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A Note on Bonus Scales

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Abstract

We revisit the optimal bonus scales introduced by Norberg (Norberg, R., 1976, *Scandinavian Actuarial Journal* (2): 92-107), Borgan, Hoem, and Norberg (Borgan, O., J. Hoem, and R. Norberg, 1981, *Scandinavian Actuarial Journal* (2): 165-178), and Gilde and Sundt (Gilde, V., and B. Sundt, 1989, *Scandinavian Actuarial Journal* (1): 13-22) and underline some potential problems of the linear scales. As a possible solution we propose the use of geometric scales.

Introduction

Bonus-malus systems, namely Markovian systems, are widely used in automobile insurance. As it is well known, see Lemaire (1995), the basic idea of these systems is to divide the policies into a finite number of classes, numbered from 1 to K, to place each policy in one of these classes during each insurance period, supposed of constant length (usually a year) and to determine the class where the policy will be in the next year based only on the present class and on the number of claims reported to the insurer. We assume that the classes are numbered from the least to the most dangerous one.

In 1976, Norberg developed optimal credibility premium scales for Markovian *bonus-malus* systems, with given transition rules, under an infinite horizon approach and assuming the minimization of the expected squared difference between the true net premium and the premium paid by the policyholder. In 1981, Borgan, Hoem, and Norberg generalized this result to a finite horizon approach.

These approaches, as mentioned by the authors, can both lead to a sequence of non-monotonic premiums, which is unacceptable from the practical point of view, since a policyholder who moves to a better class should not have to suffer an increase in his premium. Gilde and Sundt, a few years later, tackled this problem by constraining the optimization problem, assuming that the increase between consecutive classes of the system is always the same, i.e., the premium sequence is linear. However, as

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we will show in the present article, this approach can lead to another flaw: in some situations, not unrealistic, premiums given by the linear scale become negative, which is, obviously, unacceptable.

To solve this problem we propose the use of a geometric scale, i.e., the premium increase between consecutive classes is always given by the same percentage of the premium.

Using a similar framework, one could use other constraints that lead to other premium scales. Amongst the various alternatives, the geometric scale presents two qualities: its interpretation is very simple and it can be very easily combined with the other rating factors when using generalized linear models with a logarithmic link function. However, the premiums given by the geometric scale are not financially balanced and, consequently, we have to correct the results given by the constrained minimization. Note that the correction is straightforward.

The structure of the article is as follows: We start by presenting the model and the results obtained by Norberg (1976), Borgan, Hoem, and Norberg (1981), and Gilde and Sundt (1989). Then, we introduce the geometric scales. In the last section, we present two examples to illustrate our approach.

THE MODEL

Let us consider a Markovian *bonus* system with K classes. After a year with r claims in class j the policy is transferred to class $T(j, r)$. This function represents the transition rules associated to the *bonus* system and is assumed to be nondecreasing in its second argument. Let $\boldsymbol{\pi}$ be a vector whose components $\pi(j)$, $j = 1, 2, \ldots, K$, represent the premium in class j and let k be the initial class. In this context we can represent the *bonus* system by the triplet $S = (T, \boldsymbol{\pi}, k)$.

Let M_n be the number of claims reported in period n. We assume that $M_1, M_2, \ldots$ are conditionally independent given the value of an unknown risk parameter θ, regarded as the outcome of a positive random variable Θ with distribution function $U(.)$. Finally, $Z_{S,n}$ denotes the *bonus* class in period n for a given policy.

Norberg (1976), assuming that $\{M_n\}_{n=1,2,\ldots}$ are identically distributed, shows that the premium scale leading to the minimization of the expected quadratic error using the long-run distribution of the Markov chain, i.e., which minimizes

$$Q_0(S) = \mathrm{E}[\mathrm{E}(M \mid \Theta) - \pi(Z_{S,0})]^2, \tag{1}$$

where the index 0 stands for the long-run distribution and M has the same distribution as M_n, is given by

$$\pi_0(j) = \mathrm{E}[\mathrm{E}[M \mid \Theta] \mid Z_{S,0} = j] = \frac{\displaystyle\int \mathrm{E}[M \mid \theta]\, p_{T,\theta}(j)\, dU(\theta)}{p_T(j)}, \qquad j = 1, \ldots, K, \tag{2}$$

where $\mathbf{p}_{T,\theta} = [p_{T,\theta}(j)]$ is the conditionally long-run distribution and $\mathbf{p}_T = [p_T(j)]$ is the long-run distribution.

Borgan, Hoem, and Norberg (1981) generalize this approach. They construct a nonasymptotic criterion, by minimizing

$$Q(S) = \sum_{n=0}^{\infty} w_n Q_n(S) = \sum_{n=0}^{\infty} w_n \mathrm{E}[\mathrm{E}(M_n \mid \Theta) - \pi(Z_{S,n})]^2, \tag{3}$$

where $\mathbf{w} = [w_0, w_1, \ldots]$ is a vector of nonnegative weights adding 1 and where w_0 is the weight given to the long-run distribution. The solution, named Bayes scale, is

$$\pi_{\mathrm{B}}(j) = \frac{\sum_{n=0}^{\infty} w_n \int \mathrm{E}[M_n \mid \theta] p_{T,\theta}^{(n)}(j)\, dU(\theta)}{p_S(j)}, \qquad j = 1, \ldots, K, \tag{4}$$

where $\mathbf{p}_{S,\theta}^{(n)} = [p_{S,\theta}^{(n)}(j)]$ is the probability that, given θ, the policy stands in class j in period n and

$$p_S(j) = \sum_{n=0}^{\infty} w_n \int p_{T,\theta}^{(n)}(j)\, dU(\theta), \qquad j = 1, \ldots, K. \tag{5}$$

Note that the last scale generalizes the former, given by (2), considering that $\{M_n\}$ are independently and identically distributed (i.i.d.) and that $w_0 = 1$.

From the statistical point of view, these scales are optimal for a quadratic loss. However, in some practical circumstances they can lead to some annoyances: the variation of the premiums in consecutive classes of the system can be quite irregular and, in some cases, a more dangerous class of the system can pay less than a less dangerous one. Even if the situation could be understood in mathematical terms, it would be unacceptable by the policyholders.

As a solution to these annoyances Gilde and Sundt (1989) introduce linear scales, i.e., they minimize (3) subject to the constraints that $\pi(j) = a + bj, j = 1, \ldots, K$.

The solution is now $\pi_{\mathrm{L}}(j) = a_{\mathrm{L}} + b_{\mathrm{L}}\ j = 1, \ldots, K$, with

$$\begin{cases} b_{\mathrm{L}} = \dfrac{\sum_{j=1}^{K} j\pi_{\mathrm{B}}(j) p_S(j) - \sum_{j=1}^{K} j p_S(j) \sum_{j=1}^{K} \pi_{\mathrm{B}}(j) p_S(j)}{\sum_{j=1}^{K} j^2 p_S(j) - \left(\sum_{j=1}^{K} j p_S(j)\right)^2}, \\ a_{\mathrm{L}} = \sum_{j=1}^{K} \pi_{\mathrm{B}}(j) p_S(j) - b_{\mathrm{L}} \sum_{j=1}^{K} j p_S(j). \end{cases} \tag{6}$$

As we have numbered the classes of the *bonus* system from the least to the most dangerous one, we expect b_L to be positive. This would be the case if

$$\sum_{j=1}^{K} j\pi_B(j)p_S(j) - \sum_{j=1}^{K} jp_S(j)\sum_{j=1}^{K}\pi_B(j)p_S(j) > 0, \tag{7}$$

that is to say, if the covariance between $\boldsymbol{\pi}_B$ and Z_S, where $Z_S = \sum_{n=0}^{\infty} w_n Z_{S,n}$, is positive. This must be the case for all the acceptable systems and we will assume that (7) holds. But even assuming (7) these scales can, for some very heterogeneous situations, lead to a big setback: negative premiums for the best classes of the system can be obtained, i.e., $a_L + b_L$ can be negative.

GEOMETRIC SCALES

The basic idea of the geometric scales is similar to the linear scales. We minimize (3) under constraints of the form $\pi(j) = ab^j$, $j = 1, \ldots, K$, with a and b positive.

Note that using (4) and (5), we can write (3) as

$$\begin{aligned} Q(S) &= \sum_{n=0}^{\infty} w_n \int \sum_{j=1}^{K} [\mathrm{E}(M_n \mid \theta) - \pi(j)]^2 p_{S,\theta}^{(n)}(j)\, dU(\theta) \\ &= \sum_{n=0}^{\infty} w_n \int \sum_{j=1}^{K} \mathrm{E}^2(M_n \mid \theta) p_{S,\theta}^{(n)}(j)\, dU(\theta) \\ &\quad + \sum_{j=1}^{K} (\pi(j))^2 p_S(j) - 2\sum_{j=1}^{K} \pi_B(j)\pi(j)p_S(j) \\ &= \text{Constant } + \sum_{j=1}^{K} (\pi(j) - \pi_B(j))^2 p_S(j). \end{aligned} \tag{8}$$

From the last expression in (8) it is clear that the linear scale is nothing more than the result of a weighted least square regression between the Bayes scale and the indices j.

To calculate the geometric scale we will now minimize

$$\Delta_{Q_G}(S) = \sum_{j=1}^{K} (ab^j - \pi_B(j))^2 p_S(j) \tag{9}$$

that corresponds to a weighted nonlinear least squares regression. Differentiating (9) with respect to a and b and equating to zero we get the first order conditions

$$a = \frac{\sum_{j=1}^{K} b^j \pi_B(j) p_S(j)}{\sum_{j=1}^{K} b^{2j} p_S(j)} \tag{10}$$

and

$$f(b) = 0, \tag{11}$$

with

$$f(b) = \sum_{j=1}^{K} b^j \pi_{\mathrm{B}}(j) p_S(j) \sum_{j=1}^{K} j b^{2j} p_S(j) - \sum_{j=1}^{K} j b^j \pi_{\mathrm{B}}(j) p_S(j) \sum_{j=1}^{K} b^{2j} p_S(j). \tag{12}$$

Note that $f(b)$ is a polynomial of order $3K - 1$ and that the coefficient of b^{3K-1} is negative, namely $-\pi_{\mathrm{B}}\ (K-1) p_S\ (K-1) p_S\ (K)$. Furthermore $f(1) > 0$, according to (7). Consequently, as $f(1) > 0$ and $\lim_{b\to\infty} f(b) = -\infty$, $f(b)$ must have at least one root greater than 1. We were not able to prove the uniqueness of the real solution for $b > 1$, but we could not find a counterexample. Anyway, if the root is unique it is a minimum, otherwise we would compare the relative minima. Let the optimal solution be $(a_{\mathrm{G}}, b_{\mathrm{G}})$ and the optimal scale be $\boldsymbol{\pi}_{\mathrm{G}}$.

Note that the expected value of the premiums (relative to $\mathbf{p}_S$) under both the Bayes scale and the linear scale is equal to $\sum_{n=0}^{\infty} w_n \mathrm{E}[M_n]$, i.e., both systems are balanced. This does not happen with the geometric scale. If we require a balanced system we should minimize (9) subject to

$$\sum_{j=1}^{K} a b^j p_S(j) = \sum_{j=1}^{K} \pi_{\mathrm{B}}(j) p_S(j). \tag{13}$$

In this case the first order conditions are

$$\begin{cases} a = \dfrac{\sum_{j=1}^{K} \pi_{\mathrm{B}}(j) p_S(j)}{\sum_{j=1}^{K} b^j p_S(j)}, \\ \sum_{j=1}^{K} [a b^j - \pi_{\mathrm{B}}(j)] j b^{j-1} p_S(j) \sum_{j=1}^{K} b^j p_S(j) = \sum_{j=1}^{K} [a b^j - \pi_{\mathrm{B}}(j)] b^j p_S(j) \sum_{j=1}^{K} j b^{j-1} p_S(j). \end{cases} \tag{14}$$

The discussion about the uniqueness of the solution of (14) is similar to the discussion about the uniqueness of (7), being now the degree of the polynomial $3K - 2$. Let the optimal solution be $(a_{\mathrm{g}}, b_{\mathrm{g}})$ and the optimal scale be $\boldsymbol{\pi}_{\mathrm{g}}$.

It is straightforward to generalize the geometric scales to *bonus* systems with path-dependent rules or to models where the policyholders can move from one company to another; see Centeno and Andrade e Silva (2001, 2002) for the presentation of those systems.

EXAMPLES

In the following examples we consider a system with $K = 15$, $k = 10$, and transition rules given by

$$T(j,r) = \begin{cases} \max(j-1,1) & \text{if } r = 0, \\ \min(j+3r, K) & \text{if } r > 0, \end{cases}$$

that is a policy goes up three classes for each reported claim and goes down one class for each claim free year.

The number of claims $\{M_n\}$ are, conditionally on $\Theta = \theta$, i.i.d. and Poisson distributed with mean θ. The structural distribution $U(\theta)$ is given by a Gamma distribution and

$$\mathbf{w} = [0.30\ 0.12\ 0.10\ 0.09\ 0.08\ 0.07\ 0.07\ 0.06\ 0.06\ 0.05].$$

In the first example we consider a very heterogeneous situation: $E[\Theta] = 0.12$ and $Var[\Theta] = 0.039$. The Bayes scale is given in column 3 of Table 1. As we can see $\pi_B(4) > \pi_B(5)$, which is not acceptable. For the linear scale we obtained $a_L = -0.0404$ and $b_L = 0.0247$, which implies that $\pi_L(1) < 0$. The geometric scale (balanced) is given in the fourth column.

In this situation only the geometric scale is acceptable. The main difference between the Bayes and the geometric scales is found in the first classes of the system. The expected quadratic loss of both systems is very similar. $Q(S) = 0.02229$ for the Bayes scale while $Q(S) = 0.02283$ for the geometric (balanced) scale.

TABLE 1
Optimal Bonus Scales When $E(\Theta) = 0.12$ and $Var(\Theta) = 0.039$

j	$p_S(j)$	$\pi_B(j)$	$\pi_g(j)$
1	0.21120	0.03807	0.01776
2	0.03968	0.05126	0.02288
3	0.04833	0.05392	0.02947
4	0.05116	0.05992	0.03795
5	0.05347	0.05727	0.04888
6	0.06416	0.07145	0.06295
7	0.07634	0.07914	0.08108
8	0.08805	0.08853	0.10443
9	0.10551	0.10607	0.13450
10	0.13892	0.13409	0.17323
11	0.02076	0.25703	0.22311
12	0.02196	0.30592	0.28736
13	0.02319	0.37630	0.37011
14	0.02192	0.48007	0.47668
15	0.03536	0.61228	0.61394

TABLE 2
Optimal Bonus Scales When $E(\Theta) = 0.12$ and $Var(\Theta) = 0.00085$

j	$p_S(j)$	$\pi_B(j)$	$\pi_L(j)$	$\pi_g(j)$
1	0.18430	0.08122	0.07039	0.07030
2	0.03929	0.09510	0.07923	0.07658
3	0.04835	0.09751	0.08807	0.08343
4	0.05343	0.10223	0.09691	0.09089
5	0.05035	0.10131	0.10575	0.09902
6	0.06825	0.10879	0.11459	0.10787
7	0.08207	0.11247	0.12343	0.11752
8	0.09389	0.11603	0.13227	0.12803
9	0.11373	0.12210	0.14111	0.13948
10	0.14929	0.12944	0.14995	0.15195
11	0.02975	0.17816	0.15879	0.16554
12	0.02795	0.19207	0.16763	0.18034
13	0.02458	0.21288	0.17646	0.19647
14	0.01662	0.25541	0.18530	0.21403
15	0.01816	0.28162	0.19414	0.23317

Table 2 shows the results for a more homogeneous case: $E[\Theta] = 0.12$ and $Var[\Theta] = 0.00085$.

In this case the Bayes scale continues to suffer from the same problem. The linear and the geometric scales are both acceptable, but the expected quadratic error of the geometric scale is smaller: 0.00687, 0.00733, and 0.00714 for the Bayes, linear, and geometric scales, respectively.

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