Lyapunov function in the Hyper Complex Phase Space

Roman Voliansky[1[0000-0001-5674-7646]], Nina Volianska[1[0000-0001-5996-2341]], Valeriy Kuznetsov[2[0000-0003-4165-1056]], Aleksandr Sadovoi[3[0000-0002-3284-2646]] , Vitaliy Kuznetsov[4[0000-0002-8169-4598]], Yevheniia Kuznetsova[4[0000-0003-2224-8747]], Oleksandr Ostapchuk[5[0000-0003-3397-2423]]

[1]Dniprovsk State Technical University, 2, Dniprobudivska Str., Kamyanske, 51918, Ukraine
[2]Railway Research Institute_2, Chlopickiego str., 50, Warsaw, Poland,
[3]Dnipro University of Technology, 19, Karl Marx Av., Dnipro, 49000, Ukraine
[4]National metallurgical academy of Ukraine, Gagarina avenue, 4, Dnipro, Ukraine
[5]Igor Sikorsky Kyiv Polytechnic Institute, Kyiv, Ukraine

Abstract. The paper deals with the development of background for defining Lyapunov functions for a wide range of linear dynamical objects. This background is based on assuming that the Lyapunov function is redundant energy in the considered object and this energy is dissipated only during controlled motion. We assume the full derivative of the Lyapunov function for an autonomous motion of the control objects equals zero and we use its summands to define linear algebraic equations. The solution of these equations allows us to find unknown terms of the Lyapunov function. The use of these terms, while the Lyapunov equation is being written down, shows that the left-hand expression in the Lyapunov equation is equal to the zero matrix. Thus, we avoid subjective assuming of quadratic form terms in the right-hand of the Lyapunov equation.
We extend the proposed approach to the class dynamical system with uncertainty. This extension is performed by using interval methods, which allow defining object motions for minimal and maximal values of parameters. We show that for the control object, which parameters are not exactly known, one should consider two equations of object motions, which correspond to its trajectories on the boundaries of the intervals. Lyapunov functions are defined for these boundary trajectories.
Since such an approach increases the number of the considered equations we offer to decrease them by using hyper-complex numbers while object equations are written down.

Keywords: Lyapunov Function, Hyper-Complex Domain, Uncertain Parameters.

1 Introduction

Lyapunov methods are an extremely powerful background to analyze and synthesize stable object dynamic [1,2,3]. This background is based on the so-called Lyapunov functions [4], which have specific features that make it possible to study regular and chaotic dynamics in a simple way [5,6].

The wide use of Lyapunov functions explains by their strong mathematical definition and physical sense. Nowadays Lyapunov functions are considered from the energy viewpoint as some energies, which are stored, transformed, and dissipated in the considered objects during their motions [7,8,9]. The main advantage of these functions is the possibility to study the stability of any simple and complex dynamical objects.

This fact makes preconditions to use these functions why complex renewable energy systems dynamics are being analyzed [10,11] and stability conditions are defined.

Today Lyapunov functions are defined as non-quadratic [12] and quadratic [13,14,15] functions. A wide class of mathematical functions describes the first ones. The use of functions from this class allows us to study and form unique motions of various control objects and systems. However, the main known drawback of these functions is neglecting of the physical meaning of Lyapunov functions and it requires physical and mathematical justification of these functions usage and when this drawback will be avoided one can find usage of these functions are very powerful.

Classical Lyapunov functions are quadratic functions which properties are approved by Sylvester criterion [16] and there are a lot of methods to construct these functions [17,18,19,20]. Analysis of these methods shows subjectivism while terms of these functions are being defined. It makes causes different results for the same object, which depends on the experience of the researcher and his preferences.

In our paper, we offer to avoid this drawback and construct a strong Lyapunov function. We study the autonomous motion of the control object and study the full derivative of the Lyapunov function to achieve our goal.

Our paper is organized as follows: at first, we show the proposed method for the generalized linear object and closed-loop control system with exactly known parameters. Then we consider the definition of Lyapunov function for the control object with uncertainty. At third, we offer to transform the Lyapunov function into a compact form by using hyper-complex numbers, then we show an example of defining the Lyapunov function for the 2nd order dynamical object with uncertain parameters and complex state-space variables. At last, we make a conclusion.

2 Method

2.1 Lyapunov Functions for the Linear Objects which Dynamics are Given in the Real State Space

Let us consider matrix linear differential equation in the operator form.

$$s\mathbf{X} = \mathbf{AX} + \mathbf{BU}; \quad \mathbf{Y} = \mathbf{CX} + \mathbf{DU} , \tag{1}$$

here s=d/dt is a Lagrange differential operator, **X** is a n-th sized vector of the state variables, Y is m-th sized vector of the observed variables and **U** is k-th sized vector of input signals. We define above-mentioned vectors as follows

$$\mathbf{X} = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix}^T; \quad \mathbf{Y} = \begin{pmatrix} y_1 & y_2 & \dots & y_m \end{pmatrix}^T; \quad \mathbf{U} = \begin{pmatrix} u_1 & u_2 & \dots & u_k \end{pmatrix}^T, \tag{2}$$

A is nxn-th sized and **B** is kxn-th sized matrices of objects parameters

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}; \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1k} \\ b_{21} & b_{22} & \dots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nk} \end{pmatrix}, \tag{3}$$

C is nxm-th sized and **D** is nxk-th sized matrices of observability

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{pmatrix}; \quad \mathbf{D} = \begin{pmatrix} d_{11} & d_{12} & \dots & d_{1k} \\ d_{21} & d_{22} & \dots & d_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ d_{m1} & d_{m2} & \dots & d_{mk} \end{pmatrix}. \tag{4}$$

We use (1) to define all possible motions of linear objects with time-independent parameters. It is clear that these motions depend on input signals **U** and one can observe them by output vector **Y**.

Since (1) are described with both differential and algebraic matrix equations, its analysis as well as definition of any interrelations between inputs, state variables, and outputs are based on the joint solution of these equations. One can find that it is quite inconvenient to perform this solution while objects dynamics are analyzed.

That is why we offer transforming (1) into differential form by solving second equation for X and substituting results into the first one. After performing such transformation, we get following equation

$$s\mathbf{Y} = \mathbf{A}_Y \mathbf{Y} + \left(\mathbf{B}_Y + s\mathbf{B1} \right) \mathbf{U}, \tag{5}$$

$$\mathbf{A}_Y = \mathbf{CAC}^{-1}; \quad \mathbf{B}_Y = \mathbf{CB} - \mathbf{CAC}^{-1}\mathbf{D}; \quad \mathbf{B1} = \mathbf{D}. \tag{6}$$

Contrary to (1) which is defined dynamics of the considered linear objects by all state variables, (5) is defined these dynamics by using only measured variables. It is quite clear that in the mostly cases (5) has lower than (1) order. Thus, its solution are simpler than solution of initial equation.

Nevertheless, different equations are used to define objects dynamics, one can find that theirs dynamics are controlled ones if non-zero **U** vectors are used and autonomous ones otherwise. Since (1) and (5) define controllable motions of the considered objects, one can use these equations to write down autonomous-motion equations as follows

$$s\mathbf{X} = \mathbf{AX}; \tag{7}$$

$$s\mathbf{Y} = \mathbf{A}_Y \mathbf{Y}. \tag{8}$$

It is a well-known fact, that Lyapunov function shows redundant objects energies and during their motions from initial coordinates in state space to desired ones, these energies should be dissipated if objects motions are stable. The classical definition of Lyapunov function is quadratic function

$$V = \mathbf{N}^T \mathbf{K} \mathbf{N}, \tag{9}$$

here **N** is a vector of object coordinates, **K** is a square symmetric matrix.

If one assumes that energy dissipation occurs only for controllable motions, he can conclude that there is no any energy change while object moves free. This assumption allows us to write down following rule for full derivative of Lyapunov function

$$sV = \nabla \mathbf{N}^T \mathbf{K} \mathbf{N} s \mathbf{N} = 0. \tag{10}$$

We use (7) and (8) to write down full and reduced Lyapunov functions

$$V_X = \mathbf{X}^T \mathbf{K}_X \mathbf{X}, \tag{11}$$

$$V_Y = \mathbf{Y}^T \mathbf{K}_Y \mathbf{Y}, \tag{12}$$

and theirs derivatives

$$sV_X = \nabla \mathbf{X}^T \mathbf{K}_X \mathbf{X} \mathbf{A} \mathbf{X} = 0, \tag{13}$$

$$sV_Y = \nabla \mathbf{Y}^T \mathbf{K}_Y \mathbf{Y} \mathbf{A}_Y \mathbf{Y} = 0. \tag{14}$$

Solution of (13) and (14) gives us possibility to define unknown matrices $\mathbf{K}_X$ and $\mathbf{K}_Y$. It is clearly understood, that elements of these matrices depend only objects parameters and contrary to well-known stationary Lyapunov equation [21] do not depend on any empiric quadratic form and is reduced to following form

$$\mathbf{AK} + \mathbf{KA}^T = \mathbf{0}. \tag{15}$$

It should be mentioned that the proposed approach can be used for the defining Lyapunov functions for the control objects as well as for the defining these functions for the closed-loop control systems.

Since not all state variables can be measured we use (5) in this case and substitute into this equation control law which can be generalized as follows

$$\mathbf{U} = -\mathbf{WZ}, \tag{16}$$

$$\mathbf{W} = \begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1p} \\ w_{21} & w_{22} & \cdots & w_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ w_{k1} & w_{k2} & \cdots & w_{kp} \end{pmatrix}; \quad \mathbf{Z} = \begin{pmatrix} \mathbf{Y_0} \\ \mathbf{Y} \end{pmatrix}. \tag{17}$$

Here we assume that control laws can be defined by using p-n integrals from the controlled variable. These integrals extend vector of the measured variables and give us possibility to rewrite its by using two sub-vectors measured variables **Y** and integrals of controlled variable $\mathbf{Y_0}$.

If one extends matrix A by defining interrelations between components of $\mathbf{Y_0}$ vector, he can rewrite (5) as follows

$$s\mathbf{Z} = \mathbf{A}_Z\mathbf{Z}, , \tag{18}$$

$$\mathbf{A_Z} = \left(\mathbf{A}_{Ye} - \mathbf{B}_Y\mathbf{W}\right)^{-1}\left(\mathbf{E} + \mathbf{B1W}\right), \tag{19}$$

here A_{Ye} is pxp-th sized extended matrix of closed-loop system parameters, **E** is identity matrix.

If one takes into account (18), he can rewrite Lyapunov function (12) in such a way

$$V_Z = \mathbf{Z}^T\mathbf{K}_Z\mathbf{Z} \tag{20}$$

and define it terms by solving following equation

$$sV_z = \nabla\mathbf{Z}^T\mathbf{K}_Z\mathbf{Z}\mathbf{A}_Z\mathbf{Z} = 0 \ . \tag{21}$$

It is clear that function (20) is defined by using extended vector of the closed-loop system state variables. Comparison of (11), (12) and (20) as well as (13), (14) and (21) shows that form of the Lyapunov functions and their derivatives are not changed. One can find changings only in the number of summands of Lyapunov functions. This fact allows us in the future to consider only one case closed-loop system for example and use obtained results for study control object as well.

2.2 Lyapunov Functions for the Linear Objects which Uncertain Parameters

Above-considered approach are considered for the linear objects which parameters are exactly known. But parameters of the most of technical systems are changed during system operations, moreover some of these parameters cannot be defined exactly due to some uncertainties.

That is why we expand proposed approach on the class of the linear dynamical objects with uncertain parameters. Such class of object we define by using interval differential equations [22-24]

$$s\left[\mathbf{Z}_{\mathbf{min}}, \mathbf{Z}_{\mathbf{max}}\right] = \left[\mathbf{A}_{Z\mathbf{min}}, \mathbf{A}_{Z\mathbf{max}}\right]\left[\mathbf{Z}_{\mathbf{min}}, \mathbf{Z}_{\mathbf{max}}\right], \tag{22}$$

here $\mathbf{A_{min}}$, and $\mathbf{A_{max}}$ are square matrices with minimal and maximal object parameters, $\mathbf{Z_{min}}$, and $\mathbf{Z_{max}}$ are vectors with minimal and maximal values of object variables.

Real object motions (18) is always between lower and upper object trajectories which are described by following equations

$$s\mathbf{Z}_{\mathbf{min}} = \mathbf{A}_{Z\mathbf{min}}\mathbf{Z}_{\mathbf{min}}; \quad s\mathbf{Z}_{\mathbf{max}} = \mathbf{A}_{Z\mathbf{max}}\mathbf{Z}_{\mathbf{max}} \ . \tag{23}$$

These equations give us possibility to define interval Lyapunov function

$$\left[V_{Z\,min}, V_{Z\,max}\right] = \left[\mathbf{Z}_{\mathbf{min}}^{T}, \mathbf{Z}_{\mathbf{max}}^{T}\right]\left[\mathbf{K}_{\mathbf{Zmin}}, \mathbf{K}_{\mathbf{Zmax}}\right]\left[\mathbf{Z}_{\mathbf{min}}, \mathbf{Z}_{\mathbf{max}}\right] = \left[\mathbf{Z}_{\mathbf{min}}^{T}\mathbf{K}_{\mathbf{Zmin}}\mathbf{Z}_{\mathbf{min}}, \mathbf{Z}_{\mathbf{max}}^{T}\mathbf{K}_{\mathbf{Zmax}}\mathbf{Z}_{\mathbf{max}}\right] \quad (24)$$

and its derivative

$$s\left[V_{z\,min}, V_{z\,min}\right] = \nabla\left[\mathbf{Z}_{\mathbf{min}}^{T}, \mathbf{Z}_{\mathbf{max}}^{T}\right]\left[\mathbf{K}_{\mathbf{Zmin}}, \mathbf{K}_{\mathbf{Zmax}}\right]\left[\mathbf{Z}_{\mathbf{min}}, \mathbf{Z}_{\mathbf{max}}\right]\left[\mathbf{A}_{Z\mathbf{min}}, \mathbf{A}_{Z\mathbf{max}}\right]\times \times\left[\mathbf{Z}_{\mathbf{min}}, \mathbf{Z}_{\mathbf{max}}\right] = \left[\nabla\mathbf{Z}_{\mathbf{min}}^{T}\mathbf{K}_{\mathbf{Zmin}}\mathbf{Z}_{\mathbf{min}}\mathbf{A}_{Z\mathbf{min}}\mathbf{Z}_{\mathbf{min}}, \nabla\mathbf{Z}_{\mathbf{max}}^{T}\mathbf{K}_{\mathbf{Zmax}}\mathbf{Z}_{\mathbf{max}}\mathbf{A}_{Z\mathbf{max}}\mathbf{Z}_{\mathbf{max}}\right] = \mathbf{0} \quad (25)$$

Thus, one can solve (25) to define terms of the interval Lyapunov function (24).

It should be mentioned that terms of function (24) and parameters of object (22) allows us to write down interval Lyapunov equation

$$\left[\mathbf{A}_{\mathbf{Zmin}}, \mathbf{A}_{\mathbf{Zmax}}\right]\left[\mathbf{K}_{\mathbf{Zmin}}, \mathbf{K}_{\mathbf{Zmax}}\right] + \left[\mathbf{K}_{\mathbf{Zmin}}, \mathbf{K}_{\mathbf{Zmax}}\right]\left[\mathbf{A}_{\mathbf{Zmin}}, \mathbf{A}_{\mathbf{Zmax}}\right]^{T} = \mathbf{0} \quad (26)$$

or in extended form

$$\begin{aligned} &\mathbf{A}_{\mathbf{Zmin}}\mathbf{K}_{\mathbf{Zmin}} + \mathbf{K}_{\mathbf{Zmin}}\mathbf{A}_{\mathbf{Zmin}}^{T} = \mathbf{0}; \\ &\mathbf{A}_{\mathbf{Zmax}}\mathbf{K}_{\mathbf{Zmax}} + \mathbf{K}_{\mathbf{Zmax}}\mathbf{A}_{\mathbf{Zmax}}^{T} = \mathbf{0}. \end{aligned} \quad (27)$$

One can use (25) to define Lyapunov function (24) which shows lower and upper redundant energies for dynamical object with exactly not known parameters. These energies allows us to refine first Lyapunov methods by replacing domain which is bounded by some hyper-sphere in state space with smaller domain which has lower and upper boundaries. The stable autonomous object moves only in this domain.

2.3 Lyapunov Functions for the Linear Objects which Uncertain Parameters in Hyper-Complex Phase Space

The use of interval methods to describe dynamic of objects with uncertainty twice increase order of the used differential equations and one has to solve two matrix equations (21) instead of one to find Lyapunov function. Thus, we can claim increasing computational complexity of dynamical system analysis.

This complexity can be reduced if one describe dynamic of the control system by hyper-complex state variables

$$\mathbf{Z} = \mathbf{Z}_{0} + \sum_{i=1}^{q}\mathbf{Z}_{i}e_{i}\ , \quad (28)$$

here q is a dimension of the hyper-complex space, e_i is a i-th imaginary unit, and $\mathbf{Z_i}$ is a i-th vector of object variables

and object parameters

$$\mathbf{A_Z} = \mathbf{A}_{\mathbf{Z}0} + \sum_{i=1}^{q} \mathbf{A}_{\mathbf{Z}i} e_i \ , \tag{29}$$

This approach allows us to use 2p-q equations instead of initial 2p equations. Thus it is very useful while dynamic of multichannel object are defined.

Equations (28) and (29) make it possible to rewrite (22) as follows

$$\begin{aligned} &s\left[\mathbf{Z}_{0\min} + \sum_{i=1}^{q}\mathbf{Z}_{i\min}e_i, \mathbf{Z}_{0\max} + \sum_{i=1}^{q}\mathbf{Z}_{i\max}e_i\right] = \left[\mathbf{A}_{\mathbf{Z}0\min} + \sum_{i=1}^{q}\mathbf{A}_{\mathbf{Z}i\min}e_i,\right. \\ &\left.\mathbf{A}_{\mathbf{Z}0\max} + \sum_{i=1}^{q}\mathbf{A}_{\mathbf{Z}i\max}e_i\right]\left[\mathbf{Z}_{0\min} + \sum_{i=1}^{q}\mathbf{Z}_{i\min}e_i, \mathbf{Z}_{0\max} + \sum_{i=1}^{q}\mathbf{Z}_{i\max}e_i\right]. \end{aligned} \tag{30}$$

It is necessary to say that equation (30) can be defined by using well-known hyper-complex numbers and one can define its own numbers too. In this case, own definition and mathematical sense for imaginary units should be given and own hyper-complex algebra should be defined as well. Here we do not consider such definitions but we assume that scalar and matrix sum and product algebraic operations for hyper-complex numbers (28) and (29) are defined and we can use them to rewrite interval Lyapunov function (24) and its derivative as follows (25)

$$\begin{aligned} &\left[V_{Z\,min}, V_{Z\,max}\right] = \left[\left(\mathbf{Z}_{0\min}^{T} + \sum_{i=1}^{q}\mathbf{Z}_{i\min}^{T}e_i\right)\left(\mathbf{K}_{\mathbf{Z}\min} + \sum_{i=1}^{q}\mathbf{K}_{\mathbf{Z}i\min}e_i\right)\left(\mathbf{Z}_{0\min} + \sum_{i=1}^{q}\mathbf{Z}_{i\min}e_i\right),\right. \\ &\left.\left(\mathbf{Z}_{0\max}^{T} + \sum_{i=1}^{q}\mathbf{Z}_{i\max}^{T}e_i\right)\left(\mathbf{K}_{\mathbf{Z}\max} + \sum_{i=1}^{q}\mathbf{K}_{\mathbf{Z}i\max}e_i\right)\left(\mathbf{Z}_{0\max} + \sum_{i=1}^{q}\mathbf{Z}_{i\max}e_i\right)\right] \end{aligned} \tag{31}$$

$$\begin{aligned} &s\left[V_{z\,min}, V_{z\,min}\right] = \left[\nabla\left(\mathbf{Z}_{0\min}^{T} + \sum_{i=1}^{q}\mathbf{Z}_{i\min}^{T}e_i\right)\left(\mathbf{K}_{\mathbf{Z}\min} + \sum_{i=1}^{q}\mathbf{K}_{\mathbf{Z}i\min}e_i\right)\left(\mathbf{Z}_{0\min} + \sum_{i=1}^{q}\mathbf{Z}_{i\min}e_i\right)\times\right. \\ &\times\left(\mathbf{A}_{\mathbf{Z}0\min} + \sum_{i=1}^{q}\mathbf{A}_{\mathbf{Z}i\min}e_i\right)\left(\mathbf{Z}_{0\min} + \sum_{i=1}^{q}\mathbf{Z}_{i\min}e_i\right), \nabla\left(\mathbf{Z}_{0\max}^{T} + \sum_{i=1}^{q}\mathbf{Z}_{i\max}^{T}e_i\right)\times \\ &\times\left(\mathbf{K}_{\mathbf{Z}\max} + \sum_{i=1}^{q}\mathbf{K}_{\mathbf{Z}i\max}e_i\right)\left(\mathbf{Z}_{0\max} + \sum_{i=1}^{q}\mathbf{Z}_{i\max}e_i\right)\left(\mathbf{A}_{\mathbf{Z}0\max} + \sum_{i=1}^{q}\mathbf{A}_{\mathbf{Z}i\max}e_i\right)\times \\ &\left.\times\left(\mathbf{Z}_{0\max} + \sum_{i=1}^{q}\mathbf{Z}_{i\max}e_i\right)\right] = \mathbf{0}. \end{aligned} \tag{32}$$

Analysis of (31) and (32) shows that Lyapunov function and its derivative are defined in the hyper-complex domain and they can be considered as some vectors. We consider vector Lyapunov function as vector which components define redundant energy in each channel of multi-channel control system as well as energy exchange processes between channels. For interval object function (31) defines two vectors. These vectors correspond minimal and maximal values of redundant energy.

3 Result and Discussion

3.1 Definition the Lyapunov Function for the Generalized Linear Plant in the Complex Space

Let us show the benefits of the proposed approach by defining Lyapunov function for the 2nd order dual channel object, which autonomous dynamic is given by following equations

$$\begin{aligned} s\left(y_{1\Re}+jy_{1\Im}\right)&=\left(a_{11\Re}+ja_{11\Im}\right)\left(y_{1\Re}+jy_{1\Im}\right)+\left(a_{12\Re}+ja_{12\Im}\right)\left(y_{2\Re}+jy_{2\Im}\right);\\ s\left(y_{2\Re}+jy_{2\Im}\right)&=\left(a_{21\Re}+ja_{21\Im}\right)\left(y_{1\Re}+jy_{1\Im}\right)+\left(a_{22\Re}+ja_{22\Im}\right)\left(y_{2\Re}+jy_{2\Im}\right). \end{aligned} \tag{33}$$

We use complex numbers here to reduce mathematical model of the considered object and we take into account cross links between channels as well.

We think that controlled variable is y_1 and we extend (33) by integral of this variable

$$\begin{aligned} s\left(y_{0\Re}+jy_{0\Im}\right)&=\left(y_{1\Re}+jy_{1\Im}\right);\\ s\left(y_{1\Re}+jy_{1\Im}\right)&=\left(a_{11\Re}+ja_{11\Im}\right)\left(y_{1\Re}+jy_{1\Im}\right)+\left(a_{12\Re}+ja_{12\Im}\right)\left(y_{2\Re}+jy_{2\Im}\right);\\ s\left(y_{2\Re}+jy_{2\Im}\right)&=\left(a_{21\Re}+ja_{21\Im}\right)\left(y_{1\Re}+jy_{1\Im}\right)+\left(a_{22\Re}+ja_{22\Im}\right)\left(y_{2\Re}+jy_{2\Im}\right). \end{aligned} \tag{34}$$

Moreover, we assume a_{11} parameters is not exactly known and we replace it with interval

$$\mathbf{a_{11}}=\left[a_{11\Re min}+ja_{11\Im min},a_{11\Re max}+ja_{11\Im max}\right] \tag{35}$$

as follows

$$\begin{aligned} &s\left(y_{0\Re}+jy_{0\Im}\right)=\left(y_{1\Re}+jy_{1\Im}\right);\\ &s\left(y_{1\Re}+jy_{1\Im}\right)=\left[a_{11\Re min}+ja_{11\Im min},a_{11\Re max}+ja_{11\Im max}\right]\left(y_{1\Re}+jy_{1\Im}\right)+\\ &+\left(a_{12\Re}+ja_{12\Im}\right)\left(y_{2\Re}+jy_{2\Im}\right);\\ &s\left(y_{2\Re}+jy_{2\Im}\right)=\left(a_{21\Re}+ja_{21\Im}\right)\left(y_{1\Re}+jy_{1\Im}\right)+\left(a_{22\Re}+ja_{22\Im}\right)\left(y_{2\Re}+jy_{2\Im}\right). \end{aligned} \tag{36}$$

It is clear that trajectories on lower and upper boundaries are defined in a similar way. So, we can define Lyapunov function only for one of them

$$
\begin{aligned}
&s\left(y_{0\Re\,min}+jy_{0\Im\,min}\right)=\left(y_{1\Re\,min}+jy_{1\Im\,min}\right);\\
&s\left(y_{1\Re\,min}+jy_{1\Im\,min}\right)=\left(a_{11\Re\,min}+ja_{11\Im\,min}\right)\left(y_{1\Re\,min}+jy_{1\Im\,min}\right)+\\
&+\left(a_{12\Re\,min}+ja_{12\Im\,min}\right)\left(y_{2\Re\,min}+jy_{2\Im\,min}\right);\\
&s\left(y_{2\Re\,min}+jy_{2\Im\,min}\right)=\left(a_{21\Re\,min}+ja_{21\Im\,min}\right)\left(y_{1\Re\,min}+jy_{1\Im\,min}\right)+\\
&+\left(a_{22\Re\,min}+ja_{22\Im\,min}\right)\left(y_{2\Re\,min}+jy_{2\Im\,min}\right),
\end{aligned}
\tag{37}
$$

We define minimal Lyapunov function for lower boundary in following way

$$
\begin{aligned}
&V_{min}=V_{00\,min}\left(y_{0\Re\,min}+jy_{0\Im\,min}\right)^2+2V_{01\,min}\left(y_{0\Re\,min}+jy_{0\Im\,min}\right)\left(y_{1\Re\,min}+jy_{1\Im\,min}\right)+\\
&+2V_{02\,min}\left(y_{0\Re\,min}+jy_{0\Im\,min}\right)\left(y_{2\Re\,min}+jy_{2\Im\,min}\right)+V_{11\,min}\left(y_{1\Re\,min}+jy_{1\Im\,min}\right)^2+\\
&+2V_{12\,min}\left(y_{1\Re\,min}+jy_{1\Im\,min}\right)\left(y_{2\Re\,min}+jy_{2\Im\,min}\right)+V_{22\,min}\left(y_{2\Re\,min}+jy_{2\Im\,min}\right)^2
\end{aligned}
\tag{38}
$$

and its full derivative can be defined in such a way

$$
\begin{aligned}
&sV_{min}=\left(\left(2a_{11\,min}V_{01\,min}+2a_{21}V_{02\,min}+2V_{00\,min}\right)\left(y_{1\Re\,min}+jy_{1\Im\,min}\right)+\right.\\
&\left.+\left(2a_{12}V_{01\,min}+2a_{22}V_{02\,min}\right)\left(y_{2\Re\,min}+jy_{2\Im\,min}\right)\right)\left(y_{0\Re\,min}+jy_{0\Im\,min}\right)+\\
&+\left(2a_{11\,min}V_{11\,min}+2a_{21}V_{12\,min}+2V_{01\,min}\right)\left(y_{1\Re\,min}+jy_{1\Im\,min}\right)^2+\\
&+\left(2a_{11\,min}V_{12\,min}+2a_{12}V_{11\,min}+2a_{21}V_{22\,min}+2a_{22}V_{12\,min}+2V_{02\,min}\right)\times\\
&\times\left(y_{2\Re\,min}+jy_{2\Im\,min}\right)\left(y_{1\Re\,min}+jy_{1\Im\,min}\right)+\left(2a_{12}V_{12\,min}+2a_{22}V_{22\,min}\right)\times\\
&\times\left(y_{2\Re\,min}+jy_{2\Im\,min}\right)^2
\end{aligned}
\tag{39}
$$

Derivative (39) gives us possibility to write down equations to define unknown terms of Lyapunov function (38)

$$
\begin{aligned}
&2\left(a_{11\Re\,min}+ja_{11\Im\,min}\right)V_{01\,min}+2\left(a_{21\Re}+ja_{21\Im}\right)V_{02\,min}+2V_{00\,min}=0;\\
&2\left(a_{12\Re}+ja_{12\Im}\right)V_{01\,min}+2\left(a_{22\Re}+ja_{22\Im}\right)V_{02\,min}=0;\\
&2\left(a_{11\Re\,min}+ja_{11\Im\,min}\right)V_{11\,min}+2\left(a_{21\Re}+ja_{21\Im}\right)V_{12\,min}+2V_{01\,min}=0;\\
&2\left(a_{11\Re\,min}+ja_{11\Im\,min}\right)V_{12\,min}+2\left(a_{12\Re}+ja_{12\Im}\right)V_{11\,min}+2\left(a_{21\Re}+ja_{21\Im}\right)V_{22\,min}+\\
&+2\left(a_{22\Re}+ja_{22\Im}\right)V_{12\,min}+2V_{02\,min}=0;\\
&2\left(a_{12\Re}+ja_{12\Im}\right)V_{12\,min}+2\left(a_{22\Re}+ja_{22\Im}\right)V_{22\,min}=0
\end{aligned}
\tag{40}
$$

Solutions of (40) allows us to write following expression for terms of Lyapunov function (38)

$$V_{00\,min}=\frac{\left(\left(a_{11\Re\,min}+ja_{11\Im\,min}\right)\left(a_{22\Re}+ja_{22\Im}\right)-\left(a_{12\Re}+ja_{12\Im}\right)\left(a_{21\Re}+ja_{21\Im}\right)\right)^{2}}{\left(a_{12\Re}+ja_{12\Im}\right)^{2}};$$
$$V_{01\,min}=\frac{\left(a_{22\Re}+ja_{22\Im}\right)\left(\left(a_{12\Re}+ja_{12\Im}\right)\left(a_{21\Re}+ja_{21\Im}\right)-\left(a_{11\Re\,min}+ja_{11\Im\,min}\right)\left(a_{22\Re}+ja_{22\Im}\right)\right)}{\left(a_{12\Re}+ja_{12\Im}\right)^{2}};$$
$$V_{02\,min}=\frac{\left(\left(a_{11\Re\,min}+ja_{11\Im\,min}\right)\left(a_{22\Re}+ja_{22\Im}\right)-\left(a_{12\Re}+ja_{12\Im}\right)\left(a_{21\Re}+ja_{21\Im}\right)\right)}{\left(a_{12\Re}+ja_{12\Im}\right)};$$
$$V_{11\,min}=\frac{\left(a_{22\Re}+ja_{22\Im}\right)^{2}}{\left(a_{12\Re}+ja_{12\Im}\right)^{2}};V_{12\,min}=-\frac{\left(a_{22\Re}+ja_{22\Im}\right)}{\left(a_{12\Re}+ja_{12\Im}\right)};V_{22\,min}=1. \quad (41)$$

If one replace min-index with max-index in (41), he defines Lyapunov function for trajectories on upper boundary.

It is clear that function (38) with terms (41) is defined in complex domain. Nevertheless, since terms of this function obey of Sylvester criterion

$$\Delta_{3}=\begin{vmatrix}V_{00}&V_{01}&V_{02}\\V_{01}&V_{11}&V_{12}\\V_{02}&V_{12}&V_{22}\end{vmatrix}=0;\Delta_{2}=\begin{vmatrix}V_{11}&V_{12}\\V_{12}&V_{22}\end{vmatrix}=0;\Delta_{1}=V_{22}=1>0 \quad (42)$$

this function can be considered as the quadratic one.

3.2 Lyapunov Function for the AlphaBot-like Mobile Robot

Let us assume that dynamic of AlphaBot-like mobile robot is described as follows [25]

$$\begin{aligned}&pX=V\cos\Theta;\quad pY=V\sin\Theta;\quad p\Theta=\Omega;\\&pV=a_{12}\Omega^{2}+b_{11}V+b_{12}\Omega+q_{11}u_{1}+q_{12}u_{2};\\&p\Omega=a_{22}V\Omega+b_{21}V+b_{22}\Omega+q_{21}u_{1}+q_{22}u_{2},\end{aligned} \quad (43)$$

here X, Y, and Θ are robot linear and angular positions on the plane, V and Ω are its linear and angular speeds, u_1 and u_2 are left and right motor torques.

One can define factors in these equations in such a way

$$\begin{aligned}&b_{11}=-\frac{a_{13}a_{31}}{a_{33}}-\frac{a_{14}a_{41}}{a_{44}};b_{12}=-\frac{a_{13}a_{32}}{a_{33}}-\frac{a_{14}a_{42}}{a_{44}};q_{11}=-\frac{a_{13}m_{3}}{a_{33}};q_{12}=-\frac{a_{14}m_{4}}{a_{44}};\\&b_{21}=-\frac{a_{23}a_{31}}{a_{33}}-\frac{a_{24}a_{41}}{a_{44}};b_{22}=-\frac{a_{23}a_{32}}{a_{33}}-\frac{a_{24}a_{42}}{a_{44}};q_{21}=-\frac{a_{23}m_{3}}{a_{33}};q_{22}=-\frac{a_{24}m_{4}}{a_{44}};\end{aligned} \quad (44)$$

$$a_{12}=\frac{Ma}{m};a_{13}=\frac{zc}{mR_k};a_{14}=\frac{zc}{mR_k};a_{22}=-\frac{Ma}{J'};a_{23}=\frac{zcL}{J'R_k};a_{24}=-\frac{zcL}{J'R_k};$$

$$a_{31}=-\frac{zc}{L_aR_k};a_{32}=-\frac{zcL}{L_aR_k};a_{33}=-\frac{R_a}{L_a};m_3=m_4=\frac{1}{L_a};$$

$$a_{41}=-\frac{zc}{L_aR_k};a_{42}=\frac{zcL}{L_aR_k};a_{44}=-\frac{R_a}{L_a},$$

where M is a robot mass and m is a mass of robot platform with wheels, J' is an robot inertia, R_k is a wheel radius, a is a distance between wheels and center of the platform, 2L is a distance between wheels, z is gear ratio, c is motor constant, R_a and L_a are drives resistance and inductance.

It is clear that equations (43) are fifth order nonlinear ordinary differential equations. We reduce order of these equations by using complex numbers

$$\begin{aligned}&pR=V\cos\Theta+jV\sin\Theta;\quad p\Theta=\Omega;\\&pV=a_{12}\Omega^2+b_{11}V+b_{12}\Omega+q_{11}u_1+q_{12}u_2;\\&p\Omega=a_{22}V\Omega+b_{21}V+b_{22}\Omega+q_{21}u_1+q_{22}u_2,\end{aligned}\tag{45}$$

here R is a robot vector position.

Nevertheless (45) are still nonlinear differential equations and we use interval methods to transform them into interval linear-like form. We perform such transformation by considering each nonlinear function in detail.

At first, we turn our attention in harmonic functions in first equation. It is clear that in real domain these function are bounded ones. This fact allows us to rewrite first equation as follows

$$p\mathbf{R}=\mathbf{V}+j\mathbf{V},\quad \mathbf{V}=\left[V_{min},V_{max}\right],\tag{46}$$

here V_{max} is a robot maximal speed.

Equation (46) defines dynamic of complex part of the considered object.

We leave second equation unchanged and replace nonlinearities in third and fourth equations with theirs piecewise linear interval approximations. One can find example of such an approximations in fig.1

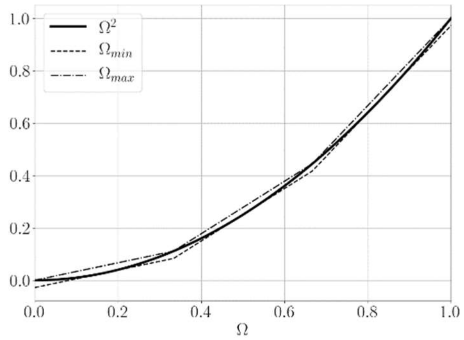

Fig.1 Piecewise linear interval approximation of the 1D nonlinear function

This approximation make it possible to replace the nonlinear function with its interval representation

$$\Omega^2 \in \mathbf{\Omega 2} = \left[c_{1min}\Omega_{min} + c_{0min}, c_{1max}\Omega_{max} + c_{0max} \right], \tag{47}$$

where c_{imin},c_{imax} are minima and maximal values of piecewise linear boundaries of the considered nonlinear function.

Let us rewrite third equation of (45) by using interval (47)

$$p\mathbf{V} = a_{12}\mathbf{\Omega 2} + b_{11}\mathbf{V} + b_{12}\mathbf{\Omega} + q_{11}u_1 + q_{12}u_2; \tag{48}$$

here $\mathbf{\Omega} = \left[\Omega_{min}, \Omega_{max} \right]$.

One can rewrite (48) in the extended form as follows

$$\begin{aligned} pV_{min} &= \left(a_{12}c_{1min} + b_{12} \right)\Omega_{min} + a_{12}c_{0min} + b_{11}V_{min} + q_{11}u_1 + q_{12}u_2; \\ pV_{max} &= \left(a_{12}c_{1max} + b_{12} \right)\Omega_{max} + a_{12}c_{0max} + b_{11}V_{max} + q_{11}u_1 + q_{12}u_2. \end{aligned} \tag{49}$$

The fourth equation of can be rewritten in a similar way by defining plane boundaries for the multiplication function.

$$p\mathbf{\Omega} = a_{22}\mathbf{V\Omega} + b_{21}\mathbf{V} + b_{22}\mathbf{\Omega} + q_{21}u_1 + q_{22}u_2, \tag{50}$$

where

$$V\Omega \in \mathbf{V\Omega} = \left[d_{1min}\Omega_{min} + d_{2min}V_{min} + d_{0min}, d_{1max}\Omega_{max} + d_{2max}V_{max} + d_{0max} \right]. \tag{51}$$

It is clear that one can rewrite (50) in extended form as follows

$$\begin{aligned} p\Omega_{min} &= \left(a_{22}d_{1min} + b_{22} \right)\Omega_{min} + \left(d_{2min} + b_{21} \right)V_{min} + d_{0min} + q_{21}u_1 + q_{22}u_2; \\ p\Omega_{max} &= \left(a_{22}d_{1max} + b_{22} \right)\Omega_{max} + \left(d_{2max} + b_{21} \right)V_{max} + d_{0max} + q_{21}u_1 + q_{22}u_2. \end{aligned} \tag{52}$$

Analysis (49) and (52) shows that these equations are real ones, thus we use them to define dynamic of real part of the considered object.

We use (49) and (52) to write interval free-motion equations for the considered robot

$$p\mathbf{R} = g_{13}\mathbf{V};\quad p\mathbf{\Theta} = g_{24}\mathbf{\Omega};$$
$$p\mathbf{V} = \mathbf{g_{33}V} + \mathbf{g_{34}\Omega};\quad p\mathbf{\Omega} = \mathbf{g_{43}V} + \mathbf{g_{44}\Omega}, \tag{53}$$

here g_{ij} factors are defined by following expressions

$$g_{13} = 1+1j;\quad g_{24} = 1;\quad g_{33} = b_{11};\mathbf{g_{40}} = \left[d_{0min}, d_{0max}\right];$$
$$\mathbf{g_{34}} = \left[a_{12}c_{1min} + b_{12}, a_{12}c_{1max} + b_{12}\right];\mathbf{g_{30}} = \left[a_{12}c_{0min}, a_{12}c_{0max}\right]; \tag{54}$$
$$\mathbf{g_{43}} = \left[d_{2min} + b_{21}, d_{2max} + b_{21}\right];\mathbf{g_{44}} = \left[a_{22}d_{1min} + b_{22}, a_{22}d_{1max} + b_{22}\right].$$

Equations (53) describe mobile robot motion in the interval form. This form defines robot motion as the set of trajectories, which are bounded by solution of equations (53). Since g_{11} factor is defined in the complex space, we claim that all possible robot motions are defined in the complex space as well.

To define interval Lyapunov function for the interval dynamical object (53) we rewrite it into the formal form

$$p\mathbf{y_1} = g_{13}\mathbf{y_3};\quad p\mathbf{y_2} = g_{24}\mathbf{y_4};\quad p\mathbf{y_3} = \mathbf{g_{33}y_3} + \mathbf{g_{34}y_4};\quad p\mathbf{y_4} = \mathbf{g_{43}y_3} + \mathbf{g_{44}y_4}, \tag{55}$$

here

$$\mathbf{y_1} = \mathbf{R} - \mathbf{R}^*;\mathbf{y_2} = \mathbf{\Theta} - \mathbf{\Theta}^*;\mathbf{y_3} = \mathbf{V} - \mathbf{V}^*;\mathbf{y_4} = \mathbf{\Omega} - \mathbf{\Omega}^*,. \tag{56}$$

where "*"-symbol shows desired values of the robot state variables.

We define Lyapunov function as following interval quadratic function

$$\mathbf{V} = \mathbf{v_{11}y_1^2} + \mathbf{2v_{12}y_1y_2} + \mathbf{2v_{13}y_1y_3} + \mathbf{2v_{14}y_1y_4} + \mathbf{v_{22}y_2^2} + \mathbf{2v_{23}y_2y_3} +$$
$$+\mathbf{2v_{24}y_2y_4} + \mathbf{v_{33}y_3^2} + \mathbf{2v_{34}y_3y_4} + \mathbf{v_{44}y_4^2}. \tag{57}$$

We assume that function (57) shows redundant robot energy and this energy is not dissipated during robot free motion. This no dissipation rule means that the full derivative of (57) equal to zero

$$\mathbf{pV} = \left(\left(\mathbf{2}g_{13}\mathbf{v_{11}} + \mathbf{2g_{33}v_{13}} + \mathbf{2g_{43}v_{14}}\right)\mathbf{y_3} + \left(\mathbf{2}g_{24}\mathbf{v_{12}} + \mathbf{2g_{34}v_{13}} + \mathbf{2g_{44}v_{14}}\right)\mathbf{y_4}\right)\mathbf{y_1} +$$
$$+\left(\left(\mathbf{2}g_{13}\mathbf{v_{12}} + \mathbf{2g_{33}v_{23}} + \mathbf{2g_{43}v_{24}}\right)\mathbf{y_3} + \left(\mathbf{2}g_{24}\mathbf{v_{22}} + \mathbf{2g_{34}v_{23}} + \mathbf{2g_{44}v_{24}}\right)\mathbf{y_4}\right)\mathbf{y_2} +$$
$$+\left(\mathbf{2}g_{13}\mathbf{v_{13}} + \mathbf{2g_{33}v_{33}} + \mathbf{2g_{43}v_{34}}\right)\mathbf{y_3^2} + \left(\mathbf{2}g_{24}\mathbf{v_{24}} + \mathbf{2g_{34}v_{34}} + \mathbf{2g_{44}v_{44}}\right)\mathbf{y_4^2} + \tag{58}$$
$$+\left(\mathbf{2}g_{13}\mathbf{v_{14}} + \mathbf{2}g_{24}\mathbf{v_{23}} + \mathbf{2g_{33}v_{34}} + \mathbf{2g_{34}v_{33}} + \mathbf{2g_{43}v_{44}} + \mathbf{2g_{44}v_{34}}\right)\mathbf{y_4y_3} = 0$$

Derivative (58) can be used to write down equations for defining Lyapunov factors

$$
\begin{aligned}
&\mathbf{2}g_{13}\mathbf{v}_{11}+\mathbf{2g}_{33}\mathbf{v}_{13}+\mathbf{2g}_{43}\mathbf{v}_{14}=0; \quad \mathbf{2}g_{24}\mathbf{v}_{12}+\mathbf{2g}_{34}\mathbf{v}_{13}+\mathbf{2g}_{44}\mathbf{v}_{14}=0;\\
&\mathbf{2}g_{13}\mathbf{v}_{12}+\mathbf{2g}_{33}\mathbf{v}_{23}+\mathbf{2g}_{43}\mathbf{v}_{24}=0; \quad \mathbf{2}g_{24}\mathbf{v}_{22}+\mathbf{2g}_{34}\mathbf{v}_{23}+\mathbf{2g}_{44}\mathbf{v}_{24}=0;\\
&\mathbf{2}g_{13}\mathbf{v}_{13}+\mathbf{2g}_{33}\mathbf{v}_{33}+\mathbf{2g}_{43}\mathbf{v}_{34}=0; \quad \mathbf{2}g_{24}\mathbf{v}_{24}+\mathbf{2g}_{34}\mathbf{v}_{34}+\mathbf{2g}_{44}\mathbf{v}_{44}=0;\\
&\mathbf{2}g_{13}\mathbf{v}_{14}+\mathbf{2}g_{24}\mathbf{v}_{23}+\mathbf{2g}_{33}\mathbf{v}_{34}+\mathbf{2g}_{34}\mathbf{v}_{33}+\mathbf{2g}_{43}\mathbf{v}_{44}+\mathbf{2g}_{44}\mathbf{v}_{34}=0.
\end{aligned} \tag{59}
$$

The solution of (59) allows us to define unknown Lyapunovs factors

$$
\begin{aligned}
&\mathbf{v}_{11}=\frac{\mathbf{g}_{33}^2+\mathbf{2g}_{33}\mathbf{g}_{43}+\mathbf{g}_{43}^2}{g_{13}^2}; \quad \mathbf{v}_{12}=\frac{\mathbf{g}_{33}\mathbf{g}_{34}+\mathbf{g}_{33}\mathbf{g}_{44}+\mathbf{g}_{34}\mathbf{g}_{43}+\mathbf{g}_{43}\mathbf{g}_{44}}{g_{24}g_{13}};\\
&\mathbf{v}_{13}=-\frac{\mathbf{g}_{33}+\mathbf{g}_{43}}{g_{13}}; \quad \mathbf{v}_{14}=-\frac{\mathbf{g}_{33}+\mathbf{g}_{43}}{g_{13}}; \quad \mathbf{v}_{22}=\frac{\mathbf{g}_{34}^2+\mathbf{2g}_{34}\mathbf{g}_{44}+\mathbf{g}_{44}^2}{g_{24}^2};\\
&\mathbf{v}_{23}=-\frac{\mathbf{g}_{34}+\mathbf{g}_{44}}{g_{24}}; \quad \mathbf{v}_{24}=-\frac{\mathbf{g}_{34}+\mathbf{g}_{44}}{g_{24}}; \quad v_{33}=1; \quad v_{34}=1; \quad v_{44}=1.
\end{aligned} \tag{60}
$$

If one takes into account (54) factors (60) can be rewritten in such a way

$$
\begin{aligned}
&\mathbf{v}_{11}=-j\frac{\mathbf{g}_{33}^2+\mathbf{2g}_{33}\mathbf{g}_{43}+\mathbf{g}_{43}^2}{2};\\
&\mathbf{v}_{12}=\frac{\mathbf{g}_{33}\mathbf{g}_{34}+\mathbf{g}_{33}\mathbf{g}_{44}+\mathbf{g}_{34}\mathbf{g}_{43}+\mathbf{g}_{43}\mathbf{g}_{44}}{2}-j\frac{\mathbf{g}_{33}\mathbf{g}_{34}+\mathbf{g}_{33}\mathbf{g}_{44}+\mathbf{g}_{34}\mathbf{g}_{43}+\mathbf{g}_{43}\mathbf{g}_{44}}{2};\\
&\mathbf{v}_{13}=-\frac{\mathbf{g}_{33}+\mathbf{g}_{43}}{2}+j\frac{\mathbf{g}_{33}+\mathbf{g}_{43}}{2}; \quad \mathbf{v}_{14}=-\frac{\mathbf{g}_{33}+\mathbf{g}_{43}}{2}+j\frac{\mathbf{g}_{33}+\mathbf{g}_{43}}{2};\\
&\mathbf{v}_{22}=\mathbf{g}_{34}^2+\mathbf{2g}_{34}\mathbf{g}_{44}+\mathbf{g}_{44}^2;\\
&\mathbf{v}_{23}=-\mathbf{g}_{34}-\mathbf{g}_{44}; \quad \mathbf{v}_{24}=-\mathbf{g}_{34}-\mathbf{g}_{44}; \quad v_{33}=1; \quad v_{34}=1; \quad v_{44}=1.
\end{aligned} \tag{61}
$$

Expressions (61) define factors of the Lyapunov function (57). It is clear that these factors are both complex and real. Real factors are used to define redundant energy in real part of the considered object and complex ones define redundant energy in its complex part.

4 Conclusion

Analysis of matrix Lyapunov equation shows that left hand-expression in this equation rises maximum if quadratic matrix in right-hand expression is zero matrix. This fact allows us to define the terms of the Lyapunov function by making equals to zero its full derivative for the autonomous object. This approach allows us to define Lyapunov functions for a wide range of linear control objects with both exactly known and uncertain parameters.

Objects with parametric uncertainty are described by interval methods and these methods define the interval Lyapunov function. The redundant energy of an object with uncertainty is always between lower and upper boundaries of the interval Lyapunov function.

It is possible to reduce the number of variables, which are used to define the Lyapunov function, by using hyper-complex numbers. Lyapunov functions which are designed by using the considered approach, are quadratic functions and obey the Sylvester criterion.

We see two possibilities to use the proposed background. At first, one can use it us the base to study stability of various complex single-channel and multi-channel electrotechnical and electromechanical devices or systems, where these devices are used. One of them are system with one or several renewable energy sources, which coordinated operation is a very important factor. In this case, the use of Lyapunov functions give us possibility to determine the operation modes of systems and define their parameters by solving corresponding parametric optimization problems.

The second way of using proposed backgrounds is using of defined Lyapunov functions while closed-loop control systems are being designed. The terms of these functions can be used as various control laws terms. Constructed in such a way control systems for electric drives and electric generators are asymptotically stable systems and thus it can be easy integrated in any technological process, including and processes of energy production. The use of interval methods gives us possibility to design robust control systems and the usage of hyper-complex numbers allows us to construct multichannels closed-loop control systems. These multi-channel systems can be implemented in an easy way by using modern microcontrollers, which can be programmed for operation with hyper-complex numbers.

References

1. A. Kumar and S. K. Bhagat, "Study of Stability Analysis using Lyapunov Function for IEEE-9 Bus System," 2019 IEEE International Conference on Electrical, Computer and Communication Technologies (ICECCT), Coimbatore, India, 2019, pp. 1-6, doi: 10.1109/ICECCT.2019.8869078.
2. X. Zhang, J. Huang, Y. Sun, X. Tong and Z. Ma, "Lyapunov Stability Constraining Solution of the Cascaded Inverter Based on Model Predictive Current Control," 2018 Chinese Automation Congress (CAC), Xi'an, China, 2018, pp. 2369-2373, doi: 10.1109/CAC.2018.8623485.
3. O. A. Rehman and N. Iqbal, "Power system stability enhancement using Lyapunov theory," 2016 International Conference on Emerging Technologies (ICET), Islamabad, Pakistan, 2016, pp. 1-5, doi: 10.1109/ICET.2016.7813255.
4. X. Liu and Z. Wang, "Construction of the Lyapunov function for a class of H∞ norm problem," 2019 Chinese Control And Decision Conference (CCDC), Nanchang, China, 2019, pp. 1108-1111, doi: 10.1109/CCDC.2019.8833256.
5. C. Ning and Q. Liu, "Indefinite Lyapunov Functions for Input-to-State Stability of Impulsive Time-Delay Systems," 2018 37th Chinese Control Conference (CCC), Wuhan, China, 2018, pp. 654-657, doi: 10.23919/ChiCC.2018.8483927.
6. H. Zhang, X. Ban, F. Wu and X. Huang, "Piecewise Lyapunov function based stability analysis of fuzzy parameter varying systems," 2017 36th Chinese Control Conference (CCC), Dalian, 2017, pp. 4245-4250, doi: 10.23919/ChiCC.2017.8028024.

7. B. Maschke, R. Ortega and A. J. Van Der Schaft, "Energy-based Lyapunov functions for forced Hamiltonian systems with dissipation," in IEEE Transactions on Automatic Control, vol. 45, no. 8, pp. 1498-1502, Aug. 2000, doi: 10.1109/9.871758.
8. Y. Z. Sun, Y. H. Song and X. Li, "Novel energy-based Lyapunov function for controlled power systems," in IEEE Power Engineering Review, vol. 20, no. 5, pp. 55-57, May 2000, doi: 10.1109/39.841351.
9. V. Utkin, "Mechanical energy-based Lyapunov function design for twisting and super-twisting sliding mode control," in IMA Journal of Mathematical Control and Information, vol. 32, no. 4, pp. 675-688, Dec. 2015, doi: 10.1093/imamci/dnu010.
10. Y. Deng, Y. He, D. Tian, W. He and P. Ding, "Analysis of the stability control of wind turbines generator system based on the Lure Lyapunov function," International Conference on Renewable Power Generation (RPG 2015), Beijing, 2015, pp. 1-6, doi: 10.1049/cp.2015.0429.
11. W. Junqiang and X. Meiqing, "Stochastic Stability Analysis of the Power Systems Based on Lyapunov Function," 2018 2nd IEEE Conference on Energy Internet and Energy System Integration (EI2), Beijing, China, 2018, pp. 1-9, doi: 10.1109/EI2.2018.8581992.
12. S. Asadi, A. Khayatian, M. Dehghani and N. Vafamand, "Simultaneous fault reconstruction of TS fuzzy systems using robust sliding mode observer and non-quadratic stability analysis," 2017 Iranian Conference on Electrical Engineering (ICEE), Tehran, 2017, pp. 823-828, doi: 10.1109/IranianCEE.2017.7985152.
13. R. Voliansky, O. Sadovoi and N. Volianska, "Defining of Lyapunov Functions for the Generalized Nonlinear Object," 2018 IEEE 5th International Conference on Methods and Systems of Navigation and Motion Control (MSNMC), Kiev, 2018, pp. 222-228, doi: 10.1109/MSNMC.2018.8576315.
14. A. Sferlazza and L. Zaccarian, "Linear flux observers for induction motors with quadratic Lyapunov certificates," 2016 IEEE 25th International Symposium on Industrial Electronics (ISIE), Santa Clara, CA, USA, 2016, pp. 167-172, doi: 10.1109/ISIE.2016.7744884.
15. M. Pasquini and D. Angeli, "On piecewise quadratic Lyapunov functions for piecewise affine models of gene regulatory networks," 2018 IEEE Conference on Decision and Control (CDC), Miami, FL, USA, 2018, pp. 1071-1076, doi: 10.1109/CDC.2018.8618671.
16. M. Comanescu, "A speed adaptive sensorless flux observer for the induction motor drive using Sylvester criterion design," 2016 IEEE Applied Power Electronics Conference and Exposition (APEC), Long Beach, CA, USA, 2016, pp. 2759-2763, doi: 10.1109/APEC.2016.7468254.
17. N. Q. Thuong, "Algorithm for Constructing Lyapunov Functions for Assessing the Stability of UAV's Motion by the Method of Statistical Synthesis," 2018 Engineering and Telecommunication (EnT-MIPT), Moscow, Russia, 2018, pp. 201-204, doi: 10.1109/EnT-MIPT.2018.00052.
18. K. H. Mohammedali, F. Fadhel and N. A. Ahmad, "Modified variable gradient method for constructing the Lyapunov function using nonlinear programming method," 2015 10th Asian Control Conference (ASCC), Kota Kinabalu, Malaysia, 2015, pp. 1-6, doi: 10.1109/ASCC.2015.7244616.
19. X. Zhicai, Q. Liang, Y. Shi and Q. Yahui, "On constructing Lyapunov function for formation control of Multi-agent Systems with directed topology," 2020 Chinese Control And Decision Conference (CCDC), Hefei, China, 2020, pp. 2284-2288, doi: 10.1109/CCDC49329.2020.9164376.
20. P. Giesl, "Construction of a local and global Lyapunov function using radial basis functions," in IMA Journal of Applied Mathematics, vol. 73, no. 5, pp. 782-802, Oct. 2008, doi: 10.1093/imamat/hxn018.

21. J. Yan, X. Xiao, H. Li, J. Zhang, J. Yan and M. Liu, "Noise-Tolerant Zeroing Neural Network for Solving Non-Stationary Lyapunov Equation," in IEEE Access, vol. 7, pp. 41517-41524, 2019, doi: 10.1109/ACCESS.2019.2907746.
22. Snehashish Chakraverty; Nisha Mahato; Perumandla Karunakar; Tharasi Dilleswar Rao, "Differential Equations with Interval Uncertainty," in Advanced Numerical and Semi-Analytical Methods for Differential Equations, Wiley, 2019, pp.197-208, doi: 10.1002/9781119423461.ch19.
23. Kuzenkov, O., Serdiuk, T., Kuznetsova, A., Tryputen, M., Kuznetsov, V., Kuznetsova, Y., Tryputen, M. Mathematical model of dynamics of homomorphic objects (2019) CEUR Workshop Proceedings, 2516, pp. 190-205.
24. Kuzenkov, O., Kuznetsov, V., Tryputen, N. Analysis of phase trajectories of the third - Order dynamic objects (2019) 2019 IEEE 2nd Ukraine Conference on Electrical and Computer Engineering, UKRCON 2019 - Proceedings, статья № 8879819, pp. 1235-1243. DOI: 10.1109/UKRCON.2019.8879819
25. R. Voliansky, O. Sadovoi, Y. Sokhina, I. Shramko and N. Volianska, "Sliding Mode Interval Controller for the Mobile Robot," 2019 XIth International Scientific and Practical Conference on Electronics and Information Technologies (ELIT), Lviv, Ukraine, 2019, pp. 76-81.