(0; n), n = 1, 2, \dots\}$.

5. On the moments of the number of claims before recovery

In this section we assume that the moment generating function exists for some strictly positive argument, in the classical model, so the adjustment coefficient R exists. Consider the surplus process ongoing even if ruin as occurred at some instant.

We first do a *re-cap* and take again the work by Gerber (1990), i.e., consider the particular case of the surplus process with $u = 0$. For this particular process, let $\tilde{K}$ denote the number of claims occurring before the first upcrossing of the process at positive level x, whether or not ruin has previously occurred. Gerber (1990) showed that the moment generating function of $\tilde{K}$ is given by

$$E[\mathrm{e}^{s\tilde{K}}] = \mathrm{e}^{f(s)x}, \tag{16}$$

where $f(s)$ is a function such that

$$s = \ln \frac{\lambda + cf(s)}{\lambda m(f(s))}, \tag{17}$$

and $-a < f(s) \le 0$ and $s \le 0$. This follows from the fact that $f(0) = 0$, the derivatives of both the numerator and the denominator in (17) are positive. The numerator is zero for $f(s) = -a$, and the denominator is always positive. If there are no restrictions in the range of $f(s)$ in expression (17), we see that the numerator equals the denominator for $f(s) = 0$ or $f(s) = R$, and that for values of $f(s) < 0$ ($f(s) > -a$) or $f(s) > R$, the fraction is between 0 and 1, and so the logarithm is negative. If we take the first two derivatives of the cumulant generating function, $\ln E[\mathrm{e}^{s\tilde{K}}] = f(s)x$, and evaluate at them at $f(0) = 0$ we get easily

$$E[\tilde{K}] = \frac{\lambda x}{c\delta(0)} \quad \text{and} \quad V[\tilde{K}] = \frac{\lambda x(c^2 + \lambda^2\sigma_X^2)}{[c\delta(0)]^3},$$

where $\sigma_X^2 = V[X_i] = b_2 - b_1^2$.

Consider now the general model with initial surplus $u \ge 0$. If we now follow what is developed by Egídio dos Reis (1993), Section 3, and take the expected value of (16) with respect to the conditional distribution of the severity of ruin, given $T < \infty$, we get the moment generating function of the number of claims occurring during a negative surplus K, given that ruin occurs. We denote this moment generating function (mgf) as $M_K(u, s)$. Hence

$$M_K(u; s) = M_{Y_\mathrm{c}}(u; f(s)), \tag{18}$$

where $f(s)$ is defined as for (17). This mgf is simply got by setting $x = Y_\mathrm{c}$ and taking expectations. If we take the first two derivatives of the cumulant generating function $\phi(s) = \ln M_{K_\mathrm{c}}(u, s) = \ln M_{Y_\mathrm{c}}(u, f(s))$, we get

$$\phi'(s) = s'(f)^{-1}\frac{\mathrm{d}}{\mathrm{d}f}\ln M_{Y_\mathrm{c}}(u, f), \qquad \phi''(s) = -\frac{s''(f)}{s'(f)^2}\phi'(s) + s'(f)^{-2}\frac{\mathrm{d}^2}{\mathrm{d}f^2}\ln M_{Y_\mathrm{c}}(u, f).$$

Evaluating at $f(0) = 0$, knowing that

$$s'(0) = \frac{c\delta(0)}{\lambda} \quad \text{and} \quad s''(0) = -\frac{c^2 + \lambda^2\sigma_X^2}{\lambda^2},$$

we get

$$E[K_c] = \frac{\lambda E[Y_c]}{c\delta(0)} = \lambda E[\tilde{T}_c|u],$$

$$V[K_c] = \frac{\lambda}{[c\delta(0)]^2}\left(\frac{c^2 + \lambda^2\sigma_X^2}{c\delta(0)} E[Y_c|u] + \lambda V[Y_c|u]\right) = \lambda\left(\frac{1 + \psi(0)}{1 - \psi(0)} E[\tilde{T}_c|u] + \lambda V[\tilde{T}_c|u]\right).$$

For the expressions for the moments $E[\tilde{T}_c|u]$ and $V[\tilde{T}_c|u]$ please see Egídio dos Reis (1993). If we look at the expected value formula for K we see that it equals the mean claim occurrence for the process times the expected negative surplus duration per unit time. Expressions for the moments of the severity random variable can be got from Lin and Willmot (2000). Easy expressions for the same moments when $u = 0$ and limiting ones when $u \to \infty$ can be found in Egídio dos Reis (1993, 2000). This means that evaluation of the moments for K_c is available, provided they exist.

Furthermore, given what is explained in the previous section concerning the distributions of K and M with $u = 0$, moments of the number of claims up to ruin are also available, for this particular case.

6. Examples

For illustration we computed some examples and give some insight on the shapes of the proper probability functions $p(u, n)/\psi(u)$ and $q(u, n)$. For some cases it is straightforward to use the built-in functions of the package `Mathematica`, or similar, to produce figures for these probability functions. Namely, if we consider the Gamma(α, β) family for the claim size distributions. If we consider for instance a Pareto distribution it is easy to produce figures for the case $u = 0$, but for positive initial surplus we need other numerical procedures which are beyond the scope of this work. We consider three examples for the individual claim size distributions: Exponential(1), Gamma(2, 2) and Pareto(2, 1), all with mean one. For the Pareto we only consider the computation of $q(0, n) = p(0, n + 1)/\psi(0)$. In this case we recall that the variance does not exist. In all the cases we put $\lambda = 1$ and $c = 1.2$. We produce some graphs for the probability functions.

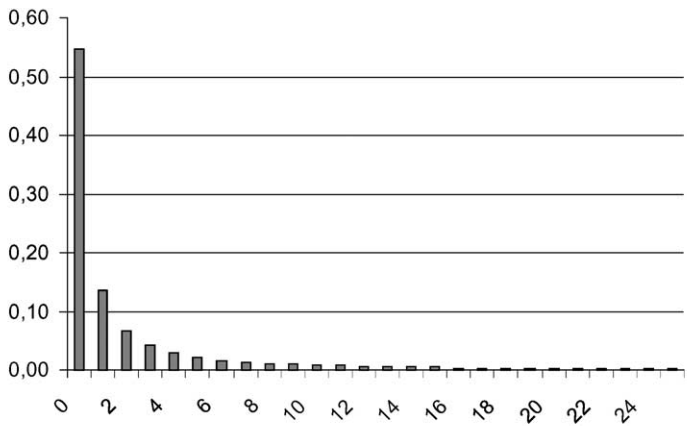

Fig. 1. $q(0, n) = p(0; n + 1)/\psi(0)$, Exponential(1), $\lambda = 1$, $c = 1.2$.

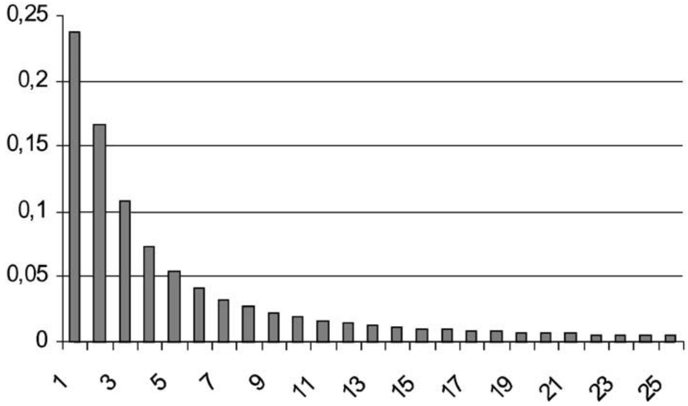

Fig. 2. $p(1;n)/\psi(1)$, Exponential(1), $\lambda = 1$, $c = 1.2$.

Example 1 (Exponential(1)). $B(x) = 1 - \exp\{-x\}$. We have that $\psi(u) = 1.2^{-1}\exp\{-u/6\}$, $g(u,x)/\psi(u)$ is independent of u and is again Exponential(1). We considered different values for the initial surplus, namely: $u = 0, 1, 2, 5, 10$. We show in Figs. 1–5 graphs for $q(0,n)$ and $p(u,n)/\psi(u)$. We also refer to the graphs shown by Stanford and Stroiński (1994), they show the cumulative probabilities. Since in this case the conditional distribution of the severity of ruin $g(u,x)/\psi(u)$ is independent of u, $q(u,n) = q(0,n)$, there is no need to show more graphs for this probability function.

We underline the following features from the graphs: for low values of the initial surplus ($u = 1, 2$) the conditional probabilities $p(u,n)/\psi(u)$ decrease with n and the probability of getting ruined on the first claim is quite high. On the other hand, if we think of recovery, given that ruin has occurred, it is very likely that recovery happens *fast*, in terms of claims occurrence (high probability of recovery with no claims). For bigger values of the initial surplus the shapes of $p(u,n)/\psi(u)$ change. Relatively speaking the probability of getting ruined on early claims ($u = 5, 10$) becomes smaller. An important feature we get from the figures is the long/thick tail of the probability functions and the strong and positive skewness. The greater the u the thicker is the tail.

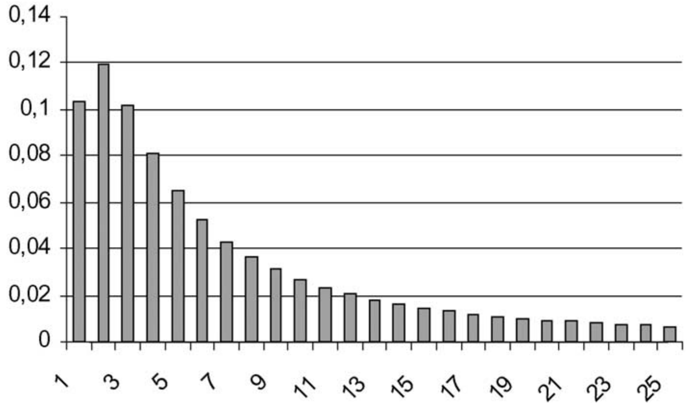

Fig. 3. $p(2;n)/\psi(2)$, Exponential(1), $\lambda = 1$, $c = 1.2$.

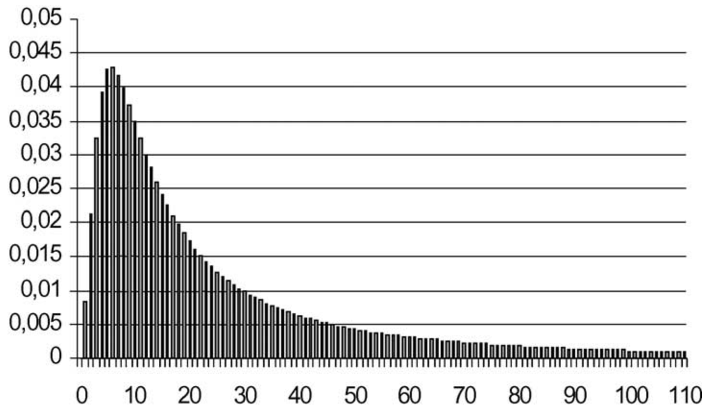

Fig. 4. $p(5; n)/\psi(5)$, Exponential(1), $\lambda = 1$, $c = 1.2$.

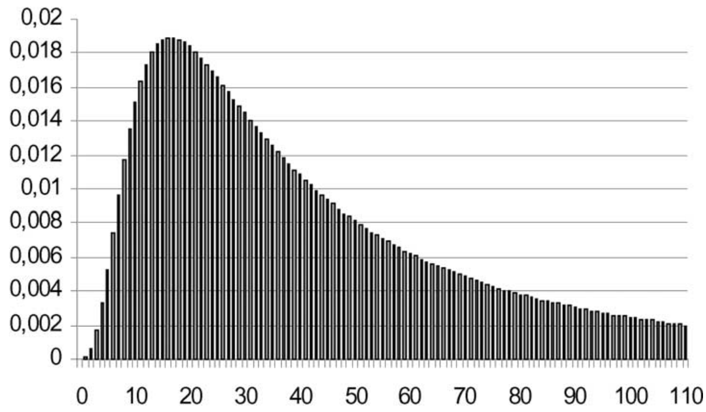

Fig. 5. $p(10; n)/\psi(10)$, Exponential(1), $\lambda = 1$, $c = 1.2$.

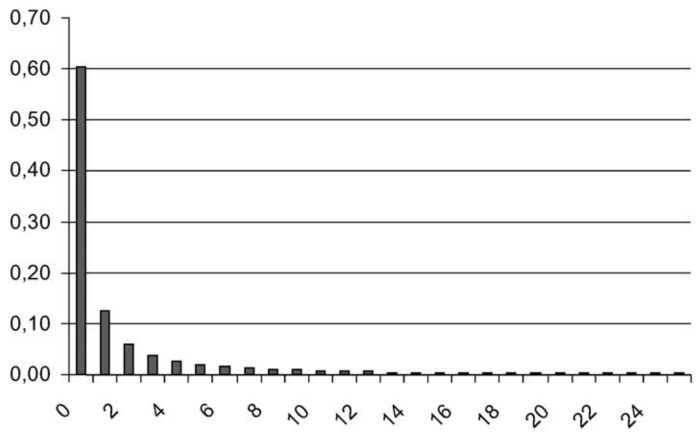

Fig. 6. $q(0, n) = p(0; n + 1)/\psi(0)$, Gamma(2, 2), $\lambda = 1$, $c = 1.2$.

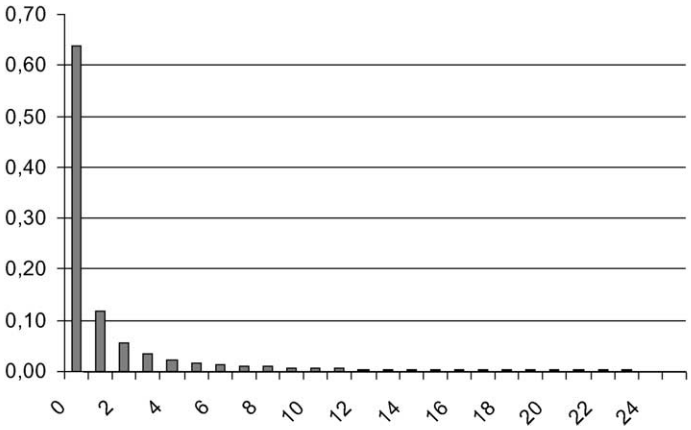

Fig. 7. $q(5, n)$, Gamma(2, 2), $\lambda = 1$, $c = 1.2$.

Example 2 (Gamma(2, 2)). $B(x) = 1 - (1 + 2x)\,\mathrm{e}^{-2x}$. We have that $\psi(u) = -0.010092\,\mathrm{e}^{-2.96841u} + 0.919183\,\mathrm{e}^{-0.122502\,u}$, $g(u, x)/\psi(u) = 1 - \mathrm{e}^{-2x} - A(u)2x\,\mathrm{e}^{-2x}$, where

$$A(u) = \frac{0.599745 + 0.309346\,\mathrm{e}^{-2.845908u}}{1.83837 - 0.0201841\,\mathrm{e}^{-2.845908u}},$$

i.e., $g(u, x)/\psi(u)$ is a mixture of an Exponential(1) and a Gamma(2, 2) [please see Egídio dos Reis (1993)]. We considered the same values of u as in the previous example. We show in Figs. 6–11 graphs for $q(u, n)$ and $p(u, n)/\psi(u)$. The features we see from the graphs in this example are similar to the previous case. A comment is needed for the function $q(u, n)$: unlike $p(u, n)/\psi(u)$ its shape does not change and there is little sensitivity of the probability function to a change in the initial surplus.

Example 3 (Pareto(2, 1)). $B(x) = 1 - (1 + x)^{-2}$. For this example we just computed $p(0; n + 1)/\psi(0) = q(0, n)$. Fig. 12 plots this probability function. In the graph two aspects are noticeable: Comparing to the previous examples the probability of getting ruined in the first claim is smaller but on the other hand the right tail is thicker.

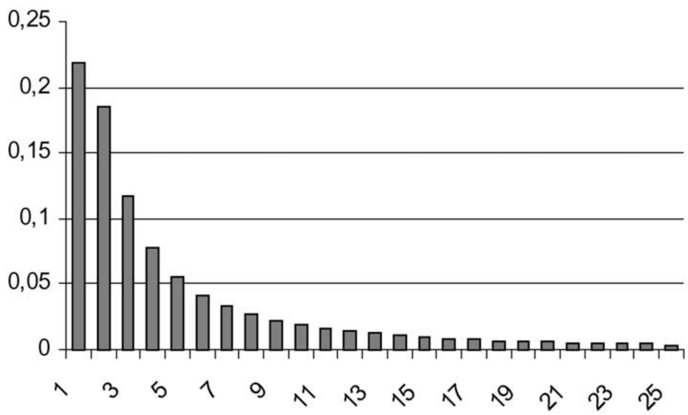

Fig. 8. $p(1; n)/\psi(1)$, Gamma(2, 2), $\lambda = 1$, $c = 1.2$.

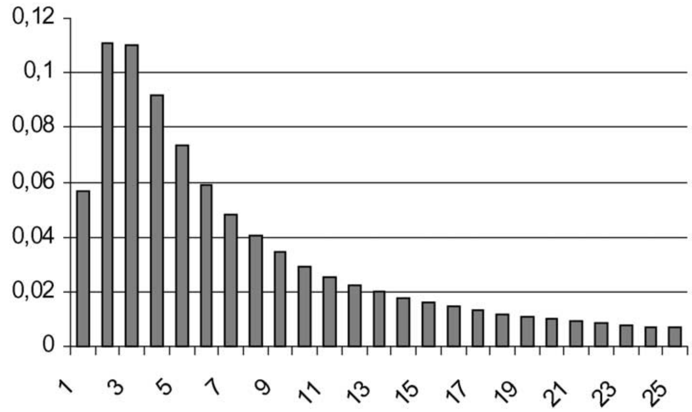

Fig. 9. $p(2; n)/\psi(2)$, Gamma(2, 2), $\lambda = 1$, $c = 1.2$.

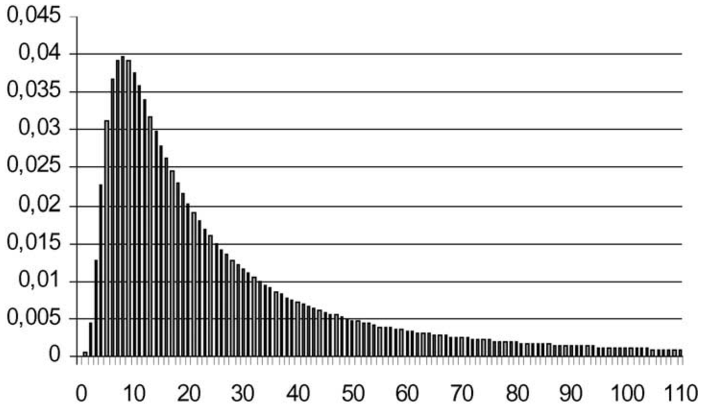

Fig. 10. $p(5; n)/\psi(5)$, Gamma(2, 2), $\lambda = 1$, $c = 1.2$.

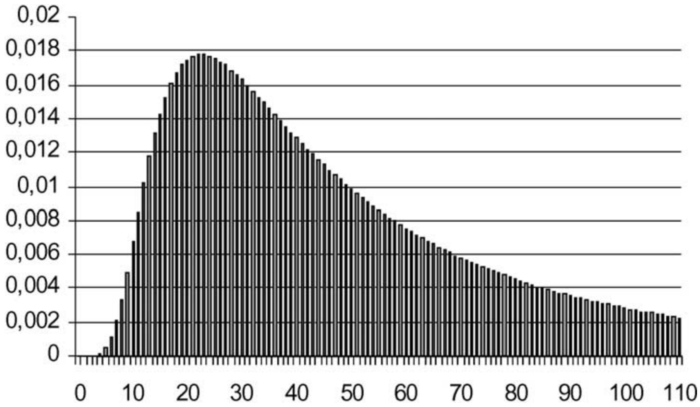

Fig. 11. $p(10; n)/\psi(10)$, Gamma(2, 2), $\lambda = 1$, $c = 1.2$.

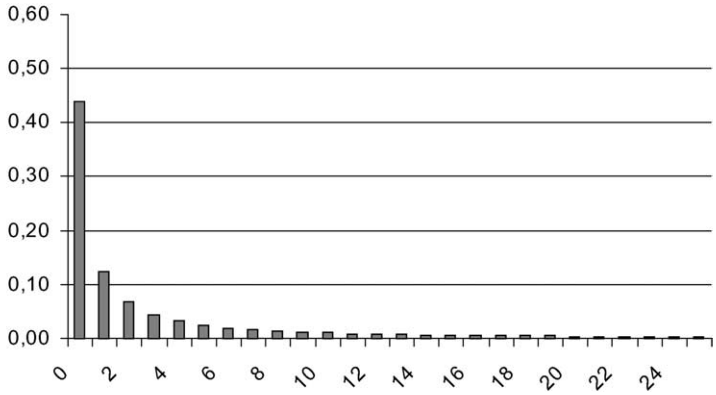

Fig. 12. $q(0, n) = p(0; n + 1)/\psi(0)$, Pareto(2,1), $\lambda = 1$, $c = 1.2$.

As a final remark on previous examples, the shapes of the probability functions $p(u, n)/\psi(u)$ remind the shapes of the conditional density functions of the time to ruin random variable [please see Cardoso and Egídio dos Reis (2002), Dickson and Waters (2002)]. The shape of $q(u, n)$ has to do with the distribution of the time to recovery.

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