Multiplicity and concentration results for local and fractional NLS equations with critical growth

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Abstract

Goal of this paper is to study the following singularly perturbed nonlinear Schrödinger equation

$$\varepsilon^{2s}(-\Delta)^s v + V(x)v = f(v), \quad x \in \mathbb{R}^N,$$

where $s \in (0,1)$, $N \geq 2$, $V \in C(\mathbb{R}^N, \mathbb{R})$ is a positive potential and f is assumed critical and satisfying general Berestycki-Lions type conditions. When $\varepsilon > 0$ is small, we obtain existence and multiplicity of semiclassical solutions, relating the number of solutions to the cup-length of a set of local minima of V; in particular we improve the result in [37]. Furthermore, these solutions are proved to concentrate in the potential well, exhibiting a polynomial decay. Finally, we prove the previous results also in the limiting local setting $s = 1$ and $N \geq 3$, with an exponential decay of the solutions.

Keywords: Nonlinear Schrödinger equation, Fractional Laplacian, Critical exponent, Singular perturbation, Spike solutions, Cup-length

MSC: 35A15, 35B25, 35B33, 35Q55, 35R11, 47J30, 58E05

Contents

1 Introduction

Following Feynman's path integral approach to quantum mechanics, Laskin [42] developed a new extension of the fractality concept deriving the fractional nonlinear Schrödinger (fNLS for short) equation

$$i\hbar\partial_t\psi = \hbar^{2s}(-\Delta)^s\psi + V(x)\psi - f(\psi), \quad (x,t) \in \mathbb{R}^N \times (0,+\infty). \tag{1.1}$$

Here $s \in (0,1)$, $N > 2s$, the symbol $(-\Delta)^s\psi = \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}(\psi))$ denotes the fractional Laplacian defined via Fourier transform $\mathcal{F}$ on the spatial variable, $\hbar$ designates the usual Planck constant, V is a real potential and f is a Gauge invariant nonlinearity, i.e. $f(e^{i\theta}\rho) = e^{i\theta}f(\rho)$ for any ρ, $\theta \in \mathbb{R}$. The wave function $\psi = \psi(x,t)$ represents the quantum mechanical probability amplitude for a given unit mass particle to have position x at time t, under the confinement due to the potential V. We refer to [42, 43] for a detailed discussion on the physical motivation of the fNLS equation, and we highlight that several applications in the physical sciences could be mentioned, ranging from the description of boson stars to water wave dynamics, from image reconstruction to jump processes in finance (see [27] and references therein).

Special solutions of the equation (1.1) are given by the standing waves, i.e. factorized functions $\psi(x,t) = e^{\frac{i\mu t}{\hbar}}v(x)$ with $\mu \in \mathbb{R}$. Regarding $\hbar > 0$ as a small quantity, these standing waves are usually called *semiclassical* states since the transition from quantum physics to classical physics is somehow described letting $\hbar \to 0$: roughly speaking, when $s = 1$ the centers of mass $q_\varepsilon = q_\varepsilon(t)$ of the soliton solutions in (1.1), under suitable assumptions and initial conditions, converge as $\hbar \to 0$ to the solution of the Newton's equation of motion

$$\ddot{q}(t) = -\nabla V(q(t)), \quad t \in (0,+\infty); \tag{1.2}$$

for $s \in (0,1)$ a suitable power-type modification of equation (1.2) is needed. Here, considering small $\hbar$ roughly means that the size of the support of the soliton in (1.1) is considerably smaller than the size of the potential V; for details we refer to [11, 34, 41, 9], and to [51] for the fractional case.

Without loss of generality, shifting μ to 0 and denoting $\hbar \equiv \varepsilon$, the search for semiclassical states leads to the investigation of the following nonlocal equation

$$\varepsilon^{2s}(-\Delta)^s v + V(x)v = f(v), \quad x \in \mathbb{R}^N \tag{1.3}$$

where V is positive and $\varepsilon > 0$ is small. Setting $u := v(\varepsilon\cdot)$, we observe that (1.3) can be rewritten as

$$(-\Delta)^s u + V(\varepsilon x)u = f(u), \quad x \in \mathbb{R}^N, \tag{1.4}$$

thus the equation

$$(-\Delta)^s U + m_0 U = f(U), \quad x \in \mathbb{R}^N \tag{1.5}$$

becomes a formal limiting equation, as $\varepsilon \to 0$, of (1.4). Solutions of (1.3) usually exhibit concentration phenomena as $\varepsilon \to 0$: by *concentrating solutions* we mean a family v_ε of solutions of (1.3) which converges, up to rescaling, to a ground state of (1.5) and whose maximum points converge to some point $x_0 \in \mathbb{R}^N$ given by the topology of V (see Theorem 1.1 for a precise statement). This point x_0 reveals, generally, to be a critical point of V - i.e. an equilibrium of (1.2) - as shown in [57, 30].

In the *subcritical* case, that is when the growth at infinity of the function f is strictly slower than $|t|^{2_s^*}$, with

$$2_s^* := \frac{2N}{N-2s}$$

fractional Sobolev critical exponent, the semiclassical analysis of local NLS equations has been largely investigated starting from the seminal papers [32, 47]: here the authors implement a Lyapunov-Schmidt dimensional reduction argument to gain existence of solutions for homogeneous sources, relying on the nondegeneracy of the ground states of the limiting problem (1.5). Successively, variational techniques have been implemented to gain both existence and multiplicity, see [49, 57, 3, 26, 21, 4, 12, 14, 20] and references therein. As regards the fractional subcritical case, we confine to mention [25, 30, 2, 31, 52, 18, 5, 19] and references therein.

In the present paper we aim to study (1.3) in a *critical* setting, when $N \geq 2$. Namely, we focus on possibly degenerate local minima of V, that is V satisfies

(V1) $V \in C(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N)$, $\underline{V} := \inf_{\mathbb{R}^N} V > 0$,

(V2) there exists a bounded domain $\Omega \subset \mathbb{R}^N$ such that

$$m_0 := \inf_{\Omega} V < \inf_{\partial\Omega} V,$$

with set of local minima

$$K := \{x \in \Omega \mid V(x) = m_0\}, \tag{1.6}$$

and we assume general Berestycki-Lions assumptions on f, i.e.

(f1) $f \in C(\mathbb{R}, \mathbb{R})$, and $f \in C^{0,\gamma}_{loc}(\mathbb{R}, \mathbb{R})$ for some $\gamma \in (1-2s, 1)$ if $s \in (0, 1/2]$,

(f2) $f(t) \equiv 0$ for $t \leq 0$,

(f3) $\lim_{t\to 0} \frac{f(t)}{t} = 0$,

(f4) $\lim_{t\to+\infty} \frac{f(t)}{t^{2_s^*-1}} = a > 0$, where $2_s^* = \frac{2N}{N-2s}$,

(f5) for some $C > 0$ and $\max\{2_s^* - 2, 2\} < p < 2_s^*$, i.e. satisfying

$$p \in \begin{cases} \left(\dfrac{4s}{N-2s}, \dfrac{2N}{N-2s}\right) & N \in (2s, 4s), \\ \left(2, \dfrac{2N}{N-2s}\right) & N \geq 4s, \end{cases} \tag{1.7}$$

it results that

$$f(t) \geq a t^{2_s^*-1} + C t^{p-1} \quad \text{for } t \geq 0.$$

See also Remark 1.3 for some weakening and comments on the assumptions (V1), (f1) and (f5). Notice that the stronger condition on p in the first line of (1.7) is verified, whenever $N \geq 2$, only if $N = 2$ and $s \in (\frac{1}{2}, 1]$, or $N = 3$ and $s \in (\frac{3}{4}, 1]$. We point out that the condition $C > 0$ in (f4) is of key importance: indeed, for pure critical nonlinearities of the type

$$f(t) = |t|^{2_s^*-2} t,$$

the limiting problem (1.5) exhibits a quite different scenario.

Some physical models related to assumptions (f1)–(f5) arise, for example, in nonlinear optics [44]. See also [58, 50, 29].

The existence of a solution in a critical, fractional setting, in the case of local minima (V1)-(V2) and general Berestycki-Lions assumptions (f2)–(f5), has been faced in [40] by assuming $V \in C^1(\mathbb{R}^N)$, and moreover in [36] by means of penalization methods.

Inspired by [49], multiplicity of solutions of (1.3) in the case of global minima of V was studied in [54] for power-type nonlinearities. Moreover, in [45] the authors consider functions of the type

$$f(t) = g(t) + |t|^{2_s^*-2}t, \tag{1.8}$$

where g is subcritical and satisfies a monotonicity condition which allows to implement the Nehari manifold tool, and they relate the number of solutions to the Lusternik-Schnirelmann category of the set of global minima.

Existence of multiple solutions for local minima of V has been investigated, in the spirit of [26], by [37] with sources of the type (1.8), where now g satisfies also an Ambrosetti-Rabinowitz condition: this assumption enables to employ Mountain Pass and Palais-Smale arguments, combined with a penalization scheme. Again, the authors are able to find $\mathrm{cat}(K)$ solutions, where K is the set of local minima of V and $\mathrm{cat}(K)$ denotes its Lusternik-Schnirelmann category.

In the present paper we prove a multiplicity result for equation (1.3) under almost optimal assumptions of f, showing the concentration of the solutions around local minima of V.

In particular, we prove the following result.

Theorem 1.1 *Assume $s \in (0,1)$, $N \geq 2$ and that* (V1)-(V2), (f1)–(f5) *hold. Let K be defined by* (1.6)*. Then, for small $\varepsilon > 0$ equation* (1.3) *has at least* $\mathrm{cupl}(K) + 1$ *positive solutions, which belong to $C^{0,\sigma}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for some $\sigma \in (0,1)$. Moreover, each of these sequences v_ε concentrates in K as $\varepsilon \to 0$: namely, there exist $x_\varepsilon \in \mathbb{R}^N$ global maximum points of v_ε, such that*

$$\lim_{\varepsilon \to 0} d(x_\varepsilon, K) = 0$$

and

$$\frac{C'}{1 + |\frac{x - x_\varepsilon}{\varepsilon}|^{N+2s}} \leq v_\varepsilon(x) \leq \frac{C''}{1 + |\frac{x - x_\varepsilon}{\varepsilon}|^{N+2s}} \quad \text{for } x \in \mathbb{R}^N$$

where $C', C'' > 0$ are uniform in $\varepsilon > 0$. Finally, for every sequence $\varepsilon_n \to 0^+$ there exists a ground state solution U of (2.12) *such that, up to a subsequence,*

$$v_{\varepsilon_n}(\varepsilon_n \cdot + x_{\varepsilon_n}) \to U \quad \text{as } n \to +\infty$$

in $H^s(\mathbb{R}^N)$ and locally on compact sets.

Here $\mathrm{cupl}(K)$ denotes the *cup-length* of K defined by the Alexander-Spanier cohomology with coefficients in some field $\mathbb{F}$ (see Definition 4.1). This topological tool denotes the geometric complexity of the set K, and it was successfully implemented also in [4, 20, 18, 19]: the idea of exploiting the topological configuration of the problem, in a singularly perturbed framework, goes back to the work [24] and it has been widely used to get multiplicity of solutions.

Remark 1.2 *Notice that the cup-length of a set K is strictly related to the Lusternik-Schnirelmann category of K. Indeed, if K is a contractible set (e.g. a point) or it is finite, then*

$$\mathrm{cupl}(K) + 1 = \mathrm{cat}(K) = 1;$$

if $K = S^{N-1}$ is the $N-1$ dimensional sphere in $\mathbb{R}^N$, then

$$\mathrm{cupl}(K) + 1 = \mathrm{cat}(K) = 2;$$

if $K = T^N$ is the N-dimensional torus, then

$$\mathrm{cupl}(K) + 1 = \mathrm{cat}(K) = N + 1.$$

However in general

$$\mathrm{cupl}(K) + 1 \leq \mathrm{cat}(K)$$

(see [22, Sections 2.8 and 9.23] for some examples where the strict inequality is attained).

Remark 1.3 *As observed in [19, 20], assumption* (V1) *in Theorem 1.1 can be relaxed without assuming the boundedness of V (see also [12, 14]). Moreover, the condition*

$$p > \max\{2_s^* - 2, 2\}$$

in (f5) *can be relaxed in $p > 2$ by paying the cost of considering a sufficiently large $C \gg 0$; see for instance [53, 36]. Finally, we remark that* (f1), *instead of the mere continuity of f, is needed only to get a Pohozaev identity by means of the regularity of solutions (see [13, Proposition 1.1]).*

We highlight that Theorem 1.1 extends the existence results in [36, 45] to a multiplicity result, and it improves the multiplicity theorem in [37], since we do not assume monotonicity nor Ambrosetti-Rabinowitz conditions on the nonlinearity. Moreover, no nondegeneracy and global conditions on V are considered.

The idea of the present paper is the following: first, we gain compactness and uniform L^∞-bounds on the set of ground states of the critical limiting problem (1.5); to this aim we employ a Moser's iteration argument adapted to the fractional framework, without the use of the s-harmonic extension, and appropriate for weak solutions. The criticality of the problem, as well as the absence of a chain rule, make the argument more delicate. The gained uniformity allows then the introduction of a suitable truncation on the nonlinearity f; the new truncated function reveals thus to be subcritical.

Therefore, we can apply to the truncated problem the approach of [19]: we employ a penalization argument on a neighborhood of expected solutions, perturbation of the ground states of a limiting problem, and this neighborhood results to be invariant under the action of a deformation flow. Compactness is restored also by the use of a new fractional center of mass, which engages a seminorm stronger than the usual Gagliardo one; the topological machinery between two level sets of the associated indefinite energy functional is then built also through the use of a Pohozaev functional. The number of solutions is thus related to the cup-length of K and these solutions are proved to exhibit a polynomial decay and to converge to a ground state of the limiting equation. This last convergence allows finally to prove that these solutions solve the original critical problem (1.3).

We point out that the techniques employed in [19] cannot be applied directly to the critical framework: indeed, the embedding of $H^s(\mathbb{R}^N)$ in $L^{2_s^*}(\mathbb{R}^N)$ is not compact, even if we reduce to radially symmetric functions or to bounded domains; in particular, the criticality obstructs the convergence of truncated Palais-Smale sequences related to the penalized functional, which is a key point in [19]. Moreover, the regularity results given by [23], exploited in the concentration and in the decay of the solutions, do not apply; in particular, L^∞-bounds and compactness of the set of ground states of the limiting problem have to be specifically investigated.

We highlight that the conclusions of Theorem 1.1 hold also for $s = 1$ and $N \geq 3$, as we state in Theorem 5.1. Regarding this local framework, Theorem 5.1 is the critical counterpart of the result in [20]: again, we point out that the arguments exploited in the subcritical setting of [20] cannot be directly implemented in our framework, because of the lack of compactness. In the critical case, previous results were given by [1, 62, 6]: in particular we extend here the existence result in [59] to a multiplicity result, and we improve the multiplicity theorem in [56] in the sense that we do not need to work with global minima of V nor we need monotonicity on f. In this setting, the solutions decay exponentially and enjoy more regularity. Notice that in such a case (f1) means f merely continuous.

Remark 1.4 *In [37, 56] the multiplicity of solutions is related to* $\mathrm{cat}(K)$*, where*

$$\mathrm{cat}(K) \geq \mathrm{cupl}(K) + 1.$$

In those papers, indeed, the monotonicity of the map $t \mapsto \frac{f(t)}{t}$ *implies the boundedness from below of the functional restricted to the Nehari manifold: thus, the tool of the (absolute) Lusternik-Schnirelmann category can be implemented in order to get* $\mathrm{cat}(K)$ *solutions.*

Under our general assumptions (f1)–(f5)*, we can not rely on the boundedness from below of the functional, and thus, in order to control both from above and below the energy, we need more sophisticated tools, such as the* relative *category. On the other hand, for any interval* $I \subset \mathbb{R}$ *and any neighborhood* K_d *of* K*, considered the inclusion*

$$j : (I \times K, \partial I \times K) \to (I \times K_d, \partial I \times K_d)$$

the key relation

$$\mathrm{cat}(j) \geq \mathrm{cat}(K),$$

essential in the estimation of the relative category of two subleves of the indefinite functional (see [20, Remark 4.3]) does not generally hold [22, Remark 7.47]. On the other hand, the same relation for the cup-length

$$\mathrm{cupl}(j) \geq \mathrm{cupl}(K)$$

holds true, as proved in [20, Lemma 5.5] (see also [33, Proposition 3.5]). Here $\mathrm{cat}(j)$ *denotes the category of the inclusion* j*, which is a standard generalization of the relative category*

$$\mathrm{cat}(A, B) \equiv \mathrm{cat}\big(id : (A, B) \to (A, B)\big)$$

with $B \subset A$*; similarly for* $\mathrm{cupl}(j)$ *(see [8] for precise definitions).*

Thus, we need to take advantage of the (relative) cup-length in order to get a bound on the number of solutions.

The paper is organized as follows. In Section 2 we recall some notions on the fractional Sobolev space, and then we obtain compactness of the set of ground states and a crucial L^∞-bound on the critical limiting problem. In Section 3 we use this uniform estimate to introduce a truncation which brings the problem back to the subcritical case, and in Section 4 we prove Theorem 1.1. Finally, in Section 5 we deal with the local case.

2 Uniform L^∞-bound

Let $N \geq 2$ and $s \in (0,1)$. For every $u \in L^2(\mathbb{R}^N)$ we define the Gagliardo seminorm [27, Section 2]

$$[u]_s^2 := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy$$

and the fractional Sobolev space

$$H^s(\mathbb{R}^N) := \{ u \in L^2(\mathbb{R}^N) \mid [u]_s < +\infty \}$$

endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^N)}^2 := \|u\|_2^2 + [u]_s^2, \quad u \in H^s(\mathbb{R}^N);$$

here $\|\cdot\|_q$ denotes the $L^q(\mathbb{R}^N)$-Lebesgue norm for $q \in [1,+\infty]$. Moreover, for every $u \in H^s(\mathbb{R}^N)$ we define the fractional Laplacian

$$(-\Delta)^s u(x) := C(N,s) \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dx$$

where the integral is in the principal value sense and $C(N,s)$ is a positive constant. We have the following relation between the Gagliardo seminorm and the L^2-norm of the fractional Laplacian [27, Proposition 3.6]

$$[u]_s^2 = C'(N,s) \left\| (-\Delta)^{s/2} u \right\|_2^2, \quad u \in H^s(\mathbb{R}^N), \tag{2.9}$$

with $C'(N,s) > 0$, and moreover by polarization

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{N+2s}} dx\, dy = C'(N,s) \int_{\mathbb{R}^N} (-\Delta)^{s/2} u\, (-\Delta)^{s/2} v\, dx \tag{2.10}$$

for every u, $v \in H^s(\mathbb{R}^N)$. We recall that the immersion

$$H^s(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$$

is continuous [27, Theorem 3.5] for $q \in [2, 2_s^*]$, where $2_s^* = \frac{2N}{N-2s}$, and compact for $q \in (2, 2_s^*)$ if we restrict to the subspace of radially symmetric functions [46]. We highlight that the embedding is not compact for $q = 2_s^*$ even on bounded subsets of $\mathbb{R}^N$. Finally we have that there exists a best Sobolev embedding constant $\mathcal{S} > 0$ such that [27, Theorem 6.5]

$$\|u\|_{2_s^*} \leq \mathcal{S}^{-\frac{1}{2}} \left\| (-\Delta)^{s/2} u \right\|_2, \quad u \in H^s(\mathbb{R}^N). \tag{2.11}$$

Let us recall some crucial results on the limiting critical problem (1.5), that is

$$(-\Delta)^s U + m_0 U = f(U), \quad x \in \mathbb{R}^N. \tag{2.12}$$

Define the energy C^1-functional $\mathcal{L} : H^s(\mathbb{R}^N) \to \mathbb{R}$

$$\mathcal{L}(U) := \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}U|^2\,dx + \frac{m_0}{2}\int_{\mathbb{R}^N} U^2\,dx - \int_{\mathbb{R}^N} F(U)\,dx, \quad U \in H^s(\mathbb{R}^N)$$

and the related *least energy*

$$E_m := \inf\big\{\mathcal{L}(U) \mid U \in H^s(\mathbb{R}^N)\setminus\{0\},\ \mathcal{L}'(U) = 0\big\}.$$

Moreover we define the *Mountain Pass level*

$$C_{mp} := \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} \mathcal{L}(\gamma(t))$$

with

$$\Gamma := \big\{\gamma \in C\big([0,1],\, H^s(\mathbb{R}^N)\big) \mid \gamma(0) = 0,\ \mathcal{L}(\gamma(1)) < 0\big\}.$$

Finally we introduce the following *minimization problem*

$$C_{min} := \inf\big\{\mathcal{T}(U) \mid U \in H^s(\mathbb{R}^N),\ \mathcal{V}(U) = 1\big\} \tag{2.13}$$

where

$$\mathcal{T}(U) := \int_{\mathbb{R}^N} |(-\Delta)^{s/2}U|^2\,dx, \quad \mathcal{V}(U) := \int_{\mathbb{R}^N}\Big(F(U) - \frac{m_0}{2}U^2\Big)\,dx.$$

Notice that $\mathcal{L} = \frac{1}{2}\mathcal{T} - \mathcal{V}$. The following collection of results states the equivalence of the previous problems and the existence of a solution.

Proposition 2.1 *There exists a ground state solution for the problem* (2.12)*, that is a function U which solves the equation and such that*

$$\mathcal{L}(U) = E_m.$$

Moreover, every ground state is also a Mountain Pass solution and (up to scaling) also a solution for the minimization problem (2.13)*, and viceversa; in addition the following relations hold*

$$E_m = C_{mp},$$

$$E_m = \frac{s}{N}(2_s^*)^{-\frac{N}{2_s^* s}}(C_{min})^{\frac{N}{2s}}, \tag{2.14}$$

and every ground state is positive. Finally, recalled that $\mathcal{S}$ is the best Sobolev constant for the embedding (2.11)*, we have that the following upper bound holds*

$$C_{min} < \left(\frac{2_s^*}{a}\right)^{\frac{2}{2_s^*}}\mathcal{S} \tag{2.15}$$

where $a > 0$ appears in assumptions (f4)-(f5).

Proof. The positivity is a straightforward consequence of assumption (f2). Existence of a ground state solution can be achieved through the use of (2.15) and minimization of C_{min} as classically made by [10]. The equivalence with the Mountain Pass formulation is instead discussed as in [39]. We refer to [40, Proposition 2.4] for the precise statement and to [60, Section 4.1 and Remark 1.2] for details.

Moreover, as observed in Remark 1.3, to get the existence of a ground state, the restriction on the range of p in assumption (f5) can be substituted, by arguing as in [54, Lemma 3.3], with the request that C is sufficiently large (see also [36] and references therein). ∎

Thanks to Proposition 2.1 we can define

$$\widehat{S} := \big\{U \in H^s(\mathbb{R}^N) \setminus \{0\} \mid U \text{ ground state solution of (2.12)}, \, U(0) = \max_{\mathbb{R}^N} U\big\}.$$

We observe that, by the fractional version of the Pólya-Szegő inequality [48] (see also [13, Theorem 1.2]), every minimizer of C_{min} (i.e. every ground states of (2.12)) is actually radially symmetric decreasing up to a translation. Thus, the request in $\widehat{S}$ for U to have a maximum in zero is equivalent to the radial symmetry of U; that is

$$\widehat{S} = \big\{U \in H^s(\mathbb{R}^N) \setminus \{0\} \mid U \text{ radially symmetric ground state solution of (2.12)}\big\}. \tag{2.16}$$

Proposition 2.2 *Every $U \in \widehat{S}$ satisfies the Pohozaev identity, i.e.*

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2} U|^2 \, dx - 2_s^* \int_{\mathbb{R}^N} \Big(F(U) - \frac{m_0}{2} U^2\Big) \, dx = 0. \tag{2.17}$$

Moreover, the set $\widehat{S}$ is compact.

Proof. Once one observes that $U \in L^\infty(\mathbb{R}^N)$, whose proof is an easy adaptation of Proposition 2.3 below (focusing on a single $U \in \widehat{S}$), the proof of (2.17) is gained by means of regularity results and explicit computations on the s-harmonic extension problem; the arguments can be easily adapted from [13, Proposition 1.1] to the critical case.

Let us show the boundedness of $\widehat{S}$. For any $U \in \widehat{S}$, the embedding (2.11) and the Pohozaev identity (2.17) lead to

$$\|U\|_{2_s^*} \leq \mathcal{S}^{-\frac{1}{2}} \left\| (-\Delta)^{s/2} U \right\|_2 = \mathcal{S}^{-\frac{1}{2}} \frac{N}{s} \mathcal{L}(U) = \mathcal{S}^{-\frac{1}{2}} \frac{N}{s} E_m;$$

moreover the equation (2.12) and the assumptions (f3)-(f4) imply

$$\left\| (-\Delta)^{s/2} U \right\|_2^2 + m_0 \|U\|_2^2 = \int_{\mathbb{R}^N} f(U) U \, dx \leq \delta \|U\|_2^2 + C_\delta \|U\|_{2_s^*}^{2_s^*}$$

for $\delta < m_0$ and some $C_\delta > 0$. The combination of the two bounds leads to the claim.

Let thus focus on compactness; we use some ideas from [61]. Let U_n be a sequence in $\widehat{S}$; by (2.16) we assume $(U_n)_n \subset H^s_r(\mathbb{R}^N)$, where $H^s_r(\mathbb{R}^N)$ denotes the subset of $H^s(\mathbb{R}^N)$ consisting in radially symmetric functions. We recall that $H^s_r(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N)$ for $q \in (2, 2_s^*)$. By the boundedness of $\widehat{S}$ we can assume $U_n \rightharpoonup U$ in $H^s_r(\mathbb{R}^N)$. Set

$$\sigma := \left(\frac{1}{2_s^*} C_{min}\right)^{\frac{1}{2s}}$$

and

$$V_n := U_n(\sigma\cdot), \quad V := U(\sigma\cdot)$$

we have, by exploiting the Pohozaev identity, that V_n are solutions of the minimization problem (2.13), that is

$$\mathcal{T}(V_n) = C_{min}, \quad \mathcal{V}(V_n) = 1.$$

Thus we have $V_n \rightharpoonup V$ in $H^s_r(\mathbb{R}^N)$, and hence $V_n \to V$ in $L^q(\mathbb{R}^N)$, $q \in (2, 2^*_s)$, and $V_n \to V$ almost everywhere. By the lower semicontinuity of the norm we obtain

$$\mathcal{T}(V) \leq C_{min}; \tag{2.18}$$

hence, to conclude the proof, it is sufficient to show that $\mathcal{V}(V) = 1$, since this implies also that $U = V(\sigma^{-1}\cdot)$ lies in $\widehat{S}$.

Set

$$W_n := V_n - V$$

we have by the Brezis-Lieb Lemma (since $(-\Delta)^{s/2}V_n \rightharpoonup (-\Delta)^{s/2}V$ in the Hilbert space $L^2(\mathbb{R}^N)$)

$$\begin{aligned} \mathcal{T}(W_n) &= \mathcal{T}(V_n) - \mathcal{T}(V) + o(1) \\ &= C_{min} - \mathcal{T}(V) + o(1) \qquad (2.19) \\ &\leq C_{min} + o(1). \qquad (2.20) \end{aligned}$$

Moreover, rewrite $\mathcal{V}(W_n)$ as

$$\mathcal{V}(W_n) = \int_{\mathbb{R}^N} \left(F(W_n) - \frac{a}{2^*_s} W_n^2 \right) dx + \frac{a}{2^*_s} \|W_n\|_{2^*_s}^{2^*_s} - \frac{m_0}{2} \|W_n\|_2^2 . \tag{2.21}$$

Again by the Brezis-Lieb Lemma (since $V_n \rightharpoonup V$ in $L^q(\mathbb{R}^N)$, $q = 2, 2^*_s$ and $V_n \to V$ almost everywhere) we have

$$\|W_n\|_q^q = \|V_n\|_q^q - \|V\|_q^q + o(1), \quad q = 2, 2^*_s. \tag{2.22}$$

Set

$$g(t) := f(t) - at^{2^*_s - 1}$$

we have that g is subcritical at infinity by (f4), and superlinear in zero by (f3); thus, set $G(t) := \int_0^t g(\tau)d\tau$, by classical arguments (see e.g. [17, Lemma 2.4]) we have

$$\int_{\mathbb{R}^N} G(W_n)\,dx = o(1), \quad \int_{\mathbb{R}^N} G(V_n)\,dx = \int_{\mathbb{R}^N} G(V)\,dx + o(1). \tag{2.23}$$

Therefore by (2.21)–(2.23) we obtain

$$\begin{aligned} \mathcal{V}(W_n) &= \mathcal{V}(V_n) - \mathcal{V}(V) + o(1) \\ &= 1 - \mathcal{V}(V) + o(1). \qquad (2.24) \end{aligned}$$

Finally, through a simple scaling argument, we observe that

$$\mathcal{T}(u) \geq C_{min}(\mathcal{V}(u))^{\frac{2}{2^*_s}} \quad \text{for every } \mathcal{V}(u) \geq 0. \tag{2.25}$$

We pass to prove that $\mathcal{V}(V) = 1$ by contradiction.
Case $\mathcal{V}(V) > 1$. In this case, by (2.25) we have

$$\mathcal{T}(V) \geq C_{min}(\mathcal{V}(V))^{\frac{2}{2_s^*}} > C_{min}$$

which contradicts (2.18).
Case $\mathcal{V}(V) < 0$. Then, by (2.24) we have that

$$\mathcal{V}(W_n) \geq 1 - \frac{1}{2}\mathcal{V}(V) > 1 \quad \text{for } n \gg 0.$$

Thus, by (2.25) we obtain

$$\mathcal{T}(W_n) \geq C_{min}(\mathcal{V}(W_n))^{\frac{2}{2_s^*}} \geq C_{min}\left(1 - \frac{1}{2}\mathcal{V}(V)\right)^{\frac{2}{2_s^*}}$$

which contradicts (2.20).
Case $\mathcal{V}(V) \in (0,1)$. Again by (2.24) we have that

$$\mathcal{V}(W_n) \geq \frac{1}{2}\left(1 - \mathcal{V}(V)\right) > 0 \quad \text{for } n \gg 0.$$

Thus by (2.19), (2.25) and (2.24) we gain

$$\begin{aligned}
C_{min} &= \lim_n \big(\mathcal{T}(W_n) + \mathcal{T}(V)\big) \geq C_{min} \lim_n \left((\mathcal{V}(W_n))^{\frac{2}{2_s^*}} + (\mathcal{V}(V))^{\frac{2}{2_s^*}}\right) \\
&= C_{min}\left((1 - \mathcal{V}(V))^{\frac{2}{2_s^*}} + (\mathcal{V}(V))^{\frac{2}{2_s^*}}\right) \\
&> C_{min}\big((1 - \mathcal{V}(V)) + \mathcal{V}(V)\big) = C_{min}
\end{aligned}$$

which is an absurd.
Case $\mathcal{V}(V) = 0$. By (2.24) we have

$$\mathcal{V}(W_n) = 1 + o(1), \tag{2.26}$$

and thus by (2.25) $\mathcal{T}(W_n) \geq C_{min}(1 + o(1))^{\frac{2}{2_s^*}}$. This, combined with (2.20), gives

$$\mathcal{T}(W_n) = C_{min} + o(1). \tag{2.27}$$

Combining (2.26), (2.21) and (2.23) we obtain

$$1 + o(1) = \mathcal{V}(W_n) = \frac{a}{2_s^*}\left\|W_n\right\|_{2_s^*}^{2_s^*} - \frac{m_0}{2}\left\|W_n\right\|_2^2$$

that is

$$\begin{aligned}
\left\|W_n\right\|_{2_s^*}^{2_s^*} &= \frac{2_s^*}{a} + \frac{2_s^* m_0}{2a}\left\|W_n\right\|_2^2 + o(1) \\
&\geq \frac{2_s^*}{a} + o(1). \qquad (2.28)
\end{aligned}$$

By (2.27), the Sobolev embedding (2.11) and (2.28) we gain

$$C_{min} + o(1) = \mathcal{T}(W_n) = \left\|(-\Delta)^{s/2} W_n\right\|_2^2 \geq \mathcal{S}\left\|W_n\right\|_{2_s^*}^2 \geq \mathcal{S}\left(\frac{2_s^*}{a} + o(1)\right)^{\frac{2}{2_s^*}}.$$

Letting $n \to +\infty$ we finally have

$$C_{min} \geq \left(\frac{2_s^*}{a}\right)^{\frac{2}{2_s^*}} \mathcal{S}$$

which is in contradiction with (2.15). This concludes the proof. ∎

As a key property to employ the truncation argument, and to detect a handy neighborhood of approximating solutions, we have the following result (see also [28]).

Proposition 2.3 *The following bound holds*

$$\sup_{U \in \widehat{S}} \|U\|_\infty < \infty.$$

Proof. Assume by contradiction that there exists $(U_n)_n \subset \widehat{S}$ such that $\|U_n\|_\infty \to +\infty$ as $n \to +\infty$. By the compactness of $\widehat{S}$ in Proposition 2.2 we may assume that U_n is positive and convergent in $H^s(\mathbb{R}^N)$. If we prove that

$$\sup_n \|U_n\|_\infty < +\infty$$

we get a contradiction and conclude the proof. We use a Moser's iteration argument in a critical, fractional framework, appropriate for weak solutions.

We already know that U_n is bounded in $L^{2_s^*}(\mathbb{R}^N)$ by (2.11). Let us introduce $\gamma > 1$, to be fixed, and an arbitrary $T > 0$, and set a *γ-linear truncation at T*

$$h(t) \equiv h_{T,\gamma}(t) := \begin{cases} 0 & \text{if } t \leq 0, \\ t^\gamma & \text{if } t \in (0, T], \\ \gamma T^{\gamma-1} t - (\gamma - 1) T^\gamma & \text{if } t > T. \end{cases}$$

We have that $h \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, it is positive (increasing and convex) and by direct computations it satisfies the following properties

$$0 \leq h(t) \leq |t|^\gamma, \quad t \in \mathbb{R}, \tag{2.29}$$

$$0 \leq t h'(t) \leq \gamma h(t), \quad t \in \mathbb{R}, \tag{2.30}$$

$$\lim_{T \to +\infty} h_{T,\gamma}(t) = t^\gamma, \quad t \geq 0. \tag{2.31}$$

The goal is to estimate $\|h(U_n)\|_{2_s^*}$ and give thus a bound of U_n in $L^{2_s^*\gamma}(\mathbb{R}^N)$, where $2_s^*\gamma > 2_s^*$. In order to handle the weak formulation of the notion of solution we introduce

$$\tilde{h}(t) := \int_0^t (h'(r))^2 \, dr, \quad t \in \mathbb{R}$$

and observe that $\tilde{h} \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ is positive, increasing and convex. In particular

$$\tilde{h}'(t) = (h'(t))^2, \quad t \in \mathbb{R} \tag{2.32}$$

by definition and

$$(\tilde{h}(t) - \tilde{h}(s)) \leq \tilde{h}'(t)(t - s), \quad t,\, s \in \mathbb{R} \tag{2.33}$$

by convexity, and we gain also the Lipschitz continuity

$$|\tilde{h}(t)-\tilde{h}(r)| \le \left\|\tilde{h}'\right\|_\infty |t-r|, \quad t,r\in\mathbb{R}.$$

Combining the definition of $\tilde{h}$, (2.30) and (2.29) we obtain

$$0 \le \tilde{h}(t) \le \left\|h'\right\|_\infty |t|^\gamma, \quad t\in\mathbb{R}. \tag{2.34}$$

Finally, by a direct application of Jensen inequality we gain

$$|h(t)-h(s)|^2 \le \big(\tilde{h}(t)-\tilde{h}(s)\big)(t-s), \quad t,\, s\in\mathbb{R}. \tag{2.35}$$

We observe that $\tilde{h}(U_n)\in H^s(\mathbb{R}^N)$: indeed, by (2.9) and the convexity (2.33) we have

$$\left\|(-\Delta)^{s/2}\tilde{h}(U_n)\right\|_2 \le \left\|\tilde{h}'\right\|_\infty \left\|(-\Delta)^{s/2}U_n\right\|_2 < \infty;$$

moreover, since 2_s^* is the best summability exponent, if we assume

$$1<\gamma\le \frac{2_s^*}{2} \tag{2.36}$$

by (2.34) we obtain also

$$\tilde{h}(U_n) \le \left\|h'\right\|_\infty U_n^\gamma \in L^2(\mathbb{R}^N).$$

We use now (2.11) and combine (2.9), (2.35) and (2.10) to obtain

$$\begin{aligned}
\|h(U_n)\|_{2_s^*}^2 &\le \mathcal{S}^{-1}\left\|(-\Delta)^{s/2}h(U_n)\right\|_2^2 \\
&= (C'(N,s))^{-1}\mathcal{S}^{-1}\int_{\mathbb{R}^{2N}} \frac{|h(U_n(x))-h(U_n(y))|^2}{|x-y|^{N+2s}}\,dx\,dy \\
&\le (C'(N,s))^{-1}\mathcal{S}^{-1}\int_{\mathbb{R}^{2N}} \frac{\big(\tilde{h}(U_n(x))-\tilde{h}(U_n(y))\big)\big(U_n(x)-U_n(y)\big)}{|x-y|^{N+2s}}\,dx\,dy \\
&= \mathcal{S}^{-1}\int_{\mathbb{R}^N} (-\Delta)^{s/2}U_n\,(-\Delta)^{s/2}\tilde{h}(U_n)\,dx.
\end{aligned}$$

Since $\tilde{h}(U_n)\in H^s(\mathbb{R}^N)$ we can choose it as a test function in the equation and gain

$$\|h(U_n)\|_{2_s^*}^2 \le \mathcal{S}^{-1}\int_{\mathbb{R}^N}\big(f(U_n)-m_0U_n\big)\tilde{h}(U_n)\,dx.$$

By the assumptions (f3)-(f4) we find a sufficiently large constant $C=C(m_0)>0$ such that

$$f(t)\le \frac{m_0}{2}t + Ct^{2_s^*-1}, \quad t\ge 0$$

and thus, thanks to the positivity of $\tilde{h}(U_n)$,

$$\|h(U_n)\|_{2_s^*}^2 \le C\mathcal{S}^{-1}\int_{\mathbb{R}^N} U_n^{2_s^*-1}\tilde{h}(U_n)\,dx.$$

Now we use (2.33) (with $s=0$), (2.32), and (2.30)

$$\|h(U_n)\|_{2_s^*}^2 \le C\mathcal{S}^{-1}\int_{\mathbb{R}^N} U_n^{2_s^*-1}U_n\tilde{h}'(U_n)\,dx \tag{2.37}$$

$$\begin{aligned} &\leq C\mathcal{S}^{-1}\int_{\mathbb{R}^N} U_n^{2_s^*-1}U_n(h'(U_n))^2\,dx \\ &\leq \gamma^2 C\mathcal{S}^{-1}\int_{\mathbb{R}^N} U_n^{2_s^*-2}(h(U_n))^2\,dx. \end{aligned} \tag{2.38}$$

Let now $R>0$ to be fixed; splitting the right-hand side of (2.38) and by using the Hölder inequality we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} U_n^{2_s^*-2}(h(U_n))^2\,dx = \\ &= \int_{U_n\leq R} U_n^{2_s^*-2}(h(U_n))^2\,dx + \int_{U_n>R} U_n^{2_s^*-2}(h(U_n))^2\,dx \\ &\leq R^{2_s^*-2}\|h(U_n)\|_2^2 + \left(\int_{U_n>R} U_n^{2_s^*}\,dx\right)^{\frac{2_s^*-2}{2_s^*}} \|h(U_n)\|_{2_s^*}^2. \end{aligned}$$

The convergence of U_n in $L^{2_s^*}(\mathbb{R}^N)$ implies that the sequence is dominated by some function in $L^{2_s^*}(\mathbb{R}^N)$; thus by the Dominated Convergence Theorem we can find a sufficiently large $R=R(\gamma,m_0,\mathcal{S}^{-1})$ such that

$$\left(\int_{U_n>R} U_n^{2_s^*}\,dx\right)^{\frac{2_s^*-2}{2_s^*}} < \frac{1}{2}\frac{1}{\gamma^2 C\mathcal{S}^{-1}}, \quad \text{uniformly for } n\in\mathbb{N}.$$

Thus, plugging this information into (2.38), and absorbing the second piece on the right-hand side into the left-hand side, we obtain by (2.29)

$$\|h(U_n)\|_{2_s^*}^2 \leq 2\gamma^2 C\mathcal{S}^{-1}R^{2_s^*-2}\|h(U_n)\|_2^2 \leq 2\gamma^2 C\mathcal{S}^{-1}R^{2_s^*-2}\|U_n\|_{2\gamma}^{2\gamma}.$$

Recalled that $h=h_{T,\gamma}$, by (2.31) and Fatou's Lemma we have

$$\begin{aligned} \|U_n\|_{2_s^*\gamma}^{2\gamma} &= \left(\int_{\mathbb{R}^N}\liminf_{T\to+\infty} h_{T,\gamma}^{2_s^*}(U_n)\,dx\right)^{\frac{2}{2_s^*}} \leq \left(\liminf_{T\to+\infty}\int_{\mathbb{R}^N} h_{T,\gamma}^{2_s^*}(U_n)\,dx\right)^{\frac{2}{2_s^*}} \\ &\leq 2\gamma^2 C\mathcal{S}^{-1}R^{2_s^*-2}\|U_n\|_{2\gamma}^{2\gamma}. \end{aligned}$$

By our choice (2.36) of γ we gain that $U_n\in L^{2_s^*\gamma}(\mathbb{R}^N)$, which was the claim. We want now to employ an iteration argument; since this last inequality reveals not to be suitable, we exploit again (2.38). Thus applying again Fatou's Lemma to (2.38) and using (2.29) we obtain

$$\|U_n\|_{2_s^*\gamma}^{2\gamma} \leq \gamma^2 C\mathcal{S}^{-1}\|U_n\|_{2_s^*+2(\gamma-1)}^{2_s^*+2(\gamma-1)}$$

where we observe that

$$2\gamma < 2_s^*+2(\gamma-1) < 2_s^*\gamma.$$

To get an iteration we set

$$\begin{cases} \gamma_0 := \dfrac{2_s^*}{2} \\ 2_s^*+2(\gamma_{i+1}-1) := 2_s^*\gamma_i \end{cases}$$

so that

$$\gamma_i - 1 = \frac{2_s^*}{2}(\gamma_{i-1}-1) = \left(\frac{2_s^*}{2}\right)^{i+1}\left(\frac{2_s^*}{2}-1\right).$$

We now reiterate the argument proving that, for each $i \in \mathbb{N}$, $(U_n)_n \subset L^{2_s^*\gamma_i}(\mathbb{R}^N)$ and

$$\|U_n\|_{2_s^*\gamma_i}^{2\gamma_i} \leq \gamma_i^2 C\mathcal{S}^{-1} \|U_n\|_{2_s^*+2(\gamma_i-1)}^{2_s^*+2(\gamma_i-1)}$$

that is

$$\|U_n\|_{2_s^*\gamma_i}^{2\gamma_i} \leq \gamma_i^2 C\mathcal{S}^{-1} \|U_n\|_{2_s^*\gamma_{i-1}}^{2_s^*\gamma_{i-1}}$$

or equivalently

$$\|U_n\|_{2_s^*\gamma_i}^{\frac{2\gamma_i}{\gamma_i-1}} \leq (\gamma_i^2 C\mathcal{S}^{-1})^{\frac{1}{\gamma_i-1}} \|U_n\|_{2_s^*\gamma_{i-1}}^{\frac{2\gamma_{i-1}}{\gamma_{i-1}-1}}.$$

Notice that the iteration is possible since at each step the equivalent of (2.36) holds, that is by our choice of γ_i we have

$$1 < \gamma_i \leq \frac{2_s^*\gamma_{i-1}}{2}.$$

Thus we have

$$\|U_n\|_{2_s^*\gamma_i}^{\frac{2\gamma_i}{\gamma_i-1}} \leq \left(\prod_{k=1}^{i}(\gamma_k^2 C\mathcal{S}^{-1})^{\frac{1}{\gamma_k-1}}\right) \|U_n\|_{2_s^*\gamma_0}^{\frac{2\gamma_0}{\gamma_0-1}}.$$

Notice that

$$\gamma_i \to +\infty \quad \text{as } i \to +\infty,$$

and by direct computations

$$\begin{aligned}\prod_{k=1}^{i}(\gamma_k^2 C\mathcal{S}^{-1})^{\frac{1}{\gamma_k-1}} \quad &\leq \quad \left(C\mathcal{S}^{-1}\frac{2_s^*-2}{2}\right)^{\frac{2}{2_s^*-1}\sum_{k=1}^{\infty}\left(\frac{2}{2_s^*}\right)^{k+1}} \left(\frac{2_s^*}{2}\right)^{\frac{2}{2_s^*-1}\sum_{k=1}^{\infty}(k+1)\left(\frac{2}{2_s^*}\right)^{k+1}} \\ &=: \quad C_0 \quad \text{uniformly for } i \in \mathbb{N}.\end{aligned}$$

Thus, recalled that $\|\cdot\|_p \to \|\cdot\|_\infty$ as $p \to +\infty$ we obtain

$$\|U_n\|_\infty \leq C_0 \|U_n\|_{2_s^*\gamma_0}^{\frac{\gamma_0}{\gamma_0-1}}$$

which leads to the claim. ∎

3 The truncated problem

In virtue of Proposition 2.3, let

$$M := \sup_{U\in\widehat{S}} \|U\|_\infty + 1.$$

We preliminary observe that we can find a $t_0 \in [0, M]$ such that

$$F(t_0) > \frac{1}{2}m_0 t_0^2. \tag{3.39}$$

Indeed fixed a whatever $U \in \widehat{S}$, by the Pohozaev identity (2.17) we have (notice that $(-\Delta)^{s/2}U$ cannot identically vanish)

$$\int_{\mathbb{R}^N} \left(F(U) - \frac{m_0}{2}U^2\right) dx = \frac{1}{2_s^*} \left\|(-\Delta)^{s/2}U\right\|_2^2 > 0$$

and thus there exists an $x_0 \in \mathbb{R}^N$ such that

$$F(U(x_0)) > \frac{m_0}{2} U(x_0)^2;$$

setting $t_0 := U(x_0) \in [0, M]$ we have the claim.

We thus set

$$k := \sup_{t \in [0,M]} f(t) \in (0, +\infty),$$

where we observe that the strict positivity is due to the fact that $F(t_0) > 0$. Moreover we define the *truncated* nonlinearity $f_k : \mathbb{R} \to \mathbb{R}$

$$f_k(t) := \min\{f(t), k\}, \quad t \in \mathbb{R}.$$

We have the following properties on $f_k : \mathbb{R} \to \mathbb{R}$:

- $f_k(t) \leq f(t)$ for each $t \in \mathbb{R}$,
- $f_k(t) = f(t)$ whenever $|t| \leq M$,
- $f_k(U) = f(U)$ for every $U \in \widehat{S}$.

Notice that the same relations hold also for F and

$$F_k(t) := \int_0^t f_k(\tau) d\tau.$$

We have that f_k is subcritical, i.e. f_k satisfies assumptions (f1)–(f3) together with

(fk4) $\lim_{t \to +\infty} \frac{f_k(t)}{t^q} = 0$ for some $q \in (1, 2_s^* - 1)$,

(fk5) $F_k(t_0) > \frac{1}{2} m_0 t_0^2$ for some $t_0 > 0$;

here $q \in (1, 2_s^* - 1)$ is however fixed and $t_0 \in [0, M]$ is the one appearing in (3.39); notice that t_0 does not depend on k.

Consider now the truncated problem

$$\varepsilon^{2s} (-\Delta)^s v + V(x) v = f_k(v), \quad x \in \mathbb{R}^N \tag{3.40}$$

and the corresponding limiting truncated problem

$$(-\Delta)^s U + m_0 U = f_k(U), \quad x \in \mathbb{R}^N. \tag{3.41}$$

Notice again that, since f_k satisfies (f2), all the ground states of (3.41) are positive. Thus define

$$\widehat{S}_k := \big\{ U \in H^s(\mathbb{R}^N) \setminus \{0\} \mid U \text{ ground state solution of (3.41)}, U(0) = \max_{\mathbb{R}^N} U \big\}.$$

We have that the following key relation holds.

Proposition 3.1 *It results that $\widehat{S} = \widehat{S}_k$. Moreover, the least energy levels coincide.*

Proof. Let us denote by $\mathcal{L}_k$, Γ_k, $\mathcal{V}_k$, $E_m^k = C_{mp}^k$, C_{min}^k the quantities of the truncation problem analogous to the ones introduced in Section 2 for the critical problem.

First observe that, by $\mathcal{L}_k \geq \mathcal{L}$, we have $\Gamma_k \subset \Gamma$ and

$$C_{mp}^k \geq C_{mp}; \tag{3.42}$$

moreover for any $V \in \widehat{S}$ we have also $\mathcal{L}_k'(V) = 0$, and hence

$$\min_{V\in\widehat{S}} \mathcal{L}_k(V) \geq \min_{\mathcal{L}_k'(V)=0} \mathcal{L}_k(V) = E_m^k. \tag{3.43}$$

Let now $U \in \widehat{S}$. We have by (3.42) and (3.43)

$$C_{mp}^k \geq C_{mp} = \mathcal{L}(U) = E_m = \min_{V\in\widehat{S}} \mathcal{L}(V) = \min_{V\in\widehat{S}} \mathcal{L}_k(V) \geq E_m^k.$$

Therefore

$$\mathcal{L}_k(U) = \mathcal{L}(U) = C_{mp}^k = E_m^k$$

which, together with $\mathcal{L}_k'(U) = \mathcal{L}'(U) = 0$, gives that $U \in \widehat{S}_k$. Hence $\widehat{S} \subset \widehat{S}_k$. As a further consequence we gain

$$E_m^k = E_m. \tag{3.44}$$

We show now that $\widehat{S}_k \subset \widehat{S}$. By (3.44), (2.14) and the analogous relation on the subcritical problem, we have

$$C_{min}^k = C_{min},$$

thus, by rescaling, it is sufficient to prove that every minimizer of C_{min}^k is also a minimizer of C_{min}. Let thus U be a minimizer for C_{min}^k, i.e. $\mathcal{T}(U) = C_{min}^k$ and $\mathcal{V}_k(U) = 1$. Since $\mathcal{T}(U) = C_{min}$, it suffices to prove that $\mathcal{V}(U) = 1$. By definition, we have

$$\mathcal{V}(U) \geq \mathcal{V}_k(U) = 1.$$

On the other hand, set $\theta := (\mathcal{V}(U))^{\frac{1}{N}}$ we obtain, by scaling, that $\mathcal{V}(U(\theta\cdot)) = 1$ and thus

$$\mathcal{T}(U) = C_{min} \leq \mathcal{T}(U(\theta\cdot)) = \theta^{-\frac{N+2s}{N}} \mathcal{T}(U)$$

from which we achieve

$$\mathcal{V}(U) \leq 1.$$

This concludes the proof. ∎

4 Proof of Theorem 1.1

Before proving the main result, we first recall the definition of *cup-length* (see e.g. [16, 33, 22] and references therein).

Definition 4.1 *Let A be a topological space, and let $\mathbb{F}$ be a whatever field. Denote by*

$$H^*(A) = \bigoplus_{q\geq 0} H^q(A)$$

the Alexander-Spanier cohomology with coefficients in $\mathbb{F}$ (see [35] and references therein). Let

$$\smile : H^*(A) \times H^*(A) \to H^*(A)$$

be the cup-product. The cup-length *of A is defined by*

$$\begin{aligned}\mathrm{cupl}(A) := \max\big\{l \in \mathbb{N} \mid \ & \exists \alpha_0 \in H^*(A),\ \exists \alpha_i \in H^{q_i}(A),\ q_i \geq 1,\ \textit{for } i = 1\dots l,\\ & \textit{s.t. } \alpha_0 \smile \alpha_1 \smile \cdots \smile \alpha_l \neq 0 \ \textit{ in } H^*(A)\big\};\end{aligned}$$

if such $l \in \mathbb{N}$ does not exist but $H^*(A)$ is nontrivial, we have $\mathrm{cupl}(A) := 0$*, otherwise we set* $\mathrm{cupl}(A) := -1$*.*

When A is not connected, a slightly different definition (which makes the cup-length additive) can be found in [7]. See Example 1.2 and Remark 1.4 for some computation and comparison with the notion of category.

Proof of Theorem 1.1.

Step 1. We first look at the truncated problem (3.40). Indeed, by [19, Theorem 1.1 and Theorem 1.4] we obtain the existence of $\mathrm{cupl}(K)+1$ sequences of solutions of (3.40) satisfying the properties of Theorem 1.1 for $\varepsilon > 0$ small. To highlight the ideas behind the result, for the reader's convenience we sketch here an outline of the proof. We omit the dependence on the value k to avoid cumbersome notation.

We work with $u = v(\varepsilon\cdot)$. Through a compact slight perturbation of the set $\widehat{S}_k$ (see [19, Section 3.2]), still called $\widehat{S}_k$, we first define, for each $r > 0$, a non-compact neighborhood of $\widehat{S}_k$

$$S(r) := \Big\{ u = U(\cdot - y) + \varphi \in H^s(\mathbb{R}^N) \mid U \in \widehat{S}_k,\ y \in \mathbb{R}^N,\ \varphi \in H^s(\mathbb{R}^N),\ \|\varphi\|_{H^s(\mathbb{R}^N)} < r \Big\}.$$

To detect information on its elements we define a *minimal radius* map $\widehat{\rho} : H^s(\mathbb{R}^N) \to \mathbb{R}_+$

$$\widehat{\rho}(u) := \inf \Big\{ \|u - U(\cdot - y)\|_{H^s(\mathbb{R}^N)} \mid U \in \widehat{S}_k,\ y \in \mathbb{R}^N \Big\}, \quad u \in H^s(\mathbb{R}^N),$$
$$u \in S(r) \implies \widehat{\rho}(u) < r,$$

and, for some suitable ρ_0, $R_0 > 0$, a *barycentric* map $\Upsilon : S(\rho_0) \to \mathbb{R}^N$

$$\Upsilon(u) := \frac{\int_{\mathbb{R}^N} y\, d(y,u) dy}{\int_{\mathbb{R}^N} d(y,u) dy}, \quad u \in S(\rho_0),$$
$$u = U(\cdot - y) + \varphi \in S(\rho_0) \implies |\Upsilon(u) - y| \le 2R_0;$$

the *density* map $d(y,u)$ appearing in the center of mass is defined by

$$d(y,u) := \psi \left(\inf_{U \in \widehat{S}_k} \|u - U(\cdot - y)\|_{B_{R_0}(y)} \right), \quad (y,u) \in \mathbb{R}^N \times S(\rho_0),$$

where ψ is a suitable cut-off function (see [19, Lemma 3.7]) and the norm involved is a modification of the usual $H^s(B_{R_0}(y))$-norm, made through the use of a stronger seminorm which takes into account the tails of the functions, i.e.

$$\|u\|_A^2 := \int_A u^2\, dx + \int_A \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy, \quad u \in H^s(\mathbb{R}^N),\ A \subset \mathbb{R}^N.$$

Then, in order to localize solutions in Ω (introduced in (V2)), we introduce a suitable penalization on the functional $I_\varepsilon : H^s(\mathbb{R}^N) \to \mathbb{R}$

$$I_\varepsilon(u) := \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2\, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(\varepsilon x) u^2\, dx - \int_{\mathbb{R}^N} F_k(u)\, dx, \quad u \in H^s(\mathbb{R}^N)$$

associated with the rescaled equation

$$(-\Delta)^s u + V(\varepsilon x) u = f_k(u), \quad x \in \mathbb{R}^N, \tag{4.45}$$

and we call this penalized functional $J_\varepsilon : H^s(\mathbb{R}^N) \to \mathbb{R}$ (see [19, Section 4.1]). Then, we restrict our attention to a *neighborhood of expected solutions*

$$\mathcal{X}_{\varepsilon,\delta} := \big\{u \in S(\rho_0) \mid \varepsilon\Upsilon(u) \in K_d,\ J_\varepsilon(u) < E_m + R(\delta, \widehat{\rho}(u))\big\},$$

where K_d is a suitable neighborhood of K, $R(\delta, \widehat{\rho}(u))$ is a suitable u-dependent radius and $\delta > 0$ is chosen sufficiently small (see [19, Section 4.3]). On $\mathcal{X}_{\varepsilon,\delta}$, for ε small, since the nonlinearity is subcritical we succeed in proving delicate ε-independent gradient estimates for J_ε, a truncated Palais-Smale-type condition, and the existence of a deformation flow $\eta : [0,1] \times \mathcal{X}_{\varepsilon,\delta} \to \mathcal{X}_{\varepsilon,\delta}$; moreover, we prove that each solution of $J'_\varepsilon(u) = 0$ is also a solution of the original problem $I'_\varepsilon(u) = 0$ (see [19, Theorem 4.7, Corollary 4.9, Proposition 4.10 and Lemma 4.11]).

To find multiple solutions we build two continuous maps satisfying

$$I \times K \overset{\Phi_\varepsilon}{\to} \mathcal{X}_{\varepsilon,\delta}^{E_m+\hat{\delta}} \overset{\Psi_\varepsilon}{\to} I \times K_d \quad \text{and} \quad \partial I \times K \overset{\Phi_\varepsilon}{\to} \mathcal{X}_{\varepsilon,\delta}^{E_m-\hat{\delta}} \overset{\Psi_\varepsilon}{\to} (I \setminus \{1\}) \times K_d,$$

where $I \subset \mathbb{R}$ is a suitable neighborhood of 1, $\widehat{\delta} \in (0, \delta)$ and the superscript denotes the intersection with the sublevels of J_ε. These maps are defined by

$$\Phi_\varepsilon(t, y) := U_0\left(\tfrac{\cdot - y/\varepsilon}{t}\right), \quad (t, y) \in I \times K,$$
$$\Psi_\varepsilon(u) := \big(T(P_{m_0}(u)), \varepsilon\Upsilon(u)\big), \quad u \in \mathcal{X}_{\varepsilon,\delta}^{E_m+\hat{\delta}},$$

where $U_0 \in \widehat{S}_k$ is fixed, T is a truncation over the interval I, and P_{m_0} is a Pohozaev functional defined by

$$P_{m_0}(u) := \left(\frac{2N}{N-2s} \frac{\int_{\mathbb{R}^N} F_k(u)\,dx - \frac{m_0}{2}\|u\|_2^2}{\big\|(-\Delta)^{s/2}u\big\|_2^2}\right)_+^{\frac{1}{2s}}, \quad u \in H^s(\mathbb{R}^N) \setminus \{0\};$$

notice that $P_{m_0}(U) = 1$ for every $U \in \widehat{S}_k$ (see [19, Section 4.4]). The composition $\Psi_\varepsilon \circ \Phi_\varepsilon$ results being homotopic to the identity, and this leads to the existence of at least $\mathrm{cupl}(K) + 1$ solutions by the following chain of inequalities involving the relative category and the relative cup-length (see [19, Section 5])

$$\begin{aligned}
\#\big\{u \text{ solutions of } (4.45)\big\} \ &\geq\ \#\Big\{u \in \mathcal{X}_{\varepsilon,\delta} \mid J'_\varepsilon(u) = 0,\ E_m - \hat{\delta} \leq J_\varepsilon(u) \leq E_m + \hat{\delta}\Big\} \\
&\geq\ \mathrm{cat}\Big(\mathcal{X}_{\varepsilon,\delta}^{E_m+\hat{\delta}}, \mathcal{X}_{\varepsilon,\delta}^{E_m-\hat{\delta}}\Big) \geq \mathrm{cupl}\Big(\mathcal{X}_{\varepsilon,\delta}^{E_m+\hat{\delta}}, \mathcal{X}_{\varepsilon,\delta}^{E_m-\hat{\delta}}\Big) + 1 \\
&\geq\ \mathrm{cupl}(\Psi_\varepsilon \circ \Phi_\varepsilon) + 1 \geq \mathrm{cupl}(K) + 1.
\end{aligned}$$

Finally, uniform L^∞-bounds and $C^{0,\sigma}$-regularity are proved through the use of recent fractional De Giorgi classes [23] (see [19, Section 5.1]).

Step 2. Rescaling back again, for each of these sequences v_ε of solutions of (3.40), called $x_\varepsilon \in \mathbb{R}^N$ a global maximum point of v_ε, by [19, Theorem 1.4] (and some contradiction argument through subsequences) we obtain

$$\lim_{\varepsilon \to 0} d(x_\varepsilon, K) = 0$$

and

$$\frac{C'}{1 + |\frac{x - x_\varepsilon}{\varepsilon}|^{N+2s}} \leq v_\varepsilon(x) \leq \frac{C''}{1 + |\frac{x - x_\varepsilon}{\varepsilon}|^{N+2s}} \quad \text{for } x \in \mathbb{R}^N \tag{4.46}$$

where $C', C'' > 0$ are uniform in $\varepsilon > 0$. Moreover, for every sequence $\varepsilon_n \to 0^+$ there exist $U \in \widehat{S}_k$ and an $x_0 \in \mathbb{R}^N$ such that, up to subsequences,

$$v_{\varepsilon_n}(\varepsilon_n \cdot + x_{\varepsilon_n}) \to U(\cdot + x_0), \quad \text{as } n \to +\infty \tag{4.47}$$

in $H^s(\mathbb{R}^N)$ and locally on compact sets. For further details we refer to [19].

Step 3. Notice that by Proposition 3.1 we have $U \in \widehat{S}$, thus $U(\cdot + x_0)$ is a ground state of (2.12). We prove now that v_ε are solutions of the original equation, which is given by

$$\|v_\varepsilon\|_\infty < M \quad \text{definitely for } \varepsilon \text{ small.} \tag{4.48}$$

Assume by contradiction that (4.48) does not hold: thus there exists a sequence $\varepsilon_n \to 0$ such that

$$\|v_{\varepsilon_n}\|_\infty \geq M \quad \text{for each } n \in \mathbb{N}.$$

By the previous Step, there exists an $U \in \widehat{S}_k$ and an $x_0 \in \mathbb{R}^N$ such that, up to subsequence, (4.47) holds. In particular, by the pointwise convergence we obtain

$$\|v_{\varepsilon_n}\|_\infty = v(x_{\varepsilon_n}) \to U(x_0) \leq \|U\|_\infty < M$$

which implies

$$\|v_{\varepsilon_n}\|_\infty < M$$

definitely for $n \gg 0$, which is an absurd. Thus (4.48) holds. As a consequence

$$f_k(v_\varepsilon) = f(v_\varepsilon)$$

and hence v_ε are solutions of the original problem (1.3), satisfying the desired properties. ∎

Remark 4.2 *We point out that the found solutions are perturbations of ground states of the truncated limiting problem* (3.41) *which are, on the other hand, coinciding with the ground states of the critical limiting problem* (2.12) *thanks to Proposition 3.1. One may think to search directly the solutions as perturbation of functions in the compact set $\widehat{S}$, as made in [19], but actually the direct approach in a critical setting reveals several problems, such as the convergence of the Palais-Smale sequences. A different and direct approach is given in [6] by means of Concentration-Compactness techniques, in the assumptions that f satisfies monotonicity and Ambrosetti-Rabinowitz conditions.*

5 The local case

The arguments presented in Theorem 1.1 apply, with suitable modifications, also to local nonlinear Schrödinger equations. We give here some details. Conditions (f1)–(f5) rewrite in the local case $s = 1$ as

(f1') $f \in C(\mathbb{R}, \mathbb{R})$,

(f2') $f(t) \equiv 0$ for $t \leq 0$,

(f3') $\lim_{t\to 0} \frac{f(t)}{t} = 0$,

(f4') $\lim_{t\to +\infty} \frac{f(t)}{t^{2^*-1}} = a > 0$, where $2^* := \frac{2N}{N-2}$, (f5') for some $C > 0$ and $\max\{2^* - 2, 2\} < p < 2^*$, i.e. satisfying

$$p \in \begin{cases} (4,6) & N = 3, \\ \left(2, \dfrac{2N}{N-2}\right) & N \geq 4, \end{cases}$$

it results that

$$f(t) \geq a t^{2^*-1} + C t^{p-1} \quad \text{for } t \geq 0.$$

See also Remark 1.3 for some weakening and comments on the assumption (f5').

Theorem 5.1 *Suppose $N \geq 3$ and that* (V1)-(V2), (f1')–(f5') *hold. Let K be defined by* (1.6)*. Then, for small $\varepsilon > 0$ the equation*

$$-\varepsilon^2 \Delta v + V(x) v = f(v), \quad x \in \mathbb{R}^N$$

has at least $\mathrm{cupl}(K) + 1$ *positive solutions, which belong to $C^{1,\sigma}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for some $\sigma \in (0,1)$. Moreover, each of these sequences v_ε concentrates in K as $\varepsilon \to 0$: namely, there exist $x_\varepsilon \in \mathbb{R}^N$ global maximum points of v_ε, such that*

$$\lim_{\varepsilon \to 0} d(x_\varepsilon, K) = 0$$

and

$$v_\varepsilon(x) \leq C' \exp\Big(- C'' \Big| \frac{x - x_\varepsilon}{\varepsilon} \Big| \Big) \quad \textit{for } x \in \mathbb{R}^N \tag{5.49}$$

where $C', C'' > 0$ are uniform in $\varepsilon > 0$. Finally, for every sequence $\varepsilon_n \to 0^+$ there exists a ground state solution U of

$$-\Delta U + m_0 U = f(U), \quad x \in \mathbb{R}^N,$$

such that, up to a subsequence,

$$v_{\varepsilon_n}(\varepsilon_n \cdot + x_{\varepsilon_n}) \to U \quad \textit{as } n \to +\infty$$

in $H^s(\mathbb{R}^N)$ and locally on compact sets.

Proof. The arguments of the previous sections apply mutatis mutandis. Indeed, we define in the same way the set of ground states $\widehat{S}$, which turns to be nonempty [61] and compact. Moreover to get the uniform $L^\infty(\mathbb{R}^N)$ bound, one can easily adapt the proof of Proposition 2.3 after observing that by the chain rule it holds

$$|\nabla h(U)|^2 = \nabla U \cdot \nabla \tilde{h}(U), \quad U \in H^1(\mathbb{R}^N),$$

where we recall that $\tilde{h}' = (h')^2$. Then the truncation machinery can be set in motion, and one can prove $\widehat{S}_k = \widehat{S}$. Existence, multiplicity and decay of solutions of the truncated problem are given by [20, Theorem 1.1 and Remark 1.3]; the regularity is instead a consequence of standard elliptic estimates [55, Appendix B]. ∎

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