Lisbon School of Economics & Management
Universidade de Lisboa

MASTER IN

ACTUARIAL SCIENCE

MASTER´S FINAL WORK

INTERNSHIP REPORT

DEVELOPMENT OF A CALCULATION TOOL FOR MAKING PENSION PROJECTIONS IN THE SCOPE OF OCCUPATIONAL DEFINED CONTRIBUTION PENSION SCHEMES

FREDERICO JORGE MACHADO PINHEIRO

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SUPERVISION

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Abstract

The ageing of the population is leading to reforms in Social Security systems with negative impact on the levels of retirement income. One way to minimize this impact is to reinforce the role of complementary pension schemes. Pension projections can be an important tool to support individuals in their decision-making about the savings for retirement and have been a part of several initiatives at the European Union level.

This report is the result of an internship at the Insurance and Pension Funds Supervisory Authority (ASF) and focuses on the development of a calculation tool for making pension projections in the scope of occupational defined contribution pension schemes.

The work aims to study the potential performance of different investment strategies using an Economic Scenario Generator framework and evaluate the impact on the retirement income that such investment strategies produce, considering also different assumptions with regard to mortality tables and discount rates applied in the calculation of annuities.

The model developed considers three main risk factors: (1) financial market risk, which includes uncertainty over return on investment, inflation and interest rates; (2) labour risk, originated from uncertainty over real wage growth path; (3) demographic risk, as a result of increasing life expectancy.

Keywords: Pension Projection, Economic Scenario Generator, Real-world Valuation, Kalman filter, Projected Lifetable, Retirement Income

Resumo

O envelhecimento da população tem conduzido a reformas nos sistemas públicos de pensões, com impacto negativo nos níveis de rendimento de reforma, sendo que uma das formas de minimizar esse impacto é o reforço do papel dos sistemas complementares de pensões. As projeções relativas aos benefícios de reforma podem constituir uma importante ferramenta para a tomada de decisão dos indivíduos, em relação ao investimento das suas poupanças direcionadas à reforma, e têm sido objeto de várias iniciativas a nível europeu.

Este relatório resulta de um estágio realizado na Autoridade de Supervisão de Seguros e Fundos de Pensões (ASF). O estudo centra-se no desenvolvimento de uma ferramenta de cálculo de projeção de benefícios de reforma no âmbito dos planos profissionais de contribuição definida.

O trabalho procura estudar o potencial desempenho de diferentes estratégias de investimento recorrendo a um Gerador de Cenários Económicos (ESG) e avaliar o impacto nos benefícios de reforma que essas estratégias produzem, considerando também diferentes pressupostos em relação às tábuas de mortalidade e taxas de desconto para o cálculo de anuidades.

O modelo desenvolvido considera três tipos de riscos: (1) risco financeiro, fruto da incerteza do retorno dos investimentos, inflação e taxas de juro; (2) risco laboral, relacionado com a incerteza da evolução salarial; (3) risco demográfico, decorrente do aumento da longevidade.

Palavras-chave: Projeção de Benefícios de Reforma, Gerador de Cenários Económicos, Avaliação Real face ao Risco, Filtro de Kalman, Tábuas de Mortalidade, Benefícios de Reforma

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1 Introduction

In the European Union, the pension landscape across countries is very diverse but, in general terms, pension systems can be divided into three pillars: the first pillar or the public system; the second pillar, which consists of occupational pension schemes; and finally the third pillar, corresponding to personal pension schemes.

In Portugal, the main source of retirement income is the pension provided by the public Social Security system, which takes the form of a pay-as-you-go (PAYG) type of system. As in other European countries, the ageing of the population, due to an increase in life expectancy and lower birth rates, is placing the Social Security system under increasing financial pressure, leading to the need of introducing reforms with negative implications on the levels of retirement income provided by the public system. Indeed, according to recent projections presented in the 2021 Ageing Report of the European Commission[1], it is expected that, in Portugal, the replacement rate from public pensions, which expresses the average new pension as a share of the average gross wage at retirement, will decrease from 74% in 2019 to 41,4% in 2070. One way to minimize the impact of the reduction of retirement income from the public pension system is to reinforce the role of complementary pension schemes.

There are different measures that a country can implement to encourage participation and increase the coverage of complementary pension schemes, as well as to ensure the adequacy of retirement outcomes from these schemes. These include, in particular, raising the individuals' awareness to the importance of planning for retirement and to promote individuals' active engagement with their pensions. In this context, pension projections with the aim of providing, at least, some indication of the foreseeable levels of future retirement benefits can be an important tool to support individuals in their decision-making about savings for retirement. In fact, in recent years, pension projections have been a part of several initiatives at the European Union level.

From a regulatory perspective, one of the main initiatives in the pensions sector was the publication of the Directive (EU) 2016/2341, commonly known as the IORP II Directive, which provides an updated legislative framework regarding the institutions for occupational retirement provision (IORPs). One of the main objectives of this directive is to ensure that IORPs provide clear and adequate information to pension schemes' members and beneficiaries, including regular information on projected levels of retirement benefits, via the so-called Pension Benefit Statement (PBS), which IORPs should make available to all members on an annual basis.

In the scope of personal schemes, the Regulation (EU) 2019/1238 on a Pan-European Personal Pension Product (PEPP), published in 2019, created a legislative framework for a new individual pension product, aiming to all EU citizens, with a harmonized set of key features, offering savers more choice and more competitive products. The PEPP may be offered by financial institutions

[1] 2021 Ageing Report of the European Commission

from different sectors, including life insurance companies and IORPs that are authorized, under national law, to manage personal schemes. Similar to the IORP II Directive, the regulation requires the provision of standardized information, namely an annual statement on PEPP benefits for savers, including information on pension benefit projections. The PEPP legal framework will enter into application on 22 March 2022, enabling the launch of the first products after that date.

Apart from these regulatory initiatives, in the end of 2020, the European Commission has asked the European Insurance and Occupational Pensions Authority (EIOPA) for technical advice on the development of best practices for setting up national pension tracking systems[2], which broadly consists of a tool, based on pension projections, that will provide individuals with an overview of their future retirement income, including their entitlements from all pension schemes in which they participate.

While pension projections can be provided for all types of schemes, i.e., from more traditional defined benefit (DB) plans to pure defined contribution (DC) plans, there tends to be more uncertainty around DC or hybrid types of plans, a combination of DB and DC, as in DB plans the retirement income is usually based on a pre-defined formula, for instance, based on the years of service and past salaries. In DC plans, the sponsor and/or individuals contribute some amount of money (fixed or a percentage) to an account to fund the retirement income. In this case, the accumulated value that could be converted into retirement income is not guaranteed. Hybrid plans often include a small portion of guaranteed retirement income.

As referred above, the retirement benefits from DC plans contain a high level of uncertainty. During the accumulation phase, two sets of risk factors, dependent on the economic and financial conditions, can be identified: (1) the financial market conditions, that have an impact on the savings accumulated in the DC account; and (2) labour market conditions, which include the employment prospect and the real wage growth path, in case contributions are based on salaries. In the decumulation phase, if the accumulated amount is converted into a life annuity, there is also uncertainty stemming from the pricing of annuities, namely the mortality tables and discount rates that are applied.

The main objective of this work is to develop a calculation tool for making pension projections in the scope of occupational DC pension schemes. The choice to focus on DC schemes relates to the fact that in recent years there has been an increasing trend towards this type of schemes, in which risks are born by individuals. There are several reasons that explain this shift in the type of pension schemes, one of which being the employer’s difficulties to meet the obligations and bear the financial costs of providing DB plans, due to financial market conditions, e.g., low interest rate environment, increasing volatility, and demographic changes, such as higher life expectancy.

The model that has been developed takes into account three main risk factors: (1) financial market risk, which includes uncertainty over return on investment, inflation and interest rates; (2) labour

[2] Call for advice to EIOPA on pension tools

risk, originated from uncertainty over real wage growth path; (3) demographic risk, as a result of increasing life expectancy.

Based on the stochastic models presented in EIOPA (2020) for the assessment of the risk and performance of PEPP products, the present work uses an economic scenario generator, as described in Chapter 3, to study the potential performance of different investment strategies and evaluate the impact on the retirement income that such investment strategies produce, considering also different assumptions with regard to mortality tables and discount rate applied in the calculation of annuities.

The structure of this work is the following. In Chapter 2, the principles and good practices that should be applied when making pension projection are presented; In Chapters 3 and 4, the mathematical theory applied is provided, for both the accumulation and decumulation phases, respectively; Chapter 5 comprehends the estimation process. Chapter 6 has the application; and finally Chapter 7 contains the conclusions.

2 Principles and Good Practices for Making Pension Projections

In what concerns the Portuguese pension funds sector, a new law was published on 2020 (Law no. 27/2020, of 23rd of July), approving a new legal regime for the constitution and operation of pension funds and pension fund management entities and transposing the IORP II Directive into the national legal framework. Among the set of requirements established by this new regime, our attention is focused on the Pension Benefit Statement (PBS), in special, the provision of pension benefit projections. The legal regime specifies the fundamental and supplementary information that should be given to the members and beneficiaries of the pension funds. In particular, pension fund management entities should draw up a PBS containing key personal and generic information about the pension scheme, which should be provided to the members, at least, on an annual basis.

The scope of the PBS covers all types of schemes. By providing, among other elements, an overview of the current situation, informing the member of the accrued entitlements or accumulated capital, and the changes occurred between years, and an estimation of the level of benefits that the member could receive at retirement, the PBS enables insight into the retirement savings with the goal of helping members to make informed decisions to achieve the expected retirement income. For instance, where possible, the members can take pro-active decisions to change the level of contributions or the choice of investment profile.

Without prejudice to the power that the Insurance and Pension Funds Supervisory Authority (ASF) has to issue further requirements on information provision, the applicable legal framework is quite flexible on how pension projections should be made. It establishes that the pension fund management entities should provide information about the benefit projections based on the expected retirement age, income level and contribution period, and should include a warning that the final value of benefits can be different from those projections. Indeed, considering that pension projections, especially when they are performed for a long time period, are subject to forecasting error, it is important that the uncertainty surrounding the results is clearly communicated to the members, either by using appropriate disclaimers or by showing a range of possible results, instead of presenting one single scenario.

The disclosure of the methodology and assumptions can also contribute to improve communication. In this regard, the legal provisions establish that the PBS should indicate where and how to obtain additional information, when applicable, on the assumptions used for expressing amounts in annuities, namely the discount rate and the mortality table.

In what concerns the scenarios to be used, the legal regime states that, if projections use economic scenarios, the information should include a best estimate scenario and an unfavourable scenario, taking into account the nature of the pension scheme. It leaves the choice on whether to use a deterministic or stochastic approach to the entities managing the pension fund. The use of a stochastic approach is more complex than the deterministic one, as it requires the use of more sophisticated models to produce a high number of scenarios, being more time and resource consuming. However, it allows the simulation of a large variety of possible outcomes and to attach probabilities to the scenarios, e.g., by calculating percentiles. A set of risk and performance indicators, as presented in Chapter 6, can also be determined using stochastic projections. These indicators can be used to assess whether the investment strategies' risk-reward profile is in line with members' retirement goals and risk tolerance. The deterministic approach has the advantage of being easier to implement and to explain to members but is more dependent on the assumptions set.

A key element of pension projections, especially for DC schemes, is the set of the underlying assumptions, which according to the legal provisions should be chosen in the most realistic way possible, considering an appropriate time horizon and should be reviewed regularly. The assumptions can be divided into economic and/or financial assumptions, features that are specific to the pension fund(s) or scheme and data related to the members.

For the accumulation phase, the main economic and/or financial assumptions used are typically related to investments' return, volatility and correlations of assets classes. When benefits depend on inflation, assumptions on the inflation rate are also required.

With regard to the specificities of the pension fund(s) or schemes, the asset allocation and its evolution over time should be taken into account, as well as information on the costs that are applicable.

The data related to the individual member that are needed for making pension projections generally include: (1) the level of contributions over time (employer and/or member contributions); (2) salary: the projection of salary path; (3) the current age of the member; (4) the expected retirement age.

For the decumulation phase, if annuities are calculated, at least assumptions on the discount rate and the mortality table are needed. When choosing these assumptions, especially for younger members, one has to take into account that the annuity rates that can be used as reference at the time the pension projections are performed may not be an appropriate estimation for the technical basis that will be used to price annuities in 30 or 40 years into the future. Therefore, the inclusion of the evolution of life expectancy, e.g., by using dynamic life tables, might provide a more realistic view of the projected retirement income stream that the member will receive in the future. Similarly, different scenarios should also be considered for the interest rate.

In what concerns the presentation of the results of pension projections to members, the legal provisions do not prescribe the pay-out form under which the outcomes should be presented (i.e., lump sum payment, annuity or a combination of both). In practice, considering that for occupational pension schemes in the Portuguese pension funds sector there is a rule stating that at least 2/3 of the benefits resulting from employer contributions have to be received as an annuity (for employee contributions the member can choose the pay-out option), both the value of accumulated assets and the monthly income can be considered useful information for members.

It is also worth noting that good practices in the scope of the PBS published by EIOPA, consistently with more recent proposals by IOPS[3], tends to favour the display of results in real terms (i.e., adjusted for the effects of inflation), in order to help members to better understand their purchasing power after retirement. For this, assumptions on the evolution of inflation are needed. Good practices proposed by IOPS also refer that replacement rates may be presented.

[3] IOPS – Good Practices for designing, presenting and supervising pension projections, 2021

3 Economic Scenario Generator

An economic scenario generator (ESG) is a computer-based model used to produce simulations of the joint behaviour of financial market and economic variables. The primary goal of ESG is to generate future economic scenarios to evaluate the potential outcomes and their likelihood, giving an extremely useful insight into future risks (SOA, 2016).

An ESG application can be valuated applying risk-neutral (market consistent) models and real-world models. The first is mainly concerned with mathematical relationships among financial instruments while the latter is concerned with potential paths of economic variables capturing market dynamics and risks. An example of risk-neutral valuation is the valuation of guarantees or the pricing of complex financial derivatives. The simulation of interest rates and of the investment portfolios' return, as presented in this work, are an example of real-world valuation.

The design and components of an ESG model can vary significantly with the goals of the specific application. For instance, pension providers can use ESG to evaluate different funding strategies and investment performances on assets.

The calibration of real-world ESG models is a forward-looking procedure, that requires a view of the future economic development and expert judgement to determine the accuracy of the scenarios that result from the parameterization process. The parameters are calibrated to be consistent with historical dynamics of economic variables. In practice, the process can be divided into four steps: (1) Estimation of the model parameters; (2) Simulation of the model; (3) Comparison of simulated statistics with key parameterization targets; (4) Adjustment of the parameters. This process is repeated until a satisfactory fit is achieved. The Kalman filtering and the Maximum likelihood estimation are some of the methods that could be used in the calibration of ESG models and have been applied in this work, to the interest rate model and to the inflation rate model, respectively.

The ESG model applied in this work is composed of risk-free interest rate model, an equity market model and, regarding economic variables, an inflation and a real wage growth path models.

3.1 Nominal Interest Rate Model

The interest rate model is a key component of most ESG models, being the core of this work. It is used to generate the price of the risk-free bond and calculate the bond investment return. The short rate is applied as a parameter in the equity model.

3.1.1 Notation and specification

The price of a zero-coupon bond and the yield to maturity are related. It can be shown that the price per unit of a zero-coupon bond, at time t, with maturity at time T, and assuming continuous compound (Björk, 2020 and Frederico, 2010) is equal to

$$P(t,T) = e^{-y(t,T)(T-t)} , \tag{3.1}$$

where $y(t,T)$ is the continuous yield to maturity from t to T, with $T-t$ being the time to maturity. We can see $P(t,T)$ as a discounting factor, where the pay-off of one unit of currency at time T equals $P(t,T)$ at time t. Rewriting equation (3.1), we obtain the yield rate over the time period $[t,T]$

$$y(t,T) = -\frac{\log P(t,T)}{(T-t)}. \tag{3.2}$$

With $y(t,T)$ a zero-coupon yield curve can be constructed, showing the relationship between bonds yields and time to maturity. The yield curve is also called the term structure of interest rates.
The instantaneous forward rate is given by

$$f(t,T) = -\frac{\partial}{\partial T}\log P(t,T)\,. \tag{3.3}$$

We define the (instantaneous) short rate as the instantaneous forward rate when $t \to T$, e.g., when the time to maturity tends to zero, as follows:

$$r(t) = f(t,t) = -\lim_{T\to t}\frac{\log P(t,T)}{T-t} = -\frac{\partial}{\partial T}\log P(t,T)\,|_{T=t}\,. \tag{3.4}$$

The short rate, $r(t)$, is not directly observed in the market, a proxy must be used. The price of a zero-coupon bond can be defined as function of the short rate (as we can see in the next section). The interest rate model allows simulating the short rate and forecasting the term structure of interest rates.

$r(t)$ was modelled using the G2++ model. Presented by Brigo & Mercurio (2006), the G2++ model has two correlated Gaussian factors and a deterministic function, defined to fit the current interest rate term structure. Being a model of two factors, it allows to also include the slope of the interest rate term structure, while one factor models only allow to deal with parallel shocks of that structure. The two factors provide the model greater flexibility to better capture the market shape of the interest rate curve. In addition, the model has the advantage to allow for negative interest rates, which is the case of the current low yield environment. Throughout this section a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathcal{M}\right)$, where $\mathcal{M}$ is either the risk-neutral measure $\mathbb{Q}$ or the real-world measure $\mathbb{P}$, as appropriate, will be our setting.

The dynamic of the short rate under a risk-neutral measure $\mathbb{Q}$ can be represented by

$$r(t) = x(t) + y(t) + \varphi(t)\,, \quad r(0) = r_0\,. \tag{3.5}$$

The two factors $\{x(t): t \geq 0\}$ and $\{y(t): t \geq 0\}$ stochastic differential equation can be written

$$dx(t) = -ax(t)dt + \sigma dW_1^{\mathbb{Q}}(t)\,, \quad x(0) = 0\,, \tag{3.6}$$
$$dy(t) = -by(t)dt + \eta dW_2^{\mathbb{Q}}(t)\,, \quad y(0) = 0\,, \tag{3.7}$$

where a, b, σ, η are positive parameters and $r_0 = \varphi(0)$. The (W_1^Q, W_2^Q) are correlated Wiener processes under the risk-neutral measure $\mathbb{Q}$. The correlation parameter, ρ, is defined by $dW_1^Q(t)\, dW_2^Q = \rho dt$, with -1 $\leq \rho \leq$ 1. The deterministic function, $\varphi(t)$, allows the model to fit the initial market term structure perfectly.

We can define the expectation and the variance of $r(t)$ conditional to the information up to time $s < t$. Let $\mathcal{F}_s$ be the sigma-field containing information up to (and including) s. The explicit solution given the information set $\mathcal{F}_s$ is

$$r(t)|\mathcal{F}_s = x(s)e^{-a(t-s)} + y(s)e^{-b(t-s)} + \sigma \int_s^t e^{-a(t-s)}\, dW_1(u) + \eta \int_s^t e^{-b(t-s)} dW_2(u) + \varphi(t), \quad (3.8)$$

which means that $r(t)$, conditioned by $\mathcal{F}_s$, follows a normal distribution with expectation and variance given by

$$E[r(t)|\mathcal{F}_s] = x(s)e^{-a(t-s)} + y(s)e^{-b(t-s)} + \varphi(t)\,, \quad (3.9)$$

$$V[r(t)|\mathcal{F}_s] = \frac{\sigma^2}{2a}\left(1 - e^{-2a(t-s)}\right) + \frac{\eta^2}{2b}\left(1 - e^{-2b(t-s)}\right) + 2\rho\frac{\sigma\eta}{a+b}\left(1 - e^{-(a+b)(t-s)}\right). \quad (3.10)$$

Following Brigo & Mercurio (2006), we derive the analytical expression of the price of a zero-coupon bond under Q. $P(t,T)$ can be defined as a function of the short rate,

$$P(t,T) = E_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\right]. \quad (3.11)$$

Defining the random variable

$$I(t,T) = \int_t^T [x(u) + y(u)]du\ , \quad (3.12)$$

$I(t,T)$, conditioned by $\mathcal{F}_s$, follows a normal distribution with conditional expectation $M(t,T)$ and conditional variance $V(t,T)$, given by

$$M(t,T) = \frac{1 - e^{-a(T-t)}}{a}x(t) + \frac{1 - e^{-b(T-t)}}{b}y(t) \quad (3.13)$$

and

$$\begin{aligned} V(t,T) = &\frac{\sigma^2}{a^2}\left[(T-t) + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right] + \\ &+\frac{\eta^2}{b^2}\left[(T-t) + \frac{2}{b}e^{-b(T-t)} - \frac{1}{2b}e^{-2b(T-t)} - \frac{3}{2b}\right] + \\ &+ \\ &+2\rho\frac{\sigma\eta}{ab}\left[(T-t) + \frac{e^{-a(T-.t)} - 1}{a} + \frac{e^{-b(T-.t)} - 1}{b} - \frac{e^{-(a+b)(T-.t)} - 1}{a+b}\right]. \end{aligned} \quad (3.14)$$

As $x(t)$ and $y(t)$ are normally distributed and $\varphi(t)$ is deterministic, the integral in (3.11) is normally distributed with mean $\mu = M(t,T) + \int_t^T \varphi(u)du$ and variance $\sigma^2 = V(t,T)$. Hence, $e^{\int_t^T r_s ds}$ is lognormally distributed with $E[e^{\int_t^T r_s ds}] = e^{\mu + \frac{1}{2}\sigma^2}$. Therefore, we obtain for $P(t,T)$,

$$P(t,T) = \exp\left[-\int_t^T \varphi(u)du - \frac{1-e^{-a(T-t)}}{a}x(t) - \frac{1-e^{-b(T-t)}}{b}y(t) + \frac{1}{2}V(t,T)\right]. \quad (3.15)$$

If, for each maturity T, the discount factor $P(0,T)$ matches with the market discount factor $P^M(0,T)$, the model fits the currently observed term structure of discounted factors. We have, using (3.6), (3.7) and (3.15),

$$P^M(0,T) = exp\left[-\int_0^T \varphi(u)du + \frac{1}{2}V(0,T)\right]. \quad (3.16)$$

From the instantaneous forward rate, as defined in (3.3), it follows that

$$f^M(0,T) = -\frac{\partial \log P^M(0,T)}{\partial T}\ . \quad (3.17)$$

With (3.16) and (3.17), we obtain

$$f^M(0,T) = \varphi(T) - \frac{\partial}{\partial T}\frac{1}{2}V(0,T)\ . \quad (3.18)$$

The derivative of $V(0,T)$ with respect to T is

$$\frac{\partial V(0,T)}{\partial T} = \frac{\sigma^2}{a^2}(1-e^{-aT})^2 + \frac{\eta^2}{b^2}(1-e^{-bT}) + 2\rho\frac{\sigma\eta}{ab}(1-e^{-aT})(1-e^{-bT})\ . \quad (3.19)$$

Hence, we get the expression of the deterministic function from (3.18) and (3.19),

$$\varphi(T) = f^M(0,T) + \frac{\sigma^2}{2a^2}(1-e^{-aT})^2 + \frac{\eta^2}{2b^2}(1-e^{-bT}) + \rho\frac{\sigma\eta}{ab}(1-e^{-aT})(1-e^{-bT})\ . \quad (3.20)$$

Considering that

$$exp\left[-\int_t^T \varphi(u)du\right] = \exp\left[-\int_0^T \varphi(u)du\right] exp\left[\int_0^t \varphi(u)du\right] =$$

$$= \frac{P^M(0,T)\ exp\left[-\frac{1}{2}V(0,T)\right]}{P^M(0,t)\ exp\left[-\frac{1}{2}V(0,t)\right]} = \frac{P^M(0,T)}{P^M(0,t)}\ exp\left\{-\frac{1}{2}[V(0,T)+V(0,t)]\right\}\ , \quad (3.21)$$

then, the price of a zero-coupon bond is

$$P(t,T) = \frac{P^M(0,T)}{P^M(0,t)}\ exp\left\{\frac{1}{2}[V(t,T) - V(0,T) + V(0,t)] - \frac{1-e^{-a(T-t)}}{a}x(t) - \frac{1-e^{-b(T-t)}}{b}y(t)\right\}. (3.22)$$

Rewriting the price of the zero-coupon bond in the framework of affine term structure models, as Duffie & Kan (1996), we obtain

$$P(t,T) = \mathcal{A}(t,T) \exp\left[-\mathcal{B}_x(t,T)x(t) - \mathcal{B}_y(t,T)y(t)\right], \tag{3.23}$$

where

$$\mathcal{A}(t,T) = \frac{P^M(0,T)}{P^M(0,t)} \, exp\left\{\frac{1}{2}[V(t,T) - V(0,T) + V(0,t)]\right\}, \tag{3.24}$$

$$\mathcal{B}_x(t,T) = \frac{1 - e^{-a(T-t)}}{a}, \qquad \mathcal{B}_y(t,T) = \frac{1 - e^{-b(T-t)}}{b}. \tag{3.25}$$

To perform real-world scenario projections, G2++ model must be regarded under the real-world measure $\mathbb{P}$. According to EIOPA (2020), this is done by using a constant, independent market price of risk, to preserve the structure of risk neutral with an additional constant drift term.

The change of measure can be made using the Girsanov's theorem (Girsanov, 1960),

$$dW_i^{\mathbb{P}} = -\lambda_i \, dt + dW_i^{\mathbb{Q}}, \quad i = 1,2, \tag{3.26}$$

where λ_i is the market price of risk. Under real-world measure $\mathbb{P}$, the two factors $x(t)$ and $y(t)$ can be written

$$dx(t) = \big(\lambda_1\sigma - ax(t)\big)dt + \sigma dW_1^{\mathbb{P}}(t), \quad x(0) = 0; \tag{3.27}$$

$$dy(t) = \big(\lambda_2\sigma - by(t)\big)dt + \eta dW_2^{\mathbb{P}}(t), \quad y(0) = 0. \tag{3.28}$$

3.1.3 Estimation – Kalman filter

In this section we present the methodology to estimate the parameters of the model using a Kalman filter. It is an algorithm to study the relationship between a series of possible noisy observed measurements (yields) and the theoretical predictions of those measurements based on unobserved (latent) state variables. It is defined by two equations, the *observation equation* and the *state equation* (see De Jong, 2000).

The *observation equation* is derived from the price of the zero-coupon bond of the risk-neutral model, combining (3.2) and (3.23); and (3.24) and (3.25):

$$y_t = A_t + B\,F_t + e_t, \quad with\; e_t | \mathcal{F}_{t-1} \sim N(0, H_t), \tag{3.29}$$

where

$$y_t = \begin{bmatrix} y_t(\tau_1) \\ \vdots \\ y_t(\tau_k) \end{bmatrix}, A_t = \begin{bmatrix} -\dfrac{log[\mathcal{A}(t,\mathrm{t}+\tau_1)]}{\tau_1} \\ \vdots \\ -\dfrac{log[\mathcal{A}(t,\mathrm{t}+\tau_\mathrm{k})]}{\tau_k} \end{bmatrix}, B = \begin{bmatrix} \dfrac{\mathcal{B}_x(t,\mathrm{t}+\tau_1)}{\tau_1} & \dfrac{\mathcal{B}_y(t,\mathrm{t}+\tau_1)}{\tau_1} \\ \vdots & \vdots \\ \dfrac{\mathcal{B}_x(t,\mathrm{t}+\tau_\mathrm{k})}{\tau_k} & \dfrac{\mathcal{B}_y(t,\mathrm{t}+\tau_\mathrm{k})}{\tau_k} \end{bmatrix}, F_t = \begin{Bmatrix} x(t) \\ y(t) \end{Bmatrix}, \tag{3.30}$$

in which $y_t(\tau_i)$ is the yield rate at time t of maturity τ_i. Measurement errors e_t are normally distributed with zero mean and variance matrix H_t. Following De Jong (2000), the measurement errors are restricted to be equal for all maturities and constant over time, being H_t a diagonal matrix with value h (whose range is defined in Chapter 5).

In the second equation, the *state equation*, the evolution of the factors is modelled by a vector auto-regressive process with one lag, VAR(1), i.e.,

$$F_{t+\Delta t} = C + \Phi F_t + \upsilon_t, \qquad with\ \upsilon_t|\mathcal{F}_{t-1} \sim N(0, Q)\ , \tag{3.31}$$

where Δt is the time interval between two consecutive observations and υ_t represents the errors, normally distributed with zero mean and variance matrix Q. The transition equation is given under the real-world measure $\mathbb{P}$, including the market prices of risk. The vectors $C + \Phi F_t$ and Q represent, respectively, the conditional expected value and the conditional variance of the solution of the stochastic differential equations for $x(t)$ and $y(t)$, see (3.27) and (3.28). Hence,

$$E_t[F_{t+\Delta t}] = \begin{bmatrix} \frac{\lambda_1}{a}(1 - e^{-a\Delta t}) \\ \frac{\lambda_1\eta\rho + \lambda_2\eta\sqrt{1-\rho^2}}{a+b}(1 - e^{-b\Delta t}) \end{bmatrix} + \begin{bmatrix} e^{-a\Delta t} & 0 \\ 0 & e^{-b\Delta t} \end{bmatrix} F_t = C + \Phi F_t\ , \tag{3.32}$$

$$V[F_{t+\Delta t}] = \begin{bmatrix} \frac{\sigma^2}{2a}(1 - e^{-2a\Delta t}) & \frac{\sigma\eta\rho}{a+b}\left(1 - e^{-(a+b)\Delta t}\right) \\ \frac{\sigma\eta\rho}{a+b}\left(1 - e^{-(a+b)\Delta t}\right) & \frac{\eta^2}{2b}(1 - e^{-2b\Delta t}) \end{bmatrix} = Q\ . \tag{3.33}$$

The unconditional mean and variance are used for the initial set, resulting in

$$F_0 = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } Q_0 = \begin{bmatrix} \frac{\sigma^2}{2a} & \frac{\sigma\eta\rho}{a+b} \\ \frac{\sigma\eta\rho}{a+b} & \frac{\eta^2}{2b} \end{bmatrix}. \tag{3.34}$$

The Kalman filter is an algorithm which involves consecutive cycles of predicting the state of an observed variable based on the model, comparing the prediction with the historical observed data and updating the parameters to reach the optimal predictive. It can be summarized in the following steps.

1. The prediction step starts by producing an optimal estimate of the state vector F_t given the information at time $t-1$, F_{t-1}. Hence,

$$F_{t|t-1} = E_{t-1}[F_t] = C + \Phi\hat{F}_{t-1}\ ; \tag{3.35}$$

2. It also estimates the variance matrix of F_t

$$P_{t|t-1} = E_{t-1}\left[\left(F_t - \hat{F}_{t|t-1}\right)\left(F_t - \hat{F}_{t|t-1}\right)'\right] = \Phi P_{t-1}\Phi' + Q\ ; \tag{3.36}$$

3. The updating step uses the information that becomes available at time t. The observable yield rate (y_t) can be compared to the prediction made at time $t-1$. The prediction error is given by

$$\xi_t = y_t - y_{t|t-1}\,, \qquad with \;\; y_{t|t-1} = A + B\,F_{t|t-1}\,; \tag{3.37}$$

4. The conditional variance of the prediction error is then

$$V_t = BP_{t|t-1}B' + H\,; \tag{3.38}$$

5. We use the prediction error to update the estimate of the state variables. We add the prediction error times a factor K_t, named Kalman gain, to the initial estimate $F_{t|t-1}$,

$$\hat{F}_t = F_{t|t-1} + K_t\xi_t\,; \tag{3.39}$$

6. The Kalman gain represents the weight of the additional information available between $t-1$ and t,

$$K_t = P_{t|t-1}B'V_t^{-1}\,; \tag{3.40}$$

7. The conditional variance of the state variables is updated,

$$\hat{P}_t = (I - K_tB)P_{t|t-1}\,. \tag{3.41}$$

We compute the loglikelihood function by repeating the seven steps for each discrete time step of the dataset. Considering the measurement prediction errors are Gaussian, we can write the loglikelihood function

$$l(y_1, \dots, y_n; \theta) = -\frac{nKlog(2\pi)}{2} - \frac{1}{2}\sum_{t=1}^{n}(\log|V_t|| + \xi_t'V_t'\xi_t)\,. \tag{3.42}$$

3.1.4 Model simulation

From the Cholesky decomposition (Haastrecht et al., 2009) applied to the correlation matrix, the interest rate model factors can be discretized by formulas

$$x(t+dt) = x(t)e^{-adt} + \frac{\lambda_1\sigma}{a}(1 - e^{-adt}) + \sqrt{\frac{\sigma^2}{2a}(1 - e^{-2adt})}\,Z_x \tag{3.43}$$

$$y(t+dt) = y(t)e^{-adt} + \frac{\lambda_2\eta}{b}(1 - e^{-bdt}) + \sqrt{\frac{\eta^2}{2b}(1 - e^{-2bdt})}\,\left(\rho Z_x + \sqrt{1-\rho^2}Z_y\right), \tag{3.44}$$

where Z_x and Z_y follow the standard normal distribution. Adding the two factors to the deterministic function, calculated using Nelson-Siegel-Svensson parameters (Svensson, 1994), the short rate is simulated. Next, the calculation of the price of the zero-coupon bond follows, applying (3.23). Then, the bond investment return can be computed, using a rolling down strategy.

3.2 Equity Index Model

3.2.1 Notation and specification

The Geometric Brownian motion (GBm) was selected to model the development of the equity index. The model can be described by two parameters: the volatility, σ, and the equity risk premium, λ_{eq}. The risk-free rate used, $r(t)$, is the one calculated by the nominal interest rate model. The stochastic differential equation of the price of the index, S_t, is given by

$$dS_t = \left(r(t) + \lambda_{eq} \right) S_t \, dt + \sigma \, S_t dW \, . \tag{3.45}$$

To solve the stochastic differential equation (see Björk, 2020), we start by defining the change of variable, $X_t = e^{S_t}$ or $S_t = \ln(X_t)$. Applying the Itô formula to $f(t,x) = \ln(x)$, we get

$$dX_t = \frac{\partial f(t,x)}{\partial t} dt + \frac{\partial f(t,x)}{\partial x} dS_t + \frac{1}{2}\frac{\partial^2 f(t,x)}{\partial^2 x} (dS_t)^2 \tag{3.46}$$

and

$$dX_t = \left(r(t) + \lambda_{eq} - \frac{1}{2}\sigma^2 \right) dt + \sigma dW_t \, . \tag{3.47}$$

The solution for X_t is

$$X_t = X_0 + \int_0^t \left(r(u) + \lambda_{eq} - \frac{1}{2}\sigma^2 \right) dt + \int_0^t \sigma dW_u \, . \tag{3.48}$$

Then, we get the solution for S_t, the price of the equity index,

$$S_t = S_0 \exp\left[\left(r(t) + \lambda_{eq} - \frac{\sigma^2}{2} \right) t + \sigma \, dW_t \right] . \tag{3.49}$$

Since S_t is log-normally distributed, its expected value and variance are given by

$$E[S_t] = S_o e^{(r(t)+\lambda_{eq})t} \tag{3.50}$$

and

$$V[S_t] = S_0^2 e^{2(r(t)+\lambda_{eq})t} \left(e^{\sigma^2 t} - 1 \right) . \tag{3.51}$$

Knowing the prices of the equity index, we can compute the annual equity return by

$$Ret_t = \frac{S_t - S_{t-1}}{S_{t-1}} \, . \tag{3.52}$$

The equity risk premium, λ_{eq} , is estimated following Damodaran method (Damodaran, 2020), explained in the next section.

3.2.2 Estimation – Damodaran method

The market risk premium, i.e., the price of risk in equity markets, is a key metric to assess the overall market.

The expected return, $E[R_m]$, i.e., the rate of return expected by investors, can be written as the sum of the risk-free rate (R_f) and a risk premium that rewards the risk (λ_{eq}),

$$E[R_m] = R_f + \lambda_{eq} . \tag{3.53}$$

There are three approaches to estimate equity risk premiums: (1) survey of investors to get an idea of their expectations about equity return in the future; (2) use the past returns earned as expectation of future return; (3) attempt to estimate an implied premium based on the market rates related to current prices. In this work, we follow the third approach.

The basis of the method is the Dividend Discounted model (DDM), that establishes the value of equity as the present value of expected dividends from the investment. Damodaran (2021) proposes an expansion of the model that considers the potential dividends instead of the actual dividends. Adding stock buybacks to aggregate dividend paid gives a better measure of total cash flow to equity. The general formula of the value of equity can be written as

$$Value\ of\ Equity = \sum_{t=1}^{N} \frac{E[FCFE_t]}{(1+E[R_m])^t} + \frac{E[FCFE_{N+1}]}{(E[R_m]-g_N)(1+E[R_{m]})^N} , \tag{3.54}$$

where N is the number of years of high growth, $E[FCFE_t]$ is the Expected Free Cash Flow to equity (potential dividends) in year t, $E[R_m]$ is the rate of return expected by investors and g_N is the stable growth (after year N).

Following the assumptions considered in the reference document from EIOPA (2020), the Expected Free Cash Flow is computed using the long-term growth EPS forecast, g, the sum of the dividend yield and the buyback yield, γ, and the price of the index, P_0 , at time $t = 0$. We consider a constant growth rate for five years followed by a perpetuity with growth rate equal to risk-free rate (R_f).

Rewriting (3.54), we have

$$\begin{aligned} Value\ of\ Equity = & \frac{\gamma P_0}{(1+E[R_m])} + \frac{\gamma(1+g)P_0}{(1+E[R_m])^2} + \frac{\gamma(1+g)^2 P_0}{(1+E[R_m])^3} + \frac{\gamma(1+g)^3 P_0}{(1+E[R_m])^4} \\ & + \frac{\gamma(1+g)^4 P_0}{(1+E[R_m])^5} + \frac{\gamma(1+g)^4\left(1+R_f\right)P_0}{\left(E[R_m]-R_f\right)(1+E[R_m])^5}, \end{aligned} \tag{3.55}$$

where $Value\ of\ Equity$ is the present value of the index. To compute the equity risk premium, we calculate the expected return, $E[R_m]$, by imposing

$$P_0 = Value\ of\ Equity. \tag{3.56}$$

The values of parameters and results from the estimation can be seen in detail in Chapter 5.

3.2.3 Model simulation

Using the short rate from the interest rate simulated model, the simulation of the equity price follows the discretization

$$S(t+dt) = S(t)\, exp\left[\left(r(t) + \lambda_{eq} - \frac{1}{2}\sigma^2 \right) dt + \sigma\sqrt{dt}\, \mathrm{Z}_w \right], \qquad (3.57)$$

where Z_w is a standard normal random variable. The equity return is computed using (3.52).

3.3 Inflation Model

3.3.1 Notation and specification

In this work, inflation rates follow one factor Vasicek process. The model was proposed by Vasicek (1977) and is a particular case of Hull-White model (Hull & White, 1993) with time dependent drift and diffusion parameters. It is a mean reverting stochastic model which ensures that the interest rates adhere to a long run reference level.

The corresponding stochastic differential equation is

$$di_t = k(\theta - i_t)\, dt + \sigma dW_t\,, \qquad i(0) = i_0\,, \qquad (3.58)$$

where i_t is the inflation rate at time t, k is the speed of mean reversion, θ is the level of mean reversion, σ is the volatility and W_t is the Wiener process.

The stochastic differential equation can be solved by the following method. Considering the variable change $i_t = z_t e^{-kt}$ (or $z_t = i_t e^{kt}$) and applying the Itô formula to $f(t,x) = xe^{kt}$, it follows that

$$dz_t = \frac{\partial f(t,x)}{\partial t} dt + \frac{\partial f(t,x)}{\partial x} di_t + \frac{1}{2}\frac{\partial^2 f(t,x)}{\partial^2 x}(di_t)^2\,. \qquad (3.59)$$

The solution for z_t is

$$z_t = i_0 + \theta(e^{kt} - 1) + \sigma \int_0^t e^{ks}\, dw_s\,. \qquad (3.60)$$

Therefore, the solution of the Vasicek model is

$$i_t = \theta + (i_0 - \theta)e^{-kt} + \sigma e^{-kt} \int_0^t e^{ks} dW_s\,. \qquad (3.61)$$

Since the random part of the solution $\int_0^t f(s) dW_s$ has a deterministic function, $f(s) = e^{ks}$, this is a Gaussian process. The expected value is then

$$E[i_t] = \theta + (i_0 - \theta)e^{-kt} \qquad (3.62)$$

and, using the Itô isometry, we have for the covariance

$$Cov[i_t, i_s] = \sigma^2 e^{-\kappa(t+s)} E\left[\left(\int_0^t e^{kr} dW_r\right)\left(\int_0^s e^{ks} dW_s\right)\right]. \quad (3.63)$$

Then, the variance is given by

$$V[i_t] = \frac{\sigma^2}{2k}(1 - e^{-2kt}). \quad (3.64)$$

When $t \to \infty$, the distribution of i_t converges to $N(\theta, \frac{\sigma^2}{2k})$, and we obtain the stationary distribution.

3.3.2 Estimation – Maximum likelihood estimation

With the mean and variance results from the previous section and the probability density function of the normal distribution, we can derive an expression of the loglikelihood function for the Vasicek model (Fergusson & Platen, 2015).

The probability density function (pdf) of the normal distribution is given by

$$f(x; \mu; \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad \sigma > 0. \quad (3.65)$$

Considering a sample $x = \{x_1, x_2, \dots, x_n\}$, the loglikelihood function is

$$L(\theta) = -\frac{n}{2}\log(\sigma^2) - \frac{n}{2}\log(2\pi) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2. \quad (3.66)$$

Let us assume we have a time series $I = \{I_{t_0}, I_{t_1}, \dots, I_{t_n}\}$ consisting of n+1 different points with equidistant time partition $dt = t_i - t_{i-1}$ in a given time interval $t_0, t_1, \dots, t_n$. Rearranging equation (3.62), it can be deduced in discrete terms, that the expected value and the variance of $I_{t_i}$is given by

$$E[I_{t_i}] = I_{t_{i-1}} e^{-\kappa dt} + \theta(1 - e^{-\kappa t}) \quad (3.67)$$

and

$$V[I_{t_i}] = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa dt}). \quad (3.68)$$

The loglikelihood function $L(\theta)$, inserting (3.67) and (3.68) in (3.66), is given by

$$L(\theta) = L(\kappa; \theta; \sigma^2) = -\frac{n}{2}\log\left[\frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa dt})\right] - \frac{n}{2}\log(2\pi)$$
$$-\frac{k}{\sigma^2(1 - e^{-2\kappa dt})}\sum_{i=1}^{n}\left[I_{t_i} - I_{t_{i-1}} e^{-kdt} - \theta(1 - e^{-\kappa dt})\right]^2. \quad (3.69)$$

Using the loglikelihood function, we can derive an estimator for each parameter κ, θ and σ^2. The full proof of the derivation can be seen in Fergusson & Platen (2015).

The maximum loglikelihood estimators are given by:

$$\hat{\kappa} = -\frac{1}{dt}\log\left[\frac{n\sum_{i=1}^{n} I_{t_i}I_{t_{i-1}} - \sum_{i=1}^{n} I_{t_i}\sum_{i=1}^{n} I_{t_{i-1}}}{n\sum_{i=1}^{n}\sum_{i=1}^{n} {I_{t_i}}^2 - \left(\sum_{i=1}^{n} I_{t_{i-1}}\right)^2}\right]; \quad (3.70)$$

$$\hat{\theta} = \frac{1}{n\left(1-e^{-\hat{k}dt}\right)}\left(\sum_{i=1}^{n} I_{t_i} - e^{\hat{k}dt}\sum_{i=1}^{n} I_{t_{i-1}}\right); \quad (3.71)$$

$$\hat{\sigma}^2 = \frac{2\hat{k}}{n\left(1-e^{-2\hat{k}dt}\right)}\sum_{i=1}^{n}\left[I_{t_i} - I_{t_{i-1}}e^{-\hat{k}dt} - \hat{\theta}\left(1-e^{-\hat{k}dt}\right)\right]^2 . \quad (3.72)$$

From historical data of inflation rates and the future projection of inflation rates, we estimate the parameters of the Vasicek model.

3.3.3 Model simulation

The inflation rate model is simulated using the discretization given by

$$i(t+dt) = i(t) + \kappa\big(\theta - i(t)\big)dt + \sigma\sqrt{dt}\, Z_s\, , \quad (3.73)$$

where Z_s is a normal (0,1) random variable.

3.4 Real Wage Growth Model

3.4.1 Notation and specification

Labour market risk, in particular employment and wages, have an impact on the value of the contributions and consequently on the asset accumulation and retirement income. Contributions to DC plans depend, among other elements, on the length of employment and the wage growth path. In the present work, only the real wage growth rate will be modelled since we assume an uninterrupted career path (no unemployment).

The real wage growth path can be very different among individuals, depending on their socio-economic situation, such as educational level, income and occupation. Studies by Bosworth et al. (2000) and by Antolin et al. (2010) conclude that there are three main career paths for real wages: (1) paths that reach a plateau at the end of the career (high real-wage gains); (2) paths where the plateau is reached earlier, between 45 and 55 (medium real-wage gains) and then real wage path falls; (3) flat real wages paths during the whole careers (a minority).

Real wage growth is modelled via formulas with random parameters, which allow to consider flat wages along the career, rising wages along entire career or wages with plateau before end of career.

The model that can be found in the document EIOPA (2020) follows a quadratic equation with age

$$wage = a\,(max - age)^2 + b\, , \quad (3.74)$$

where coefficient a follows a uniform distribution between 0.15 and 0.011, coefficient max follows a uniform distribution between 52 and 69, which represents the age when the real wage reaches the plateau, and coefficient b can be found by solving equation (3.74) after fixing the parameters for wage and age (for instance, in the example presented in Chapter 6 a wage equal to 100 and age equal 30 were considered).

3.4.2 Model simulation

The real wage growth model is simulated using formula (3.74), multiplied by 10 to obtain an initial wage of 1000 €, at age 30, in line with the example presented in Chapter 6

$$wage(t) = 10[a\ (\max - age(t))^2 + b]\ . \tag{3.75}$$

4 Construction of Projected Lifetables

In this chapter, we present the mathematical models that allow the construction of projected lifetables for pension funds population. We start by presenting the Poisson Lee-Carter model to build the dynamics mortality table. Then, we use the Denuit-Goderniaux method to close the lifetable. At last, we apply the relational model to establish the relationship between general population and the pension funds population in order the obtain the projected lifetable for the latter population.

4.1 Notation and Specification

We analyze the changes in mortality as function of age x and time t, following the notation of Brouhns et al. (2002). Hence, $\mu_x(t)$ denotes the force of mortality at age x during calendar year t, and D_{xt} will denote the number of deaths reported at age x during year t, from an exposure-to-risk E_{xt}.

The probability of death of an individual age x during year t is q_x and the probability of survival till age $x+1$ is $p_x = 1 - q_x$. The central mortality rate is given by

$$m_x(t) = \frac{D_{xt}}{E_{xt}} \; . \tag{4.1}$$

The life expectancy of an individual age x in year t, $e_x(t)$, can be approximated by

$$e_x(t) = \frac{1}{2} + \sum_{k \geq 1} \left[\prod_{j=0}^{k-1} p_{x+j}(t+j) \right]. \tag{4.2}$$

The present value of a whole life annuity paying 1 per year to an individual age x in year t, $a_x(t)$, can be written as

$$a_x(t) = \sum_{k \geq 0} \left[\prod_{j=0}^{k} p_{x+t}(t+j) \right] \upsilon^{k+1} \; , \tag{4.3}$$

where $\nu = (1+i)^{-1}$ is the discount factor with respect to yearly interest rate i.

Assuming that the force of mortality is constant within time and age interval but can vary between intervals, we obtain

$$\mu_{x+\tau}(t) = \mu_x(t), \qquad \text{for } 0 \leq \tau \leq 1 \; . \tag{4.4}$$

Therefore, we get

$$q_x(t) = 1 - e^{-\mu_x(t)}. \tag{4.5}$$

Thatcher (1999) proved that $m_x(t) \approx \mu_{x+\frac{1}{2}}(t)$ and from (4.4) we can assume $m_x(t) = \mu_x(t)$. Then, the force of mortality can be written

$$\mu_x(t) = \frac{D_{xt}}{E_{xt}}. \tag{4.6}$$

4.2 Poisson Lee-Carter Model

The classical Lee-Carter model (Lee & Carter, 1992) allows mortality projection specifying a log-bilinear form for the force of mortality $\mu_x(t)$

$$\ln \hat{\mu}_x(t) = \alpha_x + \beta_x k_t + \epsilon_x(t) \ , \qquad \text{with } \epsilon_x(t) \sim N(0, \sigma_\epsilon^2) \ , \tag{4.7}$$

where $\hat{\mu}_x(t)$ represents the observed force of mortality at age x in year t, $\epsilon_x(t)$ are the homoscedastic centered error terms and parameters α_x, β_x and k_t have the following interpretation: α_x represents the average mortality of each age over time, β_x denotes the age-specific pattern of mortality change and k_t represents time trend of mortality.

The parameters are subject to two constraints, to ensure a unique solution

$$\sum_t \kappa_t = 0 \quad \text{and} \quad \sum_x \beta_x = 1 \ . \tag{4.8}$$

Following Brouhns et al. (2002), the parameters are fitted to a matrix of age-specific observed of force of mortality using singular value decomposition (SVD). The parameters are obtained minimizing

$$\sum_{x,t} (\ln \hat{\mu}_x(t) - \alpha_x + \beta_x k_t)^2 \tag{4.9}$$

After the parameter’s estimation, Lee & Carter (1992) use an ARIMA(0,1,0) times series model, to perform projections and forecast the time trend of mortality, κ_t^* :

$$\kappa_t^* = \mu + \kappa_{t-1} + \varepsilon_t \tag{4.10}$$

Brouhns et al. (2002) develop an extension of the Lee-Carter model, where $\epsilon_x(t)$ from (4.7) is replaced with a random variable that follows a Poisson, which is justified by the assumption that the number of deaths can be considered a counting random variable. We obtain

$$D_{xt} \sim Poisson\big(E_{xt}\mu_x(t)\big) \ , \quad \text{with } \mu_x(t) = \exp(\alpha_x + \beta_x k_t) \ . \tag{4.11}$$

The parameters have the same meanings as before and are subject to the constraints (4.8) as the classical Lee-Carter model.

The model estimation is made by maximizing the loglikelihood,

$$L(\alpha;\beta;\kappa) = \sum_{x,t} \{D_{xt}(\alpha_x + \beta_x\kappa_t) - E_{xt}\exp(\alpha_x + \beta_x\kappa_t)\} + constant\,. \tag{4.12}$$

Since $\beta_x\kappa_t$ is a bilinear term, we cannot use a generalized linear model. Brouhns et al. (2002) propose an iterative method, based on Newton-Raphson algorithm, for estimation of log-linear models with bilinear terms, that was first developed by Goodman (1979).

The time trend projection is computed using ARIMA(0,1,0) like the classical Lee-Carter model.

Using the estimates of α_x and β_x and the forecast of κ_t, we can generate the mortality rates, $q_x(t)$, and force of mortality, $\mu_x(t)$,

$$q_x(t) = 1 - e^{-\mu_x(t)}\,, t > t_k\,, \tag{4.13}$$

$$\mu_x(t) = \exp(\alpha_x + \beta_x\kappa_t^*)\ . \tag{4.14}$$

where κ_t^* is the forecasted parameter from κ_t.

4.3 Denuit-Goderniaux Method

As stated in Coelho et al. (2008), data for older people lack the required quality demanded for the construction of complete lifetables. The solution is to resort to models that describe the appropriate mortality behaviour observed in these ages. There are a variety of methods to model older ages, Coelho et al. (2008) conclude that the Denuit and Goderniaux method give the best overall result.

Therefore, we use the Denuit and Goderniaux method (Denuit & Goderniaux, 2005). The method is based on a logarithm quadratic regression,

$$\ln q_x(t) = a_t + b_t x + c_t x^2 + \varepsilon_{xt}, \qquad \mathcal{E}_{xt} \sim N(0,\sigma^2)\ , \tag{4.15}$$

fitted separately to each calendar year *t* and to a given age period, and imposing two constraints

$$q_{x_{máx}} = 1 \quad \text{and} \quad q'_{x_{máx}} = 0\,, \tag{4.16}$$

where $q'_{x_{máx}}$ is the first derivative with respect to age *x*, a, b and c are parameters to be estimated by OLS (Ordinary least squares method) and $x_{máx}$ is a pre-defined highest attainable age. The first constraint imposes a maximum age for human life. The second one guarantees no decreasing death probabilities at older ages. Both guarantee a concave mortality curve with horizontal tangency at $x_{máx}$. Inserting (4.16) into (4.15), we get

$$\ln q_{\mathrm{x}}(\mathrm{t}) = \mathrm{c_t}(\mathrm{x_{max}} - \mathrm{x})^2 + \varepsilon_{\mathrm{xt}}\,,\ \ \varepsilon_{\mathrm{xt}} \sim \mathrm{N}(0,\sigma^2)\,. \tag{4.17}$$

4.4 Relational Models

When the data from the population under study is inadequate or scarce, one can apply relational models to relate the population under study (in our case, pension funds population) with a reference population. We will use the Cox proportional-hazard model (Cox,1972) based on previous work by Pateiro (2013), where the model provided the best fit for Portuguese pension funds data.

The Cox proportional-hazard model assumes that the force of mortality of the population under study ($\mu_{x,t}^{est}$) is proportional to the reference population ($\mu_{x,t}^{ref}$), with the proportional factor independent of age. We have then

$$\mu_{x,t}^{est} = a\,\mu_{x,t}^{ref}\,, \qquad \mu_{x,t}^{j} = \frac{D_{x,t}^{j}}{E_{x,t}^{j}}, j = est, ref \tag{4.18}$$

The parameter a is estimated applying a linear regression.

5 Estimation Results

5.1 ESG Models Estimation Results

5.1.1 Interest Rate Model

The "All euro area central government bond yield curve" of ECB[4] was used to estimate the model parameters. To cover the short, medium and long run, we selected maturities of 1, 10 and 30 years of the daily spot rates from 4 January 2016 to 30 December 2020 (five years). The Nelson-Siegel-Svensson parameters of the first day (4 January 2016) were used to compute the deterministic function, that depends on the forward rate.

The estimation is based on minimizing the negative loglikelihood using the differential evolution algorithm (current-to-p-best), as presented in EIOPA (2020).

The differential evolution is an algorithm of global optimization (Zhang et al., 2009), that belongs to the family of evolutionary computing algorithms. It starts with an initial population of candidate solutions. Resorting to iterations, these candidates are improved by introducing mutations into the population and retaining the fitter candidates' solutions that minimize an objective function. In this work, the candidate solutions are defined by the lower and upper bounds of the parameters and the objective function is the negative likelihood function given by the Kalman filter.

The bound limits imposed to the parameters are $a \in [0,1], b \in [0,1], \sigma \in [0,1],\ \eta \in [0,1], \rho \in [-1,1], \lambda_1 \in [0,0.02], \lambda_2 \in [0,0.02],\ h \in [0.0001,0.001]$, and the likelihood function is the one given by (3.42). Setting the initial values for the parameters within the bound limits, an initial likelihood function is computed and used as starting value for the differential evolution algorithm. The process stops when convergence is achieved. Since it is an iterative process, the procedure is repeated numerous times and using different initial values to assess the quality of the convergence. The results can be seen in the following table

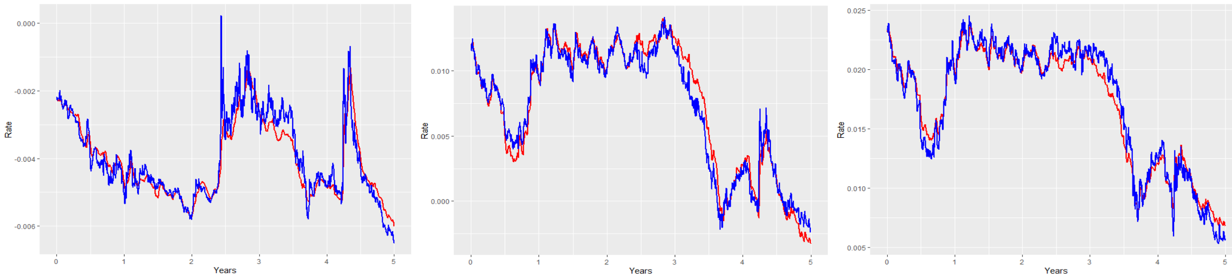
Table I - Estimated parameters for the interest rate model (4-01-2016 to 30-12-2020)

a	b	σ	η	ρ	λ_1	λ_2	h	loglike
0.12894325	0.09633414	0.04966171	0.04937197	-0.9995214	0.000171686	0.019103359	0.00082782	-21125.38

The two factors of the model have near perfect negative correlation. Only one of the market price of risk (λ_2) has a significant value and the volatility parameters (σ and η) have very low values. The comparison between the estimated yields and the observed ones is presented below. There are differences between the estimated and the observed values but, in general, the estimated curves follow the same path as the observed curves.

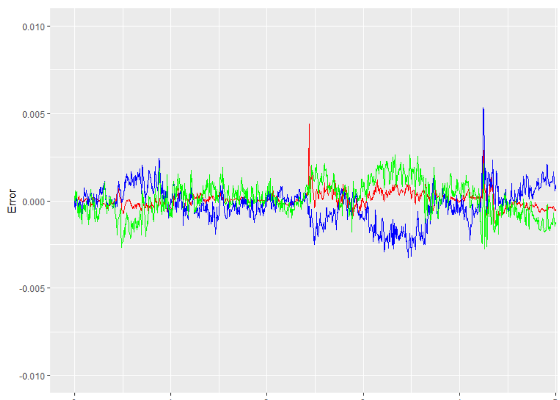
[4] ECB – Euro area yield curves

Figure 1 - Estimated (red) vs observed yields (blue) for maturities 1-year (left), 10-year (center) and 30-year (right)



The error of the estimation, extracted from the Kalman filter, is mostly within - 25 and + 25 basis points, as can be seen in the next figure.

Figure 2 - Estimation error for 1-year (red), 10-year (blue) and 30-year (green)



5.1.2 Equity Model

The index STOXX Europe 600 was used for the estimation without considering any country-specific risk premium (EIOPA, 2020). The estimation of the equity risk premium is based on the Damodaran method presented in Chapter 3.

For this purpose, the 10-year yield rate of the ECB's "All euro area central government bond yield curve" on the reference date of 30 December 2020 (i.e., first day of the simulation) was considered, being equal to $R_f = -0.194\%$. The long-term growth EPS forecast, g, is a weighted average of the average growth rate of the next six years, where the values of 2021 and 2022 were provided by Refinitiv[5] and for the following years were given by the risk-free rate, as presented in the table below, from which g=8.450% was derived.

Table II - EPS forecast

Year	2021	2022	2023 to 2026
EPS forecast (%)	25.000	21.100	-0.194

[5] REFINITIV – Financial Technology, Data and Expertise

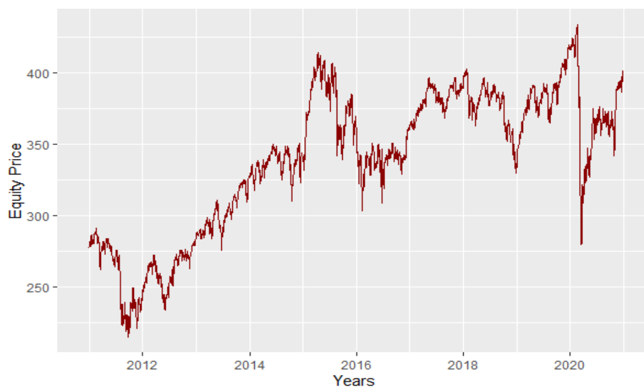
The dividend and buyback yield was provided by Bloomberg, whose estimated value is γ=4.83%. The closed price of the index is used in the estimation to solve (3.56), and on the reference date its value was P_0=400.25. The estimates of the parameters needed to compute the equity risk premium value are in the following table.

Table III - Equity risk premium parameters

Parameter	$\boldsymbol{R_f}$	$\boldsymbol{P_0}$	$\boldsymbol{g}$	$\boldsymbol{\gamma}$
Value	-0.194%	400.25	8.450%	4.83%

Imposing $P_0 = Value\ of\ Equity$ in (3.55) and using EXCEL solver, we obtain $E[R_m]$. From (3.53), we have that the equity risk premium $\lambda_{eq} = E[R_m] - R_f$ = 6,45%.

Figure 3 - STOXX Europe 600 closed price



The yearly close price of the index, from the end of 2010 until the end of 2020 (figure above), was used to estimate the volatility of the GBm process, σ, considering the annualized standard deviation of the last ten years as a proxy for the volatility of the equity model. Therefore, the estimated parameters for the equity index model are as summarised below.

Table IV - Equity model estimated parameters

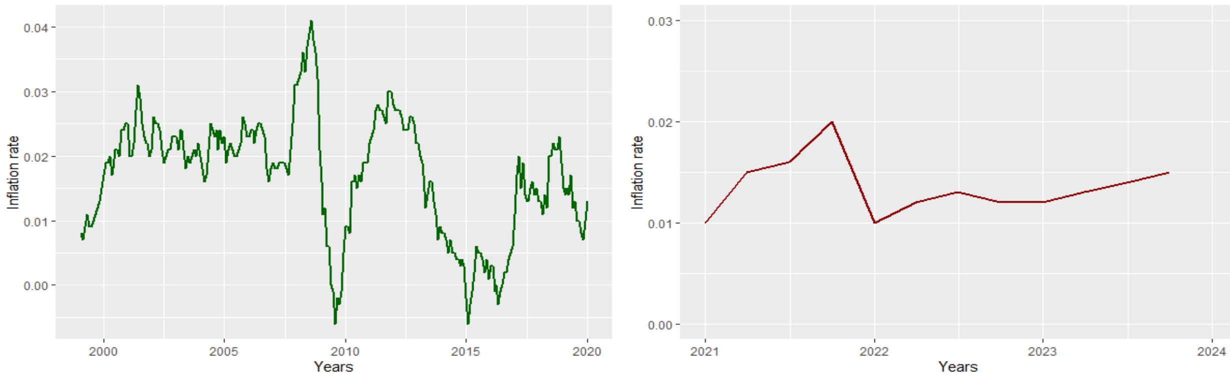
$\boldsymbol{\sigma}$	$\boldsymbol{\lambda_{eq}}$	$\boldsymbol{P_0}$	$\boldsymbol{r(t)}$
15,65%	6,45%	400.25	Short rate

5.1.3 Inflation Model

The parameter θ, which represents the mean at long-run, is given by the ECB target inflation of 2%. For the estimation of σ, the monthly Yo-Yo from HICP[6] (1999-2020) time series was considered, where the estimate of $\hat{\sigma}$ equals the standard deviation of the time series. The initial value of the inflation rate model, i_0, is the first value of monthly Yo-Yo from HICP (1999-2020) time series.

[6] HICP – ECB Statistical Data Warehouse

Figure 4 - Yo-Yo monthly inflation rates (1999-2020) on the right and inflation projection (2021-2023) on the left



The macro-economic inflation projection made by the European Commission was used to estimate the speed to the mean reversion, k. As stated in Chapter 3, the estimation is done by maximizing the loglikelihood function. In this case, we already have the values of $\hat{\theta}$ and $\hat{\sigma}$ and only need to estimate k, applying (3.70). Results are in Table V.

Table V - Inflation model estimated parameters

$\hat{\theta}$	$\hat{\sigma}$	$\hat{k}$	i_0
0.02	0.009138541	0.497069207	0.008

5.2 Projected Lifetables Estimation Results

5.2.1 Data

The construction of the projected lifetables was made using data from the Portuguese pension funds and the Portuguese general population, by age and gender. For the general population we use Human Mortality Database[7], considering ages between 0 and 90 ($x\epsilon\{0,90\}$) and years between 1970 and 2018 ($t\epsilon\{1970,2018\}$). For the pension schemes population, we use the data provided by ASF. After pension funds data analysis, we choose ages between 60 and 90 since it is the age interval with sufficient quality to conduct the mortality analysis. The Portuguese pension schemes are mainly composed of members within this age interval (see Pateiro, 2013).

5.2.2 Poisson Lee-Carter Model

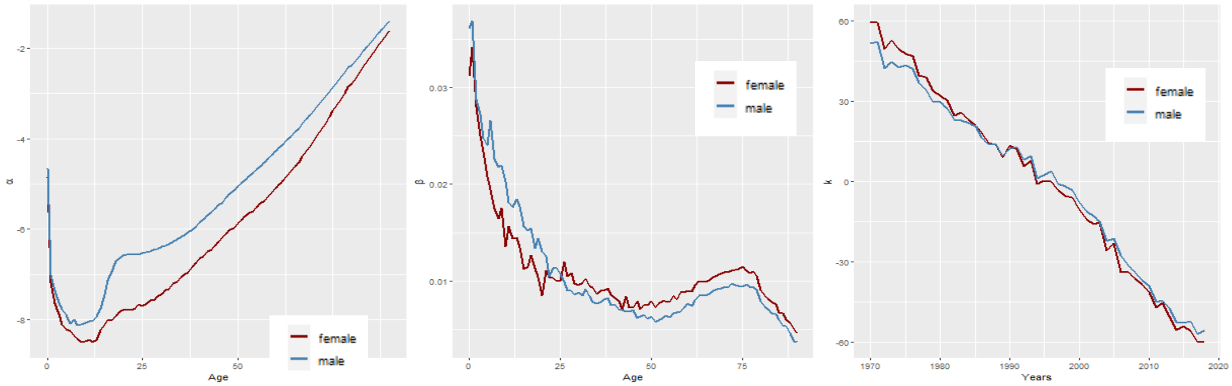
The estimation, for Portuguese population, is implemented using R software and follows three steps.

1. Estimation of the model parameters (α, β, κ), where $\alpha = \{\alpha_x, x = (0, \ldots, 90)\}$, $\beta = \{\beta_x, x = 0, \ldots, 90\}$ and $\kappa = \{\kappa_t, t = 1970, \ldots, 2018\}$.

The estimated parameters of Poisson Lee-Carter model are in the following graphs, whose shapes are similar to the ones obtained by Pateiro (2013).

[7] Human Mortality Database

Figure 5 – Males (blue) and females (red) Poisson Lee-Carter model parameters: α (left), β (middle) and k (right).



2. Estimation of ARIMA (0,1,0) model parameters using maximum loglikelihood estimation (MLE). The results are presented in the following table.

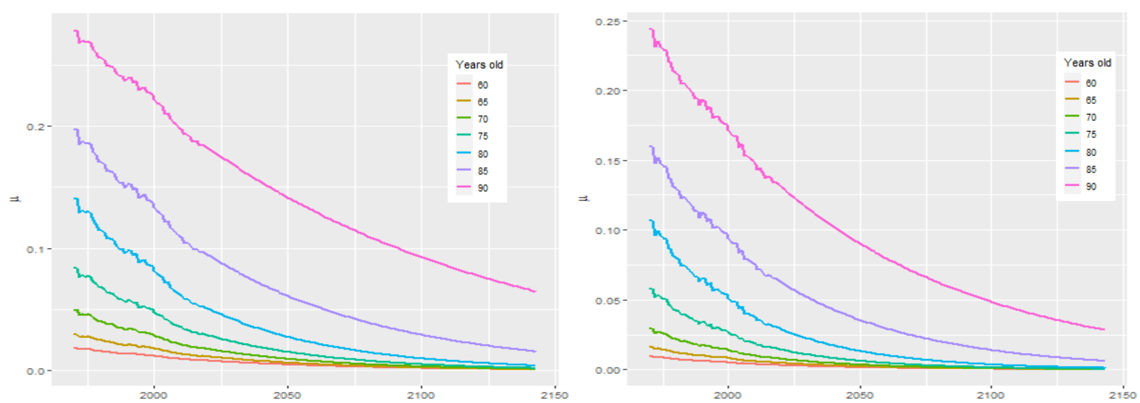
Table VI – ARIMA (0,1,0) parameters

	Males	**Females**
Drift, μ	-2.230012	-2.492037
Variance, σ^2	8.124448	11.921601

3. Projection of κ_t parameter over 125 years, from 2018 till 2143 (see Appendix A).

Finally, applying (4.14), we built two matrices with projected values of the force of mortality obtained by Poisson Lee-Carter model, $[u^{ref}_{F,x}(t)]_{91x126}$ for females and $[\mu^{ref}_{M,x}(t)]_{91x126}$ for males, where $x \epsilon \{0,1, \dots 90\}$ and $t \epsilon \{2018, \dots, 2143\}$.

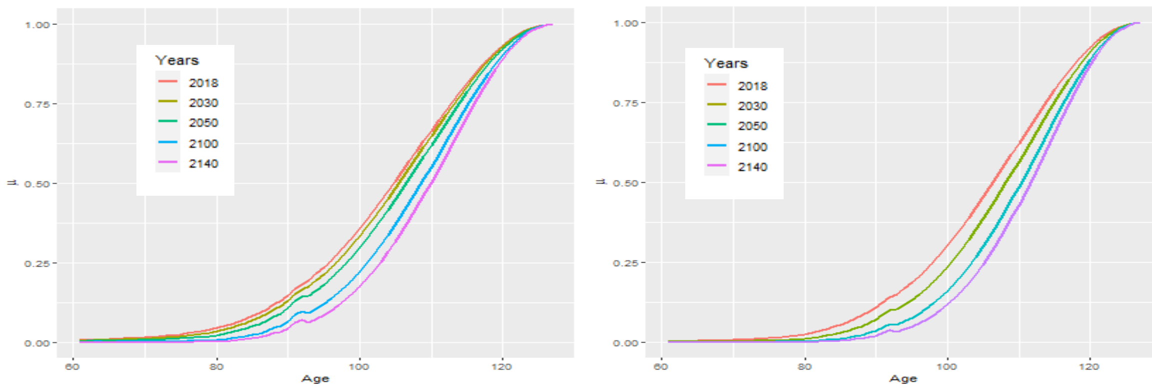
Figure 6 – Force of mortality μ for $x = 60, \dots, 90$ and $t \epsilon \{2018, \dots, 2143\}$.



5.2.3 Denuit and Goderniaux Method

We start by applying (4.13) to $[u^{ref}_{F,x}(t)]_{91x126}$ and $[\mu^{ref}_{M,x}(t)]_{91x126}$ in order to obtain the mortality rates matrices, $[q^{ref}_{F,x}(t)]_{91x126}$ and $[q^{ref}_{M,x}(t)]_{91x126}$. The closing of the lifetables was done by applying equation (4.17) with $x_{max} = 125$ as limit age, assuming that it will not be exceeded. A separate log-quadratic regression is fitted to each calendar year t and to ages (x) 90 and over. As a result, we obtain mortality rate matrices for each age $x \epsilon \{0, \ldots, 125\}$ and each calendar year $t \epsilon \{2018, \ldots, 2143\}$, $Q_F^{ref} = [q_{F,x}(t)]_{126x126}$ and $Q_M^{ref} = [q_{M,x}(t)]_{126x126}$, for female and male, respectively. Then, we use (4.14) to calculate $\mu_F^{ref} = [\mu_{F,x}(t)]_{126x126}$ and $\mu_M^{ref} = [\mu_{M,x}(t)]_{126x126}$.

Figure 7 – Closing of Projected Lifetables for males (left) and females (right)



5.2.4 Relational Model

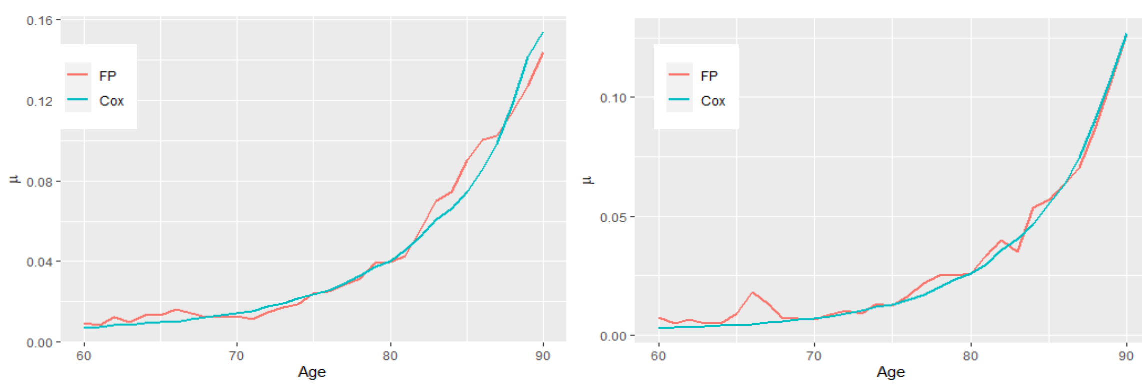
In this section, we calculate the projected lifetables for the pension schemes population (FP), using Cox proportional hazard for ages $x \epsilon \{60, \ldots, 90\}$, and years $t \epsilon \{2016, 2017, 2018\}$, from the most recent available data. The results from the linear regression using (4.18) are in Table VII.

Table VII – Estimates of the Cox proportional-hazard model parameters

	Males	Females
Parameter α	0.6969615	0.748895
Standard error σ_ε	0.0133271	0.012929

In Figure 8, we can see the difference between the force of mortality for FP data and the one obtained using the Cox proportional-hazard model.

Figure 8 - Force of mortality μ of FP and Cox model for males (left) and females (right)



Using the relational model, the computation of the probability of death matrices for the pension funds is straightforward, $Q_F^{FP} = [q_{F,x}(t)]_{126x126}$ and $Q_M^{FP} = [q_{M,x}(t)]_{126x126}$.

5.2.5 Comparison with other Lifetables

To assess the differences between the projected lifetables that were calculated and the standard lifetables (i.e., constructed using a specific time interval and not considering future longevity gains) that are commonly used by the Portuguese market, a comparison with the mortality tables TV 88/90 and GKF 95 is presented below. According to the latest data available from the ASF these are the tables most commonly used, respectively, to value pension liabilities (RSSSFP, 2019 - ASF[8]) and life annuities insurance obligations. As indicators, the life expectancy at age 65 and the actuarial value of annuities at ages 60, 65, and 70, are calculated considering the year 2018 (reference date). As we can see in Table VIII, the life expectancy assuming projected lifetables is greater, especially for the female population, with differences between 6 and 5 years, in comparison to TV 88/90 and GKF 95, respectively.

Table VIII - Life expectancy at age 65 (e_{65}) of projected lifetables (PL) and standard lifetables

Age	PL Female	PL Male	TV 88/90	GKF 95
65	25.9	22.4	19.8	20.8

With respect to the effects on actuarial value of life annuities, we compute these considering three discount rates for the projected and standard lifetables. The results are in Table IX and Table X.

Table IX - Actuarial value of annuities using projected lifetables (PL) and standard lifetables for three discount rates

	i_1=1%				i_2=2%				i_3=3%			
Age	PL Female	PL Male	TV 88/90	GKF 95	PL Female	PL Male	TV 88/90	GKF 95	PL Female	PL Male	TV 88/90	GKF 95
60	26.0	22.9	20.6	21.4	22.3	19.8	18.1	18.8	19.3	17.4	16.1	16.6
65	22.1	19.2	17.2	18.0	19.3	17.0	15.4	16.0	17.1	15.2	13.9	14.4
70	18.1	15.6	13.8	14.6	16.2	14.1	12.6	13.2	14.5	12.8	11.6	12.0

In line with the conclusions for life expectancy, the differences between projected and standard lifetables are more visible with respect to the female population and are more significant when comparing to TV 88/90 than to GKF 95. Table X gives the same results, in %.

Table X - Actuarial value of annuities for standard lifetables as a % of the corresponding value for projected lifetables (PL) with three discount rates

	i_1=1%				i_2=2%				i_3=3%			
Age	TV 88/90 / PL Female	TV 88/90 / PL Male	GKF 95 / PL Female	GKF 95 / PL Male	TV 88/90 / PL Female	TV 88/90 / PL Male	GKF 95 / PL Female	GKF 95 / PL Male	TV 88/90 / PL Female	TV 88/90 / PL Male	GKF 95 / PL Female	GKF 95 / PL Male
60	79.2	90.0	82.3	93.4	81.2	91.4	84.3	94.9	83.4	92.5	86.0	95.4
65	77.8	89.6	81.4	93.8	79.8	90.6	82.9	94.1	81.3	91.4	84.2	94.7
70	76.2	88.5	80.7	93.6	77.8	89.4	81.5	93.6	80.0	90.6	82.8	93.8

[8] Relatório do Sector Segurador e Fundos de Pensões – 2019 (ASF)

6 Application

The calculation tool was developed using R and Excel for making pension projections for DC type of pension schemes, without financial guarantees. The objective of this chapter is to present a practical example of the pension projection and produce a risk and performance analysis, considering both the accumulation and decumulation phases. We start by describing the inputs that an user can introduce in the calculation tool, followed by the specification of the simulation. Afterwards, the risk and performance measures used to assess the results of the projection are presented.

6.1 Inputs to the Calculation Tool

The calculation tool foresees a set of open fields, allowing the user to choose the key assumptions of the projection.

In terms of inputs related to the accumulation phase, the user can introduce the following assumptions: (1) Age at the beginning of the projection; (2) Retirement age; (3) Member's contribution rate; (4) Annual fee that is charged to the pension fund. In relation to the decumulation phase, the user can choose the mortality table and the discount rate to be used in the conversion to an annuity. It is implicitly assumed that the annuity is a constant lifelong annuity.

The tool allows the user to introduce two types of portfolios, the rebalanced and the lifecycle portfolios, in both cases considering a mix between bonds and equities. The rebalanced portfolio represents an investment strategy ensuring that the level of risk is kept within a certain desirable range, by allowing the user to define the lower and upper bounds of equity weights throughout the projection horizon. On the other hand, in the lifecycle portfolio the user can set the weight of equities for each year of the projection, applying an investment strategy that reduces the share in risky assets (equity) as the member approaches the retirement age. It is used to mitigate the risk of a reduction in retirement income, in case a negative shock in equity markets occurs near retirement age.

In addition, it is also possible not to use the stochastic models for making the projections but to introduce directly assumptions on the future return rate, inflation rate and/or wage growth (i.e., following a deterministic scenario).

In the application of the stochastic models, 10 000 Monte Carlo simulations are generated. Each one represents one possible scenario during the accumulation phase for the bond and equity returns, inflation rate and real wage growth rate. The reference data is the first date of the simulation, 30-12-2020, and the scenarios are simulated considering a monthly interval ($dt = 1/12$). The results from Monte Carlo simulation are in Appendix B.

6.2 Risk Profile and Performance Assessment

In what concerns the outcomes of the projections, apart from the results in terms of values (in euros), the tool produces a set of indicators which allows an analysis of the risk profile and performance assessment.

- **Retirement Income Assessment**

The replacement rate is one of the measures commonly used by regulators and policy makers to assess the adequacy of retirement income. It corresponds to the ratio between the first estimated monthly benefit and the last salary of the member:

$$RR = \frac{Estimated\ monthly\ benefit}{Wage}. \tag{6.1}$$

As it measures the percentage of the worker income that will be replaced by the expected outcome of a particular pension scheme, this indicator can be used as a benchmark by the member to help track how the scheme is doing in comparison to expectations.

- **Probability of recoup capital**

This risk measure corresponds to the probability of the investment strategy achieving at least the sum of all contribution at retirement age. It is computed as the proportion of the number of scenarios where the lump sum is greater than the total contributions.

- **Expected shortfall when not recouping capital**

The expected shortfall measures the average difference between the lump sum and total contributions, conditional on not recouping the capital. The greater the expected shortfall, the greater the risk that members will get a lump sum far lower than the sum of all contributions.

- **Risk of getting a low lump sum**

The lump sum distribution gives information about the different levels of outcomes the members can achieve and can be used to assess the investment performance, e.g., by calculating different percentiles.

- **Expected and median lump sum**

The potential performance of the investment strategy can be measured by the median or the expected value of the distribution of the lump sum. The median represents a more robust measure since it is less sensitive to extreme values than the mean.

- **Probability to reach a given goal**

The performance of the investment strategy can also be measured by comparing it with specific rates of return, which represent the level of ambition. For the results presented in section 6.3, similar to the analysis performed in EIOPA (2020), the ultimate forward rate (UFR) published by

EIOPA, and equal to 3,75% in 2020, was used as a proxy for the long-term risk-free rate, although the calculation tool allows the user to introduce other rates as benchmark. To compare the potential performance with the UFR, the tool calculates the proportion of the number of scenarios where the lump sum is equal to or higher than the lump sum resulting from an annual rate of return equal to the UFR.

- **Joint risk-performance assessment**

The combination of risk and potential performance measures can be applied to assess the risk-performance profile of the investment strategies. With both dimensions, the difference between investment strategies becomes more visible. In section 6.3.5, we will use the standard deviation of returns as risk measure and the mean of returns over total contribution as the performance measure.

6.3 Case Study

6.3.1 Inputs

For the case study, the inputs for the accumulation and decumulation phases, specific to a theoretical member, are presented in Table XI.

Table XI - Inputs of the accumulation and decumulation phases

Age	Retirement age	Initial wage	Contribution rate	Fee	Time interval
30	70	1000	10%	1%	Month

Mortality table	Discount rate 1	Discount rate 2
Projected lifetable	1%	3%

We have considered two rebalanced and two lifecycle portfolios, in an attempt to capture different investment strategies and assess the respective risk and potential performance. The lower and upper bounds for the rebalanced portfolios (RB1 and RB2) are the ones in Table XII. The first 20 years has bounds limits with higher equity exposure than the following 20 years.

Table XII - Rebalanced portfolios equity weights (%)

Portfolio	Years	Lower limit	Upper limit
RB1	From year 0 to 20	20 %	40 %
	From year 21 to 40	10 %	30 %
RB2	From year 0 to 20	30 %	60 %
	From year 21 to 40	20 %	50 %

The lifecycle portfolios (LC1 and LC2) have fixed equity weights which decrease throughout the years. In this case, it was decided to reduce the equity weight in the last 20 years, at 1% rate.

Table XIII - Lifecycle portfolios equity weight (%)

Portfolio	From year 0 to 20	From year 21 to 40
LC1	35 %	reduces 1% each year
LC2	45 %	reduces 1% each year

In the following, where applicable, the results are presented in terms of the mean and also considering three scenarios based on the percentiles of the distributions obtained: (1) Unfavourable scenario: 15th percentile; (2) Intermediate scenario: median; (3) Favourable scenario, 85th percentile. Apart from the results for the rebalanced and lifecycle portfolios and the UFR scenario, for comparison purposes, a scenario considering only the contributions paid (i.e., with 0% of return) is also presented (identified as 'CONTRIB').

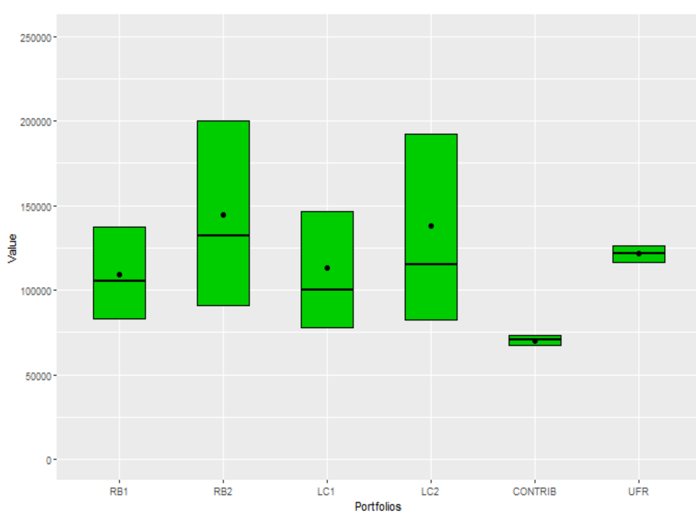
6.3.2 Lump Sum Distribution

The results obtained for the lump sum distribution can be seen in the following table and figure. For more optimistic scenarios, the higher lump sum is given by RB2 portfolio followed by the LC2 portfolio, which are the ones with higher equity exposure. In particular, the RB2 portfolio is the only one that achieves a mean and median lump sum greater than the one obtained by a portfolio with average return of UFR per year. The favourable scenario for all four portfolios gives a lump sum with average return higher than UFR. Regarding the unfavourable scenario, all portfolios achieve more positive results than the one considering only the contributions made, which means that the probability of a member receiving less than the amount of contributions must be low.

Table XIV - Lump sum distribution results (in euros)

Portfolio	Mean	15th perc	Median	85th perc
RB1	109 493	82 972	105 496	137 425
LC1	113 341	78 173	99 999	146 793
RB2	144 472	91 264	132 326	200 012
LC2	138 271	82 562	114 863	192 147
CONTRIB	70 343	67 162	70 402	73 384
UFR	121 390	116 642	121 421	125 928

Figure 9 - Lump sum distribution (in euros)[9]



[9] In the boxplots, the black points represent the mean, the top bar the 85th percentile, the middle bar the median and the bottom bar the 15th percentile.

Using the CONTRIB portfolio as reference, we can compare the results of the different portfolios, which can be seen in Table XV.

Table XV - Lump sum distribution results as function of total contributions (in percentage)

Portfolio	Mean	15th perc	Median	85th perc
RB1	156	118	150	194
LC1	161	112	142	208
RB2	205	130	188	284
LC2	196	118	163	271
CONTRIB	100	100	100	100
UFR	173	171	172	174

6.3.3 Annuities

Following the good practices described in Chapter 2, as a way to help members to better understand their purchasing power after retirement, the annuities results for this example are displayed in nominal and real terms (i.e., adjusted for the effects of inflation). The values presented are for a whole life annuity, assuming that the member starts receiving the annuity at year 2060, annually. In each case, the value of the annual payment made by the whole life annuity is obtained dividing the lump sum results of the portfolios by the present value of a whole life annuity paying 1 per year, considering two discount rates (i_1=1% and i_2=3%) and the projected lifetables. This is the reason why payments increase when the discount rate increases, since annuities are less expensive at higher discount rates.

Table XVI – Annuities annual payments (€) for male and female at discount rate i_1=1%

	Male				Female			
Portfolio	Mean	15th perc	Median	85th perc	Mean	15th perc	Median	85th perc
RB1	6 366	4 824	6 134	7 990	5 250	3 978	5 058	6 589
LC1	6 590	4 545	5 814	8 535	5 434	3 748	4 794	7 038
RB2	8 400	5 306	7 694	11 629	6 927	4 376	6 344	9 589
LC2	8 039	4 800	6 678	11 172	6 629	3 958	5 507	9 212
CONTRIB	4 090	3 905	4 093	4 267	3 373	3 220	3 375	3 518
UFR	7 058	6 782	7 060	7 322	5 820	5 592	5 821	6 037

Table XVII – Annuities annual payments (€) for male and female at discount rate i_2=3%

	Male				Female			
Portfolio	Mean	15th perc	Median	85th perc	Mean	15th perc	Median	85th perc
RB1	7 980	6 047	7 689	10 016	6 724	5 095	6 478	8 439
LC1	8 261	5 697	7 288	10 699	6 960	4 800	6 141	9 014
RB2	10 530	6 652	9 644	14 578	8 872	5 604	8 126	12 282
LC2	10 078	6 017	8 372	14 004	8 491	5 070	7 054	11 799
CONTRIB	5 127	4 895	5 131	5 348	4 320	4 124	4 323	4 506
UFR	8 847	8 501	8 850	9 178	7 454	7 163	7 456	7 733

The inflation-adjusted values for the annuities are presented in the following tables. We observe a significant reduction of the annuities' values, showing that, in the long-term, inflation adjustment can indeed have a significant effect on the adequacy of retirement income.

Table XVIII - Annuities annual payments (€) for male and female at discount rate i_1=1% (inflation-adjusted)

	Male				Female			
Portfolio	**Mean**	**15th perc**	**Median**	**85th perc**	**Mean**	**15th perc**	**Median**	**85th perc**
RB1	2 988	2 186	2 871	3 814	2 464	1 803	2 368	3 145
LC1	3 093	2 069	2 731	4 056	2 551	1 706	2 252	3 345
RB2	3 943	2 434	3 583	5 513	3 251	2 007	2 954	4 546
LC2	3 774	2 200	3 141	5 281	3 112	1 814	2 590	4 355
CONTRIB	1 820	1 615	1 812	2 027	1 501	1 332	1 494	1 672
UFR	3 313	2 919	3 290	3 708	2 732	2 407	2 713	3 058

Table XIX - Annuities annual payments (€) at discount rate i_2=3% (inflation-adjusted)

	Male				Female			
Portfolio	**Mean**	**15th perc**	**Median**	**85th perc**	**Mean**	**15th perc**	**Median**	**85th perc**
RB1	3 746	2 741	3 600	4 781	3 156	2 309	3 033	4 028
LC1	3 877	2 594	3 423	5 085	3 267	2 185	2 884	4 284
RB2	4 943	3 051	4 491	6 911	4 165	2 571	3 784	5 823
LC2	4 730	2 758	3 937	6 620	3 986	2 324	3 317	5 578
CONTRIB	2 282	2 024	2 272	2 541	1 923	1 706	1 914	2 141
UFR	4 153	3 659	4 124	4 649	3 499	3 083	3 474	3 917

Using the CONTRIB portfolio as reference, we can compare the results of the different portfolios, which can be seen in Table XX and Table XXI, for nominal and inflation-adjusted values, respectively. Since the values are independent of gender and discount rate only two tables are presented.

Table XX – Annual annuities payments in % of total annual contributions (in percentage).

Portfolio	Mean	15th perc	Median	85th perc
RB1	156	118	150	194
LC1	161	112	142	208
RB2	205	130	188	284
LC2	196	118	163	271
CONTRIB	100	100	100	100
UFR	173	171	172	174

Table XXI – Annual annuities payments in % of total annual contributions inflation-adjusted (in percentage)

Portfolio	Mean	15th perc	Median	85th perc
RB1	164	124	158	206
LC1	170	117	150	220
RB2	217	137	198	300
LC2	207	124	172	288
CONTRIB	100	100	100	100
UFR	182	175	182	189

6.3.4 Replacement Rate

The replacement rates are presented separately for a male and a female member due to the application of projected lifetables, considering two discount rates (i_1=1% and i_2=3%).

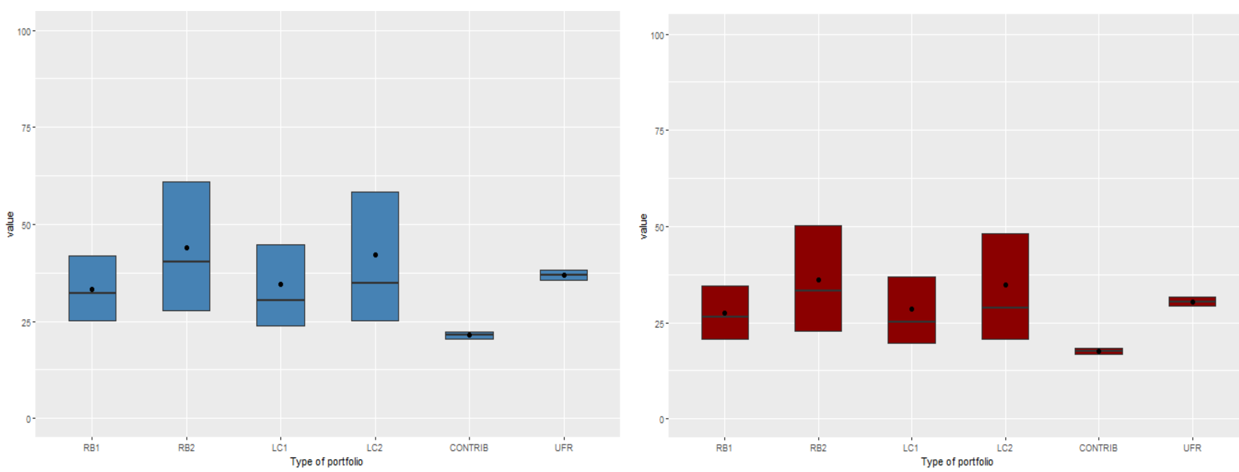
We obtain the replacement rate dividing the value of the annuities payments by the average final wage, according to (6.1).

For i_1=1%, a significant dispersion in the results can be observed. In the favourable scenario, the retirement rate can go up to near 61% and 50% for the RB2 portfolio, while in the unfavourable scenario it only achieves approximately 28% and 23%, for male and female, respectively. In the expected scenario (given by the median), only RB2 achieves a replacement rate higher than the one obtained considering an average rate of UFR per year (3,75%). Only in the favourable scenario all four portfolios achieve an average rate higher than UFR.

Table XXII - Replacement rate considering i_1=1% (percentage of final wage)

	Male				Female			
Portfolio	**Mean**	**15th perc**	**Median**	**85th perc**	**Mean**	**15th perc**	**Median**	**85th perc**
RB1	33%	25%	32%	42%	27%	21%	26%	34%
LC1	34%	24%	30%	45%	28%	20%	25%	37%
RB2	44%	28%	40%	61%	36%	23%	33%	50%
LC2	42%	25%	35%	58%	35%	21%	29%	48%
CONTRIB	21%	20%	21%	22%	18%	17%	18%	18%
UFR	37%	35%	37%	38%	30%	29%	30%	32%

Figure 10 - Replacement rate distribution considering discount rate i_1=1% (males in blue and females in red)



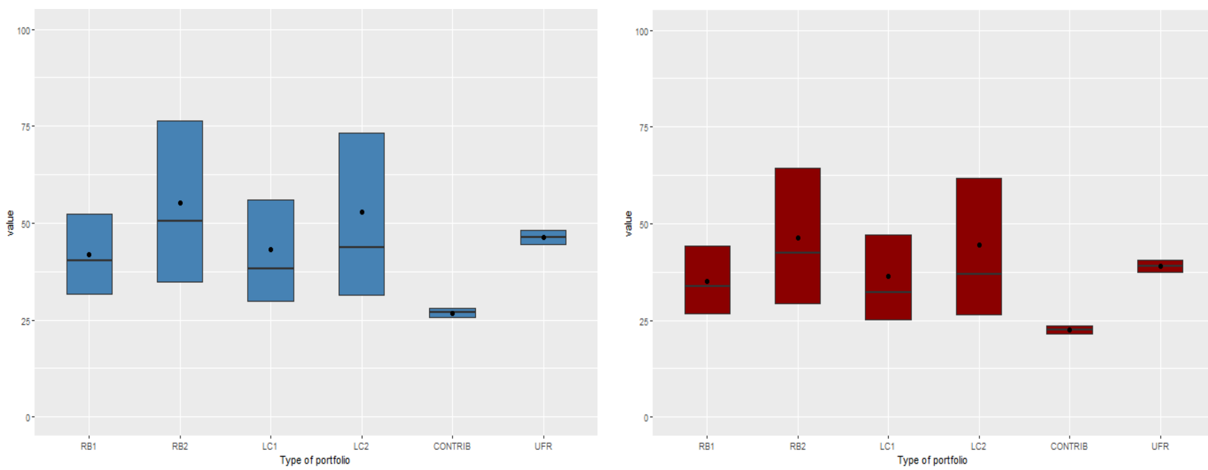
In the case of i_2=3%, the RB2 portfolio obtains the highest replacement rate, varying between 35% and 76%, for the unfavourable and favourable scenarios, respectively, considering the male mortality table. All portfolios have more disperse replacement rates than the ones aaobtained with i_1=1%, Similar to the results with i_1=1%, in the expected scenario only RB2 achieve a replacement rate higher than the one obtained by a portfolio with average return of UFR per year (3,75%).

As expected, for both discount rates (i_1 and i_2), the portfolios with higher equity exposure allow to achieve higher replacement rates but also present more dispersion, i.e., more uncertainty.

Table XXIII - Replacement rate considering i_2=3% (percentage of final wage)

	Males				Females			
Portfolio	**Mean**	**15th perc**	**Median**	**85th perc**	**Mean**	**15th perc**	**Median**	**85th perc**
RB1	42%	32%	40%	52%	35%	27%	34%	44%
LC1	43%	30%	38%	56%	36%	25%	32%	47%
RB2	55%	35%	50%	76%	46%	29%	43%	64%
LC2	53%	31%	44%	73%	44%	27%	37%	62%
CONTRIB	27%	26%	27%	28%	23%	22%	23%	24%
UFR	46%	44%	46%	48%	39%	37%	39%	40%

Figure 11 - Replacement rate distribution considering i_2=3% (males in blue and females in red)



The replacement rates calculated based on inflation-adjusted values are similar to the ones obtained without considering inflation, as expected. This is because the inflation affects equally both the numerator and denominator of the replacement rate, i.e., the annuity value and the last wage, respectively.

6.3.5 Risk and Performance Analysis

- **Risk analysis**

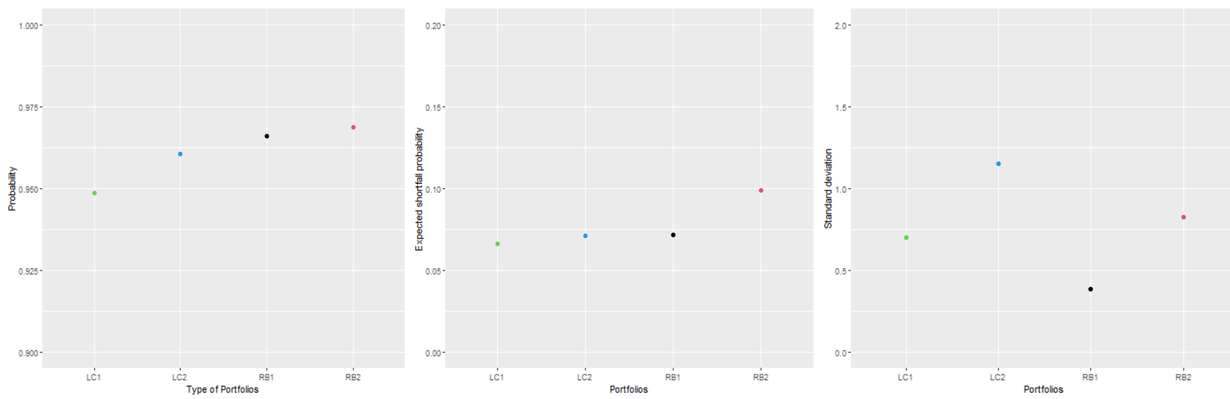
The risk analysis starts by assessing the probability of the portfolios achieving a minimum return greater than the sum of all contribution made to the pension scheme. Secondly, it analyzes what is the size of the expected potential loss if the lump sum does not achieve at least the sum of all contributions, given by the expected shortfall. Finally, the standard deviation over the total contributions is computed and analyzed.

All four portfolios chosen have a probability greater than 90% of recouping the contributions made to the pension scheme. The rebalanced portfolios obtain better results than the lifecycle portfolios, but the difference is not significant (Figure 12, left).

The expected shortfall is greater for the RB2 portfolio, which is the one with highest equity exposure, while the lower expected shortfall is given by the lifecycle portfolio with the lowest equity exposure (Figure 12, center). The difference on the expected shortfall between rebalanced portfolios is greater than the observed for lifecycle portfolios. The expected shortfall depends on the equity exposure but also on the type of portfolio.

Regarding the standard deviation, computed over the total contributions, the highest value is obtained for LC2. When comparing portfolios with similar equity exposure, we observe that the lifecycle portfolios have higher standard deviation than the rebalanced portfolios. That can be seen by comparing RB2 with LC2, and RB1 with LC1. If we assume that the standard deviation is a proxy of the risk of a portfolio, we can conclude that the rebalanced portfolios have less risk than the lifecycle portfolios, for the same equity exposure (Figure 12, right).

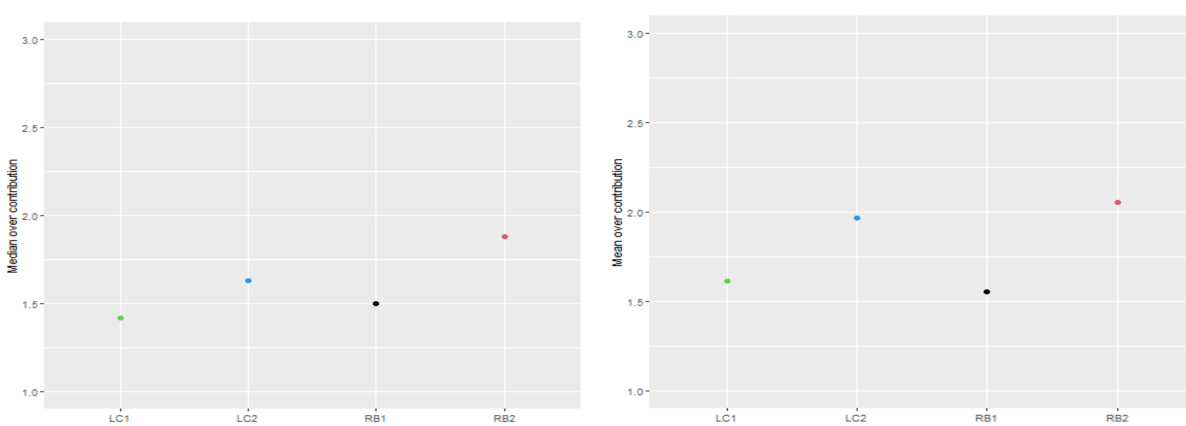
Figure 12 - Probability of lump sum greater than total contribution (left), expected shortfall (center) and standard deviation over contributions (right)



- **Performance analysis**

As expected, the mean is more sensitive to extreme values than the median, see Figure 13. However, in terms of ranking of the portfolios, the mean and the median produce similar results, i.e., higher equity exposure produces higher values for all four portfolios.

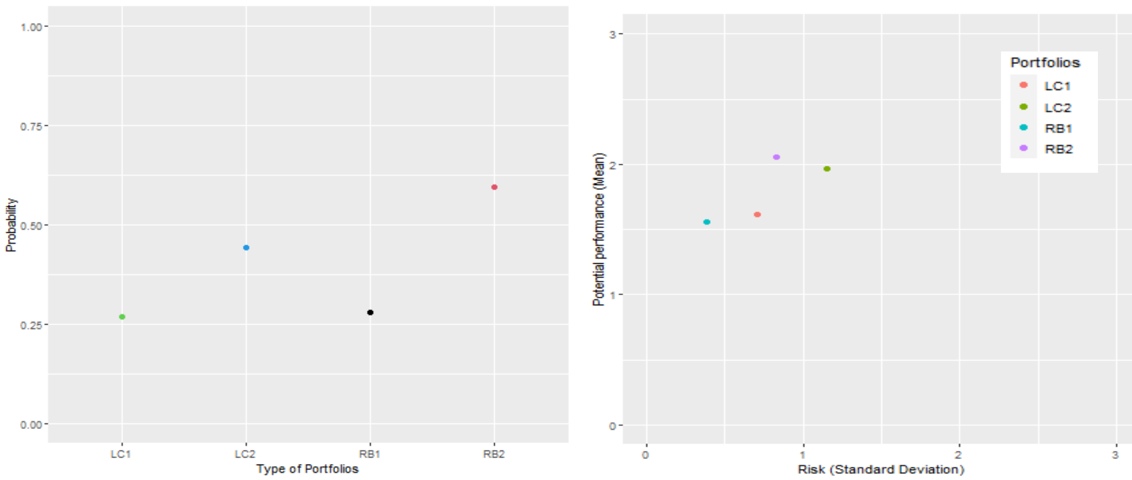
Figure 13 – Median (left) and Mean (right) over the total contributions



To measure the performance of an investment strategy, we can compare its lump sum distribution with a reference rate, in this case, with UFR. As can be seen in Figure 14 (left), the probability of the lump sum being greater than a portfolio with average return of UFR per year increases as equity exposure increases. Portfolios with very low equity exposure have a very low probability of reaching an average return of 3.75%. The portfolio with higher probability is RB2 with 58% of reaching an average return of 3.75%, while RB1 and LC1 portfolios have a probability near 25%.

When computing jointly the mean and the standard deviation it is obvious the relationship between risk and potential performance (Figure 14, right), with higher return (measured by mean) related to higher risk (measured by the standard deviation).

Figure 14 - Probability of lump sum greater than UFR (left) and joint risk-performance analysis (means vs standard deviation)



7 Conclusions

The main objective of this work was to develop a model for making pension projections for DC type of schemes. The results from the case study provide an illustration of the type of analysis that can be conducted with the calculation tool that was built and shows the impact that different inputs and scenarios have on the results.

Two key points should be highlighted, especially considering the long-term nature of the projections. First, the estimation process of the stochastic models is vital and challenging, requiring a view of the future economic development and expert judgement to determine the accuracy of the scenarios. It should be reviewed as economic and financial environment changes. Second, the choice of mortality tables applied can lead to significant differences in annuity values and using projected lifetables allows to better take into account the expected future mortality improvements.

The current low interest rate environment and market volatility are reflected in the results of the practical case. To achieve medium to high level of replacement rates, the equity exposure needs to be significant, which increases the dispersion and therefore the uncertainty around the final outputs.

Some improvements could be introduced in the model to enable a more realistic simulation. In particular, given that in this work the calculation of the lump sum only considers the investment return from government bonds and equity index, adding investment return from corporate bonds through a credit risk model would result in a more complete representation of the real investment allocation of pension funds' assets. On the other hand, regarding projected lifetables, further studies could include possible adaptations to the model for instance, to consider that past observed mortality improvements will eventually slowdown in the long-term, e.g., by making assumptions on long-term improvement rate and stipulating how the results from the model will converge to that long-term assumption.

It is important to mention the impact of Covid-19 pandemic. Covid-19 caused excess mortality both directly and indirectly by increasing deaths from other diseases. Further studies should be carried to assess the impact on mortality and longevity assumptions.

In general, this work allowed to better understand how to make pension projections and the challenges that such projections create in terms of calculations and presentation of the results.

Throughout the present internship at ASF, it become clear the importance of studying both ESG and projected lifetables, given their relevance for pension projections and retirement income analysis.

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Appendix A. Projected Lifetables

Table XXIV - Poisson Lee-Carter parameters estimation for male (left) and female (right)

x	α_x	β_x	x	α_x	β_x	t	k_t
0	-4.853	0.031	45	-6.271	0.007	1970	59.625
1	-7.179	0.034	46	-6.177	0.008	1971	59.529
2	-7.638	0.028	47	-6.099	0.007	1972	49.399
3	-7.877	0.025	48	-6.007	0.008	1973	52.769
4	-8.100	0.023	49	-5.967	0.007	1974	49.434
5	-8.203	0.021	50	-5.865	0.008	1975	47.691
6	-8.242	0.019	51	-5.797	0.007	1976	46.868
7	-8.353	0.017	52	-5.714	0.008	1977	39.751
8	-8.430	0.016	53	-5.647	0.008	1978	38.914
9	-8.498	0.018	54	-5.598	0.008	1979	33.823
10	-8.469	0.014	55	-5.511	0.008	1980	32.126
11	-8.447	0.016	56	-5.431	0.008	1981	30.271
12	-8.497	0.014	57	-5.368	0.008	1982	24.775
13	-8.433	0.015	58	-5.257	0.009	1983	25.738
14	-8.220	0.013	59	-5.191	0.009	1984	22.990
15	-8.096	0.011	60	-5.081	0.009	1985	20.942
16	-8.003	0.011	61	-5.006	0.009	1986	18.215
17	-8.008	0.013	62	-4.919	0.010	1987	14.468
18	-7.914	0.011	63	-4.818	0.010	1988	13.856
19	-7.827	0.010	64	-4.712	0.010	1989	8.995
20	-7.779	0.008	65	-4.616	0.010	1990	13.400
21	-7.762	0.011	66	-4.514	0.010	1991	12.177
22	-7.790	0.010	67	-4.404	0.010	1992	5.611
23	-7.743	0.010	68	-4.282	0.011	1993	7.743
24	-7.662	0.010	69	-4.168	0.011	1994	-0.842
25	-7.690	0.010	70	-4.033	0.011	1995	0.494
26	-7.649	0.012	71	-3.923	0.011	1996	-0.052
27	-7.564	0.010	72	-3.794	0.011	1997	-3.396
28	-7.575	0.011	73	-3.668	0.011	1998	-5.200
29	-7.490	0.010	74	-3.517	0.011	1999	-5.862
30	-7.428	0.010	75	-3.395	0.011	2000	-10.536
31	-7.342	0.010	76	-3.262	0.011	2001	-13.960
32	-7.322	0.010	77	-3.135	0.011	2002	-15.739
33	-7.202	0.010	78	-2.994	0.011	2003	-15.246
34	-7.155	0.009	79	-2.860	0.010	2004	-25.739
35	-7.050	0.009	80	-2.777	0.009	2005	-22.933
36	-7.008	0.009	81	-2.662	0.008	2006	-33.678
37	-6.947	0.009	82	-2.525	0.008	2007	-33.557
38	-6.832	0.009	83	-2.405	0.008	2008	-36.119
39	-6.739	0.009	84	-2.284	0.008	2009	-38.230
40	-6.651	0.008	85	-2.162	0.007	2010	-41.229
41	-6.601	0.008	86	-2.039	0.007	2011	-46.882
42	-6.492	0.007	87	-1.931	0.006	2012	-45.324
43	-6.449	0.008	88	-1.821	0.006	2013	-50.166
44	-6.363	0.007	89	-1.709	0.005	2014	-55.404
45	-6.271	0.007	90	-1.610	0.005	2015	-53.987
						2016	-55.648
						2017	-59.884
						2018	-59.993

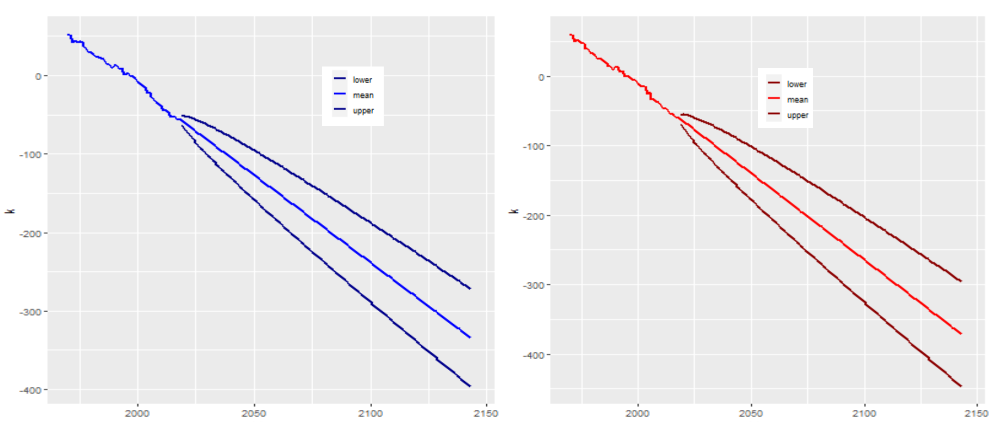
x	α_x	β_x	x	α_x	β_x	t	k_t
0	-4.667	-4.667	45	0.007	-5.468	1970	51.571
1	-7.012	-7.012	46	0.006	-5.399	1971	52.080
2	-7.352	-7.352	47	0.006	-5.310	1972	42.311
3	-7.631	-7.631	48	0.006	-5.215	1973	44.646
4	-7.759	-7.759	49	0.006	-5.140	1974	42.477
5	-7.887	-7.887	50	0.006	-5.059	1975	43.544
6	-8.078	-8.078	51	0.006	-4.990	1976	42.080
7	-7.988	-7.988	52	0.006	-4.909	1977	37.041
8	-8.117	-8.117	53	0.006	-4.816	1978	34.071
9	-8.108	-8.108	54	0.006	-4.757	1979	29.645
10	-8.076	-8.076	55	0.006	-4.673	1980	29.581
11	-8.038	-8.038	56	0.007	-4.606	1981	27.385
12	-8.039	-8.039	57	0.007	-4.513	1982	23.119
13	-7.921	-7.921	58	0.007	-4.441	1983	22.928
14	-7.740	-7.740	59	0.007	-4.361	1984	21.948
15	-7.448	-7.448	60	0.008	-4.270	1985	20.640
16	-7.126	-7.126	61	0.007	-4.192	1986	16.659
17	-6.932	-6.932	62	0.008	-4.107	1987	13.720
18	-6.706	-6.706	63	0.008	-4.019	1988	14.021
19	-6.624	-6.624	64	0.008	-3.947	1989	9.740
20	-6.586	-6.586	65	0.009	-3.844	1990	12.344
21	-6.563	-6.563	66	0.009	-3.770	1991	12.835
22	-6.566	-6.566	67	0.009	-3.670	1992	8.000
23	-6.565	-6.565	68	0.009	-3.570	1993	9.609
24	-6.558	-6.558	69	0.009	-3.480	1994	1.306
25	-6.527	-6.527	70	0.009	-3.373	1995	2.560
26	-6.510	-6.510	71	0.009	-3.287	1996	3.844
27	-6.486	-6.486	72	0.010	-3.180	1997	-0.821
28	-6.456	-6.456	73	0.010	-3.075	1998	-1.764
29	-6.442	-6.442	74	0.009	-2.970	1999	-3.259
30	-6.387	-6.387	75	0.009	-2.864	2000	-7.808
31	-6.367	-6.367	76	0.010	-2.758	2001	-11.115
32	-6.335	-6.335	77	0.009	-2.651	2002	-12.742
33	-6.282	-6.282	78	0.009	-2.540	2003	-14.987
34	-6.220	-6.220	79	0.009	-2.433	2004	-22.199
35	-6.173	-6.173	80	0.008	-2.376	2005	-21.201
36	-6.129	-6.129	81	0.007	-2.276	2006	-27.512
37	-6.072	-6.072	82	0.007	-2.171	2007	-31.037
38	-6.010	-6.010	83	0.007	-2.062	2008	-33.681
39	-5.936	-5.936	84	0.007	-1.964	2009	-36.899
40	-5.858	-5.858	85	0.006	-1.870	2010	-39.023
41	-5.787	-5.787	86	0.005	-1.763	2011	-44.821
42	-5.698	-5.698	87	0.005	-1.668	2012	-44.182
43	-5.619	-5.619	88	0.005	-1.574	2013	-47.265
44	-5.549	-5.549	89	0.004	-1.477	2014	-52.303
45	-5.468	-5.468	90	0.004	-1.401	2015	-52.711
						2016	-52.140
						2017	-56.766
						2018	-55.469

Table XXV - k_t parameters projection values

FEMALE					
t*	k_t*	t*	k_t*	t*	k_t*
2019	-62.485	2061	-167.150	2103	-271.816
2020	-64.977	2062	-169.642	2104	-274.308
2021	-67.469	2063	-172.134	2105	-276.800
2022	-69.961	2064	-174.626	2106	-279.292
2023	-72.453	2065	-177.118	2107	-281.784
2024	-74.945	2066	-179.610	2108	-284.276
2025	-77.437	2067	-182.103	2109	-286.768
2026	-79.929	2068	-184.595	2110	-289.260
2027	-82.421	2069	-187.087	2111	-291.752
2028	-84.913	2070	-189.579	2112	-294.244
2029	-87.405	2071	-192.071	2113	-296.736
2030	-89.897	2072	-194.563	2114	-299.228
2031	-92.389	2073	-197.055	2115	-301.720
2032	-94.881	2074	-199.547	2116	-304.212
2033	-97.373	2075	-202.039	2117	-306.704
2034	-99.865	2076	-204.531	2118	-309.196
2035	-102.357	2077	-207.023	2119	-311.688
2036	-104.849	2078	-209.515	2120	-314.180
2037	-107.341	2079	-212.007	2121	-316.673
2038	-109.833	2080	-214.499	2122	-319.165
2039	-112.325	2081	-216.991	2123	-321.657
2040	-114.818	2082	-219.483	2124	-324.149
2041	-117.310	2083	-221.975	2125	-326.641
2042	-119.802	2084	-224.467	2126	-329.133
2043	-122.294	2085	-226.959	2127	-331.625
2044	-124.786	2086	-229.451	2128	-334.117
2045	-127.278	2087	-231.943	2129	-336.609
2046	-129.770	2088	-234.435	2130	-339.101
2047	-132.262	2089	-236.927	2131	-341.593
2048	-134.754	2090	-239.419	2132	-344.085
2049	-137.246	2091	-241.911	2133	-346.577
2050	-139.738	2092	-244.403	2134	-349.069
2051	-142.230	2093	-246.895	2135	-351.561
2052	-144.722	2094	-249.388	2136	-354.053
2053	-147.214	2095	-251.880	2137	-356.545
2054	-149.706	2096	-254.372	2138	-359.037
2055	-152.198	2097	-256.864	2139	-361.529
2056	-154.690	2098	-259.356	2140	-364.021
2057	-157.182	2099	-261.848	2141	-366.513
2058	-159.674	2100	-264.340	2142	-369.005
2059	-162.166	2101	-266.832	2143	-371.497
2060	-164.658	2102	-269.324		

MALE					
t*	k_t*	t*	k_t*	t*	k_t*
2019	-57.699	2061	-151.360	2103	-245.020
2020	-59.929	2062	-153.590	2104	-247.250
2021	-62.159	2063	-155.820	2105	-249.480
2022	-64.389	2064	-158.050	2106	-251.710
2023	-66.619	2065	-160.280	2107	-253.940
2024	-68.849	2066	-162.510	2108	-256.170
2025	-71.079	2067	-164.740	2109	-258.400
2026	-73.309	2068	-166.970	2110	-260.630
2027	-75.539	2069	-169.200	2111	-262.860
2028	-77.769	2070	-171.430	2112	-265.090
2029	-79.999	2071	-173.660	2113	-267.320
2030	-82.229	2072	-175.890	2114	-269.550
2031	-84.459	2073	-178.120	2115	-271.780
2032	-86.689	2074	-180.350	2116	-274.011
2033	-88.919	2075	-182.580	2117	-276.241
2034	-91.149	2076	-184.810	2118	-278.471
2035	-93.380	2077	-187.040	2119	-280.701
2036	-95.610	2078	-189.270	2120	-282.931
2037	-97.840	2079	-191.500	2121	-285.161
2038	-100.070	2080	-193.730	2122	-287.391
2039	-102.300	2081	-195.960	2123	-289.621
2040	-104.530	2082	-198.190	2124	-291.851
2041	-106.760	2083	-200.420	2125	-294.081
2042	-108.990	2084	-202.650	2126	-296.311
2043	-111.220	2085	-204.880	2127	-298.541
2044	-113.450	2086	-207.110	2128	-300.771
2045	-115.680	2087	-209.340	2129	-303.001
2046	-117.910	2088	-211.570	2130	-305.231
2047	-120.140	2089	-213.800	2131	-307.461
2048	-122.370	2090	-216.030	2132	-309.691
2049	-124.600	2091	-218.260	2133	-311.921
2050	-126.830	2092	-220.490	2134	-314.151
2051	-129.060	2093	-222.720	2135	-316.381
2052	-131.290	2094	-224.950	2136	-318.611
2053	-133.520	2095	-227.180	2137	-320.841
2054	-135.750	2096	-229.410	2138	-323.071
2055	-137.980	2097	-231.640	2139	-325.301
2056	-140.210	2098	-233.870	2140	-327.531
2057	-142.440	2099	-236.100	2141	-329.761
2058	-144.670	2100	-238.330	2142	-331.991
2059	-146.900	2101	-240.560	2143	-334.221
2060	-149.130	2102	-242.790		

Figure 15 - k_t parameter projection confidence interval for male (left) and female (right)



Appendix B. Case Study

Figure 16 - Annual bond return (left) and annual equity return (right) simulation results

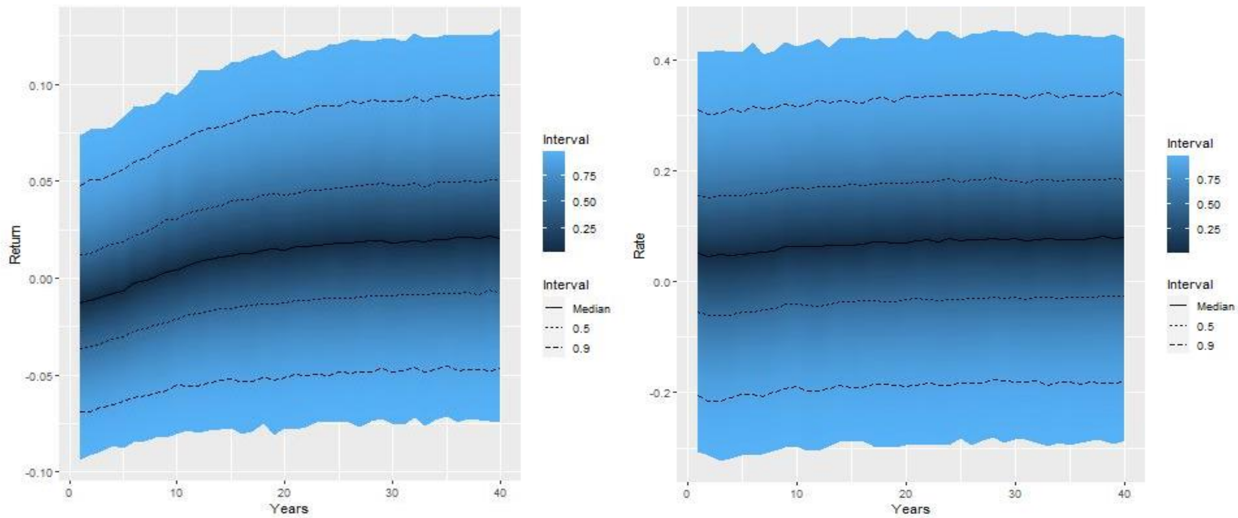


Figure 17 - Monthly wage (left) and annual inflation rate (right) simulation results

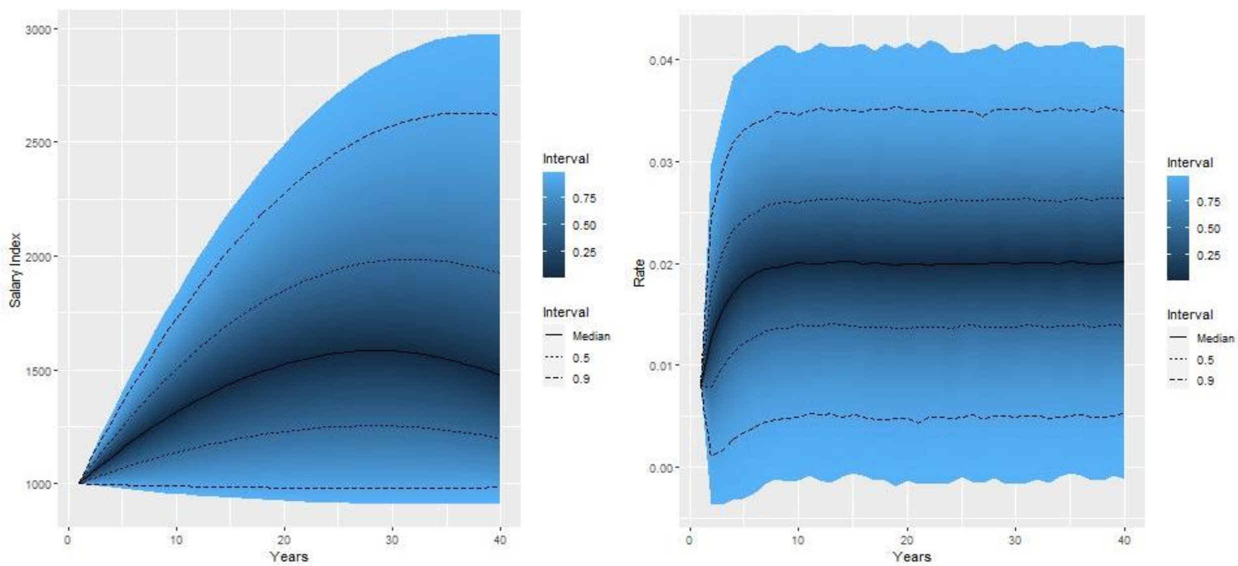


Figure 18 - Equity weight (%) result for all four portfolios

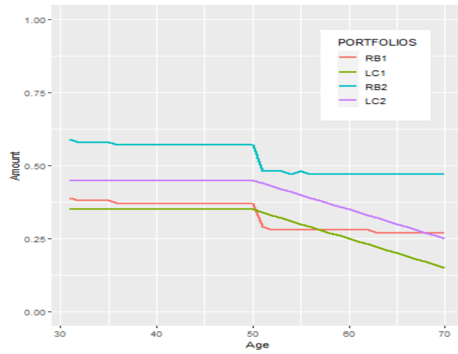


Figure 19 – Unfavourable (left), expected (center) and favourable scenario results

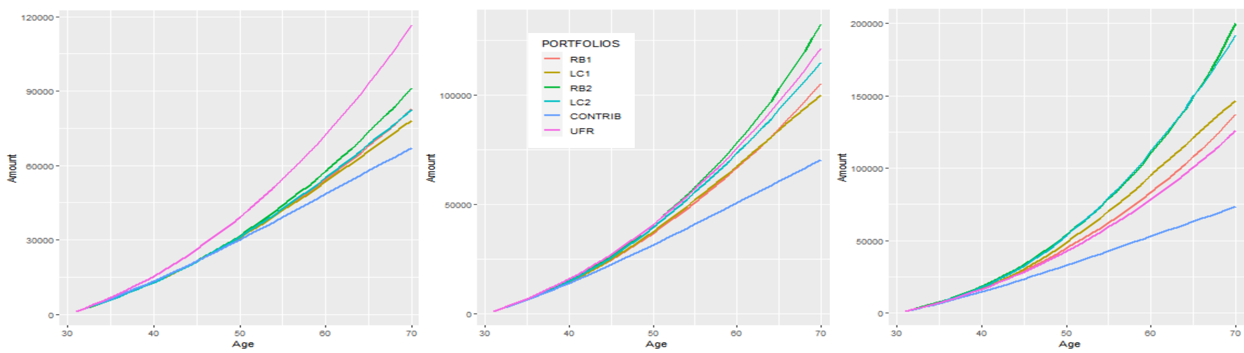


Figure 20 - Rebalanced portfolios result – RB1 (left) and RB2 (right)

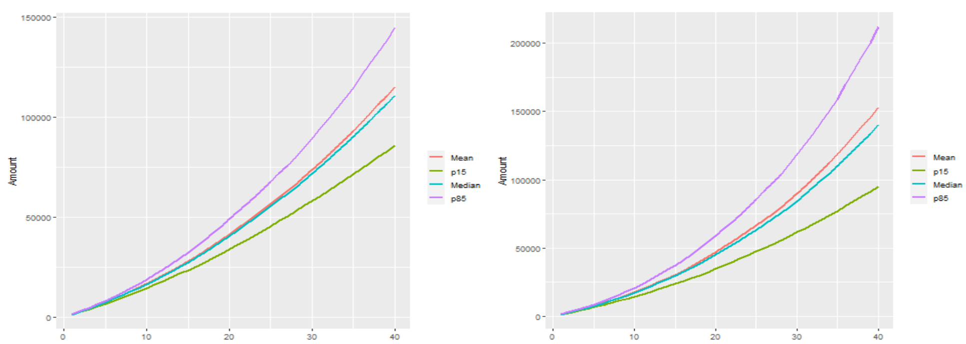


Figure 21 - Lifecycle portfolios results – LC1 (left) and LC2 (right)

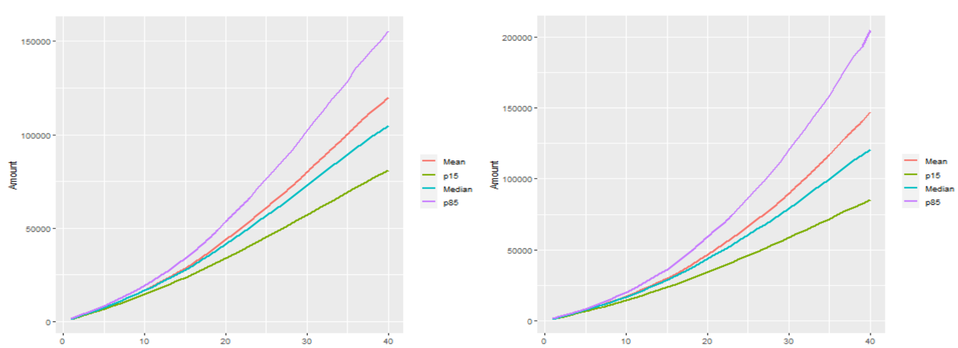


Table XXVI - Portfolios accumulated value along the years

	Rebalanced Portfolio 1 (RB1)					Rebalanced Portfolio 2 (RB2)					Lifecycle 1 (LC1)					Lifecycle 2 (LC2)				
y	w[10]	Mean	p15	Median	p85	w	Mean	p15	Median	p85	w	Mean	p15	Median	p85	w	Mean	p15	Median	p85
1	38	1208	1165	1206	1252	58	1214	1152	1211	1278	35	1207	1167	1205	1248	45	1210	1161	1208	1261
2	37	2462	2331	2455	2596	57	2488	2299	2474	2682	35	2459	2334	2452	2587	45	2472	2319	2461	2628
3	37	3767	3516	3754	4024	57	3828	3466	3803	4198	35	3763	3517	3749	4013	45	3793	3498	3771	4096
4	37	5129	4732	5106	5533	57	5241	4670	5200	5826	35	5125	4734	5096	5525	45	5181	4706	5141	5664
5	37	6550	5981	6514	7127	57	6732	5903	6666	7558	35	6549	5981	6501	7125	45	6640	5946	6574	7332
6	36	8040	7269	7991	8809	56	8312	7195	8223	9430	35	8047	7262	7977	8824	45	8183	7229	8089	9127
7	36	9608	8638	9540	10593	56	9991	8561	9860	11439	35	9627	8630	9525	10637	45	9820	8603	9672	11062
8	36	11237	10006	11134	12504	56	11754	9947	11555	13616	35	11279	9986	11124	12602	45	11539	9978	11330	13158
9	36	12941	11454	12807	14482	56	13614	11380	13340	15903	35	13014	11435	12795	14627	45	13354	11425	13056	15346
10	36	14759	12931	14603	16618	56	15629	12868	15298	18391	35	14880	12903	14613	16862	45	15320	12887	14933	17757
11	36	16635	14499	16440	18801	56	17722	14491	17323	20980	35	16814	14467	16455	19165	45	17366	14483	16898	20294
12	36	18604	16100	18344	21204	56	19942	16127	19418	23882	35	18859	16061	18393	21732	45	19541	16088	18907	23088
13	36	20666	17752	20387	23634	56	22290	17802	21700	26887	35	21021	17709	20469	24383	45	21851	17739	21083	26030
14	36	22816	19489	22504	26169	56	24767	19602	24071	29994	35	23297	19444	22624	27154	45	24297	19501	23392	29112
15	36	25076	21295	24748	28943	56	27390	21471	26583	33477	35	25707	21235	24926	30241	45	26899	21340	25829	32570
16	36	27445	23170	26973	31857	56	30189	23411	29197	37051	35	28271	23101	27242	33444	45	29690	23213	28265	36150
17	36	29918	25099	29415	34877	56	33133	25423	31960	41007	35	30977	25017	29758	36960	45	32646	25191	30982	40280
18	36	32438	27007	31869	37873	56	36144	27460	34821	44878	35	33749	26953	32322	40478	45	35681	27150	33709	44117
19	36	35083	29050	34397	41273	56	39348	29506	37813	49371	35	36709	28981	34979	44488	45	38943	29184	36589	48695
20	36	37853	31078	37025	44760	56	42755	31676	40918	54138	35	39870	31030	37703	48844	45	42452	31320	39574	53746
21	28	40621	33277	39706	48150	48	46231	33991	44108	58859	34	42959	33133	40501	52904	44	45948	33506	42585	58548
22	28	43370	35434	42390	51512	47	49663	36420	47466	63073	33	46040	35208	43265	56736	43	49438	35613	45713	63295
23	27	46216	37633	45125	55106	47	53271	38793	50771	68054	32	49242	37387	46085	61090	42	53103	37797	48837	68350
24	27	49153	39935	47965	58599	46	57076	41276	54271	73154	31	52585	39637	49032	65398	41	56984	40154	52074	73649
25	27	52215	42175	50900	62511	46	61078	43830	58039	78833	30	56048	41904	52076	70269	40	61040	42561	55592	79510
26	27	55330	44634	53899	66130	46	65174	46646	61784	84128	29	59526	44242	55078	74630	39	65147	45124	59049	84964
27	27	58595	46974	57034	70486	46	69556	49409	65739	90416	28	63226	46480	58227	79684	38	69585	47540	62469	91333
28	27	61850	49308	60283	74618	46	73929	51751	69803	96594	27	66840	48680	61410	84610	37	73956	49820	66060	97922
29	27	65231	51941	63454	79024	46	78576	54739	73907	102772	26	70597	51190	64462	89630	36	78581	52362	69792	104391
30	27	68768	54381	66774	83539	46	83526	57688	78242	110549	25	74504	53597	67642	95448	35	83472	55030	73490	112099
31	27	72390	57168	70231	88107	46	88654	60790	82667	117413	24	78466	56086	70863	100434	34	88511	57784	77160	118898
32	26	76072	59903	73718	92696	46	93855	64116	87432	124621	23	82304	58632	74210	105486	33	93444	60567	81197	125832
33	26	79850	62425	77205	97803	46	99330	66928	92390	132989	22	86247	60892	77469	110733	32	98639	62979	85436	133484
34	26	83721	64984	80988	102970	46	104935	70101	97264	140715	21	90111	63292	80662	116048	31	103804	65720	89059	141183
35	26	87747	67962	84807	108302	46	110926	73710	102747	149509	20	94137	65809	83872	121484	30	109344	68693	93536	149723
36	26	91872	70841	88787	113276	46	117091	77192	108372	158176	19	98066	68375	87360	126495	29	114867	71459	98000	156952
37	26	96183	73740	92800	119257	46	123695	80544	114092	167627	18	102154	70788	90727	131891	28	120809	73988	102213	165750
38	26	100471	76613	96953	125085	46	130265	83881	120032	177774	17	105952	73264	93982	137378	27	126494	76950	106497	174836
39	26	104912	79598	100914	131105	45	137203	87560	125852	188601	16	109722	75815	96944	141706	26	132373	79678	110412	182580
40	26	109493	82972	105496	137425	45	144472	91264	132326	200012	15	113341	78173	99999	146793	25	138271	82562	114863	192147

[10] Equity weight of the portfolio