Characterizations and representations of left and right hybrid (b,c)-inverses in rings

Huihui Zhu[a], Liyun Wu[a], Fei Peng[a], Pedro Patrício[b]

[a] *School of Mathematics, Hefei University of Technology, Hefei 230009, China.*
[b] *CMAT-Centro de Matemática and Departamento de Matemática, Universidade do Minho, Braga 4710-057, Portugal.*

Abstract

Let R be an associative ring with unity 1 and let $a,b,c \in R$. In this paper, several characterizations for left and right hybrid (b,c)-inverses of a are derived. Moreover, their formulae are given by regularities of certain element. Then, we give characterizations of right (b,c)-inverses and right annihilator (b,c)-inverses of the product of three elements. Finally, relations among the right hybrid (b,c)-inverses of paq, right (qb,c)-inverses of aq and right annihilator (b,cp)-inverses of aq are given.

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1. Introduction

In 2012, Drazin [2] introduced a type of outer generalized inverses, called "hybrid (b,c)-inverses" in rings, this can be seen as an extension of outer generalized inverses for n by n complex matrices, given by Ben-Israel and Greville ([1]). Recently, Drazin [4] renamed the original notion "hybrid (b,c)-inverses" to "right hybrid (b,c)-inverses". Also, he introduces the notion of left hybrid (b,c)-inverses. Moreover, an existence criterion of right hybrid (b,c)-inverses, stated without proof in [2, page 1922], now is explicitly given. Since the hybrid (b,c)-inverse was introduced, few papers focus on this kind of generalized inverses.

Email addresses: `hhzhu@hfut.edu.cn` (Huihui Zhu), `wlymath@163.com` (Liyun Wu), `pfmath@163.com` (Fei Peng), `pedro@math.uminho.pt` (Pedro Patrício)

In this paper, we mainly investigate characterizations and representations of right hybrid (b, c)-inverses in rings. The paper is organized as follows. In Section 2, several existence criteria and representations for right (left) hybrid (b, c)-inverses are given. As a special case, the existence criteria for the inverse along an element [5] is obtained. In section 3, we present existence criteria of right (b, c)-inverse and right annihilator (b, c)-inverse of the product of three elements. Furthermore, we derive relations among right hybrid (b, c)-inverse of paq, right (qb, c)-inverse of pa and right annihilator (b, cp)-inverse of aq in rings.

Let R be an associate ring with unity 1. Let us now recall several definitions of generalized inverses. An element $a \in R$ is called (von Neumann) regular if there exists x in R such that $a = axa$. Such an x is called an inner inverse or a 1-inverse of a, and is denoted by a^-. We use the symbol $a\{1\}$ to denote the set of all inner inverses of a. For any $a \in R$, the right annihilator and left annihilator of a are defined by $a^0 = \{s : s \in R \text{ and } as = 0\}$ and ${}^0a = \{t : t \in R \text{ and } ta = 0\}$, respectively.

Let $a, b, c \in R$. An element a is hybrid (b, c)-invertible [2] if there exists some $y \in R$ such that $yay = y$, $yR = bR$ and $y^0 = c^0$. Such an y is called the hybrid (b, c)-inverse of a. It is unique if it exists. The hybrid (b, c)-inverse now is called the right hybrid (b, c)-inverse [4] by Drazin. The present definition is more suitable since it is just defined by right ideals and right annihilators. Dually, Drazin defined the left hybrid (b, c)-inverse of a by the existence of $z \in R$ such that $zaz = z$, $Rz = Rc$ and ${}^0z = {}^0b$. The left hybrid (b, c)-inverse of a is unique if it exists. By symbols ${}_ha^{(b,c)}$ and $a_h^{(b,c)}$ we denote the left hybrid (b, c)-inverse and the right hybrid (b, c)-inverse of a, respectively.

2. Left and right hybrid (b, c)-inverses

In this section, we derive existence criteria of left (right) hybrid (b, c)-inverse in a ring R. Herein, we first give several lemmas, which play an important role in the sequel.

Lemma 2.1. [4, Theorem 2.2] *Let $a, b, c \in R$. Then a is right hybrid (b, c)-invertible if and only if $c \in cabR$ and $(cab)^0 \subseteq b^0$.*

Lemma 2.2. [6, Proposition 2.1] *Let $a, b, c, y \in R$. Then the following conditions are equivalent:*

(i) *y is the right hybrid (b, c)-inverse of a.*

(ii) *$yab = b$, $cay = c$, $yR \subseteq bR$ and $c^0 \subseteq y^0$.*

Lemma 2.3. *Let $a, b, c \in R$. Then the following conditions are equivalent*:

(i) $(cab)^0 \subseteq b^0$.

(ii) $(ca)^0 \cap bR = \{0\}$.

(iii) $(cab)^0 \subseteq (b - cab)^0$.

Proof. (i) $\Rightarrow$ (ii) Suppose $(cab)^0 \subseteq b^0$. For any $x \in (ca)^0 \cap bR$, we have $cax = 0$ and $x = by$ for some $y \in R$. Consequently, $caby = 0$, i.e. $y \in (cab)^0 \subseteq b^0$ and $x = by = 0$.

(ii) $\Rightarrow$ (iii) For any $s \in (cab)^0$, i.e. $cabs = 0$, we have $bs \in (ca)^0 \cap bR = \{0\}$, and hence $(b - cab)s = 0$, as required.

(iii) $\Rightarrow$ (i) As $(cab)^0 \subseteq (b - cab)^0$, then for any $t \in (cab)^0$, $(b - cab)t = 0$ implies $bt = 0$ since $cabt = 0$. Thus, $t \in b^0$ and $(cab)^0 \subseteq b^0$. $\square$

Dually, we have the following result ${}^0(cab) \subseteq {}^0c \Leftrightarrow {}^0(ab) \cap Rc = \{0\} \Leftrightarrow {}^0(cab) \subseteq {}^0(c - cab)$ for any $a, b, c \in R$.

Lemma 2.4. [4, Lemma 5.3] *Let $c, h \in R$. Then*

(i) $c \in chR$ *if and only if* $R = hR + c^0$.

(ii) $hR \cap c^0 = \{0\}$ *if and only if* $(ch)^0 \subseteq h^0$.

Lemma 2.5. *Let $a, b, c \in R$ and let a be right hybrid (b, c)-invertible. Then b and ab are both regular. In particular, if $c = cabw$ for some $w \in R$, then $(ab)^- a \in b\{1\}$ and $w \in ab\{1\}$.*

Proof. As a is right hybrid (b, c)-invertible, then, by Lemma 2.1, $c = cabw$ for some $w \in R$. We hence have $cab(1 - wab) = 0$. From $(cab)^0 \subseteq b^0$, it follows that $b = bwab$ and $ab = abwab$. So, $w \in ab\{1\}$ and $wa = (ab)^- a \in b\{1\}$. $\square$

It was proved in [6, Theorem 2.8] that a is right hybrid (b, c)-invertible if and only if ab is group invertible, under the hypothesis $a^0 = b^0 = c^0$. The following example 2.7 can illustrates this fact.

It follows from Lemma 2.5 that a is right hybrid (b, c)-invertible implies that ab is regular. Motivated by [6, Theorem 2.8], we next consider a further characterization of the existence of the right hybrid (b, c)-inverse by the regularity of ab, avoiding its group inverse without additional assumptions.

Theorem 2.6. *Let $a, b, c \in R$. Then the following conditions are equivalent*:

(i) *a is right hybrid (b, c)-invertible.*

(ii) $(ca)^0 \cap bR = \{0\}$ *and* $c \in cabR$.

(iii) $(cab)^0 \subseteq (b - cab)^0$ *and* $c \in cabR$.
(iv) $(ca)^0 \cap bR = \{0\}$ *and* $R = abR + c^0$.
(v) $(cab)^0 \subseteq (b - cab)^0$ *and* $R = abR + c^0$.
(vi) $(ca)^0 \cap bR = \{0\}$ *and* $R = abR \oplus c^0$.
(vii) $(cab)^0 \subseteq (b - cab)^0$ *and* $R = abR \oplus c^0$.
(viii) *ab is regular,* $c = cab(ab)^-$ *and* $(ca)^0 \cap bR = \{0\}$.
In this case, $a_h^{(b,c)} = b(ab)^-$, *where $(ab)^-$ is an inner inverse of ab.*

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii)$\Leftrightarrow$ (iv)$\Leftrightarrow$ (v) follow from Lemmas 2.1 and 2.3.

(i) $\Rightarrow$ (vi) From Lemma 2.3 and Lemma 2.4, it suffices to verify $abR \cap c^0 = \{0\}$. For any $s \in abR \cap c^0$, we have $cs = 0$ and $s = abt$ for some $t \in R$. Hence, $cabt = 0$ and $bt \in (ca)^0 \cap bR = \{0\}$. So, $s = 0$, as required.

(vi) $\Rightarrow$ (i) Clearly, (vi) implies (iv), and (vi) $\Rightarrow$ (iv) has been proved.

(i) $\Rightarrow$ (vii) is similar to (i) $\Rightarrow$ (vi).

(i) $\Rightarrow$ (viii) Suppose that a is hybrid (b, c)-invertible. Then, by Lemma 2.5, ab is regular, and $c = cabw$ for some $w \in R$, then $w \in ab\{1\}$. Therefore, $c = cab(ab)^-$.

The condition $(ca)^0 \cap bR = \{0\}$ follows from Lemma 2.1 and 2.3.

(viii) $\Rightarrow$ (i) As ab is regular, then $ca(b - b(ab)^-ab) = 0$ and $b - b(ab)^-ab \in (ca)^0 \cap bR = \{0\}$. Set $y = b(ab)^-$. Then $b = yab$, $c = cab(ab)^- = cay$ and $yR \subseteq bR$. Next, it is sufficient to show $c^0 \subseteq y^0$ by [6, Proposition 2.1]. Let $cv = 0$ for any $v \in R$. Then $0 = cayv = cab(ab)^-v$, whence $yv = b(ab)^-v \in (ca)^0 \cap bR = \{0\}$, that is, $v \in y^0$ and $c^0 \subseteq y^0$. Hence, is right hybrid (b, c)-invertible by Lemma 2.2. Moreover, $a_h^{(b,c)} = b(ab)^-$. □

From the proof of Theorem 2.6, one can see that if a is right hybrid (b, c)-invertible then b and ab are regular. However, c may not be regular in general. The following example illustrates this fact.

Example 2.7. Let $R = \mathbb{Z}$ be the ring of all integers. Take $a = 1$, $b = 1$, $c = 2 \in R$, then a is right hybrid $(1, 2)$-invertible, Indeed, $y = 1$ is the right hybrid $(1, 2)$-inverse of a as $yab = 1 \cdot 1 \cdot 1 = 1 = b$, $cay = 2 \cdot 1 \cdot 1 = 2 = c$, $yR = R = bR$ and $c^0 = y^0$. However, there exists no $x \in R$ such that $2 \cdot x \cdot 2 = 4x = 2$ holds. Hence, c is not regular.

Following [2], a is (b, c)-invertible if there exists some $y \in R$ such that $yab = b$, $cay = c$, $y \in bR \cap Rc$. Example 2.7 above also shows that the right hybrid (b, c)-invertible element may not be (b, c)-invertible in general as $Ry = \mathbb{Z} \nsubseteq 2\mathbb{Z} = Rc$.

Suppose a is right hybrid (b,c)-invertible. If c is regular, then ca and cab are both regular. Indeed, if c is regular, then a is (b,c)-invertible. The regularity of ca and cab follows immediately from the proof of [2, Theorem 2.2]. Conversely, the regularity of either ca or cab implies the regularity of c, under the assumption that a is right hybrid (b,c)-invertible. Indeed, if ca is regular, then there exists some $s \in R$ such that $ca = caxca$. Note that $c = cay$. Then post-multiplying $ca = caxca$ by y gives $c = caxc$. If cab is regular, then $cab = cabtcab$ for some $t \in R$. By Lemma 2.1, we derive that c is regular as $c \in cabR$. Hence, we can conclude that if one of the sets $\{c$, ca, $cab\}$ is regular, then they are all regular.

We next present several characterizations of left hybrid (b,c)-inverses of a in a ring whose proofs are left to the reader.

Theorem 2.8. *Let $a,b,c \in R$. Then the following conditions are equivalent*:
(i) *a is left hybrid (b,c)-invertible.*
(ii) *$b \in Rcab$ and ${}^0(cab) \subseteq {}^0c$.*
(iii) *$b \in Rcab$ and ${}^0(ab) \cap Rc = \{0\}$.*
(iv) *$R = Rca + {}^0b$ and ${}^0(ab) \cap Rc = \{0\}$.*
(v) *$R = Rca + {}^0b$ and ${}^0(cab) \subseteq {}^0(c - cab)$.*
(vi) *$R = Rca \oplus {}^0b$ and ${}^0(ab) \cap Rc = \{0\}$.*
(vii) *$R = Rca \oplus {}^0b$ and ${}^0(cab) \subseteq {}^0(c - cab)$.*
(viii) *ca is regular, $b = (ca)^-cab$ and ${}^0(ab) \cap Rc = \{0\}$.*
In this case, ${}_ha^{(b,c)} = (ca)^-c$, where $(ca)^-$ is an inner inverse of ca.

Recall that a is hybrid (b,c)-invertible [4] if it is both left and right hybrid (b,c)-invertible. Combing with Theorem 2.6 and 2.8, we obtain the following result.

Corollary 2.9. *Let $a,b,c \in R$. Then a is hybrid (b,c)-invertible if and only if it is both (b,c)-invertible and annihilator (b,c)-invertible.*

Given any $a,d \in R$, a is called invertible along d (see [5, Definition 4]) if there exists an element $x \in R$ such that $xad = d = dax$ and $x \in dR \cap Rd$. Such an x is the inverse of a along d. It is unique if it exists, and is denoted by $a^{\| d}$. Note that the implication that $d^0 \subseteq x^0$ yields $x \in Rd$, provide that d is regular. Hence, we have

Corollary 2.10. *Let $a,d \in R$. Then the following conditions are equivalent*:

(i) a *is left hybrid* (d,d)*-invertible.*
(ii) a *is right hybrid* (d,d)*-invertible.*
(iii) a *is invertible along* d*.*
In this case, ${}_ha^{(d,d)} = a_h^{(d,d)} = a^{\|d}$.

The group inverse of $a \in R$ is defined as an $x \in R$ satisfying $ax = xa$, $xax = x$ and $axa = a$. Such an x is unique if it exists, and is denoted by $a^{\#}$. Note that $a^{\|a}$ exists if and only if $a^{\#}$ exists. By Corollary 2.10, one can get ${}_ha^{(a,a)}=a_h^{(a,a)}=a^{\#}$.

It follows from [5, Theorem 7] that a is invertible along d if and only if $dR = daR$ and $(da)^{\#}$ exists if and only if $Rd = Rad$ and $(ad)^{\#}$ exists. In this case, $a^{\|d} = d(ad)^{\#} = (da)^{\#}d$. The following result gives necessary and sufficient conditions of the existence of $a^{\|d}$. Moreover, $a^{\|d} = d(ad)^- = (da)^-d$, where $(ad)^-$, $(da)^-$ are inner inverses of ad and da, respectively.

Corollary 2.11. *Let* $a, d \in R$*. Then the following conditions are equivalent*:
(i) a *is invertible along* d*.*
(ii) $(da)^0 \cap dR = \{0\}$ *and* $d \in dadR$*.*
(iii) $(dad)^0 \subseteq (d - dad)^0$ *and* $d \in dadR$*.*
(iv) $(da)^0 \cap dR = \{0\}$ *and* $R = adR + d^0$*.*
(v) $(dad)^0 \subseteq (d - dad)^0$ *and* $R = adR + d^0$*.*
(vi) $(da)^0 \cap dR = \{0\}$ *and* $R = adR \oplus d^0$*.*
(vii) $(dad)^0 \subseteq (d - dad)^0$ *and* $R = adR \oplus d^0$*.*
(viii) ${}^0(ad) \cap Rd = \{0\}$ *and* $R = Rda + {}^0d$*.*
(ix) ${}^0(dad) \subseteq {}^0(d - dad)$ *and* $R = Rda + {}^0d$*.*
(x) ${}^0(ad) \cap Rd = \{0\}$ *and* $R = Rda \oplus {}^0d$*.*
(xi) ${}^0(dad) \subseteq {}^0(d - dad)$ *and* $R = Rda \oplus {}^0d$*.*
(xii) ad *is regular,* $d = dad(ad)^-$ *and* $(da)^0 \cap dR = \{0\}$*.*
(xiii) ad *is regular,* $Rd \subseteq Rad(ad)^-$ *and* $(da)^0 \cap dR = \{0\}$*.*
(xiv) da *is regular,* $d = (da)^-dad$ *and* ${}^0(ad) \cap Rd = \{0\}$*.*
(xv) da *is regular,* $dR \subseteq (da)^-daR$ *and* ${}^0(ad) \cap Rd = \{0\}$*.*
In this case, $a^{\|d} = d(ad)^- = (da)^-d$*, where* $(ad)^-$*,* $(da)^-$ *are inner inverses of* ad *and* da*, respectively.*

Proof. (xii) ⇒ (xii) is clear.

(xiii) ⇒ (xii) As $Rd \subseteq Rad(ad)^-$, then $d = xad(ad)^-$ for some $x \in R$, and $d = xad(ad)^-ad(ad)^- = dad(ad)^-$.

(xiv) ⇔ (xv) can be referred to(xii) ⇔ (xiii). □

Remark 2.12. In Corollary 2.11 above, it should be noted that the formula of $a^{\|d}$ can be expressed by some inner inverses $(ad)^-$ of ad, which need not be the group inverse of ad. Such as, let $R = M_2(\mathbb{R})$ be the ring of all 2 by 2 matrices over the field of real numbers $\mathbb{R}$. Take $A = D = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \in R$, then, $A^{\|D} = (AD)^\# = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ by the fact that $A^{\|A}$ exists if and only if $A \in R^\#$. Note that all inner inverses of AD can be written as the form $\begin{bmatrix} 1 & 1-x \\ * & * \end{bmatrix}$, where $x \in \mathbb{R}$. Take $(AD)^- = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \neq (AD)^\#$, then $D(AD)^- = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = A^{\|D}$.

3. Left and right hybrid (b, c)-inverses of the product of three elements

In this section, we mainly investigate the existence criterion of the right hybrid (b, c)-inverse of paq in R, where $p, a, q, b, c \in R$. Several characterizations of the left hybrid (b, c)-inverse of paq can be also given similarly. Here, we omit them.

Following [3], a is called right (b, c)-invertible if there exists $y \in bR$ such that $cay = c$. Such a y is called a right (b, c)-inverse of a. Moreover, it is shown that such y exists if and only if $c \in cabR$. In particular, it there is an element $s \in R$ such that $c = cabs$ then bs is a right (b, c)-inverse of a.

An element $a \in R$ is called right annihilator (b, c)-invertible if there exists some $x \in R$ such that $xab = b$ and $c^0 \subseteq x^0$. By Lemma 2.2, one knows that a is right hybrid (b, c)-invertible if it is both right (b, c)-invertible and right annihilator (b, c)-invertible.

We next derive the characterization of right (b, c)-inverses and right annihilator (b, c)-inverses of the product of paq. Herein, an auxiliary lemma is firstly given.

Lemma 3.1. *Let $a, s, t, p \in R$. Then*

(i) *If $a^0 = s^0$, then $(at)^0 = (st)^0$.*

(ii) *If p is right invertible and $(ap)^0 = (sp)^0$, then $a^0 = s^0$.*

Proof. (i) For any $x \in (at)^0$, we have $atx = 0$ and $tx \in a^0 = s^0$, which imply $stx = 0$, i.e. $x \in (st)^0$, and consequently $(at)^0 \subseteq (st)^0$. A dual proof gives $(st)^0 \subseteq (at)^0$.

(ii) As $p \in R$ is right invertible, then there exists some $p' \in R$ such that $pp' = 1$. Applying (i), the result follows. □

Theorem 3.2. *Let $p, a, q, b, c \in R$. Then the following conditions are equivalent :*

(i) *paq is right (b, c)-invertible with inverses $x \in R$.*

(ii) *pa is right (qb, c)-invertible with inverses $y \in R$.*

In this case, $x = bs$ and $y = qx$, where $s \in R$ satisfies $y = qbs$.

PROOF. It is known that paq is right (b, c)-invertible if and only if $c \in cpaqbR$ if and only if pa is right (qb, c)-invertible. the result follows.

As x is a right (b, c)-invertible of paq, then $cpaqx = c$ and $x \in bR$, and consequently, $cpaqx = c$ and $qx \in qbR$, i.e. qx is a right (qb, c)-inverse of pa.

If pa is right (qb, c)-invertible with inverses z, then $cpay = c$ and $y \in qbR$, which give $y = qbs$ for some $s \in R$. We hence have $cpaqbs = c$, it is not difficult to verify that $x = bs$ is a right (b, c)-inverse of paq. □

Theorem 3.3. *Let $p, a, q, b, c \in R$ with p invertible. Then the following conditions are equivalent :*

(i) *paq is right annihilator (b, c)-invertible with inverse $x \in R$.*

(ii) *aq is right annihilator (b, cp)-invertible with inverse $z \in R$.*

In this case, $x = zp^{-1}$ and $z = xp$.

PROOF. (i) ⇒ (ii) Let $x \in R$ be a right annihilator (b, c)-inverse of paq. Then $xpaqb = b$ and $c^0 \subseteq x^0$. Set $z = xp$, then we $zaqb = b$ and $(cp)^0 \subseteq (xp)^0 = z^0$ by Lemma 3.1, hence xp is a right annihilator (b, cp)-inverse of aq.

(ii) ⇒ (i) As $z \in R$ is a right annihilator (b, cp)-inverse of aq, then $zaqb = b$ and $(cp)^0 \subseteq z^0$. Set $x = zp^{-1}$. Then $xpaqb = zp'paqb = zaqb = b$, and applying Lemma 3.1, it follows $x^0 = (zp')^0 \supseteq (cpp')^0 = c^0$, as required. □

Combing Theorem 3.2 and Theorem 3.3, one can get the following characterizations about right hybrid (b, c)-inverses of paq.

Theorem 3.4. *Let $p, a, q, b, c \in R$. Then the following conditions are equivalent :*

(i) *paq is right hybrid (b, c)-invertible.*

(ii) *pa is right (qb, c)-invertible and aq is right annihilator (b, cp)-invertible.*

In particular, if $x \in R$ is the right hybrid (b, c)-inverse of paq, $y \in R$ is a right (qb, c)-inverse of pa and $z \in R$ is a right annihilator (b, cp)-inverse of aq, then $x = bs$, $y = qx$ and $z = xp$, where $s \in R$ satisfies $y = qbs$.

Proof. (i) $\Rightarrow$ (ii) Applying Theorem 3.2 and Theorem 3.3.

(ii) $\Rightarrow$ (i) As pa is right (qb, c)-invertible, then $c \in cpaqbR$. If aq is right annihilator (b, cp)-invertible with right annihilator (b, cp)-inverse z, then $zaqb = b$ and $(cp)^0 \subseteq z^0$, and consequently, $(cpaqb)^0 \subseteq b^0$, indeed, for any $t \in (cpaqb)^0$, we have $cpaqbt = 0$ and $aqbt \in (cp)^0 \subseteq z^0$, hence $0 = zaqbt = bt$, which guarantees $t \in b^0$ and $(cpaqb)^0 \subseteq b^0$. It follows from Lemma 2.1 that paq is right hybrid (b, c)-invertible.

The formula $y = qx$ and $z = xp$ can be obtained by Theorem 3.2 and Theorem 3.3. The representation of the right hybrid (b, c)-inverse of paq, i.e. $a_h^{(b,c)} = bs$ follows from [6, Proposition 2.3]. $\square$

Given any $a, b, c \in R$, then a is right (b, c)-invertible implies that it is right (b, qc)-invertible, for any $q \in R$. Indeed, if y is the right (b, c)-inverse of a, then $cay = c$ and $y \in bR$. Hence, one can get $qcay = qc$ and a is right (b, qc)-invertible. Similarly, if a is right annihilator (b, c)-invertible, then it is also right annihilator (bq, c)-invertible.

Applying Theorem 3.4, then paq is right hybrid (b, c)-invertible guarantees that pa is right (qb, qc)-invertible and aq is right annihilator (bp, cp)-invertible. We next show, under some conditions, that the converse statement also holds.

Theorem 3.5. *Let $p, a, q, b, c \in R$. Suppose $b \in bpR$ and $c \in Rqc$. Then the following conditions are equivalent:*

(i) *paq is right hybrid (b, c)-invertible.*

(ii) *pa is right (qb, qc)-invertible and aq is right annihilator (bp, cp)-invertible.*

In particular, if $x \in R$ is the right hybrid (b, c)-inverse of paq, $y \in R$ is a right (qb, qc)-inverse of pa and $z \in R$ is a right annihilator (bp, cp)-inverse of aq, then $x = bs$, $y = qx$ and $z = xp$, where $s \in R$ satisfies $y = qbs$.

Proof. It suffices to prove (ii) $\Rightarrow$ (i).

Suppose that aq is right annihilator (bp, cp)-invertible. Then by a direct calculation, we have $(cpaqbp)^0 \subseteq (bp)^0$. It follows from $b \in bpR$ that $b = bpp'$ for some $p' \in R$, and hence $(cpaqb)^0 = (cpaqbpp')^0 \subseteq (bpp')^0 = b^0$. As pa is right (qb, qc)-invertible, then $qc \in qcpaqbR$, which implies $c \in cpaqbR$. Applying Lemma 2.1, it follows that paq is right hybrid (b, c)-invertible. In virtue of [6, Proposition 2.3], $x = bs$ is the right hybrid (b, c)-inverse of paq, where $s \in R$ satisfies $y = qbs$.

Suppose that p and q are invertible in Theorem 3.5. Then we have the following corollary.

Corollary 3.6. *Let $p, a, q, b, c \in R$ and let p and q be invertible. Then the following conditions are equivalent*:

(i) *paq is right hybrid (b, c)-invertible.*

(ii) *pa is right (qb, qc)-invertible and aq is right annihilator (bp, cp)-invertible.*

In particular, if $x \in R$ is the right hybrid (b, c)-inverse of paq, $y \in R$ is a right (qb, qc)-inverse of pa and $z \in R$ is a right annihilator (bp, cp)-inverse of aq, then $x = bs$, $y = qx$ and $z = xp$, where $s \in R$ satisfies $y = qbs$.

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References

References

[1] A. Ben-Israel, T. N. E. Greville, Generalized inverses: Theory and Applications, 2nd ed. Springer, New York(2003).

[2] M.P. Drazin, A class of outer generalized inverses, Linear Algebra Appl. 436 (2012) 1909-1923.

[3] M.P. Drazin, Left and right generalized inverses, Linear Algebra Appl. 510 (2016) 64-78.

[4] M.P. Drazin, Hybrid (b, c)-inverses and three finiteness properties in rings and semigroups, for submission.

[5] X. Mary, On generalized inverses and Green's relations, Linear Algebra Appl. 434 (2011) 1836-1844.

[6] H.H. Zhu, Further results on several generalized inverses, Comm. Algebra 46 (2018) 3388-3396.

[7] H.H. Zhu, J.L. Chen, P. Patrício, Further resulst on the inverse along an element in semigroups and rings, Linear Multilinear Algebra 64 (2016) 393-403.