General Quadratic Term Structures of Bond, Futures and Forward Prices

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Abstract

For finite dimensional factor models, the paper studies *general quadratic term structures*. These term structures include as special cases the affine term structures and the Gaussian quadratic term structures, previously studied in the literature. We show, however, that there are other, non-Gaussian, quadratic term structures and derive sufficient conditions for the existence of these *general quadratic term structures* for bond, futures and forward prices.

As *forward prices* are martingales under the T-forward measure, their term structure equation depends on properties of bond prices' term structure. We exploit the connection with the bond prices term structure and show that even in quadratic short rate settings we can have affine term structures for forward prices.

Finally, we show how the study of futures prices is naturally embedded in a study of forward prices and show that the difference between the two prices have to do with the correlation between bond prices and the price process of the underlying to the forward contract and this difference may be deterministic in some (non-trivial) stochastic interest rate settings.

Key words: term structure, bond price, futures price, forward price, affine term structure, quadratic term structure.

JEL Classification: E43, G13

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Contents

List of key notation

- General Quadratic Term Structure (GQTS)

$$\begin{array}{rl} \text{Bond prices:} & \ln H_p(t,z,T) = A_p(t,T) + B_p^*(t,T)z + z^*C_p(t,T)z \\ \text{Forward prices:} & \ln H_f(t,z,T) = A_f(t,T) + B_f^*(t,T)z + z^*C_f(t,T)z \\ \text{Futures prices:} & \ln H_F(t,z,T) = A_F(t,T) + B_F^*(t,T)z + z^*C_F(t,T)z \end{array}$$

$$\text{Any:} \qquad \ln H(t,z,T) = A(t,T) + B^*(t,T)z + z^*C(t,T)z$$

- General Quadratic Boundary Condition (GQBC)

$$\begin{aligned} h(z,T) &= H(T,z,T) \\ \ln h(z,T) &= a(T) + \mathbf{b}^*(T)z + z^*\mathbf{c}(T)z \end{aligned}$$

- General Quadratic Short rate (GQRS)

$$r(t,Z(t)) = Z(t)^*Q(t)Z(t) + g(t)^*Z(t) + f(t)$$

- General Quadratic Q-dynamics of factors

$$dZ(t) = \alpha(t,Z(t))dt + \sigma(t,Z(t))dW(t)$$

for Z m-dimensional and W n-dimensional

$$\begin{aligned} \alpha(t,z) &= \mathbf{d}(t) + \mathbf{E}(t)z \\ \sigma(t,z)\sigma(t,z)^* &= \mathbf{k_0}(t) + \sum_{u=1}^{m} \mathbf{k_u}(t)z_u + \sum_{u,k=1}^{m} z_u\mathbf{g_{uk}}(t)z_k. \end{aligned}$$

and by definition

$$\mathbf{K}(t) = \begin{pmatrix} \mathbf{k_1}(t) \\ \mathbf{k_2}(t) \\ \vdots \\ \mathbf{k_m}(t) \end{pmatrix}, \quad \mathbf{G}(t) = \begin{pmatrix} \mathbf{g_{11}}(t) & \mathbf{g_{12}}(t) & \cdots & \mathbf{g_{1m}}(t) \\ \mathbf{g_{21}}(t) & \mathbf{g_{22}}(t) & \cdots & \mathbf{g_{2m}}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{g_{m1}}(t) & \mathbf{g_{m2}}(t) & \cdots & \mathbf{g_{mm}}(t) \end{pmatrix}.$$

1 Introduction

The literature on term structures has been a rich one, however it has mainly focused on the study of *bond prices*' term structures, much less frequently on the term structure of *futures prices* and rarely on the term structure of *forward prices*.

The *bond prices*' literature is extremely vast from both theoretic and empirical sides. The two most studied classes of term structures are the so-called affine term structures (ATS) and the Gaussian-quadratic term structures (Gaussian-QTS).

Since the well-known papers of Vasiček [23] and of Cox, Ingersoll and Ross [9] on ATS of bond prices, and in particular after some more recent empirical work, there seems to be the belief that ATS are nice from a computational point of view, but lack the flexibility needed to explain market data.[1]

Gaussian-QTS introduced by Longstaff [19] and Beaglehole and Tenney [2] try to introduce the needed flexibility, unfortunately at the cost of imposing a Gaussian dynamics on the stochastic state variables that enter the term structures. Despite this fact the empirical evidence seems to show that these term structures have better fitting properties (see, for instance [1]).

In this paper we show how it is possible to extend the literature to (even more flexible) non-Gaussian quadratic settings.

On the *futures prices*' side the theoretic literature has focused mainly on ATS (extended sometimes to include also jump processes) and on option valuation (see, for example, [5] and [17]). The empirical literature, on the other hand, has pointed out the fact that some term structures of futures prices are also not affine, and specially when dealing with commodity futures, ATS are not flexible enough (see, for instance [13] and [20]).

In contrast to futures prices, term structures of *forward prices* have been almost ignored by the literature. The main exception is Björk and Landén [5] who devote a small section to the study ATS of forward prices in an affine interest rate setting. The main focus of their study is, however, futures prices and no attempt is maid to exploit the connection between the two prices in stochastic settings. This paper is particularly inspired by that study on what concerns forward prices but it generalizes it in three different ways: firstly, by considering a more general stochastic setting of interest rates, secondly by studying general quadratic term structures (that include the ATS as a special case) and finally by exploiting the relationship between the term structures of forward prices and futures prices.

The aims of this study are: the development of a theory for *general quadratic term structures* for bond prices, futures prices (or any other martingale under the risk-neutral measure) and forward prices (or any other martingale under forward measures) and to understand the connections between these three prices in a(stochastic) *general quadratic short rate settings.*

The main contributions of the paper are as follows:

- In Section 2 we present the general framework for the term structures dynamics. We model the entire term structures *a la* Heath-Jarrow-Morton. We define exactly what we mean by *general quadratic term structures* (GQTS) and show how they include both affine term structures (ATS) and the Gaussian-quadratic term structures (Gaussian-QTS) previously studied in the literature as special cases.

[1] The literature on ATS of bond prices is so vast that we refer to recent surveys (for instance, [1], [10], [18], [21]) for an updated list of references.

- In Section 3 we derive sufficient conditions for GQTS of *bond prices*. We show that the generalization to non-Gaussian settings depends on an *a priori* classification of factors. Our main concern is in distinguishing factors that may have a quadratic impact on a GQTS from those that have at most a linear impact. We argue that for *quadratic factors* we need a deterministic volatility structure, but for *linear factors* we can allow for a more general (stochastic) functional form and study the implications for this more general volatility structure in terms of possible correlations between *linear* and *quadratic* factors. Finally, we argue that, by imposing a deterministic volatility structure for the (entire) state variable, the previous Gaussian-QTS literature either impose a (non-needed) add hoc assumption or implicitly assume that all factors are *quadratic*, and in that sense has studied only pure quadratic term structures (PQTS).

- In Section 4 we study GQTS of *futures prices*. We show that the adequate classification of factors for futures prices is similar, in spirit, to the one used when studying bond prices and derive both sufficient conditions and a way to explicitly compute GQTS of futures prices. Finally, we compare these GQTS to the ones previously studied in the futures prices literature.

- Section 5 focus on the study of *forward prices* and their relations with both bond and futures prices. Forward prices are martingales under forward measures and their term structure equation has the particularity that it depends on the term structure of bond prices. We show which term structures of bond prices are compatible with GQTS for forward prices. In particular we prove that ATS of forward prices are consistent with a volatility restricted GQTS for bond prices and QTS of forward prices are consistent with a non-restricted GQTS of bond prices.

  As to the connection with futures prices, we show how the study futures prices term structures is in some sense included in the study of forward prices term structures and analyze the differences between the two term structures. We are able to characterize qualitatively the difference between the two term structures for any *general quadratic short rate* (GQRS) setting and to quantify it in one (still non-trivial) particular case where we show that futures and forward prices differ only by a multiplicative deterministic term (an adjustment factor) for which we give a close form solution.

- Finally, in section 6 we exemplify our theory GQTS for bond, forward and futures prices. Our examples include a number of factor models that have been proposed in the literature, as well as a couple of new models.

2 Basic Concepts

Our main goal is the study of *general quadratic term structures* (GQTS) when those term structures can be expressed as functions of a finite dimensional state process Z. In particular we will look at prices of *term* contracts (with maturity T), like zero-coupon bond prices $p(t,T)$, futures prices $F(t,T)$, and forward prices $f(t,T)$.

For that we consider a m-dimensional (column) vector of factors Z and that, at time t, the bond, futures or forward price with maturity T is given by some deterministic function, so that we have:

$$\begin{aligned} p(t,T) &= H_p(t,Z(t),T) \\ F(t,T) &= H_F(t,Z(t),T) \end{aligned}$$

$$f(t,T) \quad = \quad H_f(t,Z(t),T).$$

In the above functions it is natural to view t and $Z(t)$ as variables and T (the maturity of the prices) as a parameter.

For a fixed t and z, the mappings $T \to H_p(t,z,T)$, $T \to H_F(t,z,T)$ and $T \to H_f(t,z,T)$ are then the term structure of bond, futures and forward prices, respectively.

We now establish exactly what we mean by a general quadratic term structure (GQTS).

Definition 2.1 *The term structure H is said to be **generally quadratic** if it has the form*[2]

$$\ln H(t,z,T) = A(t,T) + B^*(t,T)z + z^*C(t,T)z \tag{1}$$

where $C_{(m\times m)}$ is symmetric and not necessarily different from $\mathbf{0}$, $B_{(m\times 1)}$ and $A_{(1\times 1)}$ are matrices of deterministic and smooth functions.

Note that requiring C to be symmetric is not restrictive. Any non-symmetric quadratic form can always be rewritten in an equivalent symmetric way with the advantage that the symmetric representation is unique. Since later on we will be interested in determining C the uniqueness property will be useful.

In terms of the specific notation of each price we have a GQTS if:

- Bond prices
$$\ln H_p(t,z,T) = A_p(t,T) + B_p^*(t,T)z + z^*C_p(t,T)z \tag{2}$$
- Futures prices
$$\ln H_F(t,z,T) = A_F(t,T) + B_F^*(t,T)z + z^*C_F(t,T)z \tag{3}$$
- Forward prices
$$\ln H_f(t,z,T) = A_f(t,T) + B_f^*(t,T)z + z^*C_f(t,T)z \tag{4}$$

Based on Definition 2.1 we identify a few special cases of a GQTS.[3]

Definition 2.2 *General quadratic term structure as in (1) have the following special cases.*

- ***Pure quadratic term structures*** *(PQTS)*

  Whenever all $Z_i \in Z$ show up in the quadratic term $z^*C(t,T)z$ at least for some t,T, i.e. when

  $\forall\ i, \quad \exists\ t,T \quad s.t. \quad C_i(t,T) \neq \mathbf{0}.$

  In this case all factors in Z have a quadratic impact[4]*, so there will be only* quadratic factors

[2]Whenever we will refer to a property of term structures that does not depend on the specificities of bond, forward of futures prices we will not use any subindex. Instead we will denote the term structure by a generic function $H(t,Z(t),T)$.

[3]$(\cdot)_i$ stands for the ith-row of a matrix.

[4]We note that, in this sense a factor Z_i is has a *quadratic impact* both if z_i^2 turn up in the term structure or if z_iz_j does for some j. In other words, a *quadratic factor* will be any factor showing up in $z^*C_p(t,T)z$.

- ***Quadratic term structures*** *(QTS)*

  Whenever there exist a $Z_i, Z_j \in Z$ such that, Z_i never shows up in the quadratic term $z^*C(t,T)z$, but Z_j does at least for some t,T, i.e.

  $\exists\ i \quad s.t. \quad C_i(t,T) = \mathbf{0} \quad \forall\ t,T \quad \textbf{and} \quad \exists\ t,T \text{ and } j \quad s.t. \quad C_j(t,T) \neq \mathbf{0}.$

  In this case there will be factors having a quadratic impact and factors having at most a linear impact on the term structure. So, there will be both linear factors *and* quadratic factors.

- ***Affine term structures*** *(ATS)*

  Whenever none of the factors $Z_i \in Z$ ever shows up on the quadratic term $z^*C(t,T)z$, i.e.,

  $\forall\ t,T \quad C(t,T) = \mathbf{0}.$

  In that case all factors are linear factors *and have at most a linear impact.*

From the above definition is now obvious that GQTS include non-pure quadratic terms structures (QTS).

Finally, we note that considering the same state process Z for all prices represents no limitation to the analysis. Any irrelevant factor Z_i for, say, the futures prices will not show up in the term structure and we would have $B_{F_i}(t,T) = 0 \quad C_{F_i}(t,T) = \mathbf{0} \quad \forall\ t,T$.

Remark 2.1 *Assuming that the state variable Z in all three term structures is the same represents no limitations for the analysis but has the obvious consequence that some of the factors in Z may be redundant for some of the term structures.*

From the specification of any term contract, the market provide us also with an *a priori* given boundary function h that give us the maturity values, possibly as a function of our factors Z. So we have

$$H(T, Z(T), T) = h(T, Z(T)).$$

The most well know boundary function is that of the bond prices, since, by definition at maturity we have $p(T,T) = 1$, which means that we have $h_p(T,z) = 1$ for all T, hence $A_p(T,T) = 0$, $B_p(T,T) = \mathbf{0}$ and $C_p(T,T) = \mathbf{0}$.

In the case of futures and forward prices the boundary functions h_F and h_f will equal the value process of the underlying (to the futures or forward contract) at maturity[5]. As before this will give us some boundary conditions for $A_F(T,T)$, $B_F(T,T)$,$C_F(T,T)$ or $A_f(T,T)$, $B_f(T,T)$,$C_f(T,T)$, but also will allow us to identify the **natural factors**.

To illustrate this point, let us for simplicity take the case of *ATS of futures prices* (i.e. $C_F(t,T) = \mathbf{0}$ for all t,T, in (3)). In that case we have

$$H_F(T,z,T) = e^{A_F(T,T)+B_F^*(T,T)z} \Leftrightarrow h_F(T,z) = e^{A_F(T,T)+B_F^*(T,T)z}$$

and if we are looking at the term structure of futures prices on, say, a stock with spot price S, then we also necessarily have $h_F(T, Z(T)) = S(T)$. Hence ,

$$S(T) = e^{A_F(T,T)+B_F^*(T,T)Z(T)}$$

[5]In particular, when we are dealing with futures and forward contracts on the same underlying we, obviously have, $h_F(T,z) = h_f(T,z)$.

and we see that the price process S will **not** be a *natural factor* to consider, that is we always have $S \neq Z_i$ for all i). In fact, the only transformation of S that may arise as a natural factor is $\ln S$, still, this will only happen if for some i, $Z_i = \ln S$ and in that case we have $A_F(T,T) = 0$, $B_{Fi}(T,T) = 1$ and $B_{Fj}(T,T) = 0$ for all $j \neq i$.

In a more general case, if $\ln S = a + \mathbf{b}^* z$, *natural factors* are all Z_i factors for which $\mathbf{b}_i \neq 0$ give us a natural factor and the limiting values then become $A_F(T,T) = a$ and $B_F(T,T) = \mathbf{b}$.

Remark 2.2 *Given a specific shape for futures (or forward) prices' term structures the boundary functions h_F (or h_f) will not only provide* ***boundary values****, but also allow us to identify the* ***natural factors****, that must belong to our vector of factors Z.*

Before we go on with the analysis let us set the scene.

Assumption 2.1 *We assume that zero-coupon bond prices are of the form*

$$p(t,T) = H_p(t, Z(t), T) \tag{5}$$

where H_p is a smooth function with the boundary condition

$$H_p(T, z, T) = h_p(T, z) = 1. \tag{6}$$

Furthermore, we assume that futures prices $F(t,T)$, can be written on the following form

$$F(t,T) = H_F(t, Z(t), T) \tag{7}$$

where H_f is also a smooth function with the boundary condition

$$H_F(T, z, T) = h_F(T, z). \tag{8}$$

for an a priori given function h_F.

Likewise, we assume that forward prices $f(t,T)$, can be written on the following form

$$f(t,T) = H_f(t, Z(t), T) \tag{9}$$

where H_f is also a smooth function with the boundary condition

$$H_f(T, z, T) = h_f(T, z). \tag{10}$$

for an a priori given function h_f.

We will also consider that our m-dimensional factor model, under the martingale measure Q, is driven by an n-dimensional Wiener process W.

Assumption 2.2 sets some notation about the dynamics of the factors Z under the martingale measure Q.

Assumption 2.2 *The dynamics of Z, under the Q-measure are given by*

$$dZ(t) = \alpha(t, Z(t))dt + \sigma(t, Z(t))dW(t) \tag{11}$$

where $\alpha(t,z)$ is a $m \times 1$ vector and $\sigma(t,z)$ is a $m \times n$ matrix, and W is a n-dimensional Wiener process.

Note that by considering a n-dimensional Wiener process we implicitly take W to be a column vector of n *independent* scalar Wiener processes. Since we can transform any system with correlated Wiener processes into an equivalent system with uncorrelated ones, this assumption of independence between the elements of W is not restrictive in any sense as stated in the next Remark.[6]

Remark 2.3 *It is always possible to transform a system of* ***correlated*** *Winer processes*

$$dZ(t) = (\cdots)dt + \hat{\sigma}(t, Z(t))d\bar{W}(t) \tag{12}$$

for $\bar{W}$ d-dimensional with possibly correlated elements, into a system of ***independent*** *ones*

$$dZ(t) = (\cdots)dt + \sigma(t, Z(t))dW(t) \tag{13}$$

for W n-dimensional with independent elements.

Furthermore, the following relation describe the connection the two Wiener processes

$$\bar{W} = \delta W$$

where δ is a $(d \times n)$ matrix of deterministic constants such that the length of its rows is one ($\|(\delta)_i\| = 1$) and $\delta\delta^* = \rho$, for ρ the correlation matrix of the elements in $\bar{W}$.

The example below illustrates this in a very simple case.

Example 1 *Consider the following dynamics of two factors Z_1 and Z_2:*

$$d\begin{pmatrix} Z_1(t) \\ Z_2(t) \end{pmatrix} = (\cdots)\,dt + \underbrace{\begin{pmatrix} \sigma_1(t, Z(t)) & 0 \\ 0 & \sigma_2(t, Z(t)) \end{pmatrix}}_{\hat{\sigma}(t,Z(t))} d\begin{pmatrix} \bar{W}_1(t) \\ \bar{W}_2(t) \end{pmatrix}$$

with $d\bar{W}_1(t)d\bar{W}_2(t) = \rho dt$.

This same system can equivalently be rewritten as

$$d\begin{pmatrix} Z_1(t) \\ Z_2(t) \end{pmatrix} = (\cdots)\,dt + \underbrace{\begin{pmatrix} \sigma_1(t, Z(t)) & 0 \\ \rho\sigma_2(t, Z(t)) & \sigma_2(t, Z(t))\sqrt{1-\rho^2} \end{pmatrix}}_{\sigma(t,Z(t))} d\begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix}$$

where we have $dW_1(t)dW_2(t) = 0$.

For a better understanding of some future results it is important to stress now that in our formulation two correlated factors will be driving by *at least* on common scalar Wiener process. Recovering Example 1, both Z_1 and Z_2 are driving by W_1 (the common Wiener process in this case).

Having defined the exact setup we can go on with the analysis.

Applying Itô to equation (5), (7) and (9) and using the dynamics for the factors in (11) we can find the dynamics of the zero-coupon bond prices, futures prices and forward prices under the martingale measure Q. Lemma 2.1 give these dynamics

[6] For a textbook treatment of this equivalence, see for example, Björk [3].

Lemma 2.1 *Assume that the dynamics of Z are as in (11), then,*

- *if the zero-coupon bond prices are given by (5), their Q-dynamics are described by*

$$dp(t,T) = \left\{ \frac{\partial H_p}{\partial t} + \sum_{i=1}^{m} \frac{\partial H_p}{\partial z_i}\alpha_i + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 H_p}{\partial z_i \partial z_j}\sigma_i\sigma_j^* \right\} dt + \sum_{i=1}^{m} \frac{\partial H_p}{\partial z_i}\sigma_i dW(t)$$

- *if the futures prices are given by (7), their Q-dynamics are (likewise) described by*

$$dF(t,T) = \left\{ \frac{\partial H_F}{\partial t} + \sum_{i=1}^{m} \frac{\partial H_F}{\partial z_i}\alpha_i + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 H_F}{\partial z_i \partial z_j}\sigma_i\sigma_j^* \right\} dt + \sum_{i=1}^{m} \frac{\partial H_F}{\partial z_i}\sigma_i dW(t)$$

- *if the forward prices are given by (9), their Q-dynamics are (likewise) described by*

$$df(t,T) = \left\{ \frac{\partial H_f}{\partial t} + \sum_{i=1}^{m} \frac{\partial H_f}{\partial z_i}\alpha_i + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 H_f}{\partial z_i \partial z_j}\sigma_i\sigma_j^* \right\} dt + \sum_{i=1}^{m} \frac{\partial H_f}{\partial z_i}\sigma_i dW(t)$$

where $*$ stands for transpose, $(.)_i$ for the i-th row of a vector/matrix. All partial derivatives should be evaluates at $(t, Z(t), T)$, and all α_i and σ_i at $(t, Z(t))$.

Using the above Lemma and the fact that zero-coupon bonds are traded assets and hence have, under the martingale measure Q, the risk-free short rate r as its local rate of return, we recover the following standard term structure equation for bond prices.

Result 2.2 *Suppose the zero-coupon bond prices are given by (5) and Assumption 2.2 hold. Then H_p satisfies the following differential equation*

$$\left\{ \begin{array}{rcl} \frac{\partial H_p}{\partial t} + \sum_{i=1}^{m} \frac{\partial H_p}{\partial z_i}\alpha_i + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 H_p}{\partial z_i \partial z_j}\sigma_i\sigma_j^* & = & rH_p \\ H_p(T,z,T) & = & 1 \end{array} \right. \tag{14}$$

where $*$ stands for transpose, $(.)_i$ for the i-th row of a vector/matrix. All partial derivatives should be evaluates at (t, z, T), and all α_i and σ_i at (t, z).

In order to derive the term structure equation of futures and forward prices, we now recall, a well-known fact (see for instance [3]).

Result 2.3 *Given any T-claim $\mathcal{X}$ futures prices are martingales under the risk neutral measure, Q, and forward prices are martingales under the T-forward measure, Q^T.*

$$F(t,T)) = E_t^Q\left[\mathcal{X}\right] \qquad\qquad f(t,T) = E_t^T\left[\mathcal{X}\right].$$

Using the fact that futures prices are martingales under the risk-neutral martingale measure Q, i.e.

$$F(t,T) = H_F(t, Z(t), T) = E_t^Q\left[h_F(T, Z(T))\right],$$

we can also recover the term structure equation for futures prices in [5].

Result 2.4 ***(Björk and Landén)*** *Suppose the future prices are given by (7) and Assumption 2.2 holds. Then H_F satisfies the following differential equation*

$$\begin{cases} \dfrac{\partial H_F}{\partial t} + \displaystyle\sum_{i=1}^{m} \dfrac{\partial H_F}{\partial z_i}\alpha_i + \dfrac{1}{2}\sum_{i,j=1}^{m} \dfrac{\partial^2 H_F}{\partial z_i \partial z_j}\sigma_i\sigma_j^* & = & 0 \\ \\ H_F(T,z,T) & = & h_F(T,z) \end{cases} \tag{15}$$

where $*$ stands for transpose, $(.)_i$ for the i-th row of a vector/matrix. All partial derivatives should be evaluates at (t,z,T), and all α_i and σ_i at (t,z).

In our basic setup we can use the above Result 2.3 to write

$$f(t,T) = H_f(t,Z(t),T) = E_t^T\left[h_f(T,Z(T))\right]$$

and get a differential equation for the forward prices function H_f. Proposition 2.5 give us that result.

Proposition 2.5 *Suppose the zero-coupon bond prices are given by (5), the forward prices are given by (9) and Assumption 2.2 holds. Then H_f satisfies the following differential equation*

$$\begin{cases} \dfrac{\partial H_f}{\partial t} + \displaystyle\sum_{i=1}^{m} \dfrac{\partial H_f}{\partial z_i}\alpha_i + \dfrac{1}{2}\sum_{i,j=1}^{m} \dfrac{\partial^2 H_f}{\partial z_i \partial z_j}\sigma_i\sigma_j^* + \sum_{i,j=1}^{m} \dfrac{\partial H_f}{\partial z_i}\sigma_i\left(\dfrac{\partial H_p}{\partial z_j}\dfrac{1}{H_p}\sigma_j^*\right) & = & 0 \\ \\ H_f(T,z,T) & = & h_f(T,z) \end{cases} \tag{16}$$

where $*$ stands for transpose, $(.)_i$ for the i-th row of a vector/matrix. All partial derivatives should be evaluates at (t,z,T), and all α_i and σ_i at (t,z).

Proof. Using the dynamics under Q of Lemma 2.1 and noting that in this case the bond prices volatility, σ_p is given by

$$\sigma_p^* = \sum_{j=1}^{m} \frac{\partial H_p}{\partial z_j}\frac{1}{H_p}\sigma_j^*$$

we can change to the Q^T measure using

$$dW = \sum_{j=1}^{m} \frac{\partial H_p}{\partial z_j}\frac{1}{H_p}\sigma_j^* dt + dW^T.$$

The dynamics under the Q^T forward measure of $f(t,T)$ become

$$\begin{aligned} df(t,T) \quad = \quad & \left\{\frac{\partial H_f}{\partial t} + \sum_{i=1}^{m} \frac{\partial H_f}{\partial z_i}\alpha_i + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 H_f}{\partial z_i \partial z_j}\sigma_i\sigma_j^* + \sum_{i,j=1}^{m} \frac{\partial H_f}{\partial z_i}\sigma_i\left(\frac{\partial H_p}{\partial z_j}\frac{1}{H_p}\sigma_j^*\right)\right\} dt \\ & + \sum_{i=1}^{m} \frac{\partial H_f}{\partial z_i}\sigma_i dW^T(t), \end{aligned}$$

and the result follows from the fact that the forward prices with maturity T are martingales under the Q^T forward measure, so its drift term (which shows up in the LHS of equation (16)) must be zero. □

Comparing equations (14), (15) and (16) one soon realizes that *forward prices* have the more complex term structure equation. Indeed, the term structure equation for forward prices does not depend only on the properties of its own term structure H_f and of the factors dynamics (trough its drift α and volatility σ), but also on properties of bond prices term structures. This dependence shows up in the last term of (16): $\sum_{i,j=1}^{m} \frac{\partial H_f}{\partial z_i} \sigma_i \left(\frac{\partial H_p}{\partial z_j} \frac{1}{H_p} \sigma_j^* \right)$.

This is however not surprising, in fact it goes in line with our intuition: since bond prices are the *numeraire* under forward measures and forward prices are martingales under those measures, bond prices term structures should, of course, play some role in the forward prices term structure.

Indeed, any study of forward prices term structures need to be done under some interest rate setting and the difference between futures and forward prices results from the role that bond prices have on the forward prices term structure. One of our goals in this paper is to study GQTS of forward prices in an *as general as possible* interest rate setting. For reasons that will become clear later on, that setting will be what we call a *general quadratic short rate* (GQSR) setting .

Since results from *bond prices* term structures will be needed when dealing with forward prices and the because the intuition that results from the analysis of *futures prices* will be extremely valuable in the study of forward prices, we start by studying bond and futures prices.

The next two sections handle, respectively, bond and futures prices and can be read independently of one another.

3 Bond Prices Term Structures in a GQSR

3.1 General Setting

The term structure of bond prices have been the main object of study among the term structure literature. Both ATS and Gaussian-QTS models have been exploited: Vasiček [23] Cox, Ross and Ingersoll [9], Brown and Schaefer [8] and Duffie and Kan [11] on the ATS side, and Longstaff [19], Beaglehole and Tenney [2], Jamshidian [17], Boyle and Tian [6] and Gombani and Runggaldier [15] on the Gaussian-QTS side, are among the most important[7].

In this section we will extend the existing results on quadratic term structures (QTS) of bond prices. We will show that QTS have not been studied in the most general possible setting and how it is possible to extend it to include non-Gaussian quadratic term structures.

The generalization results from an *a priori* distinction between types of factors, those with at most a linear impact on the bond prices term structure (*linear factors*) from those that may have a quadratic impact (*quadratic factors*), and from allowing a more flexible factor dynamics for the first type of factors.

[7] For survey studies and an almost exhaustive list of references in the subject see Ahn, Dottmar and Gallant [1], Dai and Singleton [10], Leippold and Wu [18] or Rebonato [21].

Our results will coincide with those of previous literature whenever there is (or we assume that there is) only one type of factors. When all factors are *quadratic factors* (or assumed to be so), we show that indeed a Gaussian setting for the factor dynamics is the most general setting, and we recover the results on Gaussian-QTS. In that sense, one can argue, that the previous literature is on *pure* term structures.[8] When all factors are *linear factors*, we recover the results on ATS.

We now recall that the term structure of bond prices, H_p in (5), is said to be **generally quadratic** if we have

$$\ln H_p(t,z,T) = A_p(t,T) + B_p(t,T)^* z + z^* C_p(t,T) z \tag{17}$$

where $C_{p_{(m\times m)}}$, symmetric, and not necessarily different from $\mathbf{0}$, $B_{p_{(m\times 1)}}$ and $A_{p_{(1\times 1)}}$ are matrices of smooth and deterministic functions.

We will present the exact *sufficient conditions* for existence of a GQTS for bond prices, and provide an explicit way to compute that term structure. As in any term structure study, these will be conditions on the functional form of the short rate and on the factor dynamics.

Let us, therefore, start by establishing some notation for the short rate, defining what we call a generally quadratic short rate (GQSR) setting.

Definition 3.1 *A* ***general quadratic short rate*** *setting is defined by a short rate, r, with the following functional quadratic form*

$$r(t,Z(t)) = Z(t)^* Q(t) Z(t) + g(t)^* Z(t) + f(t) \tag{18}$$

where $Q(t)_{(m\times m)}$, symmetric[9], and not necessarily different from $\mathbf{0}$, $g(t)_{(m\times 1)}$ and $f(t)_{1\times 1}$ are matrices of smooth and deterministic functions.

The next preliminary Lemma tell us that whenever we have a GQTS for bond prices we also have a GQSR for the short rate. Or, to put it differently, that a GQSR is a *necessary condition* for a GQTS of bond prices.

Lemma 3.1 *If we have a GQTS for bond prices, then we have GQSR for the short rate, and the following connection exists between the functional matrices in (18) and (17),*

$$Q(t) = -\frac{\partial C_p}{\partial T}(t,t) \quad g(t) = -\frac{\partial B_p}{\partial T}(t,t) \quad f(t) = -\frac{\partial A_p}{\partial T}(t,t).$$

Proof. Recall from (5) that we have $p(t,T) = H_p(t,Z(t),T)$. Then we can use the relation between bond prices $p(t,T)$ and instantaneous forward rates of interest, $f_r(t,T)$, to conclude

$$f_r(t,T) = -\frac{\partial \ln H_p}{\partial T}(t,Z(t),T).$$

[8]Indeed, the existing results on QTS do not include the results on ATS as a special case (all studies impose a *deterministic* matrix $\sigma\sigma^*$ which does not have as a special case the *linear in z* condition we are used from the ATS literature). This could lead an unaware reader to the puzzling conclusion that ATS are not QTS with $C_p(t,T) = \mathbf{0}$. The puzzle gets solved when one realizes that previous studies on QTS are indeed on PQTS and, by definition ATS are not a particular case of PQTS (since in this case we have by construction $C_p(t,T) \neq \mathbf{0}$). GQTS will include as special cases ATS, QTS (and hence PQTS), and are, thus, also in this sense the natural object to study.

[9]Recall, from the previous section arguments, that the symmetry assumption both for C_p in (17) or for Q in (18) is not restrictive in any way since any non-symmetric quadratic form can always be rewritten is an equivalent symmetric way that has the advantage of being unique.

Hence for GQTS of bond prices as in (17) we have

$$f_r(t,T) = -\left(\frac{\partial A_p}{\partial T}(t,T) + \frac{\partial B_p}{\partial T}(t,T)^* Z(t) + Z(t)^* \frac{\partial C_p}{\partial T}(t,T) Z(t)\right).$$

Since we also have $r(t) = f_r(t,t)$, then

$$r(t) = -\frac{\partial A_p}{\partial T}(t,t) - \frac{\partial B_p}{\partial T}(t,t)^* Z(t) - Z(t)^* \frac{\partial C_p}{\partial T}(t,t) Z(t),$$

and by comparison with the functional form in equation (18) one realizes that

$$Q(t) = -\frac{\partial C_p}{\partial T}(t,t) \quad g(t) = -\frac{\partial B_p}{\partial T}(t,t) \quad f(t) = -\frac{\partial A_p}{\partial T}(t,t). \qquad \square$$

We will soon show that having a GQSR is also one of the *sufficient conditions* for a GQTS of bond prices, but not the only one and, as one would guess, the others are on the functional form of $\alpha(t,z)$ and $\sigma(t,z)\sigma(t,z)^*$ in the factor dynamics (11).

Before we can present our main result we need two more definitions.

Definition 3.2 *The vector of factors Z is said to have* ***general quadratic Q-dynamics*** *if $\alpha(t,z)$ and $\sigma(t,z)$ in (11) are such that*

$$\alpha(t,z) = \mathbf{d}(t) + \mathbf{E}(t)z \tag{19}$$

$$\sigma(t,z)\sigma(t,z)^* = \mathbf{k_0}(t) + \sum_{u=1}^{m} \mathbf{k_u}(t) z_u + \sum_{u,k=1}^{m} z_u \mathbf{g_{uk}}(t) z_k \tag{20}$$

where $\mathbf{d}$, $\mathbf{E}$, $\mathbf{k_0}$, $\mathbf{k_u}$ and $\mathbf{g_{uk}}$ for $\mathbf{u},\mathbf{k} = \mathbf{1},\cdots,\mathbf{m}$ are matrices of deterministic smooth functions.

We also define for future reference

$$\mathbf{K}(t) = \begin{pmatrix} \mathbf{k_1}(t) \\ \mathbf{k_2}(t) \\ \vdots \\ \mathbf{k_m}(t) \end{pmatrix}, \quad \mathbf{G}(t) = \begin{pmatrix} \mathbf{g_{11}}(t) & \mathbf{g_{12}}(t) & \cdots & \mathbf{g_{1m}}(t) \\ \mathbf{g_{21}}(t) & \mathbf{g_{22}}(t) & \cdots & \mathbf{g_{2m}}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{g_{m1}}(t) & \mathbf{g_{m2}}(t) & \cdots & \mathbf{g_{mm}}(t) \end{pmatrix}. \tag{21}$$

In a setting with GQSR and Z with general quadratic Q-dynamics we can classify the factors as follows.

Definition 3.3 *Given a GQSR as in (18) and the general quadratic Q-dynamics for Z (so that (19) and (20) hold for $\alpha(t,z)$ and $\sigma(t,z)$ in (11)).*

- *Z_i is a* ***$\mathbf{Z^{(q)}}$-factor*** *if it satisfies* ***at least one*** *of the following requirements:*
  - *(i) it has a quadratic impact on the short rate of interest $r(t)$, i.e., there exists t such that $Q_i(t) \neq \mathbf{0}$;*
  - *(ii) it has a quadratic impact on the functional form of the matrix $\sigma(t,z)\sigma(t,z)^*$, i.e., there exist k and t such that $\mathbf{g_{ik}}(t) \neq \mathbf{0}$;*

(iii) it affects the drift term of the factors satisfying (i) or (ii), i.e., for Z_j satisfying (i) or (ii) we have $\mathbf{E}_{ji}(t) \neq 0$, at least for some t.[10]

- *Z_i is a $\mathbf{Z^{(l)}}$**-factor** if it does not satisfy (i)-(iii).*

The reader may wonder about the motivation for the classification of factors in Definition 3.3, and may guess that it should somehow be related to different impacts on bond prices term structures. After presenting the main result on GQTS for bond prices we will be able to give that motivation and to show that, indeed, this classification has to do with the impact that the factors may end up having on the bond prices term structure.

Before that, however, we want to show the implications of Definition 3.3 in terms of the shapes of the matrices Q, $\mathbf{E}$ and $\mathbf{G}$ and to stress that the shapes (22) bellow are **not** the result of any assumption, rather they hold *by definition.*

Remark 3.1 *We note that given Definition 3.3,*

- *it is always possible to reorder the vector of factors Z and its correspondent value vector, so that we have*
$$Z = \begin{pmatrix} Z^{(q)} \\ Z^{(l)} \end{pmatrix} \qquad z = \begin{pmatrix} z^{(q)} \\ z^{(l)} \end{pmatrix}.$$
- *with this reordering of factors we have, **by definition**, the following shapes for Q in (18) and for $\mathbf{E}$ and $\mathbf{G}$ in (19) and (21), respectively*

$$Q(t) = \begin{pmatrix} Q^{(qq)}(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \mathbf{E}(t) = \begin{pmatrix} \mathbf{E^{(qq)}}(t) & \mathbf{0} \\ \mathbf{E^{(lq)}}(t) & \mathbf{E^{(ll)}}(t) \end{pmatrix} \quad \mathbf{G}(t) = \begin{pmatrix} \mathbf{G^{(qq)}}(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{22}$$

3.2 Main result on Bond prices

Theorem 3.2 *Suppose that Assumption 2.1 and 2.2 are in force. Furthermore suppose that we are in a GQSR, so that (18) hold and that Z has a general quadratic Q-dynamics, (i.e., that α and σ from the factor dynamics (11), satisfy (19)-(21)).*

Finally assume that the factors are reordered as $Z = \begin{pmatrix} Z^{(q)} \\ Z^{(l)} \end{pmatrix}$ (using Definition 3.3), and that the following restrictions apply to $\mathbf{K}$ and $\mathbf{G}$ in (21):

$$\mathbf{k_u}(t) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{k_u^{(ll)}}(t) \end{pmatrix} \quad \forall\, t, \forall\, \mathbf{u} \tag{23}$$

$$\mathbf{g_{uk}}(t) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{g_{uk}^{(ll)}}(t) \end{pmatrix} \quad \forall\, t \text{ and } \forall\, \mathbf{u}, \mathbf{k} \text{ s.t. } z_u, z_k \in \mathbf{z^{(q)}}. \tag{24}$$

[10] We see that in the most general drift dynamics, that is, when the drift of all factors depend linearly in all other factors, as long as there is one $Z^{(q)}$-factor, all factors would also be of the type $Z^{(q)}$. So, one can say that a *necessary condition* for existence of more than one type of factors is that the drift of factors meeting (i) and (ii) in Definition 3.3 does not depend on all remaining factors.

Then the term structure of bond prices is generally quadratic, i.e. H_p from (5) can be written on the form (17) and A_p, B_p and C_p can be obtained by solving the following system of ordinary differential equations.

$$\begin{cases} \dfrac{\partial A_p}{\partial t} + \mathbf{d}(t)^* B_p + \dfrac{1}{2} B_p^* \mathbf{k_0}(t) B_p + tr\left\{ (C_p) \mathbf{k_0}(t) \right\} = f(t) \\ A_p(T,T) = 0 \end{cases} \tag{25}$$

$$\begin{cases} \dfrac{\partial B_p}{\partial t} + \mathbf{E}(t)^* B_p + 2C_p \mathbf{d}(t) + \dfrac{1}{2} \bar{B_p}^* \mathbf{K}(t) B_p + 2C_p \mathbf{k_0}(t) B_p = g(t) \\ B_p(T,T) = 0 \end{cases} \tag{26}$$

$$\begin{cases} \dfrac{\partial C_p}{\partial t} + C_p \mathbf{E}(t) + \mathbf{E}(t)^* C_p + 2C_p \mathbf{k_0}(t) C_p + \dfrac{1}{2} \bar{B_p}^* \mathbf{G}(t) \bar{B_p} = Q(t) \\ C_p(T,T) = 0 \end{cases} \tag{27}$$

where C_p has the special form $C_p = \begin{pmatrix} C_p^{(qq)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ and A_p, B_p and $C_p^{(qq)}$ should be evaluated at (t,T).

$\mathbf{E}$, $\mathbf{d}$, $\mathbf{k_0}$, $\mathbf{K}$, $\mathbf{G}$ *are the same as in (19)-(21), and*

$$\bar{B_p} = \begin{pmatrix} B_p & 0 & \cdots & 0 \\ 0 & B_p & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & B_p \end{pmatrix}. \tag{28}$$

Proof. We need to show that $H_p(t,z,T)$ from (17) where A_p, B_p and C_p solve (25)-(27), solves the PDE (14) that uniquely characterizes the bond prices in this setting.

Taking partial derivatives

$$\begin{aligned} \frac{\partial H_p}{\partial t} &= \left[\frac{\partial A_p}{\partial t} + \frac{\partial B_p}{\partial t}^* z + z^* \frac{\partial C_p}{\partial t} z \right] H_p \\ \frac{\partial H_p}{\partial z_i} &= \left[B_{p_i} + C_{p_i} z + C_{p_i}^* z \right] H_p \\ \frac{\partial^2 H_p}{\partial z_i \partial z_j} &= \left[C_{p_{ij}} + C_{p_{ji}} \right] H_p + \left[B_{p_i} + C_{p_i} z + C_{p_i}^* z \right] \left[B_{p_j} + C_{p_j} z + C_{p_j}^* z \right] H_p \end{aligned}$$

so the PDE (14) reduces in this case to

$$\begin{cases} \dfrac{\partial A_p}{\partial t} + \dfrac{\partial (B_p)}{\partial t} z + z^* \dfrac{\partial C_p}{\partial t} z + \displaystyle\sum_{i=1}^{m} \left[(B_p)_i + (C_p)_i z + (C_p^*)_i z \right] \alpha_i \\ \quad + \dfrac{1}{2} \displaystyle\sum_{i,j=1}^{m} \left[(B_p)_i + (C_p)_i z + (C_p^*)_i z \right] \sigma_i \sigma_j^* \left[(B_p)_j + (C_p)_j z + (C_p^*)_j z \right] \\ \qquad + \dfrac{1}{2} \displaystyle\sum_{i,j=1}^{m} \left((C_p)_{ij} + (C_p)_{ji} \right) \sigma_i \sigma_j^* = r \\ \exp\left\{ A_p(T,T) + B_p(T,T)^* z + z^* C_p(t,T) z \right\} = 1 \end{cases} \tag{29}$$

Substituting r, α and $\sigma\sigma^*$ from (18), (19) and (20), respectively, and using the restrictions (23)-(24), the PDE (29) becomes always a separable equation equivalent to (25)-(27).

If all $Z_i \in Z^{(q)}$, we know, that $\mathbf{K}(t) = \mathbf{0}$ and $\mathbf{G}(t) = \mathbf{0}$ for all t. Hence, equation (29) becomes separable up to quadratic terms of z. For this case we note that there is more than one way to write the quadratic terms of (29) in vector notation, but only one way compatible with our non-restrictive assumption of a symmetric matrix C_p. Symmetry of C_p also allow us to get a simplified version for the ODE (27). It is easy to show that given the shapes (22) for the matrices Q, $\mathbf{E}$ and $\mathbf{G}$, $C_p^{(ll)}(t,T) = \mathbf{0}$ and $C_p^{(ql)}(t,T) = C_p^{(lq)*}(t,T) = \mathbf{0}$ for all t,T always solve (27).

If there exist $Z^{(l)}$ factors, equation (29) will still be separable but it will have terms up to order four in the state variable z. The third and fourth order terms in z will result from the terms $\frac{\partial^2 H_p}{\partial z_i \partial z_j}$ where both $Z_i, Z_j \in Z^{(l)}$, so will only affect $C_p^{(ll)}$ and $C_p^{(ll)}(t,T) = \mathbf{0}$ for all t,T is always a solution for those conditions. Lower order terms will depend on the entire matrix C_p, however the restrictions imposed on the matrices Q, $\mathbf{E}$ and $\mathbf{G}$ in (22) once again guarantee that $C_p^{(ll)}(t,T) = \mathbf{0}$ and $C_p^{(ql)}(t,T) = C_p^{(lq)*}(t,T) = \mathbf{0}$ for all t,T are also always a solution to (27). □

Theorem 3.2 give us both sufficient conditions for GQTS of bond prices - first part, and an explicit way to compute them- second part. Since these sufficient conditions are somehow hidden in previous definitions and the intuition lost in the formulas we state this conditions verbally.

The **sufficient conditions** for a GQTS of bond prices can be stated in the following way.

- To have an interest rate model that guarantees a GQSR (as in (18)).
- To have Z-factors with general quadratic Q-dynamics (as in (19)-(20)) **restricted** to guarantee deterministic $\sigma^{(q)}\sigma^{(q)*}$, $\sigma^{(q)}\sigma^{(l)*}$ (and hence, by symmetry also $\sigma^{(l)}\sigma^{(q)*}$). That is, that the volatility structure of $Z^{(q)}$-factors, or of any factors correlated with them, is deterministic.

Using these sufficient conditions (and the implications of Definition 3.3 *per se*), the next Remark restated in a more visual way the first part of Theorem 3.2.

Remark 3.2 *Assume that an short rate model can be described by*

$$r(t, Z(t)) = Z(t)^* \begin{pmatrix} Q^{(qq)}(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} Z(t) + g(t)^* Z(t) + f(t),$$

and that the factors Z have Q-dynamics given by

$$dZ(t) = \alpha(t,z)dt + \sigma(t, Z(t))dW(t),$$

where α and $\sigma\sigma^*$ satisfy

$$\alpha(t,z) = \mathbf{d}(t) + \begin{pmatrix} \mathbf{E^{(qq)}}(t) & \mathbf{0} \\ \mathbf{E^{(lq)}}(t) & \mathbf{E^{(ll)}}(t) \end{pmatrix} z$$

$$\sigma(t,z)\sigma(t,z)^* \quad = \quad \begin{pmatrix} \mathbf{k_0^{(qq)}}(t) & \mathbf{k_0^{(ql)}}(t) \\ \mathbf{k_0^{(lq)}}(t) & \mathbf{k_0^{(ll)}}(t) + \sum_{i=1}^{m} \mathbf{k_u^{(ll)}}(t) z_u + \sum_{z_u, z_k \in \mathbf{z^{(q)}}} z_u \mathbf{g_{uk}^{(ll)}}(t) z_k \end{pmatrix} \tag{30}$$

Then we have a GQTS for bond prices as in (17).

3.2.1 Understanding $Z^{(q)}$ and $Z^{(l)}$ factors

We are now under the conditions of better understanding the classification of factors in Definition 3.3. Let us first formalize the kind of impact a factor can have in a GQTS.

Definition 3.4 *A given factor Z_i is called **quadratic** and is said to have a **quadratic impact** on a GQTS*

$$\ln H(t,z,T) = A(t,T) + B(t,T)z + z * C(t,T)z$$

if

$$\exists\ t, T \ s.t.\ C_{p_i}(t,T) \neq \mathbf{0}. \tag{31}$$

*A factor Z_j is called **linear** and is said to have at most a **linear impact** on a GQTS if it does not satisfy (31).*[11]

The impact of $Z^{(l)}$-factors is an immediate consequence of Theorem 3.2 that follows from $C_p^{(ll)}(t,T) = \mathbf{0}$ and $C_p^{(ql)}(t,T) = C_p^{(lq)*}(t,T) = \mathbf{0}$ for all t,T.

Corollary 3.3 ***(Linear factors)*** *The $Z^{(l)}$-factors are linear factors in a GQTS of bond prices.*

It is, thus, particularly comforting to note that since $\sigma\sigma^*$ can only depend quadraticly on $Z^{(q)}$-factors (check (30)), in models where there are only $Z^{(l)}$- factors the quadratic term of $\sigma\sigma^*$ disappear and we recover the well known result of a linear $\sigma\sigma^*$ for ATS.

It would now be interesting to show that the $Z^{(q)}$-factors actually have a quadratic impact. As we will soon see this seems to be the case for almost all models, but to show that for any $Z_i \in Z^{(q)}$, $C_{p_i}(t,T) \neq \mathbf{0}$ for some t,T is not a trivial task. Non-arbitrage arguments give us, however an easy partial answer.

Lemma 3.4 *A factor $Z_i \in Z^{(q)}$ for which $Q_i(t) \neq 0$ at least for some t has a quadratic impact in the bond prices term structure.*

Proof. It follows immediately from Lemma 3.1 that $Q_{ij}(t) \neq 0 \quad \Rightarrow \quad C_{p_{ij}}(t,T) \neq 0,\ \forall\ T.$ □

For those $Z^{(q)}$ factors that do not affect quadraticly the short rate ($Z_i \in Z^{(q)}$ with $Q_i(t) = \mathbf{0}$ for all t) the answer is, however, less trivial.

[11] One could argue that we could skip the "at most" expression, because a factor that does have a quadratic impact and that does not have a linear impact on a GQTS does not show at all in the term structure and can be taken out of the vector Z. This argument is indeed true, but for more complex term structures, such those of forward prices, we will need to consider the possibility of including factors in Z that do not show up in bond prices term structures but they do in forward prices term structures, and still compute bond prices term structures using the techniques here presented so our definitions have to include such a situation.

They are the factors that satisfy only (ii) or (iii) in Definition 3.3 and their concrete impact can only be accessed by studying the solution of the ODE (27). Definition 3.3 guarantees that for all such $Z_i \in Z^{(q)}$ and at least for some t, T, the ODE $C_{p_i}(t,T)$ is a Riccati equation with at least one non-zero independent term and that thus, in general, $C_{p_i}(t,T) \neq \mathbf{0}$. However, under some pathological situations when a factor Z_i affects quadraticly the volatility for many linear factors, and/or when it affects the drift of many quadratic factors, it could (in theory) be that the many non-zero independent terms on the Riccati equation for $C_{p_i}(t,T)$ would cancel each other. In practice this situation is only likely to occur by including in the model redundant factors or pathological constructions. The next assumption imposes the needed regularity condition on the model that guarantee that any $Z^{(q)}$-factor will, indeed have a quadratic impact.

Assumption 3.1 *For any i, k, t, T such that*

$$E_{ji}(t) C_{p_{jk}}(t,T) \neq 0 \quad \textit{for some } j$$

or

$$B_{p_u}(t,T) \mathbf{g_{ik}}_{uu}(t) B_{p_u}(t,T) \neq 0 \quad \textit{for some } u$$

the following ***regularity condition***[12] *hold*

$$\sum_j E_{ji}(t) C_{jk}(t,T) + \sum_{u,v} B_{p_u}(t,T) \mathbf{g_{ik}}_{uv}(t) B_{p_v}(t,T) \neq 0.$$

Proposition 3.5 ***(Quadratic factors)*** *As long as we exclude from the analysis any redundant factors and under the regularity condition of Assumption 3.1, the $Z^{(q)}$-factors are quadratic factors in a GQTS of bond prices.*

Proof. Lemma 3.4 proves that any $Z_i \in Z^{(q)}$ such that $Q_i(t) \neq \mathbf{0}$ at least for some t has a quadratic impact.

It remains to show that for all $Z_i \in Z^{(q)}$ for which $Q_i(t) = \mathbf{0}$ for all t (all remaining $Z^{(q)}$-factors), we also have, at least for some t, T, $C_{p_i}(t,T) \neq \mathbf{0}$ as a solution to (27). It follows from Definition 3.3 that, as long as we do not consider the redundant factors, for such Z_i there will always be a k, t, T such that $E_{ji}(t) C_{p_{jk}}(t,T) \neq 0$ and/or $B_{p_u}(t,T) \mathbf{g_{ik}}_{uu}(t) B_{p_u}(t,T) \neq 0$ for some u.[13] The regularity condition of Assumption 3.1 then guarantees that for that k, t, T, $C_{p_{ik}}(t,T)$ solves a Riccati equation with a non-zero independent term and that, thus $C_{p_{ik}}(t,T) \neq 0$. This guarantees the quadratic impact of Z_i according to Definition 3.4. □

3.2.2 Actually solving the system of ODEs

To actually solve the system of ODEs in (25)-(27) may seem to be a challenging task, especially because equations (26)-(27) are interrelated matrix Riccati equations[14] for B_p and C_p.

[12] For any model we could think of this regularity condition is satisfied. Nonetheless and since it is only needed to guarantee that all $Z^{(q)}$-factors have a quadratic impact, one could just ignore it and think of the $Z^{(q)}$-factors as the factors that will, *in principle*, have a quadratic impact.

[13] If Z_u is a redundant (linear) factor $\mathbf{g_{ik}}_{uu}(t) \neq 0$ does not guarantee $B_{p_u}(t,T) \mathbf{g_{ik}}_{uu}(t) B_{p_u}(t,T) \neq 0$ at least some T and hence, for this pathological case, the quadratic impact of Z_i cannot be guaranteed.

[14] For and interesting note on the importance of Riccati equations in Mathematical Finance, see Boyle, Tian and Guan [7].

The good news is that, given the very special shapes of the matrices Q, $\mathbf{E}$ and $\mathbf{G}$ in (22) and the fact that we have $C_p^{(ll)}(t,T) = \mathbf{0}$ and $C_p^{(ql)}(t,T) = C_p^{(ql)*}(t,T) = \mathbf{0}$ for all t, T, they turn out to be two *independent* Riccati equations and the strategy to solve them is as follows:

- Note that it is possible to split the vector equation (26) for B_p into two vector equations for $B_p^{(q)}$ and $B_p^{(l)}$, using $B_p = \begin{pmatrix} B_p^{(q)} \\ B_p^{(l)} \end{pmatrix}$ and $g = \begin{pmatrix} g^{(q)} \\ g^{(l)} \end{pmatrix}$.

  Moreover, replacing B_p by $B_p = \begin{pmatrix} B_p^{(q)} \\ B_p^{(l)} \end{pmatrix}$, C_p by $C_p = \begin{pmatrix} C_p^{(qq)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, Q by $Q = \begin{pmatrix} Q^{(qq)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ and simplifying, we can also write the matrix equation (27) for C_p in terms of $B_p^{(q)}$, $B_p^{(l)}$ and $C_p^{(qq)}$. Doing this we get

$$\begin{cases} \dfrac{\partial B_p^{(q)}}{\partial t} + \mathbf{E^{(qq)}}(t)^* B_p^{(q)} + \mathbf{E^{(lq)}}(t)^* B_p^{(l)} + \dfrac{1}{2}\bar{B}_p^{(l)*}\mathbf{K^{(q)}}(t) B_p^{(l)} & \\ +2C_p^{(qq)}\mathbf{d^{(qq)}}(t) + 2\left[C_p^{(qq)}\mathbf{k_0^{(qq)}}(t)B_p^{(q)} + C_p^{(qq)}\mathbf{k_0^{(lq)}}(t)B_p^{(l)}\right] = g^{(q)}(t) \\ B_p^{(q)}(T,T) = 0 \end{cases} \tag{32}$$

$$\begin{cases} \dfrac{\partial B_p^{(l)}}{\partial t} + \mathbf{E^{(ll)}}(t)^* B_p^{(l)} + \dfrac{1}{2}\bar{B}_p^{(l)*}\tilde{\mathbf{K}}^{(\mathbf{ll})}(t) B_p^{(l)} = g^{(l)}(t) \\ B_p^{(l)}(T,T) = 0 \end{cases} \tag{33}$$

$$\begin{cases} \dfrac{\partial C_p^{(qq)}}{\partial t} + 2C_p^{(qq)}\mathbf{k_0^{(qq)}}(t)C_p^{(qq)} + C_p^{(qq)}\mathbf{E^{(qq)}} + \mathbf{E^{(qq)*}}C_p^{(qq)} & \\ +\tilde{\bar{B}}_p^{(l)*}\tilde{\mathbf{G}}^{(\mathbf{ll})}(t)\tilde{\bar{B}}_p^{(l)} = Q^{(qq)}(t) \\ C_p^{(qq)}(T,T) = 0 \end{cases} \tag{34}$$

  where $\bar{B}_p^{(l)} = \begin{pmatrix} B_p^{(l)} & 0 & \cdots & 0 \\ 0 & B_p^{(l)} & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & B_p^{(l)} \end{pmatrix}$ and has dimension $q \times l.q$ for q the number of $Z^{(q)}$- factors and l the number of $Z^{(l)}$ - factors, and where we take

$$\tilde{\mathbf{K}}^{(\mathbf{ll})} = \begin{pmatrix} \mathbf{k_1^{(ll)}} \\ \mathbf{k_2^{(ll)}} \\ \vdots \\ \mathbf{k_m^{(ll)}} \end{pmatrix} \qquad \tilde{\mathbf{G}}^{(\mathbf{ll})} = \begin{pmatrix} \mathbf{g_{11}^{(ll)}}(t) & \mathbf{g_{12}^{(ll)}} & \cdots & \mathbf{g_{1q}^{(ll)}} \\ \mathbf{g_{21}^{(ll)}}(t) & \mathbf{g_{22}^{(ll)}}(t) & \cdots & \mathbf{g_{2q}^{(ll)}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{g_{q1}^{(ll)}} & \mathbf{g_{q2}^{(ll)}} & \cdots & \mathbf{g_{qq}^{(ll)}} \end{pmatrix}.$$

- Note that the ODE (33) for $B_p^{(l)}$ only depends on $B_p^{(l)}$ itself.

That the ODE (34) depends on $C_p^{(qq)}$ and $B_p^{(l)}$ but not on $B_p^{(q)}$.

And, finally, that the ODE (32) depend on all three functions $B_p^{(q)}$, $B_p^{(l)}$ and $C_p^{(qq)}$.

Given this, the next steps are obvious.

1) Solve equation (33) and get the solution for $B_p^{(l)}$.
2) Use the solution for $B_p^{(l)}$ to solve (34) and get the solution for $C_p^{(qq)}$.
3) Finally, use both solutions for $B_p^{(l)}$ and $C_p^{(qq)}$ to solve for (32) and get the solution for $B_p^{(q)}$.

This is equivalent to solve the ODEs (26)-(27). We can then insert the solutions for B_p and C_p into equation (25) and simply integrate to obtain A_p.

The examples in Section 6.1 gives an illustration of technique described above.

3.3 On the factor dynamics conditions *per se*

3.3.1 Linear Drift

From a careful reading of the proof of Theorem 3.2, one can realize that whenever there are both types of factors, we could, in principle, allow the drift α of $Z^{(l)}$-factors to depend also quadraticly on some of the factors z, since the PDE (29) would still be separable in a way that would not compromise the existence of solution.

Concretely a drift of the form,

$$\alpha(t,z) = \mathbf{d}(t) + \begin{pmatrix} \mathbf{E^{(qq)}}(t) & \mathbf{0} \\ \mathbf{E^{(lq)}}(t) & \mathbf{E^{(ll)}}(t) \end{pmatrix} \begin{pmatrix} z^{(q)} \\ z^{(l)} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & z^{(q)*} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{F^{(ll)}}(t) \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ z^{(q)} \end{pmatrix}$$

could, theoretically, be considered.

This possibility is however ruled out in Definition 3.2 because by allowing a quadratic drift we **cannot** guarantee existence of a unique global solution [15] for the SDE (11), since there would be **no** K such that

$$||\alpha(t,x) - \alpha(t,y)|| \leq K||x-y||,$$

therefore, the most general form for the drift term α is that it is linear in z (as in (19)).

3.3.2 Quadratic Variance

We can, nonetheless, allow for quadratic $\sigma(t,z)\sigma(t,z)^*$ (as in (20)), since this essentially means that $\sigma(t,z)$ is a linear function of z, and, therefore, we still can find a K such that

$$||\sigma(t,x) - \sigma(t,y)|| \leq K||x-y||.$$

Interesting questions to pose at this point are:

[15] For a textbook treatment of solutions of SDEs see, for instance, [3].

1) What do the conditions on $\sigma(t,z)\sigma(t,z)^*$ really imply for the structure of $\sigma(t,z)$ itself?

2) What is the equivalent to given shape of $\sigma(t,z)$ in terms of volatility $\hat{\sigma}(t,z)$ of a model is correlated[16] Wiener processes?

In previous literature little effort has been spent in trying to answer the above questions, the only exception is Duffie and Kan [12], that answer the *first* for affine $\sigma\sigma^*$. The next two Propositions answer *both* questions for a quadratic $\sigma\sigma^*$, that is when

$$\sigma(t,z)\sigma(t,z)^* = \mathbf{k_0}(t) + \sum_{u=1}^{m} \mathbf{k_u}(t) z_u + \sum_{u,k=1}^{m} z_u \mathbf{g_{uk}}(t) z_k. \tag{35}$$

Proposition 3.6 *Taking $\sigma(t,z)\sigma(t,z)^*$ to be of the form (35) is, under non degeneracy conditions and possible reordering of indices, the same as taking $\sigma(t,z)$ in (11) to be of the form*

$$\sigma(t,z) = \Sigma(t)U(t,z), \tag{36}$$

where $\Sigma(t)$ is a $(m \times n)$ deterministic matrix and $U(t,z)$ is a $(n \times n)$ matrix with the specific from

$$U(t,z) = \begin{pmatrix} \sqrt{u_1(t,z)} & 0 & \cdots & 0 \\ 0 & \sqrt{u_2(t,z)} & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & 0 & \sqrt{u_n(t,z)} \end{pmatrix} \tag{37}$$

where

$$u_i(t,z) = e_i(t) + f_i(t)z + z^* g_i(t) z \tag{38}$$

with e_i a scalar deterministic function and f_i a deterministic row-vector function.

Proof. This is a straight forward generalization of a similar result from Duffie and Kan [11] on the implications for the matrix $\sigma(t,z)$ of requiring a linear functional form of $\sigma\sigma^*$ (the well-known ATS condition). □

Proposition 3.6 shows that each column of the matrix $\sigma(t,z)$ can only be dependent on the square root of one particular quadratic combination of factors, otherwise $\sigma(t,z)\sigma(t,z)^*$ would have elements of the form $\sqrt{u_i}\sqrt{u_j}$ which, for $u_i \neq u_j$, would not be linear in z. That is, the matrix $\sigma(t,z)$ in (11) needs to be of the following form

$$\sigma(t,z) = \begin{pmatrix} s_{11}\sqrt{u_1(t,z)} & \cdots & s_{1n}\sqrt{u_n(t,z)} \\ s_{21}\sqrt{u_1(t,z)} & \cdots & s_{2n}\sqrt{u_n(t,z)} \\ \vdots & & \vdots \\ s_{m1}\sqrt{u_1(t,z)} & \cdots & s_{mn}\sqrt{u_n(t,z)} \end{pmatrix}. \tag{39}$$

The implications from (36)-(38) are quite strong since by letting each column of $\sigma(t,z)$ depend at most on one particular square root function, it implies that the term associated with each of the n elements of the multi-dimensional Wiener process (since each column of σ multiply by a different element of W) can also depend at most on one particular square root function.

[16] Recall Remark 2.3 and Example 1.

This is just an algebraic fact that, however, together with the fact that the matrix $\sigma(t, z)$ is the volatility when we consider a multi-dimensional Wiener process (i.e. independent scalar Wiener processes) may turn out to be quite restrictive.[17] Note that given an original model with correlated Wiener processes, its transformed version with multi-dimensional Wiener process will incorporate the original correlation between scalar Wiener processes in a functionally more complex matrix $\sigma(t, z)$ and so restrictions of this matrix lead to restrictions on possible correlations on the original scalar Wiener processes.

Taking these two observations simultaneously it is easy to see that correlations can only be allowed between factors driven by the same u_i function.

We recover Example 1.

Example 1 *The following original dynamics of two factors Z_1 and Z_2:*

$$d\begin{pmatrix} Z_1(t) \\ Z_2(t) \end{pmatrix} = (\dots)\,dt + \underbrace{\begin{pmatrix} \sigma_1\sqrt{Z_1(t)} & 0 \\ 0 & \sigma_2\sqrt{Z_2(t)} \end{pmatrix}}_{\hat{\sigma}(t,Z(t))} d\begin{pmatrix} \bar{W}_1(t) \\ \bar{W}_2(t) \end{pmatrix} \tag{40}$$

with $d\bar{W}_1(t)d\bar{W}_2(t) = \rho dt$, has the following transformed form

$$d\begin{pmatrix} Z_1(t) \\ Z_2(t) \end{pmatrix} = (\dots)\,dt + \underbrace{\begin{pmatrix} \sigma_1\sqrt{Z_1(t)} & 0 \\ \rho\sigma_2\sqrt{Z_2(t)} & \sigma_2\sqrt{Z_2(t)}\sqrt{1-\rho^2} \end{pmatrix}}_{\sigma(t,Z(t))} d\begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix} \tag{41}$$

with $dW_1(t)dW_2(t) = 0$.

*Since $\sigma(t, Z(t))$ in (41) does not have the form (39) for any $\rho \neq 0$, we immediately see that the model (40) when we allow for correlation between W_1 and W_2 is **not** in accordance with (35).*

When the two Wiener processes are not correlated (i.e. if $d\bar{W}_1(t)d\bar{W}_2(t) = 0dt$), then we have the form (36)-(38) with $g_i = \mathbf{0}$ for all i, consequently the matrix $\sigma(t, z)$ is linear in z and in accordance with (35).

Proposition 3.7 tell us what are the conditions in terms of the volatility structure $\hat{\sigma}(t, z)$ of a possibly correlated system that would guarantee a multidimensional representation with a volatility structure $\sigma(t, z)$ of the form (39).

Proposition 3.7 *Under non degeneracy conditions and possible reordering of terms we have a transformed matrix $\sigma(t, z)\sigma(t, z)^*$ quadratic in z (of the form (35)), if and only if,*

- *the volatility structure of the factors $\hat{\sigma}(t, z)$ in a system with correlated Wiener processes (as in equation (12)) can be written as*

$$\hat{\sigma}(t, z) = \hat{\Sigma}(t)\hat{U}(t, z) \tag{42}$$

where $\hat{\Sigma}(t)$ is a deterministic $(m \times d)$ matrix and $\hat{U}(t, z)$ a $(d \times d)$ matrix with the same form as (37)-(38),

and

[17] Recall the comment just made after Example 1 (in Section 2), that any two correlated factors, in the assumed formulation must be driven by at least one common (scalar) Wiener process.

- *for any possible i and j we only allow for correlations among the scalar wiener processes $\bar{W}_i$ and $\bar{W}_j$ in $\bar{W} = \begin{pmatrix} \bar{W}_1 \\ \vdots \\ \bar{W}_d \end{pmatrix}$ if we have $u_i(t,z) = u_j(t,z)$ for that i and j.*

Proof. It follows from Proposition 3.6 and the equivalence between the representation (12) and (13) in Remark 2.3. □

To understand the idea behind Proposition 3.7, note that since, $U(t,z)$ in (42) is of the same form as in (36), the difference between $\hat{\sigma}(t,z)$ and $\sigma(t,z)$ comes essentially from the matrices $\hat{\Sigma}(t)$ and $\Sigma(t)$. $\hat{\Sigma}(t)$ tend to be a better behaved diagonal matrix as $\Sigma(t)$ has to include possible correlations in off-diagonal cells. Since, the basic structure is maintained when going from the original to the transformed dynamics of the factors Z, the very specific shape in (39) only allows for very restrictive correlations among the elements of originally correlated Wiener processes $\bar{W}$.

Example 2 *A five-factor model with possibly correlated Wiener processes*

$$dZ(t) = (\cdots)dt + \begin{pmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2\sqrt{Z_1(t)} & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 0 \\ 0 & 0 & 0 & \sigma_4\sqrt{Z_1^2(t) + Z_3(t)} & 0 \\ 0 & 0 & 0 & 0 & \sigma_5\sqrt{Z_1(t)} \end{pmatrix} d \begin{pmatrix} \bar{W}_1 \\ \bar{W}_2 \\ \bar{W}_3 \\ \bar{W}_4 \\ \bar{W}_5 \end{pmatrix}$$

has a volatility structure $\hat{\sigma}(t,z)$ that can be rewritten as

$$\hat{\sigma}(t,z) = \underbrace{\begin{pmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 0 \\ 0 & 0 & 0 & \sigma_4 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5 \end{pmatrix}}_{\hat{\Sigma}} \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \sqrt{z_1} & 0 & \cdots & \vdots \\ \vdots & 0 & 1 & 0 & \vdots \\ \vdots & \vdots & 0 & \sqrt{z_1^2 + z_3} & 0 \\ 0 & \cdots & \cdots & 0 & \sqrt{z_1} \end{pmatrix} \tag{43}$$

From (43) we see that the only correlations we can allow for, according to Proposition 3.7, are between $\bar{W}_1$ and $\bar{W}_3$ on the one hand, and between $\bar{W}_2$ and $\bar{W}_5$ on the other.

The two propositions above study the shape of σ (and $\hat{\sigma}$) when we want to guarantee a general quadratic shape of $\sigma\sigma^*$ as in (35).

We know, however from the previous analysis, that the form (35) does not guarantee a GQTS and only a restricted version of it does. Considering the needed restrictions we now analyze

$$\sigma(t,z)\sigma(t,z)^* = \begin{pmatrix} \mathbf{k_0^{(qq)}}(t) & \mathbf{k_0^{(ql)}}(t) \\ \mathbf{k_0^{(lq)}}(t) & \mathbf{k_0^{(ll)}}(t) + \sum_{i=1}^{m} \mathbf{k_u^{(ll)}}(t) z_u + \sum_{z_u, z_k \in \mathbf{z^{(q)}}} z_u \mathbf{g_{uk}^{(ll)}}(t) z_k \end{pmatrix}. \tag{44}$$

The exact shape of σ (and $\hat{\sigma}$) that is needed in this case should now be obvious and are presented in the next Corollary (the proof is omitted as it follows immediately from Propositions 3.6 and 3.7).

Corollary 3.8 *Given the classification of factors in Definition 3.3, imposing the structure (44) on $\sigma(t,z)\sigma(t,z)^*$ is equivalent to require, under non degeneracy conditions, that:*

- *the volatility structure, $\sigma(t,z)$ in (11), is of the following form*

$$\sigma(t,z) = \begin{pmatrix} \Sigma_A^{(q)}(t) & 0 \\ \Sigma_A^{(l)}(t) & \Sigma_B^{(l)}(t) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & U(t,z) \end{pmatrix}$$

  for some division of the elements in the multidimensional Wiener process into $W = \begin{pmatrix} W_A \\ W_B \end{pmatrix}$ and where $\Sigma_{\cdot}^{(\cdot)}$ are deterministic matrices and $U(t,z)$ has the same form as in (37)-(38).

or equivalently,

- *the volatility structure of the original model with possibly correlated Wiener processes, $\hat{\sigma}(t,z)$ in (12), can be written as*

$$\hat{\sigma}(t,z) = \begin{pmatrix} \hat{\Sigma}_A^{(q)}(t) & 0 \\ \hat{\Sigma}_A^{(l)}(t) & \hat{\Sigma}_B^{(l)}(t) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \hat{U}(t,z) \end{pmatrix}$$

  for some division of $\bar{W} = \begin{pmatrix} \bar{W}_A \\ \bar{W}_B \end{pmatrix}$ and where $\hat{\Sigma}^{(\cdot)}_{\cdot}$ are deterministic matrices and $\hat{U}(t,z)$ has the same form as in (37)-(38).

  and

  for any i and j we only allow for correlations among the scalar wiener processes $\bar{W}_i$ and $\bar{W}_j$ in $\bar{W} = \begin{pmatrix} \bar{W}_1 \\ \vdots \\ \bar{W}_m \end{pmatrix}$ if both belong to the group $\bar{W}_A$, or if we have $u_i(t,z) = u_j(t,z)$ for that i and j.

The results on this subsection help us to check, by inspection, if any model satisfy the volatility sufficient condition for a GQTS.

Remark 3.3 *An important consequence of the results here presented is that* ***any*** *factor (even a linear factor) that is correlated with a quadratic factor must also have a deterministic volatility structure.*

3.4 Important special cases

The most important special case of a GQTS of bond prices are the *Guassian-QTS* and the *ATS* previously studied in the literature. In any of these cases the classification of factors becomes irrelevant.

3.4.1 Gaussian Quadratic Term Structures

In a Gaussian-QTS model all the factors have deterministic volatility,

$$\sigma(t,z)\sigma(t,z)^* = \mathbf{k_0}(t) \quad \forall t,$$

hence there is no need to distinguish between factors to check if the volatility structure is under the conditions of Theorem 3.2. Only in models where at least one of the factors is non-Gaussian we need to check if this is a $Z^{(l)}$-factor. In a completely Gaussian setting the volatility conditions are always satisfied.

For Gaussian-QTS we can, therefore, restated Theorem 3.2 without any allusion to the classification of factors.

Corollary 3.9 (Gaussian-QTS)

Suppose that Assumption 2.1 and 2.2 are in force. Furthermore suppose that we are in a GQSR setting, so that (18) and $Q(t) \neq 0$ for some t.

Finally, assume that α and σ from the factor dynamics (11) are of the following form:

$$\begin{aligned} \alpha(t,z) &= \mathbf{d}(t) + \mathbf{E}(t)z \\ \sigma(t,z)\sigma(t,z)^* &= \mathbf{k_0}(t) \end{aligned}$$

where $\mathbf{d}$, $\mathbf{E}$, $\mathbf{k_0}$ are matrices of deterministic smooth functions.

Then the term structure of bond prices is generally quadratic, i.e. H_p from (5) can be written on the form (17). A_p, B_p and C_p in (17) solve the following system of ordinary differential equations.

$$\begin{cases} \dfrac{\partial A_p}{\partial t} + \mathbf{d}(t)^* B_p + \dfrac{1}{2} B_p^* \mathbf{k_0}(t) B_p + tr\left\{(C_p)\mathbf{k_0}(t)\right\} = f(t) \\ A_p(T,T) = 0 \end{cases} \tag{45}$$

$$\begin{cases} \dfrac{\partial B_p}{\partial t} + \mathbf{E}(t)^* B_p + 2C_p\mathbf{d}(t) + 2C_p\mathbf{k_0}(t)B_p = g(t) \\ B_p(T,T) = 0 \end{cases} \tag{46}$$

$$\begin{cases} \dfrac{\partial C_p}{\partial t} + C_p\mathbf{E}(t) + \mathbf{E}(t)^* C_p + 2C_p\mathbf{k_0}(t)C_p = Q(t) \\ C_p(T,T) = 0 \end{cases} \tag{47}$$

The Gaussian structure of the model stops to be an *assumption* and becomes a *needed condition* when all the factors are of the type $Z^{(q)}$. So, for pure quadratic term structures (PQTS) the system of ODEs to be solve is also (45)-(47) not because we have an addoc assumption but because otherwise $\sigma\sigma^*$ would not have the required shape.

The previous literature on quadratic term structures can, thus, be seen as **either** a literature on Gaussian-QTS, where a non-needed add hoc assumption is introduced, **or** as a literature on PQTS that has not considered the possibility that some factors may have only a linear impact on the term structure. In either case the results are those of Corollary 3.9 and are obtainef from Theorem 3.2 by setting $\mathbf{K}(t) = \mathbf{0}$ and $\mathbf{G}(t) = \mathbf{0}$ for all t.

In Section 6 we give examples of both pure (thus Gaussian) and non-pure QTS.

3.4.2 Affine Term Structures

From Corollary 3.3 we know that if there are only $Z^{(l)}$-factors we have an ATS of bond prices.

Thus, *sufficient conditions* to guarantee ATS of bond prices are: an affine setting of interest rates **and** a linear matrix $\sigma(t,z)\sigma(t,z)^*$. Otherwise there would be at least one $Z^{(q)}$-factor (for one of the reasons (i) or (ii) in Definition 3.3) having a quadratic impact in bond prices term structure.

So, $Q(t) = \mathbf{0}$ and $\mathbf{G}(t) = \mathbf{0}$ for all t, guarantees an ATS and Theorem 3.2 give us the right result when we include this fact. Also in this case Theorem 3.2 can be restated without any allusion to the classification of factors (there is only one type of factor, anyway).

Corollary 3.10 (ATS)

Suppose that Assumption 2.1 and 2.2 are in force. Furthermore suppose that we are in an affine interest rate setting, that is (18) hold with $Q(t) = \mathbf{0}$.

Finally assume that α and σ from the factor dynamics (11) are of the following form:

$$\begin{aligned} \alpha(t,z) &= \mathbf{d}(t) + \mathbf{E}(t)z \\ \sigma(t,z)\sigma(t,z)^* &= \mathbf{k_0}(t) + \sum_{u=1}^{m} \mathbf{k_u}(t) z_u \end{aligned}$$

where $\mathbf{d}$, $\mathbf{E}$, $\mathbf{k_0}$ and $\mathbf{k_u}$ for all $\mathbf{u}$, are matrices of deterministic smooth functions.

Then the term structure of bond prices is affine, i.e. H_p from (5) can be written on the form (17) with $C_p = \mathbf{0}$ and A_p, B_p solve the following system of ordinary differential equations.

$$\begin{cases} \dfrac{\partial A_p}{\partial t} + \mathbf{d}(t)^* B_p + \dfrac{1}{2} B_p^* \mathbf{k_0}(t) B_p + tr\left\{C_p)\mathbf{k_0}(t)\right\} = f(t) \\ A_p(T,T) = 0 \end{cases} \tag{48}$$

$$\tag{49}$$

$$\begin{cases} \dfrac{\partial B_p}{\partial t} + \mathbf{E}(t)^* B_p + \dfrac{1}{2} \bar{B_p}^* \mathbf{K}(t) B_p = g(t) \\ B_p(T,T) = 0 \end{cases} \tag{50}$$

where $\mathbf{K}$ $\bar{B_p}$ are defined as in (21) and (28) respectively.

3.5 On higher order term structures

After studying GQTS, which include all term structures up to the order two, one may wonder about higher order term structures. We will see later on that while GQTS for bond prices are natural to assume when dealing with ATS of forward prices, general cubic term structures for bond prices (GCTS) could be reasonable to assume when dealing with QTS of forward prices.

These GCTS would include all term structures up to the order three, so they would include the GQTS previously studied as well as cubic term structures (CTS).

Cubic Term Structures are, however, not nice objects of study for two reasons:

- *They are computationally harder to deal with.* The reason is that as long as we leave the quadratic setting, the second-order derivatives $\dfrac{\partial^2 \ln H_p}{\partial z_i \partial z_j}$ are of a higher order than $\ln H_p$ itself, which makes almost impossible (it is possible only in very pathological/lucky cases) that the term structure equation will have a solution.

- *They are in general, not consistent with the no-arbitrage assumption.* Filipovic [14] who focus, on maximal degree problems for term structures shows that actually, only ATS and QTS (i.e., term structures up to the second order) are usually well behaved (consistent[18]), and that term structures of higher orders are, in general, not consistent with the no-arbitrage assumption.

Given these problems of CTS, the most general interest rate setting we are going to consider when dealing with *forward prices* is the *general quadratic short rate setting*- GQSR.

Before dealing with (the more complex) *forward prices*, we will study GQTS of *futures prices*.

4 Generally Quadratic Term Structures for Futures Prices

4.1 General Setting

Recall from Definition 2.1 that a GQTS for futures prices H_F has the form

$$\ln H_F(t,z,T) = A_F(t,T) + B_F^*(t,T)z + z^* C_F(t,T)z \tag{51}$$

where $C_{F(m\times m)}$ symmetric and not necessarily different from $\mathbf{0}$, $B_{F(m\times 1)}$ and $A_{F(1\times 1)}$ are matrices of deterministic and smooth functions.

And, from Definition 3.2, that a vector of factors Z is said to have general quadratic Q- dynamics if $\alpha(t,z)$ and $\sigma(t,z)$ in (11) are such that

$$\alpha(t,z) = \mathbf{d}(t) + \mathbf{E}(t)z \tag{52}$$

$$\sigma(t,z)\sigma(t,z)^* = \mathbf{k_0}(t) + \sum_{u=1}^{m} \mathbf{k_u}(t) z_u + \sum_{u,k=1}^{m} z_u \mathbf{g_{uk}}(t) z_k \tag{53}$$

where $\mathbf{d}$, $\mathbf{E}$, $\mathbf{k_0}$, $\mathbf{k_u}$ and $\mathbf{g_{uk}}$ for $\mathbf{u},\mathbf{k} = \mathbf{1},\cdots,\mathbf{m}$ are matrices of deterministic smooth functions. We also defined

$$\mathbf{K}(t) = \begin{pmatrix} \mathbf{k_1}(t) \\ \mathbf{k_2}(t) \\ \vdots \\ \mathbf{k_m}(t) \end{pmatrix}, \quad \mathbf{G}(t) = \begin{pmatrix} \mathbf{g_{11}}(t) & \mathbf{g_{12}}(t) & \cdots & \mathbf{g_{1m}}(t) \\ \mathbf{g_{21}}(t) & \mathbf{g_{22}}(t) & \cdots & \mathbf{g_{2m}}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{g_{m1}}(t) & \mathbf{g_{m2}}(t) & \cdots & \mathbf{g_{mm}}(t) \end{pmatrix}. \tag{54}$$

Since the boundary condition h_F for futures prices is not as easy as the one for bond prices (where we always had $h_p(T,z) = 1$ for all T, z), we now need a new definition.

[18] *Consistency* in this context, as discussed in Björk and Christensen [4], means that the interest rate model will produce forward curves belonging to the parameterized family. Filipovic [14] proves, under certain regularity conditions, that if one represents the forward rate as a time-separable polynomial function of a diffusion state vector, that the maximal consistent order of the polynomial is *two*.

Definition 4.1 *A term structure is said to have a* ***generally quadratic boundary condition*** *(GQBC) if the boundary function h is of the form*

$$\ln h(T,z) = a(T) + \mathbf{b}(T)^* z + z^* \mathbf{c}(T) z \tag{55}$$

where $\mathbf{c}(T)_{(m\times m)}$, symmetric, and not necessarily different from $\mathbf{0}$, $\mathbf{b}(T)_{(m\times 1)}$ and $\mathbf{a}(T)_{1\times 1}$ are matrices of smooth and deterministic functions.[19]

Moreover, for all i such that $\mathbf{b}_i(T) \neq 0$ or $\mathbf{c}_i(T) \neq \mathbf{0}$, Z_i is said to be a ***natural factor****.*

We note that, if the boundary function h is not of the form (55), then we cannot have a GQTS because the definition of a GQTS would fail at maturity.

In the particular case of futures prices we, thus have a GQBC if

$$\ln h_F(T,z) = a_F(T) + \mathbf{b}_F(T)^* z + z^* \mathbf{c}_F(T) z, \tag{56}$$

and a *necessary condition* for a GQTS of futures prices is that (56) hold.

Following the same strategy as before, we now to give the relevant classification of factors for futures prices.

Definition 4.2 *Given a general quadratic Q-dynamics for Z (so that (52) and (53) hold for the α and σ in (11)), and a GQBC as in (56).*

- *Z_i is a* ***$\bar{\mathbf{Z}}^{(\mathbf{q})}$-factor*** *if it satisfies* ***at least one*** *of the following requirements:*
  - *(i) it has a quadratic impact on the boundary condition h_F, i.e., there exists T such that $\mathbf{c}_F(T)_i \neq \mathbf{0}$;*
  - *(ii) it has a quadratic impact on the functional form of the matrix $\sigma(t,z)\sigma(t,z)^*$, i.e., there exist $\mathbf{k}$ and t such that $\mathbf{g}_{\mathbf{ik}}(t) \neq \mathbf{0}$;*
  - *(iii) it affects the drift terms of the factors satisfying in (i) or (ii), i.e., for Z_j satisfying (i) or (ii) we have $\mathbf{E}(t)_{ji} \neq \mathbf{0}$ at least for some t.*
- *Z_i is a* ***$\bar{\mathbf{Z}}^{(\mathbf{l})}$-factor*** *if it does not satisfy (i)-(iii).*

As before, from the classification of factors it results, by definition, special shapes for the matrices involved in this classification.

Remark 4.1 *We note that given Definition 4.2,*

- *it is always possible to reorder the vector of factors Z and its correspondent value vector, so that we have*

$$Z = \begin{pmatrix} \bar{Z}^{(q)} \\ \bar{Z}^{(l)} \end{pmatrix} \qquad z = \begin{pmatrix} \bar{z}^{(q)} \\ \bar{z}^{(l)} \end{pmatrix}.$$

- *with this reordering of factors we have,* ***by definition****, the following shapes for $\mathbf{c}_F$ in (18) and for $\mathbf{E}$ and $\mathbf{G}$ in (19) and (21), respectively*

$$\mathbf{c}_F(t) = \begin{pmatrix} \mathbf{c}^{(\mathbf{qq})}{}_F(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \mathbf{E}(t) = \begin{pmatrix} \mathbf{E}^{(\mathbf{qq})}(t) & \mathbf{0} \\ \mathbf{E}^{(\mathbf{lq})}(t) & \mathbf{E}^{(\mathbf{ll})}(t) \end{pmatrix} \quad \mathbf{G}(t) = \begin{pmatrix} \mathbf{G}^{(\mathbf{qq})}(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{57}$$

[19] Definition 4.1 can also be applied to bond prices. In this case the "underlying asset" is the non-risky asset that pays 1 at maturity and $a(T) = 1$, $\mathbf{b}(T) = \mathbf{0}$ and $\mathbf{c}(T) = \mathbf{0}$ for all T, satisfy (55).

4.2 Main result on Futures Prices

Theorem 4.1 *Suppose that Assumption 2.1 and 2.2 are in force. Furthermore suppose that Z follows a general quadratic Q-dynamics, (i.e., that α and σ from the factor dynamics (11), satisfy (52)-(53)) and that we have a GQBC, so that the boundary condition h_F from (8), has the quadratic form in (56).*

Finally assume that the factors are reordered as $Z = \begin{pmatrix} \bar{Z}^{(q)} \\ \bar{Z}^{(l)} \end{pmatrix}$ (using Definition 4.2), and that the following restrictions apply to k_u and g_{uk} in (53):

$$\mathbf{k_u}(t) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{k_u^{(ll)}}(t) \end{pmatrix} \quad \forall\, \mathbf{u}, \forall\, t \tag{58}$$

$$\mathbf{g_{uk}}(t) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{g_{uk}^{(ll)}}(t) \end{pmatrix} \quad \forall\, \mathbf{u}, \mathbf{k} : (z_u \wedge z_k) \in \bar{\mathbf{z}}^{(\mathbf{q})}, \forall\, t \tag{59}$$

Then the term structure of futures prices is generally quadratic, i.e. H_F from (7) can be written on the form (51)and A_F, B_F and C_F can be obtained by solving the following system of ordinary differential equations.

$$\begin{cases} \dfrac{\partial A_F}{\partial t} + \mathbf{d}(t)^* B_F + \dfrac{1}{2} B_F^* \mathbf{k_0}(t) B_F + tr\left\{C_F \mathbf{k_0}(t)\right\} = 0 \\ A_F(T,T) = a_F(T) \end{cases} \tag{60}$$

$$\begin{cases} \dfrac{\partial B_F}{\partial t} + \mathbf{E}(t)^* B_F + 2C_F \mathbf{d}(t) + \dfrac{1}{2} \bar{B_F}^* \mathbf{K}(t) B_F + 2C_F \mathbf{k_0}(t) B_F = \mathbf{0} \\ B_F(T,T) = \mathbf{b}_F(T) \end{cases} \tag{61}$$

$$\begin{cases} \dfrac{\partial C_F}{\partial t} + C_F \mathbf{E}(t) + \mathbf{E}(t)^* C_F + 2C_F \mathbf{k_0}(t) C_F + \bar{B_F}^* \mathbf{G}(t) \bar{B_F} = \mathbf{0} \\ C_F(T,T) = \mathbf{c}_F(T) \end{cases} \tag{62}$$

where C_F has the special form $C_F = \begin{pmatrix} C_F^{(qq)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ and A_F, B_F, $C_F^{(qq)}$ should be evaluated at (t,T).

$\mathbf{E}$ *and* $\mathbf{d}$ *are the same as in (52),* $\mathbf{k_0}$, $\mathbf{K}$ *and* $\mathbf{G}$ *the same as in (53) (with the same notation as in (54)), and where $\bar{B_F}$ follows the same idea as the definition in (28) and have dimension $(m^2 \times m)$.*

Proof. We need to show that $H_F(t,z,T)$ from (51) where A_F, B_F and C_F solve (60)-(62), solves the PDE (15) that uniquely characterizes the futures prices in this setting.

Taking partial derivatives

$$\begin{aligned} \frac{\partial H_F}{\partial t} &= \left[\frac{\partial A_F}{\partial t} + \frac{\partial B_F}{\partial t}^* z + z^* \frac{\partial C_F}{\partial t} z\right] H_F \\ \frac{\partial H_F}{\partial z_i} &= \left[B_{Fi} + C_{Fi} z + C_{Fi}^* z\right] H_F \\ \frac{\partial^2 H_F}{\partial z_i \partial z_j} &= \left[C_{Fij} + C_{Fji}\right] H_F + \left[B_{Fi} + C_{Fi} z + C_{Fi}^* z\right] \left[B_{Fj} + C_{Fj} z + C_{Fj}^* z\right] H_F \end{aligned}$$

so the PDE (15) reduces in this case to

$$\begin{cases} \frac{\partial A_F}{\partial t} + \frac{\partial (B_F)}{\partial t} z + z^* \frac{\partial C_F}{\partial t} z + \sum_{i=1}^{m} \left[(B_F)_i + (C_F)_i z + (C_F^*)_i z \right] \alpha_i \\ \quad + \frac{1}{2} \sum_{i,j=1}^{m} \left[(B_F)_i + (C_F)_i z + (C_F^*)_i z \right] \sigma_i \sigma_j^* \left[(B_F)_j + (C_F)_j z + (C_F^*)_j z \right] \\ \qquad + \frac{1}{2} \sum_{i,j=1}^{m} \left((C_F)_{ij} + (C_F)_{ji} \right) \sigma_i \sigma_j^* \quad = \quad 0 \\ A_F(T,T) + B_F(T,T)^* z + z^* C_F(t,T) z \quad = \quad h_F(t,z) \end{cases} \tag{63}$$

Substituting α, $\sigma\sigma^*$ and h_F from (52), (53) and (56) respectively, and using the restrictions (58)-(59), the PDE (63) becomes always a separable equation equivalent to (60)-(62). If all $Z_i \in \bar{Z}^{(q)}$, we know, that $\mathbf{K}(t) = \mathbf{0}$ and $\mathbf{G}(t) = \mathbf{0}$ for all t. Hence, equation (63) becomes separable up to quadratic terms of z. It is easy to show that given the shapes (57) for the matrices $\mathbf{c}_F$, $\mathbf{E}$ and $\mathbf{G}$, $C_F^{(ll)}(t,T) = \mathbf{0}$ and $C_F^{(ql)}(t,T) = C_F^{(lq)*}(t,T) = \mathbf{0}$ for all t, T always solve (62). If there exist $\bar{Z}^{(l)}$ factors, equation (63) will still be separable but it will have terms up to order four in the state variable z. The third and fourth order terms in z will result from the terms $\frac{\partial^2 H_F}{\partial z_i \partial z_j}$ where both $Z_i, Z_j \in \bar{Z}^{(l)}$, so will only affect $C_p^{(ll)}$ and $C_p^{(ll)}(t,T) = \mathbf{0}$ for all t, T is always a solution for those conditions. Lower order terms will depend on the entire matrix C_p, however the restrictions imposed on the matrices $\mathbf{c}_F$, $\mathbf{E}$ and $\mathbf{G}$ in (57) once again guarantee that $C_F^{(ll)}(t,T) = \mathbf{0}$ and $C_F^{(ql)}(t,T) = C_F^{(lq)*}(t,T) = \mathbf{0}$ for all t, T are also always a solution to (62). □

The system of ODEs for futures prices, (60)-(62), is of the same level of difficulty as the system (25)-(27) for bond prices. And here as there it is possible to split the interrelated Riccati equations (61)-(62) into simpler ODEs for $B_F^{(q)}$, $B_F^{(l)}$ and $C_F^{(qq)}$ that can be solved in iterative order.[20]

In terms of the *impacts*[21] of various factors on the futures prices term structure, it follows from $C_F^{(ll)}(t,T) = \mathbf{0}$ and $C_F^{(ql)}(t,T) = C_F^{(lq)*}(t,T) = \mathbf{0}$ for all t, T that the $\bar{Z}^{(l)}$ factors have a linear impact.

Corollary 4.2 ***(Linear factors)*** *The $\bar{Z}^{(l)}$-factors are linear factors in a GQTS of futures prices.*

Moreover, it follows from this corollary that the matrices $\mathbf{c}_F$ and $\mathbf{G}$ play quite and important role in determining the type of terms structure. If $\mathbf{c}_F(T) = \mathbf{0}$ and $\mathbf{G}(t) = \mathbf{0}$ for all t, T, there will be no $\bar{Z}^{(q)}$-factors and the terms structure will be affine. In other words, *necessary conditions* for existence of a quadratic term structure are that either $\mathbf{c}_F(T) \neq \mathbf{0}$ at least for some T or $\mathbf{G}(t) = \mathbf{0}$ at least for some t.

Besides this, some $\bar{Z}^{(q)}$-factors have obviously a quadratic impact.

Lemma 4.3 *A factor $Z_i \in \bar{Z}^{(q)}$ for which ${\mathbf{c}_F}_i(T) \neq \mathbf{0}$ at least for some T, has a quadratic impact in the futures prices term structure.*

[20] Recall the ideas presented in Section 3.2.2.

[21] Recall Definition 3.4

Proof. It follows from $C_{F_i}(T,T) = \mathbf{c}_{F_i}(T)$ and the definition of quadratic impact. □

The remaining $\bar{Z}^{(q)}$-factors will also have a quadratic impact as long as we exclude redundant factors from the analysis and consider that some (not at all restrictive) regularity condition is satisfied.

Assumption 4.1 *For any i, k, t, T such that*

$$E_{ji}(t)C_{F_{jk}}(t,T) \neq 0 \quad \textit{for some } j$$

or

$$B_{F_u}(t,T)\mathbf{g_{ik}}_{uu}(t)B_{F_u}(t,T) \neq 0 \quad \textit{for some } u$$

the following ***regularity condition*** *hold*

$$\sum_j E_{ji}(t)C_{F_{jk}}(t,T) + \sum_{u,v} B_{F_u}(t,T)\mathbf{g_{ik}}_{uv}(t)B_{F_v}(t,T) \neq 0.$$

Proposition 4.4 ***(Quadratic factors)*** *As long as we exclude from the analysis any redundant factors and under the regularity condition of Assumption 4.1, the $\bar{Z}^{(q)}$-factors are quadratic factors in a GQTS of futures prices.*

Proof. Lemma 4.3 proves that any $Z_i \in \bar{Z}^{(q)}$ such that $\mathbf{c}_{F_i}(T) \neq \mathbf{0}$ at least for some T has a quadratic impact. It remains to show that for all $Z_i \in \bar{Z}^{(q)}$ for which $\mathbf{c}_{F_i}(T) \neq \mathbf{0}$ for all T (all remaining $\bar{Z}^{(q)}$-factors), we also have, at least for some t, T, $C_{F_i}(t,T) \neq \mathbf{0}$ as a solution to (62). It follows from Definition 4.2 that, as long as we do not consider the redundant factors, for such Z_i there will always be a k, t, T such that $E_{ji}(t)C_{F_{jk}}(t,T) \neq 0$ and/or $B_{F_u}(t,T)\mathbf{g_{ik}}_{uu}(t)B_{F_u}(t,T) \neq 0$ for some u. The regularity condition of Assumption 4.1 then guarantees that for that k, t, T, $C_{F_{ik}}(t,T)$ solves a Riccati equation with a non-zero independent term and that, thus $C_{F_{ik}}(t,T) \neq 0$. This guarantees the quadratic impact of Z_i according to Definition 3.4. □

4.3 Important Special Cases

As far as our knowledge goes, quadratic term structures of *futures prices* have not been previously studied. Affine term structures, on the other hand, have (even allowing for the possibility of jump in the dynamics of the factors, see [5]).

Here we present as special cases of *Gaussian-QTS* and the *ATS*. The results follow immediately either by taking $\mathbf{K}(t) = \mathbf{0}$ and $\mathbf{G}(t) = \mathbf{0}$ for all t (to get the Gaussian-QTS result) or by setting $\mathbf{c}_F(t) = \mathbf{0}$ and $\mathbf{G}(t) = \mathbf{0}$ (to get the ATS) and by rewriting Theorem 4.1 so that it does not make reference to an unnecessary (for these two particular cases) classification of factors.

Corollary 4.5 (Gaussian-QTS)

Suppose that Assumption 2.1 and 2.2 are in force. Furthermore suppose that we have a GQBC so that (56) hold.

Finally, assume that α and σ from the factor dynamics (11) are of the following form:

$$\begin{aligned}\alpha(t,z) &= \mathbf{d}(t) + \mathbf{E}(t)z \\ \sigma(t,z)\sigma(t,z)^* &= \mathbf{k_0}(t)\end{aligned}$$

where $\mathbf{d}$, $\mathbf{E}$, $\mathbf{k_0}$ are matrices of deterministic smooth functions.

Then the term structure of futures prices is generally quadratic, i.e. H_F from (9) can be written on the form (51) and A_F, B_F and C_F solve the following system of ordinary differential equations.

$$\left\{\begin{aligned}\frac{\partial A_F}{\partial t} + \mathbf{d}(t)^* B_F + \frac{1}{2}B_F^*\mathbf{k_0}(t)B_F + tr\left\{(C_F)\mathbf{k_0}(t)\right\} &= 0 \\ A_F(T,T) &= a_F(T)\end{aligned}\right. \tag{64}$$

$$\left\{\begin{aligned}\frac{\partial B_F}{\partial t} + \mathbf{E}(t)^* B_F + 2C_F\mathbf{d}(t) + 2C_F\mathbf{k_0}(t)B_F &= \mathbf{0} \\ B_F(T,T) &= \mathbf{b}_F(T)\end{aligned}\right. \tag{65}$$

$$\left\{\begin{aligned}\frac{\partial C_F}{\partial t} + C_F\mathbf{E}(t) + \mathbf{E}(t)^*C_F + 2C_F\mathbf{k_0}(t)C_F &= \mathbf{0} \\ C_F(T,T) &= \mathbf{c}_F(T)\end{aligned}\right. \tag{66}$$

Corollary 4.6 (ATS)

Suppose that Assumption 2.1 and 2.2 are in force. Furthermore suppose that we are in an affine boundary condition, that is (56) hold with $\mathbf{c}_F(T) = \mathbf{0}$ for all T.

Finally assume that α and σ from the factor dynamics (11) are of the following form:

$$\begin{aligned}\alpha(t,z) &= \mathbf{d}(t) + \mathbf{E}(t)z \\ \sigma(t,z)\sigma(t,z)^* &= \mathbf{k_0}(t) + \sum_{u=1}^{m}\mathbf{k_u}(t)z_u\end{aligned}$$

where $\mathbf{d}$, $\mathbf{E}$, $\mathbf{k_0}$ and $\mathbf{k_u}$ for all $\mathbf{u}$, are matrices of deterministic smooth functions.

Then the term structure of futures prices is affine, i.e. H_F from (9) can be written on the form (51) with $C_F(t,T) = \mathbf{0}$ for all t,T and A_F, B_F solve the following system of ordinary differential equations.

$$\left\{\begin{aligned}\frac{\partial A_F}{\partial t} + \mathbf{d}(t)^* B_F + \frac{1}{2}B_F^*\mathbf{k_0}(t)B_F + tr\left\{C_F)\mathbf{k_0}(t)\right\} &= 0 \\ A_F(T,T) &= a_F(T)\end{aligned}\right. \tag{67}$$

$$\tag{68}$$

$$\left\{\begin{aligned}\frac{\partial B_F}{\partial t} + \mathbf{E}(t)^* B_F + \frac{1}{2}\bar{B_F}^*\mathbf{K}(t)B_F &= 0 \\ B_F(T,T) &= \mathbf{b}_F(T)\end{aligned}\right. \tag{69}$$

Many of the results proved for bond prices and futures prices will be extremely useful in providing the right intuition for the more complex situation we face when dealing with *forward prices*. Moreover, since forward prices are martingales under the forward measures (and bond prices are numeraires under those measures), the term structure of bond prices will play a special role in their term structure. As we will show, it is exactly this role of bond prices that make the term structure of futures and forward prices on a same underlying differ.

5 General Quadratic Term Structures for Forward prices

5.1 General Setting

In this section we will be looking at GQTS of forward prices,

$$\ln H_f(t,z,T) = A_f(t,T) + B_f^*(t,T)z + z^* C_f(t,T)z \tag{70}$$

where $C_{f_{(m\times m)}}$ symmetric and not necessarily different from $\mathbf{0}$, $B_{f_{(m\times 1)}}$ and $A_{f_{(1\times 1)}}$ are matrices of deterministic and smooth functions.

Already taking into account the specific shape of the forward prices term structures in (70), the term structure equation for forward prices previously derived (recall equation (16)), can be rewritten in terms of A_f, B_f, C_f, the elements characterizing the factor dynamics (α and σ), and the properties of the term structure of bond prices H_p.

Lemma 5.1 *Suppose that the Z dynamics, as before, are given by (11). Suppose, furthermore, that the term structure of forward prices is generally quadratic so that equation (70) hold. Then the differential equation (16) can be written in the following terms*

$$\begin{cases} \frac{\partial A_f}{\partial t} + \frac{\partial B_f}{\partial t} z + z^* \frac{\partial C_f}{\partial t} z + \sum_{i=1}^{m} \left[B_{f_i} + 2C_{f_i} z \right] \alpha_i + \frac{1}{2} \sum_{i,j=1}^{m} 2C_{f_{ij}} \sigma_i \sigma_j^* \\ \qquad + \frac{1}{2} \sum_{i,j=1}^{m} \left[B_{f_i} + 2C_{f_i} z \right] \sigma_i \sigma_j^* \left[B_{f_j} + 2C_{f_j} z \right] \\ \qquad + \sum_{i,j=1}^{m} \left[B_{f_i} + 2C_{f_i} z \right] \sigma_i \sigma_j^* \frac{\partial H_p}{\partial z_j} \frac{1}{H_p} = 0 \\ H_f(T,z,T) = h_f(T,z) \end{cases} \tag{71}$$

where $*$ stands for transpose, $(\cdot)_i$ for the i-th row in a vector/matrix and all partial derivatives should be evaluates at (t,z,T), and all α_i and σ_i at (t,z_i).

Proof. If we have $H_f(t,z,T) = \exp\{A_f(t,T) + B_f(t,T)^* z + z^* C_f(t,T) z\}$, for symmetric C_f then we have

$$\begin{aligned} \frac{\partial H_f}{\partial t} &= \left[\frac{\partial A_f}{\partial t} + \frac{\partial B_f}{\partial dt}^* z + z^* \frac{\partial C_f}{\partial t} z \right] H_f \\ \frac{\partial H_f}{\partial z_i} &= \left[B_{f_i} + 2C_{f_i} z \right] H_f \\ \frac{\partial^2 H_f}{\partial z_i \partial z_j} &= \left[2C_{f_{ij}} \right] H_f + \left[B_{f_i} + 2C_{f_i} z \right] \left[B_{f_j} + 2C_{f_j} z \right] H_f \end{aligned}$$

Substituting this partial derivatives into (16) and canceling the H_f present in all terms of the LHS give us the result. □

In equation (71) we see clearly how the term structure of forward prices is linked to the term structure H_p of bond prices, through the terms

$$\left[B_{f_i} + 2C_{f_i} z \right] \sigma_i \sigma_j^* \frac{\partial H_p}{\partial z_j} \frac{1}{H_p} \qquad \forall i,j. \tag{72}$$

Thus any model for forward prices must necessarily also include a model for interest rates.[22] In this study we are interested in finding an *as general as possible* interest rate setting to study GQTS of forward prices. The exact reason why a *general quadratic short rate* (GQSR) setting turn out to be the relevant one can now be fundamented.

Given the form of equation (71), it is natural to think in terms of separation of variables, i.e., to think of conditions that would guarantee that each of the terms in (71) are polynomials of z.

This will lead us to the usual conditions on α, $\sigma\sigma^*$ and the boundary condition h_f but this time also to conditions on the functional form of H_p, indirectly via the terms (72).

Before we go on we recall that we have a general quadratic Q-dynamics if α and σ in (11) are such that

$$\alpha(t,z) = \mathbf{d}(t) + \mathbf{E}(t)z \tag{73}$$

$$\sigma(t,z)\sigma(t,z)^* = \mathbf{k_0}(t) + \sum_{u=1}^{m} \mathbf{k_u}(t)z_u + \sum_{u,k=1}^{m} z_u \mathbf{g_{uk}}(t) z_k \tag{74}$$

where $\mathbf{d}$, $\mathbf{E}$, $\mathbf{k_0}$, $\mathbf{k_u}$ and $\mathbf{g_{uk}}$ for $\mathbf{u}, \mathbf{k} = \mathbf{1}, \cdots, \mathbf{m}$ are matrices of deterministic smooth functions and

$$\mathbf{K}(t) = \begin{pmatrix} \mathbf{k_1}(t) \\ \mathbf{k_2}(t) \\ \vdots \\ \mathbf{k_m}(t) \end{pmatrix}, \quad \mathbf{G}(t) = \begin{pmatrix} \mathbf{g_{11}}(t) & \mathbf{g_{12}}(t) & \cdots & \mathbf{g_{1m}}(t) \\ \mathbf{g_{21}}(t) & \mathbf{g_{22}}(t) & \cdots & \mathbf{g_{2m}}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{g_{m1}}(t) & \mathbf{g_{m2}}(t) & \cdots & \mathbf{g_{mm}}(t) \end{pmatrix}. \tag{75}$$

And that, according to Definition 4.1, on the functional form of the boundary function, we have a GQBC for the term structure of forward prices if

$$\ln h_f(T,z) = a_f(T) + \mathbf{b}_f(T)z + z * \mathbf{c}_f z. \tag{76}$$

Equations (73), (74) and (76) give us the usual conditions for α, $\sigma\sigma^*$ and h_f.

We now look more carefully on the role of bond prices via the terms (72). Concretely we need the terms (72) to be also polynomials of z and we see that this can happen only if **either** they are all (i.e. for all i, j) zero and we get a trivial polynomial, **or** if $\dfrac{\partial H_p}{\partial z_j}\dfrac{1}{H_p}$ is itself a polynomial of z and the term structure of bond prices is an exponential of polynomials of z.

We look at each of these hypothesis.

A *sufficient condition* for all terms (72) to be zero is that the term structure of bond prices is deterministic. Then for all j such that $\dfrac{\partial H_p}{\partial z_j}\dfrac{1}{H_p} \neq 0$, i.e. for all Z_j in the bond prices term structure, we have $\sigma_j^*(t) = \mathbf{0}$. This give us the classical result that in a deterministic interest rate setting the measures Q and Q^T are the same and, thus, futures and forward prices are also the same. For the purposes of this section this is however an uninteresting case[23] and we exclude it in the next assumption.

Assumption 5.1 *The term structure of bond prices is* ***stochastic****.*

[22] For instance, in [5], the study of ATS of forward prices is done in an affine interest rate setting.

[23] In deterministic interest rate settings the futures' results of the previous section apply trivially to forward prices.

Another trivial circumstance, when we would also have all terms (72) equal to zero, is when all factors related to the bond prices term structure are not *connected* in any sense to the factors related to underlying of the forward contract. That is, when there is full separability between the two sets of factors.[24] In full generality this is, of course, hard to formalize. Nonetheless, making use of the fact that bond prices and futures prices term structures can be independently determined we can also exclude this case from the analysis (Assumption 5.2).

Definition 5.1 *Given a forward contract on some underlying $\mathcal{X}$ and an interest rate setting. We say that*

- *Z_i is a* $\mathbf{Z^{(u)}}$***-factor*** *if it shows up in the futures price term structure.*[25]
- *Z_i is a* $\mathbf{Z^{(p)}}$***-factor*** *if it shows up in the bond price term structure.*

Assumption 5.2 *We assume that at least one of the following conditions hold:*

- $Z^{(u)} \cap Z^{(p)} \neq \emptyset$.
- $\sigma_i(t)\sigma_j^*(t) \neq 0$ *form some* $Z_i \in Z^{(u)}$ *and* $Z_j \in Z^{(p)}$ *and for some* t.

Given Assumptions 5.1 and 5.2, we exclude the possibility that all terms (72) are zero, and hence the condition for separability of the forward prices term structure equation (71) is now that

$$\frac{\partial H_p}{\partial z_j} \frac{1}{H_p} \qquad \forall j$$

have to be polynomial in z, and thus the term structure of bond prices to be exponential of a polynomials of z.

This is very good news, not only the GQTS for bond prices previously studied are a natural candidate, but given that exponential term structures of order higher than two suffer from consistency and computational problems, they are (except of pathological cases) the most general setting that can be considered.[26]

Moreover, since for a GQTS of forward prices we need to require general quadratic Q-dynamics anyway (from the standard requirements on α and $\sigma\sigma^*$ for separability of (71)), the only additional condition[27] that has to be included to guarantee as well a GQTS for bond prices is on the functional form of the short-rate r.

As we will see, general quadratic short rates mix very well with general quadratic forward prices, and GQTS of bond prices and forward prices may be coupled in great variety. In particular we will see below that, perhaps surprisingly, a GQTS of bond prices may be coupled with an ATS of forward prices.

In order to exclude from the analysis the (pathological) cases where higher-order term structures of bond prices may exist, we state the needed short rate setting in the form of an assumption.

[24] Also in this case there would be equivalence between futures and forward prices and the results of the previous sections would apply.

[25] Considering the same underlying $\mathcal{X}$ and hence that $h_F(t,z) = H_f(T,z)$ for all T, z.

[26] Recall discussion in Section 3.5. If higher order term structures would not suffer from such consistency problems, cubic bond prices term structures could (at least theoretically and with caution and a different classification of factors) be considered since $\frac{\partial H_p}{\partial z_j}\frac{1}{H_p}$ would then be a polynomial of degree 2.

[27] Besides a careful classification of factors, see Definition 5.2.

Assumption 5.3 *We are in a **GQSR** setting (recall Definition 3.1), that is, the short rate of interest is of the following form*

$$r(t, Z(t)) = Z(t)^* Q(t) Z(t) + g(t)^* Z(t) + f(t) \tag{77}$$

where $Q(t)_{(m\times m)}$, symmetric, and not necessarily different from $\mathbf{0}$, $g(t)_{(m\times 1)}$ and $f(t)_{1\times 1}$ are matrices of smooth and deterministic functions.

We can now give the adequate classification of factors for dealing with GQTS of *forward prices.*

This classification, which at first sight may look strange, can be justified using the intuition from both bond prices and futures prices term structures. One can guess the classification will have to do with the impact that the various factors may have on the term structure of forward prices, and this time also on their impact on bond prices term structures (since whenever bond prices affect forward prices we need then to have also a GQTS).

Definition 5.2 *Given a general quadratic Q-dynamics for Z (so that (73) and (74) hold for the α and σ in (11)), a GQBC as in (76), and a GQSR*[28] *as in (77).*

- *Z_i is a $\tilde{\mathbf{Z}}^{(\mathbf{q})}$**-factor** if it satisfies **at least one** of the following requirements:*
  - *(i) it has a quadratic impact on the boundary condition h_f, i.e., there exists T such that $\mathbf{c_f}_i(T) \neq \mathbf{0}$;*
  - *(ii) it has a quadratic impact on the short rate of interest $r(t)$, i.e., there exists t such that $Q_i(t) \neq \mathbf{0}$;*
  - *(iii) it has a quadratic impact on the functional form of the matrix $\sigma(t,z)\sigma(t,z)^*$, i.e., there exists k and t such that $\mathbf{g_{ik}}(t) \neq \mathbf{0}$;*
  - *(iv) it affects the drift term of factors satisfying (i), (ii) or (iii) i.e., for Z_j satisfying (i), (ii) or (iii) we have $\mathbf{E}_{ji}(t) \neq 0$, at least for some t.*
- *Z_i is a $\tilde{\mathbf{Z}}^{(\mathbf{l})}$**-factor** if it **does not** satisfy (i)-(iv).*

Note that considering a futures contract on a same underlying as our forward contract, (i.e. $h_F = h_f$) and the previous two classifications of factors, in Definitions 3.3 and 4.2, the following hold

$$\tilde{Z}^{(q)} = Z^{(q)} \cup \bar{Z}^{(q)} \qquad \text{and} \qquad \tilde{Z}^{(l)} = Z^{(l)} \cap \bar{Z}^{(l)}.$$

The classification of the factors in Definition 5.2 have, thus, some implications for many matrices in our standard setup.

Remark 5.1 *We note that given Definition 5.2:*

- *it is always possible to reorder the vector of factors Z, so that we have*

$$Z = \begin{pmatrix} \tilde{Z}^{(q)} \\ \tilde{Z}^{(l)} \end{pmatrix}$$

[28] If Assumptions 5.1 and 5.2 do not hold, we do not have to have a GQSR, and this classification of factors reduces to that of Definition 4.2.

- *with this reordering of factors we have,* ***by definition****, the following shapes for* $\mathbf{E}$ *and* $\mathbf{G}$ *in (73) and (75),*

$$\mathbf{E}(t) = \begin{pmatrix} \mathbf{E}^{(\mathbf{qq})}(t) & \mathbf{0} \\ \mathbf{E}^{(\mathbf{lq})}(t) & \mathbf{E}^{(\mathbf{ll})}(t) \end{pmatrix} \quad \mathbf{G}(t) = \begin{pmatrix} \mathbf{G}^{(\mathbf{qq})}(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{78}$$

and for $\mathbf{c}_f$ *and* $\mathbf{Q}$ *in (76) and (77)*

$$\mathbf{c}_f(T) = \begin{pmatrix} \mathbf{c^{(qq)}}_f(T) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad Q(t) = \begin{pmatrix} Q^{(qq)}(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{79}$$

5.2 Main result on Forward prices

Theorem 5.2 *Suppose that Assumptions 2.1 and 2.2 are in force. Furthermore suppose that* Z *has general quadratic Q-dynamics, (i.e., that* α *and* σ *from the factor dynamics (11), satisfy (73)-(74)), that we have a GQBC, so that the boundary condition* h_f *from (10), has the quadratic form in (76) and that we are in a GQSR setting (so that (77) hold).*

Finally assume that the factors are reordered as $Z = \begin{pmatrix} \tilde{Z}^{(q)} \\ \tilde{Z}^{(l)} \end{pmatrix}$ *(using Definition 5.2), and that the following restrictions apply to* $\mathbf{k_u}$ *and* $\mathbf{g_{uk}}$ *in (74):*

$$\mathbf{k_u}(t) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{k_u^{(ll)}}(t) \end{pmatrix} \qquad \forall\ \mathbf{u} \text{ and } \forall\ t \tag{80}$$

$$\mathbf{g_{uk}}(t) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{g_{uk}^{(ll)}}(t) \end{pmatrix} \qquad \forall\ t \text{ and } \forall\ \mathbf{u},\mathbf{k} \text{ s.t. } z_u, z_k \in \tilde{\mathbf{z}}^{(\mathbf{q})}. \tag{81}$$

Then the term structure of forward prices is generally quadratic, i.e. H_f *from (9) can be written on the form (70) and* A_f*,* B_f *and* C_f *can be obtained by solving the following system of ordinary differential equations.*

$$\begin{cases} \dfrac{\partial A_f}{\partial t} + \mathbf{d}(t)^* B_f + \dfrac{1}{2} B_f^* \mathbf{k_0}(t) B_f + B_f^* \mathbf{k_0}(t) B_p + tr\left\{C_f \mathbf{k_0}(t)\right\} = 0 \\ A_f(T,T) = a_f(T) \end{cases} \tag{82}$$

$$\begin{cases} \dfrac{\partial B_f}{\partial t} + \mathbf{E}(t)^* B_f + 2C_f \mathbf{d}(t) + \dfrac{1}{2} \bar{B_f}^* \mathbf{K}(t) B_f + 2C_f \mathbf{k_0}(t) B_f \\ \qquad + \bar{B_f}^* \mathbf{K}(t) B_p + 2C_f \mathbf{k_0}(t) B_p + 2C_p \mathbf{k_0}(t) B_f = \mathbf{0} \\ B_f(T,T) = \mathbf{b}_f(T) \end{cases} \tag{83}$$

$$\begin{cases} \dfrac{\partial C_f}{\partial t} + C_f \mathbf{E}(t) + \mathbf{E}(t)^* C_f + 2C_f \mathbf{k_0}(t) C_f + \dfrac{1}{2} \bar{B_f}^* \mathbf{G}(t) \bar{B_f} \\ \qquad + 4C_f \mathbf{k_0}(t) C_p + \bar{B_f}^* \mathbf{G}(t) \bar{B_p} = \mathbf{0} \\ C_f(T,T) = \mathbf{c}_f(T) \end{cases} \tag{84}$$

where C_f *has the special form* $C_f = \begin{pmatrix} C_f^{(qq)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$*,* A_f*,* B_f*,* $C_f^{(qq)}$ *should be evaluated at* (t,T) *and* B_p*,* C_p *solve (26)-(27).*

$\mathbf{E}$ *and* $\mathbf{d}$ *are the same as in (73),* $\mathbf{k_0}$*,* $\mathbf{K}$ *and* $\mathbf{G}$ *the same as in (74)-(75), and where* $\bar{B}_f$ *and* $\bar{B}_p$ *follows the same idea as in (28) and have dimension* $(m^2 \times m)$*.* A_f*,* B_f*,* C_f

Proof. We need to show that $H_f(t, z, T)$ from (70) where A_f, B_f and C_f solve (82)-(84), solves the PDE (71) that uniquely characterizes the forward prices in this setting. From the functional form of α, $\sigma\sigma^*$ and r, in (73),(74), and (77), respectively, Theorem 3.2 guarantee a GQTS for bond prices with $C_p^{(ll)}(t,T) = \mathbf{0}$ and $C_p^{(lq)}(t,T) = C_p^{(ql)}(t,T) = \mathbf{0}$ for all t, T. This follows form the fact that $Z^{(q)} \subset \tilde{Z}^{(q)}$. Using the definition of a GQTS of bond prices we also get $\frac{\partial H_p}{\partial z_j}\frac{1}{H_p} = (B_p)_j + 2(C_p)_j z$. Taking all this into account and using the restrictions (80)(81), the PDE (71) becomes always a separable equation equivalent to (82)-(84). If all $Z_i \in \tilde{Z}^{(q)}$, we know, that $\mathbf{K}(t) = \mathbf{0}$ and $\mathbf{G}(t) = \mathbf{0}$ for all t. Hence, equation (71) becomes separable up to quadratic terms of z. For this case we note that there is more than one way to write the quadratic terms of (71) in vector notation, but only one way compatible with our non-restrictive assumption of a symmetry for the matrices C_f and C_p. If there exist $\tilde{Z}^{(l)}$ factors, equation (71) will still be separable but up to terms of order four in the state variable z. The third and the fourth order terms in z will result from the terms $\frac{\partial^2 H_f}{\partial z_i \partial z_j}$ and $\frac{\partial H_f}{\partial z_i}\sigma_i\sigma_j^*\frac{\partial H_p}{\partial z_j}\frac{1}{H_p}$ when both $Z_i, Z_j \in \tilde{Z}^{(l)}$. $C_f^{(ll)}(t,T) = \mathbf{0}$ solves those conditions for all t, T. The restrictions (80)-(81) imposed on the matrices $\mathbf{K}$ and $\mathbf{G}$, together with the restrictions on $\mathbf{E}$, $\mathbf{G}$, $\mathbf{c}_f$ and Q (check (78) and (79)) guarantee not only that $C_f^{(ll)}(t,T) = \mathbf{0}$ for all t, T is also solution for lower order terms, but that $C_f^{(ql)}(t,T) = C_f^{(lq)*}(t,T) = \mathbf{0}$ for all t, T is also a solution to (84). □

The system of ODEs (82)-(84) seems quite complex, but once again, in most practical situations it is possible to decompose it in much easier smaller systems. It is also important to note that B_p and C_p are independently obtained. So the natural steps to compute a *forward prices* term structure are:

1) obtain B_p and C_p from solving (26)-(27) and substitute the solutions into (82)-(84),

2) split the ODEs (83)-(84) into simpler ODEs for $B_f^{(l)}$, $B_f^{(q)}$ and $C_f^{(qq)}$ and iteratively solve them.[29]

3) substitute the solutions into (82) and integrate to obtain A_f.

Important consequences of Theorem 5.2 are the following.

Corollary 5.3 ***(Linear Factors)*** *The* $\tilde{Z}^{(l)}$*-factors are linear factors in a GQTS of forward prices.* [30]

Corollary 5.4 *Necessary conditions of a QTS of forward prices are* $\mathbf{c}_f(T) \neq \mathbf{0}$ *for some* T *or* $\mathbf{G}(t) \neq \mathbf{0}$ *for some* t*.*

Corollary 5.3 follows immediately from $C_f^{(ll)}(t,T) = \mathbf{0}$ and $C_f^{(ql)}(t,T) = C_f^{(ql)*}(t,T) = \mathbf{0}$ for all t, T. Corollary 5.4 can easily be checked by taking $\mathbf{c}_f(T) = \mathbf{0}$ and $\mathbf{G}(t) = \mathbf{0}$ in (84) and noting that $C_f(t,T) = \mathbf{0}$ solves (84), for all t, T.

[29] Recall the procedure in Section 3.2.2.

[30] Recall Definition 3.4.

It follows that factors satisfying requirement (ii) of Definition 5.2, i.e., factors having a quadratic impact on the short rate r, do not necessarily have a quadratic impact on the *forward prices* term structure. The next Remark is a direct consequence of Corollary 5.4.

Remark 5.2 *For some models (those with $\mathbf{c}_f(T) = \mathbf{0}$ for all T), a volatility restricted (by $\mathbf{G}(t) = \mathbf{0}$ for all t) GQTS for bond prices is consistent with and ATS of forward prices.*

A concrete example of a model with an ATS for forward prices but a QTS for bond prices in now presented. In the examples section we explicitly compute both term structures.

Example 3 *Consider the model*

$$\begin{aligned} dZ_1(t) &= [\beta_1(t) - \alpha_1 Z_1(t)]\, dt + \sigma_1 dW_1(t) \\ dZ_2(t) &= [\beta_2(t) - \alpha_2 Z_2(t)]\, dt + \sigma_2 dW_2(t) \end{aligned}$$

where α_1, α_2, σ_1 and σ_2 are deterministic constants and $\beta_1(\cdot)$, $\beta_2(\cdot)$, $a_1(\cdot)$, $a_2(\cdot)$, $a_3(\cdot)$, $k_0(\cdot)$ and $k_3(\cdot)$ deterministic functions of time and W_1,W_2 are independent Wiener processes.

The following relations between the factors above, the underlying of the forward contract S, and the short rate r.

$$\ln S(t) = Z_1(t) \qquad r(t) = \frac{1}{2}\left[Z_1^2(t) + Z_2^2(t)\right]$$

First, note that in this model we have general quadratic Q-dynamics with $\mathbf{K}(t) = \mathbf{0}$ and $\mathbf{G}(t) = \mathbf{0}$ for all t.

Secondly, Z_1 and Z_2 are $Z^{(q)}$-factors according to Definition 3.3, so the term structure of bond prices will actually be quadratic in both these factors (Proposition 3.5).

Finally, since Z_1 show only linearly in the boundary condition $\ln S = Z_1$ (and Z_2 does not show up) we have $\mathbf{c}_f(T) = \mathbf{0}$ for all T and (Corollary 5.4) and ATS for forward prices.

From what have just been said one easily realizes that not all $\tilde{Z}^{(q)}$ factors will have a *quadratic impact*[31] on the *forward prices* term structure. In fact the $\tilde{Z}^{(q)}$-factors should be interpreted has the factors that, under the some regularity conditions, will have **either** a quadratic impact only on the term structure of bond prices, **or** a quadratic impact only on the term structure of forward prices, **or** on both term structures.

The only easy answer in given in the next Lemma.

Lemma 5.5 *A factor $Z_i \in \bar{\bar{Z}}^{(q)}$ for which $\mathbf{c}_{f_i}(T) \neq \mathbf{0}$ at least for some T, has a quadratic impact in the futures prices term structure.*

Proof. It follows from $C_{F_i}(T,T) = \mathbf{c}_{F_i}(T)$ and the definition of quadratic impact. □

To understand the exact role of bond prices is the same as to understand the difference between futures and forward prices term structures. The task is not an easy one and it is complicated by the various relations that may exist among various types of factors and because the same factors may be of different types depending on the specific term structure we are looking at.

[31] Recall Definition 3.4.

In the next section we study the difference between futures and forward prices term structures in the (stochastic) GQSR setting. Most results will be, however, of a qualitative nature. Only in a very concrete situation it is possible to quantify this difference.

5.3 Connection between Bond, Futures and Forward prices

As is well known forward prices (with maturity T) are martingales under the T-forward measure, while futures prices are martingales under the risk-neutral measure Q. Loosely speaking, this means that except for deterministic interest rates or full separability between the factors related to the underlying of a forward contract ($Z^{(u)}$ in Definition 5.1) and to the short rate ($Z^{(p)}$ in Definition 5.1), we have no reason to believe that the are equal.

Despite this fact, and with the exception of the obvious equivalence between the two prices in these extreme settings, there has been few studies in the literature studying the relation between the two prices. We now show how the study of GQTS structures of futures prices is somehow included in the study of GQTS of forward prices and analyze their relations in a *general quadratic short rate*(GQRS) setting.

Our motivation comes from the fact that the second part of Theorem 5.2 is equivalent to the second part of Theorem 4.1 if we "delete" the terms dependent on the bond price term structure (i.e. terms with B_p or C_p). Thus the comparison of the systems of ODEs (60)-(62) and (82)-(84) should be useful in understanding the difference between futures and forward prices.

The only point of caution has to do with the fact that the classification of factors in both theorems are not the same. We note that in forward prices term structures we typically have to deal with more factors than in futures prices, since we also have to model the term structure of bond prices. In some situations there may be a GQTS for *futures prices* but not for *forward prices*. The following example may help to clarify this point.

Example 4 *Consider the following two-factor model*

$$dZ(t) = (\cdots)dt + \begin{pmatrix} \sqrt{k_{01}(t) + k_1(t)Z_1(t)} & 0 \\ 0 & \sqrt{k_{02}(t) + k_2(t)Z_2(t)} \end{pmatrix} dW(t)$$

where we have $r(t) = \frac{1}{2}\left[Z_1(t)^2 + Z_1(t)^2\right]$ *and the spot price, of the underlying to the futures and forward contract, is given by* $S(t) = e^{b_1 Z_1(t) + b_2 Z_2(t)}$.

In this model we have a linear boundary function h_F and $\sigma(t,z)\sigma(t,z)^*$, is also obviously linear in z, so this model is compatible with an ATS for futures (there are only $\bar{Z}^{(l)}$ factors according to Definition 4.2).

However, since the term structure of bond prices will be quadratic in both Z_1 and Z_2 (both Z_1 and Z_2 are $Z^{(q)}$ hence $\tilde{Z}^{(q)}$ factors, Definitions 3.3 and 5.2), and the volatility structure is not deterministic for both factors, we are ***not*** *under the conditions for a GQTS for bond prices and hence also* ***not*** *under the conditions for an or GQTS for forward prices.*

In the situations when futures prices have a GQTS but forward prices do not, the comparison the ODEs systems (60)-(62) and (82)-(84) cannot help us, your goal is this section is to compare the term structure of futures and forward prices when **both** are GQTS.

Assumption 5.4 *We are under the conditions for a GQTS for forward prices.*

A direct consequence of Assumption 5.4 is that we also have GQTS for bond prices and for futures prices. As we will see the following classification of factors will be useful.

Definition 5.3 *Consider that we are under the conditions for a GQTS of bond prices and for a GQTS of futures prices. Then we have the following classification of factors.*[32]

- Z_i *is a* $\mathbf{Z^{(qp)}}$***-factor*** *if it has a quadratic impact on the term structure of bond prices.*
- Z_i *is a* $\mathbf{Z^{(lp)}}$***-factor*** *if it a linear impact on the term structure of bond prices.*
- Z_i *is a* $\mathbf{Z^{(p)}}$***-factor*** *if* $Z_i \in \left[Z^{(qp)} \cup Z^{(lp)}\right]$.

- Z_i *is a* $\mathbf{Z^{(qu)}}$***-factor*** *if it has a quadratic impact on the term structure of futures prices.*
- Z_i *is a* $\mathbf{Z^{(lu)}}$***-factor*** *if it has a linear impact on the term structure of futures prices.*
- Z_i *is a* $\mathbf{Z^{(u)}}$***-factor*** *if* $Z_i \in \left[Z^{(qu)} \cup Z^{(lu)}\right]$.

Before we go on we note that given the *a priori* classification of factors for bond and futures prices (Definitions 3.3 and 4.2, respectively), and under some regularity conditions, we know from Propositions 3.5 and 4.4 which factors will have a linear or a quadratic impact.

So, we do not need to solve any system of ODEs to identify all the factors $Z^{(qp)}$, $Z^{(lp)}$, $Z^{(qu)}$ and $Z^{(lu)}$.

We also note that these classifications are not mutually exclusive. The following example may help to illustrate this point.

Example 5 *Consider the following naive 5-factor model*

$$\begin{aligned}
dZ_1(t) &= [\beta_1 + \alpha_1 Z_1(t)]\, dt + \sigma_1 dW_1(t) \\
dZ_2(t) &= [\beta_2 + \alpha_2 Z_3(t)]\, dt + \sigma_2 dW_2(t) \\
dZ_3(t) &= [\beta_3 + \alpha_3 Z_3(t)]\, dt + \sigma_3 dW_3(t) \\
dZ_4(t) &= [\beta_4 + \alpha_4 Z_4(t)]\, dt + \sigma_4 \sqrt{Z_2(t)^2 + Z_4(t)} dW_4(t) \\
dZ_5(t) &= [\beta_5 + \alpha_5 Z_5(t)]\, dt + \sigma_5 dW_5(t) \\
r(t) &= Z_1(t)^2 + Z_2(t)^2 \\
h_f(T, Z(t)) &= Z_4(t) + Z_5(t)^2
\end{aligned}$$

where all W_i *are independent Wiener processes.*

According to Definition 5.3 we have the following classification of factors:

$$Z^{(qu)} = \{Z_5\} \quad Z^{(lu)} = \{Z_4, Z_2\} \quad Z^{(qp)} = \{Z_1, Z_2, Z_3\} \quad Z^{(lp)} = \emptyset$$

We now define various types of correlation concepts.

[32] Note that the $Z^{(p)}$ and $Z^{(u)}$ factors are the same here as in Definition 5.1.

Definition 5.4 *Any two stochastic Z_i and Z_j are said to be **correlated** in a general quadratic Q-dynamics if $\sigma\sigma^*$ have the form in (74) and we have at least one of the following conditions satisfied:*

Deterministic correlation:	*there exist a t such that*	$\mathbf{k_0}_{ij}(t) \neq 0.$
Linear correlation:	*there exists a t and an $\mathbf{u}$ such that*	$\mathbf{k_u}_{ij}(t) \neq 0.$
Quadratic correlation:	*there exists a t and $\mathbf{u}, \mathbf{k}$ such that*	$\mathbf{g_{uk}}_{ij}(t) \neq 0.$

The use of the name correlated factors in Definition 5.4 is justified by noting that off-diagonal terms in a volatility matrix will imply nonzero correlation between factors. We note that in the way we define correlation any stochastic factor is, by definition, correlated to itself but that a deterministic process is not. This turns out to be a crucial point since only stochastic factors may play a role in the difference between futures and forward prices.

The next example may help to clarify these concepts.

Example 6 *Consider the following three-factor model*

$$d\begin{pmatrix} Z_1(t) \\ Z_2(t) \\ Z_3(t) \\ Z_4(t) \end{pmatrix} = \left[\mathbf{d}(t) + \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_4 \end{pmatrix} \begin{pmatrix} Z_1(t) \\ Z_2(t) \\ Z_3(t) \\ Z_4(t) \end{pmatrix}\right] dt + \sigma(t, Z(t))dW_t$$

with

$$\begin{aligned} \sigma(t, Z(t))\sigma^*(t, Z(t)) \quad = \quad & \overbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 & 0 & 0 \\ 0 & \rho_{23}\sigma_2\sigma_3 & \sigma_3^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}^{\mathbf{k_0}} + \overbrace{\begin{pmatrix} \sigma_1^2 & 0 & 0 & \delta\sigma_1\sigma_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \delta\sigma_1\sigma_4 & 0 & 0 & \sigma_4^2 \end{pmatrix}}^{\mathbf{k_1}} Z_3(t) \\ & + \overbrace{\begin{pmatrix} \sigma_1^2 & 0 & 0 & \delta\sigma_1\sigma_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \delta\sigma_1\sigma_4 & 0 & 0 & \sigma_4^2 \end{pmatrix}}^{\mathbf{g_{22}}} Z_2^2(t). \end{aligned}$$

In this case Z_2 and Z_3 are deterministicly correlated, while Z_1 and Z_4 are both linearly (via $\mathbf{k_3}$) and quadraticly correlated (via $\mathbf{g_{22}}$).

We are now in conditions to our main result on the comparison between futures and forward prices GQTS.

Proposition 5.6 *Consider a given underlying T-claim $\mathcal{X}$. Given Definition 5.3, there will be differences between the term structures of forward and futures prices for $\mathcal{X}$ only if $Z^{(p)}$-factors are correlated with $Z^{(u)}$-factors.*

Moreover, if $Z^{(p)}$-factors are correlated with $Z^{(u)}$-factors, the following hold

- *the term structure of futures and forward prices will differ in the* ***quadratic term****, i.e. $C_F(t,T) \neq C_f(t,T)$ for some t,T, if at least one of the following conditions hold:*
  - *(i) $Z^{(qp)}$-factors are deterministicly correlated with $Z^{(qu)}$-factors*
  - *(ii) $Z^{(lp)}$-factors are quadraticly correlated with $Z^{(lu)}$-factors.*
- *the term structure of futures and forward prices will differ in the* ***linear term****, i.e. $B_F(t,T) \neq B_f(t,T)$ for some t,T, if at least one of the following conditions hold:*
  - *(iii) $C_F(t,T) \neq C_f(t,T)$ at least for some t,T*
  - *(iv) $Z^{(qp)}$-factors are deterministicly correlated with $Z^{(lu)}$-factors*
  - *(v) $Z^{(qu)}$-factors are deterministicly correlated with $Z^{(lp)}$-factors*
  - *(vi) $Z^{(lp)}$-factors are linearly correlated with $rZ^{(lf)}$-factors.*
- *the term structure of futures and forward prices will differ in the* ***deterministic*** *term, i.e. $A_F(t,T) \neq A_f(t,T)$ for some t,T, if at least one of the following conditions hold:*
  - *(vii) $C_F(t,T) \neq C_f(t,T)$ at least for some t,T*
  - *(viii) $B_F(t,T) \neq B_f(t,T)$ at least for some t,T*
  - *(ix) $\bar{Z}^{(lp)}$ are deterministicly correlated with $\bar{Z}^{(lu)}$-factors.*

Proof. Consider that B_p, C_p, B_F and C_F solve (26), (27),(61) and (62), respectively. If any of the conditions (i)-(ii) hold, C_F is not a solution for (84), since $C_f\mathbf{k_0}(t)C_p \neq \mathbf{0}$, or $\bar{B_f}^*\mathbf{G}(t)\bar{B}_p \neq \mathbf{0}$ at least for some t,T and under those conditions the equations (62) and (84) are not the same. Otherwise C_f and C_F have the same ODE so $C_f = C_F$ solves (84). If any of the conditions (iii)-(vi) hold, B_F is not a solution for (83), since $C_F \neq C_f$, or $\bar{B_f}^*\mathbf{K}(t)B_p \neq \mathbf{0}$, or $C_f\mathbf{k_0}(t)B_p \neq \mathbf{0}$ or $C_p\mathbf{k_0}(t)B_f \neq \mathbf{0}$ at least for some t,T and under those conditions the equations (61) and (83) are not the same. Otherwise B_f and B_F have the same ODE so $B_f = B_F$ solves (83). If any of the conditions (vii)-(ix) hold, A_F is not a solution for (82), since $C_F \neq C_f$ or $B_F \neq B_f$ or $B_f^*\mathbf{k_0}(t)B_p \neq \mathbf{0}$ at least for some t,T and under those conditions the equations (60) and (82) are not the same. Otherwise A_f and A_F have the same ODE so $A_f = A_F$ solves (82). □

Unfortunately Proposition 5.6 give us only **qualitative** differences. Given the complexity of the matrix system of ODEs it is impossible to quantify the differences when they occur in the quadratic or the linear terms. However, if the difference occurs **only** in the deterministic term we can explicitly compute it. In that case we say that the forward and futures prices differ by a deterministic *adjustment term.*

Proposition 5.7 *Consider the classification of factors of Definition 5.3. If the only correlations (as defined by Definition 5.4) between $Z^{(p)}$ and $Z^{(u)}$ factors are via deterministic correlations of $Z^{(lp)}$ and $Z^{(lu)}$ factors, then the term structures of futures and forward prices differ by a deterministic adjustment term $D(t,T)$, and we have*

$$H_f(t,z,T) = D(t,T)H_F(t,z,T) \quad \text{with} \quad D(t,T) = e^{\left\{\int_t^T B_f(s,T)^*\mathbf{k_0}(s)B_p(s,T)ds\right\}}.$$

Proof. Proposition 5.6 show that the impact is only deterministic. The exact form of the adjustment term $D(t,T)$ follows form the fact that both (82) and (60) case be solved by simple integration and in we have

$$A_f(t,T) = A_F(t,T) + \int_t^T B_f(s,T)^* \mathbf{k_0}(s) B_p(s,T) ds. \qquad \square$$

To give an example of this use of Proposition 5.7 we can simply say that (as long as its conditions are satisfied) one can extend results from futures prices to forward prices by simply considering the adjustment term $D(t,T)$.

On what the conditions themselves are concerned, even though they may look like quite restrictive given the generality used in previous sections, in practice they are actually not that restrictive: most multi-factor models previously studied in the literature assume constant volatility structure for the factors (so only deterministic correlations) and most studies are preformed in an affine interest rate setting. Note that this two conditions are even more restrictive than those of Proposition 5.7).

5.4 Important Special Cases

For completeness we present here the corollaries on *Gaussian-QTS* and *ATS* for forward prices. As referred before, the term structure of *forward prices* has received much less attention then the term structure of bond prices, or even of futures prices.

On the theoretical literature, the exceptions is Björk and Landén [5] who present a result on ATS of forward prices in an affine interest rate setting. Below we show how their result can be recovered from the ATS corollary. [33] Quadratic term structures of forward prices, pure or non-pure, have (to our knowledge) not been studied previously.

Corollary 5.8 ***(Gaussian-QTS)*** *Suppose that Assumptions 2.1 and 2.2 are in force. Furthermore suppose that we are in a GQSR, so that (77), hold and that we have a GQBC, so that (76) hold.*

Finally, assume that the functions α and σ from the factor dynamics (11) are of the following form:

$$\begin{aligned} \alpha(t,z) &= \mathbf{d}(t) + \mathbf{E}(t)z \\ \sigma(t,z)\sigma(t,z)^* &= \mathbf{k_0}(t) \end{aligned}$$

where $\mathbf{d}$, $\mathbf{E}$, $\mathbf{k_0}$ are matrices of deterministic smooth functions.

Then the term structure of forward prices is generally quadratic, i.e. H_f from (9) can be written on the form (70) and A_f, B_f and C_f can be obtained by solving the following system of ordinary differential equations.

$$\left\{ \begin{aligned} \frac{\partial A_f}{\partial t} + \mathbf{d}(t)^* B_f + \frac{1}{2} B_f^* \mathbf{k_0}(t) B_f + B_f^* \mathbf{k_0}(t) B_p + tr\left\{C_f \mathbf{k_0}(t)\right\} &= 0 \\ A_f(T,T) &= a_f(T) \end{aligned} \right. \qquad (85)$$

[33] They study ATS in situations where the bond prices have themselves and ATS, but they allow the factor dynamics to be driven by a multidimensional wiener process as well as by a general marked point process. Since in this paper we only consider the factor dynamics to be driven by a multidimension Wiener process we recover their result only to the extent that the general marked point process is not considered.

$$\begin{cases} \dfrac{\partial B_f}{\partial t} + \mathbf{E}(t)^* B_f + 2C_f \mathbf{d}(t) + 2C_f \mathbf{k_0}(t) B_f + 2C_p \mathbf{k_0}(t) B_f &= \mathbf{0} \\ B_f(T,T) &= \mathbf{b}_f(T) \end{cases} \tag{86}$$

$$\begin{cases} \dfrac{\partial C_f}{\partial t} + C_f \mathbf{E}(t) + \mathbf{E}(t)^* C_f + 2C_f \mathbf{k_0}(t) C_f + 4C_f \mathbf{k_0}(t) C_p &= \mathbf{0} \\ C_f(T,T) &= \mathbf{c}_f(T) \end{cases} \tag{87}$$

where A_f, B_f, C_f should be evaluated at (t,T) and B_p and C_p solve (26)-(27).

Corollary 5.9 ***(ATS)*** *Suppose that Assumptions 2.1 and 2.2 are in force. Furthermore suppose that we are in a GQSR, so that (77) hold and that we have an affine boundary condition h_f, so that (76) hold with $\mathbf{c}_f(T) = \mathbf{0}, \forall\ T$.*

Finally, assume that the factors are reordered as $Z = \begin{pmatrix} Z^{(q)} \\ Z^{(l)} \end{pmatrix}$ (using Definition 3.3)[34]*, and that α and σ from the factor dynamics (11) are of the following form:*

$$\alpha(t,z) = \mathbf{d}(t) + \mathbf{E}(t) z$$

$$\sigma(t,z)\sigma(t,z)^* = \begin{pmatrix} \mathbf{k_0^{(qq)}}(t) & \mathbf{k_0^{(ql)}}(t) \\ \mathbf{k_0^{(lq)}}(t) & \mathbf{k_0^{(ll)}}(t) + \sum_{u=1}^{m} \mathbf{k_u^{(ll)}}(t) z_u \end{pmatrix}$$

where $\mathbf{d}$, $\mathbf{E}$, $\mathbf{k_0}$ and $\mathbf{k_u}, \forall\ u$, are matrices of deterministic smooth functions.

Then the term structure of forward prices is affine, i.e. H_p from (9) can be written on the form (70) with $C_f(t,T) = \mathbf{0}$, for all t,T and where A_f and B_f can be obtained by solving the following system of ordinary differential equations.

$$\begin{cases} \dfrac{\partial A_f}{\partial t} + \mathbf{d}(t)^* B_f + \dfrac{1}{2} B_f^* \mathbf{k_0}(t) B_f + B_f^* \mathbf{k_0}(t) B_p &= 0 \\ A_f(T,T) &= a_f(T) \end{cases} \tag{88}$$

$$\begin{cases} \dfrac{\partial B_f}{\partial t} + \mathbf{E}(t)^* B_f + \dfrac{1}{2} \bar{B_f}^* \mathbf{K}(t) B_f + \bar{B_f}^* \mathbf{K}(t) B_p + 2C_p \mathbf{k_0}(t) B_f &= \mathbf{0} \\ B_f(T,T) &= \mathbf{b}_f(T) \end{cases} \tag{89}$$

A_f, B_f should be evaluated at (t,T) and B_p and C_p solve (26)-(27).

If we have an affine short rate (that is if (77) holds with $Q(T) = \mathbf{0}$ for all possible t) and linear α and $\sigma\sigma^*$, we have an ATS for bond prices (Corollary 3.10) and hence we know that $C_p(t,T) = \mathbf{0}$, for all t and T. In this case there is also no need to classify factors. Using this fact we can recover Björk and Landén [5] for ATS of forward prices in an affine interest rate setting (just take $C_p = \mathbf{0}$ in (88)-(89) and take $Z^{(q)} = \emptyset$).

It is, however, important to stress that some quadratic short rate settings are consistent with an ATS for forward prices.[35]

[34] Note that the only reason to reorder factors have to do in this case with the possibility of having quadratic factors on the bond prices term structure, so the relevant reordering is that of bond prices.

[35] So, ATS of bond prices is *sufficient* for an ATS of forward prices but **not** necessary.

We now apply the result on GQTS of *bond, forward* and *futures* prices and their connections to some models previously studied in the literature and some original models.

6 Examples of General Quadratic Term Structures

In this section we selected some factor models to exemplify the theory of GQTS for bond, futures and forward prices and their relations. Some of the models have been proposed in the literature, others, in particularly the ones concerning quadratic term structures of forward and futures prices, are new.

All the models are given directly under the martingale measure Q.

6.1 GQTS of bond prices

6.1.1 Example 1 - PQTS

Consider the following naive PQTS model for the short interest rate

$$\begin{aligned} dZ_1(t) &= [\beta_1(t) - \alpha_1 Z_1(t)]\, dt + \sigma_1 dW_1(t) \\ dZ_2(t) &= [\beta_2(t) - \alpha_2 Z_2(t)]\, dt + \sigma_2 dW_2(t) \\ dW_1(t)dW_2(t) &= 0dt \\ \\ r(t) &= \frac{1}{2}\left[Z_1^2(t) + Z_2^2(t)\right] \end{aligned}$$

where α_1, α_2, σ_1 and σ_2 are deterministic constants and $\beta_1(\cdot)$ and $\beta_2(\cdot)$ deterministic functions of time.

We have a GQSR with

$$Q(t) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \qquad g(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad f(t) = 0.$$

Both factors are $Z^{(q)}$-factors (according to Definition 3.3) and we have a general quadratic Q-dynamics for the factors, i.e., (19) and (20) hold with

$$\begin{aligned} \mathbf{d}(t) &= \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix}, \quad \mathbf{E}(t) = \begin{pmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix} \\ \mathbf{k_0}(t) &= \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}, \quad \mathbf{k_u}(t) = \mathbf{0}, \quad \mathbf{g_{uk}}(t) = \mathbf{0}. \end{aligned}$$

Hence, we are under the conditions for a PQTS for bond prices, and to obtain A_p, B_p and C_p we need to solve the system (45)-(47).

For the model above C_p solves,

$$\left\{ \begin{aligned} \frac{\partial C_p}{\partial t} + 2\begin{pmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix} C_p + 2C_p^* \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} C_p &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \\ C_p(T,T) &= 0 \end{aligned} \right. .$$

We can immediately see that $C_p^{(12)} = C_p^{(21)} = 0$ is part of the solution, so we can solve the simpler ODE

$$\begin{cases} \frac{\partial}{\partial t}\begin{pmatrix} C_p^{(11)} & 0 \\ 0 & C_p^{(22)} \end{pmatrix} + 2\begin{pmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix}\begin{pmatrix} C_p^{(11)} & 0 \\ 0 & C_p^{(22)} \end{pmatrix} & \\ \quad +2\begin{pmatrix} C_p^{(11)} & 0 \\ 0 & C_p^{(22)} \end{pmatrix}\begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}\begin{pmatrix} C_p^{(11)} & 0 \\ 0 & C_p^{(22)} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \\ \begin{pmatrix} C_p^{(11)} & 0 \\ 0 & C_p^{(22)} \end{pmatrix}(T,T) = 0 \end{cases}$$

and we just need to solve two scalar Riccati equations.

The final solution of C_p is then,

$$\begin{aligned} C_p(t,T) &= \begin{pmatrix} C_p^{(11)}(t,T) & 0 \\ 0 & C_p^{(22)}(t,T) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1-e^{2\gamma_1(T-t)}}{2(\alpha_1+\gamma_1)\left(e^{2\gamma_1(T-t)}-1\right)+4\gamma_1} & 0 \\ 0 & \frac{1-e^{2\gamma_2(T-t)}}{2(\alpha_2+\gamma_1)\left(e^{2\gamma_2(T-t)}-1\right)+4\gamma_1} \end{pmatrix} \end{aligned} \tag{90}$$

and $\gamma_i = \sqrt{\alpha_i^2 + \sigma_i^2}$ for $i = 1, 2$.

With these solutions we can go on and solve the ODE for B_p and A_p.

$$\begin{cases} \frac{\partial B_p}{\partial t} + \begin{pmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix} B_p + 2C_p\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + 2C_p\begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} B_p = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ B_p(T,T) = 0 \end{cases}$$

Which simplifies to

$$\begin{cases} \frac{\partial B_p}{\partial t} + \begin{pmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix} B_p + 2\begin{pmatrix} C_p^{(11)}(t,T) & 0 \\ 0 & C_p^{(22)}(t,T) \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} & \\ \quad +2\begin{pmatrix} C_p^{(11)}(t,T) & 0 \\ 0 & C_p^{(22)}(t,T) \end{pmatrix}\begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} B_p = g(t) \\ B_p(T,T) = 0 \end{cases}$$

and the solution is given by

$$B_p^{(i)}(t,T) = 2\int_t^T e^{\int_t^s \left(\alpha_i - 2C_p^{(ii)}(u,T)\right)du}\beta_i(s)C_p^{(ii)}(s,T)ds \qquad i = 1, 2 \tag{91}$$

Finally we can obtain A_p by simple integration of (45), that is

$$A_p(t,T) = \sum_{i=1}^{m}\left\{\int_t^T \beta_i(s)B_p^{(i)}(s,T)ds + \frac{1}{2}\sigma_i^2\int_t^T \left[B_p^{(i)}(s,T\right]^2 ds + \sigma_i^2\int_t^T C_p^{(ii)}(s,T)ds\right\}$$

6.1.2 Example 2 - A naive non-pure QTS

Here we show in a very simple example that we can allow for non deterministic volatility structures whenever we can (*a priori*) classify factors.

Consider the following (naive) two-factor model

$$\begin{aligned} dZ_1(t) &= \beta_1(t)dt + \sigma_1 dW_1(t) \\ dZ_2(t) &= \beta_2(t)dt + \sqrt{Z_1(t)+Z_2(t)}dW_2(t) \\ \\ r(t) &= Z_1^2(t) + Z_2(t) \end{aligned}$$

where σ_1 is a deterministic constant and $\beta_1(\cdot)$, $\beta_2(\cdot)$ are deterministic functions of time.

We have a GQSR with

$$Q(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad g(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad f(t) = 0.$$

And, from Definition 3.3, we can conclude that Z_1 is a $Z^{(q)}$-factor and Z_2 is a $Z^{(l)}$-factor.

We also see that both drift and volatility conditions are satisfied with

$$\begin{aligned} \mathbf{d}(t) &= \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \quad \mathbf{E}(t) = \mathbf{0} \\ \mathbf{k_0}(t) &= \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{k_u}(t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ for } u = 1,2, \quad \mathbf{g_{uk}}(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ for } u,k = 1,2. \end{aligned}$$

Hence bond prices have a QTS and A_p, B_p and C_p solve

$$\begin{cases} \dfrac{\partial A_p}{\partial t} + \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} B_p + \dfrac{1}{2} B_p^* \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & 0 \end{pmatrix} B_p + tr\left\{ C_p \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & 0 \end{pmatrix} \right\} = 0 \\ A_p(T,T) = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial B_p}{\partial t} + 2C_p \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \dfrac{1}{2} \bar{B}_p{}^* \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} B_p + 2C_p \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & 0 \end{pmatrix} B_p = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ B_p(T,T) = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial C_p}{\partial t} + 2C_p \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & 0 \end{pmatrix} C_p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ C_p(T,T) = 0 \end{cases}$$

It is easy to see that in this case $C_p^{(11)}$ solves the scalar Riccati equation

$$\begin{cases} \dfrac{\partial C_p^{(11)}}{\partial t} + 2\sigma_1^2 \left(C_p^{(11)} \right)^2 = 1 \\ C_p^{(11)}(T,T) = 0 \end{cases}$$

whose solution is given by

$$C_p^{(11)} = \frac{1 - e^{2\gamma(T-t)}}{\gamma\left(e^{2\gamma(T-t)} - 1\right) + 2\gamma}$$

And for the remaining cells in C_p,(i.e. for $(ij) \neq (11)$), $C_p^{(ij)} = 0$.

Each entry of B_p then solves

$$\begin{cases} \dfrac{\partial B_p^{(1)}}{\partial t} + 2C_p^{(11)}\left(\beta\right)_1 + \dfrac{1}{2}\left(B_p^{(2)}\right)^2 + 2C_p^{(11)}\sigma_1^2 B_p^{(1)} & = & 0 \\ B_p^{(1)}(T,T) & = & 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial B_p^{(2)}}{\partial t} + \dfrac{1}{2}\left(B_p^{(2)}\right)^2 & = & 1 \\ B_p^{(2)}(T,T) & = & 0 \end{cases}$$

We can solve first for $B_p^{(2)}$ and then for $B_p^{(1)}$ to get

$$B_p^{(1)}(t,T) = \int_t^T e^{\sigma_1^2 \int_t^s 2C_p(11)(u,T)du} \left[2C_p^{(11)}(s,T)\beta_1(s) - \frac{1}{2}\left(B_p^{(2)}(s,T)\right)^2\right] ds$$

$$B_p^{(2)}(t,T) = \frac{\sqrt{2}\left(1 - e^{\sqrt{2}(T-t)}\right)}{\left(e^{\sqrt{2}(T-t)} - 1\right) + 2}$$

Finally one easily sees that A_p is given by

$$A_p(t,T) = \sum_{i=1}^{2}\left\{\int_t^T \beta_i(s)B_p^{(i)}(s,T)ds\right\} + \frac{1}{2}\sigma_1^2\int_t^T \left(B_p^{(1)}(s,T)\right)^2 ds + \sigma_1^2\int_t^T C_p^{(11)}(s,T)ds$$

More complex GQTS of bond prices can also be computed.

6.2 Examples of GQTS for Forward and Futures prices

6.2.1 Example 3 - Schwartz spot price with Vasiček short rate

We consider that the spot price S is driven by a Schwartz [22] type of model and that the short rate process follows the Hull and White extension of the Vasiček [16] model. We furthermore consider that both processes are deterministicly correlated.

So, we have

$$\begin{aligned} dS(t) &= \left[\beta(t) + \alpha(t)\ln S(t)\right] S(t)dt + \sigma_s(t)S(t)d\bar{W}_s(t) \\ dr(t) &= \left[b(t) - ar(t)\right] dt + \sigma_r d\bar{W}_r(t) \end{aligned}$$

where we have $d\bar{W}_s(t)d\bar{W}_r(t) = \rho(t)dt$.

The parameters a and σ_r are consider to be deterministic constants but all others are allowed to be deterministic functions of time.

The factors that we consider are $Z = \begin{pmatrix} Y \\ r \end{pmatrix}$, for $Y = \ln S$, and we have that $h_f(T, Z(T)) = e^{Y(T)} = S(T)$.

Hence our state variable dynamics in a multi-dimensional framework are:

$$dZ(t) = \left[\begin{pmatrix} \beta(t) - \frac{1}{2}\sigma_s^2(t) \\ b(t) \end{pmatrix} + \begin{pmatrix} \alpha(t) & 0 \\ 0 & -a \end{pmatrix} Z(t) \right] dt + \begin{pmatrix} \sigma_s(t) & 0 \\ \rho(t)\sigma_r & \sqrt{1-\rho(t)}\sigma_r \end{pmatrix} dW(t).$$

Given the dynamics of Z and recalling that for the Hull and White extension of the Vasiček model of the short rate allow for an ATS of bond prices with $B_p(t,T) = \begin{pmatrix} 0 \\ \frac{1}{a}\left\{e^{-a(T-t)} - 1\right\} \end{pmatrix}$.

We can also check that (using Definition 4.2) we have only $\bar{Z}^{(l)}$ factors and so we will also have an ATS for futures prices.

For ATS A_F and B_F satisfy the system (67)-(69), which in our case becomes

$$\begin{cases} \dfrac{\partial A_F}{\partial t} + \left(\beta(t) - \frac{1}{2}\sigma_s^2(t) \quad b(t) \right) B_F + \dfrac{1}{2} B_F^* \begin{pmatrix} \sigma_s^2(t) & \rho(t)\sigma_s(t)\sigma_r \\ \rho(t)\sigma_s(t)\sigma_r & \sigma_r^2 \end{pmatrix} B_f &= 0 \\ A_F(T,T) &= 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial B_F}{\partial t} + \begin{pmatrix} \alpha(t) & 0 \\ 0 & -a \end{pmatrix} B_F &= 0 \\ B_F(T,T) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}.$$

The solutions are given by

$$B_F(t,T) = \begin{pmatrix} B_F^{(y)}(t,T) \\ B_F^{(r)}(t,T) \end{pmatrix}$$

where

$$B_F^{(y)}(t,T) = \exp\left\{ \int_t^T \alpha(s)ds \right\} \qquad B_F^{(r)}(t,T) = 0$$

and

$$A_F(t,T) = \int_t^T \left[\beta(s) - \frac{1}{2}\sigma_s(s) \right] B_F^{(y)}(s,T)ds + \frac{1}{2}\int_t^T \sigma^2(s) \left[B_F^{(y)}(s,T) \right]^2 ds.$$

The term structure of *forward prices* could likewise be obtained by noting that both Y and r are $\tilde{Z}^{(l)}$ factors (according to Definition 5.2) and solving the system of ODEs (88)-(89).

Alternatively, we could use Proposition 5.7 (there are only *deterministic* correlations and $Y \in Z^{lu}$ while $r \in Z^{lp}$) and compute forward prices term structures using the adjustment term. In our particular case

$$D(t,T) = e^{\frac{\sigma_r}{a} \int_t^T \rho(s)\sigma_s(s) B_f^{(y)}(s,T) \left\{ e^{-a(T-s)} - 1 \right\} ds}.$$

For specified $\beta(\cdot)$, $\alpha(\cdot)$, $\rho(\cdot)$ and $\sigma_s(\cdot)$ exact solutions can, in principle, be obtained. *For instance*, for the particular case where β, α, ρ and σ_s are deterministic constants we have the exact expressions

$$B_F(t,T) = B_f(t,T) = \begin{pmatrix} e^{\alpha(T-t)} \\ 0 \end{pmatrix}$$

and

$$
\begin{aligned}
A_F(t,T) &= \frac{\beta - \frac{1}{2}\sigma_s^2}{\alpha}\left\{e^{\alpha(T-t)} - 1\right\} + \frac{\sigma_s^2}{2\alpha}\left\{e^{2\alpha(T-t)} - 1\right\} \\
A_f(t,T) &= A_F(t,T) + \frac{\rho\sigma_s\sigma_r}{a(\alpha-a)\alpha}\left\{\alpha\left[e^{(\alpha-a)(T-t)} - 1\right] - (\alpha-a)\left[e^{\alpha(T-t)} - 1\right]\right\}.
\end{aligned}
$$

6.2.2 Example 4 -Two-factor model with QTS for bond Prices

Consider the following model

$$
\begin{aligned}
dZ_1(t) &= [\beta_1(t) - \alpha_1 Z_1(t)]\,dt + \sigma_1 dW_1(t) \\
dZ_2(t) &= [\beta_2(t) - \alpha_2 Z_2(t)]\,dt + \sigma_2 dW_2(t) \\
dW_1(t)dW_2(t) &= 0dt \\
\\
S(t) &= e^{Z_1(t)} \\
r(t) &= \frac{1}{2}\left[Z_1^2(t) + Z_2^2(t)\right]
\end{aligned}
$$

where α_1, α_2, σ_1 and σ_2 are deterministic constants and $\beta_1(.)$ and $\beta_2(.)$ deterministic functions of time.

Taking $Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ we have then

$$
dZ(t) = \left[\begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix} + \begin{pmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix} Z(t)\right] dt + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} dW(t) \qquad (92)
$$

Note that in terms of the interest rate setting we have exactly the model of **Example 1 - PQTS**, so we can use the solution for B_p and C_p in (90) and (91), respectively.

In this case, despite the fact that we have a QTS for bond prices, we have a linear boundary function and a deterministic volatility structure so will have ATS for both futures and forward prices.

Moreover since $Z_1 \in \left[Z^{(qp)} \cap Z^{lu}\right]$ and is (obviously) deterministicly correlated to itself (recall Definitions 5.3 and 5.4), we know that the term structures of futures and forward prices will differ in both the deterministic and the linear terms.

Taking the more complex case of forward prices, A_f and B_f should solve system of ODEs (88)-(89), which in the particular case of our model becomes

$$
\left\{
\begin{aligned}
&\frac{\partial A_f}{\partial t} + \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix} B_f + \frac{1}{2} B_f^* \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} B_f \\
&\qquad + B_f^* \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \begin{pmatrix} B_p^{(1)}(t,T) \\ B_p^{(1)}(t,T) \end{pmatrix} = 0 \\
&\qquad A_f(T,T) = 0
\end{aligned}
\right.
$$

$$
\begin{cases}
\dfrac{\partial B_f}{\partial t} + 2\begin{pmatrix} C_p^{(11)}(t,T) & 0 \\ 0 & C_p^{(22)}(t,T) \end{pmatrix}\begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} B_f & \\
\qquad\qquad\qquad\qquad + \begin{pmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix} B_f & = 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad B_f(T,T) & = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{cases}
$$

and the solutions are given by

$$B_f(t,T) = \begin{pmatrix} B_f^{(1)}(t,T) \\ 0 \end{pmatrix}$$

where

$$B_f^{(1)}(t,T) = \exp\left\{\alpha_1(T-t) - 2\sigma_1^2 \int_t^T C_p^{(11)}(s,T)ds\right\}$$

and

$$A_f(t,T) = \int_t^T \beta_1(s)B_f^{(1)}(s,T)ds + \frac{1}{2}\sigma_1^2 \int_t^T \left[B_f^{(1)}(s,T)\right]^2 ds + \sigma_1^2 \int_t^T B_f^{(1)}(s,T)B_p^{(1)}(s,T)ds.$$

with $B_p^{(i)}$ and $C_p^{(ii)}$ from equations (91)-(90). Given the dependence B_f on C_p, we must rely on numerical integration to obtain both B_f and A_f.

The futures price term structure is in this case is given by (note that both terms are different)

$$B_F(t,T) = \begin{pmatrix} e^{\alpha_1(T-t)} \\ 0 \end{pmatrix} \quad A_F(t,T) = \int_t^T \beta_1(s)e^{\alpha_1(T-s)}ds + \frac{1}{2}\sigma_1^2 \int_t^T \left[e^{\alpha_1(T-s)}\right]^2 ds.$$

6.2.3 Example 5 - A QTS for forward and futures prices

Consider a model specified by the following equations

$$
\begin{aligned}
dZ_1(t) &= \beta_1(t)dt + \sigma_1(t)dW_1(t) \\
dZ_2(t) &= \beta_2(t)dt + \sigma_2(t)dW_2(t) \\
dr(t) &= [b(t) - ar(t)]\,dt + \sigma_r dW_3(t) \\
W_1, W_2 \quad &\text{and} \quad W_3 \text{ are independent Wiener processes.} \\
\\
\ln S(t) &= q_1\left[Z_1(t)\right]^2 + q_2\left[Z_2(t)\right]^2 + g_1 Z_1(t) + g_2 Z_2(t) + f(t)
\end{aligned}
$$

The parameters a,σ_r, q_1, q_2, g_1, g_2 are consider to be deterministic constants but all others are allowed to be deterministic functions of time.

The factors that we consider are $Z = \begin{pmatrix} Z_1 \\ Z_2 \\ r \end{pmatrix}$ and as boundary condition we have we have that

$$h_f(T, Z(T)) = e^{q_1[Z_1(T)]^2 + q_2[Z_2(T)]^2 + g_1 Z_1(T) + g_2 Z_2(T) + f(T)} = S(T).$$

So

$$dZ(t) = \left[\begin{pmatrix} \beta_1(t) \\ \beta_2(t) \\ b(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix} Z(t) \right] dt + \begin{pmatrix} \sigma_1(t) & 0 & 0 \\ 0 & \sigma_2(t) & 0 \\ 0 & 0 & \sigma_r \end{pmatrix} dW(t)$$

Given the dynamics of Z and recalling that the Hull and White extension of the Vasiček model of the short rate allows for an ATS of bond prices with

$$B_p(t,T) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{a}\left\{ e^{-a(T-t)} - 1 \right\} \end{pmatrix}.$$

Since $Z^{(p)} = \{r\}$ and $Z^{(u)} = \{Z_1, Z_2\}$ (check Definition 5.3), and the factors $Z^{(p)}$ are not correlated to the factors $Z^{(u)}$, the term structures of futures and forward prices will be the same. So we have

$$A_F(t,T) = A_f(t,T) \quad B_F(t,T) = B_f(t,T) \quad C_F(t,T) = C_f(t,T).$$

Picking the forward prices system[36] we have:

$$\begin{cases} \dfrac{\partial A_f}{\partial t} + \begin{pmatrix} \beta_1(t) & \beta_2(t) & b(t) \end{pmatrix} B_f + \dfrac{1}{2} B_f^* \begin{pmatrix} \sigma_1(t) & 0 & 0 \\ 0 & \sigma_2(t) & 0 \\ 0 & 0 & \sigma_r \end{pmatrix} B_f \\ \qquad + B_f^* \begin{pmatrix} \sigma_1(t) & 0 & 0 \\ 0 & \sigma_2(t) & 0 \\ 0 & 0 & \sigma_r \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \frac{1}{a}\left\{ e^{-a(T-t)} - 1 \right\} \end{pmatrix} \\ \qquad + tr \left\{ C_f \begin{pmatrix} \sigma_1(t) & 0 & 0 \\ 0 & \sigma_2(t) & 0 \\ 0 & 0 & \sigma_r \end{pmatrix} \right\} = 0 \\ A_f(T,T) = f(T) \end{cases}$$

$$\begin{cases} \dfrac{\partial B_f}{\partial t} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix} B_f + 2C_f \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \\ b(t) \end{pmatrix} + 2C_f \begin{pmatrix} \sigma_1(t) & 0 & 0 \\ 0 & \sigma_2(t) & 0 \\ 0 & 0 & \sigma_r \end{pmatrix} B_f \\ \qquad + 2C_f \begin{pmatrix} \sigma_1(t) & 0 & 0 \\ 0 & \sigma_2(t) & 0 \\ 0 & 0 & \sigma_r \end{pmatrix} B_p = 0 \\ B_f(T,T) = \begin{pmatrix} g_1 \\ g_2 \\ 0 \end{pmatrix} \end{cases}$$

[36]The easier system to choose would be the futures price. The choice for the forward price system is only a pedagogical one to see how the *extra-term* will all be zero.

$$
\begin{cases}
\dfrac{\partial C_f}{\partial t} + 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix} C_f + 2C_f \begin{pmatrix} \sigma_1(t) & 0 & 0 \\ 0 & \sigma_2(t) & 0 \\ 0 & 0 & \sigma_r \end{pmatrix} C_f & = & 0 \\
C_f(T,T) & = & \begin{pmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{cases}
$$

The solutions are given by

$$
C_f(t,T) = \begin{pmatrix} C_p^{(11)}(t,T) & 0 & 0 \\ 0 & C_p^{(22)}(t,T) & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

and

$$
B_f(t,T) = \begin{pmatrix} B_f^{(1)}(t,T) \\ B_f^{(2)}(t,T) \\ 0 \end{pmatrix}
$$

where for $i = 1, 2$,

$$
C_f^{(ii)}(t,T) = \frac{q_i}{1 - 2q_i \int_t^T \sigma_i^2(s)ds}
$$

$$
B_f^{(i)}(t,T) = 2\int_t^T e^{2\int_t^s C_f^{(ii)}(u,T)du} C_f^{(ii)}(s,T)\beta_i(s)ds
$$

and

$$
A_f(t,T) = \sum_{i=1}^{2} \left\{ \int_t^T \beta_i(s) B_f^{(i)}(s,T)ds + \frac{1}{2}\sigma_i^2 \int_t^T \left[B_f^{(i)}(s,T)\right]^2 ds + \sigma_i^2 \int_t^T C_f^{(ii)}(s,T)ds \right\}
$$

For specified $\beta_1(\cdot)$,$\beta_2(\cdot)$, $\sigma_1(\cdot)$, $\sigma_2(\cdot)$ and $f(\cdot)$ exact solutions can, in principle, be obtained.

For instance, for the particular case where β_1,β_2, σ_1, σ_2 and f are deterministic constants we have the exact expressions for and $i = 1, 2$

$$
C_f^{(ii)}(t,T) = \frac{q_i}{1 - \delta_i(T-t)}
$$

$$
B_f^{(i)}(t,T) = 2\beta_i \left[q_1(T-t) + \frac{1}{\sigma_1^2}(1-\delta_i)\ln(1-\delta_i) \right]
$$

and

$$
A_f(t,T) = M_1(t,T) + M_2(t,T)
$$

where

$$
\begin{aligned}
M_i(t,T) \;=\; & \frac{\beta_i^2}{\sigma_i^2}\left[\left(2(T-t) - \delta_i(T^2-t^2)\right)L_i(t,T) - \delta_i(T-t) - \delta_i t^2 L_i(t,T) - \delta_i(T^2-t^2)\right] \\
+ \; & 2\beta_i\sigma_i^2\left[\frac{4T^3}{3} - \frac{t^3}{3} - T^2 t - T^2 - t^2\right] + 2\beta_i q_i T\left[(2T - \delta_i T)L_i(t,T) - (T^2-t^2)\delta_i\right] \\
- \; & 2\beta_i q_i\left[\left(t^2 - \delta_i T t^2 + \frac{2t^3}{3}\delta_i\right)L_i(t,T) + \left(T^2 - \delta_i\frac{T^3}{3}\right)L_i(t,T) + \left(\frac{T^3}{3} - \frac{t^3}{3}\right)\delta_i\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \quad 2\beta_i q_i \left[\left(\frac{t^3}{2} - \frac{t^2}{2} + \frac{t}{4} - \frac{1}{4} \right) L_i(t,T) \right] - 2\frac{\beta_i}{\sigma_i^2} \left[\frac{(1-\delta_i(T_t))^3}{3\delta_i} L_i(t,T)^2 + \frac{2\,(1-\delta_i(t,T))^3}{27\delta_i} \right] \\
&+ \quad 2\frac{\beta_1}{\sigma_1^2} \left[\frac{2\,(1-\delta_i)^3}{9\delta_i} L_i(t,T) + \frac{2}{27\delta_i} \right] - \frac{1}{2} L_i(t,T)
\end{aligned}
$$

and

$$
L_i(t,T) = \ln\left(1-\delta_i(t,T)\right) \qquad \text{with} \qquad \delta_i = 2q_i\sigma_i^2.
$$

7 Concluding remarks

This paper investigates the term structures of *bond, futures* and *forward prices* when we assume that these prices are functions of a finite dimensional state process. It generalizes previous studies by considering non-Gaussian quadratic term structures. This generalization relies on the *a priori* separation of factors into *quadratic* and *linear* factors.

The **G**eneralized **Q**uadratic **T**erm **S**tructure have as special cases the affine and the Gaussian-quadratic term structures previously studied in the literature.

We show that unless all factors are of the quadratic type, the requirement of a deterministic volatility structure is not needed for all factors. Still on volatility conditions we devote some effort to try to understand all the implications that different conditions for different types of factors have in terms or their possible correlations.

Motivated by the fact that forward prices are martingales under measures where bond prices are *numeraires*, we exploit the connection between bond prices and forward prices. We show, on the one hand, that bond prices will only play a role in forward prices term structures trough correlations. And, on the other hand, that if they do play a role, volatility-restricted GQTS of bond prices are consistent with ATS of forwards while its non-restricted version is only consistent with QTS of forward prices.

Finally we show that, in some sense, the study of GQTS of futures prices in included in the study of GQTS of forward prices. Their difference is related to the impact that bond prices may have on forward prices term structures. We *qualify* that difference in a quadratic short rate setting and give a *quantification* for situations where the two term structures differ only by a deterministic adjustment term.

The examples' section highlight the applicability in practice of the theoretical results derived, but, obviously their field of application was not exhausted, and it concretization to market situations is left for future research.

We conclude by noting that the GQTS results here presented can be (almost directly) applied to study term structures of *any* martingale either under the risk neutral or under the forward measure (not just futures and forward prices).

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