Chapter 19
On Comparison Results for Neutral Stochastic Differential Equations of Reaction-Diffusion Type in $L_2(\mathbb{R}^d)$

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Abstract In the present paper, we establish a comparison result for solutions to the Cauchy problems for two stochastic integro-differential equations of reaction-diffusion type with delay. On this subject number of authors have obtained their comparison results. We deal with the Cauchy problems for two stochastic integro-differential equations of reaction-diffusion type with delay. Except drift and diffusion coefficients, our equations include also one integro-differential term. Basic difference of our case from the case of all earlier investigated problems is presence of this term. Presence of this term turns this equation into a nonlocal neutral stochastic equation of reaction-diffusion type. Nonlocal deterministic equations of this type are well known in literature and have wide range of applications. Such equations arise, for instance, in mechanics, electromagnetic theory, heat flow, nuclear reactor dynamics, and population dynamics. These equations are used in modeling of phytoplankton growth, distant interactions in epidemic models and nonlocal consumption of resources. We introduce a concept of mild solutions to our problems and state and prove a comparison theorem for them. According to our result, under certain assumptions on coefficients of equations under consideration, their solutions depend on the transient coefficients in a monotone way.

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V. A. Sadovnichiy, M. Z. Zgurovsky (eds.), *Modern Mathematics and Mechanics*, Understanding Complex Systems, https://doi.org/10.1007/978-3-319-96755-4_19

19.1 Introduction

In the given paper we study the following Cauchy problems for neutral partial stochastic integro-differential equations of reaction-diffusion type

$$d\Big(u_i(t,x) + \int_{\mathbb{R}^d} b_i(t,x,u_i(t-r,\xi),\xi)d\xi\Big) = \big(\Delta_x u_i(t,x) + f_i(t,u_i(t-r,x),x)\big)dt$$

$$+\,\sigma(t,x)dW(t,x),\; 0 < t \le T,\, x \in \mathbb{R}^d,\, i \in \{1,2\}, \tag{19.1}$$

$$u_i(t,x) = \phi_i(t,x),\; -r \le t \le 0,\, x \in \mathbb{R}^d,\, r > 0,\, i \in \{1,2\}, \tag{19.2}$$

where $T > 0$ is fixed, $\Delta_x \equiv \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ is d-measurable Laplacian in the space variables, W is a Q-Wiener process, $f_i : [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $i \in \{1,2\}$, $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ and $b_i : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $i \in \{1,2\}$, are some given functions to be specified later, $\phi_i : [-r,0] \times \mathbb{R}^d \to \mathbb{R}$, $i \in \{1,2\}$, are initial-datum functions. For solutions u_1 and u_2 of these problems we prove a comparison theorem. According to our result, if $f_1 \ge f_2$, then $u_1 \ge u_2$ with probability one.

A problem of comparison of solutions to stochastic differential equations in finite-dimensional case has firstly arised in [10]. A comparison theorem for equation of the form $d\xi(t) = f(t,\xi(t))dt + \sigma(t,\xi(t))d\beta(t)$, where β is standard one-dimensional Brownian motion, has been obtained in this work. According to this theorem, under certain assumptions, a solution of the equation above is monotonously non-decreasing function from "drift" coefficient f. A more general presentation of the comparison theorem is given in [11, 12]. Variations of the result from [10] have been the proposed in [2, 3, 5, 6, 8, 9, 13]. In [4] this theorem for solutions to stochastic differential equations with a multidimensional Wiener process and stochastic partial differential equations has been obtained. In [7] a comparison result for solutions to the Cauchy problem for stochastic differential equations with a Q-Wiener processes in Hilbert space is presented. The main goal of the given work is to prove a comparison theorem for solutions of problem (19.1)–(19.2), using the idea from this work. This result plays an important role when studying the existence of solutions to the Cauchy problem for stochastic differential equations of reaction-diffusion type with non-Lipschitz conditions on "drift" coefficients.

This paper is organised as follows. Firstly, in Sect. 19.2, we introduce a statement of the problem and formulate our main result. Then we represent a few necessary facts, needed for the treatment in the subsequent sections. These auxiliary results of independent interest are gathered in Sect. 19.3. Section 19.4 is devoted to the proof of the main theorem.

19.2 Problem Definition

Throughout the paper let $(\Omega, \mathscr{F}, \mathbf{P})$ be a complete probability space, $L_2(\mathbb{R}^d)$ denotes real Hilbert space with the norm $\|g\|_{L_2(\mathbb{R}^d)} = \left(\int\limits_{\mathbb{R}^d} g^2(x)dx \right)^{\frac{1}{2}}$. Let $\{e_n(x), n \in \{1, 2, \dots\}\}$ be an orthonormal basis on $L_2(\mathbb{R}^d)$ such that

$$\sup_{n \in \{1,2,\dots\}} \quad ess \sup_{x \in \mathbb{R}^d} |e_n(x)| \leq 1.$$

We now define $L_2(\mathbb{R}^d)$-valued Q-Wiener process $W(t, x) = W(t, \cdot), t \geq 0, x \in \mathbb{R}^d$, as follows

$$W(t, \cdot) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n(\cdot)\beta_n(t), t \geq 0,$$

where $\{\beta_n(t), n \in \{1, 2, \dots\}\} \subset \mathbb{R}$ are independent standard one-dimensional Brownian motions on $t \geq 0$, $\{\lambda_n, n \in \{1, 2, \dots\}\}$ is a sequence of positive numbers such that $\lambda = \sum\limits_{n=1}^{\infty} \lambda_n < \infty$. Let $\{\mathscr{F}_t, t \geq 0\}$ be a normal filtration on $\mathscr{F}$. We assume that $W(t, \cdot), t \geq 0$, is a Q-Wiener process with respect to a filtration $\{\mathscr{F}_t, t \geq 0\}$, i.e.,

- $W(t, \cdot), t \geq 0$, is $\mathscr{F}_t$-measurable;
- the increments $W(t + h, \cdot) - W(t, \cdot)$ are independent of $\mathscr{F}_t$ for all $h > 0$ and $t \geq 0$.

Let the following conditions be true

(1) $f_i : [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}, i \in \{1, 2\}, \sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}, b_i : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \times \mathbb{R}^d \to \mathbb{R}, i \in \{1, 2\}$, are measurable functions with respect to all of their variables.

(2) The initial-data functions $\phi_i(t, x, \omega) : [-r, 0] \times \mathbb{R}^d \times \Omega \to L_2(\mathbb{R}^d), i \in \{1, 2\}$, are $\mathscr{F}_0$-measurable random functions, independent of $W(t, x), t \geq 0, x \in \mathbb{R}^d$, with almost surely continuous paths and such that

$$\sup_{-r \leq t \leq 0} \mathbf{E}\|\phi_i(t, \cdot)\|^2_{L_2(\mathbb{R}^d)} < \infty, i \in \{1, 2\}.$$

(3) $b_i, i \in \{1, 2\}$, are uniformly continuous in the first argument and satisfy the Lipschitz condition in the third argument of the form

$$|b_i(t, x, u, \xi) - b_i(t, x, v, \xi)| \leq l(t, x, \xi)|u - v|,$$

$$0 \leq t \leq T, \{x, \xi\} \subset \mathbb{R}^d, \{u, v\} \subset \mathbb{R}, i \in \{1, 2\},$$

where $l\colon [0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty)$ is such that

$$\sup_{0\le t\le T}\int_{\mathbb{R}^d}\sqrt{\int_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx} < \infty,$$

$$\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx < \frac{1}{4}. \tag{19.3}$$

(4) There exists a function $\chi\colon \mathbb{R}^d\times\mathbb{R}^d\to[0,\infty)$, satisfying the following conditions

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\chi(x,\xi)d\xi dx < \infty,$$

$$\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\chi(x,\xi)d\xi\right)^2 dx < \infty,$$

such that

$$\sup_{0\le t\le T}|b_i(t,x,0,\xi)| \le \chi(x,\xi),\, 0\le t\le T,\, \{x,\xi\}\subset\mathbb{R}^d,\, i\in\{1,2\}. \tag{19.4}$$

(5) There exists a function $\eta\colon [0,T]\times\mathbb{R}^d\to[0,\infty)$ with

$$\sup_{0\le t\le T}\int_{\mathbb{R}^d}\eta^2(t,x)dx < \infty,$$

such that the following linear-growth and Lipschitz conditions are valid for f_i, $i\in\{1,2\}$,

$$|f_i(t,u,x)| \le \eta(t,x)+L|u|,\, 0\le t\le T,\, u\in\mathbb{R},\, x\in\mathbb{R}^d,\, i\in\{1,2\}, \tag{19.5}$$

$$|f_i(t,u,x)-f_i(t,v,x)| \le L|u-v|,\, 0\le t\le T,\, u\in\mathbb{R},\, x\in\mathbb{R}^d,\, i\in\{1,2\}.$$

(6) The next condition holds true for σ

$$\sup_{0\le t\le T}\|\sigma(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} < \infty.$$

(7) For $\nabla_x b_i$ and $D_x^2 b_i$, $i \in \{1,2\}$, the following linear-growth condition with respect to the third argument is true

$$|\nabla_x b_i(t,x,u,\xi)| + \|D_x^2 b_i(t,x,u,\xi)\| \le \psi(t,x,\xi)\big(1+|u|\big),$$
$$0 \le t \le T, \{x,\xi\} \subset \mathbb{R}^d, u \in \mathbb{R}, i \in \{1,2\},$$

and for $D_x^2 b_i$, $i \in \{1,2\}$, – the following Lipschitz condition

$$\|D_x^2 b_i(t,x,u,\xi) - D_x^2 b_i(t,x,v,\xi)\| \le \psi(t,x,\xi)|u-v|,$$
$$0 \le t \le T, \{x,\xi\} \subset \mathbb{R}^d, \{u,v\} \subset \mathbb{R}, i \in \{1,2\}, \tag{19.6}$$

where $\psi\colon [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ is such that

$$\sup_{0 \le t \le T} \int\limits_{\mathbb{R}^d} \left(\int\limits_{\mathbb{R}^d} \psi(t,x,\xi)d\xi \right)^2 dx < \infty,$$

$$\sup_{0 \le t \le T} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \psi^2(t,x,\xi)d\xi dx < \infty,$$

and besides for any point $x_0 \in \mathbb{R}^d$ there is its neighborhood $B_\delta(x_0)$ and a nonnegative function φ such that

$$\sup_{0 \le t \le T} \varphi(t,\cdot,x_0,\delta) \in L_2(\mathbb{R}^d) \cap L_1(\mathbb{R}^d), \delta > 0,$$

$$|\psi(t,x,\xi) - \psi(t,x_0,\xi)| \le \varphi(t,\xi,x_0,\delta)|x-x_0|,$$
$$0 \le t \le T, |x-x_0| < \delta, \xi \in \mathbb{R}^d.$$

Next, we introduce the notion of a mild solution to the problem (19.1)–(19.2).

Remark 19.1 From now on we use the notation $S(t)g(\cdot)$, $g \in L_2(\mathbb{R}^d)$, to denote the convolution

$$\big(S(t)g(\cdot)\big)(x) = \int\limits_{\mathbb{R}^d} \mathscr{K}(t,x-\xi)g(\cdot)d\xi, \, x \in \mathbb{R}^d, \, g \in L_2(\mathbb{R}^d).$$

It is known from semi-group theory that

$$\|(S(t)g(\cdot))(x)\|^2_{L_2(\mathbb{R}^d)} \le \|g(x)\|^2_{L_2(\mathbb{R}^d)}.$$

Here

$$\mathcal{K}(t,x)=\begin{cases}\frac{1}{(4\pi t)^{\frac{d}{2}}}\exp\left\{-\frac{|x|^2}{4t}\right\},\ t>0,\ x\in\mathbb{R}^d,\\ 0,\ t<0,\ x\in\mathbb{R}^d,\end{cases}$$

denotes the fundamental solution (source function, diffusion kernel) of the heat equation.

For convenience denote $u\equiv u_i$, $\phi\equiv\phi_i$, $b\equiv b_i$, $f\equiv f_i$, $i\in\{1,2\}$.

Definition 19.1 A continuous random process $u(t,x,\omega)\colon[-r,T]\times\mathbb{R}^d\times\Omega\to L_2(\mathbb{R}^d)$ is called a **mild solution (solution)** to (19.1)–(19.2) provided

1. It is $\mathcal{F}_t$-measurable for almost all $-r\le t\le T$.
2. It satisfies the integral equation

$$\begin{aligned}u(t,x)&=\int\limits_{\mathbb{R}^d}\mathcal{K}(t,x-\xi)\Bigg(\phi(0,\xi)+\int\limits_{\mathbb{R}^d}b(0,\xi,\phi(-r,\zeta),\zeta)d\zeta\Bigg)d\xi\\&-\int\limits_{\mathbb{R}^d}b(t,x,u(t-r,\xi),\xi)d\xi\\&-\int\limits_0^t\Bigg(\Delta_x\int\limits_{\mathbb{R}^d}\mathcal{K}(t-s,x-\xi)\Bigg(\int\limits_{\mathbb{R}^d}b(s,\xi,u(s-r,\zeta),\zeta)d\zeta\Bigg)d\xi\Bigg)ds\\&+\int\limits_0^t\int\limits_{\mathbb{R}^d}\mathcal{K}(t-s,x-\xi)f(s,u(s-r,\xi),\xi)d\xi ds\\&+\int\limits_0^t\sum_{n=1}^{\infty}\sqrt{\lambda_n}\Bigg(\int\limits_{\mathbb{R}^d}\mathcal{K}(t-s,x-\xi)\sigma(s,\xi)e_n(\xi)d\xi\Bigg)d\beta_n(s),\\&0<t\le T,\ x\in\mathbb{R}^d,\end{aligned}\tag{19.7}$$

$$u(t,x)=\phi(t,x),\ -r\le t\le 0,\ x\in\mathbb{R}^d,\ r>0.\tag{19.8}$$

3. It satisfies the condition

$$\mathbf{E}\int\limits_0^T\|u(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}dt<\infty.$$

Remark 19.2 It is assumed in the definition above that all the integrals from (19.7) are well defined.

The following is the comparison theorem.

Theorem 19.1 (Comparison Theorem) *Suppose assumptions* **(1)–(7)** *are satisfied. Let*

1) the initial-datum functions satisfy the condition

$$\phi_1(t,x) \geq \phi_2(t,x),\ 0 \leq t \leq T,\ x \in \mathbb{R}^d;$$

2) the functions b_i, $i \in \{1,2\}$, satisfy the conditions

$$b_1(0,x,\phi_2(-r,\xi),\xi) = b_2(0,x,\phi_2(-r,\xi),\xi),\ \{x,\xi\} \subset \mathbb{R}^d,$$

$$b_1(0,x,\phi_1(-r,\xi),\xi) = b_2(0,x,\phi_1(-r,\xi),\xi),\ \{x,\xi\} \subset \mathbb{R}^d,$$

$$b_1(0,x,\phi_1(-r,\xi),\xi) = b_1(0,x,\phi_2(-r,\xi),\xi),\ \{x,\xi\} \subset \mathbb{R}^d,$$

$$b_1(t,x,u,\xi) \leq b_2(t,x,u,\xi),\ 0 \leq t \leq T,\ \{x,\xi\} \subset \mathbb{R}^d,\ u \in \mathbb{R};$$

3) the functions f_i, $i \in \{1,2\}$, satisfy the conditions

$$f_1(t,u,x) \geq f_2(t,u,x),\ 0 \leq t \leq T,\ u \in \mathbb{R},\ x \in \mathbb{R}^d.$$

Let one of the following conditions be true

M1) *b_1 is monotonously non-increasing, f_1 is monotonously non-decreasing with respect to u;*

M2) *b_2 is monotonously non-increasing, f_2 is monotonously non-decreasing with respect to u.*

Then for all $0 \leq t \leq T$ the solutions of (19.1)–(19.2) *satisfy the inequality*

$$u_1(t,x) \geq u_2(t,x),\ x \in \mathbb{R}^d,$$

with probability one.

19.3 Preliminaries

This section is the toolbox of the results that will be used in the proof of Theorem 19.1.

19.3.1 Comparison Theorem for Finite-Dimensional Case

In order to prove our main result we need a finite-dimensional comparison theorem for the following Cauchy problems for two neutral stochastic integro-differential equations

$$d\left(u_i(t,x) + \int\limits_{\mathbb{R}^d} b_i(t,x,u_i(\alpha(t),\xi),\xi)d\xi\right) = f_i(t,u_i(\alpha(t),x),x)dt$$

$$+\sigma(t,x)d\beta(t), 0 < t \le T, x \in \mathbb{R}^d, i \in \{1,2\}, \tag{19.9}$$

$$u_i(t,x) = \phi_i(t,x), \; -r \le t \le 0, x \in \mathbb{R}^d, r > 0, i \in \{1,2\}, \tag{19.10}$$

where β is one-dimensional real-valued Brownian motion, $\alpha\colon [0,T] \to [-r,\infty)$ is a delay function.

Concerning coefficients of this problem we impose the following conditions

(1) $\alpha\colon [0,T] \to [-r,\infty)$ belongs to $C^1([0,T])$ with $\alpha' \ge 1, \alpha(t) \le t$;

(2) $f_i\colon [0,T]\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}, i \in \{1,2\}, \sigma\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}, b_i\colon [0,T]\times\mathbb{R}^d\times\mathbb{R}\times \times \mathbb{R}^d \to \mathbb{R}$, $i \in \{1,2\}$, are measurable with respect to all of their variables functions;

(3) the initial-datum functions $\phi_i(t,x,\omega)\colon [-r,0]\times\mathbb{R}^d\times\Omega \to L_2(\mathbb{R}^d), i \in \{1,2\}$, are $\mathscr{F}_0$-measurable random functions and such that

$$\sup_{-r\le t\le 0} \mathbf{E}\|\phi_i(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} < \infty, i \in \{1,2\};$$

(4) $b_i, i \in \{1,2\}$, satisfy the Lipshitz condition in the third argument of the form

$$|b_i(t,x,u,\xi) - b_i(t,x,v,\xi)| \le l(t,x,\xi)|u-v|,$$

$$0 \le t \le T, \{x,\xi\} \subset \mathbb{R}^d, \{u,v\} \subset \mathbb{R}, i \in \{1,2\},$$

where $l\colon [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \to [0,\infty)$ is such that

$$\sup_{0\le t\le T} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx < \frac{1}{4};$$

(5) there exists a function $\chi : \mathbb{R}^d\times\mathbb{R}^d \to [0,\infty)$, satisfying the following condition

$$\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d} \chi(x,\xi)d\xi\right)^2 dx < \infty,$$

such that

$$\sup_{0\le t\le T} |b_i(t,x,0,\xi)| \le \chi(x,\xi),\ 0\le t\le T,\ \{x,\xi\}\subset\mathbb{R}^d,\ i\in\{1,2\}.$$

For notational simplicity denote $u\equiv u_i,\ \phi\equiv\phi_i,\ b\equiv b_i,\ f\equiv f_i,\ i\in\{1,2\}$.

Definition 19.2 A continuous random process $u(t,x,\omega)\colon [-r,T]\times\mathbb{R}^d\times\Omega\to$ $\to L_2(\mathbb{R}^d)$ is called a **solution** to (19.9)–(19.10) provided

1. It is $\mathscr{F}_t$-measurable for almost all $-r\le t\le T$.
2. It satisfies the following integral equation

$$u(t,x)=\phi(0,x)+\int\limits_{\mathbb{R}^d} b(0,x,\phi(-r,\xi),\xi)d\xi-\int\limits_{\mathbb{R}^d} b(t,x,u(\alpha(t),\xi),\xi)d\xi$$
$$+\int\limits_0^t f(s,u(\alpha(s),x),x)ds+\int\limits_0^t \sigma(s,x)d\beta(s),\ 0\le t\le T,\ x\in\mathbb{R}^d, \tag{19.11}$$

$$u(t,x)=\phi(t,x),\ -r\le t\le 0,\ x\in\mathbb{R}^d,\ r>0. \tag{19.12}$$

3. It satisfies the condition

$$\mathbf{E}\int\limits_0^T \|u(t,\ \cdot)\|^2_{L_2(\mathbb{R}^d)}dt<\infty.$$

Earlier we have stated and proved the following two theorems.

Theorem 19.2 (Existence Theorem) *Suppose assumptions* **(1)**–**(5)** *and conditions* **(5)**, **(6)** *from Sect. 19.2 are valid. Then* (19.11)–(19.12) *has a unique solution.*

Theorem 19.3 (Finite-Dimensional Comparison Theorem) *Suppose conditions of existence theorem above are valid and*

1) the initial-datum functions satisfy the condition

$$\phi_1(t,x)\ge\phi_2(t,x),\ 0\le t\le T,\ x\in\mathbb{R}^d;$$

2) the functions b_i, $i\in\{1,2\}$, satisfy the conditions

$$b_1(0,x,\phi_2(-r,\xi),\xi)=b_2(0,x,\phi_2(-r,\xi),\xi),\ \{x,\xi\}\subset\mathbb{R}^d,$$
$$b_1(0,x,\phi_1(-r,\xi),\xi)=b_2(0,x,\phi_1(-r,\xi),\xi),\ \{x,\xi\}\subset\mathbb{R}^d,$$
$$b_1(0,x,\phi_1(-r,\xi),\xi)=b_1(0,x,\phi_2(-r,\xi),\xi),\ \{x,\xi\}\subset\mathbb{R}^d,$$
$$b_1(t,x,u,\xi)\le b_2(t,x,u,\xi),\ 0\le t\le T,\ \{x,\xi\}\subset\mathbb{R}^d,\ u\in\mathbb{R};$$

3) the functions f_i, $i \in \{1, 2\}$, *satisfy the conditions*

$$f_1(t, u, x) \geq f_2(t, u, x),\ 0 \leq t \leq T,\ u \in \mathbb{R},\ x \in \mathbb{R}^d.$$

Let one of the following conditions be true

M1) b_1 *is monotonously non-increasing,* f_1 *is monotonously non-decreasing with respect to* u*;*

M2) b_2 *is monotonously non-increasing,* f_2 *is monotonously non-decreasing with respect to* u.

Then for all $0 \leq t \leq T$ *the solutions of* (19.9)–(19.10) *satisfy the inequality*

$$u_1(t, x) \geq u_2(t, x),\ x \in \mathbb{R}^d,$$

with probability one.

19.3.2 Approximation Properties

During the proof we will need also auxiliary results of independent interest for the following Cauchy problem

$$\frac{\partial z(t, x)}{\partial t} = Az(t, x),\ t > 0,\ x \in \mathbb{R}^d, \tag{19.13}$$

$$z(0, x) = g(x),\ x \in \mathbb{R}^d, \tag{19.14}$$

where $A\colon L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ is a monotone operator. The next theorem is true.

Theorem 19.4 ([1], p. 25) *For any* $g \in L_2(\mathbb{R}^d)$ *there exists a unique solution* z *to* (19.13)–(19.14)*, belonging to* $C^1([0, \infty) \times \mathbb{R}^d) \cap ([0, \infty) \times \mathbb{R}^d)$*, and besides for* $t > 0$

$$\|z(t, \cdot)\|_{L_2(\mathbb{R}^d)} \leq \|g(\cdot)\|_{L_2(\mathbb{R}^d)},$$
$$\left\|\frac{\partial z(t, \cdot)}{\partial t}\right\|_{L_2(\mathbb{R}^d)} = \|Az(t, \cdot)\|_{L_2(\mathbb{R}^d)} \leq \|Ag(\cdot)\|_{L_2(\mathbb{R}^d)}. \tag{19.15}$$

Lemma 19.1 ([1], p. 22) *Let* $z_N \in C^1([0, \infty) \times \mathbb{R}^d)$ *be a solution to the following Yosida approximating equation*

$$\frac{\partial z_N(t, x)}{\partial t} = A_N z_N(t, x),\ t > 0,\ x \in \mathbb{R}^d,$$

where A_N, $N \in \{1, 2, \ldots\}$, *is Yosida approximation of operator* A. *Then* $\|z_N(t,\ \cdot)\|_{L_2(\mathbb{R}^d)}$ *and* $\left\| \frac{\partial z_N(t,\ \cdot)}{\partial t} \right\|_{L_2(\mathbb{R}^d)}$ *are monotonously non-increasing on* $t > 0$.

The following approximative property is valid.

Lemma 19.2 *There exists Yosida approximation of operator* $A = \Delta_x$ *by a sequence* $\{A_N, N \in \{1, 2, \ldots\}\}$ *of linear bounded operators* $A_N \colon L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ *and the following conditions are true*

1) for each $N \in \{1, 2, \ldots\}$ *there exists a constant* $C_N > 0$ *such that*

$$\|A_N\|^2_{\mathscr{L}(L_2(\mathbb{R}^d), L_2(\mathbb{R}^d))} \leq C_N; \tag{19.16}$$

2) for each $g \in L_2(\mathbb{R}^d)$ *the following equality is true*

$$\lim_{N \to \infty} \left\| \big((A_N - A)\, g(\,\cdot\,) \big)(x) \right\|^2_{L_2(\mathbb{R}^d)} = 0,\ x \in \mathbb{R}^d; \tag{19.17}$$

3) operators A_N, $N \in \{1, 2, \ldots\}$, *generate semigroup* $\{S_N(t), N \in \{1, 2, \ldots\}\}$ *of operators* $S_N(t) \colon L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ *with the following properties*

a) for an arbitrary $x \in \mathbb{R}^d$ *there exists* $N_0 = N_0(x) \in \{1, 2, \ldots\}$ *such that for all* $N \geq N_0(x)$ $\big(S_N(t - s)g(\,\cdot\,)\big)(x) \geq 0$, $0 \leq s \leq t \leq T$, $x \in \mathbb{R}^d$, $g \in L_2(\mathbb{R}^d)$, $g \geq 0$;

b)

$$\lim_{N \to \infty} \sup_{0 \leq s \leq t \leq T} \left\| \big((S_N(t - s) - S(t - s))\, g(\,\cdot\,) \big)(x) \right\|^2_{L_2(\mathbb{R}^d)} = 0,$$
$$x \in \mathbb{R}^d,\ g \in L_2(\mathbb{R}^d). \tag{19.18}$$

19.4 Proof of Theorem 19.1

1. From now on $x \in \mathbb{R}^d$ is supposed to be fixed. Let fix an arbitrary $M \in \{1, 2, \ldots\}$ and define by $W_M(t,\ \cdot)$ Q_M-Wiener process

$$W_M(t,\ \cdot\,) = \sum_{j=1}^{M} \sqrt{\lambda_j} e_j(\cdot) \beta_j(t),\ 0 \leq t \leq T.$$

Let us consider the following Cauchy problems

$$u_i^{N,M}(t,\cdot)=\phi_i(0,\cdot)+\int_{\mathbb{R}^d} b_i(0,\cdot,\phi_i(-r,\xi),\xi)d\xi-\int_{\mathbb{R}^d} b_i(t,\cdot,u_i^{N,M}(t-r,\xi),\xi)d\xi$$
$$+\int_0^t \big(A_N u_i^{N,M}(s,\cdot)+f_i(s,u_i^{N,M}(s-r,\cdot),\cdot)\big)ds+\int_0^t \sigma(s,\cdot)dW_M(s,\cdot),$$
$$0<t\le T,\, i\in\{1,2\},\, N\in\{1,2,\ldots\}, \tag{19.19}$$

$$u_i^{N,M}(t,\cdot)=\phi_i(t,\cdot),\ -r\le t\le 0, r>0, i\in\{1,2\}, N\in\{1,2,\ldots\}, \tag{19.20}$$

where $\{A_N, N\in\{1,2,\ldots\}\}$ are operators from Lemma 19.2. Denote $u\equiv u_i$, $\phi\equiv\phi_i$, $b\equiv b_i$, $f\equiv f_i$, $i\in\{1,2\}$, for simplicity. A continuous $\mathscr{F}_t$-measurable for almost all $-r\le t\le T$ random process $u^{N,M}:[-r,T]\times\Omega\to L_2(\mathbb{R}^d)$ is called a **solution** to (19.19)–(19.20) provided

$$u^{N,M}(t,\cdot)=S_N(t)\Big(\phi(0,\cdot)+\int_{\mathbb{R}^d} b(0,\cdot,\phi(-r,\xi),\xi)d\xi\Big)$$
$$-\int_{\mathbb{R}^d} b(t,\cdot,u^{N,M}(t-r,\xi),\xi)d\xi$$
$$-\int_0^t A_N S_N(t-s)\Big(\int_{\mathbb{R}^d} b(s,\cdot,u^{N,M}(s-r,\xi),\xi)d\xi\Big)ds$$
$$+\int_0^t S_N(t-s)f(s,u^{N,M}(s-r,\cdot),\cdot)ds+\int_0^t\sigma(s,\cdot)dW_M(s,\cdot),$$
$$0<t\le T,\, N\in\{1,2,\ldots\}, \tag{19.21}$$

$$u^{N,M}(t,\cdot)=\phi(t,\cdot),\ -r\le t\le 0, r>0, N\in\{1,2,\ldots\}, \tag{19.22}$$

and $\mathbf{E}\int_0^T\|u(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}dt<\infty$. Since operators $\{A_N, N\in\{1,2,\ldots\}\}$ are bounded, (19.21)–(19.22) possesses a unique up to equivalence solution. Fix additionally $N\in\{1,2,\ldots\}$ and write u_i instead of $u_i^{N,M}$, $i\in\{1,2\}$, for notational simplicity. Let us prove that $u_1(t,\cdot)\ge u_2(t,\cdot)$, $0\le t\le T$, almost surely.

2. Let us fix $n \in \{1, 2, \ldots\}$, put $t_k = \frac{kT}{N}$, $k \in \{0, \ldots, n\}$, with $t_{k+1} - t_k = \frac{T}{n} < r$, $-r < 0 \le t_1 \le r \le 2t_1 \le 2r \le \ldots$, and consider the next equations

$$z_i^{0,n}(t, \cdot) = \phi_i(0, \cdot) + \int_{\mathbb{R}^d} b_i(0, \cdot, \phi_i(-r, \xi), \xi)d\xi - \int_{\mathbb{R}^d} b_i(t, \cdot, z_i^{0,n}(t-r, \cdot), \xi)d\xi$$

$$+ \int_0^t \sigma(s, \cdot)dW_M(s, \cdot), 0 \le t \le t_1, i \in \{1, 2\}, \tag{19.23}$$

$$v_i^{0,n}(t, \cdot) = z_i^{0,n}(t_1, \cdot) + \int_0^t \big(A_N v_i^{0,n}(s, \cdot) + f_i(s, v_i^{0,n}(s-r, \cdot), \cdot)\big)ds,$$

$$0 \le t \le t_1, i \in \{1, 2\}, \tag{19.24}$$

and

$$z_i^{k,n}(t, \cdot) = v_i^{k-1,n}(t_k, \cdot) + \int_{\mathbb{R}^d} b_i(t_k, \cdot, z_i^{k-1,n}(t_k - r, \xi), \xi)d\xi$$

$$- \int_{\mathbb{R}^d} b_i(t, \cdot, z_i^{k,n}(t-r, \xi), \xi)d\xi$$

$$+ \int_{t_k}^t \sigma(s, \cdot)dW_M(s, \cdot), t_k \le t \le t_{k+1}, k \in \{1, \ldots, n-1\}, i \in \{1, 2\}, \tag{19.25}$$

$$v_i^{k,n}(t, \cdot) = z_i^{k,n}(t_{k+1}, \cdot) + \int_{t_k}^t \big(A_N v_i^{k,n}(s, \cdot) + f_i(s, v_i^{k,n}(s-r, \cdot), \cdot)\big)ds,$$

$$t_k \le t \le t_{k+1}, k \in \{1, \ldots, n-1\}, i \in \{1, 2\}, \tag{19.26}$$

$$z_i^{k,n}(t, \cdot) = v_i^{k,n}(t, \cdot) = \phi_i(t, \cdot), \ -r \le t \le 0, k \in \{0, \ldots, n-1\}, i \in \{1, 2\}.$$

3. Define $z_i^n\colon [0,T]\times\Omega\to L_2(\mathbb{R}^d)$, $v_i^n\colon [0,T]\times\Omega\to L_2(\mathbb{R}^d)$, $i\in\{1,2\}$, as follows

$$z_i^n(t,\cdot)=z_i^{k,n}(t,\cdot),\, t_k\le t<t_{k+1}, k\in\{0,\dots,n-1\}, i\in\{1,2\}, \tag{19.27}$$

$$v_i^n(t,\cdot)=v_i^{k,n}(t,\cdot),\, t_k< t\le t_{k+1}, k\in\{0,\dots,n-1\}, i\in\{1,2\}, \tag{19.28}$$

$$z_i^n(T,\cdot)=v_i^n(T,\cdot), i\in\{1,2\},$$

$$z_i^n(t,\cdot)=v_i^n(t,\cdot)=\phi_i(t,\cdot),\ -r\le t\le 0, i\in\{1,2\}.$$

Taking into account identities for $v_i^{k-1,n}(t_k,\cdot)$, $z_i^{k-1,n}(t_k,\cdot)$, $k\in\{1,\dots,n-1\}$, $i\in\{1,2\}$, and $z_i^{0,n}(t_1,\cdot)$, $i\in\{1,2\}$, one easily verifies

$$v_i^{k-1,n}(t_k,\cdot)=\underbrace{z_i^{k-1,n}(t_k,\cdot)}+\int_{t_{k-1}}^{t_k}\big(A_N v_i^{k-1,n}(s,\cdot)+f_i(s,v_i^{k-1,n}(s-r,\cdot),\cdot)\big)ds$$

$$=\underbrace{\overbrace{v_i^{k-2,n}(t_{k-1},\cdot)}+\int_{\mathbb{R}^d} b_i(t_{k-1},\cdot,z_i^{k-2,n}(t_{k-1}-r,\xi),\xi)d\xi}$$

$$\underbrace{-\int_{\mathbb{R}^d} b_i(t_k,\cdot,z_i^{k-1,n}(t_k-r,\xi),\xi)d\xi+\int_{t_{k-1}}^{t_k}\sigma(s,\cdot)dW_M(s,\cdot)}$$

$$+\int_{t_{k-1}}^{t_k}\big(A_N v_i^{k-1,n}(s,\cdot)+f_i(s,v_i^{k-1,n}(s-r,\cdot),\cdot)\big)ds=\overbrace{z_i^{k-2,n}(t_{k-1},\cdot)}$$

$$\overbrace{+\int_{t_{k-2}}^{t_{k-1}}\big(A_N v_i^{k-2,n}(s,\cdot)+f_i(s,v_i^{k-2,n}(s-r,\cdot),\cdot)\big)ds}$$

$$+\int_{\mathbb{R}^d} b_i(t_{k-1},\cdot,z_i^{k-2,n}(t_{k-1}-r,\xi),\xi)d\xi$$

$$-\int_{\mathbb{R}^d} b_i(t_k,\cdot,z_i^{k-1,n}(t_k-r,\xi),\xi)d\xi+\int_{t_{k-1}}^{t_k}\sigma(s,\cdot)dW_M(s,\cdot)$$

$$+\int_{t_{k-1}}^{t_k}\big(A_N v_i^{k-1,n}(s,\cdot)+f_i(s,v_i^{k-1,n}(s-r,\cdot),\cdot)\big)ds=\dots$$

$$= v_i^{0,n}(t_1, \cdot) + \int_{\mathbb{R}^d} b_i(t_1, \cdot, \phi_i(t_1 - r, \xi), \xi)d\xi - \int_{\mathbb{R}^d} b_i(t_k, \cdot, z_i^{k-1,n}(t_k - r, \xi), \xi)d\xi$$

$$+ \int_{t_1}^{t_2} \sigma(s, \cdot)dW_M(s, \cdot) + \int_{t_2}^{t_3} \sigma(s, \cdot)dW_M(s, \cdot) + \ldots + \int_{t_{k-2}}^{t_{k-1}} \sigma(s, \cdot)dW_M(s, \cdot)$$

$$+ \int_{t_{k-1}}^{t_k} \sigma(s, \cdot)dW_M(s, \cdot) + \int_{t_1}^{t_2} \big(A_N v_i^{1,n}(s, \cdot) + f_i(s, v_i^{1,n}(s - r, \cdot), \cdot)\big)ds$$

$$+ \int_{t_2}^{t_3} \big(A_N v_i^{2,n}(s, \cdot) + f_i(s, v_i^{2,n}(s - r, \cdot), \cdot)\big)ds + \ldots$$

$$+ \int_{t_{k-2}}^{t_{k-1}} \big(A_N v_i^{k-2,n}(s, \cdot) + f_i(s, v_i^{k-2,n}(s - r, \cdot), \cdot)\big)ds$$

$$+ \int_{t_{k-1}}^{t_k} \big(A_N v_i^{k-1,n}(s, \cdot) + f_i(s, v_i^{k-1,n}(s - r, \cdot), \cdot)\big)ds = \ldots$$

$$= \phi_i(0, \cdot) + \int_{\mathbb{R}^d} b_i(0, \cdot, \phi_i(-r, \xi), \xi)d\xi - \int_{\mathbb{R}^d} b_i(t_1, \cdot, \phi_i(t_1 - r, \xi), \xi)d\xi$$

$$+ \int_0^{t_1} \sigma(s, \cdot)dW_M(s, \cdot) + \int_0^{t_1} \big(A_N v_i^{0,n}(s, \cdot) + f_i(s, v_i^{0,n}(s - r, \cdot), \cdot)\big)ds$$

$$+ \int_{\mathbb{R}^d} b_i(t_1, \cdot, \phi_i(t_1 - r, \xi), \xi)d\xi - \int_{\mathbb{R}^d} b_i(t_k, \cdot, z_i^{k-1,n}(t_k - r, \xi), \xi)d\xi$$

$$+ \int_{t_1}^{t_2} \sigma(s, \cdot)dW_M(s, \cdot) + \int_{t_2}^{t_3} \sigma(s, \cdot)dW_M(s, \cdot) + \ldots + \int_{t_{k-2}}^{t_{k-1}} \sigma(s, \cdot)dW_M(s, \cdot)$$

$$+ \int_{t_{k-1}}^{t_k} \sigma(s, \cdot)dW_M(s, \cdot) + \int_{t_1}^{t_2} \big(A_N v_i^{1,n}(s, \cdot) + f_i(s, v_i^{1,n}(s - r, \cdot), \cdot)\big)ds$$

$$+ \int_{t_2}^{t_3} \big(A_N v_i^{2,n}(s, \cdot) + f_i(s, v_i^{2,n}(s - r, \cdot), \cdot)\big)ds + \ldots$$

$$+\int_{t_{k-2}}^{t_{k-1}}\big(A_N v_i^{k-2,n}(s,\,\cdot)+f_i(s,v_i^{k-2,n}(s-r,\,\cdot),\,\cdot)\big)ds$$

$$+\int_{t_{k-1}}^{t_k}\big(A_N v_i^{k-1,n}(s,\,\cdot)+f_i(s,v_i^{k-1,n}(s-r,\,\cdot),\,\cdot)\big)ds,\ i\in\{1,2\}. \tag{19.29}$$

After substitution (19.29) into (19.25) and, using (19.27), we obtain for $t_k \le t < t_{k+1}$, $k \in \{1, \ldots, n-1\}$,

$$z_i^{k,n}(t,\,\cdot)=z_i^n(t,\,\cdot)=\phi_i(0,\,\cdot)+\int_{\mathbb{R}^d} b_i(0,\,\cdot,\phi_i(-r,\xi),\xi)d\xi$$

$$-\int_{\mathbb{R}^d} b_i(t_k,\,\cdot,z_i^n(t_k-r,\xi),\xi)d\xi+\int_0^{t_k}\sigma(s,\,\cdot)dW_M(s,\cdot)$$

$$+\int_0^{t_k}\big(A_N v_i^n(s,\,\cdot)+f_i(s,v_i^n(s-r,\,\cdot),\,\cdot)\big)ds+\int_{\mathbb{R}^d} b_i(t_k,\,\cdot,z_i^n(t_k-r,\xi),\xi)d\xi$$

$$-\int_{\mathbb{R}^d} b_i(t,\,\cdot,z_i^n(t-r,\xi),\xi)d\xi+\int_{t_k}^{t}\sigma(s,\,\cdot)dW_M(s,\cdot)$$

$$=\phi_i(0,\,\cdot)+\int_{\mathbb{R}^d} b_i(0,\,\cdot,\phi_i(-r,\xi),\xi)d\xi-\int_{\mathbb{R}^d} b_i(t,\,\cdot,z_i^n(t-r,\xi),\xi)d\xi$$

$$+\int_0^{t_k}\big(A_N v_i^n(s,\,\cdot)+f_i(s,v_i^n(s-r,\,\cdot),\,\cdot)\big)ds+\int_0^{t}\sigma(s,\,\cdot)dW_M(s,\cdot),\ i\in\{1,2\}.$$

Similarly, taking into account (19.28), we get from (19.26) for $t_k < t \le t_{k+1}$, $k \in \{1, \ldots, n-1\}$,

$$v_i^n(t,\,\cdot)=v_i^{k,n}(t,\,\cdot)=\phi_i(0,\,\cdot)+\int_{\mathbb{R}^d} b_i(0,\,\cdot,\phi_i(-r,\xi),\xi)d\xi$$

$$-\int_{\mathbb{R}^d} b_i(t_{k+1},\,\cdot,z_i^n(t_{k+1}-r,\xi),\xi)d\xi+\int_0^{t_k}\big(A_N v_i^n(s,\,\cdot)+f_i(s,v_i^n(s-r,\,\cdot),\,\cdot)\big)ds$$

$$+\int_0^{t_{k+1}} \sigma(s,\cdot)dW_M(s,\cdot) + \int_{t_k}^{t} \big(A_N v_i^n(s,\cdot) + f_i(s, v_i^n(s-r,\cdot),\cdot)\big)ds$$

$$= \phi_i(0,\cdot) + \int_{\mathbb{R}^d} b_i(0,\cdot,\phi_i(-r,\xi),\xi)d\xi - \int_{\mathbb{R}^d} b_i(t_{k+1},\cdot,z_i^n(t_{k+1}-r,\xi),\xi)d\xi$$

$$+\int_0^{t} \big(A_N v_i^n(s,\cdot)+f_i(s, v_i^n(s-r,\cdot),\cdot)\big)ds+\int_0^{t_{k+1}} \sigma(s,\cdot)dW_M(s,\cdot),\ i\in\{1,2\}.$$

Thus, one easily verifies for $z_i^n(t,\cdot)$, $v_i^n(t,\cdot)$, $i\in\{1,2\}$,

$$z_i^n(t,\cdot) = \phi_i(0,\cdot) + \int_{\mathbb{R}^d} b_i(0,\cdot,\phi_i(-r,\xi),\xi)d\xi - \int_{\mathbb{R}^d} b_i(t,\cdot,z_i^n(t-r,\xi),\xi)d\xi$$

$$+\int_0^{t_k} \big(A_N v_i^n(s,\cdot) + f_i(s, v_i^n(s-r,\cdot),\cdot)\big)ds + \int_0^{t} \sigma(s,\cdot)dW_M(s,\cdot),$$

$$t_k \le t < t_{k+1},\ k\in\{0,\dots,n-1\},\ i\in\{1,2\}, \tag{19.30}$$

$$v_i^n(t,\cdot) = \phi_i(0,\cdot)+\int_{\mathbb{R}^d} b_i(0,\cdot,\phi_i(-r,\xi),\xi)d\xi-\int_{\mathbb{R}^d} b_i(t_{k+1},\cdot,z_i^n(t_{k+1}-r,\xi),\xi)d\xi$$

$$+\int_0^{t} \big(A_N v_i^n(s,\cdot) + f_i(s, v_i^n(s-r,\cdot),\cdot)\big)ds + \int_0^{t_{k+1}} \sigma(s,\cdot)dW_M(s,\cdot),$$

$$t_k < t \le t_{k+1},\ k\in\{0,\dots,n-1\},\ i\in\{1,2\}, \tag{19.31}$$

$$z_i^n(t,\cdot) = v_i^n(t,\cdot) = \phi_i(t,\cdot),\ -r\le t\le 0,\ i\in\{1,2\}.$$

4. Now let us show that

$$z_1^n(t,\cdot) \ge z_2^n(t,\cdot), \tag{19.32}$$

$$v_1^n(t,\cdot) \ge v_2^n(t,\cdot), \tag{19.33}$$

almost surely for any $0\le t\le T$.

Let us prove (19.32) for $0 \le t \le t_1$. Invoking Theorem 19.3, one obtains

$$z_1^n(t,\cdot) = \phi_1(0,\cdot) + \int_{\mathbb{R}^d} b_1(0,\cdot,\phi_1(-r,\xi),\xi)d\xi - \int_{\mathbb{R}^d} b_1(t,\cdot,z_1^n(t-r,\xi),\xi)d\xi$$

$$+ \sum_{j=1}^{M} \sqrt{\lambda_j} e_j(\cdot) \int_0^t \sigma(s,\cdot)d\beta_j(s) \ge \phi_2(0,\cdot) + \int_{\mathbb{R}^d} b_2(0,\cdot,\phi_2(-r,\xi),\xi)d\xi$$

$$- \int_{\mathbb{R}^d} b_2(t,\cdot,z_2^n(t-r,\xi),\xi)d\xi + \sum_{j=1}^{M} \sqrt{\lambda_j} e_j(\cdot) \int_0^t \sigma(s,\cdot)d\beta_j(s) = z_2^n(t,\cdot),$$

$$0 \le t < t_1. \tag{19.34}$$

Similarly we obtain for $z_i^{0,n}(t_1,\cdot)$, $i \in \{1,2\}$,

$$z_1^{0,n}(t_1,\cdot) = \phi_1(0,\cdot) + \int_{\mathbb{R}^d} b_1(0,\cdot,\phi_1(-r,\xi),\xi)d\xi - \int_{\mathbb{R}^d} b_1(t_1,\cdot,z_1^{0,n}(t_1-r,\xi),\xi)d\xi$$

$$+ \sum_{j=1}^{M} \sqrt{\lambda_j} e_j(\cdot) \int_0^{t_1} \sigma(s,\cdot)d\beta_j(s) \ge \phi_2(0,\cdot) + \int_{\mathbb{R}^d} b_2(0,\cdot,\phi_2(-r,\xi),\xi)d\xi$$

$$- \int_{\mathbb{R}^d} b_2(t_1,\cdot,z_2^{0,n}(t_1-r,\xi),\xi)d\xi + \sum_{j=1}^{M} \sqrt{\lambda_j} e_j(\cdot) \int_0^{t_1} \sigma(s,\cdot)d\beta_j(s) = z_2^{0,n}(t_1,\cdot).$$

Now let us prove (19.33) for $0 \le t \le t_1$. Since this inequality is obvious for $t = 0$, we will show it for $0 < t \le t_1$. Since we have $-r < t - r \le t_1 - r \le 0$ for $0 < t \le t_1$, then $z_i^n(t-r,\cdot) = \phi_i(t-r,\cdot)$, $0 < t \le t_1$, $i \in \{1,2\}$, and $f_2(s,\phi_1(s-r,\cdot),\cdot) \ge f_2(s,\phi_2(s-r,\cdot),\cdot)$, according to **M2**. Then we get for $v_1^n(t,\cdot) - v_2^n(t,\cdot)$, $0 < t \le t_1$, from (19.31), taking into account conditions of the theorem, the following identity

$$v_1^n(t,\cdot) - v_2^n(t,\cdot) = \left(z_1^{0,n}(t_1,\cdot) - z_2^{0,n}(t_1,\cdot)\right) + \int_0^t A_N\left(v_1^n(s,\cdot) - v_2^n(s,\cdot)\right)ds$$

$$+ \int_0^t \left(f_1(s,\phi_1(s-r,\cdot),\cdot) - f_2(s,\phi_1(s-r,\cdot),\cdot)\right)ds$$

$$+ \int_0^t \left(f_2(s,\phi_1(s-r,\cdot),\cdot) - f_2(s,\phi_2(s-r,\cdot),\cdot)\right)ds.$$

Since S_N supposed to be positivity preserving, the equality above can be rewritten in the following manner

$$v_1^n(t,\cdot) - v_2^n(t,\cdot) = S_N(t)\big(z_1^{0,n}(t_1,\cdot) - z_2^{0,n}(t_1,\cdot)\big)$$
$$+ \int_0^t S_N(t-s)\big(f_1(s,\phi_1(s-r,\cdot),\cdot) - f_2(s,\phi_1(s-r,\cdot),\cdot)\big)ds$$
$$+ \int_0^t S_N(t-s)\big(f_2(s,\phi_1(s-r,\cdot),\cdot) - f_2(s,\phi_2(s-r,\cdot),\cdot)\big)ds \geq 0.$$

Thus we have

$$v_1^n(t,\cdot) \geq v_2^n(t,\cdot), 0 < t \leq t_1. \tag{19.35}$$

This estimate implies (19.33) for $0 \leq t \leq t_1$.

It remains to show that $z_1^n(t_1,\cdot) \geq z_2^n(t_1,\cdot)$. Since

$$v_i^n(t_1,\cdot) = z_i^n(t_1,\cdot), i \in \{1,2\},$$

then we obviously obtain from (19.35)

$$z_1^n(t_1,\cdot) \geq z_2^n(t_1,\cdot).$$

This estimate and relation (19.34), in turn, give (19.32) for any $0 \leq t \leq t_1$.

Let us prove (19.32) for $t_1 \leq t \leq t_2$. First let estimate $z_1^n(t,\cdot) - z_2^n(t,\cdot)$, $t_1 \leq t < t_2$. Since we have $-r \leq t_1 - r \leq t - r < t_2 - r = 2t_1 - r \leq t_1$, i.e. $-r \leq t - r \leq t_1$ for $t_1 \leq t < t_2$, then $z_1^n(t-r,\cdot) \geq z_2^n(t-r,\cdot)$, $v_1^n(t-r,\cdot) \geq v_2^n(t-r,\cdot)$, $t_1 \leq t < t_2$, and, according to **M2**, $f_2(t, v_1^n(t-r,\cdot),\cdot) \geq f_2(t, v_2^n(t-r,\cdot),\cdot)$, $b_2(t,\cdot,z_2^n(t-r,\xi),\xi) \geq b_2(t,\cdot,z_1^n(t-r,\xi),\xi)$, $t_1 \leq t < t_2$, $\xi \in \mathbb{R}^d$. Then we obtain for $z_1^n(t,\cdot) - z_2^n(t,\cdot)$, $t_1 \leq t < t_2$, from (19.30)

$$z_1^n(t,\cdot) - z_2^n(t,\cdot) = \big(\phi_1(0,\cdot) - \phi_2(0,\cdot)\big)$$
$$+ \int_{\mathbb{R}^d} \big(b_1(0,\cdot,\phi_1(-r,\xi),\xi) - b_2(0,\cdot,\phi_2(-r,\xi),\xi)\big)d\xi$$
$$+ \int_{\mathbb{R}^d} \big(b_2(t,\cdot,z_2^n(t-r,\xi),\xi) - b_2(t,\cdot,z_1^n(t-r,\xi),\xi)\big)d\xi$$

$$+\int_{\mathbb{R}^d} \big(b_2(t,\,\cdot\,, z_1^n(t-r,\xi),\xi) - b_1(t,\,\cdot\,, z_1^n(t-r,\xi),\xi)\big)d\xi$$

$$+\int_0^{t_1} A_N\big(v_1^n(s,\,\cdot\,) - v_2^n(s,\,\cdot\,)\big)ds$$

$$+\int_0^{t_1} \big(f_1(s, v_1^n(s-r,\,\cdot\,),\,\cdot\,) - f_2(s, v_1^n(s-r,\,\cdot\,),\,\cdot\,)\big)ds$$

$$+\int_0^{t_1} \big(f_2(s, v_1^n(s-r,\,\cdot\,),\,\cdot\,) - f_2(s, v_2^n(s-r,\,\cdot\,),\,\cdot\,)\big)ds.$$

Since S_N supposed to be positivity preserving, the equality above can be rewritten in the following manner

$$z_1^n(t,\,\cdot\,) - z_2^n(t,\,\cdot\,) = S_N(t)\big(\phi_1(0,\,\cdot\,) - \phi_2(0,\,\cdot\,)\big)$$

$$+ S_N(t)\int_{\mathbb{R}^d} \big(b_2(t,\,\cdot\,, z_2^n(t-r,\xi),\xi) - b_2(t,\,\cdot\,, z_1^n(t-r,\xi),\xi)\big)d\xi$$

$$+ S_N(t)\int_{\mathbb{R}^d} \big(b_2(t,\,\cdot\,, z_1^n(t-r,\xi),\xi) - b_1(t,\,\cdot\,, z_1^n(t-r,\xi),\xi)\big)d\xi$$

$$+\int_0^{t_1} S_N(t-s)\big(f_1(s, v_1^n(s-r,\,\cdot\,),\,\cdot\,) - f_2(s, v_1^n(s-r,\,\cdot\,),\,\cdot\,)\big)ds$$

$$+\int_0^{t_1} S_N(t-s)\big(f_2(s, v_1^n(s-r,\,\cdot\,),\,\cdot\,) - f_2(s, v_2^n(s-r,\,\cdot\,),\,\cdot\,)\big)ds \geq 0.$$

The last inequality holds because of the conditions of the theorem.

Hence it follows that

$$z_1^n(t,\,\cdot\,) \geq z_2^n(t,\,\cdot\,),\, t_1 \leq t < t_2. \tag{19.36}$$

Now let us prove (19.33) for $t_1 \le t \le t_2$. Since estimate for $t = t_1$ follows from (19.35), we will show it for $t_1 < t \le t_2$. We derive from (19.31)

$$
\begin{aligned}
v_1^n(t,\cdot) - v_2^n(t,\cdot) &= \big(\phi_1(0,\cdot) - \phi_2(0,\cdot)\big) \\
&+ \int_{\mathbb{R}^d} \big(b_1(0,\cdot,\phi_1(-r,\xi),\xi) - b_2(0,\cdot,\phi_2(-r,\xi),\xi)\big)d\xi \\
&+ \int_{\mathbb{R}^d} \big(b_2(t_2,\cdot,z_2^n(t_2-r,\xi),\xi) - b_2(t_2,\cdot,z_1^n(t_2-r,\xi),\xi)\big)d\xi \\
&+ \int_{\mathbb{R}^d} \big(b_2(t_2,\cdot,z_1^n(t_2-r,\xi),\xi) - b_1(t_2,\cdot,z_1^n(t_2-r,\xi),\xi)\big)d\xi \\
&+ \int_0^t \big(f_1(s,v_1^n(s-r,\cdot),\cdot) - f_2(s,v_1^n(s-r,\cdot),\cdot)\big)ds \\
&+ \int_0^t \big(f_2(s,v_1^n(s-r,\cdot),\cdot) - f_2(s,v_2^n(s-r,\cdot),\cdot)\big)ds.
\end{aligned}
$$

Since S_N supposed to be positivity preserving, the equality above can be rewritten as

$$
\begin{aligned}
v_1^n(t,\cdot) - v_2^n(t,\cdot) &= S_N(t)\big(\phi_1(0,\cdot) - \phi_2(0,\cdot)\big) \\
&+ S_N(t)\int_{\mathbb{R}^d} \big(b_2(t_2,\cdot,z_2^n(t_2-r,\xi),\xi) - b_2(t_2,\cdot,z_1^n(t_2-r,\xi),\xi)\big)d\xi \\
&+ S_N(t)\int_{\mathbb{R}^d} \big(b_2(t_2,\cdot,z_1^n(t_2-r,\xi),\xi) - b_1(t_2,\cdot,z_1^n(t_2-r,\xi),\xi)\big)d\xi \\
&+ \int_0^t S_N(t-s)\big(f_1(s,v_1^n(s-r,\cdot),\cdot) - f_2(s,v_1^n(s-r,\cdot),\cdot)\big)ds \\
&+ \int_0^t S_N(t-s)\big(f_2(s,v_1^n(s-r,\cdot),\cdot) - f_2(s,v_2^n(s-r,\cdot),\cdot)\big)ds \ge 0.
\end{aligned}
$$

Hence,

$$
v_1^n(t,\cdot) \ge v_2^n(t,\cdot),\, t_1 < t \le t_2.
$$

Thus, (19.33) is proved for $t_1 \le t \le t_2$.

It remains to show that $z_1^n(t_2, \cdot) \geq z_2^n(t_2, \cdot)$. Since

$$v_i^n(t_2, \cdot) = z_i^n(t_2, \cdot), i \in \{1, 2\},$$

then

$$z_1^n(t_2, \cdot) \geq z_2^n(t_2, \cdot).$$

The inequality above and estimate (19.36) give (19.32) for $t_1 \leq t \leq t_2$.

5. We will prove here that there exists $C_n > 0$ such that

$$\sup_{0\leq t\leq T} \mathbf{E}\|z_i^n(t, \cdot)\|^2_{L_2(\mathbb{R}^d)} \leq C_n, i \in \{1, 2\}, \tag{19.37}$$

$$\sup_{0\leq t\leq T} \mathbf{E}\|v_i^n(t, \cdot)\|^2_{L_2(\mathbb{R}^d)} \leq C_n, i \in \{1, 2\}, \tag{19.38}$$

where v_i^n, $i \in \{1, 2\}$, are defined from (19.31), z_i^n, $i \in \{1, 2\}$, – from (19.30). In order to prove (19.37) it is sufficient to show that

$$\sup_{t_j\leq t\leq t_{j+1}} \mathbf{E}\|z_i^{j,n}(t, \cdot)\|^2_{L_2(\mathbb{R}^d)} \leq c_n, j \in \{0, \dots, n-1\}, i \in \{1, 2\}, \tag{19.39}$$

for some $c_n > 0$. It is sufficient to prove (19.39) for $j \in \{0, 1\}$, because for $j \in$ $\in \{2, \dots, n-1\}$ the proof is similar.

5.1. Let us estimate $\sup_{0\leq t\leq t_1} \mathbf{E}\|z_i^{0,n}(t, \cdot)\|^2_{L_2(\mathbb{R}^d)}$, $i \in \{1, 2\}$. From (19.23) we derive

$$\begin{aligned}
\sup_{0\leq t\leq t_1} \mathbf{E}\|z_i^{0,n}(t, \cdot)\|^2_{L_2(\mathbb{R}^d)} &\leq 4\mathbf{E}\|\phi_i(0, \cdot)\|^2_{L_2(\mathbb{R}^d)} \\
&+ 4\mathbf{E}\left\| \int_{\mathbb{R}^d} |b_i(0, \cdot, \phi_i(-r, \xi), \xi)|d\xi \right\|^2_{L_2(\mathbb{R}^d)} \\
&+ 4 \sup_{0\leq t\leq t_1} \mathbf{E}\left\| \int_{\mathbb{R}^d} |b_i(t, \cdot, \phi_i(t-r, \xi), \xi)|d\xi \right\|^2_{L_2(\mathbb{R}^d)} \\
&+ 4 \sup_{0\leq t\leq t_1} \mathbf{E}\left\| \int_0^t \sigma(s, \cdot)dW_M(s, \cdot) \right\|^2_{L_2(\mathbb{R}^d)} \\
&= 4\mathbf{E}\|\phi_i(0, \cdot)\|^2_{L_2(\mathbb{R}^d)} + \sum_{j=1}^{3} S_j^{(Z^0)}, i \in \{1, 2\}.
\end{aligned} \tag{19.40}$$

Let us now estimate each of $S_j^{(Z^0)}$, $j \in \{1, 2, 3\}$, separately. We have, taking into account conditions of the theorem,

$$S_1^{(Z^0)} = 4\mathbf{E}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|b(0, x, \phi_i(-r, \xi), \xi)|d\xi\right)^2 dx$$

$$\leq 8\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(0, x, \xi)d\xi dx\right)\mathbf{E}\|\phi_i(-r, \cdot)\|^2_{L_2(\mathbb{R}^d)}$$

$$+ 8\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\chi(x, \xi)d\xi\right)^2 dx, i \in \{1, 2\},$$

$$S_2^{(Z^0)} = 4\sup_{0\leq t\leq t_1}\mathbf{E}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|b_i(t, x, \phi_i(t - r, \xi), \xi)|d\xi\right)^2 dx$$

$$\leq 8\left(\sup_{0\leq t\leq T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(t, x, \xi)d\xi dx\right)$$

$$\times \sup_{-r\leq t\leq 0}\mathbf{E}\|\phi_i(t, \cdot)\|^2_{L_2(\mathbb{R}^d)} + 8\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\chi(x, \xi)d\xi\right)^2 dx, i \in \{1, 2\},$$

$$S_3^{(Z^0)} = 4\sup_{0\leq t\leq t_1}\mathbf{E}\int_{\mathbb{R}^d}\left(\sum_{j=1}^{M}\sqrt{\lambda_j}\left(\int_0^t\sigma(s, x)d\beta_j(s)\right)e_j(x)\right)^2 dx$$

$$\leq 4M\int_{\mathbb{R}^d}\left(\sum_{j=1}^{M}\lambda_j\int_0^{t_1}\sigma^2(s, x)ds\right)dx$$

$$\leq 4M\left(\sum_{j=1}^{M}\lambda_j\right)\frac{T}{n}\sup_{0\leq t\leq T}\|\sigma(t, \cdot)\|^2_{L_2(\mathbb{R}^d)}.$$

With the help of these estimates we get from (19.40)

$$\sup_{0\leq t\leq t_1}\mathbf{E}\|z_i^{0,n}(t, \cdot)\|^2_{L_2(\mathbb{R}^d)}\leq 4\mathbf{E}\|\phi_i(0, \cdot)\|^2_{L_2(\mathbb{R}^d)}+8\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(0, x, \xi)d\xi dx\right)$$

$$\times \mathbf{E}\|\phi_i(-r, \cdot)\|^2_{L_2(\mathbb{R}^d)} + 16\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\chi(x, \xi)d\xi\right)^2 dx$$

$$+ 8\Big(\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\Big)\sup_{-r\le t\le 0}\mathbf{E}\|\phi_i(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}$$

$$+ 4M\Big(\sum_{j=1}^{M}\lambda_j\Big)\frac{T}{n}\sup_{0\le t\le T}\|\sigma(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} = C_n^{(Z^0)},\, i\in\{1,2\}. \tag{19.41}$$

5.2. Let us estimate $\mathbf{E}\|v_i^{0,n}(t_1,\cdot)\|^2_{L_2(\mathbb{R}^d)}$, $i \in \{1,2\}$. We derive from (19.24), taking into account (19.41),

$$\mathbf{E}\|v_i^{0,n}(t_1,\cdot)\|^2_{L_2(\mathbb{R}^d)} \le 3\mathbf{E}\|z_i^{0,n}(t_1,\cdot)\|^2_{L_2(\mathbb{R}^d)} + 3\mathbf{E}\Big\|\int_0^{t_1} A_N v_i^{0,n}(s,\cdot)ds\Big\|^2_{L_2(\mathbb{R}^d)} ds$$

$$+ 3\mathbf{E}\Big\|\int_0^{t_1}|f_i(s,\phi_i(s-r,\cdot),\cdot)|ds\Big\|^2_{L_2(\mathbb{R}^d)}$$

$$\le 3\big(C_n^{(Z^0)} + S_1^{(V^0)} + S_2^{(V^0)}\big),\, i\in\{1,2\}. \tag{19.42}$$

In order to estimate $S_1^{(V^0)}$ let us take (19.16) into account. We obtain

$$S_1^{(V^0)} = \mathbf{E}\int_{\mathbb{R}^d}\Big(\int_0^{t_1} A_N v_i^{0,n}(s,x)ds\Big)^2 dx \le \frac{T}{n}\mathbf{E}\int_0^{t_1}\|A_N v_i^{0,n}(s,\cdot)\|^2_{L_2(\mathbb{R}^d)}ds$$

$$\le \frac{C_N T}{n}\int_0^{t_1}\mathbf{E}\|v_i^{0,n}(s,\cdot)\|^2_{L_2(\mathbb{R}^d)},\, i\in\{1,2\}.$$

In order to estimate $S_2^{(V^0)}$ let us take (19.5) into account. We obtain

$$S_2^{(V^0)} = \mathbf{E}\int_{\mathbb{R}^d}\Big(\int_0^{t_1}|f_i(s,\phi_i(s-r,x),x)|ds\Big)^2 dx$$

$$\le \frac{2T}{n}\mathbf{E}\int_0^{t_1}\int_{\mathbb{R}^d}\big(\eta^2(s,x) + L^2\phi_i^2(s-r,x)\big)dxds$$

$$\le \frac{2T}{n}\left(T \sup_{0\le t\le T}\int_{\mathbb{R}^d}\eta^2(t,x)dx + L^2\int_{-r}^{t_1-r}\mathbf{E}\|\phi_i(s-r,\cdot)\|^2_{L_2(\mathbb{R}^d)}d(s-r)\right)$$

$$\le \frac{2T}{n}\left(T \sup_{0\le t\le T}\int_{\mathbb{R}^d}\eta^2(t,x)dx + rL^2\sup_{-r\le t\le 0}\mathbf{E}\|\phi_i(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}\right), i\in\{1,2\}.$$

With the help of the obtained estimates it follows from (19.42) and from Bellman-Gronwalls inequality

$$\mathbf{E}\|v_i^{0,n}(t_1,\cdot)\|^2_{L_2(\mathbb{R}^d)} \le \Bigg[3C_n^{(Z^0)} + \frac{6T}{n}\left(T\sup_{0\le t\le T}\int_{\mathbb{R}^d}\eta^2(t,x)dx + rL^2\sup_{-r\le t\le 0}\mathbf{E}\|\phi_i(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}\right)\Bigg] \times \exp\left\{\frac{3C_NT}{n}\cdot\frac{T}{n}\right\}, i\in\{1,2\}. \tag{19.43}$$

5.3. Let estimate $\sup_{t_1\le t\le t_2}\mathbf{E}\|z_i^{1,n}(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}$, $i \in \{1,2\}$. Applying (19.25) and (19.43), we conclude

$$\sup_{t_1\le t\le t_2}\mathbf{E}\|z_i^{1,n}(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}$$

$$\le 6\mathbf{E}\|v_i^{0,n}(t_1,\cdot)\|^2_{L_2(\mathbb{R}^d)} + 6\mathbf{E}\left\|\int_{\mathbb{R}^d}|b_i(t_1,\cdot,\phi_i(t_1-r,\xi),\xi)|d\xi\right\|^2_{L_2(\mathbb{R}^d)}$$

$$+2\sup_{t_1\le t\le t_2}\left\|\int_{\mathbb{R}^d}|b_i(t,\cdot,z_i^{1,n}(t-r,\xi),\xi)|d\xi\right\|^2_{L_2(\mathbb{R}^d)}$$

$$+6\sup_{t_1\le t\le t_2}\mathbf{E}\left\|\int_{t_1}^{t}\sigma(s,\cdot)dW_M(s,\cdot)\right\|^2_{L_2(\mathbb{R}^d)}$$

$$\le 6\Bigg[3C_n^{(Z^0)}+\frac{6T}{n}\left(T\sup_{0\le t\le T}\int_{\mathbb{R}^d}\eta^2(t,x)dx + rL^2\sup_{-r\le t\le 0}\mathbf{E}\|\phi_i(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}\right)\Bigg]$$

$$\times\exp\left\{\frac{3C_NT}{n}\cdot\frac{T}{n}\right\} + \sum_{j=1}^{3}S_j^{(Z^1)}, i\in\{1,2\}. \tag{19.44}$$

Let us estimate each of $S_j^{(Z^1)}$, $j \in \{1, 2, 3\}$, separately. We conclude

$$S_1^{(Z^1)} = 6\mathbf{E}\int\limits_{\mathbb{R}^d} \left(\int\limits_{\mathbb{R}^d} |b_i(t_1, x, \phi_i(t_1 - r, \xi), \xi)| d\xi \right)^2 dx$$

$$\leq 12\left(\sup_{0\leq t\leq T} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t, x, \xi) d\xi dx \right)$$

$$\times \sup_{-r\leq t\leq 0} \mathbf{E}\|\phi_i(t, \cdot)\|^2_{L_2(\mathbb{R}^d)} + 12\int\limits_{\mathbb{R}^d} \left(\int\limits_{\mathbb{R}^d} \chi(x, \xi) d\xi \right)^2 dx, i \in \{1, 2\},$$

$$S_2^{(Z^1)} = 2\sup_{t_1\leq t\leq t_2} \mathbf{E}\int\limits_{\mathbb{R}^d} \left(\int\limits_{\mathbb{R}^d} |b_i(t, x, z_i^{1,n}(t - r, \xi), \xi)| d\xi \right)^2 dx$$

$$\leq 4\left(\sup_{0\leq t\leq T} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t, x, \xi) d\xi dx \right)$$

$$\times \left(\sup_{-r\leq t\leq 0} \mathbf{E}\|\phi_i(t, \cdot)\|^2_{L_2(\mathbb{R}^d)} + \sup_{0\leq t\leq t_1} \mathbf{E}\|z_i^{0,n}(t, \cdot)\|^2_{L_2(\mathbb{R}^d)} \right)$$

$$+ 4\int\limits_{\mathbb{R}^d} \left(\int\limits_{\mathbb{R}^d} \chi(x, \xi) d\xi \right)^2 dx$$

$$\leq 4\left(\sup_{0\leq t\leq T} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t, x, \xi) d\xi dx \right)\left(\sup_{-r\leq t\leq 0} \mathbf{E}\|\phi_i(t, \cdot)\|^2_{L_2(\mathbb{R}^d)} + C_n^{(Z^0)} \right)$$

$$+ 4\int\limits_{\mathbb{R}^d} \left(\int\limits_{\mathbb{R}^d} \chi(x, \xi) d\xi \right)^2 dx, i \in \{1, 2\},$$

$$S_3^{(Z^1)} = 6\sup_{t_1\leq t\leq t_2} \mathbf{E}\int\limits_{\mathbb{R}^d} \left(\sum_{j=1}^{M} \sqrt{\lambda_j} \left(\int\limits_{t_1}^{t} \sigma(s, x) d\beta_j(s) \right) e_j(x) \right)^2 dx$$

$$\leq 6M\left(\sum_{j=1}^{M} \lambda_j \right) \frac{T}{n} \sup_{0\leq t\leq T} \|\sigma(t, \cdot)\|^2_{L_2(\mathbb{R}^d)}.$$

These three estimates and (19.44) finally give

$$\sup_{t_1\le t\le t_2} \mathbf{E}\|z_i^{1,n}(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} \le 6\Bigg[3C_n^{(Z^0)} + \frac{6T}{n}\Bigg(T\sup_{0\le t\le T}\int_{\mathbb{R}^d}\eta^2(t,x)dx$$

$$+ rL^2\sup_{-r\le t\le 0}\mathbf{E}\|\phi_i(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}\Bigg)\Bigg]$$

$$\times\exp\left\{\frac{3C_NT}{n}\cdot\frac{T}{n}\right\}+16\Bigg(\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\Bigg)\sup_{-r\le t\le 0}\mathbf{E}\|\phi_i(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}$$

$$+16\int_{\mathbb{R}^d}\Bigg(\int_{\mathbb{R}^d}\chi(x,\xi)d\xi\Bigg)^2dx + 4\Bigg(\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\Bigg)C_n^{(Z^0)}$$

$$+6M\Bigg(\sum_{j=1}^{M}\lambda_j\Bigg)\frac{T}{n}\sup_{0\le t\le T}\|\sigma(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} = C_n^{(Z^1)},\ i\in\{1,2\}.$$

Equation (19.39) and, obviously, (19.37) is proved. Equation (19.38) is proved in a similar way.

6. Next we will prove that there exists some $C^{(U)}(T)>0$ such that

$$\sup_{0\le t\le T}\mathbf{E}\|u_i(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} \le C^{(U)}(T),\ i\in\{1,2\}. \tag{19.45}$$

In order to do it we need to estimate $\sup_{0\le t\le T}\mathbf{E}\|u_i(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}$, $i\in\{1,2\}$, from (19.19). We have

$$\sup_{0\le t\le T}\mathbf{E}\|u_i(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} \le 10\mathbf{E}\|\phi_i(0,\cdot)\|^2_{L_2(\mathbb{R}^d)}$$

$$+10\mathbf{E}\Bigg\|\int_{\mathbb{R}^d}|b_i(0,\cdot,\phi_i(-r,\xi),\xi)|d\xi\Bigg\|^2_{L_2(\mathbb{R}^d)}$$

$$+10\sup_{0\le t\le T}\mathbf{E}\Bigg\|\int_0^t A_Nu_i(s,\cdot)ds\Bigg\|^2_{L_2(\mathbb{R}^d)}$$

$$+10\sup_{0\le t\le T}\mathbf{E}\Bigg\|\int_0^t |f_i(s,u_i(s-r,\cdot),\cdot)|ds\Bigg\|^2_{L_2(\mathbb{R}^d)}$$

$$+10\sup_{0\le t\le T}\mathbf{E}\Bigg\|\int_0^t \sigma(s,\cdot)dW_M(s,\cdot)\Bigg\|^2_{L_2(\mathbb{R}^d)}$$

$$+ 2 \sup_{0\le t\le T} \mathbf{E}\left\| \int_{\mathbb{R}^d} |b_i(t,\ \cdot, u_i(t-r,\xi),\xi)|d\xi \right\|^2_{L_2(\mathbb{R}^d)}$$

$$= 10\mathbf{E}\|\phi_i(0,\ \cdot)\|^2_{L_2(\mathbb{R}^d)} + \sum_{j=1}^{5} S_j^{(U)},\ i \in \{1,2\}. \tag{19.46}$$

Taking into account previous calculations, we obtain

$$S_1^{(U)} = 10\mathbf{E} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |b_i(0,x,\phi_i(-r,\xi),\xi)|d\xi \right)^2 dx$$

$$\le 20 \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(0,x,\xi)d\xi dx \right) \mathbf{E}\|\phi_i(-r,\ \cdot)\|^2_{L_2(\mathbb{R}^d)}$$

$$+ 20 \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \chi(x,\xi)d\xi \right)^2 dx,\ i \in \{1,2\},$$

$$S_2^{(U)} = 10 \sup_{0\le t\le T} \mathbf{E} \int_{\mathbb{R}^d} \left(\int_0^t A_N u_i(s,x)ds \right)^2 dx$$

$$\le 10 C_N T \int_0^T \sup_{0\le s\le t} \mathbf{E}\|u_i(s,\ \cdot)\|^2_{L_2(\mathbb{R}^d)} dt,\ i \in \{1,2\},$$

$$S_3^{(U)} = 10 \sup_{0\le t\le T} \mathbf{E} \int_{\mathbb{R}^d} \left(\int_0^t |f_i(s,u_i(s-r,x),x)|ds \right)^2 dx \le 20T \left(T \sup_{0\le t\le T} \int_{\mathbb{R}^d} \eta^2(t,x)dx \right.$$

$$\left. + rL^2 \sup_{-r\le t\le 0} \mathbf{E}\|\phi_i(t,\ \cdot)\|^2_{L_2(\mathbb{R}^d)} + L^2 \int_0^T \sup_{0\le s\le t} \mathbf{E}\|u_i(s,\ \cdot)\|^2_{L_2(\mathbb{R}^d)} dt \right),\ i \in \{1,2\},$$

$$S_4^{(U)} = 10 \sup_{0\le t\le T} \mathbf{E} \int_{\mathbb{R}^d} \left(\sum_{j=1}^{M} \sqrt{\lambda_j} \left(\int_0^t \sigma(s,x)d\beta_j(s) \right) e_j(x) \right)^2 dx$$

$$\le 10M \left(\sum_{j=1}^{M} \lambda_j \right) T \sup_{0\le t\le T} \|\sigma(t,\ \cdot)\|^2_{L_2(\mathbb{R}^d)},$$

$$S_5^{(U)} = 2 \sup_{0\le t\le T} \mathbf{E} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |b_i(t,x,u_i(t-r,\xi),\xi)|d\xi \right)^2 dx$$

$$\le 4 \left(\sup_{0\le t\le T} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx \right)$$

$$\times\left(\sup_{-r\le t\le 0}\mathbf{E}\|\phi_i(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}+\sup_{0\le t\le T}\mathbf{E}\|u_i(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}\right)$$

$$+4\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\chi(x,\xi)d\xi\right)^2dx,\ i\in\{1,2\}.$$

Put

$$\begin{aligned}c^{(U)}(T)&=10\mathbf{E}\|\phi_i(0,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}+20\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}l^2(0,x,\xi)d\xi dx\right)\mathbf{E}\|\phi_i(-r,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}\\&+24\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\chi(x,\xi)d\xi\right)^2dx+20T\left(T\sup_{0\le t\le T}\int_{\mathbb{R}^d}\eta^2(t,x)dx\right.\\&\left.+rL^2\sup_{-r\le t\le 0}\mathbf{E}\|\phi_i(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}\right)\\&+10M\left(\sum_{j=1}^{M}\lambda_j\right)T\sup_{0\le t\le T}\|\sigma(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}+4\left(\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx\right)\\&\times\sup_{-r\le t\le 0}\mathbf{E}\|\phi_i(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)},\ i\in\{1,2\}.\end{aligned}$$

Invoking Bellman-Gronwalls lemma and condition (19.3), we obtain from (19.46) estimate (19.45) of the form

$$\begin{aligned}\sup_{0\le t\le T}\mathbf{E}\|u_i(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}&\le\left(1-4\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx\right)^{-1}c^{(U)}(T)\\&\times\exp\left\{\left(1-4\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx\right)^{-1}\left(10C_NT+20L^2T\right)\cdot T\right\}=C^{(U)}(T),\end{aligned}$$

$i\in\{1,2\}$.

7. Due to (19.37), (19.38) and (19.45), there exists a constant $C_n>0$ such that

$$\begin{aligned}&\sup_{0\le t\le T}\mathbf{E}\|v_i^n(t,\,\cdot)-u_i(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}+\sup_{0\le t\le T}\mathbf{E}\|z_i^n(t,\,\cdot)-u_i(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}\\&\quad\le 2\sup_{0\le t\le T}\mathbf{E}\|v_i^n(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}+4\sup_{0\le t\le T}\mathbf{E}\|u_i(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}\\&\quad+2\sup_{0\le t\le T}\mathbf{E}\|z_i^n(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}\le C_n,\ i\in\{1,2\}.\end{aligned}\tag{19.47}$$

Now let us prove

$$\lim_{n\to\infty}\sup_{t_k\le t<t_{k+1}}\mathbf{E}\|z_i^n(t,\,\cdot)-u_i(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}=0,\,i\in\{1,2\}. \tag{19.48}$$

7.1. Let estimate $\mathbf{E}\|v_i^n(t,\,\cdot)-u_i(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}$, $t_k<t\le t_{k+1}$, $k\in\{0,\dots,n-1\}$, $i\in\{1,2\}$. We have

$$\begin{aligned}
&\mathbf{E}\|v_i^n(t,\,\cdot)-u_i(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}\le 2\mathbf{E}\Big\|\int_{\mathbb{R}^d}|b_i(t_{k+1},\,\cdot\,,u_i(t_{k+1}-r,\xi),\xi)\\
&-b_i(t_{k+1},\,\cdot\,,z_i^n(t_{k+1}-r,\xi),\xi)|d\xi\Big\|^2_{L_2(\mathbb{R}^d)}+10\mathbf{E}\Big\|\int_{\mathbb{R}^d}|b_i(t,\,\cdot\,,u_i(t-r,\xi),\xi)\\
&-b_i(t_{k+1},\,\cdot\,,u_i(t-r,\xi),\xi)|d\xi\Big\|^2_{L_2(\mathbb{R}^d)}+10\mathbf{E}\Big\|\int_{\mathbb{R}^d}|b_i(t_{k+1},\,\cdot\,,u_i(t-r,\xi),\xi)\\
&-b_i(t_{k+1},\,\cdot\,,u_i(t_{k+1}-r,\xi),\xi)|d\xi\Big\|^2_{L_2(\mathbb{R}^d)}+10\mathbf{E}\Big\|A_N\big(v_i^n(s,\,\cdot)-u_i(s,\,\cdot)\big)ds\Big\|^2_{L_2(\mathbb{R}^d)}\\
&+10\mathbf{E}\Big\|\int_0^t|f_i(s,v_i^n(s-r,\,\cdot),\,\cdot)-f_i(s,u_i(s-r,\,\cdot),\,\cdot)|ds\Big\|^2_{L_2(\mathbb{R}^d)}\\
&+10\mathbf{E}\Big\|\int_t^{t_{k+1}}\sigma(s,\,\cdot)dW_M(s,\,\cdot)\Big\|^2_{L_2(\mathbb{R}^d)}=\sum_{j=1}^6 S_j^{(V-U)},\,i\in\{1,2\}.
\end{aligned}\tag{19.49}$$

Let us estimate each of $S_j^{(V-U)}$, $j\in\{1,\dots,6\}$, from (19.49) separately. We conclude

$$\begin{aligned}
S_1^{(V-U)}&=2\mathbf{E}\int_{\mathbb{R}^d}\bigg(\int_{\mathbb{R}^d}|b_i(t_{k+1},x,u_i(t_{k+1}-r,\xi),\xi)-b_i(t_{k+1},x,z_i^n(t_{k+1}-r,\xi),\xi)|d\xi\bigg)^2dx\\
&\le 2\bigg(\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx\bigg)\mathbf{E}\|z_i^n(t_{k+1}-r,\,\cdot)-u_i(t_{k+1}-r,\,\cdot)\|^2_{L_2(\mathbb{R}^d)},\\
&i\in\{1,2\}.
\end{aligned}$$

In order to estimate $S_2^{(V-U)}$ we will use the unform continuity of b_i, $i\in\{1,2\}$, and Lebesgue's dominated convergence theorem. Finally we get

$$\begin{aligned}
S_2^{(V-U)}&=10\mathbf{E}\int_{\mathbb{R}^d}\bigg(\int_{\mathbb{R}^d}|b_i(t,x,u_i(t-r,\xi),\xi)-b_i(t_{k+1},x,u_i(t-r,\xi),\xi)|d\xi\bigg)^2dx\\
&=\epsilon_1(n),\,i\in\{1,2\},\ \lim_{n\to\infty}\epsilon_1(n)=0.
\end{aligned}$$

Taking into account continuity of u_i, $i \in \{1, 2\}$, we get for $S_3^{(V-U)}$

$$S_3^{(V-U)} = 10\mathbf{E}\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|b_i(t_{k+1}, x, u_i(t-r,\xi),\xi) - b_i(t_{k+1}, x, u_i(t_{k+1}-r,\xi),\xi)|d\xi\Big)^2 dx$$

$$\leq 10\Big(\sup_{0\leq t\leq T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(t, x, \xi)d\xi dx\Big)\mathbf{E}\|u_i(t-r,\,\cdot)$$

$$- u_i(t_{k+1}-r,\,\cdot)\|^2_{L_2(\mathbb{R}^d)} = \epsilon_2(n),$$

$$i \in \{1, 2\},\ \lim_{n\to\infty}\epsilon_2(n) = 0.$$

Considering $S_j^{(V-U)}$, $j \in \{4, 5, 6\}$, let us take into account proceeding analysis. We conclude

$$S_4^{(V-U)} = 10\mathbf{E}\int_{\mathbb{R}^d}\Big(\int_0^t A_N\big(u_i(s, x) - v_i^n(s, x)\big)ds\Big)^2 dx$$

$$\leq 10 C_N T\int_0^t \mathbf{E}\|u_i(s,\,\cdot) - v_i^n(s,\,\cdot)\|^2_{L_2(\mathbb{R}^d)} ds,\ i \in \{1, 2\},$$

$$S_5^{(V-U)} = 10\mathbf{E}\int_{\mathbb{R}^d}\Big(\int_0^t |f_i(s, v_i^n(s-r, x), x) - f_i(s, u_i(s-r, x), x)|ds\Big)^2 dx$$

$$\leq 10 L^2 T\int_{-r}^{t-r}\mathbf{E}\|u_i(s-r,\,\cdot) - v_i^n(s-r,\,\cdot)\|^2_{L_2(\mathbb{R}^d)} d(s-r)$$

$$\leq 10 L^2 T\int_0^t \mathbf{E}\|u_i(s,\,\cdot) - v_i^n(s,\,\cdot)\|^2_{L_2(\mathbb{R}^d)} ds,\ i \in \{1, 2\},$$

$$S_6^{(V-U)} = 10\mathbf{E}\int_{\mathbb{R}^d}\Big(\sum_{j=1}^M\sqrt{\lambda_j}\Big(\int_t^{t_{k+1}}\sigma(s, x)d\beta_j(s)\Big)e_j(x)\Big)^2 dx$$

$$\leq 10M\Big(\sum_{j=1}^M\lambda_j\Big)\frac{T}{n}\sup_{0\leq t\leq T}\|\sigma(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}.$$

Taking into account obtained estimates, we apply Bellman-Gronwalls inequality to (19.49) (it is applicated due to (19.47)) and, summing up, obtain the following estimate

$$\sup_{t_k<t\le t_{k+1}} \mathbf{E}\|v_i^n(t,\,\cdot) - u_i(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)} \le \beta_n^{(V-U)}(t_{k+1}) \\ \times \exp\{10\big(C_N + L^2\big)T^2\},\ i \in \{1,2\}, \tag{19.50}$$

with

$$\beta_n^{(V-U)}(t_{k+1}) = \epsilon_3(n) + 2\Bigg(\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\Bigg)\mathbf{E}\|z_i^n(t_{k+1}-r,\,\cdot) \\ - u_i(t_{k+1}-r,\,\cdot)\|^2_{L_2(\mathbb{R}^d)} \\ + 10M\Bigg(\sum_{j=1}^{M}\lambda_j\Bigg)\frac{T}{n}\sup_{0\le t\le T}\|\sigma(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)},\ i\in\{1,2\}, \\ \epsilon_3(n) = \min\{\epsilon_1(n), \epsilon_2(n)\}.$$

7.2. Now let us estimate $\sup_{t_k\le t<t_{k+1}} \mathbf{E}\|z_i^n(t,\,\cdot) - u_i(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)},\ k\in\{0,\dots,n-1\}$, $i\in\{1,2\}$. Since we have for difference

$$v_i^n(t_k,\,\cdot) - u_i(t_k,\,\cdot) = -\int\limits_{\mathbb{R}^d} b_i(t_k,\,\cdot,z_i^n(t_k-r,\xi),\xi)d\xi \\ + \int\limits_{\mathbb{R}^d} b_i(t_k,\,\cdot,u_i(t_k-r,\xi),\xi)d\xi \\ + \int\limits_0^{t_k}\big(A_N v_i^n(s,\,\cdot) + f_i(s,v_i^n(s-r,\,\cdot),\,\cdot)\big)ds - \int\limits_0^{t_k}\big(A_N u_i(s,\,\cdot) \\ + f_i(s,u_i(s-r,\,\cdot),\,\cdot)\big)ds, \\ i\in\{1,2\},$$

we observe

$$\sup_{t_k\le t<t_{k+1}} \mathbf{E}\|z_i^n(t,\,\cdot) - u_i(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)} = \sup_{t_k\le t<t_{k+1}} \mathbf{E}\Bigg\|v_i^n(t_k,\,\cdot) - u_i(t_k,\,\cdot) \\ + \int\limits_{\mathbb{R}^d}\big(b_i(t,\,\cdot,u_i(t-r,\xi),\xi) - b_i(t_k,\,\cdot,u_i(t_k-r,\xi),\xi)\big)d\xi$$

$$+\int_{\mathbb{R}^d}\big(b_i(t_k,\cdot,z_i^n(t_k-r,\xi),\xi)-b_i(t,\cdot,z_i^n(t-r,\xi),\xi)\big)d\xi$$

$$-\int_{t_k}^{t}\big(A_N u_i(s,\cdot)+f_i(s,u_i(s-r,\cdot),\cdot)\big)ds\Big\|^2_{L_2(\mathbb{R}^d)} \le 2\mathbf{E}\|v_i^n(t_k,\cdot)-u_i(t_k,\cdot)\|^2_{L_2(\mathbb{R}^d)}$$

$$+8\sup_{t_k\le t<t_{k+1}}\mathbf{E}\Big\|\int_{\mathbb{R}^d}|b_i(t,\cdot,u_i(t-r,\xi),\xi)-b_i(t_k,\cdot,u_i(t_k-r,\xi),\xi)|d\xi\Big\|^2_{L_2(\mathbb{R}^d)}$$

$$+8\sup_{t_k\le t<t_{k+1}}\mathbf{E}\Big\|\int_{\mathbb{R}^d}|b_i(t_k,\cdot,z_i^n(t_k-r,\xi),\xi)-b_i(t,\cdot,z_i^n(t-r,\xi),\xi)|d\xi\Big\|^2_{L_2(\mathbb{R}^d)}$$

$$+8\sup_{t_k\le t<t_{k+1}}\mathbf{E}\Big\|\int_{t_k}^{t}A_N u_i(s,\cdot)ds\Big\|^2_{L_2(\mathbb{R}^d)}+8\sup_{t_k\le t<t_{k+1}}\mathbf{E}\Big\|\int_{t_k}^{t}|f_i(s,u_i(s-r,\cdot),\cdot)|ds\Big\|^2_{L_2(\mathbb{R}^d)}$$

$$=2\mathbf{E}\|v_i^n(t_k,\cdot)-u_i(t_k,\cdot)\|^2_{L_2(\mathbb{R}^d)}+8\sum_{j=1}^{4}S_j^{(Z-U)},\ i\in\{1,2\}. \tag{19.51}$$

Let us estimate each of $S_j^{(Z-U)}$, $j\in\{1,\dots,4\}$, from (19.51) separately. Estimating as before, we obtain

$$\begin{aligned}
S_1^{(Z-U)}&=\sup_{t_k\le t<t_{k+1}}\mathbf{E}\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|b_i(t,\cdot,u_i(t-r,\xi),\xi)-b_i(t_k,\cdot,u_i(t_k-r,\xi),\xi)|d\xi\Big)^2dx\\
&\le 2\sup_{t_k\le t<t_{k+1}}\mathbf{E}\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|b_i(t,\cdot,u_i(t-r,\xi),\xi)-b_i(t,\cdot,u_i(t_k-r,\xi),\xi)|d\xi\Big)^2dx\\
&+2\sup_{t_k\le t<t_{k+1}}\mathbf{E}\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|b_i(t,\cdot,u_i(t_k-r,\xi),\xi)-b_i(t_k,\cdot,u_i(t_k-r,\xi),\xi)|d\xi\Big)^2dx\\
&=\epsilon_4(n),\ i\in\{1,2\},\ \lim_{n\to\infty}\epsilon_4(n)=0,
\end{aligned} \tag{19.52}$$

$$\begin{aligned}
S_2^{(Z-U)}&=\sup_{t_k\le t<t_{k+1}}\mathbf{E}\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|b_i(t_k,x,z_i^n(t_k-r,\xi),\xi)-b_i(t,x,z_i^n(t-r,\xi),\xi)|d\xi\Big)^2dx\\
&\le 2\sup_{t_k\le t<t_{k+1}}\mathbf{E}\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|b_i(t_k,x,z_i^n(t_k-r,\xi),\xi)-b_i(t_k,x,z_i^n(t-r,\xi),\xi)|d\xi\Big)^2dx\\
&+2\sup_{t_k\le t<t_{k+1}}\mathbf{E}\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|b_i(t_k,x,z_i^n(t-r,\xi),\xi)-b_i(t,x,z_i^n(t-r,\xi),\xi)|d\xi\Big)^2dx\\
&=\epsilon_5(n),\ i\in\{1,2\},\ \lim_{n\to\infty}\epsilon_5(n)=0,
\end{aligned} \tag{19.53}$$

$$S_3^{(Z-U)} = \sup_{t_k \le t < t_{k+1}} \mathbf{E} \int_{\mathbb{R}^d} \left(\int_{t_k}^{t} A_N u_i(s,x) ds \right)^2 dx \le \frac{C_N T^2}{n} \sup_{0 \le t < T} \mathbf{E} \| u_i(t,\cdot)\|^2_{L_2(\mathbb{R}^d)},$$
$$i \in \{1,2\}, \tag{19.54}$$

$$S_4^{(Z-U)} = \sup_{t_k \le t < t_{k+1}} \mathbf{E} \int_{\mathbb{R}^d} \left(\int_{t_k}^{t} |f_i(s, u_i(s-r,x),x)| ds \right)^2 dx \le \frac{2T}{n} \Big(T \sup_{0 \le t \le T} \int_{\mathbb{R}^d} \eta^2(t,x) dx$$
$$+ rL^2 \sup_{-r \le t \le 0} \mathbf{E} \|\phi_i(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} + L^2 T \sup_{0 \le t \le T} \mathbf{E} \|u_i(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} \Big),$$
$$i \in \{1,2\}. \tag{19.55}$$

It follows from (19.50)

$$\mathbf{E}\|v_i^n(t_k,\cdot) - u_i(t_k,\cdot)\|^2_{L_2(\mathbb{R}^d)} \le \sup_{t_{k-1} < t \le t_k} \mathbf{E}\|v_i^n(t,\cdot) - u_i(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}$$
$$\le \beta_n^{(V-U)}(t_k) \cdot \exp\{10\big(C_N + L^2\big)T^2\}, i \in \{1,2\}, \tag{19.56}$$

with

$$\beta_n^{(V-U)}(t_k) = \epsilon_3(n) + 2\Big(\sup_{0 \le t \le T} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} l^2(t,x,\xi) d\xi dx \Big) \mathbf{E}\|z_i^n(t_k - r,\cdot)$$
$$- u_i(t_k - r,\cdot)\|^2_{L_2(\mathbb{R}^d)}$$
$$+ 10M \Big(\sum_{j=1}^{M} \lambda_j \Big) \frac{T}{n} \sup_{0 \le t \le T} \|\sigma(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} \le \epsilon_3(n)$$
$$+ 2\Big(\sup_{0 \le t \le T} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} l^2(t,x,\xi) d\xi dx \Big)$$
$$\times \sup_{t_k \le t < t_{k+1}} \mathbf{E}\|z_i^n(t,\cdot) - u_i(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}$$
$$+ 10M \Big(\sum_{j=1}^{M} \lambda_j \Big) \frac{T}{n} \sup_{0 \le t \le T} \|\sigma(t,\cdot)\|^2_{L_2(\mathbb{R}^d)},$$
$$i \in \{1,2\}.$$

Thus, we finally obtain from (19.51), using estimates (19.52)–(19.56),

$$\sup_{t_k \le t < t_{k+1}} \mathbf{E}\|z_i^n(t,\,\cdot) - u_i(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)} \le C_n^{(Z-U)}(T)$$

$$\times \left(1 - 4\exp\{10\big(C_N + L^2\big)T^2\}\left(\sup_{0\le t\le T} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)\right)^{-1},$$

$$i \in \{1,2\}, \tag{19.57}$$

with

$$C_n^{(Z-U)}(T) = \epsilon(n) + \frac{8C_N T^2}{n} \sup_{0\le t<T} \mathbf{E}\|u_i(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}$$

$$+ 20M\left(\sum_{j=1}^{M}\lambda_j\right)\frac{T}{n}\sup_{0\le t\le T}\|\sigma(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}$$

$$\times \exp\{10\big(C_N + L^2\big)T^2\} + \frac{16T}{n}\Bigg(T\sup_{0\le t\le T}\int_{\mathbb{R}^d}\eta^2(t,x)dx$$

$$+ rL^2 \sup_{-r\le t\le 0} \mathbf{E}\|\phi_i(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}$$

$$+ L^2T\sup_{0\le t\le T}\mathbf{E}\|u_i(t,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}\Bigg),\ i\in\{1,2\},$$

$$\epsilon(n) = \min\{2\exp\{10\big(C_N+L^2\big)T^2\}\epsilon_3(n), \epsilon_4(n), \epsilon_5(n)\},\ \lim_{n\to\infty}\epsilon(n) = 0.$$

Since $\lim_{n\to\infty} C_n^{(Z-U)}(T) = 0$, then (19.57) clearly implies (19.48).

8. For any $0 \le t \le T$ a sequence $\{z_i^n(t,\,\cdot), n \in \{1,2,\ldots\}\}$, $i \in \{1,2\}$, contains a subsequence $\{z_i^{n_m}(t,\,\cdot), m \in \{1,2,\ldots\}\}$, $i \in \{1,2\}$, converging to $u_i(t,\,\cdot)$, $i \in \{1,2\}$, in $L_2(\mathbb{R}^d))$ almost surely. This implies

$$u_1(t,\,\cdot) \ge u_2(t,\,\cdot)$$

almost surely for $0 \le t \le T$.

9. Denote $u \equiv u_i$, $\phi \equiv \phi_i$, $b \equiv b_i$, $f \equiv f_i$, $i \in \{1, 2\}$, for brevity. Let $u^M : [-r, T] \times \Omega \to L_2(\mathbb{R}^d)$ be a continuous $\mathscr{F}_t$-measurable for almost all $-r \le t \le T$ process, defined as a unique solution to the following integral equation

$$u^M(t,\cdot) = S(t)\Big(\phi(0,\cdot) + \int_{\mathbb{R}^d} b(0,\cdot,\phi(-r,\xi),\xi)d\xi\Big) - \int_0^t AS(t-s)\Big(\int_{\mathbb{R}^d} b(s,\cdot,u^M(s-r,\xi),\xi)d\xi\Big)ds + \int_0^t S(t-s)f(s,u^M(s-r,\cdot),\cdot)ds + \int_0^t \sigma(s,\cdot)dW_M(s,\cdot),$$

$$0 < t \le T, \tag{19.58}$$

$$u^M(t,\cdot) = \phi(t,\cdot),\ -r \le t \le 0, r > 0, \tag{51*}$$

satisfying the condition

$$\mathbf{E}\int_0^T \|u^M(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} dt < \infty.$$

It remains to show that

$$\lim_{N\to\infty} \sup_{0\le t\le T} \mathbf{E}\|u_i^{N,M}(t,\cdot) - u_i^M(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} = 0, i \in \{1, 2\}, \tag{19.59}$$

$$\lim_{M\to\infty} \sup_{0\le t\le T} \mathbf{E}\|u_i^M(t,\cdot) - U_i(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} = 0, i \in \{1, 2\}, \tag{19.60}$$

where $U_i : [-r, T] \times \Omega \to L_2(\mathbb{R}^d)$, $i \in \{1, 2\}$, is a unique solution to (19.1).

9.1. Let us prove (19.59), where $u_i^{N,M}$, $i \in \{1, 2\}$, are defined from (19.19)–(19.20). In order to do it we will estimate $\sup_{0\le t\le T} \mathbf{E}\|u_i^{N,M}(t,\cdot) - u_i^M(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}$, $i \in \{1, 2\}$. We get

$$\sup_{0\le t\le T} \mathbf{E}\|u_i^{N,M}(t,\cdot) - u_i^M(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} \le 4 \sup_{0\le t\le T} \mathbf{E}\Big\|\big(S_N(t) - S(t)\big)\Big(\phi_i(0,\cdot) + \int_{\mathbb{R}^d} b_i(0,\cdot,\phi_i(-r,\xi),\xi)d\xi\Big)\Big\|^2_{L_2(\mathbb{R}^d)}$$

$$+4 \sup_{0\le t\le T} \mathbf{E}\Big\| \int_{\mathbb{R}^d} |b_i(t,\,\cdot\,, u_i^M(t-r,\xi),\xi) - b_i(t,\,\cdot\,, u_i^{N,M}(t-r,\xi),\xi)| d\xi \Big\|^2_{L_2(\mathbb{R}^d)}$$

$$+4 \sup_{0\le t\le T} \mathbf{E}\Big\| \int_0^t AS(t-s)\Big(\int_{\mathbb{R}^d} b_i(s,\,\cdot\,, u_i^M(s-r,\xi),\xi) d\xi\Big) ds$$

$$-\int_0^t A_N S_N(t-s)\Big(\int_{\mathbb{R}^d} b_i(s,\,\cdot\,, u_i^{N,M}(s-r,\xi),\xi) d\xi\Big) ds \Big\|^2_{L_2(\mathbb{R}^d)}$$

$$+4 \sup_{0\le t\le T} \mathbf{E}\Big\| \int_0^t \big(S_N(t-s) f_i(s, u^{N,M}(s-r,\,\cdot\,),\,\cdot\,)$$

$$-S(t-s) f_i(s, u^M(s-r,\,\cdot\,),\,\cdot\,)\big) ds \Big\|^2_{L_2(\mathbb{R}^d)} = \sum_{j=1}^4 S_j^{(U^N-U)},\ i\in\{1,2\}. \tag{19.61}$$

Let estimate $S_j^{(U^N-U)}$, $j\in\{1,\dots,4\}$, from (19.61) separately. Taking into account belonging $\phi_i(0,\cdot)+\int_{\mathbb{R}^d} b_i(0,\,\cdot\,,\phi_i(-r,\xi),\xi)d\xi$, $i\in\{1,2\}$, to $L_2(\mathbb{R}^d)$ and property (19.18), we conclude

$$S_1^{(U^N-U)} = 4 \sup_{0\le t\le T} \mathbf{E}\Big\| \big(S_N(t)-S(t)\big)\Big(\phi_i(0,\cdot) + \int_{\mathbb{R}^d} b_i(0,\,\cdot\,,\phi_i(-r,\xi),\xi) d\xi\Big)\Big\|^2_{L_2(\mathbb{R}^d)}$$

$$= \epsilon_1(N),\ i\in\{1,2\},\ \lim_{N\to\infty}\epsilon_1(N) = 0. \tag{19.62}$$

For $S_2^{(U^N-U)}$ we get, estimating as before,

$$S_2^{(U^N-U)} = 4 \sup_{0\le t\le T} \mathbf{E} \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} |b_i(t,x,u_i^M(t-r,\xi),\xi) - b_i(t,x,u_i^{N,M}(t-r,\xi),\xi)| d\xi\Big)^2 dx$$

$$\le 4\Big(\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\Big) \sup_{0\le t\le T} \mathbf{E}\|u_i^M(t-r,\,\cdot\,) - u_i^{N,M}(t-r,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)}$$

$$\le 4\Big(\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\Big) \sup_{0\le t\le T} \mathbf{E}\|u_i^M(t,\,\cdot\,) - u_i^{N,M}(t,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)},$$

$$i\in\{1,2\}. \tag{19.63}$$

For $S_3^{(U^N-U)}$ we conclude

$$\begin{aligned}
S_3^{(U^N-U)} &= 4 \sup_{0\le t\le T} \mathbf{E}\Big\| \int_0^t AS(t-s)\Big(\int_{\mathbb{R}^d} b_i(s,\,\cdot\,,u_i^M(s-r,\xi),\xi)d\xi\Big)ds \\
&- \int_0^t A_N S_N(t-s)\Big(\int_{\mathbb{R}^d} b_i(s,\,\cdot\,,u_i^M(s-r,\xi),\xi)d\xi\Big)ds\Big\|^2_{L_2(\mathbb{R}^d)} \\
&+ \int_0^t A_N S_N(t-s)\Big(\int_{\mathbb{R}^d} \big(b_i(s,\,\cdot\,,u_i^M(s-r,\xi),\xi) \\
&- b_i(s,\,\cdot\,,u_i^{N,M}(s-r,\xi),\xi)\big)d\xi\Big)ds\Big\|^2_{L_2(\mathbb{R}^d)} \\
&\le 8 \sup_{0\le t\le T} \mathbf{E}\Big\| \int_0^t AS(t-s)\Big(\int_{\mathbb{R}^d} b_i(s,\,\cdot\,,u_i^M(s-r,\xi),\xi)d\xi\Big)ds \\
&- \int_0^t A_N S_N(t-s)\Big(\int_{\mathbb{R}^d} b_i(s,\,\cdot\,,u_i^M(s-r,\xi),\xi)d\xi\Big)ds\Big\|^2_{L_2(\mathbb{R}^d)} \\
&+ 8 \sup_{0\le t\le T} \mathbf{E}\Big\| \int_0^t A_N S_N(t-s)\Big(\int_{\mathbb{R}^d} \big(b_i(s,\,\cdot\,,u_i^M(s-r,\xi),\xi) \\
&- b_i(s,\,\cdot\,,u_i^{N,M}(s-r,\xi),\xi)d\xi\big)\Big)ds\Big\|^2_{L_2(\mathbb{R}^d)}, i\in\{1,2\}.
\end{aligned} \tag{19.64}$$

For estimating the first term in (19.64) we will use Lemma 19.1. It follows from this lemma uniform in t convergence $z_N(t,\,\cdot\,)$ to $z(t,\,\cdot\,)$ as $N\to\infty$, where $z_N(t,\,\cdot\,) = S_N(t-s)\Big(\int_{\mathbb{R}^d} b_i(s,\,\cdot\,,u_i^M(s-r,\xi),\xi)d\xi\Big)$, $t\ge s$, $i\in\{1,2\}$, is a solution to the problem (19.13)–(19.14) of the form

$$\frac{\partial z_N(t,\,\cdot\,)}{\partial t} = A_N z_N(t,\,\cdot\,), t>s,$$

$$z_N(s,\,\cdot\,) = \int_{\mathbb{R}^d} b_i(s,\,\cdot\,,u_i^M(s-r,\xi),\xi)d\xi, i\in\{1,2\},$$

and $z(t,\cdot) = S(t-s)\Big(\int\limits_{\mathbb{R}^d} b_i(s,\cdot,u_i^M(s-r,\xi),\xi)d\xi\Big)$, $t \geq s$, $i \in \{1,2\}$, solves the problem (19.13)–(19.14) of the form

$$\frac{\partial z(t,\cdot)}{\partial t} = Az(t,\cdot), t > s,$$
$$z(s,\cdot) = \int\limits_{\mathbb{R}^d} b_i(s,\cdot,u_i^M(s-r,\xi),\xi)d\xi, i \in \{1,2\}.$$

Therefore we have from (19.64)

$$8 \sup_{0\leq t\leq T} \mathbf{E}\Big\| \int_0^t AS(t-s)\Big(\int\limits_{\mathbb{R}^d} b_i(s,\cdot,u_i^M(s-r,\xi),\xi)d\xi\Big)ds$$
$$- \int_0^t A_N S_N(t-s)\Big(\int\limits_{\mathbb{R}^d} b_i(s,\cdot,u_i^M(s-r,\xi),\xi)d\xi\Big)ds\Big\|^2_{L_2(\mathbb{R}^d)} = \epsilon_2(N),$$
$$i \in \{1,2\}, \ \lim_{N\to\infty} \epsilon_2(N) = 0.$$

For estimating the second term in (19.64) we will use Lemma 19.1, (19.15) from Theorem 19.4 and property (19.6). We conclude

$$8 \sup_{0\leq t\leq T} \mathbf{E}\Big\| \int_0^t A_N S_N(t-s)\Big(\int\limits_{\mathbb{R}^d} \big(b_i(s,\cdot,u_i^M(s-r,\xi),\xi)$$
$$- b_i(s,\cdot,u_i^{N,M}(s-r,\xi),\xi)\big)d\xi\Big)ds\Big\|^2_{L_2(\mathbb{R}^d)}$$
$$\leq 8 \sup_{0\leq t\leq T} \mathbf{E}\Big\| \int_0^t A_N\Big(\int\limits_{\mathbb{R}^d} \big(b_i(s,\cdot,u_i^M(s-r,\xi),\xi)$$
$$- b_i(s,\cdot,u_i^{N,M}(s-r,\xi),\xi)\big)d\xi\Big)ds\Big\|^2_{L_2(\mathbb{R}^d)}$$
$$\leq 8 \sup_{0\leq t\leq T} \mathbf{E}\int\limits_{\mathbb{R}^d}\Big(\int_0^t \Delta_x\Big(\int\limits_{\mathbb{R}^d} \big(b_i(s,x,u_i^M(s-r,\xi),\xi)$$
$$- b_i(s,x,u_i^{N,M}(s-r,\xi),\xi)\big)d\xi\Big)ds\Big)^2 dx$$

$$\leq 8T\mathbf{E}\int_0^T\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\|D_x^2 b_i(s,x,u_i^M(s-r,\xi),\xi)\right.$$

$$\left.-D_x^2 b_i(s,x,u_i^{N,M}(s-r,\xi),\xi)\|d\xi\right)^2 dxds$$

$$\leq 8T\mathbf{E}\int_0^T\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\psi(s,x,\xi)|u_i^M(s-r,\xi)-u_i^{N,M}(s-r,\xi)|d\xi\right)^2 dxds$$

$$\leq 8T\left(\sup_{0\leq t\leq T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi^2(t,x,\xi)d\xi dx\right)\int_{-r}^{T-r}\mathbf{E}\|u_i^M(s-r,\,\cdot)$$

$$-u_i^{N,M}(s-r,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}d(s-r)$$

$$\leq 8T\left(\sup_{0\leq t\leq T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi^2(t,x,\xi)d\xi dx\right)\int_0^T\sup_{0\leq s\leq t}\mathbf{E}\|u_i^M(s,\,\cdot)$$

$$-u_i^{N,M}(s,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}dt,$$

$i\in\{1,2\}$.

Thus we obtain from (19.64)

$$S_3^{(U^N-U)}\leq\epsilon_2(n)+8T\left(\sup_{0\leq t\leq T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi^2(t,x,\xi)d\xi dx\right)$$
$$\times\int_0^T\sup_{0\leq s\leq t}\mathbf{E}\|u_i^M(s,\,\cdot)-u_i^{N,M}(s,\,\cdot)\|^2_{L_2(\mathbb{R}^d)}dt,\ i\in\{1,2\}. \tag{19.65}$$

For estimating $S_4^{(U^N-U)}$ let apply (19.18) and belonging $\eta(t,\,\cdot)+|u_i^{N,M}(t-r,\,\cdot)|$,
$0\leq t\leq T$, $i\in\{1,2\}$, to $L_2(\mathbb{R}^d)$. We have

$$S_4^{(U^N-U)}=4\sup_{0\leq t\leq T}\mathbf{E}\int_{\mathbb{R}^d}\left(\int_0^t\big(S_N(t-s)-S(t-s)\right.$$
$$+S(t-s)\big)f_i(s,u_i^{N,M}(s-r,\,\cdot),\,\cdot)ds$$

$$- \int_0^t S(t-s) f_i(s, u_i^M(s-r,\,\cdot\,),\,\cdot\,)ds\Big)^2 dx$$

$$\leq 8 \sup_{0\leq t\leq T} \mathbf{E} \int_{\mathbb{R}^d} \Big(\int_0^t \big(S_N(t-s) - S(t-s)\big)|f_i(s, u_i^{N,M}(s-r,\,\cdot\,),\,\cdot\,)|ds\Big)^2 dx$$

$$+ 8 \sup_{0\leq t\leq T} \mathbf{E} \int_{\mathbb{R}^d} \Big(\int_0^t S(t-s)|f_i(s, u_i^{N,M}(s-r,x),x)$$

$$- f_i(s, u_i^M(s-r,x),x)|ds\Big)^2 dx$$

$$\leq 8 \sup_{0\leq t\leq T} \mathbf{E} \int_{\mathbb{R}^d} \Big(\int_0^t \big(S_N(t-s) - S(t-s)\big)\big(\eta(s,x)$$

$$+ L|u_i^{N,M}(s-r,x)|\big)ds\Big)^2 dx$$

$$+ 8L^2 \sup_{0\leq t\leq T} \mathbf{E} \int_{\mathbb{R}^d} \Big(\int_0^t S(t-s)|u_i^{N,M}(s-r,x) - u_i^M(s-r,x)|ds\Big)^2 dx$$

$$\leq 8T \sup_{0\leq t\leq T} \mathbf{E} \int_0^t \Big\| \big(S_N(t-s) - S(t-s)\big)\big(\eta(s,\,\cdot\,)$$

$$+ L|u_i^{N,M}(s-r,\,\cdot\,)|\big)\Big\|^2_{L_2(\mathbb{R}^d)} ds$$

$$+ 8L^2T \sup_{0\leq t\leq T} \mathbf{E} \int_0^t \Big\| S(t-s)\big(u_i^{N,M}(s-r,\,\cdot\,) - u_i^M(s-r,\,\cdot\,)\big)\Big\|^2_{L_2(\mathbb{R}^d)} ds$$

$$\leq 8T \sup_{0\leq t\leq T} \mathbf{E} \int_0^t \Big\| \big(S_N(t-s) - S(t-s)\big)\big(\eta(s,\,\cdot\,) + L|u_i^{N,M}(s-r,\,\cdot\,)|\big)\Big\|^2_{L_2(\mathbb{R}^d)} ds$$

$$+ 8L^2T \sup_{0\leq t\leq T} \int_{-r}^{T-r} \mathbf{E}\|u_i^{N,M}(s,\,\cdot\,) - u_i^M(s,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} ds \leq \epsilon_3(n)$$

$$+ 8L^2T \int_0^T \sup_{0\leq s\leq t} \mathbf{E}\|u_i^{N,M}(s,\,\cdot\,) - u_i^M(s,\,\cdot\,)\|^2_{L_2(\mathbb{R}^d)} dt, \ \lim_{N\to\infty} \epsilon_3(N) = 0. \tag{19.66}$$

Taking into account (19.62), (19.63), (19.65) and (19.66), we conclude

$$\sup_{0\le t\le T}\mathbf{E}\|u_i^{N,M}(t,\cdot)-u_i^M(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}\le\left(1-4\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx\right)^{-1}$$

$$\times\left[\epsilon(N)+8T\left(\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\psi^2(t,x,\xi)d\xi dx+L^2\right)\right.$$

$$\left.\times\int\limits_0^T\sup_{0\le s\le t}\mathbf{E}\|u_i^{N,M}(s,\cdot)-u_i^M(s,\cdot)\|^2_{L_2(\mathbb{R}^d)}dt\right],\ i\in\{1,2\},$$

$$\epsilon(N)=\min\{\epsilon_1(N),\epsilon_2(N),\epsilon_3(N)\},\ \lim_{N\to\infty}\epsilon(N)=0.$$

An application of Bellman-Gronwalls inequality yields

$$\sup_{0\le t\le T}\mathbf{E}\|u_i^{N,M}(t,\cdot)-u_i^M(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}\le\epsilon(N)\left(1-4\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx\right)^{-1}$$

$$\times\exp\left\{\left(1-4\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}l^2(t,x,\xi)d\xi dx\right)^{-1}\right.$$

$$\left.\times 8T\left(\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\psi^2(t,x,\xi)d\xi dx+L^2\right)\cdot T\right\},\ i\in\{1,2\}.\tag{19.67}$$

Since $\lim_{N\to\infty}\epsilon(N)=0$, then (19.67) obviously implies (19.59).

9.2. Finally let us prove (19.60). In order to do it we will estimate $\sup_{0\le t\le T}\mathbf{E}\|u_i^M(t,\cdot)-U_i(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}$, $i\in\{1,2\}$. We have

$$\sup_{0\le t\le T}\mathbf{E}\|u_i^M(t,\cdot)-U_i(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}$$

$$\le 4\sup_{0\le t\le T}\mathbf{E}\left\|\int\limits_{\mathbb{R}^d}|b_i(t,\cdot,U_i(t-r,\xi),\xi)-b_i(t,\cdot,u_i^M(t-r,\xi),\xi)|d\xi\right\|^2_{L_2(\mathbb{R}^d)}$$

$$+4\sup_{0\le t\le T}\mathbf{E}\left\|\int\limits_0^t A_N\left(\int\limits_{\mathbb{R}^d}\big(b_i(s,\cdot,u_i^M(s-r,\xi),\xi)\right.\right.$$

$$\left.\left.-b_i(s,\cdot,U_i(s-r,\xi),\xi)\big)d\xi\right)ds\right\|^2_{L_2(\mathbb{R}^d)}$$

$$+4\sup_{0\le t\le T}\mathbf{E}\left\|\int\limits_0^t S(t-s)|f_i(s,u_i^M(s-r,\cdot),\cdot)-f_i(s,U_i(s-r,\cdot),\cdot)|ds\right\|^2_{L_2(\mathbb{R}^d)}$$

$$+4\sup_{0\le t\le T}\mathbf{E}\left\|\int_0^t S(t-s)\sigma(s,\cdot)dW_M(s,\cdot)-\int_0^t S(t-s)\sigma(s,\cdot)dW(s,\cdot)\right\|^2_{L_2(\mathbb{R}^d)}$$

$$=\sum_{j=1}^4 S_j^{(u-U)},\ i\in\{1,2\}. \tag{19.68}$$

Using preceding calculations, we obtain

$$S_1^{(u-U)}=4\sup_{0\le t\le T}\mathbf{E}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|b_i(t,x,U_i(t-r,\xi),\xi)-b_i(t,x,u_i^M(t-r,\xi),\xi)|d\xi\right)^2dx$$

$$\le 4\left(\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)\sup_{0\le t\le T}\mathbf{E}\|U_i(t,\cdot)-u_i^M(t,\cdot)\|^2_{L_2(\mathbb{R}^d)},$$

$$i\in\{1,2\}, \tag{19.69}$$

$$S_2^{(u-U)}=4\sup_{0\le t\le T}\mathbf{E}\int_{\mathbb{R}^d}\left(\int_0^t \Delta_x S(t-s)\left(\int_{\mathbb{R}^d}\big(b_i(s,x,u_i^M(s-r,\xi),\xi)\right.\right.$$

$$\left.\left.-\,b_i(s,x,U_i(s-r,\xi),\xi)\big)d\xi\right)ds\right)^2dx$$

$$\le 4T\sup_{0\le t\le T}\mathbf{E}\int_0^t\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\|D_x^2 b_i(s,x,u_i^M(s-r,\xi),\xi)\right.$$

$$\left.-\,D_x^2 b_i(s,x,U_i(s-r,\xi),\xi)\|d\xi\right)^2dxds$$

$$\le 4T\left(\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi^2(t,x,\xi)d\xi dx\right)\int_0^T\sup_{0\le s\le t}\mathbf{E}\|u_i^M(s,\cdot)-U_i(s,\cdot)\|^2_{L_2(\mathbb{R}^d)}dt,$$

$$i\in\{1,2\}. \tag{19.70}$$

$$S_3^{(u-U)}=4\sup_{0\le t\le T}\mathbf{E}\int_{\mathbb{R}^d}\left(\int_0^t S(t-s)|f_i(s,u_i^M(s-r,x),x)-f_i(s,U_i(s-r,x),x)|ds\right)^2dx$$

$$\le 4L^2T\int_0^T\sup_{0\le s\le t}\mathbf{E}\|u_i^M(s,\cdot)-U_i(s,\cdot)\|^2_{L_2(\mathbb{R}^d)}dt,\ i\in\{1,2\}. \tag{19.71}$$

$$S_4^{(u-U)} = 4 \sup_{0\le t\le T} \mathbf{E} \int_{\mathbb{R}^d} \Bigg(\sum_{j=1}^{M} \sqrt{\lambda_j} \Bigg(\int_0^t S(t-s)\sigma(s,x)d\beta_j(s) \Bigg) e_j(x)$$

$$- \sum_{j=1}^{\infty} \sqrt{\lambda_j} \Bigg(\int_0^t S(t-s)\sigma(s,x)d\beta_j(s) \Bigg) e_j(x) \Bigg)^2 dx$$

$$= 4 \sup_{0\le t\le T} \mathbf{E} \int_{\mathbb{R}^d} \Bigg(\sum_{j=M+1}^{\infty} \sqrt{\lambda_j} \Bigg(\int_0^t S(t-s)\sigma(s,x)d\beta_j(s) \Bigg) e_j(x) \Bigg)^2 dx$$

$$\le 4 \Bigg(\sum_{j=M+1}^{\infty} \lambda_j \Bigg) \sup_{0\le t\le T} \mathbf{E} \int_{\mathbb{R}^d} \int_0^t \big(S(t-s)\sigma(s,x) \big)^2 ds dx$$

$$= 4 \Bigg(\sum_{j=M+1}^{\infty} \lambda_j \Bigg) \sup_{0\le t\le T} \mathbf{E} \int_0^t \| S(t-s)\sigma(s,\,\cdot\,) \|^2_{L_2(\mathbb{R}^d)} ds$$

$$\le 4T \Bigg(\sum_{j=M+1}^{\infty} \lambda_j \Bigg) \sup_{0\le t\le T} \mathbf{E} \| \sigma(t,\,\cdot\,) \|^2_{L_2(\mathbb{R}^d)}. \tag{19.72}$$

Estimates (19.69)–(19.72) with property (19.3) help us conclude from (19.68) that

$$\sup_{0\le t\le T} \mathbf{E} \| u_i^M(t,\,\cdot\,) - U_i(t,\,\cdot\,) \|^2_{L_2(\mathbb{R}^d)} \le \Bigg(1 - 4 \sup_{0\le t\le T} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} l^2(t,x,\xi) d\xi dx \Bigg)^{-1}$$

$$\times \Bigg(4T \Bigg(\sup_{0\le t\le T} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi^2(t,x,\xi) d\xi dx \Bigg) \int_0^T \sup_{0\le s\le t} \mathbf{E} \| u_i^M(s,\,\cdot\,) - U_i(s,\,\cdot\,) \|^2_{L_2(\mathbb{R}^d)} dt$$

$$+ 4L^2 T \int_0^T \sup_{0\le s\le t} \mathbf{E} \| u_i^M(s,\,\cdot\,) - U_i(s,\,\cdot\,) \|^2_{L_2(\mathbb{R}^d)} dt$$

$$+ 4T \Bigg(\sum_{j=M+1}^{\infty} \lambda_j \Bigg) \sup_{0\le t\le T} \mathbf{E} \| \sigma(t,\,\cdot\,) \|^2_{L_2(\mathbb{R}^d)} \Bigg),\ i \in \{1,2\}.$$

Finally Bellman-Gronwalls inequality leads to

$$\sup_{0\le t\le T} \mathbf{E}\|u_i^M(t,\cdot) - U_i(t,\cdot)\|^2_{L_2(\mathbb{R}^d)} \le \left(1 - 4\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)^{-1}$$
$$\times 4T\left(\sum_{j=1}^{M}\lambda_j\right)\sup_{0\le t\le T}\mathbf{E}\|\sigma(t,\cdot)\|^2_{L_2(\mathbb{R}^d)}\exp\left\{\left(1-4\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(t,x,\xi)d\xi dx\right)^{-1}\right.$$
$$\left.\times 4T\left(\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi^2(t,x,\xi)d\xi dx + L^2\right)\cdot T\right\},\ i\in\{1,2\}.$$

Recall that $\sum_{n=1}^{\infty}\lambda_n < \infty$. This fact along with estimate above implies (19.60), thereby completing the proof of the theorem. □

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