Some advances on the Erlang(n) dual risk model

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Abstract

The dual risk model assumes that the surplus of a company decreases at a constant rate over time and grows by means of upward jumps, which occur at random times and sizes. It has applications to companies with economical activities involved in research and development. This model is *dual* to the well known Cramér-Lundberg risk model with applications to insurance.

Existing results on the study of the dual model assume that the random waiting times between consecutive gains follow an exponential distribution, as in the classical Cramér–Lunderg risk model.

We generalize to other compound renewal risk models where such waiting times are Erlang(n) distributed. Using the roots of the fundamental and the generalized Lundberg's equation, we get expressions for the ruin probability and the Laplace transform of the time of ruin for an arbitrary single gain distribution. Furthermore, we compute expected discounted dividends, as well as higher moments, when the individual common gains follow a Phase–Type, PH(m), distribution.

Finally, we perform illustrations working some examples for some particular gain distributions and obtain numerical results.

Keywords: Dual risk model; Erlang(n) interarrival times; Phase–Type distribution; generalized Lundberg's equation; ruin probability; time of ruin; expected discounted dividends.

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1 Introduction

We consider the dual risk model where the surplus or equity of the company is commonly described by the equation

$$U(t) = u - ct + \sum_{i=0}^{N(t)} X_i,\ t \geq 0,\ u \geq 0, \tag{1.1}$$

where $X_0 \equiv 0$, u is the initial surplus and c is a constant meaning the rate of expenses, assumed deterministic and fixed. The gains sequence is denoted by $\{X_i\}_{i=0}^{\infty}$, it is a sequence of i.i.d. random variables with common cumulative distribution and density functions $P(x)$, $P(0) = 0$, and $p(x)$, respectively. We assume the existence of $\mu_1 = E[X_1]$. We denote the Laplace transform of $p(x)$ by $\hat{p}(s)$. The model is called *dual* as opposed to the Cramér–Lundberg risk model with applications to insurance. For simplicity we will refer this latter as the *primal* model (where c is seen as the income premium rate and X_i means the i-th single loss).

By $N(t) = \max\{k : T_0 + T_1 + \cdots + T_k \leq t,\ T_0 = 0\}$ we denote the number of gains occurring before a given time t, where the random variable T_i, denotes the interarrival time between jumps $i-1$ and i (≥ 1). We assume that $\{T_i\}_{i=0}^{\infty}$ is a sequence of i.i.d. random variables and also independent from $\{X_i\}$. We further assume that T_1 follows an Erlang(n) distribution, whose density is denoted as $k_n(t) = \lambda^n t^{n-1} e^{-\lambda t}/(n-1)!$, $t \geq 0$, $\lambda > 0$, $n \in \mathbb{N}^+$. The corresponding distribution function is denoted as $K_n(t)$.

We assume the existence of the negative loading condition, i.e. $cE(W_1) < E(X_1) \Leftrightarrow cn < \lambda\mu_1$, designated *negative* as opposed to the well known *positive loading* in the *primal* Sparre–Andersen risk model, where the condition is reversed. These conditions mean that on average gains are greater than expenses, per unit time.

This model has been of increasing interest in ruin theory in recent times. There are many possible interpretations for the model. We can look at the surplus as the amount of capital of a business engaged in research and development, where gains are random, at random instants, and costs are certain. More precisely, the company pays expenses which occur continuously along time for the research activity, gets occasional revenues according to an Erlang(n) distribution and of size driven by distribution $P(\cdot)$. Revenues can be interpreted as values of future gains from an invention or discovery, the decrease of surplus can represent costs of production, payments to employees, maintenance of equipment, etc.

Among pioneer works on the subject we can cite Cramér (1955), Takács (1967), Seal (1969), Bühlmann (1970) and Gerber (1979). Recent works include those by Avanzi *et al.* (2007), Albrecher *et al.* (2008), Avanzi and Gerber (2008), Bayraktar and Egami (2008), Cheung and Drekic (2008), Gerber and Smith (2008), Song *et al*. (2008), Yang and Zhu (2008), Avanzi (2009), Ng (2009), Ng (2010), Afonso *et al*. (2011) and Cheung (2011). Published works, particularly those concerning the dual model, deal with the the compound Poisson, or Erlang(1), dual model and the computation of discounted dividends. We particularly reference the work by Avanzi *et al.* (2007) that explains well where applications of the dual model are said to be appropriate. On this matter Bayraktar and Egami (2008) used it to model capital investments. On dividend and optimal strategies strategies we underline the works by Avanzi *et al.* (2007), Avanzi and Gerber (2008) and Avanzi (2009). The latter is an excellent review paper, see also references therein. We also underline the work by Afonso *et al.* (2011) who, among other problems, give a different view of the dividend problem calculation, by taking advantage of the relationship between the primal and the dual models.

As we said, the works particularly focusing the dual model and the problem of discounted dividends assume that interarrival times follow an exponential distribution. In our manuscript we extend many of the existing developments to a general Erlang(n) distributed interarrival times. Also, we relate some developments taken from the primal to the dual model and that might lead to further and closest relationships.

We consider now some of the basic definitions and notations for the quantities of interest developed throughout this paper. Let

$$\tau_u = \begin{cases} \min\{t > 0 : U(t) = 0 \mid U(0) = u\} \\ \infty \ \text{ if } \ U(t) \geq 0 \quad \forall t \geq 0 \end{cases}$$

be the time to ruin, $\psi(u) = P(\tau_u < \infty)$ be the ultimate ruin probability and

$$\psi(u, \delta) = E[e^{-\delta \tau_u} I(\tau_u < \infty) \mid U(0) = u]$$

be the Laplace transform of the time to ruin, where $\delta > 0$ and $I(.)$ is the indicator function. This Laplace transform can be interpreted as the expected value of one monetary unit received at the time of ruin discounted at the constant force of interest δ.

In further developments we introduce an upper barrier into the model and let b denote its level. This barrier means a dividend payment level whose i-th single amount is going to be denoted by the random variable D_i explained as follows. Each time the surplus process upcrosses level b the excess gain is paid out immediately to the capital holder as a dividend, prior to ruin. Let $\{D_i\}_{i=1}^{\infty}$ be the sequence of the dividend payments and let $D(u, b)$ be the aggregate discounted dividends, at force of interest δ and from initial surplus u. We denote by $V_k(u, b) = E[D(u, b)^k]$, $k \geq 1$, the k-th order moment of $D(u, b)$, for simplicity denote $V(u, b) = V_1(u, b)$.

In the next section we consider Lundberg's fundamental and generalized equations and their relationship concerning the primal and the dual models. In Section 3 we study the solutions of the Lundberg's equations. In Section 4 we develop an integro-differencial equation for the ultimate ruin probability and then find a solution formula for that probability. We also give some illustration examples. In Section 5 we develop similar expressions for the Laplace transform of the time to ruin. In the last section we work the problem of discounted dividends, first the expected discounted dividends and then higher moments. Here we present a general integro-differential equation for the expected discounted dividends. To solve the equation we need to particularize the distribution of the individual jumps size to the Phase-type family. At the end of this section we show a solution for higher moments.

2 The primal and the dual model

In this section we make some connections of interest between the Cramér-Lundberg insurance risk model and the dual model. We could call the first as the *classical or standard risk model* however, often the literature when referring to the classical model it means the compound Poisson risk model, which is a particular case of the Erlang(n) risk model. So, we chose to call it simply the primal model.

The **primal model** is driven by an equation similar to (1.1)

$$U_P(t) = u + ct - \sum_{i=0}^{N(t)} X_i,\ t \geq 0,\ u \geq 0$$

where $U_P(t)$ represents the surplus of a portfolio of insurance risks at time t. For convenience we keep the same notation but note that the quantities involved have different meanings, particularly c and X_i, respectively premium rate and individual claim size i. Here, it is assumed a positive loading condition, $cE(W_1) > E(X_1)$, and it brings an economical sense to the model: it is expected that the income until the next claim is greater than the size of the next claim. The net income between the $(i-1)$-th and the i-th claims is $cW_i - X_i$. In this model it is weel known the notion of the adjustment coefficient, provided that the moment generating function of X_1 exists, and is denoted by $M_X(.)$. The adjustment coefficient, denoted as R, is the unique positive real root of the equation

$$E\left[e^{-r(cW_1-X_1)}\right] = 1 \Leftrightarrow E\left[e^{-rcW_1}\right]E\left[e^{rX_1}\right] = 1 \Leftrightarrow M_X(r) = \left(1 + \left(\frac{c}{\lambda}\right) r\right)^n.$$

We note that expectation $E\left[e^{rX_1}\right]$ exists at least for $r < 0$. Note that $E\left[e^{-RcW_1}\right]$ exists, not only because in our case W_1 follows an Erlang(n) distribuition, but also because it is a Laplace transform (and we did not need the former assumption). If we look at the equation above we can regard it as the expected discounted profit for each *waiting arrival period*. So that the adjustment coefficient R, provided that it exists, makes the expected discounted profit even (considering that premium income and claim costs come together). Constant R is then seen as an interest force. That equation is known as the fundamental Lundberg's equation.

Now, let's have the same perspective for the **dual model** case and refer to equation (1.1). The fundamental Lundberg's equation is now given as

$$E\left[e^{-s(X_1-cW_1)}\right] = 1 \Leftrightarrow E\left[e^{scW_1}\right]E\left[e^{-sX_1}\right] = 1 \Leftrightarrow \hat{p}(s) = \left(1 - \left(\frac{c}{\lambda}\right) s\right)^n, \tag{2.1}$$

where the corresponding net income per *waiting arrival period* i is given by the reversed difference $X_i - cW_i$. In either case the fundamental Lundberg's equation has the same form, but here we do not have to assume the existence of the moment generating function of X_1, if we consider $s > 0$, and the definition of a similar constant to the the adjustment coefficient in the primal model is not needed, we would indeed need the existence of expectation $E\left[e^{scW_1}\right]$ if a general distribution of W_1 were considered.

A generalization of each of the above equations were introduced to the actuarial literature and became known as the generalized Lundberg's equation. They take the following form, respectively for the primal and the dual model, for a constant $\delta > 0$ (see e.g. Landriault and Willmot (2008)):

$$E\left[e^{-\delta W_1}e^{-r(cW_1-X_1)}\right] = 1 \quad \text{and} \quad E\left[e^{-\delta W_1}e^{-s(X_1-cW_1)}\right] = 1$$

In our case, with Erlang(n) interarrival times, they take the following forms, respectively:

$$\begin{aligned} M_X(r) &= \left(1 + \frac{\delta}{\lambda} + \left(\frac{c}{\lambda}\right) r\right)^n \quad \text{and} \\ \hat{p}(s) &= \left(1 + \frac{\delta}{\lambda} - \left(\frac{c}{\lambda}\right) s\right)^n. \end{aligned} \tag{2.2}$$

The positive constant δ is often regarded as an interest force and we can think of (2.1) as the limiting case of (2.2) when $\delta \to 0^+$. In the following section we discuss the solutions of both the fundamental and general Lundberg's equation.

3 Solutions of the Lundberg's equations

According to Theorem 2 and Remark 1 of Li and Garrido (2004), in a Sparre–Andersen risk model with Erlang(n) distributed interclaim times, equation (2.2) has n roots with positive real parts and equation (2.1) has $n-1$ roots with positive real parts.

In the dual model, we can use Rouché's theorem, as in Theorem 2 of Li and Garrido (2004), to prove that both equations have exactly n roots with positive real parts. Let $\rho_1(\delta), \ldots, \rho_n(\delta)$ denote these roots. Moreover, if n is an odd number only one of these roots is real, say $\rho_n(\delta)$, and if n is even there are always two real roots, say $\rho_n(\delta)$ and $\rho_{n-1}(\delta)$ such that $0 < \rho_n(\delta) < \rho_{n-1}(\delta)$. The other roots form pairs of conjugate complex numbers on each situation. We note that Remark 1 of that theorem does not totally apply to the dual model since it needs the loading condition in its point 2, which is reversed in our case.

The difference between Li and Garrido (2004)'s conclusion and ours concerning the number of roots in the limiting case $\delta \to 0^+$ lies on the loading condition. To understand this we can proceed as in Li and Garrido (2004). Let's define the function

$$h(s) = \left(\frac{\lambda}{c}\right)^n \hat{p}(s) - \left(\frac{\lambda+\delta}{c} - s\right)^n .$$

Since $h(0) < 0$ and $\lim_{s\to-\infty} h(s) = +\infty$, for a sufficiently smooth density $p(x)$ (it is sufficient that $\hat{p}(s)$ is continuous) we will have at least one negative real root, we denote the larger one by $-R(\delta)$. Also, we have

$$h'(0) = -\left(\frac{\lambda}{c}\right)^n \mu_1 + n\left(\frac{\lambda+\delta}{c}\right)^{n-1} = \left(\frac{\lambda}{c}\right)^{n-1}\left(-\frac{\lambda}{c}\mu_1\right) + n\left(\frac{\lambda+\delta}{c}\right)^{n-1} < 0,$$

due to the negative loading condition ($cn < \lambda\mu_1$) and for a sufficiently small δ. Note that $h(s)$ has a local minimum between 0 and $\rho_n(\delta)$. Therefore, $\lim_{\delta\to0^+}(-R(\delta)) = 0$ because $\lim_{\delta\to0^+} h(0) = 0$. In the limit only the root $-R(\delta)$ equals zero, all the others remain nonzero, since the $\lim_{\delta\to0^+} \rho_n(\delta) > 0$. Note that if we considered the loading to be reversed, which makes economical sense for the primal model, we would have $\lim_{\delta\to0^+} \rho_n(\delta) = 0$ [see Remark 1 of Theorem 2 in Li and Garrido (2004)].

Following Ji and Zhang (2012) we note that roots $\rho_1(\delta), \ldots, \rho_n(\delta)$ are all distinct for $\delta \geq 0$, see end of their Section 1, p. 75. This remark was originally described for the primal model, but it remains valid in the case of the dual (their equation corresponding to (2.2) although dependent of c is irrespective of the loading condition). This feature will be very important later on this manuscript.

For simplicity we will denote $\rho_i(\delta)$ by ρ_i, $i = 1, ..., n$, unless stated otherwise.

4 The ruin probability

The ultimate ruin probability in the dual risk model with exponential interarrival times, i.e. $k(t) = \lambda e^{-\lambda t}$, satisfies the following renewal equation

$$\psi(u) = e^{-\lambda t_0} + \int_0^{t_0} \lambda e^{-\lambda t} \int_0^\infty p(x)\psi(u - ct + x)dx\, dt, \tag{4.1}$$

where $t_0 = u/c$ is the time of ruin without any gain arrival. This can be found in Afonso *et al*. (2011), and is got by conditioning on the time and amount of the first jump.

Differentiating with respect to u and rearranging, we get an integro–differential equation for $\psi(u)$ given by

$$\psi(u) + \left(\frac{c}{\lambda}\right) \frac{d}{du}\psi(u) = \int_0^\infty p(x)\psi(u + x)dx.$$

We can write this equation as

$$\left(\mathcal{I} + \left(\frac{c}{\lambda}\right) \mathcal{D}\right) \psi(u) = \int_0^\infty p(x)\psi(u + x)dx, \tag{4.2}$$

where $\mathcal{I}$ is the identity operator and $\mathcal{D}$ is the differentiation operator.

We can extend the previous method for a general Erlang(n) interarrival time. Likewise, the renewal equation corresponding to (4.1) becomes

$$\psi(u) = 1 - K_n(t_0) + \int_0^{t_0} k_n(t) \int_0^\infty p(x)\psi(u - ct + x)dx\, dt. \tag{4.3}$$

The integro–differential equation analogous to (4.2) in given in the following theorem.

Theorem 4.1 *In the Erlang(n) dual risk model the ruin probability satisfies the integro–differential equation*

$$\left(\mathcal{I} + \left(\frac{c}{\lambda}\right) \mathcal{D}\right)^n \psi(u) = \int_0^\infty p(x)\psi(u + x)dx, \tag{4.4}$$

with boundary conditions

$$\psi(0) = 1 \text{ and } \left.\frac{d^i}{du^i}\psi(u)\right|_{u=0} = 0,\ i = 1, \ldots, n - 1. \tag{4.5}$$

Proof. We proceed taking successive derivatives of the ruin probability using the renewal equation (4.3). Changing the variable $u - ct = s$ the renewal equation can be rewritten in the form

$$\psi(u) = 1 - K_n\left(\frac{u}{c}\right) + \frac{1}{c}\int_0^u k_n\left(\frac{u - s}{c}\right) W(s)ds,$$

where $W(s) = \int_0^\infty \psi(s + x)p(x)dx$.

After applying the operator $(\mathcal{I} + (c/\lambda)\,\mathcal{D})$ to the ruin probability we get

$$\left(\mathcal{I} + \left(\frac{c}{\lambda}\right) \mathcal{D}\right) \psi(u) = 1 - K_{n-1}\left(\frac{u}{c}\right) + \frac{1}{c}\int_0^u k_{n-1}\left(\frac{u - s}{c}\right) W(s)ds.$$

Following an inductive argument, it is easy to show that

$$\left(\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^i\psi(u)=1-K_{n-i}\left(\frac{u}{c}\right)+\frac{1}{c}\int_0^u k_{n-i}\left(\frac{u-s}{c}\right)W(s)ds,$$

for $i=1,\ldots,n-1$. Particularly, we have

$$\left(\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^{n-1}\psi(u)=1-K_1\left(\frac{u}{c}\right)+\frac{1}{c}\int_0^u k_1\left(\frac{u-s}{c}\right)W(s)ds.$$

Applying the operator once more we get

$$\left(\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n\psi(u)=W(u).$$

This proves equation (4.4). We have been using here some very known properties of the Erlang(n) probability density function (for $n\geq 2$), namely

$$\begin{aligned}k_n'(t) &= \lambda(k_{n-1}(t)-k_n(t)),\\ k_n^{(i)}(0) &= 0,\quad i=0,\ldots,n-2,\\ k_n^{(n-1)}(0) &= \lambda^n.\end{aligned}$$

We now prove the boundary conditions. Clearly, $\psi(0)=1$. We find the remaining conditions by computing directly the derivatives of $\psi(u)$ and evaluating at $u=0$,

$$\frac{d^i}{du^i}\psi(u)=-\left(\frac{1}{c}\right)^i k_n^{(i-1)}\left(\frac{u}{c}\right)+\left(\frac{1}{c}\right)^{i+1}\int_0^u k_n^{(i)}\left(\frac{u-s}{c}\right)W(s)ds$$

for $i=1,\ldots,n-1$. Hence, we obtain

$$\left.\frac{d^i}{du^i}\psi(u)\right|_{u=0}=0,\ i=1,\ldots,n-1.$$

■

The solution for the integro-differential equation (4.4) with boundary conditions given by (4.5) is shown in the following theorem.

Theorem 4.2 *The ultimate ruin probability can be written as a combination of exponential functions*

$$\psi(u)=\sum_{k=1}^n\left[\prod_{i=1,i\neq k}^n\frac{\rho_i}{(\rho_i-\rho_k)}\right]e^{-\rho_k u}, \tag{4.6}$$

where $\rho_1,\ldots,\rho_n$ *are the only roots of the fundamental Lundberg's equation* (2.1) *which have positive real parts.*

Proof. Let's consider a general solution $f(u)$ for equation (4.4)

$$\left(\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n f(u)=\int_0^\infty p(x)f(u+x)dx. \tag{4.7}$$

We now look for particular solutions of this equation. Let $f(u) = e^{-ru}$, for some $r \in \mathbb{C}$. Then, for the left hand side of (4.7) we obtain

$$\left(\mathcal{I} + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n f(u) = \left(1 - \left(\frac{c}{\lambda}\right)r\right)^n e^{-ru},$$

hence,

$$\left(1 - \left(\frac{c}{\lambda}\right)r\right)^n = \hat{p}(r),$$

which means that r must be a root of the Fundamental Lundberg's equation (2.1).

Define the functions $f_1(u) = e^{-\rho_1 u}, \ldots, f_n(u) = e^{-\rho_n u}$. Since they are linearly independent we can write any solution of (4.7) as

$$f(u) = \sum_{i=1}^{n} a_i e^{-\rho_i u},$$

where a_i, $i = 1, ..., n$, are constants. To get a formula for $\psi(u)$ we must find the constants a_i using the boundary conditions (4.5). These can be determined by solving a system of n equations on the unknowns $a_1, ..., a_n$. In matrix form we have

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \rho_1 & \rho_2 & \cdots & \rho_n \\ \vdots & \vdots & \ddots & \vdots \\ \rho_1^{n-1} & \rho_2^{n-1} & \cdots & \rho_n^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Leftrightarrow \mathbf{a} = \mathbf{P}^{-1}\mathbf{e},$$

where $\mathbf{P} = \mathbf{P}(\rho_1, \ldots, \rho_n)$ is a Vandermonde matrix, $\mathbf{a}' = (a_1, a_2, \ldots, a_n)$ and $\mathbf{e}' = (1, 0, \ldots, 0)$. The determinant of $\mathbf{P}$ is given by

$$Det\ \mathbf{P} = \prod_{1 \leq i < j \leq n} (\rho_j - \rho_i),$$

and using Cramér's rule we get expressions for the coefficients

$$\begin{aligned} a_k &= \frac{(-1)^{k-1}(\prod_{i=1,\, i\neq k}^{n} \rho_i)(\prod_{1\leq i<j\leq n,\, i\neq k,\, j\neq k}(\rho_j - \rho_i))}{\prod_{1\leq i<j\leq n}(\rho_j - \rho_i)} \\ &= \frac{(-1)^{k-1}(\prod_{i=1,\, i\neq k}^{n} \rho_i)}{(\prod_{i=1}^{k-1}(\rho_k - \rho_i))(\prod_{j=k+1}^{n}(\rho_j - \rho_k))} = \prod_{i=1, i\neq k}^{n} \frac{\rho_i}{(\rho_i - \rho_k)}. \end{aligned}$$

■

Remarks:

1. Note that although some of the roots are complex, expression (4.6) is always a real number.

2. If we considered the positive loading condition to be reversed, recall that $\rho_n = 0$ as explained at the end of Section 3, then we would have $a_n = 1$ and all the remaining coefficents $a_k = 0$, $k = 1, ..., n-1$, therefore giving $\psi(u) = 1$ as expected.

Example 4.1 *For $n=1$ (exponential case): Gerber (1979) found that $\psi(u)=e^{-\rho u}$, where ρ is the unique positive root of the fundamental Lundberg's equation* (2.1)*.*

For $n=2$:

$$\psi(u)=\frac{\rho_2}{\rho_2-\rho_1}e^{-\rho_1 u}-\frac{\rho_1}{\rho_2-\rho_1}e^{-\rho_2 u},$$

where $\rho_1,\rho_2>0$ are real and solutions of $\left(1-\left(\frac{c}{\lambda}\right)s\right)^2=\hat{p}(s)$.

For $n=3$:

$$\begin{aligned}\psi(u) &= \frac{\rho_2\rho_3}{(\rho_3-\rho_1)(\rho_2-\rho_1)}e^{-\rho_1 u}-\frac{\rho_1\rho_3}{(\rho_3-\rho_2)(\rho_2-\rho_1)}e^{-\rho_2 u}\\ &\quad+\frac{\rho_1\rho_2}{(\rho_3-\rho_1)(\rho_3-\rho_2)}e^{-\rho_3 u},\end{aligned}$$

where ρ_1,ρ_2,ρ_3 are solutions of $\left(1-\left(\frac{c}{\lambda}\right)s\right)^3=\hat{p}(s)$; one root is real and the other two are complex conjugates.

5 The Laplace transform of the time to ruin

For the Erlang(n) case, the Laplace transform of the time to ruin satisfies the renewal equation

$$\psi(u,\delta)=(1-K_n(t_0))\,e^{-\delta t_0}+\int_0^{t_0}k_n(t)\,e^{-\delta t}\int_0^\infty p(x)\psi(u-ct+x,\delta)dxdt. \tag{5.1}$$

with $t_0=u/c$. The following theorem shows an integro–differential equation for $\psi(u,\delta)$.

Theorem 5.1 *In the Erlang(n) dual risk model the Laplace transform of the time of ruin satisfies the integro–differential equation*

$$\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n\psi(u,\delta)=\int_0^\infty p(x)\psi(u+x,\delta)dx, \tag{5.2}$$

with boundary conditions

$$\psi(0,\delta)=1,\qquad \left.\frac{d^i}{du^i}\psi(u,\delta)\right|_{u=0}=(-1)^i\left(\frac{\delta}{c}\right)^i,\ i=1,\ldots,n-1. \tag{5.3}$$

Proof. Using a similar the procedure as in Theorem 4.1 we take successive derivatives of (5.1). Then, changing variable the renewal equation can be rewritten in the form

$$\psi(u,\delta)=\left(1-K_n\left(\frac{u}{c}\right)\right)e^{-\delta(\frac{u}{c})}+\frac{1}{c}\int_0^u k_n\left(\frac{u-s}{c}\right)e^{-\delta(\frac{u-s}{c})}W_\delta(s)ds,$$

where $W_\delta(s)=\int_0^\infty\psi(s+x,\delta)p(x)dx$.

After applying the operator $\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)$ to the Laplace transform we get

$$\begin{aligned}\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)\psi(u,\delta) &= \left(1-K_{n-1}\left(\frac{u}{c}\right)\right)e^{-\delta(\frac{u}{c})}\\ &\quad+\frac{1}{c}\int_0^u k_{n-1}\left(\frac{u-s}{c}\right)e^{-\delta(\frac{u-s}{c})}W_\delta(s)ds.\end{aligned}$$

Similarly, following an inductive argument, we show that

$$\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^{i}\psi(u,\delta) = \left(1-K_{n-i}\left(\frac{u}{c}\right)\right)e^{-\delta\left(\frac{u}{c}\right)} + \frac{1}{c}\int_0^u k_{n-i}\left(\frac{u-s}{c}\right)e^{-\delta\left(\frac{u-s}{c}\right)}W_\delta(s)ds,$$

for $i=1,\dots,n-1$. In particular, we obtain

$$\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^{n-1}\psi(u,\delta) = \left(1-K_{1}\left(\frac{u}{c}\right)\right)e^{-\delta\left(\frac{u}{c}\right)} + \frac{1}{c}\int_0^u k_{1}\left(\frac{u-s}{c}\right)e^{-\delta\left(\frac{u-s}{c}\right)}W_\delta(s)ds.$$

Applying once more the operator gives

$$\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^{n}\psi(u,\delta) = W_\delta(u).$$

This proves equation (5.2).

For the boundary conditions, clearly $\psi(0,\delta)=1$. We find the remaining conditions computing directly the derivatives of $\psi(u,\delta)$ and evaluate at $u=0$,

$$\frac{d^i}{du^i}\psi(u,\delta) = \left[\left(-\frac{\delta}{c}\right)^i\left(1-K_n\left(\frac{u}{c}\right)\right)-\frac{1}{c^i}\sum_{j=1}^{i}\binom{i}{j}(-\delta)^{i-j}k_n^{(j-1)}\left(\frac{u}{c}\right)\right]e^{-\delta\left(\frac{u}{c}\right)} + \left(\frac{1}{c}\right)\int_0^u\left[\frac{1}{c^i}\sum_{j=0}^{i}\binom{i}{j}(-\delta)^{i-j}k_n^{(j)}\left(\frac{u-s}{c}\right)\right]e^{-\delta\left(\frac{u-s}{c}\right)}W_\delta(s)ds,$$

for $i=1,\dots,n-1$, so that we get $\left.\frac{d^i}{du^i}\psi(u,\delta)\right|_{u=0}=\left(-\frac{\delta}{c}\right)^i$, $i=1,\dots,n-1$. ■

The solution for $\psi(u,\delta)$ is given in the following theorem.

Theorem 5.2 *The Laplace transform of the time of ruin can be written as a combination of exponential functions*

$$\psi(u,\delta)=\sum_{k=1}^{n}\left[\prod_{i=1,i\neq k}^{n}\frac{\left(\rho_i-\frac{\delta}{c}\right)}{(\rho_i-\rho_k)}\right]e^{-\rho_k u}, \tag{5.4}$$

where $\rho_1,\dots,\rho_n$ *are the only roots of the Lundberg's equation* (2.2) *which have positive real parts.*

Proof. We use a similar procedure as in Theorem 4.2 to obtain formula (5.4). All the functions $e^{-\rho_k u}$, $k=1,...,n$, are solutions of the integro–differential equation

$$\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^{n}f(u)=\int_0^\infty p(x)f(u+x)dx. \tag{5.5}$$

Since these functions are linearly independent, we can write every solution of (5.5) as a linear combination of them. Therefore,

$$\psi(u,\delta)=\sum_{i=1}^{n}a_i e^{-\rho_i u}, \quad a_i \quad \text{constants.}$$

where constants a_i, $i=1,...n$, are solutions of the system

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \rho_1 & \rho_2 & \cdots & \rho_n \\ \vdots & \vdots & \ddots & \vdots \\ \rho_1^{n-1} & \rho_2^{n-1} & \cdots & \rho_n^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \frac{\delta}{c} \\ \vdots \\ \left(\frac{\delta}{c}\right)^{n-1} \end{pmatrix} \Leftrightarrow \mathbf{a} = \mathbf{P}^{-1}\mathbf{\Lambda},$$

in matrix form, where $\mathbf{P} = \mathbf{P}(\rho_1, \ldots, \rho_n)$ is a Vandermonde matrix, $\mathbf{a}' = (a_1, a_2, \ldots, a_n)$ and $\mathbf{\Lambda}' = (1, \delta/c, \ldots, (\delta/c)^{n-1})$.

Finally, we get expressions for the coefficients

$$\begin{aligned} a_k &= \frac{(-1)^{k-1}(\prod_{i=1,\, i\neq k}^{n}(\rho_i - \frac{\delta}{c}))(\prod_{1\leq i<j\leq n,\, i\neq k,\, j\neq k}(\rho_j - \rho_i))}{\prod_{1\leq i<j\leq n}(\rho_j - \rho_i)} \\ &= \frac{(-1)^{k-1}(\prod_{i=1,\, i\neq k}^{n}(\rho_i - \frac{\delta}{c}))}{(\prod_{i=1}^{k-1}(\rho_k - \rho_i))(\prod_{j=k+1}^{n}(\rho_j - \rho_k))} = \prod_{i=1, i\neq k}^{n} \frac{\left(\rho_i - \frac{\delta}{c}\right)}{(\rho_i - \rho_k)}. \end{aligned}$$

We note that δ/c is not a root of equation (2.2). Hence, we get the result. ■

Remarks:

1. The Laplace transform (5.4) shows an interesting form as it corresponds to Formula (2.12) found by Li (2008), concerning the primal model and applied for the first hitting time that the surplus risk process, starting from zero, upcrosses a level $u > 0$. This result enhances the duality of the two models as explained by Afonso *et al.* (2011) who worked the compound Poisson, or Erlang(1), model. We mean, the first hitting time in the primal model corresponds to the ruin time in the dual model. It is interesting that the duality features shown for the classical Erlang(1) can be extended. Note that the loading conditions in the two models are reversed. We refer to the explanations for the Lundberg's equations in Sections 2 and 3.

2. Formula (4.6) is a limiting case, as $\delta \to 0^+$, of (5.4). We couldn't transpose ot our model directly Formula (2.12) of Li (2008) derived for the primal model because its limit as $\delta \to 0^+$ would lead to a ruin probability of one. This is due to the reverse loading condition. The first hitting time in the primal model is a proper random variable as far as the time to ruin in the dual model is a defective one. Formulae (5.4) above and (2.12) from Li (2008) show the same *appearance* but parameter c have different admissible values.

Example 5.1 *For $n=1$, exponential case, Ng (2009) found that $\psi(u,\delta) = e^{-\rho_1 u}$, where ρ_1 is the positive real solution of $1 + \frac{\delta}{\lambda} - \left(\frac{c}{\lambda}\right)s = \hat{p}(s)$.*

For $n=2$:

$$\psi(u,\delta) = \frac{\rho_2 - \frac{\delta}{c}}{\rho_2 - \rho_1} e^{-\rho_1 u} - \frac{\rho_1 - \frac{\delta}{c}}{\rho_2 - \rho_1} e^{-\rho_2 u},$$

where $\rho_1, \rho_2 > 0$ are real, solutions of $\left(1 + \frac{\delta}{\lambda} - \left(\frac{c}{\lambda}\right)s\right)^2 = \hat{p}(s)$. The above formula corresponds to expression (2.1) of Dickson and Li (2012).

For $n = 3$:

$$\psi(u,\delta) = \frac{(\rho_2 - \frac{\delta}{c})(\rho_3 - \frac{\delta}{c})}{(\rho_3 - \rho_1)(\rho_2 - \rho_1)} e^{-\rho_1 u} - \frac{(\rho_1 - \frac{\delta}{c})(\rho_3 - \frac{\delta}{c})}{(\rho_3 - \rho_2)(\rho_2 - \rho_1)} e^{-\rho_2 u} + \frac{(\rho_1 - \frac{\delta}{c})(\rho_2 - \frac{\delta}{c})}{(\rho_3 - \rho_1)(\rho_3 - \rho_2)} e^{-\rho_3 u},$$

where ρ_1, ρ_2, ρ_3 are solutions of $\left(1 + \frac{\delta}{\lambda} - \left(\frac{c}{\lambda}\right) s\right)^3 = \hat{p}(s)$. One root is real and positive, the other two are complex conjugates.

6 Expected Discounted Dividends

From this section on we consider the existence of an upper dividend barrier b so that when the surplus upcrosses b the excess is paid as dividend.

6.1 An integro-differential equation

In the Poisson case, exponentially distributed interjumps arrivals, see e.g. Afonso *et al.* (2011), the expected present value of the discounted dividends, $V(u,b)$, satisfies the renewal equation, for $u \leq b$,

$$\begin{aligned} V(u,b) &= \int_0^{\frac{u}{c}} \lambda e^{-(\lambda+\delta)t} \left\{ \int_0^{b-u+ct} V(u-ct+y,b)p(y)dy \right. \\ &\quad \left. + \int_{b-u+ct}^{\infty} [y+u-ct-b+V(b,b)]\, p(y)dy \right\} dt \end{aligned}$$

Note that $V(0,b) = 0$, since at $u = 0$ ruin occurs, and that

$$V(u,b) = u - b + V(b,b), \quad \text{for} \quad u > b. \tag{6.1}$$

Changing variable, $s = u - ct$, and differentiating with respect to u we get

$$\left(\left(1 + \frac{\delta}{\lambda}\right)\mathcal{I} + \left(\frac{c}{\lambda}\right)\mathcal{D}\right) V(u,b) = W_\delta(u,b),$$

where

$$W_\delta(u,b) = \int_0^{b-u} V(u+y,b)p(y)dy + \int_{b-u}^{\infty} (y+u-b+V(b,b))p(y)dy. \tag{6.2}$$

In the Erlang(n) model ($n \geq 2$), the corresponding renewal equation is given by

$$\begin{aligned} V(u,b) &= \int_0^{\frac{u}{c}} k_n(t)e^{-\delta t} \left[\int_0^{b-u+ct} V(u-ct+y,b)p(y)dy + \right. \\ &\quad \left. \int_{b-u+ct}^{\infty} (y+u-ct-b+V(b,b))p(y)dy \right] dt. \end{aligned}$$

After a similar variable change, we can write it in the following form

$$V(u,b) = \frac{1}{c}\int_0^u k_n\left(\frac{u-s}{c}\right) e^{-\delta(\frac{u-s}{c})} W_\delta(s,b)ds. \tag{6.3}$$

The following theorem shows the final form of this equation.

Theorem 6.1 $V(u,b)$ *satisfies the integro–differential equation*

$$\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n V(u,b)=W_\delta(u,b), \tag{6.4}$$

with boundary conditions

$$\left.\frac{d^i}{du^i}V(u,b)\right|_{u=0}=0, \quad i=0,\ldots,n-1. \tag{6.5}$$

Proof. The proof follows exactly the method applied previously, taking successive derivatives of (6.3). ■

6.2 The annihilator of $p(x-u)$

Because of condition (6.1), we can not write the solutions of (6.4) as a linear combination of n exponential functions as we did before in the cases of the ruin probability and the Laplace transform of the time of ruin. Otherwise, conditions given by (6.5) would led to $V(u,b)\equiv 0$, which is a contradiction. We will need instead, more than n exponential functions, the exact number needed will depend on the nature of the distribution of the single gains, $P(x)$. However, we can apply the annihilator approach known from the theory of ordinary differential equations to find the appropriate solutions, e.g. see Zill (2012), Section 4.5.

We can rewrite $W_\delta(u,b)$ in (6.2) as

$$\begin{aligned} W_\delta(u,b) &= \int_u^b V(x,b)p(x-u)dx+\int_b^\infty (x-b+V(b,b))p(x-u)dx \\ &= \int_u^b V(x,b)p(x-u)dx+\int_b^\infty \widetilde{V}(x,b)p(x-u)dx, \end{aligned}$$

with $\widetilde{V}(x,b)=x-b+V(b,b)$. The idea is to find a linear differential operator that will annihilate $p(x-u)$ (where the variable is u), so that when we apply this operator to the integro–differential equation (6.4) we obtain a linear homogeneous differential equation of a higher degree (and the integral term vanish).

From this moment onwards we work the particular case when the single gains follow a distribution of the Phase-Type family, PH(m). Our notations and definitions are presented as usually done in this case. Denote by $\mathbf{B}=(b_{ij})_{1\le i,j\le m}$ the matrix of the transition rates between the transient states, let $\alpha'=(\alpha_1,\alpha_2,\ldots,\alpha_m)$ be the vector of the initial probabilities, $\eta'=(\eta_1,\eta_2,\ldots,\eta_m)$ the vector of the exit rates to the absorbing state, and the $1\times m$ vector $\mathbf{1}'=(1,1,\ldots,1)$. Let $\mathbf{I_m}$ denote the identity matrix of order m. It is well known for this family that the probability and distribution functions are denoted as $p(x)=\alpha'\mathbf{e}^{\mathbf{B}x}\eta$ and that $P(x)=1-\alpha'\mathbf{e}^{\mathbf{B}x}\mathbf{1}$, respectively. Let's consider the following theorem:

Theorem 6.2 *One annihilator of degree* m *for* $p(x-u)$ *is* $q_{\mathbf{B}}(-\mathcal{D})$, *where* $\mathcal{D}=\frac{d}{du}$ *denote differentiation with respect to* u *and* $q_{\mathbf{B}}(y)=Det(\mathbf{B}-y\mathbf{I_m})$ *is the characteristic polynomial of the matrix* $\mathbf{B}$.

Proof. The proof is based on the Cayley–Hamilton theorem of linear algebra, which states that every square matrix satisfies its own characteristic equation [see e.g. Lang (2010)]. ■

Example 6.1 *When we consider the exponential(β) distribution for the individual gain size, we have that $p(x) = \beta e^{-\beta x}$, then $\mathbf{B} = (-\beta)$, $\alpha' = (1)$, $\eta' = (\beta)$ and $\mathbf{1}' = (1)$. Hence,*

$$q_{\mathbf{B}}(y) = Det(\mathbf{B} - yI_1) = -\beta - y \quad \textit{and} \quad q_{\mathbf{B}}(-\mathcal{D}) = \frac{d}{du} - \beta.$$

It is easy to check that $\left(\frac{d}{du} - \beta\right) p(x-u) = 0$.

For a more general case when the individual gain size follows an Erlang(m, β) distribution we have that $p(x) = \beta^m x^{m-1} e^{-\beta x}/(m-1)!$, so that

$$\mathbf{B} = \begin{pmatrix} -\beta & \beta & \cdots & 0 & 0 \\ 0 & -\beta & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\beta & \beta \\ 0 & 0 & \cdots & 0 & -\beta \end{pmatrix},$$

$\alpha' = (1, 0, \ldots, 0)$, $\eta' = (0, 0, \ldots, 0, \beta)$ and $\mathbf{1}' = (1, 1, \ldots, 1)$. Then

$$q_{\mathbf{B}}(y) = Det(\mathbf{B} - yI_m) = (-\beta - y)^m \quad \textit{and} \quad q_{\mathbf{B}}(-\mathcal{D}) = \left(\frac{d}{du} - \beta\right)^m.$$

It is easily verified that $\left(\frac{d}{du} - \beta\right)^m p(x-u) = 0$.

Now, we want to apply $q_{\mathbf{B}}(-\mathcal{D})$ to the integro–differential equation (6.4). We consider the polynomial expression of $q_{\mathbf{B}}(-\mathcal{D})$:

$$q_{\mathbf{B}}(-\mathcal{D}) = \sum_{i=0}^{m} q_i \frac{d^i}{du^i},$$

where q_i, $i = 0, 1, ..., m$, are constants (namely $q_0 = Det(\mathbf{B})$, $q_{m-1} = Trace\,(\mathbf{B})$, $q_m = 1$). Thus, we have the following theorem:

Theorem 6.3 *After applying $q_{\mathbf{B}}(-\mathcal{D})$ to the integro–differential equation* (6.4) *we get a linear homogeneous differential equation of degree $m+n$ of the following form*

$$\begin{aligned} 0 \quad = \quad & \sum_{l=0}^{n+m} \left[\sum_{i+k=l} q_i \binom{n}{n-k} \left(1 + \frac{\delta}{\lambda}\right)^{n-k} \left(\frac{c}{\lambda}\right)^k \right] \frac{d^l}{du^l} V(u,b) \\ & + \sum_{j=0}^{m-1} \left[\sum_{k=j+1}^{m} q_k \alpha' (-\mathbf{B})^{k-j} \mathbf{1} \right] \frac{d^j}{du^j} V(u,b). \end{aligned} \tag{6.6}$$

Proof. Since, expanding the binomial, with $\frac{d^0}{du^0} = \mathcal{I}$,

$$\left(\left(1 + \frac{\delta}{\lambda}\right) \mathcal{I} + \left(\frac{c}{\lambda}\right) \mathcal{D} \right)^n V(u,b) = \sum_{k=0}^{n} \binom{n}{n-k} \left(1 + \frac{\delta}{\lambda}\right)^{n-k} \left(\frac{c}{\lambda}\right)^k \frac{d^k}{du^k} V(u,b),$$

then from one side we have

$$q_{\mathbf{B}}(-\mathcal{D}) \left(\left(1 + \frac{\delta}{\lambda}\right) \mathcal{I} + \left(\frac{c}{\lambda}\right) \mathcal{D} \right)^n V(u,b) = \sum_{l=0}^{n+m} \left[\sum_{i+k=l} q_i \binom{n}{n-k} \left(1 + \frac{\delta}{\lambda}\right)^{n-k} \left(\frac{c}{\lambda}\right)^k \right] \frac{d^l}{du^l} V(u,b),$$

and from the other side

$$
\begin{aligned}
q_{\mathbf{B}}(-\mathcal{D})W_{\delta}(u,b) &= q_{\mathbf{B}}(-\mathcal{D})\left[\int_u^b V(x,b)p(x-u)dx + \int_b^{\infty} \widetilde{V}(u,b)p(x-u)dx\right] \\
&= \int_u^b V(x,b)q_{\mathbf{B}}(-\mathcal{D})p(x-u)dx + \int_b^{\infty} \widetilde{V}(x,b)q_{\mathbf{B}}(-\mathcal{D})p(x-u)dx \\
&\quad -\sum_{k=1}^{m} q_k \sum_{j=0}^{k-1} \frac{d^j}{du^j}V(u,b)\left[\left.\frac{d^{k-1-j}}{du^{k-1-j}}p(x-u)\right|_{x=u}\right] \\
&= -\sum_{j=0}^{m-1}\left[\sum_{k=j+1}^{m} q_k \alpha'(-\mathbf{B})^{k-j}\mathbf{1}\right]\frac{d^j}{du^j}V(u,b).
\end{aligned}
$$

The result follows. ■

6.3 Expression for the expected discounted dividends

We look for solutions of (6.6) of the form

$$
V(u,b) = \sum_{k=1}^{n+m} a_k e^{-r_k u} \tag{6.7}
$$

for some coefficients a_k and some exponents r_k that are up to be determined. Replacing (6.7) in (6.6) we get

$$
\begin{aligned}
0 &= \left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I} + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n V(u,b) - W(u,b) \\
&= \sum_{l=1}^{n+m} a_l \left[\left(1+\frac{\delta}{\lambda} - \left(\frac{c}{\lambda}\right)r_l\right)^n - \hat{p}(r_l)\right]e^{-r_l u} \\
&\quad -\alpha'\left[\sum_{l=1}^{n+m} a_l e^{-r_l b}\left((r_l\mathbf{I_m} - \mathbf{B})^{-1}\mathbf{B} + \mathbf{I_m}\right) - \mathbf{B}^{-1}\right]e^{\mathbf{B}(b-u)}\mathbf{1}.
\end{aligned} \tag{6.8}
$$

Since equation (6.8) holds for any $u \geq 0$, the coefficients of $e^{-r_l u}$ and $e^{\mathbf{B}(b-u)}$ must be zero. This means that

$$
\left(1+\frac{\delta}{\lambda} - \left(\frac{c}{\lambda}\right)r_l\right)^n - \hat{p}(r_l) = 0, \quad l = 1,\ldots,n+m,
$$

so the exponents r_l, $l = 1,...,n+m$, are all the $m+n$ roots of the generalized Lundberg's equation (2.2), from those n roots have positive real parts, namely $\rho_1,\rho_2,\ldots,\rho_n$, and m roots with negative real parts, $\rho_{n+1},\rho_{n+2},\ldots,\rho_{n+m}$. Also, we must have

$$
\alpha'\left[\sum_{l=1}^{n+m} a_l e^{-r_l b}\left((r_l\mathbf{I_m} - \mathbf{B})^{-1}\mathbf{B} + \mathbf{I_m}\right) - \mathbf{B}^{-1}\right] = \mathbf{0}. \tag{6.9}
$$

This gives a homogeneous system of m equations on the $m+n$ unknown coefficients a_l. The remaining n equations that we need (to have a full system of $m+n$ equations on the $m+n$ unknowns), are the n boundary conditions (6.5).

Example 6.2 *Lets assume that the time between two consecutive jumps is Erlang(2) distributed and the individual jump amounts are Erlang($2, \beta$) distributed. Then, the negative loading condition is $c < \lambda/\beta$ and the generalized Lundberg's equation* (2.2) *is given by*

$$(\lambda + \delta - cs)^2(\beta + s)^2 = \lambda^2\beta^2. \tag{6.10}$$

Let

$$V(u, b) = \sum_{l=1}^{4} a_l e^{-\rho_l u}$$

Then the exponents ρ_l are the four roots of (6.10). *Say that ρ_1, ρ_2 are the two roots with positive real parts and ρ_3, ρ_4 are the ones with negative real parts. From the two boundary conditions* (6.5) *we get*

$$\sum_{l=1}^{4} a_l = 0, \text{ and } \sum_{l=1}^{4} a_l\rho_l = 0,$$

and from (6.9) *we get*

$$\sum_{l=1}^{4} a_l e^{-\rho_l b}\frac{\rho_l}{\rho_l + \beta} = -\frac{1}{\beta}, \quad \sum_{l=1}^{4} a_l e^{-\rho_l b}\frac{\rho_l\beta}{(\rho_l + \beta)^2} = -\frac{1}{\beta},$$

so we have a system of four equations in the four unknowns $a_1, \ldots, a_4$. In matrix form we have

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \rho_1 & \rho_2 & \rho_3 & \rho_4 \\ e^{-\rho_1 b}\frac{\rho_1}{\rho_1+\beta} & e^{-\rho_2 b}\frac{\rho_2}{\rho_2+\beta} & e^{-\rho_3 b}\frac{\rho_3}{\rho_3+\beta} & e^{-\rho_4 b}\frac{\rho_4}{\rho_4+\beta} \\ e^{-\rho_1 b}\frac{\rho_1\beta}{(\rho_1+\beta)^2} & e^{-\rho_2 b}\frac{\rho_2\beta}{(\rho_2+\beta)^2} & e^{-\rho_3 b}\frac{\rho_3\beta}{(\rho_3+\beta)^2} & e^{-\rho_4 b}\frac{\rho_4\beta}{(\rho_4+\beta)^2} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{\beta} \\ -\frac{1}{\beta} \end{pmatrix}.$$

Now, set the values for the parameters $\lambda = \beta = 1$, $c = 0.75$, $\delta = 0.02$. Then $\rho_1 = 0.423$, $\rho_2 = 1.831$, $\rho_3 = -0.063$ and $\rho_4 = -1.471$. After computing the coefficients we obtain the values of the expected discounted dividends, for $u \in \{1, 3, 5, 10, 15, 20\}$ and $b \in \{2, 3, 6, 10, 30, 40\}$, that are shown in Table 6.1. The values are quite similar to those in Table 7.1 of Afonso et al. (2011) although a little bit smaller. Also we notice that for a fixed u the value of $V(u, b)$ increases until a certain value of b and then decreases rapidly. This behavior is expected and corroborates the findings of Afonso et al. (2011) and Avanzi et al. (2007).

6.4 Higher moments of the discounted dividends

In the Erlang(n) model, the k-th ordinary moment of the discounted dividends $V_k(u, b)$ satisfies the renewal equation

$$\begin{aligned} V_k(u, b) &= \int_0^{\frac{u}{c}} k_n(t)e^{-k(\delta)t}\left[\int_0^{b-u+ct} V_k(u - ct + y, b)p(y)dy + \right. \\ &\left. \int_{b-u+ct}^{\infty} \widetilde{V}_k(u - ct + y, b)p(y)dy\right] dt, \end{aligned}$$

$u\backslash b$	2	3	6	10	30	40
1	1.049	1.301	1.856	1.781	0.526	0.279
3		4.533	6.451	6.189	1.826	0.972
5			9.374	8.993	2.653	1.412
10				13.829	4.081	2.172
15					5.647	3.006
20					7.746	4.123

Table 6.1: Expected discounted dividends

with

$$\widetilde{V}_k(x,b) = \sum_{j=0}^{k} \binom{k}{j} (x-b)^j V_{k-j}(b,b), \quad x \geq b.$$

In the above expression we assume the convention $V_0(u,b) \equiv 1$. The corresponding integro–differential equation is

$$\left(\left(1+\frac{k\delta}{\lambda}\right)\mathcal{I} + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n V_k(u,b) = W_{k\delta}(u,b), \tag{6.11}$$

with boundary conditions

$$\left.\frac{d^i}{du^i} V_k(u,b)\right|_{u=0} = 0, \quad i = 0,\ldots,n-1, \tag{6.12}$$

where

$$W_{k\delta}(u,b) = \int_u^b V_k(x,b)p(x-u)dx + \int_b^\infty \widetilde{V_k}(x,b)p(x-u)dx.$$

Assuming that the gains follow a PH(m) distribution we can apply an analogous method to find and an expression for $V_k(u,b)$, and numerical values in the same way. We apply the same annihilator $q_{\mathbf{B}}(-\mathcal{D})$ to the integro–differential equation (6.11) to obtain

$$\begin{aligned} 0 \;=\; & \sum_{l=0}^{n+m} \left[\sum_{i+j=l} q_i \binom{n}{n-j} \left(1+\frac{k\delta}{\lambda}\right)^{n-j} \left(\frac{c}{\lambda}\right)^j \right] \frac{d^l}{du^l} V_k(u,b) + \\ & \sum_{j=0}^{m-1} \left[\sum_{i=j+1}^{m} q_i \alpha' (-\mathbf{B})^{i-j} \mathbf{1} \right] \frac{d^j}{du^j} V_k(u,b).. \end{aligned} \tag{6.13}$$

Therefore we seek for solutions of (6.13) of the form

$$V_k(u,b) = \sum_{l=1}^{n+m} a_l e^{-r_l u}. \tag{6.14}$$

Replacing (6.14) in (6.13) gives

$$
\begin{aligned}
0 &= \left(\left(1+\frac{k\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n V(u,b)-W(u,b)\\
&= \sum_{l=1}^{n+m} a_l\left[\left(1+\frac{k\delta}{\lambda}-\left(\frac{c}{\lambda}\right)r_l\right)^n-\hat{p}(r_l)\right]e^{-r_l u}\\
&\quad -\alpha'\left[\sum_{l=1}^{n+m} a_l e^{-r_l b}\left((r_l\mathbf{I_m}-\mathbf{B})^{-1}\mathbf{B}+\mathbf{I_m}\right)\right.\\
&\quad \left.+\sum_{j=1}^{k} j\binom{k}{j}V_{k-j}(b,b)(-\mathbf{B})^{-j}\right]e^{\mathbf{B}(b-u)}\mathbf{1}. \qquad (6.15)
\end{aligned}
$$

Since (6.15) is valid $\forall u$ then we must have

$$
\left(1+\frac{k\delta}{\lambda}-\left(\frac{c}{\lambda}\right)r_l\right)^n-\hat{p}(r_l)=0, \quad l=1,\ldots,n+m,
$$

so the exponents r_l, $l=1,...,n+m$, are all the $m+n$ roots of the generalized Lundberg's equation (2.2), from those n roots have positive real parts, namely $\rho_1,\rho_2,\ldots,\rho_n$, and m roots with negative real parts, $\rho_{n+1},\rho_{n+2},\ldots,\rho_{n+m}$. Also, we must have

$$
\alpha'\left[\sum_{l=1}^{n+m} a_l e^{-r_l b}\left((r_l\mathbf{I_m}-\mathbf{B})^{-1}\mathbf{B}+\mathbf{I_m}\right)+\sum_{j=1}^{k} j\binom{k}{j}V_{k-j}(b,b)(-\mathbf{B})^{-j}\right]=\mathbf{0}. \qquad (6.16)
$$

Example 6.3 *We want to compute $V_2(u,b)$. Lets assume that the time between two consecutive jumps is Erlang(2) distributed and the individual jump amounts are Erlang($2,\beta$) distributed. Then, the negative loading condition is $c<\lambda/\beta$ and the generalized Lundberg's equation (2.2) is given by*

$$
(\lambda+k\delta-cs)^2(\beta+s)^2=\lambda^2\beta^2. \qquad (6.17)
$$

Let

$$
V_2(u,b)=\sum_{l=1}^{4} a_l e^{-\rho_l u}
$$

Then the exponents ρ_l are the four roots of (6.17). Say that ρ_1,ρ_2 are the two roots with positive real parts and ρ_3,ρ_4 are the ones with negative real parts. From the two boundary conditions (6.12) we get

$$
\sum_{l=1}^{4} a_l=0, \ \textit{and} \ \sum_{l=1}^{4} a_l\rho_l=0,
$$

and from (6.16) we get

$$
\sum_{l=1}^{4} a_l e^{-\rho_l b}\frac{\rho_l}{\rho_l+\beta}=-2V(b,b)\frac{1}{\beta}-\frac{2}{\beta^2}, \quad \sum_{l=1}^{4} a_l e^{-\rho_l b}\frac{\rho_l\beta}{(\rho_l+\beta)^2}=-2V(b,b)\frac{1}{\beta}-\frac{4}{\beta^2},
$$

so we have a system of four equations in the four unknowns $a_1,\ldots,a_4$.

In matrix form we have

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \rho_1 & \rho_2 & \rho_3 & \rho_4 \\ e^{-\rho_1 b}\frac{\rho_1}{\rho_1+\beta} & e^{-\rho_2 b}\frac{\rho_2}{\rho_2+\beta} & e^{-\rho_3 b}\frac{\rho_3}{\rho_3+\beta} & e^{-\rho_4 b}\frac{\rho_4}{\rho_4+\beta} \\ e^{-\rho_1 b}\frac{\rho_1\beta}{(\rho_1+\beta)^2} & e^{-\rho_2 b}\frac{\rho_2\beta}{(\rho_2+\beta)^2} & e^{-\rho_3 b}\frac{\rho_3\beta}{(\rho_3+\beta)^2} & e^{-\rho_4 b}\frac{\rho_4\beta}{(\rho_4+\beta)^2} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ -2V(b,b)\frac{1}{\beta} - \frac{2}{\beta^2} \\ -2V(b,b)\frac{1}{\beta} - \frac{4}{\beta^2} \end{pmatrix}.$$

Set values for the parameters as $\lambda = \beta = 1$, $c = 0.75$, $\delta = 0.02$. Then $\rho_1 = 0.494$, $\rho_2 = 1.853$, $\rho_3 = -0.107$ and $\rho_4 = -1.467$. After computing the coefficients we obtain the values for the standard deviation of $D(u,b)$, for $u \in \{1,3,5,10,15,20\}$ and $b \in \{2,3,6,10,30,40\}$, that are shown in Table 6.2.

$u\backslash b$	2	3	6	10	30	40
1	2.534	3.389	4.893	4.638	1.655	0.973
3		5.058	7.335	6.906	2.667	1.621
5			7.483	6.985	2.966	1.841
10				6.864	3.531	2.277
15					4.269	2.829
20					5.093	3.496

Table 6.2: Standard deviation of the discounted dividends

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