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# POISSON RANDOM MEASURES AND NONCRITICAL MULTITYPE MARKOV BRANCHING PROCESSES

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**Abstract**

We investigate noncritical multi-type Markov branching processes with immigration generated by Poisson measures. Limiting distributions are obtained when the rates of the Poisson measures are asymptotically equivalent to exponential or regularly varying functions. In particular, results analogous to a strong LLN are presented, and limiting normal distributions are obtained when the rates increase. When the rates decrease, then conditional limiting distributions are established. A stationary limiting distribution is obtained when the mean Poisson measure grows linearly. The asymptotic behaviour of the first and second moments of the processes is also investigated.

**Key words:** multitype Markov branching processes, immigration, Poisson measures, limiting distributions

**2020 Mathematics Subject Classification:** Primary 60J80, Secondary 60J85

**1. Introduction.** The main objective of this paper is to study noncritical multitype Markov branching processes with immigration governed by Poisson random measures with time-varying rates. This paper continues the results announced first in [[1]] and after that published with proofs in [[2]], where this family of processes were investigated in the critical case.

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Multitype Markov branching processes were introduced by KOLMOGOROV and DMITRIEV [3], where the terminology *branching process* officially appeared for the first time. The first branching process with immigration was proposed by SEVASTYANOV [4]. He investigated a single-type Markov branching process in which immigration occurs in accordance with a time-homogeneous Poisson process, and obtained limiting distributions in the sub-, super-, and critical cases (see also SEVASTYANOV [5]).

Many variants of branching processes with immigration were subsequently investigated. Most of these models have been reviewed by SEVASTYANOV [6] and VATUTIN and ZUBKOV [7,8]. Some of the references considered models with time-inhomogeneous immigration. In particular, we mention the work by Badalbaev, Rahimov and some of their collaborators who focused on critical processes. See RAHIMOV's monograph [9] for a review.

Branching processes with time-inhomogeneous Poisson immigration have been successfully used to describe the dynamics of cellular systems (e.g., red blood cell progenitor cells, leukemia) [10–13]. In these applications, the immigration process describes an influx of cells (e.g., stem cells) arising from a compartment that is not directly observable but remains a significant contributor to population dynamics. A key feature of cell kinetics is that this influx is often time-dependent.

This paper focuses on the asymptotic behaviour of noncritical multitype Markov branching processes that develop along an immigration process generated by Poisson measures with a rate $r(t)$ and a mean measure $R(t)$. To this end, we have organized the paper as follows. Section 2 presents the model and its basic equations. These results are used in Sections 3 and 4, where the limiting distributions are presented when the Perron–Frobenius root $\rho \neq 0$ (i.e. in the noncritical cases). Different limiting distributions are obtained depending of the rate $r(t)$ of the Poisson measures. Section 3 deals with the subcritical case $\rho < 0$ : Theorem 1 considers settings that include the case, where $r(t) = O(e^{\theta t})$, $\theta < \rho$, and Theorem 2 the case, where $r(t) \sim r_0 e^{\theta t}$, $\theta < 0$. Both theorems present conditional limiting distributions. Theorem 3 proves that a stationary limiting distribution holds when $R(t)/t \to r_0 > 0$ as $t \to \infty$. Theorem 4 shows, when $r(t) \sim r_0 e^{\theta t}$, $\theta > 0$, that the process exhibits limit behaviours analogous to a strong Law of Large Numbers (LLN) and a Central Limit Theorem (CLT) as $t \to \infty$. Finally, Theorem 5 investigates the case, where $r(t) \sim r_0 t^{\theta}$, $t \to \infty$, where $r_0 > 0$ and $\theta \in \mathbb{R}$. When $\theta < 0$, a conditional limiting distribution holds; when $\theta > 0$, a LLN and CLT are presented. In Section 4 three different types of limiting distributions are obtained in the supercritical case $\rho > 0$. Notice that the condition $\widetilde{r}(\rho) = \int_0^{\infty} e^{-\rho x} r(x) dx < \infty$ in Theorem 6 in fact includes the cases, where $r(t)$ is a regularly varying function (r.v.f.) or $r(t) = O(e^{\theta t})$, $\theta < \rho$. Then we obtain $L_2$ convergence of the process normalized by its mean. Theorems 7 and 8 investigate the cases, where $r(t) \sim r_0 e^{\theta t}$, $\theta \geq \rho$ and $r_0 > 0$, $t \to \infty$. Theorem 7 can be considered an analogue to the strong LLN while Theorem 8 presents two variants of CLT. Notice that in all theorems in Sections 3 and 4 the asymptotic behaviour of the first and second moments of the corresponding processes is also obtained.

Finally, we would like to point out that multitype Markov branching processes with nonhomogeneous Poisson immigration are considered in [14]. The authors investigated subcritical processes in the case $r(t) \sim r_0 e^{\theta t}$, $\theta \geq 0$, $r_0 > 0$, and they obtained a convergence in probability to some constant, when $\theta > 0$, or a convergence in distribution to some random variable, when $\theta = 0$. Notice that in the case $\theta > 0$ we proved *a.s.* and $L_2$ convergences and we also obtained limiting normal distributions. In fact the case $\theta = 0$ follows from our Theorem 3 and moreover we obtained also a differential equation for the limiting probability generating function (p.g.f.). In the supercritical case the authors investigated the case $r(t) \sim r_0 e^{\theta t}$, $\theta \geq \rho$, and obtained convergence in probability to some constant, when we proved *a.s.* and $L_2$ convergence and obtained limiting normal distributions. Under the condition $\widetilde{r}(\rho) < \infty$ the authors obtained only convergence of the characteristic functions.

We have to point out that the methods of proofs in [14] are quite different from the methods developed in our paper. Moreover Lemma 2 of [14] is in fact a particular case of Theorem 1 from [2] and the limiting result in the critical case of Theorem 5 of [14] has been previously obtained in Theorem 8 of [2].

**2. Model and equations.** We consider a population that consists of $d$ types of cells (individuals, particles), and evolves in accordance with an immigration process and a branching mechanism. Let $0 < T_1 < T_2 < \cdots$ be random time points arising from a Poisson random measure $\Pi(t) = \sum_{i=1}^{\infty} \mathbf{1}_{\{T_i \leq t\}}, t \geq 0$, with local intensity $r(t) > 0$ and mean measure $R(t) = \int_0^t r(x)dx$. Then, $\mathbf{P}\{\Pi(t) = n\} = e^{-R(t)} R^n(t)/n!$ for $n = 0, 1, \ldots$. Assume that $\mathbf{I}_k = (I_{k1}, \ldots, I_{kd})$, $k = 1, 2 \cdots$, are independent and identically distributed (i.i.d.) non-negative integer-valued random vectors with p.g.f. $g(\mathbf{s}) = \mathbf{E}\{\mathbf{s}^{\mathbf{I}_k}\} = \sum_{\boldsymbol{\alpha} \in \mathbf{N}^d} \mathbf{P}\{\mathbf{I}_k = \boldsymbol{\alpha}\}\mathbf{s}^{\boldsymbol{\alpha}}$, $\mathbf{s} = (s_1, \ldots, s_d)$, $|\ \mathbf{s}\ | \leq \mathbf{1}$, where $\mathbf{s}^{\boldsymbol{\alpha}} = \prod_{i=1}^{d} s_i^{\alpha_i}$ for every $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$. We consider the marked point process $\{(T_k, \mathbf{I}_k), k = 1, 2, \ldots\}$. The vector $\mathbf{I}_k$ is interpreted as the number of immigrants that join the population at time $T_k$.

Let $\mathbf{Z} = \{\mathbf{Z}_i(t) = (Z_{i1}(t), Z_{i2}(t), \ldots, Z_{id}(t)), i = 1, \ldots, d; t \geq 0\}$ be a multitype branching process where $Z_{ij}(t)$ denotes the number of type-$j$ cells at time $t$ produced by a single type-$i$ cell born at $t = 0$, where $i, j = 1, \ldots, d$, and assume that cells evolve independently of each other. Next, we introduce the p.g.f. $F_i(t; \mathbf{s}) = \mathbf{E}\{\mathbf{s}^{\mathbf{Z}_i(t)}\} = \sum_{\boldsymbol{\alpha} \in \mathbf{N}^d} \mathbf{P}\{\mathbf{Z}_i(t) = \boldsymbol{\alpha}\}\mathbf{s}^{\boldsymbol{\alpha}}$, with $F_i(0, \mathbf{s}) = s_i$, and define the vector $\mathbf{F}(t;\mathbf{s}) = (F_1(t;\mathbf{s}), F_2(t;\mathbf{s}), \ldots, F_d(t;\mathbf{s}))$.

Let $\widetilde{\mathbf{Z}} = \{\widetilde{\mathbf{Z}}_k(t) = (\widetilde{Z}_{k1}(t), \ldots, \widetilde{Z}_{kd}(t)); t \geq 0; k = 1, 2, \ldots\}$ be i.i.d. copies of $\mathbf{Z}$, but with initial conditions $\widetilde{\mathbf{Z}}_k(0) = \mathbf{I}_k$. Therefore, $\mathbf{E}\{\mathbf{s}^{\widetilde{\mathbf{Z}}_k(t)}\} = g(\mathbf{F}(t;\mathbf{s}))$ because of the independence of the individual evolutions. We assume that the sets $\widetilde{\mathbf{Z}}$ and $\mathbf{\Pi} =\{\Pi(t), t \geq 0\}$ are independent.

Define the process

$$\mathbf{Y}(t) = \sum_{k=1}^{\Pi(t)} \widetilde{\mathbf{Z}}_k(t - T_k)\mathbf{1}_{\{\Pi(t)>0\}}, \quad t \geq 0, \quad \mathbf{Y}(0) = \mathbf{0}. \tag{1}$$

Its first increment occurs when the first group of $\mathbf{I}_1$ immigrants enters the population at time $T_1$, each of which evolves independently and in accordance with a process $\mathbf{Z}$. A second group of $\mathbf{I}_2$ immigrants arrives at time $T_2$, etc. We refer to $\mathbf{Y} =\{\mathbf{Y}(t) = (Y_1(t), \ldots, Y_d(t)), t \geq 0\}$ as a *multitype branching process generated by Poisson measure or multitype branching process with non-homogeneous Poisson immigration.*

If $\Phi(t;\mathbf{s}) = \mathbf{E}\{\mathbf{s}^{\mathbf{Y}(t)}\}$, then by (1) the following presentation holds (see Theorem 1 from [2])

$$\Phi(t;\mathbf{s}) = \exp\left\{-\int_0^t r(t-x)[1 - g(\mathbf{F}(x;\mathbf{s}))]dx\right\}, \Phi(0;\mathbf{s}) = \mathbf{1}. \tag{2}$$

**Remark 1.** As it is proved in [2], the relation (2) is valid for a broad class of branching processes in which individuals evolve independently of each other. Such processes include multitype Markov, Bellman–Harris, Sevastyanov or Crump–Mode–Jagers branching models, which are described in well-known monographs [5, 15, 16].

**Remark 2.** The relation (2) is presented as Lemma 2 in [14] as mentioned in the Introduction.

Similarly for $\Phi(t,\tau;\mathbf{s}_1,\mathbf{s}_2) = \mathbf{E}\{\mathbf{s}_1^{\mathbf{Y}(t)}\mathbf{s}_2^{\mathbf{Y}(t+\tau)}\}$ one can obtain that

$$\begin{aligned}\Phi(t,\tau;\mathbf{s}_1,\mathbf{s}_2) = \exp\bigg\{&-\int_0^t r(x)[1 - g(\mathbf{F}(t-x,\tau;\mathbf{s}_1,\mathbf{s}_2))]dx \\ &-\int_t^{t+\tau} r(x)[1 - g(\mathbf{F}(t,\tau - x;\mathbf{1},\mathbf{s}_2))]dx\bigg\},\end{aligned} \tag{3}$$

where $F_i(t,\tau;\mathbf{s}_1,\mathbf{s}_2) = \mathbf{E}\{\mathbf{s}_1^{\mathbf{Z}_i(t)}\mathbf{s}_2^{\mathbf{Z}_i(t+\tau)}\}$.

From now on we consider the case where $\mathbf{Z}$ is a multitype Markov branching process; that is, the lifespan and the offspring vector of any type-$i$ cell, $\tau_i$ and $\boldsymbol{\nu}_i = (\nu_{i1}, \ldots, \nu_{id})$, are independent, $\mathbf{P}\{\tau_i \leq t\} = 1 - e^{-t/\mu_i}$, and $h_i(\mathbf{s}) = \mathbf{E}\{\mathbf{s}^{\boldsymbol{\nu}_i}\} = \sum_{\alpha\in\mathbf{N}^d} p_\alpha^i \mathbf{s}^\alpha$, $i = 1, \ldots, d$. Under these assumptions, the p.g.f. $F_i(t;\mathbf{s}) =$

$\sum_{\boldsymbol{\alpha}\in\mathbf{N}^d} \mathbf{P}\{\mathbf{Z}_i(t)=\boldsymbol{\alpha}\}\mathbf{s}^{\boldsymbol{\alpha}}$ satisfy the system of differential equations

$$(4)\qquad \frac{\partial}{\partial t}\mathbf{F}(t;\mathbf{s})=\mathbf{f}(\mathbf{F}(t;\mathbf{s})),\quad \frac{\partial}{\partial t}\mathbf{F}(t;\mathbf{s})=\sum_{i=1}^{d} f_i(\mathbf{s})\frac{\partial}{\partial s_i}\mathbf{F}(t;\mathbf{s}),\quad \mathbf{F}(0;\mathbf{s})=\mathbf{s},$$

where $f_i(\mathbf{s})=[h_i(\mathbf{s})-s_i]/\mu_i$ are the infinitesimal generating functions and $\mathbf{f}(\mathbf{s})=(f_1(\mathbf{s}),\dots,f_d(\mathbf{s}))$. Under these assumptions, $\mathbf{Y}(t)$ is a multitype Markov branching process with non-homogeneous Poisson immigration. For $1\le i,j\le d$, let $A_{ij}(t)=\mathbf{E}\{Z_{ij}(t)\}=\left.\frac{\partial F_i(t;\mathbf{s})}{\partial s_j}\right|_{\mathbf{s}=\mathbf{1}}$, and introduce the matrix of first infinitesimal characteristics $\mathbf{a}=\|a_{ij}\|_{1\le i,j\le d}$ where $a_{ij}=\left.\frac{\partial f_i(\mathbf{s})}{\partial s_j}\right|_{\mathbf{s}=\mathbf{1}}$. It is well known that $\mathbf{A}(t)=||A_{ij}(t)||_{1\le i,j\le d}=\exp(\mathbf{a}t)=\sum_{n=0}^{\infty}\frac{\mathbf{a}^n t^n}{n!}$.

We assume that $\mathbf{a}$ is a positive regular matrix with Perron–Frobenius root $\rho$. Further on we will consider the noncritical case when $\rho\neq 0$. The associated right and left eigenvectors $\mathbf{u}=(u_1,\dots,u_d)$ and $\mathbf{v}=(v_1,\dots,v_d)$ can be chosen positive, with $u_1>0$ and $v_1>0$, and normalized such that $\sum_{i=1}^{d}u_i=1$ and $\sum_{i=1}^{d}u_iv_i=1$. Define the second infinitesimal characteristics $b^i_{jk}=\left.\frac{\partial^2 f_i(\mathbf{s})}{\partial s_j\partial s_k}\right|_{\mathbf{s}=\mathbf{1}}$ for $1\le i,j,k\le d$, and immigration moments $m_i=\left.\frac{\partial g(\mathbf{s})}{\partial s_i}\right|_{\mathbf{s}=\mathbf{1}}$, $\beta_{ij}=\left.\frac{\partial^2 g(\mathbf{s})}{\partial s_i\partial s_j}\right|_{\mathbf{s}=\mathbf{1}}$. All these quantities are assumed finite when the second moments $B^i_{jk}(t)=\mathbf{E}\{Z_{ij}(t)(Z_{ik}(t)-\delta_{jk})\}=\left.\frac{\partial^2 F_i(t;\mathbf{s})}{\partial s_j\partial s_k}\right|_{\mathbf{s}=\mathbf{1}}$ are investigated, where as usual $\delta_{jk}=1$ if $j=k$, and 0 otherwise.

It is known (see for example [5,15,16]) that $A_{ij}(t)\sim u_iv_je^{\rho t}$, $t\to\infty$. Also if $\rho<0$ (subcritical case), then one gets $B^i_{jk}(t)\sim\widetilde{B}^i_{jk}e^{\rho t}$ for some constants $\widetilde{B}^i_{jk}>0$. Now using the above asymptotic results and applying Theorem VI.7.7 of Sevastyanov [5] one can obtain for $\rho<0$ and $\tau>0$ that $B^i_{jk}(t,\tau)\sim\widetilde{B}^i_{jk}(\tau)e^{\rho t}$, where $\widetilde{B}^i_{jk}(\tau)=\sum_{l=1}^{d}\widetilde{B}^i_{jl}A_{lk}(\tau)+u_iv_jA_{jk}(\tau)$. In the supercritical case ($\rho>0$) it holds that $B^i_{jk}(t)\sim\widehat{B}^iv_jv_ke^{2\rho t}$, $t\to\infty$, where $\widehat{B}^i=\sum_{l,m,n=1}^{d}D^i_lb^l_{mn}u_mu_n/|2\rho\delta_{\alpha\beta}-a_{\alpha\beta}|$ and $D^i_l$ is the algebraic complement to the element $2\rho\delta_{li}-a_{li}$ of the determinant $|2\rho\delta_{\alpha\beta}-a_{\alpha\beta}|$. Similarly for $\rho>0$ one has that $B^i_{jk}(t,\tau)=\mathbf{E}\{Z_{ij}(t)Z_{ik}(t+\tau)\}\sim\widehat{B}^iv_jv_ke^{\rho(2t+\tau)}$, $t\to\infty$.

Let $M_i(t) = \mathbf{E}\{Y_i(t)\}$ and the covariances $C_{ij}(t) = \mathbf{Cov}\{Y_i(t), Y_j(t) - \delta_{ij}\}$ and $C_{ij}(t,\tau) = \mathbf{Cov}\{Y_i(t), Y_j(t+\tau)\}$. Notice that $C_{ij}(t) = \mathbf{Cov}\{Y_i(t), Y_j(t)\}$ for $i \neq j$ and $C_{ii}(t) = \mathbf{Var}\{Y_i(t)\}$. Then from (2) and (4) one obtains

$$M_i(t) = \frac{\partial \log \Phi(t; \mathbf{s})}{\partial s_i}\Big|_{\mathbf{s}=\mathbf{1}} = \int_0^t r(t-x)\overline{A}_i(x)dx,$$

where $\overline{A}_i(x) = \sum_{k=1}^{d} m_k A_{ki}(x) \sim v_i \overline{A} e^{\rho x}, \overline{A} = \sum_{k=1}^{d} m_k u_k, x \to \infty.$

Similarly one can verify that from (2) and (4)

$$C_{ij}(t) = \frac{\partial^2 \log \Phi(t; \mathbf{s})}{\partial s_i \partial s_j}\Big|_{\mathbf{s}=\mathbf{1}} = \int_0^t r(t-x)\overline{C}_{ij}(x)dx,$$

where $\overline{C}_{ij}(x) = \sum_{k=1}^{d} m_k B_{ij}^k(x) + \sum_{k=1}^{d}\sum_{l=1}^{d} \beta_{kl} A_{ki}(x) A_{lj}(x)$ and from (3) and (4)

$$C_{ij}(t,\tau) = \frac{\partial^2 \log \Phi(t,\tau; \mathbf{s}_1, \mathbf{s}_2)}{\partial s_{1i} \partial s_{2j}}\Big|_{\mathbf{s}=\mathbf{1}} = \int_0^t r(t-x)\overline{C}_{ij}(x,\tau)dx,$$

where $\overline{C}_{ij}(x,\tau) = \sum_{k=1}^{d} m_k B_{ij}^k(x,\tau) + \sum_{k=1}^{d}\sum_{l=1}^{d} \beta_{kl} A_{ki}(x) A_{lj}(x+\tau).$

Then for $\rho < 0$ one can obtain that

$$\overline{C}_{ij}(x) \sim \widetilde{C}_{ij} e^{\rho x},\ \overline{C}_{ij}(x,\tau) \sim \widetilde{C}_{ij}(\tau) e^{\rho x}, x \to \infty,$$

where $\widetilde{C}_{ij} = \sum_{k=1}^{d} m_k \widetilde{B}_{ij}^k$ and $\widetilde{C}_{ij}(\tau) = \sum_{k=1}^{d} m_k \widetilde{B}_{ij}^k(\tau)$ and

$$\widetilde{C}_{ij}(\tau) = \sum_{k=1}^{d}\sum_{l=1}^{d} m_k \widetilde{B}_{il}^k A_{lk}(\tau) + u_i v_j \sum_{k=1}^{d} m_k A_{jk}(\tau).$$

Similarly in the supercritical case $\rho > 0$ we obtain that

$$\overline{C}_{ij}(x) \sim \overline{C}_{ij} e^{2\rho x}, \overline{C}_{ij}(x,\tau) \sim \overline{C}_{ij} e^{\rho(2x+\tau)}, x \to \infty,$$

where $\overline{C}_{ij} = v_i v_j \overline{C}, \overline{C} = \sum_{k=1}^{d} m_k \widehat{B}^k + \sum_{k=1}^{d}\sum_{l=1}^{d} \beta_{kl} u_k u_l.$

Further on it is assumed also that $b_{jk}^i < \infty$ everywhere they appear and similarly for the immigration moments $m_i < \infty$ and $\beta_{ij} < \infty$, $1 \leq i, j, k \leq d$.

**3. Asymptotic behaviour of the subcritical processes.** In this section we will consider the case $\rho < 0$.

**Theorem 1.** *Let* $\widetilde{r}(\rho) = \int_0^\infty e^{-\rho x} r(x)dx < \infty$. *Then as* $t \to \infty$ :

**(i)** $M_i(t) \sim v_i \overline{A} \widetilde{r}(\rho) e^{\rho t}$, $C_{ij}(t) \sim \widetilde{C}_{ij} \widetilde{r}(\rho) e^{\rho t}$, *for* $i, j = 1, \ldots, d$;

**(ii)** $\mathbf{P}\{\mathbf{Y}(t) > \mathbf{0}\} \sim K \overline{A} \widetilde{r}(\rho) e^{\rho t}$, $K > 0$, *and* $\mathbf{P}\{\mathbf{Y}(t) = \alpha | \mathbf{Y}(t) > \mathbf{0}\} \to \mathbf{P}^*_\alpha$, *where* $F^*(\mathbf{s}) = \sum_{\alpha \in \mathbb{N}^d} \mathbf{P}^*_\alpha \mathbf{s}^\alpha$ *is the unique solution of the equation*

$$\sum_{k=1}^d f_k(\mathbf{s}) \frac{\partial F^*(\mathbf{s})}{\partial s_k} = -\rho(1 - F^*(\mathbf{s})), F^*(\mathbf{0}) = 0. \tag{5}$$

**Theorem 2.** *Let* $r(t) \sim r_0 e^{\theta t}, \theta < 0$, *and* $t \to \infty$.

**(i)** *If* $\theta = \rho$, *then* $M_i(t) \sim r_0 v_i \overline{A} t e^{\rho t}, C_{ij}(t) \sim r_0 \widetilde{C}_{ij} t e^{\rho t}$, $\mathbf{P}\{\mathbf{Y}(t) > \mathbf{0}\} \sim r_0 K \overline{A} t e^{\rho t}$, *and* $\Psi(t; \mathbf{s}) = \mathbf{E}\{\mathbf{s}^{\mathbf{Y}(t)} | \mathbf{Y}(t) > \mathbf{0}\} \to F^*(\mathbf{s})$, *where* $F^*(\mathbf{s})$ *is the unique solution of equation* (5).

**(ii)** *If* $\theta > \rho$, *then* $M_i(t) \sim r_0 \widetilde{A}_i(\theta) e^{\theta t}$, $C_{ij}(t) \sim r_0 \widetilde{C}_{ij}(\theta) e^{\theta t}$, *where* $\widetilde{A}_i(\theta) = \int_0^\infty e^{-\theta x} \overline{A}_i(x) dx < \infty$ *and* $\widetilde{C}_{ij}(\theta) = \int_0^\infty e^{-\theta x} \overline{C}_{ij}(x) dx < \infty$; $\mathbf{P}\{\mathbf{Y}(t) > \mathbf{0}\} \sim r_0 D(\theta) e^{\theta t}$ *and* $\Psi(t; \mathbf{s}) = \mathbf{E}\{s^{\mathbf{Y}(t)} | \mathbf{Y}(t) > \mathbf{0}\} \to \Psi^*(\mathbf{s}) = 1 - D(\theta; \mathbf{s})/D(\theta)$, *where* $D(\theta; \mathbf{s}) = \int_0^\infty e^{-\theta x} [1 - g(\mathbf{F}(x; \mathbf{s}))] dx < \infty$ *and* $D(\theta) = D(\theta; \mathbf{0})$.

**Remark 3.** The case $r(t) \sim r_0 e^{\theta t}, \theta < \rho$, is treated in fact by Theorem 1 because in this case $\widetilde{r}(\rho) = \int_0^\infty e^{-\rho x} r(x) dx < \infty$.

**Theorem 3.** *If* $R(t)/t \to r_0 > 0$ *as* $t \to \infty$, *then*

**(i)** $M_i(t) \to r_0 \int_0^\infty \overline{A}_i(x) dx < \infty$, $C_{ij}(t) \to r_0 \int_0^\infty \overline{C}_{ij}(x) dx < \infty$;

**(ii)** $\mathbf{P}\{\mathbf{Y}(t) = \alpha\} \to \mathbf{P}^*_\alpha$, *where* $\Phi(t; \mathbf{s}) = \mathbf{E}\{\mathbf{s}^{\mathbf{Y}(t)}\} \to \Phi^*(\mathbf{s}) = e^{-r_0 J(\mathbf{s})}$ *and* $J(\mathbf{s}) = \int_0^\infty [1 - g(\mathbf{F}(x; \mathbf{s}))] dx < \infty$ *satisfies the equation*

$$\sum_{k=1}^d f_k(\mathbf{s}) \frac{\partial}{\partial s_k} J(\mathbf{s}) = -[1 - g(\mathbf{s})], \tag{6}$$

*which admits an unique solution.*

**Corollary 1.** *If* $d = 1$, *then it follows from* (6) *that* $J(s) = \int_s^1 \frac{1 - g(u)}{f(u)} du$.

**Remark 4.** A similar result as in Corollary 1 was obtained from Sevastyanov [4] when he investigated single type subcritical Markov branching process with homogeneous Poisson immigration.

**Remark 5.** In Theorem 4, (i) of [14] the authors obtained also a convergence in distribution to a random variable but under the stronger condition $r(t) \to r_0 > 0$ as $t \to \infty$ and without equation (6). Notice that if $r(t) \to r_0 > 0$ as $t \to \infty$, then one has $R(t) \sim r_0 t$, $t \to \infty$.

**Theorem 4.** *Let $r(t) \sim r_0 e^{\theta t}$, $r_0 > 0$, $\theta > 0$. Then as $t \to \infty$*

**(i)** $M_i(t) \sim r_0 \widetilde{A}_i(\theta) e^{\theta t}, C_{ij}(t) \sim r_0 \widetilde{C}_{ij}(\theta) e^{\theta t}, C_{ij}(t, \tau) \sim r_0 \widetilde{C}_{ij}(\theta, \tau) e^{\theta t}$, *where*

$$\widetilde{A}_i(\theta) = \int_0^\infty e^{-\theta x} \overline{A}_i(x) dx < \infty, \ \widetilde{C}_{ij}(\theta) = \int_0^\infty e^{-\theta x} \overline{C}_{ij}(x) dx < \infty$$

*and*

$$\widetilde{C}_{ij}(\theta, \tau) = \int_0^\infty e^{-\theta x} \overline{C}_{ij}(x, \tau) dx < \infty, \quad i, j = 1, \ldots, d;$$

**(ii)** $X_i(t) = \dfrac{Y_i(t)}{M_i(t)} \to 1$, *in $L_2$ and a.s., $i = 1, 2, \ldots, d$;*

**(iii)** $\mathbf{U}(t) = (U_1(t), U_2(t), \ldots, U_d(t)) \to \mathbf{N}(\mathbf{0}, \mathbf{\Sigma}(\theta))$ *in distribution, where* $U_j(t) = [Y_j(t) - M_j(t)]/\sqrt{C_{jj}(t)}$ *and* $\mathbf{\Sigma}(\theta) = ||\widetilde{\sigma}_{jk}(\theta)||$ *is the covariance matrix with* $\widetilde{\sigma}_{jk}(\theta) = \widetilde{C}_{jk}(\theta)/\sqrt{\widetilde{C}_{jj}(\theta)\widetilde{C}_{kk}(\theta)}$, $j \neq k$, *and* $\widetilde{\sigma}_{jj}(\theta) = 1 + \widetilde{A}_j(\theta)/\widetilde{C}_{jj}(\theta)$.

**Remark 6.** Theorem 4, (i) can be interpreted as an analogue of a Strong LLN and Theorem 4, (ii) as a CLT.

**Remark 7.** In Theorem 4 (iii) as a CLT from the preprint [14] a convergence in distribution to some constant is proved (which is equivalent in this case to the convergence in probability).

**Theorem 5.** *Let $r(t) \sim r_0 t^\theta, t \to \infty$, where $r_0 > 0$ and $\theta \in \mathbb{R}$. Then*

**(i)** $M_i(t) \sim r_0 A^* t^\theta$, $C_{ij}(t) \to r_0 C^*_{ij} t^\theta$, $C_{ij}(t, \tau) \to r_0 C^*_{ij}(\tau) t^\theta$, *where*

$$A^* = \int_0^\infty \overline{A}_i(x) dx < \infty, \quad C^*_{ij} = \int_0^\infty \overline{C}_{ij}(x) dx < \infty$$

*and*

$$C^*_{ij}(\tau) = \int_0^\infty \overline{C}_{ij}(x, \tau) dx < \infty, \quad i, j = 1, 2, \ldots, d.$$

**(ii)** *If $\theta < 0$, then $\mathbf{P}\{\mathbf{Y}(t) > \mathbf{0}\} \sim r_0 Q^* t^\theta$, where*

$$Q^* = \int_0^\infty [1 - g(\mathbf{1} - \mathbf{Q}(t))] dt < \infty$$

*and*

$$\Psi(t; \mathbf{s}) = \mathbf{E}\{\mathbf{s}^{\mathbf{Y}(t)} | \mathbf{Y}(t) > \mathbf{0}\} \to 1 - Q^*(\mathbf{s})/Q^*,$$

*where*

$$Q^*(\mathbf{s}) = \int_0^t [1 - g(\mathbf{1} - \mathbf{Q}(x; \mathbf{s}))] dx < \infty.$$

**(iii)** *If* $\theta > 0$, *then* $X_i(t) = \dfrac{Y_i(t)}{M_i(t)} \to 1$, *in probability and* $L_2$, *where for* $\theta > 1$ *the convergence is also a.s.,* $i = 1, 2, \dots, d$;

$\mathbf{U}(t) = (U_1(t), U_2(t), \dots, U_d(t)) \to \mathbf{N}(\mathbf{0}, \mathbf{\Sigma}^*))$ *in distribution, where* $U_j(t) = [Y_j(t) - M_j(t)]/\sqrt{C_{jj}(t)}$ *and* $\mathbf{\Sigma}^* = ||\sigma^*_{ij}||$ *is the covariance matrix with elements* $\sigma^*_{ij} = C^*_{ij}/\sqrt{C^*_{ii}C^*_{jj}}$, $i \neq j$, *and* $\sigma^*_{ii} = 1 + A^*_i/C^*_{ij}$, $i, j = 1, 2, \dots, d$.

**4. Limiting distributions for the supercritical processes.** In this section we will consider the supercritical case $\rho > 0$.

**Theorem 6.** *Let* $\widetilde{r}(\rho) = \int_0^\infty e^{-\rho x} r(x) dx < \infty$. *Then as* $t \to \infty$

**(i)** $M_i(t) \sim v_i \overline{A} \widetilde{r}(\rho) e^{\rho t}$, $C_{ij}(t) \sim v_i v_j \overline{C} \widetilde{r}(2\rho) e^{2\rho t}$, $C_{ij}(t, \tau) \sim v_i v_j \overline{C} \widetilde{r}(2\rho) e^{\rho(2t+\tau)}$, $i, j = 1, \dots, d$;

**(ii)** $X_i(t) = \dfrac{Y_i(t)}{M_i(t)}$ *converges in* $L_2$ *to a random variable* $X_i$ *where* $\mathbf{E}X_i = 1$, $\mathbf{Var}X_i = \dfrac{\widetilde{r}(2\rho)\overline{C}}{\widetilde{r}^2(\rho)\overline{A}^2}$, $i = 1, 2, \dots, d$, *and* $X_1 = X_2 = \dots = X_d$ *a.s.;*

**(iii**) *The limiting Laplace transform* $\mathbf{E}\{e^{-\sum_{j=1}^d \lambda_j X_j}\} = \psi(\overline{\lambda})$, *where* $\psi(z) = \mathbf{E}\{e^{-zX_1}\}$ *and* $\overline{\lambda} = \sum_{j=1}^d \lambda_j$, *has the presentation*

$$\psi(\overline{\lambda}) = \exp\left\{ -\int_0^\infty [1 - g(\varphi(\overline{\lambda} e^{-rx}/(\widetilde{r}(\rho)\overline{A})))] r(x) dx \right\}, \tag{7}$$

*where* $\varphi(\overline{\lambda}) = (\varphi_1(\overline{\lambda}), \dots, \varphi_d(\overline{\lambda}))$ *is the unique solution of the system of differential equations:* $\dfrac{d}{d\overline{\lambda}}\varphi(\overline{\lambda}) = \mathbf{f}(\varphi(\overline{\lambda}))/(\overline{\lambda}\rho)$, $\varphi(0) = \mathbf{1}$.

**Corollary 2.** *Consider the classical case of homogeneous Poisson immigration (where* $r(t) \equiv r_0$) *and assume the classical norming* $v_i e^{\rho t}$ *instead of* $M_i(t)$. *Then from* (7) *applying the substitution* $y = \overline{\lambda} e^{-rx}$ *one obtains that*

$$\psi^*(\overline{\lambda}) = \exp\left\{ -r_0 \rho^{-1} \int_0^{\overline{\lambda}} [1 - g(\varphi(y))] y^{-1} dy \right\}. \tag{8}$$

*Hence for the single type process* $Y(t)$, *putting* $\overline{\lambda} = \lambda$ *in* (8) *one gets just the classical result obtained by Sevastyanov* [4].

**Remark 8.** The condition $\widetilde{r}(\rho) < \infty$ implies that $r(t)$ could be a regularly varying function or in general $r(t) = O(e^{\theta t}), \theta < \rho$. Notice that it also covers the homogeneous Poisson case $r(t) \equiv r_0$ because in this case $\widetilde{r}(\rho) = r_0/\rho$. Hence it will be interesting to consider the case, where $r(t) \sim r_0 e^{\theta t}, \theta \geq \rho, r_0 > 0$.

**Remark 9.** In Theorem 3, [14], the authors obtained only the convergence of the characteristic function.

**Theorem 7.** *If $r(t) \sim r_0 e^{\theta t}, \theta \geq \rho$ and $r_0 > 0$, then*

$$X_i(t) = \frac{Y_i(t)}{M_i(t)} \to 1, t \to \infty, \quad \text{in } \ L_2 \ \text{ and a.s.}, \ i = 1, 2, \dots, d.$$

**Remark 10.** Notice that in Theorem 4, (ii) and (iii) of [14] the authors proved a convergence in distribution to some constant which is in fact equivalent in this case to the convergence in probability.

**Theorem 8.** *Let $r(t) \sim r_0 e^{\theta t}, \theta \geq \rho, r_0 > 0$, and for $j = 1, 2, \dots, d$*

$$U_j(t) = [Y_j(t) - M_j(t)]/\sqrt{C_{jj}(t)}, \quad \mathbf{U}(t) = (U_1(t), U_2(t), \dots, U_d(t)).$$

**(i)** *If $\rho \leq \theta \leq 2\rho$, then $\lim_{t\to\infty} \mathbf{U}(t) = \mathbf{U} = (U_1, U_2, \dots, U_d)$ in distribution, where $U_1 = U_2 = \dots = U_d$ a.s. and $U_1$ has $N(0,1)$ distribution.*

**(ii)** *If $\theta > 2\rho$, then $\mathbf{U}(t)$ as $t \to \infty$ has a limiting multidimensional normal distribution $N(\mathbf{0}, \mathbf{\Sigma}(\theta))$, where $\mathbf{\Sigma}(\theta)= ||\sigma_{kl}(\theta)||$ is the covariance matrix with elements $\widetilde{\sigma}_{kl}(\theta) = \widetilde{C}_{kl}(\theta)/\sqrt{\widetilde{C}_{kk}(\theta)\widetilde{C}_{ll}(\theta)}$, for $k \neq l$, and $\widetilde{\sigma}_{kk}(\theta) = 1 + \widetilde{A}_k(\theta)/\widetilde{C}_{kk}(\theta)$, $k, l = 1, 2, \dots, d$.*

**Remark 11.** Theorem 7 can be interpreted as an analogue of a Strong LLN and Theorem 8 as a CLT.

**5. Concluding remarks.** The results presented in this paper complete the study of the asymptotic behaviour of multitype Markov branching processes in the subcritical, critical and supercritical cases. The investigation of the multitype age-dependent processes is an open problem. It could be done as in the single type case for Sevastyanov age-dependent model (see [17–20]).

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