ON MULTIDIMENSIONAL POVERTY RANKINGS OF BINARY ATTRIBUTES

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Abstract. We address the problem of ranking distributions of attributes in terms of poverty, when the attributes are represented by binary variables. In order to accomplish this task, we identify a suitable notion of 'multidimensional poverty line' and characterize axiomatically the Head-Count and the Attribute-Gap poverty rankings, which are the natural counterparts of the most widely used income poverty indices.

Finally, we apply our methodology and compare our empirical results with those obtained with some other well-known poverty measures.

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1. Introduction

The present work proposes and characterizes poverty criteria in a setting where individuals are endowed with a finite set of attributes that are represented by binary variables. These attributes can be interpreted as individual opportunities or dimensions of an achievement space and are dichotomous variables: any individual either has or has not access to a given attribute. We argue that this modeling choice is able to encompass a wide range of empirically relevant analyses of poverty. In fact, while some individual attributes are dichotomous in nature (consider e.g. literacy, health insurance, individual rights or civil liberties in a given society), some other attributes for which a finer classification is possible (admitting ordinal or cardinal representations) can always be transformed into binary variables by choosing a proper threshold. Thus, the framework adopted in the present paper allows a quite comprehensive approach to the problems of multidimensional poverty measurement.

Motivation. Poverty reduction plays a prominent role in political debates all over the world, and methods and techniques to make poverty comparisons are essential tools to design and evaluate policies aimed at decreasing poverty. Indeed, since the publication of Sen's (1976) pioneering paper on poverty measurement, in the last decades a massive amount of literature has been devoted to this subject and several measures of poverty are now available. However, most of the existing works on poverty measurement focuses on income or consumption expenditures as the only relevant explanatory dimension of poverty. This approach is now widely regarded as insufficient and incomplete because various issues interact to impact on poverty such as education, health, housing, income,

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food security and access to the decision making process that goes on in politics. The problem of poverty therefore permeates many dimensions of human life which are not easily reduced to a unique feature, namely poverty is essentially a multidimensional phenomenon and the exclusive reliance on just one indicator can hide crucial aspects of economic deprivation. In that respect, many scholars, like Rawls (1971) and Sen (1985, 1991), have defended in their influential works the necessity to move from an income-based evaluation of social inequities towards the more comprehensive domain of attainable achievements, agreeing on the fact that two societies with the same distribution of monetary earnings can hardly be considered as equivalent in terms of poverty if in one of them a fraction of the population is denied a number of basic rights and liberties such as the right to vote, freedom of speech, freedom of movement, access to basic education and health care and so on. We agree on this point and we focus on the specific problem of poverty measurement when the explanatory variable is a *set* of individual attributes (rather than a scalar, as it is the case with income or consumption). As a consequence, our problem amounts to ranking different attribute profiles on the basis of poverty hence a multidimensional evaluation exercise that is by no means straightforward from a theoretical and empirical point of view.

Contents. A natural approach towards devising a multidimensional poverty ranking for attribute distributions consists in extending the familiar notion of "poverty line" and the most well-known income poverty measures to a multidimensional setting. We identify the different value systems involved in the use of different poverty criteria by using an axiomatic approach: we propose a number of properties that a *poverty ranking* on the possible distributions (profiles) of finite attribute sets should satisfy, and study their logical implications. We address the problem by referring to an abstract attribute space, where attributes may be thought of as non-welfare characteristics of agents such as basic liberties, political rights, individual freedoms, or access to certain welfare-enhancing traits. We model attributes as *binary* variables, with an assignment of attribute values to each unit of a finite population representing an attributes-sets distribution or simply an attribute profile. These three elements, i.e. a finite list of relevant binary attributes, a finite population and the related attribute profile, constitute the initial *inputs* of our model. It is worth observing here that a binary representation of attributes typically results from a previous preprocessing of ordinal or cardinal attribute-data by introducing a poverty/deprivation threshold for each attribute: our model is *reduced* -as opposed to *extensive*- in that such a preprocessing of attribute-data is taken as granted (and not explicitly mentioned).

We then proceed by following Sen's approach which divides the evaluation of poverty into two steps: (i) the *identification step*, in which the poor are identified in any given population; (ii) the *aggregation step*, in which the data about who is poor according to one (or several) variable(s) are brought together into an overall measure in order to obtain a global assessment of poverty for the relevant population.

In the unidimensional context, the *identification step* is solved by choosing a poverty line that divides the population into two sets: the poor and the non-poor. The identification of the poverty line can follow an absolute or a relative approach. While with an absolute approach the poverty line is defined in an exogenous way and it is the same across distributions, with a relative approach the poverty line in a distribution is a function of the distribution itself (e.g. the poverty line can be fixed at half the median income level in that society: see Atkinson, (2003)). In the present multidimensional context there are two different choices to be made. The first is the choice of a threshold for each relevant dimension. The second is the aggregation along the different dimensions in order to evaluate the poverty of each individual.

As for the first problem, our choice is essentially dictated by the domain we are working with: each dimension is modelled as a binary variable. One individual does have access to a specific attribute or he does not; there are not intermediate degrees levels of access to a given dimension of well-being.[1]

The second choice concerns the aggregation of the different dimensions. There are two main approaches in the existing literature: the *union approach*, which establishes that a population unit is poor if he/she is below the threshold of *at least one* dimension, and the *intersection approach* which instead regards a population unit as poor if he/she is below the threshold in *all* the relevant dimensions (Atkinson, 2003).[2] We propose a solution based on the concept of *essential attributes* as induced by a threshold T. The threshold T is supposed to be exogenous and may in fact provided by the members of a suitably appointed committee of experts who submit a profile of poverty thresholds and select a single multidimensional threshold T by applying some previously specified aggregation rule.[3] After having fixed T, we get that each population unit is either *poor/deprived* or *not poor/deprived*: the resulting identification step is therefore *'crisp'*.[4]

Thus, threshold T identifies sets of essential attributes within the universal set, and an individual is denoted as 'poor' if she does not have access to *all* the essential attributes as specified by T. Hence we follow the *union approach to identification*, but we restrict it to the essential attributes that in such a case must be considered as *complementary goods.* One possible interpretation of such a multidimensional poverty line is linked to the *essential needs* approach: having access to all essential attributes amounts to being able to satisfy all basic needs.

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[1]The treatment of each attribute as a dichotomous variable is a domain restriction -especially if referred to variables such as access to education, health care, income- and may imply a loss of information. However, it also allows a simplified, uniform treatment of variables such as access to basic liberties and rights which are usually taken to be binary, and other variables which would also admit non-binary representations.

[2]For an "intermediate" solution, based on a variable a 'suitably fixed minimal number' of deprivation see Alkire and Foster (2011).

[3]Such an aggregation rule might well be a majority-like rule as the *median* of the proposed thresholds, since the latter can be shown to be well-defined (and admits a simple axiomatization itself, see Savaglio and Vannucci (2019)).

[4]In a generalization of the present model, we extend our results to the case in which the entire profile of thresholds is considered. Each element $T_i \in T$ amounts to a block of *complementary opportunities* which is also a *substitute* for any other block $T_j \in T$. Thus each $T \in \mathcal{T}$ may also regarded as a possibly complex and nuanced judgment about *complementarity and substitutability relationships* among subsets of attributes.

Such results are available upon request from the authors.

Once we have identified the poor, we need to amalgamate information on the deprivation suffered by the poor in order to answer the question: 'when is one person poorer (richer) than another person in terms of attributes?' In other words, we have to define a criterion to compare individuals endowed with different sets of attributes.[5] In order to answer such a question, it is worth noticing first that the univariate case allows a natural *total* ordering of personal attribute profiles or endowments (e.g. income, wealth, consumption expenditure etc.). On the contrary, any (multivariate) attribute profile is a multidimensional distribution that typically admits only dominance *partial* rankings as natural and non-controversial (in fact, an individual might be better than another one in one attribute or dimension but worse in others). Thus, while we treat all the attribute sets lying above the poverty threshold as non-poor and therefore mutually indifferent, the attribute sets lying below the poverty threshold are ranked by set inclusion. Hence our poverty threshold T mimics the poverty line of the unidimensional case, but induces a (natural) *partial preorder*[6] on the elements of an attribute profile. Indeed, given such any set of essential attributes as specified by T, we distinguish an indifference class of population units that are non-poor, because T is a subset of their individual attribute sets, and a class of poor population units who lack at least one essential attribute and are possibly mutually ranked in terms of set-inclusion, arguably one of the mildest and less-controversial criteria in the literature on rankings of sets of objects (see Barbera, Bossert and Pattanaik (2004)).

As for the *aggregation step* and *final output* of our model, we collect all the information concerning the relevant aspects of poverty to yield a global assessment of poverty, and propose a characterization of two fundamental orderings: the *Head-Count* (HC) and the *Attribute-Gap* (AG) poverty rankings. Such rankings are the natural multidimensional counterparts to the most widely used income poverty measures, namely the *head count ratio* and *the income poverty gap*. Indeed, the head-count ranking is produced by counting the number of population units whose endowments fail to meet the minimum standard T. Although the HC-poverty ranking fails to record the 'total intensity of poverty', it may be quite acceptable as a measure of the number of people deprived of access to a decent living standard if we regard our threshold T as the minimum requirement to guarantee the latter.

On the other hand, if we want to assess the 'total intensity of poverty', then we may use the Attribute-Gap ranking that is produced by counting the number of extra-'total amount of attributes' each population unit should be actually endowed with in order to achieve the minimum standard, and by *summing* them. Hence, the Attribute-Gap poverty ranking aggregates information about individual poverty by counting binary gaps and summing them in order to specify how poor are the poor. This is admittedly a quite crude measure of 'poverty intensity' and has been also the target of sustained criticism. However, we maintain that such multidimensional version of the AG-poverty ranking may make much sense as a first approximation to a sound assessment of the aggregate 'intensity of poverty', whenever it is combined with suitable definitions of the attribute

[5]There is an extensive literature devoted to the problem of ranking sets of objects under different interpretations of the latter (see on this the excellent survey by Barberà, Bossert and Pattanaik (2004)).

[6]We recall that a partial preorder is a reflexive and transitive binary relation.

space and the poverty threshold. Be it as it may, it should be emphasized that *the output of our model is a poverty-ranking of opportunity attributes.* We focus on two distinct poverty-rankings, namely the poverty head-count total preorder and the poverty gap total preorder: each one of them is provided with a tight characterization by a set of simple requirements.[7] Of course, two *indices* are also obtained as a by-product, but they are *not* the target of the present analysis. It should also be noticed that the axioms introduced enjoy two key properties, namely (a) they do not include behavioural assumptions on the relevant evaluators (i.e. the experts) and (b) are sufficient to determine a *unique outcome.*

Empirics. Based on the 2016 cross-sectional component of the *European Union Survey on Income and Living Conditions* EU-SILC data, we provide an illustration of our methodology by estimating the Head-Count and the Attribute-Gap poverty rankings for some European countries.[8] We also discuss why the application of our approach leads to conclusions that partially differ both from the traditional measure of income poverty and from those resulting from two novel cut-off rankings measures: the approach used by Eurostat for measuring material deprivation and the Alkire Foster methodology (2011). Indeed, though all of these methodologies follow the *union* approach, there are notable differences in terms of poverty estimation. That is so because income poverty rankings are obtained by estimating the proportion of people whose income lies below the poverty line. On the other hand, the union approach used by Eurostat for estimating material deprivation consists in aggregating across the different attributes (items) for each individual, and then across individuals: an individual score of deprivation results from the (possibly weighted) sum of binary attributes. In the most simple case of equal item weights, deprived individuals are defined as those lacking at least a certain number of items (Guio, Marlier, Gordon, Fahmy, Nandy, and Pomati, 2016). That is precisely the estimation of poverty rankings according to the Alkire-Foster methodology (2011) which requires the identification of a "cutoff", namely the minimum level of achievement considered necessary to be not poor. In our methodology based on the concept of *essential attributes*, poverty rankings are estimated by computing the proportions of people suffering from poverty in at least one specific attribute within a set of essential attributes, which induce the multidimensional poverty line. The selected poverty line includes income, education and health status (in the form of binary variables), as essential attributes. The poverty rankings of European countries in terms of Head-Count and Attribute-Gap induced by that threshold exhibit a different and complementary picture with respect to the one obtained with the traditional measure of income poverty or with the Eurostat measure of material deprivation under the Alkire-Foster measure with cut-off equal to two.

[7] Peragine, Savaglio and Vannucci (2008) provides axiomatic characterizations of some further poverty rankings resulting from lexicographic combinations and weighted sums of the head-count and poverty-gap indices.

[8] The Statistical Agency of the EU (Eurostat) collects EU-SILC data on a regular basis and the importance of collecting these data is emphasized by the European Union as part of the European 2020 Agenda measures.

Our results suggest that the most severe poverty is suffered by the Mediterranean countries (Portugal, Greece, Spain and Italy), while the Nordic countries show the best performance, and an intermediate position is occupied by the continental and Eastern European countries. This macro picture is basically confirmed by both the Head-Count and the Attribute-Gap measures. Countries (e.g. Germany) with quite consistent percentages of poor citizens seem to be more attractive in terms of education and access to health than countries with lower level of poverty. As a consequence, our results corroborate the need of including in the poverty measures other individual attributes representing potential determinants of life opportunities.

The paper is organized as follows. The next section introduces the analytical setting and defines formally the basic problem studied in the present work. Section 3 discusses a set of axioms and contains the theoretical results of the paper: the characterization of the Head-Count and the Attribute-Gap poverty rankings. In section 4, we discuss the related literature. Section 5 presents the data and the main empirical results. Section 6 concludes with a brief discussion of the results and of directions for future research, while an appendix collects all the proofs.

2. The framework

We start by identifying a universal non-empty finite set of attributes, denoted by X. We assume that each element in X is desirable in some universal sense. Moreover, following the existing literature, we assume that attributes are *non-rival*, so that a given attribute is potentially available to everyone simultaneously, and that attributes are *excludable*, so that providing an attribute to some individuals does not necessarily imply that everyone has this attribute. Moreover, our attributes are entities that are *mutually non-exclusive*.[9]

Let $N = \{1, ..., n\}$ denote the finite set of relevant population units[10] and $\mathcal{P}[X]$ the set of all *finite* subsets of X. Elements of $\mathcal{P}[X]$ are referred to as *attribute sets*, and mappings $\mathbf{Y} = (Y_1, ..., Y_n) \in \mathcal{P}[X]^N$ as profiles of attribute sets, or simply *attribute profiles*. Hence, each individual in a society is endowed with an *attribute set* and a society is represented by an *attribute profile*.

[9]Indeed, a non-negligible part of the literature on opportunity sets does rely on an interpretation of opportunities as objects coming jointly for an agent (notably Ok (1997), Ok and Kranich (1998), Savaglio and Vannucci (2007)), while other prominent contributions do admit both a mutually-exclusive and non-mutually-exclusive interpretation of opportunities (see e.g. Kranich (1996), Herrero, Iturbe-Ormaetxe and Nieto (1998), Alcalde-Unzu and Ballester (2005, 2010), Barberà, Bossert, Pattanaik (2004, section 5)).

[10]The framework of the present paper only refers to a fixed finite set N of *population units*. Therefore, the relevant poverty rankings mentioned above only concern attribute profiles of the *same size*. That modelling choice brings about a considerable simplification of notation and axioms with respect to its possible variable-population counterparts. That is not to say, however, that the present framework can only be applied to data concerning communities (national, or otherwise) having the very same number of members. On the contrary, our model may be consistently applied to produce poverty comparisons concerning communities of *different size* as will be in fact shortly done in section 4 below. That only requires a minor massaging of data and a straightforward reinterpretation of the relevant population units, as explained again in section 5.

As previously mentioned in the Introduction, we are interested in ranking such attribute profiles in terms of poverty.[11] Then, in order to proceed with our analysis, we first need to *identify* who is poor in our framework. We therefore define a *poverty threshold* (or poverty line) as a set $T \in \mathcal{P}[X]$, which identifies a collection of *essential attributes*: an individual is said to be poor or, equivalently, to be below the poverty threshold, if her attribute set does *not* contain *all the essential attributes* i.e. all the elements of T. It is worth noticing here that we treat the essential attributes as exogenous, hence we follow the so-called *absolute* approach to poverty in the identification step. However, we can also observe that our threshold T could also be taken to be contingent on suitable profiles of attribute sets. In particular, the threshold T may be defined as the *median* of individual attribute sets of a given profile of threshold-proposals (see Savaglio and Vannucci (2019)).[12] Then, the criterion we adopt to compare individuals endowed with different attribute sets is the following: *all the (individual attribute) sets above the poverty threshold are mutually indifferent, while the sets below the poverty thresholds are ranked by set inclusion.* Therefore, the universe of the non–poor is represented by a unique indifference class and the very mild condition of set inclusion is proposed as the reference ranking rule within the poor-subpopulation.

The following example should clarify this point.

Example 1. Let $X = \{x_1, x_2, ..., x_k\}$ be the set of all attributes, $N = \{1, 2, 3\}$ the relevant population and $T = \{x_1, x_3\}$. Then, at attribute profile:

$$\mathbf{Y} = (Y_1 = \{x_1, x_2, x_4, x_5, x_6\}, Y_2 = \{x_3, x_4\}, Y_3 = \{x_1, x_2, x_3\}),$$

population units 1 and 2 are poor, while 3 is non-poor because $Y_1 \not\supseteq T$, $Y_2 \not\supseteq T$ and $Y_3 \supset T$. However, neither $Y_1 \supseteq Y_2$ or $Y_2 \supseteq Y_1$.

Formally, our starting point is a (partial) individual poverty preorder $\succcurlyeq^*_T$ on $\mathcal{P}[X]$ induced by the poverty threshold T and defined as follows: for any $Y, Z \in \mathcal{P}[X]$,

$$Y \succcurlyeq^*_T Z \text{ if and only if } [Z \cap T \supseteq Y \cap T \text{ or } Z \supseteq T],$$

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[11] Notice that our general model can also be related to behaviorally oriented notions of opportunity sets by the following interpretation. Let X be a possibly multidimensional space of relevant, observable functionings, N^* a population, $\mathbf{x} \in X^{N^*}$ the profile of achieved functionings within the population under consideration, and $\pi = \{\pi_1, ..., \pi_n\} \in \Pi(N)$ a partition of the population into a finite set $N = \{1, ..., n\}$ of types according to a fixed set of verifiable criteria. Then, the opportunity set of type $i \in N$ at $(\mathbf{x}, \pi)$ is $X_i = \{x \in X : \text{there exists } j \in \pi_i \text{ such that } x_j = x\}$.

[12] We recall that in a sequel to the present paper we consider the general case of composite thresholds consisting of multiple essential sets. The essential sets of such a composite threshold are of course mutually incomparable (in terms of set-inclusion), and mutual *substitutes*. In any case, we envisage the relevant threshold T as the outcome of an aggregation rule as applied to the thresholds proposed by the members of a panel of experts. In particular, the relevant committee can select T by computing the median of all the proposed thresholds.

The characterization results presented here easily extend to this more general framework: details are available from the authors upon request.

namely individual i endowed with the attribute set Y is at least as poor in terms of essential attributes as individual j endowed with attribute set Z whenever either the set of all essential attributes of i is a subset of the corresponding set of essential attributes of j or j is *non-poor*.

It should be emphasized here that a distinctive feature of our approach is that *complete* individual poverty rankings (let alone individual *poverty indices* as e.g. in Bossert, Chakravarty, D'Ambrosio (2013)), are *not* included among the basic data of our model. Rather, we stick to the quite uncontroversial set-inclusion *partial order* of attribute sets (a weak dominance criterion) as supplemented with an agreed upon multidimensional poverty threshold. As a result, we end up with a partial poverty preorder of (individual) attribute sets and rely on it as a common basis for individual poverty comparisons.

Remark 1. Partial preorder $\succcurlyeq_T^*$ has both a top indifference class comprising precisely the empty set, and a bottom indifference class including all supersets of T. The distance of each $Y \subseteq X$ from such a bottom indifference class may be considered as a gross numerical estimate of the *severity* of poverty attached to it. It is easily checked that -by construction- $\succcurlyeq_T^*$ is *graded*, namely all the maximal chains joining an arbitrary pair of attribute sets have the same *length*.[13] Every graded poset is equipped with an integer-valued function, called *rank function,* that preserves the ordering, i.e. in our case, $\succcurlyeq_T^*$ actually admits a *poverty* rank function $r : \mathcal{P}[X] \to \mathbb{Z}_+$ that 'preserves' $\succcurlyeq_T^*$, meaning that for every $Y, Z \in \mathcal{P}[X]$ with $Y \subset Z$, it must be the case that $r(Y) < r(Z)$. Moreover, $r(Y) = r(Z) + 1$ whenever Y is an *upper cover*[14] (or immediate successor) of Z according to $\succcurlyeq_T^*$.[15] In particular, it is also easy (and left to the reader) to check that the poverty rank of an attribute set Y according to $\succcurlyeq_T^*$ is precisely the number of attributes included in the threshold T that are missing in Y, namely it amounts to our Attribute-Gap ranking as informally described in the Introduction, and formally defined below. Thus the poverty rank function induced by $\succcurlyeq_T^*$ does indeed provide us with a meaningful numerical index of individual poverty: but notice that it is an auxiliary *derivative* notion that can be defined in a natural way on the basis of our general assumptions, not an additional primitive notion requiring further independent stipulations of its own.

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[13]In order to fully understand the structure of the poset $(\mathcal{P}[X], \subseteq)$, suppose again that the bottom indifference class is the empty set (meaning it is most poor who has a null attribute set) and the top indifference class is the set of all supersets of $\{x, y, z\}$. Then, consider the following two $\subseteq$-induced maximal chains (i.e. totally ordered sets to which no element can be added without losing the property of being totally ordered): $[\{y\}, \{x, y\}, \{x, y, z\}]$ and $[\{y\}, \{y, z\}, \{x, y, z\}]$. Both have the same *length*, seen ad the number of steps we count from the bottom to the top. If for any pair of elements the posetic structure under analysis satisfies this property then it is said to be *graded*.

[14]We recall that the covering relation of a partially ordered set is the binary relation which holds between comparable elements that are immediate neighbours. Saying that Z covers Y, written, in our case, $Y \subset Z$ means that there is no element W such that $Y \subset W \subset Z$. In particular, for the power set $\mathcal{P}[X]$ under consideration, which has the structure of a Boolean algebra, a subset Z of $\mathcal{P}[X]$ *covers* a subset Y of $\mathcal{P}[X]$ if and only if Z is obtained from Y by adding one element not in Y. In this case, the *cover* dominance relation is numerically evaluated by saying that the value of Z is equal to the value of Y plus 1.

[15]See e.g. Anderson (1987) for further details concerning the rank function of a graded poset.

The notation $\mathbf{Y}_{|T}$ will be employed in the rest of this paper to denote attribute profile $(Y_i \cap T)_{i\in N}$.

Finally, we define a *poverty ranking* of attribute profiles on X induced by threshold $T \subseteq X$ a preorder $\succcurlyeq_T$ on $\mathcal{P}[X]^N$ such that for any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$ whenever $Y_i \succcurlyeq_T^* Z_i$ for each $i \in N$.

In the present setting of attribute-set profiles two of the most widely used income poverty indices, namely the *head count ratio* and the *income poverty gap,* can be easily generalized:

Definition 1. *The Head-Count (HC) poverty ranking of attribute profiles under threshold T is the total preorder $\succcurlyeq_T^h$ on $\mathcal{P}[X]^N$ defined as follows: for any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$,*

$$\mathbf{Y} \succcurlyeq_T^h \mathbf{Z} \textit{ if and only if } h_T(\mathbf{Y}) \geq h_T(\mathbf{Z}),$$

where for each $\mathbf{W} \in \mathcal{P}[X]^N$, $h_T(\mathbf{W}) = \#H_T(\mathbf{W})$ and $H_T(\mathbf{W}) = \{i \in N : W_i \not\supseteq T\}$.

The Head-Count poverty ranking ranks two distributions on the basis of the number of individuals that are below the poverty threshold T. Hence, it captures the *incidence of poverty.* However, the head-count fails to take into account the depth or the severity of the deprivation suffered by the poor. In order to capture this aspect of the aggregate poverty, one may consider the *Attribute-Gap (AG) poverty ranking* which measures the *aggregate intensity of poverty.*

Definition 2. *The attribute-gap (AG) poverty ranking of attribute profiles under threshold T is the total preorder $\succcurlyeq_T^g$ on $\mathcal{P}[X]^N$ defined as follows: for any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$,*

$$\mathbf{Y} \succcurlyeq_T^g \mathbf{Z} \textit{ if and only if } g_T(\mathbf{Y}) \geq g_T(\mathbf{Z}),$$

where for each $\mathbf{W} \in \mathcal{P}[X]^N$, $g_T(\mathbf{W}) = \sum_{i\in H_T(\mathbf{W})} \#\{x : x \in T \setminus W_i\}$.

Thus, for each poor individual, the intensity of poverty, the "individual poverty gap", is measured by the number of essential attributes she does not have access to. That is, for each poor individual i, with attribute set W_i, the "individual poverty gap" $g_T(W_i)$ is given by the following refined cardinality-difference[16] with respect to the threshold set T: $g_T(W_i) = |\#(T) - \#(W_i \cap T)|$.

Notice that the aggregate poverty gap $g_T(\mathbf{W})$ records the number of population units that are 'poor' with respect to *some* essential attribute. Therefore, computing $g_T(\mathbf{W})$ amounts to counting the number of poor with respect to *each* essential attribute in T, and then adding those numbers across attributes. In that respect, the aggregate poverty gap may also be regarded as an alternative version of the head-count of poor. Finally, it is easily checked that $g_T(\cdot)$ is exactly the poverty rank function induced by $\succcurlyeq_T^*$ as discussed above (see Remark 1).

[16]The cardinality-difference relation was introduced and axiomatically characterized by Kranich (1996).

3. The characterization of the HC and AG poverty rankings

The axiomatic structure to be presented below will lead us to a characterization of the Head-Count and Attribute-Gap poverty rankings. Let us start by introducing some basic properties for a poverty ranking $\succcurlyeq_T$ of $\mathcal{P}[X]^N$:

Anonymity (AN). For any $\mathbf{Y} \in \mathcal{P}[X]^N$ and any permutation π of N such that $\pi\mathbf{Y} = (Y_{\pi(1)}, ..., Y_{\pi(n)})$: $\mathbf{Y} \sim_T \pi\mathbf{Y}$.

Irrelevance of Inessential Attributes (IIA). For any $\mathbf{Y} \in \mathcal{P}[X]^N$, $i \in N$, and $x \in Y_i \setminus T$: $\mathbf{Y} \sim_T (\mathbf{Y}_{-i}, Y_i \setminus \{x\})$.

Irrelevance of Poor Attribute Deletion (IPAD). For any $\mathbf{Y} \in \mathcal{P}[X]^N$, $i \in H_T(\mathbf{Y})$, $x \in Y_i$, if $\{x\} \neq T \backslash Y_i$ then $\mathbf{Y} \sim_T (\mathbf{Y}_{-i}, Y_i \setminus \{x\})$.

Dominance at Essential Profiles (DEP). For any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$ such that both $Y_i \in \{T, \emptyset\}$ and $Z_i \in \{T, \emptyset\}$ for all $i \in N$:

$$\mathbf{Y} \succ_T \mathbf{Z} \text{ if and only if } \# \{i \in N : Y_i = \emptyset\} > \# \{i \in N : Z_i = \emptyset\}.$$

The first three axioms are *invariance properties*, in the sense that they require our poverty rankings to ignore certain aspects of the attribute distributions and to focus on others. The first, *Anonymity,* is an axiom that requires a symmetric treatment of individuals, thereby preventing the relevant ranking from taking into account information concerning the identities of individuals. *Irrelevance of Inessential Attributes* says that if the attribute set of an individual i is reduced by the subtraction of an attribute which is not essential, then the new profile of attribute sets exhibits the same degree of poverty as the original profile. This axiom is reminiscent of the *focus axiom*, used in the income poverty paradigm (see also Bourguignon and Chakravarty (2002) for the multidimensional case), which requires invariance with respect to reduction in the incomes of the non-poor. However, instead of distinguishing between the poor and the non-poor, in the current scenario the basic distinction is between essential and non-essential attributes. *Irrelevance of Poor Attribute Deletion* says that if the attribute set of a poor individual i is reduced by the subtraction of an attribute, then the new profile of attribute sets exhibits the same degree of poverty as the original profile.[17]

While the previous invariance properties are useful in identifying the information that our poverty rankings should use, the last axiom is a *dominance property*, which identifies classes of transformations that have a certain effect on the poverty rankings, thereby restricting the set of poverty criteria. *Dominance at Essential Profiles* indeed considers a particular case in which two 'degenerate' profiles

[17] To illustrate the significance of IPAD, consider, for instance, a situation with one essential attribute, namely the right to have an education and an individual that has no access to it. According to our definition, this person is poor. Therefore, the possible non-essential opportunity to free access to all libraries of her town does not increase her freedom of choice (because she is not able to read), hence her possibility 'to be someone or to do something' in that respect.

are composed of either empty sets or sets coinciding with threshold T. In this special case, a profile exhibits more poverty than another one if the number of people endowed with the empty set in the former is higher than the number of individuals endowed with the empty set in the latter.

Our first proposition shows that these axioms are necessary and sufficient conditions for the characterization of the HC-poverty ranking $\succcurlyeq_T^h$:

Proposition 1. *Let $\succcurlyeq_T$ be a poverty ranking of $\mathcal{P}[X]^N$ under threshold $T \subseteq X$. Then $\succcurlyeq_T$ is the HC-ranking $\succcurlyeq_T^h$ if and only if $\succcurlyeq_T$ satisfies* AN*,* IIA*,* IPAD *and* DEP*. Moreover, such a characterization is tight.*

We now introduce two further axioms:

Strict Monotonicity with respect to Essential Deletions (SMED). For any $\mathbf{Y} \in \mathcal{P}[X]^N$, $i \in N$, and $x \in Y_i \cap T$: $(\mathbf{Y}_{-i}, Y_i \setminus \{x\}) \succ_T \mathbf{Y}$.

Independence of Balanced Essential Deletions (IBED). For any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, $i \in N$, $y \in Y_i \cap T$ and $z \in Z_i \cap T$:

$$\mathbf{Y} \succcurlyeq_T \mathbf{Z} \text{ if and only if } (\mathbf{Y}_{-i}, Y_i \setminus \{y\}) \succcurlyeq_T (\mathbf{Z}_{-i}, Z_i \setminus \{z\}).$$

Strict Monotonicity with respect to Essential Deletions is another *dominance property* which says that if the attribute set of an individual i is reduced by the subtraction of an essential attribute, then the new profile of attribute sets exhibits a higher degree of poverty than the original profile. This axiom is a direct translation in our context of the Monotonicity axiom used within the income inequality framework (see Foster (2006)). Once again, the difference relies on the fact that in the current scenario the crucial distinction is between essential and non-essential attributes rather than between poor and non-poor individuals.

Finally, *Independence of Balanced Essential Deletions* is a standard independence axiom, which concerns deletion of an essential attribute from the attribute-set of an individual i in two attribute profiles $\mathbf{Y}, \mathbf{Z}$. Such balanced deletions preserves the ranking of the two attribute profiles and imposes equal weights on the essential attributes, since all of them are *equally* all-important.

It is quite obvious that by fixing an arbitrary and more detailed preference structure and making reference to it, one might say that a certain essential attribute is more valuable than another. But this finer preferential structure would not be justified in our most parsimonious, minimalist setting. Why should we consider two essential attributes (such as the right to vote and freedom of speech or, say, access to minimal education and minimal health care) as non-trivially i.e. asymmetrically ranked, given the *strict complementarity* between them as embodied in the very definition of threshold T? It is worth emphasizing here that our talk about *complementarity* refers to the following simple point concerning $\succcurlyeq_T^*$: movements across distinct *comparable* indifference classes of that preorder (say, from attribute set Y to attribute set Z of a lower poverty rank) require that whatever attributes in T are included in Y (the attribute set of higher rank, i.e. the poorer attribute set) are retained in attribute set Z. No *substitution* between attributes of Y and Z can be contemplated because such a substitution would invariably render Y and Z mutually *incomparable* -as opposed to indifferent- with respect to $\succcurlyeq_T^*$. That 'complementarity-talk' is admittedly rather informal but we maintain that it could be properly articulated in a suitably formalized general setting.

It turns out that the first two and the last two axioms of this section are necessary and sufficient to characterize our attribute-gap criterion:

Proposition 2. *Let $\succcurlyeq_T$ be a poverty ranking of $\mathcal{P}[X]^N$ under threshold $T \subseteq X$. Then $\succcurlyeq_T$ is the AG-ranking $\succcurlyeq_T^g$ if and only if $\succcurlyeq_T$ satisfies* AN, IIA, SMED *and* IBED. *Moreover, such a characterization is tight.*

Thus, we provide two simple characterizations of the most basic poverty rankings of attribute profiles.[18] We would like to stress that, to the best of our knowledge, those results have no counterpart in the standard literature on poverty indices of income distributions, though the head-count and poverty-gap are among the most widely used criteria in the theoretical and empirical literature on poverty. We further observe that the counting cut-offs proposed by Alkire and Foster (2011) are special case of our general $\succcurlyeq_T$ rankings[19] with $T = T(k) := \{Y \subseteq X : \#(X \backslash Y) \leq k\}$.

Before proceeding with an application of our approach to the measurement of multidimensional poverty, it is worth discussing two prominent theoretical issues.

It should be emphasized that our setting is by no means a special case of the typical real-vector-based models adopted in the literature on multidimensional poverty measurement. The latter relies on a domain that is the (non-negative) $n \times k$-dimensional Euclidean vector space, while we consider (multidimensional) distributions of individual attributes that are represented by binary variables. Indeed, our attribute profiles amount to points of a $n \times k$-dimensional boolean hypercube $\mathbf{2}^{nk}$, hence we work in a Boolean-algebraic setting. As a result, our axioms are tailored to such a Boolean framework and are not a specialization of axioms used in the previous literature on multidimensional poverty. Similarly, our characterizations of the poverty rankings are consistent with our Boolean framework and are by no means specific examples of previous characterization of general class of multidimensional poverty indices in an Euclidean setting.

Finally, observe that our model may also be applied to produce poverty comparisons among societies with *different population size* as will be in fact done in the next section. Indeed, let $(X_i)_{i \in I}$ be a profile of attribute sets concerning the members of a certain community I of (finite) size $|I|$ that we want to compare to other communities of different sizes in terms of poverty. To start with, one should first rearrange the attribute sets of the given profile according to the attribute-gap induced by individual (partial) poverty preorder $\succcurlyeq_T^*$ as defined above. Then, one obtains a permuted profile $(X_{\pi(i)})_{i \in I}$ where $\pi : I \to \{1, ..., |I|\}$ is a bijective function such that $\pi(i) \geq \pi(j)$ whenever $X_i \succcurlyeq_T^* X_j$. The next step boils down to fixing the number n of relevant quantiles. It amounts to a normalization

[18] It can be easly checked that the HC-poverty ranking does not satisfy IBED and SMED, while the AG-poverty ranking does not satisfy IPAD, but it satisfies DEP.

[19] Namely, those induced by a *composite* threshold T (see Note 12 above).

of the population size which is dictated by some fine details of the statistics to be considered: if, as in the present paper (see below), percentages (of poor) with *two* decimals are considered then $n = 10^4$ so that a population unit is a fraction of community I's population comprising exactly $|I| \cdot 10^{-4}$ individuals, a *decimillile* of profile $(X_{\pi(i)})_{i \in I}$. Then, to any such decimillile we assign its *median attribute set* (or one of its median attribute sets if the median is not uniquely defined), to the effect of obtaining a new attribute profile of size $n = 10^4$ which can in turn be reshuffled by an arbitrary permutation $\tau : N \to N$ where $N = \{1, ..., n\}$. The resulting attribute profile $(X'_i)_{i \in N}$ is thus normalized precisely as required by our model and the empirical analysis discussed below.

4. Relation to the literature

The present model is related to the literature on *freedom of choice* (see Barberà, Bossert and Pattanaik (2004)). In particular, we remark here that attribute sets may in fact be equivalently regarded as subsets of a basic universal set of *non-exclusive opportunities* or as binary (or Boolean) vectors of a multidimensional binary (or Boolean) achievement space. Accordingly, an alternative interpretation of our model is in fact related to the problem of ranking different distributions of opportunity sets in terms of inequality (see Kranich (1996, 1997), Ok (1997), Herrero (1997), Herrero, Iturbe-Ormaetxe, and Nieto (1998), Ok and Kranich (1998), Arlegi and Nieto (1999), Bossert, Fleurbaey, and Van de gaer (1999) and Savaglio and Vannucci (2007)). A non negligible part of this literature considers points of 'opportunity spaces' (such as e.g. non-rivalrous rights and/or benefits) to be jointly available. We might use here that interpretation and replace the label 'attributes' with 'opportunities' that are mutually compatible objects. Now, the issue of *ranking different distributions of opportunity sets in terms of poverty* has never been addressed in that literature. Therefore, the present paper may also contribute to fill this gap.

Our paper is also related to the literature on the measurement of poverty (see, among others, Ebert and Moyes, (2002), Duclos and Makdissi, (2004)) and polarization (see, among others, Permanyer and D'Ambrosio (2013), and Wang and Tsui (2002)). In particular, it is definitively related to the literature on multidimensional poverty (see, among others, Chakravarty, Mukherjee and Ranade (1998), Bourguignon and Chakravarty (1998, 2002), Tsui (2002)), whose main aim consists in extending poverty analysis from unidimensional to multidimensional settings. In order to do that, it is usually assumed that the space of multiattribute-values amounts to a multidimensional Euclidean space. Furthermore, in this literature, the different aspects of deprivation of each individual are (generally) summarized by using a real-valued individual poverty function. As a consequence, individuals, who are characterized by several achievements besides income, are nonetheless (disputably) represented by a scalar and therefore totally ordered in terms of (multidimensional) individual poverty. But that kind of data massaging results in a massive and unnecessary information loss.

Our model departs from this very rich setting adopting a more parsimonious and possibly less controversial approach. Since individuals may be poor and non-poor in different dimensions to the effect of making highly ambiguous and disputable any attempt to rank them in terms of poverty, we argue that only certain partial rankings (e.g. dominance rankings) of the many aspects of personal deprivations can be safely assumed to be natural and non-controversial. That quite elementary consideration prompted us to rely on a single poverty (threshold induced) partial preorder of individual attributes, rather than on controversial if implicit total preorders of individual poverty.

In what follows, we briefly analyze the peculiarities of some relevant contributions on multidimensional deprivation published in the last decade, observing that all of these works propose more or less complete guidelines for multidimensional deprivation assessment protocols which *invariably differ from our own proposal in several significant respects.* Before starting, we warn the reader that we do not regard as particularly significant the choice between framing the model in an explicit *deprivation space* as derived from a primitive *attribute/achievement space,* or on the contrary sticking to the latter (as it is done in the present work). Therefore, that feature of the relevant models will be possibly mentioned but not commented upon in the ensuing review. Moreover, we distinguish between those models or *protocols* to assess multidimensional poverty of attribute profiles which are *reduced* in that single attributes are given in binary form, and those which are on the contrary *extensive* in that ordinal or cardinal attribute-data are processed into binary data by means of a system of thresholds.

Let us then start from contributions focusing on *reduced* protocols. To begin with, we should mention here Peragine, Savaglio and Vannucci (2008) which is in fact an early precursor of the present work, sharing with the latter the basic framework. It characterizes a larger set of poverty rankings, including lexicographic products and weighted products of the head-count and poverty-gap rankings and has no illustrative examples.

Dhongde, Li, Pattanaik and Xu (2016) also focuses on the case in which the only information available is whether an individual is deprived in an attribute or not, but proposes a class of multidimensional deprivation measures (as opposed to rankings). In particular, that work takes as input a finite population endowed with attributes that are binary, ordinally measurable and can be classified in terms of their relative importance for life-quality. The formal representation is a binary deprivation matrix whose rows denote the deprivation status of the population units with respect to each attribute. For every population unit, an admissible individual poverty assessment is just any *monotonic increasing function g of the weighted sum of her deprivations*, with arguments and values in the closed real unit interval. For any such individual poverty assessment g, the aggregation step is accomplished by computing the average poverty in the given population: thus, its output is a *multidimensional poverty index. The class of all such poverty indices, and some of its subclasses are characterized by a set of plausible axioms.* In particular, two remarkable subclasses of such poverty indices are characterized: the class which obtains when a *hierarchy* of *basic* and *non-basic* attributes is introduced, and the class of 'distribution-sensitive' indices whose values decrease when the poorer of two population units switches her deprived status in a certain attribute with the non-deprived status of the other unit in the same attribute, while remaining globally poorer than the latter. The first class of measures, distinguishing between basic and non-basic attributes, recalls the approach advanced in the present work whenever the threshold is given by a *unique* subset of *essential* attributes. It should be emphasized however that, even leaving aside some finer details, the output of our approach (a uniquely characterized ranking, as opposed to a class of indices) is in any case different. Dhongde, Li, Pattanaik and Xu (2016) indeed proposes a collection of multidimensional *deprivation indices*, whereas we study multidimensional *poverty orderings*. They work on a binary matrix space, we compare distributions of individual attribute-sets as endowed with a threshold that is a set of essential, non-substitute, equally-weighted items that any individual should have access in order to qualify as non-poor. Finally, we observe that some of the properties Dhongde et al.(2016) use to characterize their class of multidimensional deprivation indices do not adapt to the present setting. Concerning the latter point, in particular the Deprivation-Decreasing Switch that stipulates that the overall deprivation of a society decreases as a result of a switch in one missing attribute from a more deprived individual to a less deprived one does not hold in our framework, since such a switch could make both individuals poor because of the strong complementarities of the essential items in the selected threshold.

A most recent contribution by Aaberge, Peluso and Sligstad (2019) proposes an entirely different class of *reduced* multidimensional poverty assessment protocols. This work takes as input a finite population, a finite set of binary attributes, and provides an *entire class of multidimensional poverty indices* which apply to the set of all possible cumulative distribution functions of deprivations in the given population. Thus, no individual cross-dimensional poverty threshold is required by this approach. Arguably, it is implicitly assumed that a population unit is poor whenever it suffers deprivation with respect to at least one attribute. Under that assumption, the (implicit) poor identification function is in fact a crisp one. However, an alternative and perhaps more natural interpretation is that the underlying poor identification function is in fact a *fuzzy* one. Anyway, the resulting indices reflect and embody the utility function of a decision-maker, *under the behavioural assumption* that her preferences are continuous total preorders which satisfy the well-known *dual independence axiom.* It is shown that the induced total preorders on cumulative distribution functions of deprivations extend the Lorenz partial preorder, and that the subclasses of such indices induced by dually independent *convex* (respectively, *concave*) utility functions reflect priority of concern for the severity and the incidence of deprivation, respectively. Again, the present exercise is different: we characterize rankings as opposed to indices and we do make minimalistic assumptions (i.e. no class of preferences in the primitives of our model) on the information required to assess multidimensional poverty.

Let us now consider a few remarkable contributions dealing with *extensive* multidimensional poverty assessment protocols. Alkire and Foster (2011) takes as primary input a finite population, a finite list of *ordinal and/or cardinal* real-valued attributes, each one endowed with a deprivation threshold, and an attribute/achievement matrix for the given population. Then, guidelines for designing a large class of extensive multidimensional poverty assessment protocols (denoted as '*dual cutoff methodologies'*) are proposed and discussed. Every such 'dual cutoff methodology' requires the specification of a *poverty threshold* in the multiattribute space in order to enable the *poor identification* step: such a threshold is a *positive integer* k not larger than the number d of attributes, and denotes the minimum value of the (possibly weighted) sum of individual deprivations which are required in order to be classified as '*poor'*. Then, the final *aggregation step* consists in a real-valued *multidimensional poverty index* (or *measure*) which is defined for every multiattribute achievement matrix and is typically obtained by a suitably specified averaging of individual poverty and deprivation values.[20] While no single class of dual cutoff protocols is fully characterized in this work, some special classes are singled out and shown to satisfy a rich array of requirements which mimic and extend some well-known axioms from the literature on income inequality and poverty measurement.[21]

From all of the above, it transpires that Alkire and Foster's (2011) work is broadly speaking related to ours, but with some remarkable differences. Indeed, Alkire and Foster do not provide a characterization of the two multidimensional poverty criteria (i.e. the adjusted head-count ratio and the adjusted poverty gap), which are instead axiomatically characterized in the present work. Moreover, by fixing the minimum deprivation count (i.e. the poverty cutoff k) required to be considered poor, they treat the different dimensions in which an individual is deprived as *substitutes*. In our model, instead, all individual attributes that are considered as essential are equally important and they *complement* each other. As a possible interpretation, consider the case in which the value of the multidimensional poverty threshold has to reflect some policy goal or public decision: it can be regarded as, for instance, the result of the judgement aggregation of an expert committee that chooses by majority voting. In Alkire and Foster (2011), those experts are required to choose and fix just a number k, the minimum quantity of dimensions in which a person must be deprived in order to be considered poor: a very restrictive sort of assessment. On the contrary, in the case of our paper a committee has to choose a collection of individual essential attributes: a much more articulated job.

A refinement of Alkire-Foster's approach is offered by Datt (2019). Datt focuses on the case of *cardinal* real-valued attributes,[22] and notes that Alkire and Foster's dual cutoff poverty indices violate a natural counterpart of the Pigou-Dalton transfer axiom. Thus, he defines a generalized class of *distribution-sensitive multidimensional poverty indices* which satisfy the aforementioned

[20] It should be noticed that -somewhat disputably- such a proposal is also advanced for the case of ordinal data (including binary data).

[21] Pattanaik and Xu (2018) provide a critical evaluation of Alkire and Foster's (2011) approach to the measurement of multidimensional deprivation expressing some reservations about their methodology when individual dimensional deprivations are cardinally measurable and about their interpretation of certain measures of deprivation in terms of freedom/unfreedom.

[22] To be sure, no explicit distinction between cardinal and ordinal real-valued attributes is made by this Author, but it seems to us quite clear from context that cardinality of attributes is actually assumed in this work.

Pigou-Dalton transfer axiom, and provides an extended discussion of its properties including a cross-dimensional convexity axiom, and of the dimensional decomposition of such indices by means of the Shapley value. He does *not* provide a *characterization* of the *indices* he studies. However, a more substantial difference between his work and ours consists in the fact that the equality-enhancing operation of making transfers is a mathematically meaningless requirement in our Boolean setting and useless in order to characterize orderings. Indeed, since 1 and 0 denote having access or not to a given attribute, respectively, averaging the access among individuals (as in Datt's transfer axiom) has no theoretical or practical meaning within the present framework.

Permanyer (2019) takes as primary input a finite list of *cardinal* real-valued attributes (each one endowed with a deprivation threshold), and an attribute/achievement matrix for any given finite population. From that input, a ($[0,1]$-normalized) *deprivation gap matrix* is easily computed. Then, three distinct nested classes of *crisp* (*i.e. binary) identification functions* to single out the subpopulation of poor individuals are *characterized.*[23] The largest one is the class of *consistent identification functions* (i.e. essentially those which are *monotonic* with respect to the natural pointwise multi-deprivation partial order[24]). The intermediate class consists of those crisp identification functions which rely on a system of weights on attributes and select as poor those individuals whose weighted proportion of deprivations reach a certain positive real-valued threshold. The smallest class is the subclass of the intermediate one which results from a *uniform* system of weights on attributes.[25] A further refinement of identication functions is proposed in this work by considering an enriched input: a *partition* of attributes into *dimensions,* in order to capture at least some nontrivial, nuanced judgments of *complementarity* (between dimensions) and *substitutability* (within dimensions) of attributes. Accordingly, *two-stage* identification functions are defined by composing *weigthed counting* identification functions *within* dimensions and general *non-constant monotonic* identification functions *across* dimensions. Then, the *aggregation step* is accomplished to define *several families of multivariate poverty indices* mapping deprivation matrices into real numbers: any such family of indices is distinguished by its domain, which is given by the deprivation matrices induced by one of the classes of identification functions previously identified. Finally, each one of those *families of multivariate poverty indices* is characterized by a set of quite natural axioms: again, a quite different exercise from the one proposed in the present work.

On the contrary, Fattore (2015) takes as primary input a list of general non-binary *ordinal* attributes and builds up the resulting *basic achievement poset* as induced by the component-wise partial order defined on the multiattribute space. Moreover, a *supplementary input* is considered:

[23]Similarly, Pattanaik and Xu (2019) also studies $n \times m$ matrices of ordinal data and characterize social well-being measures that are functions $f : \mathcal{A} \to [0,1]$, where $\mathcal{A}$ is the set of such matrices. In particular, those functions f, in this exercise, are generalized means.

[24]To be sure, only non-constant monotonic identification functions are taken into consideration.

[25]Observe that the intermediate and the smallest classes of identification functions mentioned above amount to the weighted counting identification functions and the simple counting identification functions proposed by Alkire and Foster (2011) for the case of *cardinal* real-valued attributes.

it consists of a hierarchy among attributes, namely a *partial order* on the set of attributes reflecting judgments on their comparative relevance by the evaluating agency. No restrictions on that partial order or on subprotocols to generate it are suggested. Thus, arguably, the main content of Fattore (2016) is in fact a set of guidelines for an *entire family of protocols*. That is of course consistent with the overtly methodological intent of that valuable piece of work. Such an addition results in the *attribute/achievement poset*, a refinement of the basic achievement poset. Then, a (composite) *threshold consisting of the minimally-deprived positions* (i.e. the maximal points of the attribute/achievement poset which are deprived) is fixed.[26] The *identification step* amounts to computing the *degree of deprivation* of an individual attribute/achievement vector by counting the *proportion* of linearly ordered extensions of the attribute/achievement poset which classify such a vector as *inferior or equal* to *every* position of the deprivation threshold. Thus, the identification process results in a *fuzzy indicator.*[27] Furthermore, the *individual severity of deprivation* is defined by the *average distance*[28] of an individual attribute/achievement vector from the vector that covers the best vector of the deprivation threshold, according to the linearly ordered extensions of the attribute/achievement poset. Finally, the *aggregation step* amounts to averaging both degree of deprivation and severity of deprivation over the population: thus, aggregate deprivation and deprivation-severity *indices* are the main outputs of such a kind of protocol. A *fuzzy analogous* of the Head Count Ratio and the Poverty Gap of classical income poverty measures is then provided. We observe that Fattore's paper (2016) shares with the present work the basic idea that individual attribute profiles in a multidimensional distribution typically admit only dominance partial rankings as natural and non-controversial. However, Fattore does not axiomatically characterize his multidimensional poverty *indices* and presents a *fuzzy evaluation* procedure.[29]

Finally, Bossert, Chakravarty and D'Ambrosio (2013) considers as input a population of n individuals who differ in d material living conditions and represents it by a $n \times d$ deprivation matrix whose entries are 1 if the individual is poor with respect to one of the d dimensions and 0 otherwise. A *multidimensional index of poverty*, that is the weighted sum of the individual material deprivations, is then characterized. Once again, another unambiguously different exercise with respect to the present one. The following table summarizes our discussion/analysis and put in evidence the main difference and similarities between our approach and the some prominent papers on multidimensional poverty measurement.

[26]The resulting protocol is therefore *extensive*, as opposed to *reduced*.

[27]This is of course at variance with most of the relevant literature discussed here, and with the present work.

[28]Such a distance is given by the minimum path on the Hasse diagram (or covering diagram) of the attribute/achievement poset.

[29]On the contrary, our model is related both to the literature on deprivation of distributions of non-exclusive opportunity sets and to the issue of multidimensional poverty evaluation of binary vectors of a multidimensional binary (or Boolean) achievement space, but definitively not to the fuzzy theoretic literature on poverty measurement (see Lemmi and Betti (2016) for references).

TABLE 1. Summary table

AF (2011)	BCD (2013)	F (2015)	DLPX (2016)	APS (2019)	D (2019)	P (2019)	PSV (2008) and present work
E	R	E	R	R	R	E	R
OA	OA	OA	OA	OA	CA	CA	OA
CId	CId	FId	CId	FId	CId	CId	Cld
Ind	Ind	Ind	Ind	Ind	Ind	Ind	Ran
None	BC	None	BC	BC	None	BC	SC
-	BUC	-	BUC	BRC	-	BUC	BUC

Legend: *Columns*: Papers (Authors' Surnames Initials, Publication/Last Revision Date);
Rows: E/R Extended *versus* Restricted poverty assessment protocol;
OA/CA Ordinal *versus* Cardinal Attribute data;
CId/FId Crisp *versus* Fuzzy poor Identification;
Ran/Ind Ranking *versus* Index as an output of the protocol;
BC/SC Broad Characterization of a family of admissible outputs
versus Sharp Characterization of a uniquely defined output;
BUC/BRC Behaviourally Unrestricted Characterization
Behaviourally Restricted Characterization.

5. AN EMPIRICAL ILLUSTRATION

We now provide an empirical illustration of the Head-Count (HC) and the Attribute-Gap (HG) poverty rankings and compare the obtained results with those ones provided by using some of the most popular methodologies in studying poverty. To do that, we use data from the 2016 cross-sectional component of the *European Union Survey on Income and Living Conditions* (EU-SILC 2016 rev.1).[30] Individuals are identified as the units of our analysis and the results are corrected by sample weights in order to replicate the original populations in the EU countries. The threshold that we use is composed by income, education and health, attributes that we consider as complementary and essential to live a life worth living. Individuals are therefore identified as poor if they lack at least one of those attributes. We first observe that although there exists a tendency to polarization in high levels of education between the rich and the poor, there is a more stable pattern for upper secondary and post-secondary education in Europe. The centralized and egalitarian school systems in the European Union in fact reduce the cost of education for poor families, that can potentially access to the same level of education as the members of richer families. This evidence gives support for including education as an essential attribute distinct from -but possibly- complementary to income. In addition to education, health is also selected as the third essential attribute, in line with

[30]The cross-sectional component of EU-SILC is a collection of harmonized micro-data coming from comparable annual national surveys of socio-economic conditions of individuals and households in the EU countries. The survey contains information about individual and family characteristics (age, gender, education, working status), household and individual incomes and deprivation items to measure their difficulties to meet basic needs. Education level of the respondent and health status of each household member are also recorded. Cross-sectional sampling weights are assigned to each individual (household) in the sample.

the idea of identifying individuals deprived in terms of "essential opportunities".[31] Hence, we select a multidimensional poverty line that includes monetary income, education and health status, in the form of binary variables, as essential attributes.

The EU-SILC survey is quite rich in terms of universe of potential essential attributes X, going from standard of living (housing burden), economic insecurity, social connection, housing quality. For example, a social exclusion/non-monetary household deprivation indicator (ability to make ends meet) could be included in the set of essential attributes as a substitute of income. Other possible indicators refer to economic insecurity, like arrears in the payment of housing and non-housing bills as well as the repayment of other loans and credit, or inability to face unexpected financial expenses. Many indicators can be instead included the set of inessential attributes as, for instance, the inability to afford the payments for one-week annual holiday away from home, the inability to afford the participation in a leisure activity on regular basis, having a color TV, etc. Hence, the identification of the collection of essential attributes is a crucial issue.

In the present illustration, we rely on the foregoing list of items forming a multidimensional poverty threshold just to illustrate our methodology, namely:

$$T = \{\text{Income poverty}, \text{Low education}, \text{Poor health status}\}.$$

where: (i) the income poverty status is already a binary variable as officially defined in EU-SILC and it is equal to 1 if an individual is poor (in terms of monetary income) and 0 otherwise, with the cut-off point equal to 60% of the national median equalized disposable income of all persons; (ii) education is dichotomized by considering the level of education reached by the respondents (equal to 1 if respondents declare a lower-primary education and equal to 0 if they declare higher levels of education); (iii) finally, if respondents declare bad or very bad health we fix their health status variable equal to 1 and 0 if they declare fair, good or very good health. The selected indicators have been largely used in the literature (for a review of the most used indicators see Donge (2020)).

Multidimensional poverty is therefore estimated by computing the proportions of people in each country whose endowment fails to meet the multidimensional line T, that is the percentage of people suffering from poverty in at least one essential attribute within such multidimensional line Differently from the counting approach used by Eurostat that identifies as poor those individuals having a deprivation score above a fixed value,[32] here the occurrence of poverty in one essential attribute entails a condition of overall poverty: individuals are identified as poor when they are deprived in at least one essential attribute, in line with the "union criterion" as defined in Atkinson (2003). Based on that, the HD ranking for all the EU countries are estimated and shown in Table

[31]Any other possible choice of the threshold is potentially fine. The choice of the essential attributes by experts involves political issues, empirical measurability, data availability and other concerns. Such issues, however, are not addressed in the present paper.

[32]Eurostat considers deprived individuals who cannot afford at least three items out of nine, while individuals who cannot afford at least four items are defined severely deprived.

2, where rank 1 denotes the lowest level of poverty, rank 2 the second lowest level of deprivation and so on.

The "intensity of poverty" measured by the AG ranking, evaluates the impact of cumulating failures in more than one essential attribute. The AG poverty ranking aggregates information by counting the number of extra -'total amount of attributes' each population unit should be actually endowed with in order to achieve the minimum standard, and by summing them. Estimation of AG ranking requires counting the number of poor with respect to each essential attribute in T and adds those numbers across attributes:

$$\widehat{AG}_T = \frac{\sum_{i=1}^{n} S_i \cdot w_i}{size(T) \cdot N}$$

where $S_i = \{0, 1, 2, 3\}$ is the total number of essential attributes missing in individual i, with the obvious meaning, for instance, that $S_i = 2$ if individual i is deprived in two essential attributes, T is our multidimensional threshold, $size(T) = 3$ is the total number of essential attributes in T, w_i the sample weight and $N = \sum_{i=1}^{n} w_i$ the total number of individuals in the population. The resulting attribute-gap poverty ranking (see Table 2) is the empirical counterpart of the poverty ranking profile in Definition 2. The normalization with respect to $size(T) \cdot N$ is of course an isotonic transformation, that does not change the poverty ranking. Thus, we rank all the countries by estimating (i) the percentage of individuals who are poor with respect to T and (ii) how poor are the poor yielding the AG poverty ranking under the selected threshold T.

The results shown in Table 2 indicate that the most severe poverty is suffered by the Mediterranean countries (Portugal, Greece, Spain and Italy), while the Nordic countries show the best performance and an intermediate position is occupied by the Continental and Eastern European countries. This macro picture is basically confirmed by both the Head-Count and the Attribute-Gap. Driving factors beyond these differences could be both the functioning of the economies, hence the distribution of market incomes of individuals, and the differences in the systems of social protection, particularly relevant in determining the differential degree of access to health care and education. From the latter viewpoint, the ranking obtained could be partially explained by well-known differences between the European models of welfare states, with the Nordic countries characterized by generous and universalistic social protection nets and the Eastern and Mediterranean countries characterized, respectively, by more minimalistic and less efficient (and less universalistic) systems of protection.

In order to highlight the peculiarities of our methodology, we also compare our results with some obtained by applying some of the most widely used measures of poverty. Thus, we consider the traditional monetary income poverty, the Eurostat official measure of material deprivation (Eurostat, 2012)[33] and the multidimensional poverty index proposed by Alkire and Foster (2011), based on

[33]Individuals are considered materially deprived if they live in households that cannot afford at least three items on a list of nine. The nine-item list is fixed by Eurostat for all the EU countries and all the items have the same relevance.

Table 2. Head-count (HC) poverty ranking of attribute profiles under threshold T (monetary income, education and health) along with Attribute-gap (AG) poverty ranking of attribute profiles under the same threshold T, year 2015. Values are weighted and in percentages.

Country	**Head-Count poverty ranking**	**Country**	**Attribute-Gap poverty ranking**
Austria	20.12	Austria	7.53
Slovakia	21.56	Slovakia	7.72
Finland	21.99	Finland	7.89
UK	23.28	UK	8.51
Germany	23.39	Netherlands	8.86
Netherlands	23.65	Germany	9.15
Belgium	26.28	Slovenia	10.70
France	26.87	France	10.89
Slovenia	27.91	Ireland	11.12
Ireland	28.03	Belgium	11.24
Cyprus	30.88	Cyprus	13.13
Latvia	33.06	Latvia	13.72
Estonia	33.72	Estonia	14.28
Luxembourg	35.85	Luxembourg	15.43
Lithuania	36.85	Lithuania	15.97
Italy	37.92	Italy	16.57
Spain	42.32	Spain	17.90
Greece	43.48	Greece	19.17
Portugal	54.67	Portugal	27.11

the same dimensions as our approach, with threshold fixed at $k = 2$ and with an equal-weight assignment. An individual is identified as multidimensional poor if he/she simultaneously experienced deprivation in two or more of the three indicators. As for HC and AG, individuals in the sample are associated with sampling weights. Sampling weights permit inferences from individuals in the sample to the entire population from which they were drawn, allowing for a better comparison of the ranks. The corresponding rankings are shown in Table 3.

As expected the rankings are quite different, particularly if we compare our multidimensional approach to the monetary income poverty and the measure of material deprivation (see Table 2) with the divergence essentially driven by the inclusion of the access to basic services as education and health care in our analysis.

TABLE 3. Poverty ranking based on monetary income, on material deprivation as defined by Eurostat and on Alkire and Foster (2011) multidimensional measure (monetary income, education and health) with $k = 2$, year 2015. Values are weighted and in percentages.

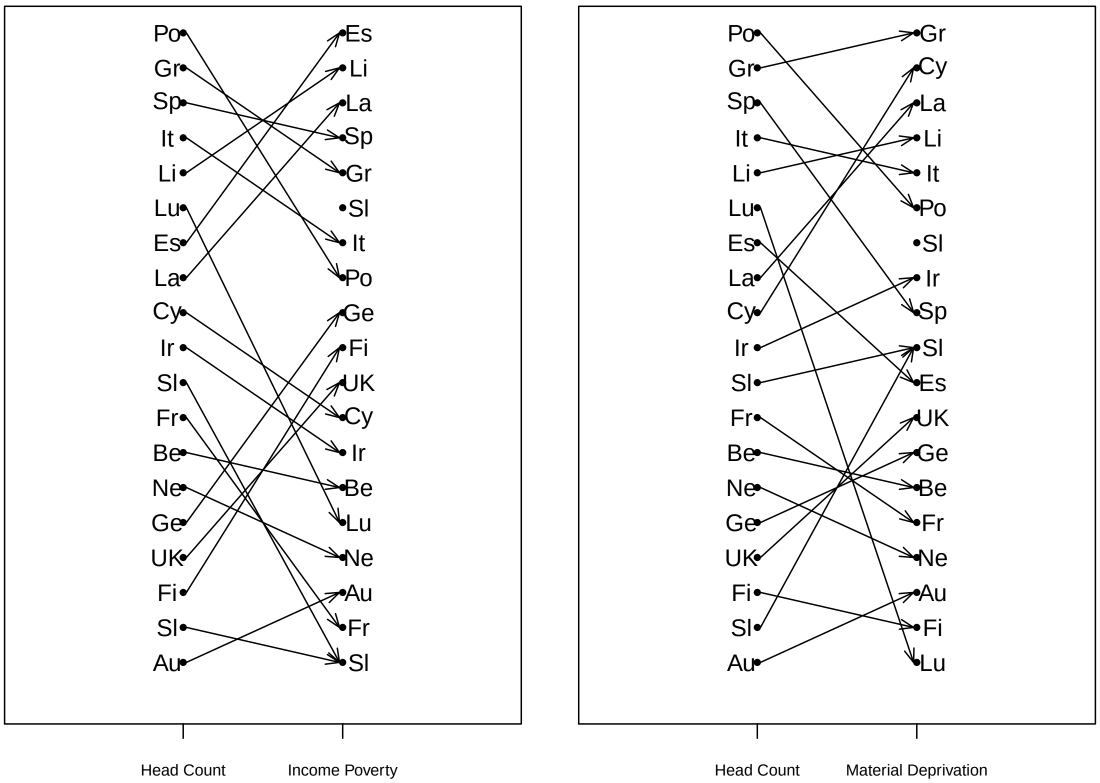
Country	**Monetary Income Poverty**	**Country**	**Material Deprivation**	**Country**	**Alkire Foster**
Slovakia	10.62	Luxembourg	4.56	Slovakia	1.58
France	11.67	Finland	7.71	Finland	1.68
Austria	12.80	Austria	7.73	UK	2.26
Netherlands	13.10	Netherlands	8.30	Austria	2.34
Luxembourg	13.97	France	10.33	Netherlands	2.74
Belgium	14.28	Belgium	10.63	Germany	3.81
Ireland	15.94	Germany	11.10	Slovenia	4.20
Cyprus	16.05	UK	12.46	Ireland	5.00
UK	16.70	Estonia	12.78	France	5.29
Finland	16.95	Slovenia	15.34	Belgium	6.45
Germany	17.03	Spain	15.78	Latvia	7.45
Portugal	18.59	Ireland	18.15	Cyprus	7.68
Italy	18.64	Slovakia	19.93	Estonia	8.50
Slovenia	18.64	Portugal	21.46	Luxembourg	9.18
Greece	20.24	Italy	22.25	Lithuania	9.68
Spain	20.90	Lithuania	27.64	Italy	10.38
Latvia	22.52	Latvia	29.98	Spain	10.49
Lithuania	23.82	Cyprus	33.74	Greece	12.53
Estonia	24.76	Greece	39.90	Portugal	22.64

Some differences (although much less evident) emerges also when the comparison is made with Alkire-Foster's index of multidimensional poverty (see Figure 2 and Table 3).

Now, to better understand those differences, we first estimated the percentage of individuals missing each single item in material deprivation[34] as well as the percentages of individuals being poor in each of the three attributes considered in our methodology and in Alkire Foster's multidimensional poverty index.

[34]Namely, ability to keep the house warm, one week of holidays, ability to afford a meal with meat, chicken and fish or a protein equivalent every second day, ability to face unexpected expenses, having a telephone, having a color TV, having a washing machine, having a car, ability to avoid arrears on mortgage, rent, utility bills or loans.

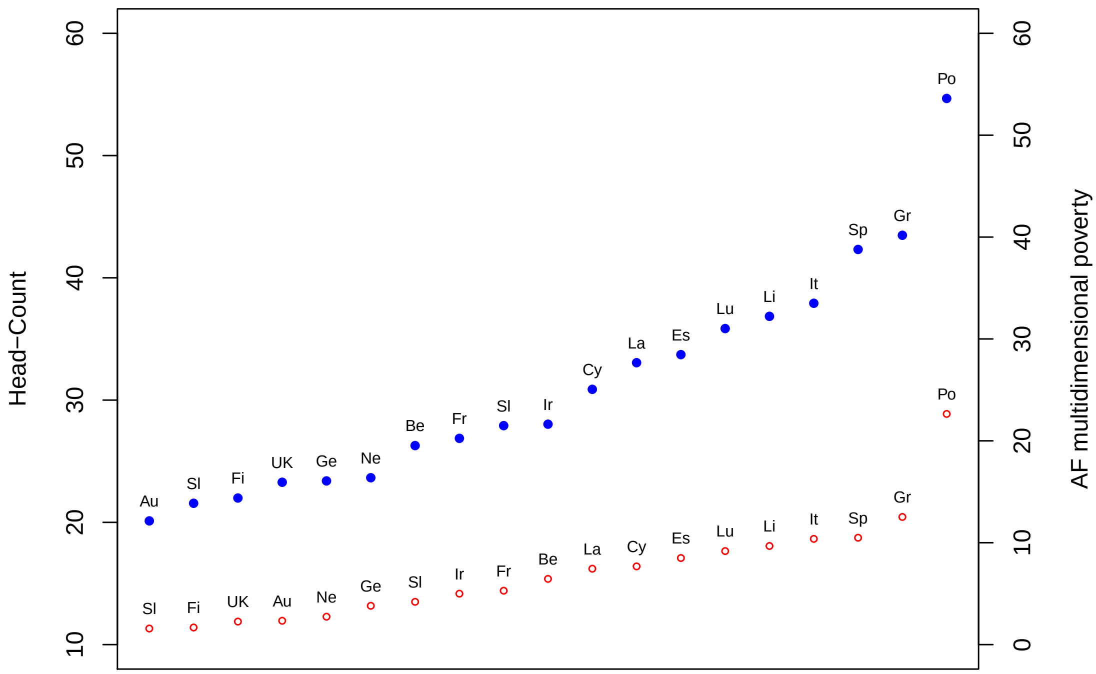
FIGURE 1. Multidimensional poverty Head-count against income poverty (left panel) and against material deprivation (right panel). Countries are ranked lowest to highest poverty scores.



The results in Table 4 give a more complete picture of what we have seen so far. Portugal, Greece, Spain and Luxembourg present the highest percentage of population with low education.[35] That explains, at least in part, the position of Luxembourg in HC and AF ranking. Quite different is the situation in the former socialist countries, like Slovenia, Slovakia, Latvia etc., in which almost all the residents completed at least lower secondary education. Since the disparity within countries in income poverty and health is not so severe, education level makes the difference in the rankings as it can be observe in Figure 1 and Figure 2. On the other hands, if we consider material deprivation, Luxembourg is the country with the lowest percentage of deprived population. In fact the percentages of population missing the deprivation items are very low, with the exception of the 'facing unexpected expenses' dimension. However, since Eurostat adopts the item-counting approach with equal weighting, facing unexpected expenses is treated as any other item (i.e. not having a car or a

[35]One of the recent challenges in Luxembourg is to increase completion rates in upper secondary education (OECD, 2016).

FIGURE 2. Multidimensional poverty head count (in blue) and Alkire Foster multidimensional poverty index (in red). Countries are ranked lowest to highest poverty scores.



week of holiday) and therefore an individual is considered poor if he/she misses at least three items out of nine, not matter which items are.

If we consider as another example Italy, the position in the HC and AF ranking is quite different (see Table 2 and Table 3). The change in the position here is due to the different methodology adopted in the two multidimensional poverty measures. Italy has a quite significant percentage of population with low education (18.68%). Now, the HC multidimensional index considers each attribute as essential and not as substitute. Therefore, if an individual is poor in education is definitively poor and Italy reaches an higher position in the HC poverty ranking with respect to the AF rank. Germany instead is a country characterized by a significant percentage of individuals below the income poverty line (17.03%), but the percentages of German people with low education and bad health are quite low (seeTable 4). This explains the difference in the ranking when we use our methodology with respect to the AF method. Similarly, in terms of missing items when we estimate the material deprivation: there are consistent percentages of Germans who cannot afford a week of holidays and cannot face unexpected expenses. Both these items are related to income, but since in the Eurostat approach all the items are equally treated and individuals are poor when they miss three out of nine items, Germany has a very different position with respect to the HC rank.

Table 4. Percentages of population being poor in each attribute considered in HC and AF multidimensional poverty indeces and percentages of population missing the deprivation items. All the values are weighted.

Country	Income poverty	bad health	low education	house warm	one week holidays	afford a meal	unexpected expenses	telephone	color TV	washing machine	car	arrears
Austria	12.80	8.79	0.99	2.61	16.59	6.96	21.19	0.11	0.61	0.31	5.67	5.81
Belgium	14.28	9.48	9.98	4.87	25.82	4.99	23.89	0.14	0.63	1.55	6.35	5.94
Cyprus	16.05	5.31	18.04	28.20	53.20	3.55	59.81	0.00	0.48	0.29	2.28	29.55
Germany	17.03	7.78	2.64	4.72	20.55	7.76	31.49	0.26	0.28	0.54	7.13	5.01
Estonia	24.76	15.68	2.39	2.19	32.26	5.26	36.39	0.27	0.21	1.33	12.59	8.55
Greece	20.24	10.37	26.90	29.21	53.35	12.04	53.18	0.66	0.44	1.51	10.56	47.41
Spain	20.90	7.44	25.34	10.51	41.61	2.55	39.12	0.25	0.12	0.21	5.56	10.76
Finland	16.95	6.72	0.00	1.70	12.88	3.09	27.38	0.00	0.36	0.64	9.45	9.18
France	11.67	7.94	13.06	5.60	23.16	7.04	30.08	0.09	0.15	0.45	2.41	7.88
Ireland	15.94	3.72	13.69	8.51	40.78	2.67	47.99	0.36	0.29	0.40	7.30	14.04
Italy	18.64	12.37	18.68	17.16	47.33	11.88	39.32	0.17	0.27	0.25	2.12	13.98
Lithuania	23.82	17.62	6.46	32.57	44.54	15.20	53.03	0.90	0.26	2.47	12.93	8.83
Luxembourg	13.97	8.63	23.68	0.93	12.85	2.09	22.22	0.04	0.26	0.24	2.10	4.61
Latvia	22.52	16.27	2.39	14.61	42.52	16.89	60.66	0.39	0.85	4.17	21.79	16.72
Netherlands	13.10	6.03	7.45	2.90	16.96	2.32	22.67	0.00	0.23	0.78	7.66	5.38
Portugal	18.59	17.93	44.80	24.47	52.04	3.46	40.31	0.93	0.32	1.37	8.14	9.24
Slovenia	18.64	13.46	0.00	5.84	30.37	6.89	43.31	0.20	0.34	0.28	3.84	18.41
Slovakia	10.62	12.24	0.29	5.58	46.74	20.07	35.66	0.77	0.30	0.76	13.04	7.03
UK	16.70	8.78	0.05	7.36	26.11	6.29	35.11	0.20	0.27	0.47	7.55	8.40

Our empirical illustration gives support to the intuition that European Countries are experiencing new and different forms of poverty, that are not necessarily based only on income or in lacking material needs.

Complementing traditional measures of income poverty and deprivation with a more general measure is therefore worth pursuing since it in fact substantially improves our understanding of societies and helps orienting strategies and policies towards those particular dimensions with high percentages of poor individuals.

6. Final remarks

The need for complementing the traditional evaluation of income poverty by a full-fledged analysis of the deprivation suffered in many dimensions of individual and social life has been forcefully defended by many scholars in the last decades. Such an extension of the scope of poverty measurement may substantially improve our understanding of poverty in any given population and may well have far-reaching policy implications. To keep the analysis as general as possible, in this paper the different dimensions have been treated in an abstract way: we have defined an attribute set as any finite subset in some arbitrary attribute space and we have attempted to outline an axiomatic theory for the measurement of poverty in a binary multidimensional framework. To the best of our knowledge, there have been no previous attempts to characterize poverty rankings of attribute profiles within that setting.

We have characterized two fundamental rankings, the Head-Count and the Attribute-Gap, which generalize to a multidimensional environment two of the best known and most widely used poverty indices, namely the *head count ratio* and *the income poverty gap.*

We also provide an empirical application of the theoretical approach developed in the paper by using the 2016 cross-sectional component of the *European Union Survey on Income and Living Conditions* (EU-SILC 2016): our results show that a more articulated picture about poverty emerges from the data than the one provided by the traditional poverty measurement. That suggests the necessity of a further improvement of our poverty assessment protocols in order to devise effective policies oriented to contrast individual deprivations.

We are of course aware of the critique of the head-count and poverty-gap measures, as formulated by Sen (1976) within the income poverty framework, and based on their inability to take into account the inequality among the poor. That critique has led to the characterization of richer families of income poverty indices (see Clark, Hemming and Ulph (1981) and Foster, Greer and Thorbecke (1984)). It would be interesting to study such an extension in our setting. Moreover, we would like to show that the approach proposed in this paper admits an extension to the case in which all attributes are treated as non-binary ordinal variables.

Finally, we have only considered comparisons of attribute profiles for a fixed population. A possible extension of our analysis would be to compare the attribute profiles with different numbers of individuals. This would make it easier to rank attribute profiles for different countries, different demographic groups, and for different time periods.[36]

7. Appendix: Proofs

Proof of Proposition 1. It is straightforward to check that $\succcurlyeq_T^h$ is a poverty ranking and does indeed satisfy AN, IIA, DEP and IPAD. Conversely, suppose $\succcurlyeq_T$ is a poverty ranking that satisfies AN, NT, IIA, and IPAD. Now, consider $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$ such that $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$. Then, by repeated application of IIA and transitivity, $\mathbf{Y}_{|T} \succcurlyeq_T \mathbf{Z}_{|T}$. Next, observe that $(T^{N \setminus H_T(\mathbf{Y})}, \emptyset^{H_T(\mathbf{Y})}) \sim_T \mathbf{Y}_{|T} \succcurlyeq_T \mathbf{Z}_{|T} \sim_T (T^{N \setminus H_T(\mathbf{Z})}, \emptyset^{H_T(\mathbf{Z})})$, by repeated application of IPAD. Let us now suppose that $h_T(\mathbf{Z}) > h_T(\mathbf{Y})$: then, by AN and DEP, $\mathbf{Z} \succ_T \mathbf{Y}$, a contradiction. Hence, $h_T(\mathbf{Y}) \geq h_T(\mathbf{Z})$, i.e. $\mathbf{Y} \succcurlyeq_T^h \mathbf{Z}$.

To prove the reverse inclusion, suppose that $\mathbf{Y} \succcurlyeq_T^h \mathbf{Z}$, i.e. $h_T(\mathbf{Y}) \geq h_T(\mathbf{Z})$. Then, consider $(T^{N \setminus H_T(\mathbf{Y})}, \emptyset^{H_T(\mathbf{Y})})$, $(T^{N \setminus H_T(\mathbf{Z})}, \emptyset^{H_T(\mathbf{Z})})$ and a permutation π of N such that $\pi(H_T(\mathbf{Z})) \subseteq \pi(H_T(\mathbf{Y}))$. By IIA, $\mathbf{Y} \sim_T (T^{N \setminus H_T(\mathbf{Y})}, \emptyset^{H_T(\mathbf{Y})})$ and $\mathbf{Z} \sim_T (T^{N \setminus H_T(\mathbf{Z})}, \emptyset^{H_T(\mathbf{Z})})$. By AN,

$$(T^{N \setminus H_T(\mathbf{Y})}, \emptyset^{H_T(\mathbf{Y})}) \sim_T (T^{\pi(N \setminus H_T(\mathbf{Y}))}, \emptyset^{\pi(H_T(\mathbf{Y}))}) \text{ and } (T^{N \setminus H_T(\mathbf{Z})}, \emptyset^{H_T(\mathbf{Z})}) \sim_T (T^{\pi(N \setminus H_T(\mathbf{Z}))}, \emptyset^{\pi(H_T(\mathbf{Z}))}).$$

[36]See, however, the last paragraph of Section 3 on this point.

Clearly, if $\pi(H_T(\mathbf{Z})) = \pi(H_T(\mathbf{Y}))$, then

$$(T^{\pi(N\setminus H_T(\mathbf{Y}))}, \emptyset^{\pi(H_T(\mathbf{Y}))}) = (T^{\pi(N\setminus H_T(\mathbf{Z}))}, \emptyset^{\pi(H_T(\mathbf{Z}))}),$$

hence, by transitivity of $\succcurlyeq_T$, $\mathbf{Y} \sim_T \mathbf{Z}$. Let us then suppose that $\pi(H_T(\mathbf{Z})) \subset \pi(H_T(\mathbf{Y}))$. By DEP, it follows that:

$$(T^{\pi(N\setminus H_T(\mathbf{Y}))}, \emptyset^{\pi(H_T(\mathbf{Y}))}) \succ_T (T^{\pi(N\setminus H_T(\mathbf{Z}))}, \emptyset^{\pi(H_T(\mathbf{Z}))}),$$

hence, in particular, $\mathbf{Y} \succ_T \mathbf{Z}$.

Morever, the characterization provided is tight. To check the validity of this claim, consider the following examples.

i) To begin with, consider the non-anonymous refinement of HG defined by the following rule: $\mathbf{Y} \succcurlyeq_T^{h_1} \mathbf{Z}$ if and only if:

a) $\mathbf{Y} \succcurlyeq_T^{h} \mathbf{Z}$ and $\{Y_i, Z_i\} \subseteq \{T, \emptyset\}$ for each $i \in N$ or

b) $\mathbf{Y} \succ_T^{h} \mathbf{Z}$ or

c) $\mathbf{Y} \sim_T^{h} \mathbf{Z}$, there exist $i, j \in N$ such $\{Y_i, Z_j\} \cap \{T, \emptyset\} = \emptyset$, and $Y_1 \not\supseteq T$.

Clearly, $\succcurlyeq_T^{h_1}$ is a poverty ranking that satisfies IIA, IIAP and DEP, but violates AN.

ii) Consider the refinement of HC defined by the following rule: $\mathbf{Y} \succcurlyeq_T^{h^*} \mathbf{Z}$ if and only if $\mathbf{Y} \succ_T^{h} \mathbf{Z}$ or $\mathbf{Y} \sim_T^{h} \mathbf{Z}$ and $\#\{i \in N : Y_i \supset T\} \leq \#\{i \in N : Z_i \supset T\}$. Such a preorder is a poverty ranking that satisfies AN, DEP and IIAP but violates IIA.

iii) Consider the universal indifference poverty ranking: i.e. $\mathbf{Y} \succcurlyeq^{I} \mathbf{Z}$ for any $\mathbf{Y}$,$\mathbf{Z} \in \mathcal{P}[X]^N$. That ranking does satisfy AN, IIA and IIAP but violates DEP.

iv) Consider the AG-refinement of HC as defined by the following rule: $\mathbf{Y} \succcurlyeq_T^{h_g} \mathbf{Z}$ if and only if either $\mathbf{Y} \succ_T^{h} \mathbf{Z}$ or ($\mathbf{Y} \sim_T^{h} \mathbf{Z}$ and $g_T(\mathbf{Y}) \geq g_T(\mathbf{Z})$). Such a preorder is a poverty ranking that satisfies AN, IIA and DEP, but fails to satisfy IIAP.

Proof of Proposition 2. It is easily checked that $\succcurlyeq_T^{g}$ is a poverty ranking and does satisfy AN, IIA, SMED and IBED. Conversely, suppose $\succcurlyeq_T$ is a poverty ranking that satisfies AN, IIA, SMED and IBED. Then, consider $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$ such that $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$. Again, by repeated application of IIA and transitivity, $\mathbf{Y}_{|T} \succcurlyeq_T \mathbf{Z}_{|T}$. Now, suppose that $g_T(\mathbf{Z}) > g_T(\mathbf{Y})$. Then, by repeated application of IBED, $\mathbf{Z}'_{|T} \sim_T \mathbf{Y}_{|T}$ for some $\mathbf{Z}'$ such that $Z'_i \subseteq Z_i$ for each $i \in N$, and $g_T(\mathbf{Z}') = g_T(\mathbf{Y})$. It follows that, by repeated application of SMED, $\mathbf{Z}_{|T} \succ_T \mathbf{Z}'_{|T}$, hence by transitivity, $\mathbf{Z}_{|T} \succ_T \mathbf{Y}_{|T}$. Thus, by repeated application of IIA and transitivity again, $\mathbf{Z} \succ_T \mathbf{Y}$, a contradiction. On the other hand, suppose that $\mathbf{Y} \succcurlyeq_T^{g} \mathbf{Z}$, i.e. $g_T(\mathbf{Y}) \geq g_T(\mathbf{Z})$, and consider $\mathbf{T} = (T, ..., T) \in \mathcal{P}[X]^N$. Of course, $\mathbf{T} \sim_T \mathbf{T}$, by reflexivity. Then, by AN and repeated application of IBED to $\mathbf{T} \sim_T \mathbf{T}$, it follows that $\mathbf{Y}' \succcurlyeq_T \mathbf{Z}$ for some $\mathbf{Y}'$ such that $Y'_i \setminus T = Y_i \setminus T$ and $Y_i \subseteq Y'_i$ for each $i \in N$, and $g_T(\mathbf{Y}') = g_T(\mathbf{Z})$. If, in particular, $g_T(\mathbf{Y}') = g_T(\mathbf{Y})$ then $\mathbf{Y}' = \mathbf{Y}$, hence $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$, and we are done. Otherwise, there exist $i \in N$ and $x \in T \cap (Y'_i \setminus Y_i)$, hence $\mathbf{Y} \succ_T \mathbf{Z}$ by transitivity and repeated application of SMED. In any case, $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$ as required. Then, we show that the characterization provided is tight.

To verify this claim consider the following examples.

i) Take the following non-anonymous refinement of the AG poverty ranking: $\mathbf{Y} \succcurlyeq_T^{g_1} \mathbf{Z}$ if and only if $\mathbf{Y} \succ_T^g \mathbf{Z}$ or ($\mathbf{Y} \sim_T^g \mathbf{Z}$, $Y_1 \not\supseteq T$ and $Z_1 \cap T \supseteq Y_1 \cap T$). That ranking satisfies IIA, SMED and IBEA but fails to satisfy AN.

ii) Consider the following refinement of the AG poverty ranking: $\mathbf{Y} \succcurlyeq_T^{g^*} \mathbf{Z}$ if and only if $\mathbf{Y} \succ_T^g \mathbf{Z}$ or ($\mathbf{Y} \sim_T^g \mathbf{Z}$ and $\sum_{i \in N} \#(Y_i \smallsetminus T) \leq \sum_{i \in N} \#(Z_i \smallsetminus T)$). That ranking satisfies AN, SMED and IBEA but fails to satisfy IIA.

iii) Consider again the universal indifference ranking: i.e. $\mathbf{Y} \succcurlyeq^I \mathbf{Z}$ for any $\mathbf{Y}$,$\mathbf{Z} \in \mathcal{P}[X]^N$. That preorder is a poverty ranking which does satisfy AN, IIA and IBEA but violates SMED.

iv) Consider the HC-refinement of the AG poverty ranking: $\mathbf{Y} \succcurlyeq_T^{g_h} \mathbf{Z}$ if and only if $\mathbf{Y} \succ_T^g \mathbf{Z}$ or ($\mathbf{Y} \sim_T^g \mathbf{Z}$ and $h_T(\mathbf{Y}) \geq h_T(\mathbf{Z})$). That poverty ranking satisfies AN, IIA, SMED but violates IBEA.

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