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EXISTENCE AND UNIQUENESS OF THE SOLUTION TO THE CAUCHY PROBLEM FOR THE STOCHASTIC REACTION-DIFFUSION DIFFERENTIAL EQUATION OF NEUTRAL TYPE

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We prove a theorem on the existence and uniqueness of a mild solution to the Cauchy problem for a stochastic differential equation of neutral type in the weighted Hilbert space.

1. Introduction

Numerous works are devoted to the problems of existence and uniqueness of the solutions of stochastic differential equations with given initial and boundary conditions in different function spaces [5, 7, 10–14] and, in particular, in Hilbert spaces [1, 3, 4, 8, 9]. Note that the nonlinear stochastic partial differential equations with delay (nonlinear ***stochastic differential equations of neutral type***) and the properties of their solutions are of especial interest. The initial-value problem for an abstract functional equation of this type in a Hilbert space was considered and a theorem on existence and uniqueness of its ***mild solution*** was proved in [9]. However, it is difficult to check the conditions of this theorem in the general form for special applied problems. Thus, the problem of determination of the coefficient conditions for the existence and uniqueness of the solution, i.e., conditions expressed via the coefficients of the equation and, hence, convenient for verification, is of high importance. This can be done only in some special cases. The present paper is devoted to the analysis of one of these cases.

The paper is organized as follows: The statement of the problem is presented in Sec. 2. Section 3 contains the required preliminary results. The main result of the paper is formulated in Sec. 4, and its proof is given in Sec. 5. In Sec. 6, we present a corollary.

2. Statement of the Problem

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space. Consider the Cauchy problem for a stochastic reaction-diffusion integrodifferential equation of neutral type

$$d\left(u(t,x) + \int\limits_{\mathbb{R}^d} b(t,x,u(\alpha(t),\xi),\xi)\,d\xi\right) = (\Delta_x u(t,x) + f(t,u(\alpha(t)),x))\,dt$$

$$+\,\sigma(t,u(\alpha(t)),x)\,dW(t,x), \quad 0 < t \le T, \quad x \in \mathbb{R}^d, \qquad (1)$$

$$u(t,x) = \phi(t,x), \quad -r \le t \le 0, \quad x \in \mathbb{R}^d, \quad r > 0,$$

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where $T > 0$ is a fixed number,

$$\Delta_x \equiv \sum_{i=1}^{d} \partial_{x_i}^2$$

is a d-dimensional Laplace operator,

$$\partial_{x_i}^2 \equiv \frac{\partial^2}{\partial x_i^2},$$

$i \in \{1, \ldots, d\}$, $W(t, x)$ is an $L_2(\mathbb{R}^d)$-valued Q-Wiener process, $\{f, \sigma\}: [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $b: [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ are given functions, $\phi: [-r, 0] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ are initial data, and $\alpha: [0, T] \to [-r, \infty)$ is a function of delay. For problem (1), we prove the theorem on existence and uniqueness of a mild solution.

3. Preliminary Results

In this section, we present the notation and some known results required in what follows. Assume that the flow of σ-algebras $\{\mathcal{F}_t, t \geq 0\}$ is generated by an $L_2(\mathbb{R}^d)$-valued Q-Wiener process

$$W(t, x) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n(x) \beta_n(t),$$

where $\{\beta_n(t), n \in \{1, 2, \ldots\}\} \subset \mathbb{R}$ are independent Brownian motions, $\{\lambda_n, n \in \{1, 2, \ldots\}\}$ is a sequence of positive numbers such that

$$\sum_{n=1}^{\infty} \lambda_n < \infty, \tag{2}$$

and $\{e_n(x), n \in \{1, 2, \ldots\}\}$ is an orthonormal basis in $L_2(\mathbb{R}^d)$ such that

$$\sup_{n \in \{1,2,\ldots\}} \operatorname{ess\,sup}_{x \in \mathbb{R}^d} |e_n(x)| \leq 1. \tag{3}$$

In what follows, we need some information from the theory of partial differential equations.

Lemma 1 [2, p. 47]. *If, in a homogeneous Cauchy problem*

$$\begin{aligned} \partial_t u(t, x) &= \Delta_x u(t, x), \quad t > 0, \quad x \in \mathbb{R}^d, \\ u(0, x) &= g(x), \quad x \in \mathbb{R}^d, \end{aligned} \tag{4}$$

the initial data g belong to $C(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$, then the solution of this problem can be represented in the form of a Poisson integral

$$u(t, x) = \int_{\mathbb{R}^d} \mathcal{K}(t, x - \xi) g(\xi) \, d\xi, \tag{5}$$

where

$$\mathcal{K}(t,x) = \begin{cases} \dfrac{1}{(4\pi t)^{\frac{d}{2}}} \exp\left\{-\dfrac{x^2}{4t}\right\}, & t > 0, \\ 0, & t < 0, \end{cases}$$

is the heat-conduction kernel and, in addition, $u \in C^\infty((0,\infty)\times\mathbb{R}^d)$.

Lemma 2. *If* g *belongs to* $L_1(\mathbb{R}^d)$, *then function (5) satisfies the following limit relations:*

$$\lim_{|x|\to\infty} u(t,x) = 0 \quad \textit{and} \quad \lim_{|x|\to\infty} \partial_t u(t,x) = 0. \tag{6}$$

Proof. The proof of the lemma follows from the theorems on differentiability of the Lebesgue integral with respect to parameter and the possibility of limit transition in this integral.

In what follows, the derivatives are understood in the ordinary sense.

Lemma 3 [6, p. 319]**.** *The derivatives of the kernel* $\mathcal{K}$ *admit the estimate*

$$|\partial_t^r \partial_x^s \mathcal{K}(t,x)| \le c_{r,s} t^{-\frac{d}{2}-r-\frac{s}{2}} \exp\left\{-\frac{c_0|x|^2}{t}\right\}, \quad c_{r,s} > 0, \quad 0 < c_0 < \frac{1}{4}.$$

Lemmas 1–3 yield the following result:

Lemma 4 [2, p. 360]**.** *If* g *belongs to* $L_1(\mathbb{R}^d)$ *and*

$$|\nabla_x g| \in L_2(\mathbb{R}^d), \quad \|D_x^2 g\| \in L_2(\mathbb{R}^d), \tag{7}$$

then the second derivatives of function (5) admit the estimate

$$\sup_{0\le t\le T} \int_{\mathbb{R}^d} (\Delta_x u(t,x))^2 \, dx = \sup_{0\le t\le T} \int_{\mathbb{R}^d} \|D_x^2 u(t,x)\|^2 \, dx \le C \int_{\mathbb{R}^d} \left\|D_x^2 g(x)\right\|^2 \, dx, \tag{8}$$

where $C > 0$ *is a constant depending only on* T, $\nabla_x \equiv \left(\partial_{x_1} \ldots \partial_{x_d}\right)^{\mathrm{T}}$,

$$D_x^2 \equiv \begin{pmatrix} \partial_{x_1}^2 & \ldots & \partial_{x_1 x_d} \\ \vdots & \ddots & \vdots \\ \partial_{x_d x_1} & \ldots & \partial_{x_d}^2 \end{pmatrix}$$

is the Hessian operator, and $\|\cdot\|$ *is the corresponding norm of the matrix.*

Definition 1. *A positive bounded function* $\rho \in L_1(\mathbb{R}^d)$ *is called an admissible weight if, for every* $T > 0$, *there exists a constant* $C_\rho(T) > 0$ *such that the estimate*

$$\int_{\mathbb{R}^d} \mathcal{K}(t, x-\xi)\rho(x)\, dx \le C_\rho(T)\rho(\xi), \quad \xi \in \mathbb{R}^d,$$

holds for any $0 \le t \le T$.

Remark 1. The functions

$$\rho(x) = \exp\{-r|x|\}, \quad r > 0, \quad \rho(x) = \frac{1}{1+|x|^r}, \quad r > d,$$

are typical examples of admissible weights.

Remark 2. Without loss of generality, we assume that $0 < \rho \le 1$.

Here and in what follows, by $L_2^\rho(\mathbb{R}^d)$ we denote a weighted Hilbert space with admissible weight and the norm

$$\|f\|_{L_2^\rho(\mathbb{R}^d)} = \sqrt{\int_{\mathbb{R}^d} |f(x)|^2 \rho(x)\,dx}.$$

Lemma 5 [16; 17, p. 188]. *The operators* $S(t)\colon L_2^\rho(\mathbb{R}^d) \to L_2^\rho(\mathbb{R}^d)$ *generating the solution of problem (4) by the rule*

$$u(t,x) = (S(t)g(\cdot))(x) = \int\limits_{\mathbb{R}^d} \mathcal{K}(t, x-\xi) g(\cdot)\,d\xi \tag{9}$$

form a (C_0)*-semigroup of operators with infinitesimal operator* Δ_x. *In this case, the inequality*

$$\|(S(t)g(\cdot))(x)\|^2_{L_2^\rho(\mathbb{R}^d)} \le C_\rho(T)\|g(x)\|^2_{L_2^\rho(\mathbb{R}^d)}, \quad 0 \le t \le T, \quad g \in L_2^\rho(\mathbb{R}^d), \tag{10}$$

is true.

Let $p \ge 2$. By $\mathfrak{B}_{p,T,\rho}$ we denote the Banach space of all $L_2^\rho(\mathbb{R}^d)$-valued random processes Φ $\mathcal{F}_t$-measurable for almost all $0 \le t \le T :\ [0,T] \times \Omega \to L_2^\rho(\mathbb{R}^d)$ and continuous in t for almost all $\omega \in \Omega$ with the norm

$$\|\Phi\|_{\mathfrak{B}_{p,T,\rho}} = \sqrt[p]{\sup_{0 \le t \le T} \mathbf{E}\|\Phi(t)\|^p_{L_2^\rho(\mathbb{R}^d)}}.$$

In the next section, we formulate a theorem on the existence and uniqueness of a mild solution of problem (1) for $0 \le t \le T$ in the space $\mathfrak{B}_{p,T,\rho}$.

4. Main Result

In what follows, we assume that the following assumptions are true:

(4.1) $\alpha\colon [0,T] \to [-r, \alpha(T)]$ is a function from $C^1([0,T])$ with $0 < \alpha' \le 1$;

(4.2) $\{f,\sigma\}\colon [0,T]\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ and $b\colon [0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ are functions measurable in the collections of their arguments;

(4.3) the initial function $\phi\colon [-r,0]\times\mathbb{R}^d\times\Omega\to\mathbb{R}$ is $\mathcal{F}_0$-measurable, independent of $W(t,x)$, $t\ge 0$, and such that

$$\sup_{-r\le t\le 0}\mathbf{E}\|\phi(t)\|^p_{L_2^\rho(\mathbb{R}^d)}<\infty. \tag{11}$$

Definition 2. *A continuous random process $u\colon [-r,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R}$ is called a mild solution of problem (1) if*

(i) it is $\mathcal{F}_t$-measurable for almost all $-r\le t\le T$;

(ii) it is a solution of the integrodifferential equation

$$u(t,x)=\int_{\mathbb{R}^d}\mathcal{K}(t,x-\xi)\left(\phi(0)+\int_{\mathbb{R}^d}b(0,\xi,\phi(-r,\zeta),\zeta)\,d\zeta\right)d\xi-\int_{\mathbb{R}^d}b(t,x,u(\alpha(t),\xi),\xi)\,d\xi$$

$$-\int_0^t\left(\Delta_x\int_{\mathbb{R}^d}\mathcal{K}(t-s,x-\xi)\left(\int_{\mathbb{R}^d}b(s,\xi,u(\alpha(s),\zeta),\zeta)\,d\zeta\right)d\xi\right)ds$$

$$+\int_0^t\int_{\mathbb{R}^d}\mathcal{K}(t-s,x-\xi)f(s,u(\alpha(s)),\xi)\,d\xi\,ds$$

$$+\int_0^t\sum_{n=1}^{\infty}\sqrt{\lambda_n}\left(\int_{\mathbb{R}^d}\mathcal{K}(t-s,x-\xi)\sigma(s,u(\alpha(s)),\xi)e_n(\xi)\,d\xi\right)d\beta_n(s),$$

$$0\le t\le T,\quad x\in\mathbb{R}^d,$$

$$u(t,x)=\phi(t,x),\quad -r\le t\le 0,\quad x\in\mathbb{R}^d,\quad r>0;$$

(iii) it satisfies the condition

$$\mathbf{E}\int_0^T\|u(t)\|^p_{L_2^\rho(\mathbb{R}^d)}\,dt<\infty.$$

The following theorem is true for the solution thus defined:

Theorem 1. *Suppose that Assumptions 4.1–4.3 are true and, in addition,* *(i) the functions $\{f, \sigma\}$ satisfy the condition of linear growth and the Lipschitz condition with respect to the second argument, i.e., there exists $L > 0$ such that*

$$|f(t,u,x)| + |\sigma(t,u,x)| \le L(1+|u|), \quad 0 \le t \le T, \quad u \in \mathbb{R}, \quad x \in \mathbb{R}^d, \tag{12}$$

$$|f(t,u,x) - f(t,v,x)| + |\sigma(t,u,x) - \sigma(t,v,x)| \le L|u-v|,$$
$$0 \le t \le T, \quad \{u,v\} \subset \mathbb{R}, \quad x \in \mathbb{R}^d; \tag{13}$$

(ii) the function b satisfies the conditions

$$\sup_{0\le t\le T} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |b(t,x,0,\xi)| d\xi \right)^2 \rho(x) dx < \infty, \tag{14}$$

$$\sup_{0\le t\le T} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |b(t,x,0,\xi)| d\xi dx < \infty, \tag{15}$$

and the Lipschitz conditions with respect to the third argument

$$|b(t,x,u,\xi) - b(t,x,v,\xi)| \le l(t,x,\xi)|u-v|, \quad 0 \le t \le T, \quad \{x,\xi\} \subset \mathbb{R}^d, \quad \{u,v\} \subset \mathbb{R}, \tag{16}$$

where the function $l\colon [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ is such that

$$\sup_{0\le t\le T} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{l^2(t,x,\xi)}{\rho(\xi)} d\xi \right) \rho(x) dx < \infty, \tag{17}$$

$$\sup_{0\le t\le T} \int_{\mathbb{R}^d} \sqrt{\int_{\mathbb{R}^d} \frac{l^2(t,x,\xi)}{\rho(\xi)} d\xi}\, dx < \infty; \tag{18}$$

(iii) for any $x \in \mathbb{R}^d$, the derivatives $\partial_{x_i} b$, $\partial_{x_i x_j} b$, $\{i,j\} \subset \{1,\ldots,d\}$ exist and, in addition, the gradient $\nabla_x b$ and the matrix $D_x^2 b$ satisfy the condition of linear growth with respect to the third argument

$$|\nabla_x b(t,x,u,\xi)| + \|D_x^2 b(t,x,u,\xi)\| \le \psi(t,x,\xi)(1+|u|),$$
$$0 \le t \le T, \quad \{x,\xi\} \subset \mathbb{R}^d, \quad u \in \mathbb{R}, \tag{19}$$

and the matrix $D_x^2 b$ satisfies the Lipschitz condition

$$\left\| D_x^2 b(t,x,u,\xi) - D_x^2 b(t,x,v,\xi) \right\| \le \psi(t,x,\xi)|u-v|,$$
$$0 \le t \le T, \quad \{x,\xi\} \subset \mathbb{R}^d, \quad \{u,v\} \subset \mathbb{R}, \tag{20}$$

where the function $\psi\colon [0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty)$ *satisfies the conditions*

$$\sup_{0\le t\le T}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\psi(t,x,\xi)d\xi\right)^2 dx<\infty, \tag{21}$$

$$\sup_{0\le t\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\psi^2(t,x,\xi)}{\rho(\xi)}d\xi\,dx<\infty; \tag{22}$$

moreover, for any point $x_0\in\mathbb{R}^d$, *there exists a neighborhood* $B_\delta(x_0)$ *and a nonnegative function* $\varphi(t,\xi,x_0,\delta)$ *such that*

$$\sup_{0\le t\le T}\frac{\varphi(t,\cdot,x_0,\delta)}{\sqrt{\rho(\cdot)}}\in L_2(\mathbb{R}^d),\ \delta\in\mathbb{R}^+, \tag{23}$$

$$|\psi(t,x,\xi)-\psi(t,x_0,\xi)|\le\varphi(t,\xi,x_0,\delta)|x-x_0|,\quad 0\le t\le T,\quad |x-x_0|<\delta,\quad \xi\in\mathbb{R}^d. \tag{24}$$

Under these conditions, if

$$\sup_{0\le t\le T}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\frac{l^2(t,x,\xi)}{\rho(\xi)}d\xi\right)\rho(x)dx\right)^{\frac{p}{2}}<\frac{1}{4^{p-1}}, \tag{25}$$

then problem (1) possesses a unique mild solution $u\in\mathfrak{B}_{p,T,\rho}$ *on* $0\le t\le T$.

5. Proof of Theorem 1

The proof of the theorem is based on the classical Banach fixed-point theorem. Thus, we consider an operator $\Psi\colon\mathfrak{B}_{p,T,\rho}\to\mathfrak{B}_{p,T,\rho}$:

$$(\Psi u)(t)=\int_{\mathbb{R}^d}\mathcal{K}(t,x-\xi)\left(\phi(0)+\int_{\mathbb{R}^d}b(0,\xi,\phi(-r,\zeta),\zeta)d\zeta\right)d\xi-\int_{\mathbb{R}^d}b(t,x,u(\alpha(t),\xi),\xi)d\xi$$

$$-\int_0^t\left(\Delta_x\int_{\mathbb{R}^d}\mathcal{K}(t-s,x-\xi)\left(\int_{\mathbb{R}^d}b(s,\xi,u(\alpha(s),\zeta),\zeta)d\zeta\right)d\xi\right)ds$$

$$+\int_0^t\int_{\mathbb{R}^d}\mathcal{K}(t-s,x-\xi)f(s,u(\alpha(s)),\xi)d\xi ds$$

$$+\int_0^t \sum_{n=1}^{\infty} \sqrt{\lambda_n} \left(\int_{\mathbb{R}^d} \mathcal{K}(t-s, x-\xi)\sigma(s, u(\alpha(s)), \xi) e_n(\xi) d\xi \right) d\beta_n(s) = \sum_{j=0}^{4} I_j(t),$$

$$0 \le t \le T, \quad x \in \mathbb{R}^d,$$

$$u(t,x) = \phi(t,x), \quad -r \le t \le T, \quad x \in \mathbb{R}^d,$$

and prove that this operator is contracting. First, we show that Ψu belongs to $\mathfrak{B}_{p,T,\rho}$ for any $u \in \mathfrak{B}_{p,T,\rho}$. To this end, we estimate

$$\|I_j(s)\|^p_{\mathfrak{B}_{p,t,\rho}} = \sup_{0\le s\le t} \mathbf{E}\|I_j(s)\|^p_{L_2^\rho(\mathbb{R}^d)}, \quad j \in \{0, \dots, 4\}.$$

Further, we estimate $\|I_0(s)\|^p_{\mathfrak{B}_{p,t,\rho}}$ as follows:

$$\|I_0(s)\|^p_{\mathfrak{B}_{p,t,\rho}} = \sup_{0\le s\le t} \mathbf{E} \left\| \int_{\mathbb{R}^d} \mathcal{K}(s, x-\xi) \left(\phi(0) + \int_{\mathbb{R}^d} b(0, \xi, \phi(-r, \zeta), \zeta) d\zeta \right) d\xi \right\|^p_{L_2^\rho(\mathbb{R}^d)}$$

$$\le 2^{p-1} \left(\sup_{0\le s\le t} \mathbf{E} \left\| \int_{\mathbb{R}^d} \mathcal{K}(s, x-\xi)\phi(0) d\xi \right\|^p_{L_2^\rho(\mathbb{R}^d)} \right.$$

$$+ \sup_{0\le s\le t} \mathbf{E} \left\| \int_{\mathbb{R}^d} \mathcal{K}(s, x-\xi) \left(\int_{\mathbb{R}^d} (b(0, \xi, \phi(-r, \zeta), \zeta) \right.\right.$$

$$\left. -b(0,\xi,0,\zeta) + b(0,\xi,0,\zeta)) d\zeta \right) d\xi \|^p_{L_2^\rho(\mathbb{R}^d)} \Big)$$

$$\le 2^{p-1} \sup_{0\le s\le t} \mathbf{E} \left\| \int_{\mathbb{R}^d} \mathcal{K}(s, x-\xi)\phi(0) d\xi \right\|^p_{L_2^\rho(\mathbb{R}^d)}$$

$$+ 4^{p-1} \sup_{0\le s\le t} \mathbf{E} \left\| \int_{\mathbb{R}^d} \mathcal{K}(s, x-\xi) \right.$$

$$\times \left. \left(\int_{\mathbb{R}^d} (b(0, \xi, \phi(-r, \zeta), \zeta) - b(0, \xi, 0, \zeta)) d\zeta \right) d\xi \right\|^p_{L_2^\rho(\mathbb{R}^d)}$$

$$+ 4^{p-1} \sup_{0\le s\le t} \mathbf{E} \left\| \int_{\mathbb{R}^d} \mathcal{K}(s, x-\xi) \left(\int_{\mathbb{R}^d} b(0,\xi,0,\zeta)d\zeta \right) d\xi \right\|^p_{L_2^\rho(\mathbb{R}^d)}$$

$$= I_0^1 + I_0^2 + I_0^3.$$

By using (10) and (11), we can estimate the quantity I_0^1 as follows:

$$I_0^1 = 2^{p-1} \sup_{0\le s\le t} \mathbf{E} \left\| \int_{\mathbb{R}^d} \mathcal{K}(s, x-\xi)\phi(0)d\xi \right\|^p_{L_2^\rho(\mathbb{R}^d)} \le 2^{p-1} C_\rho^{\frac{p}{2}}(T)\mathbf{E}\|\phi(0)\|^p_{L_2^\rho(\mathbb{R}^d)} < \infty.$$

By virtue of (10), (11), (16), and (17) and the Cauchy–Bunyakovsky inequality, for the quantity I_0^2, we find

$$I_0^2 = 4^{p-1} \sup_{0\le s\le t} \mathbf{E} \left\| \int_{\mathbb{R}^d} \mathcal{K}(s, x-\xi) \left(\int_{\mathbb{R}^d} (b(0,\xi,\phi(-r,\zeta),\zeta) - b(0,\xi,0,\zeta))d\zeta \right) d\xi \right\|^p_{L_2^\rho(\mathbb{R}^d)}$$

$$\le 4^{p-1} C_\rho^{\frac{p}{2}}(T)\mathbf{E} \left\| \int_{\mathbb{R}^d} |b(0,x,\phi(-r,\zeta),\zeta) - b(0,x,0,\zeta)|d\zeta \right\|^p_{L_2^\rho(\mathbb{R}^d)}$$

$$= 4^{p-1} C_\rho^{\frac{p}{2}}(T)\mathbf{E} \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |b(0,x,\phi(-r,\zeta),\zeta) - b(0,x,0,\zeta)|d\zeta \right)^2 \rho(x)dx \right)^{\frac{p}{2}}$$

$$\le 4^{p-1} C_\rho^{\frac{p}{2}}(T)\mathbf{E} \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{l(0,x,\zeta)}{\sqrt{\rho(\zeta)}} |\phi(-r,\zeta)| \sqrt{\rho(\zeta)} d\zeta \right)^2 \rho(x)dx \right)^{\frac{p}{2}}$$

$$\le 4^{p-1} C_\rho^{\frac{p}{2}}(T)\mathbf{E} \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{l^2(0,x,\zeta)}{\rho(\zeta)} d\zeta \right) \left(\int_{\mathbb{R}^d} \phi^2(-r,\zeta)\rho(\zeta)\, d\zeta \right) \rho(x)dx \right)^{\frac{p}{2}}$$

$$= 4^{p-1} C_\rho^{\frac{p}{2}}(T) \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{l^2(0,x,\zeta)}{\rho(\zeta)} d\zeta \right) \rho(x)dx \right)^{\frac{p}{2}} \mathbf{E} \left(\int_{\mathbb{R}^d} \phi^2(-r,\zeta)\rho(\zeta)d\zeta \right)^{\frac{p}{2}}$$

$$= 4^{p-1} C_\rho^{\frac{p}{2}}(T) \left(\int\limits_{\mathbb{R}^d} \left(\int\limits_{\mathbb{R}^d} \frac{l^2(0,x,\zeta)}{\rho(\zeta)}\, d\zeta \right) \rho(x) dx \right)^{\frac{p}{2}} \mathbf{E} \|\phi(-r)\|^p_{L_2^\rho(\mathbb{R}^d)} < \infty.$$

In view (10) and (14), for the quantity I_0^3, we can write

$$I_0^3 = 4^{p-1} \sup_{0 \le s \le t} \mathbf{E} \left\| \int\limits_{\mathbb{R}^d} \mathcal{K}(s, x-\xi) \left(\int\limits_{\mathbb{R}^d} b(0,\xi,0,\zeta) d\zeta \right) dx \right\|^p_{L_2^\rho(\mathbb{R}^d)}$$

$$\le 4^{p-1} C_\rho^{\frac{p}{2}}(T) \left\| \int\limits_{\mathbb{R}^d} |b(0,x,0,\zeta)| d\zeta \right\|^p$$

$$= 4^{p-1} C_\rho^{\frac{p}{2}}(T) \left(\int\limits_{\mathbb{R}^d} \left(\int\limits_{\mathbb{R}^d} |b(0,x,0,\zeta)| d\zeta \right)^2 \rho(x) dx \right)^{\frac{p}{2}} < \infty.$$

It follows from the established three estimates that

$$\|I_0(s)\|^p_{\mathfrak{B}_{p,t,\rho}} < \infty. \tag{26}$$

By using (11), (14), (16), (17), and the Cauchy–Bunyakovsky inequality, we estimate $\|I_1(s)\|^p_{\mathfrak{B}_{p,t,\rho}}$ as follows:

$$\|I_1(s)\|^p_{\mathfrak{B}_{p,t,\rho}} = \sup_{0 \le s \le t} \mathbf{E} \left\| \int\limits_{\mathbb{R}^d} b(s,x,u(\alpha(s),\xi),\xi) d\xi \right\|^p_{L_2^\rho(\mathbb{R}^d)}$$

$$= \sup_{0 \le s \le t} \mathbf{E} \left(\int\limits_{\mathbb{R}^d} \left(\int\limits_{\mathbb{R}^d} b(s,x,u(\alpha(s),\xi),\xi) d\xi \right)^2 \rho(x) dx \right)^{\frac{p}{2}}$$

$$\le 2^{\frac{p}{2}} \sup_{0 \le s \le t} \mathbf{E} \left(\int\limits_{\mathbb{R}^d} \left(\int\limits_{\mathbb{R}^d} |b(s,x,u(\alpha(s),\xi),\xi) - b(s,x,0,\xi)| d\xi \right)^2 \rho(x) dx \right.$$

$$\left. + \int\limits_{\mathbb{R}^d} \left(\int\limits_{\mathbb{R}^d} |b(s,x,0,\xi)| d\xi \right)^2 \rho(x) dx \right)^{\frac{p}{2}}$$

$$\leq 2^{p-1} \sup_{0\leq s\leq t} \mathbf{E}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \frac{l(s,x,\xi)}{\sqrt{\rho(\xi)}}\,|u(\alpha(s),\xi)|\sqrt{\rho(\xi)}d\xi\right)^2 \rho(x)dx\right)^{\frac{p}{2}}$$

$$+\, 2^{p-1} \sup_{0\leq s\leq t}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |b(s,x,0,\xi)|d\xi\right)^2 \rho(x)dx\right)^{\frac{p}{2}}$$

$$\leq 2^{p-1} \sup_{0\leq s\leq t}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \frac{l^2(s,x,\xi)}{\rho(\xi)}d\xi\right)\rho(x)dx\right)^{\frac{p}{2}} \mathbf{E}\left(\int_{\mathbb{R}^d} u^2(\alpha(s),\xi)\rho(\xi)d\xi\right)^{\frac{p}{2}}$$

$$+\, 2^{p-1} \sup_{0\leq s\leq t}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |b(s,x,0,\xi)|d\xi\right)^2 \rho(x)dx\right)^{\frac{p}{2}}$$

$$\leq 2^{p-1} \sup_{0\leq s\leq t}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \frac{l^2(s,x,\xi)}{\rho(\xi)}\,d\xi\right)\rho(x)\,dx\right)^{\frac{p}{2}} \sup_{0\leq s\leq t} \mathbf{E}\|u(\alpha(s))\|^p_{L_2^\rho(\mathbb{R}^d)}$$

$$+\, 2^{p-1} \sup_{0\leq s\leq t}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |b(s,x,0,\xi)|d\xi\right)^2 \rho(x)\,dx\right)^{\frac{p}{2}}.$$

Let $0 < t^* < \alpha(T)$ be such that $\alpha(t^*) = 0$. Thus, we find

$$\sup_{0\leq s\leq t} \mathbf{E}\|u(\alpha(s))\|^p_{L_2^\rho(\mathbb{R}^d)} \leq \sup_{0\leq s\leq t^*} \mathbf{E}\|u(\alpha(s))\|^p_{L_2^\rho(\mathbb{R}^d)} + \sup_{t^*\leq s\leq t} \mathbf{E}\|u(\alpha(s))\|^p_{L_2^\rho(\mathbb{R}^d)}$$

$$= \sup_{-r\leq s\leq 0} \mathbf{E}\|\phi(s)\|^p_{L_2^\rho(\mathbb{R}^d)} + \sup_{0\leq s\leq \alpha(t)} \mathbf{E}\|u(s)\|^p_{L_2^\rho(\mathbb{R}^d)}$$

$$\leq \sup_{-r\leq s\leq 0} \mathbf{E}\|\phi(s)\|^p_{L_2^\rho(\mathbb{R}^d)} + \sup_{0\leq s\leq t} \mathbf{E}\|u(s)\|^p_{L_2^\rho(\mathbb{R}^d)} < \infty.$$

This yields

$$\|I_1(s)\|^p_{\mathfrak{B}_{p,t,\rho}} < \infty. \tag{27}$$

In view of Remark 2, the Cauchy–Bunyakovsky inequality, and the Fubini theorem, we can estimate $\|I_2(s)\|^p_{\mathfrak{B}_{p,t,\rho}}$ as follows:

$$\|I_2(s)\|^p_{\mathfrak{B}_{p,t,\rho}} = \sup_{0\le s\le t} \mathbf{E} \left\| \int\limits_0^s \left(\Delta_x \int\limits_{\mathbb{R}^d} \mathcal{K}(s-\tau, x-\xi) \times \right.\right.$$

$$\left.\left. \times \left(\int\limits_{\mathbb{R}^d} b(\tau, \xi, u(\alpha(\tau), \zeta) d\zeta \right) d\xi \right) d\tau \right\|^p_{L^\rho_2(\mathbb{R}^d)}$$

$$= \sup_{0\le s\le t} \mathbf{E} \left(\int\limits_{\mathbb{R}^d} \left(\int\limits_0^s \left(\Delta_x \int\limits_{\mathbb{R}^d} \mathcal{K}(s-\tau, x-\xi) \right.\right.\right.$$

$$\left.\left.\left. \times \left(\int\limits_{\mathbb{R}^d} b(\tau, \xi, u(\alpha(\tau), \zeta), \zeta) d\zeta \right) d\xi \right) d\tau \right)^2 \rho(x) dx \right)^{\frac{p}{2}}$$

$$\le t^{\frac{p}{2}} \sup_{0\le s\le t} \mathbf{E} \left(\int\limits_0^s \int\limits_{\mathbb{R}^d} \left(\Delta_x \int\limits_{\mathbb{R}^d} \mathcal{K}(s-\tau, x-\xi) \right.\right.$$

$$\left.\left. \times \left(\int\limits_{\mathbb{R}^d} b(\tau, \xi, u(\alpha(\tau), \zeta), \zeta) d\zeta \right) d\xi \right)^2 dx d\tau \right)^{\frac{p}{2}}$$

$$\le C^{\frac{p}{2}} t^{\frac{p}{2}} \mathbf{E} \left(\int\limits_0^t \int\limits_{\mathbb{R}^d} \left\| D^2_x \int\limits_{\mathbb{R}^d} b(\tau, x, u(\alpha(\tau), \zeta), \zeta) d\zeta \right\|^2 dx d\tau \right)^{\frac{p}{2}} \tag{28}$$

provided that the conditions of Lemma 4 with

$$g(\tau, x) = \int\limits_{\mathbb{R}^d} b(\tau, x, u(\alpha(\tau), \zeta), \zeta) d\zeta$$

and

$$u(\tau, x) = \int\limits_{\mathbb{R}^d} \mathcal{K}(s-\tau, x-\xi) \left(\int\limits_{\mathbb{R}^d} b(\tau, \xi, u(\alpha(\tau), \zeta), \zeta) d\zeta \right) d\xi$$

are satisfied. We now check the validity of Lemma 4 for this function. To this end, we prove that

(i) $\int_{\mathbb{R}^d} b(\tau, \cdot, u(\alpha(\tau), \zeta), \zeta) d\zeta \in L_1(\mathbb{R}^d)$ for any $0 \le \tau \le t$ with probability 1;

(ii) condition (7) is satisfied.

1. Indeed, by virtue of (11), (15), (16), (18), and the Cauchy–Bunyakovsky inequality, we obtain

$$\mathbf{E}\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} b(\tau,x,u(\alpha(\tau),\zeta),\zeta)d\zeta\right|dx$$

$$\leq \mathbf{E}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{l(\tau,x,\zeta)}{\sqrt{\rho(\zeta)}}|u(\alpha(\tau),\zeta)|\sqrt{\rho(\zeta)}d\zeta\,dx+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|b(\tau,x,0,\zeta)|d\zeta\,dx$$

$$\leq\left(\sup_{0\leq\tau\leq t}\int_{\mathbb{R}^d}\sqrt{\int_{\mathbb{R}^d}\frac{l^2(\tau,x,\zeta)}{\rho(\zeta)}\,d\zeta}\,dx\right)\sqrt{\sup_{0\leq\tau\leq t}\mathbf{E}\|u(\alpha(\tau))\|^2_{L_2^\rho(\mathbb{R}^d)}}$$

$$+\sup_{0\leq\tau\leq t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|b(\tau,x,0,\zeta)|\,d\zeta dx$$

$$\leq\left(\sup_{0\leq\tau\leq t}\int_{\mathbb{R}^d}\sqrt{\int_{\mathbb{R}^d}\frac{l^2(\tau,x,\zeta)}{\rho(\zeta)}\,d\zeta}dx\right)\sqrt{\sup_{-r\leq\tau\leq 0}\mathbf{E}\|\phi(\tau)\|^2_{L_2^\rho(\mathbb{R}^d)}+\sup_{0\leq\tau\leq t}\mathbf{E}\|u(\tau)\|^2_{L_2^\rho(\mathbb{R}^d)}}$$

$$+\sup_{0\leq\tau\leq t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|b(\tau,x,0,\zeta)|d\zeta dx<\infty.$$

With probability 1, this yields

$$\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} b(\tau,x,u(\alpha(\tau),\zeta),\zeta)d\zeta\right|dx<\infty.$$

2. We now prove the differentiability of the function g for $x=x_0$, which is an arbitrary point from $\mathbb{R}^d$. Let $B_\delta(x_0)$ be the neighborhood from assertion (iii). Thus, by virtue of (19) and (24), we get

$$\begin{aligned}|\nabla_x b(\tau,x,u(\alpha(\tau),\zeta),\zeta)| &\leq \psi(\tau,x,\zeta)(1+|u(\alpha(\tau),\zeta)|)\\ &\leq(|\psi(\tau,x,\zeta)-\psi(\tau,x_0,\zeta)|+\psi(\tau,x_0,\zeta))(1+|u(\alpha(\tau),\zeta)|)\\ &\leq(\varphi(\tau,\zeta,x_0,\delta)|x-x_0|+\psi(\tau,x_0,\zeta))(1+|u(\alpha(\tau),\zeta)|)\\ &\leq(\delta\varphi(\tau,\zeta,x_0,\delta)+\psi(\tau,x_0,\zeta))(1+|u(\alpha(\tau),\zeta)|).\end{aligned}$$

We now show that

$$(\delta\varphi(\tau,\cdot,x_0,\delta)+\psi(\tau,x_0,\cdot))(1+|u(\alpha(\tau),\cdot)|)\in L_1(\mathbb{R}^d).$$

By virtue of the Cauchy–Bunyakovsky inequality and relations (11) and (21)–(23), we obtain

$$\mathbf{E}\int_{\mathbb{R}^d} (\delta\varphi(\tau,\zeta,x_0,\delta)+\psi(\tau,x_0,\zeta))(1+|u(\alpha(\tau),\zeta)|)d\zeta$$

$$=\delta\int_{\mathbb{R}^d}\frac{\varphi(\tau,\zeta,x_0,\delta)}{\sqrt{\rho(\zeta)}}\sqrt{\rho(\zeta)}d\zeta+\int_{\mathbb{R}^d}\psi(\tau,x_0,\zeta)d\zeta$$

$$+\delta\mathbf{E}\int_{\mathbb{R}^d}\frac{\varphi(\tau,\zeta,x_0,\delta)}{\sqrt{\rho(\zeta)}}|u(\alpha(\tau),\zeta)|\sqrt{\rho(\zeta)}d\zeta+\mathbf{E}\int_{\mathbb{R}^d}\frac{\psi(\tau,x_0,\zeta)}{\sqrt{\rho(\zeta)}}|u(\alpha(\tau),\zeta)|\sqrt{\rho(\zeta)}d\zeta$$

$$\le\delta\sqrt{\int_{\mathbb{R}^d}\frac{\varphi^2(\tau,\zeta,x_0,\delta)}{\rho(\zeta)}d\zeta}\sqrt{\int_{\mathbb{R}^d}\rho(\zeta)d\zeta}+\int_{\mathbb{R}^d}\psi(\tau,x_0,\zeta)d\zeta$$

$$+\left(\delta\sqrt{\int_{\mathbb{R}^d}\frac{\varphi^2(\tau,\zeta,x_0,\delta)}{\rho(\zeta)}d\zeta}+\sqrt{\int_{\mathbb{R}^d}\frac{\psi^2(\tau,x_0,\zeta)}{\rho(\zeta)}d\zeta}\right)\sqrt{\sup_{0\le\tau\le t}\mathbf{E}\|u(\alpha(\tau))\|^2_{L_2^{\rho}(\mathbb{R}^d)}}$$

$$\le\delta\sqrt{\int_{\mathbb{R}^d}\frac{\varphi^2(\tau,\zeta,x_0,\delta)}{\rho(\zeta)}d\zeta}\sqrt{\int_{\mathbb{R}^d}\rho(\zeta)\,d\zeta}+\int_{\mathbb{R}^d}\psi(\tau,x_0,\zeta)d\zeta$$

$$+\left(\delta\sqrt{\int_{\mathbb{R}^d}\frac{\varphi^2(\tau,\zeta,x_0,\delta)}{\rho(\zeta)}d\zeta}+\sqrt{\int_{\mathbb{R}^d}\frac{\psi^2(\tau,x_0,\zeta)}{\rho(\zeta)}d\zeta}\right)$$

$$\times\sqrt{\sup_{-r\le\tau\le0}\mathbf{E}\|\phi(\tau)\|^2_{L_2^{\rho}(\mathbb{R}^d)}+\sup_{0\le\tau\le t}\mathbf{E}\|u(\tau)\|^2_{L_2^{\rho}(\mathbb{R}^d)}}<\infty.$$

With probability 1, this leads to the required condition

$$\int_{\mathbb{R}^d}(\delta\varphi(\tau,\zeta,x_0,\delta)+\psi(\tau,x_0,\zeta))(1+|u(\alpha(\tau),\zeta)|)d\zeta<\infty.$$

Hence, by the local theorem on differentiability of an integral with respect to the parameter, there exists $\nabla_x g(\tau,x)$ and the following equality is true:

$$\nabla_x\int_{\mathbb{R}^d}b(\tau,x,u(\alpha(\tau),\zeta),\zeta)d\zeta=\int_{\mathbb{R}^d}\nabla_x b(\tau,x,u(\alpha(\tau),\zeta),\zeta)d\zeta. \tag{29}$$

We now show that

$$\nabla_x \int_{\mathbb{R}^d} b(\tau, \cdot, u(\alpha(\tau), \zeta), \zeta) d\zeta \in L_2(\mathbb{R}^d).$$

By virtue of (11), (19), (21), (22), (29), and the Cauchy–Bunyakovsky inequality, we get

$$\mathbf{E} \int\limits_{\mathbb{R}^d} \left| \nabla_x \int\limits_{\mathbb{R}^d} b(\tau, x, u(\alpha(\tau), \zeta), \zeta)\, d\zeta \right|^2 dx$$

$$\leq \mathbf{E} \int\limits_{\mathbb{R}^d} \left(\int\limits_{\mathbb{R}^d} |\nabla_x b(\tau, x, u(\alpha(\tau), \zeta), \zeta)| d\zeta \right)^2 dx$$

$$\leq \mathbf{E} \int\limits_{\mathbb{R}^d} \left(\int\limits_{\mathbb{R}^d} \psi(\tau, x, \zeta)(1 + |u(\alpha(\tau), \zeta)|) d\zeta \right)^2 dx$$

$$\leq 2 \int\limits_{\mathbb{R}^d} \left(\int\limits_{\mathbb{R}^d} \psi(\tau, x, \zeta) d\zeta \right)^2 dx + 2 \left(\int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \frac{\psi^2(\tau, x, \zeta)}{\rho(\zeta)} d\zeta dx \right) \mathbf{E}\|u(\alpha(\tau))\|^2_{L_2^\rho(\mathbb{R}^d)}$$

$$\leq 2 \sup_{0\leq\tau\leq t} \int\limits_{\mathbb{R}^d} \left(\int\limits_{\mathbb{R}^d} \psi(\tau, x, \zeta) d\zeta \right)^2 dx$$

$$+ 2 \sup_{0\leq\tau\leq t} \left(\int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \frac{\psi^2(\tau, x, \zeta)}{\rho(\zeta)} d\zeta dx \right) \mathbf{E}\|u(\alpha(\tau))\|^2_{L_2^\rho(\mathbb{R}^d)}$$

$$\leq 2 \sup_{0\leq\tau\leq t} \int\limits_{\mathbb{R}^d} \left(\int\limits_{\mathbb{R}^d} \psi(\tau, x, \zeta) d\zeta \right)^2 dx$$

$$+ 2 \left(\sup_{0\leq\tau\leq t} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \frac{\psi^2(\tau, x, \zeta)}{\rho(\zeta)} d\zeta dx \right) \sup_{0\leq\tau\leq t} \mathbf{E}\|u(\alpha(\tau))\|^2_{L_2^\rho(\mathbb{R}^d)}$$

$$\leq 2 \sup_{0\leq\tau\leq t} \int\limits_{\mathbb{R}^d} \left(\int\limits_{\mathbb{R}^d} \psi(\tau, x, \zeta) d\zeta \right)^2 dx + 2 \left(\sup_{0\leq\tau\leq t} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \frac{\psi^2(\tau, x, \zeta)}{\rho(\zeta)} d\zeta\, dx \right)$$

$$\times\left(\sup_{-r\le\tau\le 0}\mathbf{E}\|\phi(\tau)\|^2_{L_2^\rho(\mathbb{R}^d)}+\sup_{0\le\tau\le t}\mathbf{E}\|u(\tau)\|^2_{L_2^\rho(\mathbb{R}^d)}\right)<\infty,$$

whence, with probability 1, we arrive at the required condition

$$\int\limits_{\mathbb{R}^d}\left|\nabla_x\int\limits_{\mathbb{R}^d}b(\tau,x,u(\alpha(\tau),\zeta),\zeta)d\zeta\right|^2dx<\infty.$$

For $D_x^2\int_{\mathbb{R}^d}b(\tau,x,u(\alpha(\tau),\zeta),\zeta)d\zeta$, condition (7) is proved in a similar way. This implies that, in inequality (28),

$$\|I_2(s)\|^p_{\mathfrak{B}_{p,t,\rho}}\le C^{\frac{p}{2}}t^{\frac{p}{2}}\mathbf{E}\left(\int\limits_0^t\int\limits_{\mathbb{R}^d}\left\|D_x^2\int\limits_{\mathbb{R}^d}b(\tau,x,u(\alpha(\tau),\zeta),\zeta)d\zeta\right\|^2dxd\tau\right)^{\frac{p}{2}}$$

$$\le C^{\frac{p}{2}}t^{\frac{p}{2}}\mathbf{E}\left(\int\limits_0^t\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d}\left\|D_x^2b(\tau,x,u(\alpha(\tau),\zeta),\zeta)\right\|d\zeta\right)^2dxd\tau\right)^{\frac{p}{2}}$$

$$\le 2^{\frac{p}{2}}C^{\frac{p}{2}}t^{\frac{p}{2}}\mathbf{E}\left(\int\limits_0^t\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d}\psi(\tau,x,\zeta)d\zeta\right)^2dxd\tau\right.$$

$$\left.+\left(\int\limits_0^t\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\frac{\psi^2(\tau,x,\zeta)}{\rho(\zeta)}d\zeta dxd\tau\right)\|u(\alpha(\tau))\|^2_{L_2^\rho(\mathbb{R}^d)}\right)^{\frac{p}{2}}$$

$$\le 2^{p-1}C^{\frac{p}{2}}t^{\frac{p}{2}}\left(\left(\int\limits_0^t\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d}\psi(\tau,x,\zeta)d\zeta\right)^2dxd\tau\right)^{\frac{p}{2}}\right.$$

$$\left.+\left(\int\limits_0^t\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\frac{\psi^2(\tau,x,\zeta)}{\rho(\zeta)}d\zeta dxd\tau\right)^{\frac{p}{2}}\mathbf{E}\|u(\alpha(\tau))\|^p_{L_2^\rho(\mathbb{R}^d)}\right)\le 2^{p-1}C^{\frac{p}{2}}t^p$$

$$\times\left(\sup_{0\le\tau\le t}\left(\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d}\psi(\tau,x,\zeta)d\zeta\right)^2dx\right)^{\frac{p}{2}}+\sup_{0\le\tau\le t}\left(\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\frac{\psi^2(\tau,x,\zeta)}{\rho(\zeta)}d\zeta dx\right)^{\frac{p}{2}}\right.$$

$$\times\left(\sup_{-r\le\tau\le 0}\mathbf{E}\|\phi(\tau)\|^p_{L_2^\rho(\mathbb{R}^d)}+\sup_{0\le\tau\le t}\mathbf{E}\|u(\tau)\|^p_{L_2^\rho(\mathbb{R}^d)}\right)\Bigg)<\infty. \tag{30}$$

By using (10)–(12), we estimate $\|I_3(s)\|^p_{\mathfrak{B}_{p,t,\rho}}$ as follows:

$$\|I_3(s)\|^p_{\mathfrak{B}_{p,t,\rho}}=\sup_{0\le s\le t}\mathbf{E}\left\|\int_0^s\int_{\mathbb{R}^d}\mathcal{K}(s-\tau,x-\xi)f(\tau,u(\alpha(\tau)),\xi)d\xi d\tau\right\|^p_{L_2^\rho(\mathbb{R}^d)}$$

$$=\sup_{0\le s\le t}\mathbf{E}\left(\int_{\mathbb{R}^d}\left(\int_0^s\int_{\mathbb{R}^d}\mathcal{K}(s-\tau,x-\xi)f(\tau,u(\alpha(\tau),\xi),\xi)d\xi d\tau\right)^2\rho(x)dx\right)^{\frac{p}{2}}$$

$$\le t^{\frac{p}{2}}\sup_{0\le s\le t}\mathbf{E}\left(\int_0^s\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\mathcal{K}(s-\tau,x-\xi)f(\tau,u(\alpha(\tau),\xi),\xi)d\xi\right)^2\rho(x)dxd\tau\right)^{\frac{p}{2}}$$

$$\le L^pt^{\frac{p}{2}}\sup_{0\le s\le t}\mathbf{E}\left(\int_0^s\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\mathcal{K}(s-\tau,x-\xi)(1+|u(\alpha(\tau),\xi)|)d\xi\right)^2\rho(x)dxd\tau\right)^{\frac{p}{2}}$$

$$\le 2^{\frac{p}{2}}L^pt^{\frac{p}{2}}\sup_{0\le s\le t}\mathbf{E}\Bigg(\int_0^s\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\mathcal{K}(s-\tau,x-\xi)d\xi\right)^2\rho(x)dxd\tau$$

$$+\int_0^s\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\mathcal{K}(s-\tau,x-\xi)|u(\alpha(\tau),\xi)|d\xi\right)^2\rho(x)dxd\tau\Bigg)^{\frac{p}{2}}$$

$$=2^{\frac{p}{2}}L^pt^{\frac{p}{2}}\sup_{0\le s\le t}\mathbf{E}\Bigg(\int_0^s\int_{\mathbb{R}^d}\rho(x)dxd\tau$$

$$+\int_0^s\left\|\int_{\mathbb{R}^d}\mathcal{K}(s-\tau,x-\xi)|u(\alpha(\tau))|d\xi\right\|^2_{L_2^\rho(\mathbb{R}^d)}d\tau\Bigg)^{\frac{p}{2}}$$

$$\leq 2^{p-1}L^p t^{\frac{p}{2}} \left(\sup_{0\leq s\leq t} \left(\int_0^s \int_{\mathbb{R}^d} \rho(x)dxd\tau \right)^{\frac{p}{2}} \right.$$

$$\left. + \sup_{0\leq s\leq t} \mathbf{E} \left(\int_0^s \left\| \int_{\mathbb{R}^d} \mathcal{K}(s-\tau, x-\xi)|u(\alpha(\tau))|d\xi \right\|^2_{L_2^\rho(\mathbb{R}^d)} d\tau \right)^{\frac{p}{2}} \right)$$

$$\leq 2^{p-1}L^p t^{\frac{p}{2}} \left(t^{\frac{p}{2}} \left(\int_{\mathbb{R}^d} \rho(x)dx \right)^{\frac{p}{2}} + C_\rho^{\frac{p}{2}}(T)\mathbf{E} \left(\int_0^t \|u(\alpha(\tau))\|^2_{L_2^\rho(\mathbb{R}^d)} d\tau \right)^{\frac{p}{2}} \right)$$

$$\leq 2^{p-1}L^p t^{\frac{p}{2}} \left(t^{\frac{p}{2}} \left(\int_{\mathbb{R}^d} \rho(x)dx \right)^{\frac{p}{2}} + C_\rho^{\frac{p}{2}}(T)t^{\frac{p-2}{2}}\mathbf{E} \int_0^t \|u(\alpha(\tau))\|^p_{L_2^\rho(\mathbb{R}^d)} d\tau \right)$$

$$= 2^{p-1}L^p t^{\frac{p}{2}} \left(t^{\frac{p}{2}} \left(\int_{\mathbb{R}^d} \rho(x)dx \right)^{\frac{p}{2}} + C_\rho^{\frac{p}{2}}(T)t^{\frac{p-2}{2}} \right.$$

$$\left. \times \mathbf{E} \left(\int_0^{t^*} \|u(\alpha(\tau))\|^p_{L_2^\rho(\mathbb{R}^d)} d\tau + \int_{t^*}^t \|u(\alpha(\tau))\|^p_{L_2^\rho(\mathbb{R}^d)} d\tau \right) \right)$$

$$= 2^{p-1}L^p t^{\frac{p}{2}} \left(t^{\frac{p}{2}} \left(\int_{\mathbb{R}^d} \rho(x)dx \right)^{\frac{p}{2}} + C_\rho^{\frac{p}{2}}(T)t^{\frac{p-2}{2}} \right.$$

$$\left. \times \mathbf{E} \left(\int_{-r}^0 \|\phi(\alpha(\tau))\|^p_{L_2^\rho(\mathbb{R}^d)} \frac{1}{\alpha'(\tau)} d\alpha(\tau) + \int_0^{\alpha(t)} \|u(\alpha(\tau))\|^p_{L_2^\rho(\mathbb{R}^d)} \frac{1}{\alpha'(\tau)} d\alpha(\tau) \right) \right).$$

It follows from Proposition 4.1 that there exists $c > 0$ such that

$$\frac{1}{\alpha'(\tau)} \leq c, \quad 0 \leq \tau \leq t.$$

Hence,

$$2^{p-1}L^p t^{\frac{p}{2}}\left(t^{\frac{p}{2}}\left(\int\limits_{\mathbb{R}^d}\rho(x)dx\right)^{\frac{p}{2}} + C_\rho^{\frac{p}{2}}(T)t^{\frac{p-2}{2}}\mathbf{E}\left(\int\limits_{-r}^{0}\|\phi(\alpha(\tau))\|^p_{L_2^\rho(\mathbb{R}^d)}\frac{1}{\alpha'(\tau)}\,d\alpha(\tau)\right.\right.$$

$$\left.\left.+\int\limits_{0}^{\alpha(t)}\|u(\alpha(\tau))\|^p_{L_2^\rho(\mathbb{R}^d)}\frac{1}{\alpha'(\tau)}d\alpha(\tau)\right)\right)$$

$$\le 2^{p-1}L^p t^{\frac{p}{2}}\left(t^{\frac{p}{2}}\left(\int\limits_{\mathbb{R}^d}\rho(x)dx\right)^{\frac{p}{2}} + cC_\rho^{\frac{p}{2}}(T)t^{\frac{p-2}{2}}\right.$$

$$\left.\times\mathbf{E}\left(\int\limits_{-r}^{0}\|\phi(\tau)\|^p_{L_2^\rho(\mathbb{R}^d)}d\tau + \int\limits_{0}^{\alpha(t)}\|u(\tau)\|^p_{L_2^\rho(\mathbb{R}^d)}d\tau\right)\right)$$

$$\le 2^{p-1}L^p t^{\frac{p}{2}}\left(t^{\frac{p}{2}}\left(\int\limits_{\mathbb{R}^d}\rho(x)dx\right)^{\frac{p}{2}}\right.$$

$$\left.+cC_\rho^{\frac{p}{2}}(T)t^{\frac{p-2}{2}}\left(r\sup_{-r\le\tau\le0}\mathbf{E}\|\phi(\tau)\|^p_{L_2^\rho(\mathbb{R}^d)} + \alpha(t)\sup_{0\le\tau\le\alpha(t)}\mathbf{E}\|u(\tau)\|^p_{L_2^\rho(\mathbb{R}^d)}\right)\right)$$

$$\le 2^{p-1}L^p t^p\left(\left(\int\limits_{\mathbb{R}^d}\rho(x)dx\right)^{\frac{p}{2}} + cC_\rho^{\frac{p}{2}}(T)\right.$$

$$\left.\times\left(\sup_{-r\le\tau\le0}\mathbf{E}\|\phi(\tau)\|^p_{L_2^\rho(\mathbb{R}^d)} + \sup_{0\le\tau\le t}\mathbf{E}\|u(\tau)\|^p_{L_2^\rho(\mathbb{R}^d)}\right)\right) < \infty. \qquad (31)$$

Further, we estimate $\|I_4(s)\|^p_{\mathfrak{B}_{p,t,\rho}}$ for $p > 2$. By virtue of Lemma 7.2 in [18, p. 182], we obtain

$$\mathbf{E}\left\|\int\limits_0^s S(s-\tau)\sigma(\tau,u(\alpha(\tau)),\xi)dW(\tau,x)\right\|^p$$

$$\le C_p\mathbf{E}\left(\int\limits_0^s\|S(s-\tau)\sigma(\tau,u(\alpha(\tau)),\xi)\|^2_{L_2^0}d\tau\right)^{\frac{p}{2}}. \qquad (32)$$

Here, $\|\cdot\|_{L_2^0}$ is the corresponding Hilbert–Schmidt norm appearing in the structure of the stochastic integral over the Q-Wiener process [18, p. 91]. In this case, inequality (32) takes the form

$$\mathbf{E}\left\|\int_0^s S(s-\tau)\sigma(\tau,u(\alpha(\tau)),\xi)dW(\tau,x)\right\|^p$$

$$\leq C_p\mathbf{E}\left(\int_0^s\sum_{n=1}^{\infty}\lambda_n\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\mathcal{K}(s-\tau,x-\xi)\sigma(\tau,u(\alpha(\tau),\xi),\xi)e_n(\xi)d\xi\right)^2\rho(x)dx\right)d\tau\right)^{\frac{p}{2}}$$

$$\leq C_\rho^{\frac{p}{2}}(T)L^pC_p\mathbf{E}\left(\int_0^s\sum_{n=1}^{\infty}\lambda_n\left(\int_{\mathbb{R}^d}(1+|u(\alpha(\tau),x)|)^2e_n^2(x)\rho(x)dx\right)d\tau\right)^{\frac{p}{2}}$$

$$\leq 2^{\frac{p}{2}}\left(\sum_{n=1}^{\infty}\lambda_n\right)^{\frac{p}{2}}C_\rho^{\frac{p}{2}}(T)L^pC_p\left(\int_0^s\int_{\mathbb{R}^d}\rho(x)dxd\tau+\int_0^s\|u(\alpha(\tau))\|^2_{L_2^\rho(\mathbb{R}^d)}d\tau\right)^{\frac{p}{2}}.$$

The expression

$$2^{\frac{p}{2}}\left(\sum_{n=1}^{\infty}\lambda_n\right)^{\frac{p}{2}}C_\rho^{\frac{p}{2}}(T)L^pC_p$$

is denoted by A. Hence, the last expression does not exceed the following expression:

$$2^{\frac{p-2}{2}}At^{\frac{p}{2}}\left(\int_{\mathbb{R}^d}\rho(x)dx\right)^{\frac{p}{2}}+2^{\frac{p-2}{2}}A\left(\int_0^s\|u(\alpha(\tau))\|^2_{L_2^\rho(\mathbb{R}^d)}d\tau\right)^{\frac{p}{2}}$$

$$\leq 2^{\frac{p-2}{2}}At^{\frac{p}{2}}\left(\int_{\mathbb{R}^d}\rho(x)dx\right)^{\frac{p}{2}}+2^{\frac{p-2}{2}}At^{\frac{p-2}{2}}\int_0^s\|u(\alpha(\tau))\|^p_{L_2^\rho(\mathbb{R}^d)}d\tau<\infty. \tag{33}$$

For $p=2$, estimate (33) is established in the same way.

By using estimates (26), (27) (30), (31), and (33), for $u\in\mathfrak{B}_{p,T,\rho}$, we get

$$\|\Psi u\|^p_{\mathfrak{B}_{p,T,\rho}}=\left\|\sum_{j=0}^4 I_j(t)\right\|^p_{\mathfrak{B}_{p,T,\rho}}\leq 5^{p-1}\sum_{j=0}^4\|I_j(t)\|^p_{\mathfrak{B}_{p,T,\rho}}<\infty,$$

i.e., the operator Ψ maps the space $\mathfrak{B}_{p,T,\rho}$ into itself.

We now establish the property of contraction. Note that, for any $\{u, v\} \subset \mathfrak{B}_{p,t,\rho}$, we have

$$\|\Psi u - \Psi v\|^p_{\mathfrak{B}_{p,T,\rho}} = \left\| \sum_{j=1}^{4} (I_j(s)(u) - I_j(s)(v)) \right\|^p_{\mathfrak{B}_{p,T,\rho}} \leq 4^{p-1} \sum_{j=1}^{4} \|I_j(s)(u) - I_j(s)(v)\|^p_{\mathfrak{B}_{p,T,\rho}}.$$

Thus, by virtue of inequalities (27), (30), (31), and (33), we can estimate the quantities

$$\|I_j(s)(u) - I_j(s)(v)\|^p_{\mathfrak{B}_{p,T,\rho}}, \quad j \in \{1, \ldots, 4\},$$

as follows:

$$\|I_1(s)(u) - I_1(s)(v)\|^p_{\mathfrak{B}_{p,t,\rho}}$$

$$= \sup_{0 \leq s \leq t} \mathbf{E} \left\| \int_{\mathbb{R}^d} (b(s, x, u(\alpha(s), \xi), \xi) - b(s, x, v(\alpha(s), \xi), \xi)) d\xi \right\|^p_{L_2^\rho(\mathbb{R}^d)}$$

$$\leq \sup_{0 \leq s \leq t} \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{l^2(s, x, \xi)}{\rho(\xi)} d\xi \right) \rho(x) dx \right)^{\frac{p}{2}} \sup_{0 \leq s \leq t} \|u(s) - v(s)\|^p_{L_2^\rho(\mathbb{R}^d)}$$

$$= \sup_{0 \leq s \leq t} \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{l^2(s, x, \xi)}{\rho(\xi)} d\xi \right) \rho(x) dx \right)^{\frac{p}{2}} \|u - v\|^p_{\mathfrak{B}_{p,t,\rho}},$$

$$\|I_2(s)(u) - I_2(s)(v)\|^p_{\mathfrak{B}_{p,t,\rho}}$$

$$= \sup_{0 \leq s \leq t} \mathbf{E} \left\| \int_0^s \left(\Delta_x \int_{\mathbb{R}^d} \mathcal{K}(s - \tau, x - \xi) \right.\right.$$

$$\left.\left. \times \left(\int_{\mathbb{R}^d} (b(\tau, \xi, u(\alpha(\tau), \zeta), \zeta) - b(\tau, \xi, v(\alpha(\tau), \zeta), \zeta)) d\zeta \right) d\xi \right) d\tau \right\|^p_{L_2^\rho(\mathbb{R}^d)}$$

$$\leq C t^p \sup_{0 \leq \tau \leq t} \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\psi^2(\tau, x, \zeta)}{\rho(\zeta)} \, d\zeta dx \right)^{\frac{p}{2}} \sup_{0 \leq \tau \leq t} \|u(\tau) - v(\tau)\|^p_{L_2^\rho(\mathbb{R}^d)}$$

$$= Ct^p \sup_{0\le\tau\le t} \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\psi^2(\tau,x,\zeta)}{\rho(\zeta)} d\zeta dx \right)^{\frac{p}{2}} \|u-v\|^p_{\mathfrak{B}_{p,t,\rho}},$$

$$\|I_3(s)(u) - I_3(s)(v)\|^p_{\mathfrak{B}_{p,t,\rho}}$$

$$= \sup_{0\le s\le t} \mathbf{E} \left\| \int_0^s \int_{\mathbb{R}^d} \mathcal{K}(s-\tau, x-\xi)(f(\tau, u(\alpha(\tau)),\xi) - f(\tau, v(\alpha(\tau)),\xi)) d\xi d\tau \right\|^2_{L_2^\rho(\mathbb{R}^d)}$$

$$\le cC_\rho^{\frac{p}{2}}(T)L^p t^p \sup_{0\le\tau\le t} \|u(\tau)-v(\tau)\|^p_{L_2^\rho(\mathbb{R}^d)} = cC_\rho^{\frac{p}{2}}(T)L^p t^p \|u-v\|^p_{\mathfrak{B}_{p,t,\rho}},$$

$$\|I_4(s)(u) - I_4(s)(v)\|^p_{\mathfrak{B}_{p,t,\rho}}$$

$$= \sup_{0\le s\le t} \mathbf{E} \Bigg\| \int_0^s \sum_{n=1}^{\infty} \sqrt{\lambda_n} \Bigg(\int_{\mathbb{R}^d} \mathcal{K}(s-\tau, x-\xi)$$

$$\times (\sigma(\tau, u(\alpha(\tau)),\xi) - \sigma(\tau, v(\alpha(\tau)),\xi)) e_n(\xi) d\xi \Bigg) d\beta_n(\tau) \Bigg\|^p_{L_2^\rho(\mathbb{R}^d)}$$

$$\le \left(\sum_{n=1}^{\infty} \lambda_n \right)^{\frac{p}{2}} C_\rho^{\frac{p}{2}}(T) L^p C_p \left(\int_0^t \|u(\alpha(\tau)) - v(\alpha(\tau))\|^2_{L_2^\rho(\mathbb{R}^d)} d\tau \right)^{\frac{p}{2}}$$

$$\le c \left(\sum_{n=1}^{\infty} \lambda_n \right) C_\rho^{\frac{p}{2}}(T) L^p C_p t^{\frac{p}{2}} \sup_{0\le\tau\le t} \|u(\tau)-v(\tau)\|^p_{L_2^\rho(\mathbb{R}^d)}$$

$$= c \left(\sum_{n=1}^{\infty} \lambda_n \right) C_\rho^{\frac{p}{2}}(T) L^p C_p t^{\frac{p}{2}} \|u-v\|^p_{\mathfrak{B}_{p,t,\rho}}.$$

For any $\{u, v\} \subset \mathfrak{B}_{p,t,\rho}$, this yields

$$\|\Psi u - \Psi v\|^p_{\mathfrak{B}_{p,t,\rho}} \le 4^{p-1} \left(\sup_{0\le s\le t} \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{l^2(s,x,\xi)}{\rho(\xi)} d\xi \right) \rho(x) dx \right)^{\frac{p}{2}} \right.$$

$$+ Ct^p \sup_{0\le\tau\le t} \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\psi^2(\tau,x,\zeta)}{\rho(\zeta)} d\zeta dx \right)^{\frac{p}{2}}$$

$$+c\, C_\rho^{\frac{p}{2}}(T)L^p t^p + c\left(\sum_{n=1}^{\infty}\lambda_n\right) C_\rho^{\frac{p}{2}}(T)L^p C_p t^{\frac{p}{2}}\Bigg)\|u-v\|^p_{\mathfrak{B}_{p,t,\rho}}$$

$$= \gamma(t)\|u-v\|^p_{\mathfrak{B}_{p,t,\rho}}.$$

By virtue of (25), the first term in γ is smaller than 1. Thus, choosing sufficiently small $0 \le t_1 \le T$, we conclude that $0 \le \gamma(t_1) < 1$. This means that the operator Ψ given in the Banach space $\mathfrak{B}_{p,t_1,\rho}$ is contracting and, hence, possesses a unique fixed point, which is a solution $u \in \mathfrak{B}_{p,t_1,\rho}$ of the equation $\Psi u = u$. This procedure can be repeated finitely many times on the other sufficiently small segments

$$[t_1, t_2], [t_2, t_3], \dots, [t_{n-2}, t_{n-1}], [t_{n-1}, T]$$

that form $[0, T]$. As a result, we obtain the required solution as the union of solutions on these segments. This proves the theorem.

Example. We now consider the Cauchy problem

$$d\left(u(t,x) + \int_{\mathbb{R}} \sqrt{\boldsymbol{c}\rho(\xi)}\exp\left\{-t-|\xi|-2x^2\right\}\sin u(t-h,\xi)d\xi\right)$$

$$= \left(\Delta_x u(t,x) + f(t,x)\cos u(t-h,x)\right)dt + g(t,x)\sin u(t-h,x)dW(t,x), \tag{34}$$

$$0 < t \le T, \quad x \in \mathbb{R},$$

$$u(t,x) = \phi(t,x), \quad h \le t \le 0, \quad x \in \mathbb{R}, \quad h > 0,$$

where

$$0 < \boldsymbol{c} < \frac{\sqrt[p]{16}}{8\sqrt{\pi}}$$

and $\{f, g\}$ are measurable bounded functions and check the conditions of Theorem 1 for this problem.

Conditions (12)–(15) are obvious.

We now verify condition (16). Thus, we find

$$|b(t,x,u,\xi) - b(t,x,v,\xi)| = \sqrt{\boldsymbol{c}\rho(\xi)}\exp\left\{-t-|\xi|-2x^2\right\}|\sin u - \sin v|$$

$$\le \sqrt{\boldsymbol{c}\rho(\xi)}\exp\left\{-t-|\xi|-2x^2\right\}|u-v| = l(t,x,\xi)|u-v|,$$

i.e., we arrive at condition (16) with

$$l(t,x,\xi) = \sqrt{\boldsymbol{c}\rho(\xi)}\exp\{-t-|\xi|-2x^2\}.$$

We now check conditions (17) and (18) for l:

$$\sup_{0\le t\le T}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\frac{l^2(t,x,\xi)}{\rho(\xi)}d\xi\right)\rho(x)dx$$

$$=\sup_{0\le t\le T}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\frac{\boldsymbol{c}\rho(\xi)\exp\{-2t-2|\xi|-4x^2\}}{\rho(\xi)}\,d\xi\right)\rho(x)dx$$

$$=\boldsymbol{c}\left(\sup_{0\le t\le T}\exp\{-2t\}\right)\left(\int_{\mathbb{R}}\exp\{-2|\xi|\}d\xi\right)\int_{\mathbb{R}}\exp\{-4x^2\}\rho(x)dx$$

$$\le\boldsymbol{c}\int_{\mathbb{R}}\exp\{-4x^2\}dx=\frac{\boldsymbol{c}\sqrt{\pi}}{2}<\infty,$$

$$\sup_{0\le t\le T}\int_{\mathbb{R}}\sqrt{\int_{\mathbb{R}}\frac{l^2(t,x,\xi)}{\rho(\xi)}\,d\xi}\,dx$$

$$=\sup_{0\le t\le T}\int_{\mathbb{R}}\sqrt{\int_{\mathbb{R}}\frac{\boldsymbol{c}\rho(\xi)\exp\{-2t-2|\xi|-4x^2\}}{\rho(\xi)}\,d\xi}\,dx$$

$$=\sqrt{\boldsymbol{c}}\left(\sup_{0\le t\le T}\exp\{-t\}\right)\left(\int_{\mathbb{R}}\exp\left\{-2x^2\right\}dx\right)\sqrt{\int_{\mathbb{R}}\exp\{-2|\xi|\}d\xi}$$

$$=\sqrt{\frac{\boldsymbol{c}\pi}{2}}<\infty,$$

i.e., conditions (17) and (18) are satisfied.

Further, we establish the validity of conditions (19)–(25) by using the following transformations:

$$|\partial_x b(t,x,u,\xi)|=4x\exp\left\{-x^2\right\}\sqrt{\boldsymbol{c}\rho(\xi)}\exp\left\{-t-|\xi|-x^2\right\}|\sin u|$$

$$\le 2\sqrt{\frac{2}{e}}\sqrt{\boldsymbol{c}\rho(\xi)}\exp\left\{-t-|\xi|-x^2\right\}(1+|u|)$$

$$=\psi_1(t,x,\xi)(1+|u|),$$

$$|\partial_x^2 b(t,x,u,\xi)|\le 4(1+4x^2)\exp\left\{-x^2\right\}\sqrt{\boldsymbol{c}\rho(\xi)}\exp\left\{-t-|\xi|-x^2\right\}|\sin u|$$

$$\leq \frac{20}{e}\sqrt{\boldsymbol{c}\rho(\xi)}\exp\left\{-t-|\xi|-x^2\right\}(1+|u|)$$

$$= \psi_2(t,x,\xi)(1+|u|),$$

$$|\partial_x^2 b(t,x,u,\xi)-\partial_x^2 b(t,x,v,\xi)| = 4\left|1-4x^2\right|\sqrt{\boldsymbol{c}\rho(\xi)}\exp\left\{-t-|\xi|-2x^2\right\}|\sin u-\sin v|$$

$$\leq \frac{20}{e}\sqrt{\boldsymbol{c}\rho(\xi)}\exp\{-t-|\xi|-x^2\}|u-v|$$

$$= \psi_2(t,x,\xi)|u-v|,$$

i.e., we arrive at conditions (19) and (20), where

$$\psi(t,x,\xi) = \max\{\psi_1(t,x,\xi),\psi_2(t,x,\xi)\} = \frac{20}{e}\sqrt{\boldsymbol{c}\rho(\xi)}\exp\{-t-|\xi|-x^2\}.$$

We now check the validity of conditions (21), (22), and (24) for ψ:

$$\sup_{0\leq t\leq T}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\psi(t,x,\xi)d\xi\right)^2 dx$$

$$= \frac{400}{e^2}\sup_{0\leq t\leq T}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\sqrt{\boldsymbol{c}\rho(\xi)}\exp\left\{-t-|\xi|-x^2\right\}d\xi\right)^2 dx$$

$$= \frac{400\boldsymbol{c}}{e^2}\left(\sup_{0\leq t\leq T}\exp\{-2t\}\right)\left(\int_{\mathbb{R}}\sqrt{\rho(\xi)}\exp\{-|\xi|\}d\xi\right)^2\int_{\mathbb{R}}\exp\left\{-2x^2\right\}dx$$

$$\leq \frac{400\boldsymbol{c}}{e^2}\sqrt{\frac{\pi}{2}}\left(\int_{\mathbb{R}}\rho(\xi)d\xi\right)\int_{\mathbb{R}}\exp\{-2|\xi|\}d\xi$$

$$= \frac{400\boldsymbol{c}}{e^2}\sqrt{\frac{\pi}{2}}\int_{\mathbb{R}}\rho(\xi)d\xi < \infty,$$

$$\sup_{0\leq t\leq T}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\psi^2(t,x,\xi)}{\rho(\xi)}d\xi dx$$

$$= \frac{400}{e^2}\sup_{0\leq t\leq T}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\boldsymbol{c}\rho(\xi)\exp\left\{-2t-2|\xi|-2x^2\right\}}{\rho(\xi)}d\xi\, dx$$

$$= \frac{400\boldsymbol{c}}{e^2}\left(\sup_{0\le t\le T}\exp\{-2t\}\right)\left(\int_{\mathbb{R}}\exp\{-2|\xi|\}d\xi\right)\int_{\mathbb{R}}\exp\left\{-2x^2\right\}dx$$

$$= \frac{400\boldsymbol{c}}{e^2}\sqrt{\frac{\pi}{2}} < \infty,$$

i.e., we arrive at conditions (21) and (22). We fix a point $x_0 \in \mathbb{R}$ and prove that there exists a function $\varphi(t, \xi, x_0, \delta)$, $\delta \in \mathbb{R}^+$, from condition (24) satisfying condition (23):

$$|\psi(t,x,\xi) - \psi(t,x_0,\xi)| \le \frac{20}{e}\sqrt{\boldsymbol{c}\rho(\xi)}\exp\{-t-|\xi|\}(\delta+2|x_0|)|x-x_0|$$

$$= \varphi(t,\xi,x_0,\delta)|x-x_0|,$$

i.e., the function ψ satisfies condition (24) with the function

$$\varphi(t,\xi,x_0,\delta) = \frac{20}{e}\sqrt{\boldsymbol{c}\rho(\xi)}\exp\{-t-|\xi|\}(\delta+2|x_0|).$$

We now verify condition (23) for this function:

$$\sup_{0\le t\le T}\int_{\mathbb{R}}\frac{\varphi^2(t,\xi,x_0,\delta)}{\rho(\xi)}d\xi$$

$$= \frac{400}{e^2}(\delta+2|x_0|)^2\sup_{0\le t\le T}\int_{\mathbb{R}}\frac{\boldsymbol{c}\rho(\xi)\exp\{-2t-2|\xi|\}}{\rho(\xi)}d\xi$$

$$= \frac{400\boldsymbol{c}}{e^2}(\delta+2|x_0|)^2\left(\sup_{0\le t\le T}\exp\{-2t\}\right)\int_{\mathbb{R}}\exp\{-2|\xi|\}d\xi$$

$$= \frac{400\boldsymbol{c}}{e^2}(\delta+2|x_0|)^2 < \infty.$$

Condition (25) takes the form

$$\sup_{0\le t\le T}\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\frac{l^2(t,x,\xi)}{\rho(\xi)}d\xi\right)\rho(x)dx\right)^{\frac{p}{2}}$$

$$= \left(\frac{\boldsymbol{c}\sqrt{\pi}}{2}\right)^{\frac{p}{2}} < \left(\frac{\sqrt[p]{16}}{8\sqrt{\pi}}\frac{\sqrt{\pi}}{2}\right)^{\frac{p}{2}}$$

$$= \left(\frac{\sqrt[p]{16}}{16} \right)^{\frac{p}{2}} = \frac{1}{4^{p-1}},$$

which means that the conditions of Theorem 1 are satisfied for problem (34).

6. Corollary of the Theorem

In a special case of problem (1), i.e., for the initial-value problem

$$d\left(u(t,x) + \int_{\mathbb{R}^d} b(t,x,\xi)u(t-h,\xi)d\xi \right)$$
$$= (\Delta_x u(t,x) + f(t,u(t-h),x))\,dt + \sigma(t,u(t-h),x)dW(t,x), \quad 0 < t \le T, \quad x \in \mathbb{R}^d, \tag{35}$$
$$u(t,x) = \phi(t,x), \quad -h \le t \le 0, \quad x \in \mathbb{R}^d, \quad h > 0,$$

the following theorem is true:

Theorem 2. *Suppose that the following conditions are satisfied:*

(i) $\{f,\sigma\}\colon [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ *are the functions from assertion (i) in Theorem 1;*

(ii) $b\colon [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ *is a measurable function such that*

$$\sup_{0 \le t \le T} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{b^2(t,x,\xi)}{\rho(\xi)}\, d\xi \right) \rho(x)dx < \infty$$

and

$$\sup_{0 \le t \le T} \int_{\mathbb{R}^d} \sqrt{\int_{\mathbb{R}^d} \frac{b^2(t,x,\xi)}{\rho(\xi)}\, d\xi}\, dx < \infty;$$

(iii) for any $x \in \mathbb{R}^d$, *there exist the derivatives* $\partial_{x_i} b$ *and* $\partial_{x_i x_j} b$, $\{i,j\} \subset \{1,\ldots,d\}$; *moreover, the gradient* $\nabla_x b$, *and the matrix* $D_x^2 b$ *satisfy the conditions*

$$|\nabla_x b(t,x,\xi)| + \left\| D_x^2 b(t,x,\xi) \right\| \le \psi(t,x,\xi), \quad 0 \le t \le T, \quad \{x,\xi\} \subset \mathbb{R}^d,$$

where the function ψ *satisfies the conditions of assertion (iii) in Theorem 1.*

Under these conditions, if

$$\sup_{0 \le t \le T} \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{b^2(t,x,\xi)}{\rho(\xi)} d\xi \right) \rho(x)dx \right)^{\frac{p}{2}} < \frac{1}{4^{p-1}},$$

then problem (35) possesses a unique mild solution $u \in \mathfrak{B}_{p,T,\rho}$ *on* $0 \leq t \leq T$.

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