



Nonexistence of global solutions for generalized Tricomi equations with combined nonlinearity



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ABSTRACT

In the present paper, we investigate the blow-up dynamics for local solutions to the semilinear generalized Tricomi equation with combined nonlinearity. As a result, we enlarge the blow-up region in comparison to the ones for the corresponding semilinear models with either power nonlinearity or nonlinearity of derivative type. Our approach is based on an iteration argument to establish lower bound estimates for the space average of local solutions. Finally, we obtain upper bound estimates for the lifespan of local solutions as byproduct of our iteration argument.

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1. Introduction

Aim of the present work is to derive a blow-up result for the semilinear *generalized Tricomi equation* with a combined nonlinearity

$$\begin{cases} \partial_t^2 u - t^{2\ell} \Delta u = |\partial_t u|^p + |u|^q, & x \in \mathbb{R}^n, t > 0, \\ u(0, x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ \partial_t u(0, x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (1)$$

where ℓ is a positive parameter, $p, q > 1$ and ε is a positive constant describing the size of Cauchy data. More specifically, our goal is to enlarge the blow-up region in the (p, q) -plane in comparison to the ones for the corresponding models with *power nonlinearity* $|u|^q$ and *nonlinearity of derivative type* $|\partial_t u|^p$, respectively.

Over the last years, several papers have been devoted to the study of semilinear Cauchy problem associated with the *generalized Tricomi operator* $\partial_t^2 - t^{2\ell} \Delta$. In [1] the authors conjectured the critical exponent for the semilinear generalized Tricomi equation with power nonlinearity in space dimension $n \geq 2$

$$\begin{cases} \partial_t^2 u - t^{2\ell} \Delta u = |u|^p, & x \in \mathbb{R}^n, t > 0, \\ u(0, x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ \partial_t u(0, x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (2)$$

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where $\ell > 0$ and $p > 1$. More precisely, this exponent, denoted by $p_0(n; \ell)$ in the present paper, is the positive root of the quadratic equation

$$((\ell + 1)n - 1)p^2 - ((\ell + 1)n + 1 - 2\ell)p - 2(\ell + 1) = 0. \tag{3}$$

For $\ell = 0$ the exponent $p_0(n; \ell)$ coincides with the so-called *Strauss exponent* (denoted by $p_{Str}(n)$), which is the critical exponent for the semilinear wave equation with power nonlinearity.

This conjecture was made according to a blow-up result for the subcritical case $1 < p < p_0(n; \ell)$ when $n \geq 2$ proved by means of the so-called Kato’s lemma for second order ordinary differential inequalities. Then, in [2] and [3], the global existence of small data solutions in suitable weighted Sobolev spaces in space dimension $n \geq 3$ and $n = 2$, respectively, is proved for $p > p_0(n; \ell)$ in the subconformal case. Furthermore, in [3] a blow-up result is proved in the critical case $p = p_0(n; \ell)$ as well (when $n \geq 2$). Recently, in [4] the one dimensional case has been considered and it has been shown that the critical exponent is no longer a generalized Strauss exponent, rather the Kato-type exponent $1 + \frac{2}{\ell}$ (all explicit computations are done for the Tricomi operator, namely, for $\ell = 1/2$).

On the other hand, for the semilinear generalized Tricomi equation with nonlinearity of derivative type

$$\begin{cases} \partial_t^2 u - t^{2\ell} \Delta u = |\partial_t u|^p, & x \in \mathbb{R}^n, t > 0, \\ u(0, x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ \partial_t u(0, x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{4}$$

in [5] the second and the third author proved a blow-up result for $1 < p \leq p_1(n; \ell)$, where

$$p_1(n; \ell) \doteq \frac{(\ell + 1)n + 1}{(\ell + 1)n - 1} = p_{Gla}((\ell + 1)n) \quad \text{and} \quad p_{Gla}(n) \doteq 1 + \frac{2}{n - 1}$$

is the so-called *Glassey exponent*, which is, the critical exponent for the semilinear wave equation with nonlinearity of derivative type $|\partial_t u|^p$. In [6] the authors announced to have enlarged this blow-up range, employing an approach based on the test function method.

Studying the blow-up dynamics for local solutions to the semilinear Cauchy problem (1), we want to emphasize how the contemporary presence of a power nonlinearity $|u|^q$ and of a nonlinearity of derivative type $|\partial_t u|^p$ produces a larger blow-up region in the (p, q) -plane than the one that has been got by dealing individually with each of these two kinds of nonlinear terms as in the treatment of (2) or (4).

Moreover, we mention that the semilinear wave models with *combined nonlinearity* $|\partial_t u|^p + |u|^q$ (namely, (1) for $\ell = 0$) has been already studied in [7–10]. In particular, in [7,9] blow-up results are shown for $p, q > 1$ such that $(q - 1)((n - 1)p - 2) < 4$, under suitable sign assumptions for the Cauchy data. On the other hand, in [8] it is shown the optimality of the previous condition, by proving the global existence of small data solution for $p, q > 1$ such that $(q - 1)((n - 1)p - 2) \geq 4$, $p > p_{Gla}(n)$, and $q > p_{Str}(n)$ in space dimension $n = 2, 3$.

Finally, we point out that other semilinear wave models with combined nonlinearity have been investigated (from the viewpoint of the blow-up dynamics) in the *scattering producing case* [11] and in the *scale-invariant case* [12–14].

Let us introduce now the kind of solutions to (1) that we are going to consider throughout this paper.

Definition 1.1. Let $u_0, u_1 \in L^1_{loc}(\mathbb{R}^n)$ with $\text{supp } u_0, \text{supp } u_1 \subset B_R$ for some $R > 0$. We say that u is a weak solution to (1) on $[0, T)$ if

$$u \in \mathcal{C}([0, T), W^{1,1}_{loc}(\mathbb{R}^n)) \cap \mathcal{C}^1([0, T), L^1_{loc}(\mathbb{R}^n)) \cap L^q_{loc}((0, T) \times \mathbb{R}^n) \quad \text{such that} \quad \partial_t u \in L^p_{loc}((0, T) \times \mathbb{R}^n)$$

satisfies $u(0, \cdot) = \varepsilon u_0$ in $L^1_{loc}(\mathbb{R}^n)$, the support condition

$$\text{supp } u(t, \cdot) \subset B_{R+t^{\ell+1}/(\ell+1)} \quad \text{for any } t \in (0, T), \tag{5}$$

and the integral identity

$$\begin{aligned} & \int_{\mathbb{R}^n} \partial_t u(t, x) \psi(t, x) \, dx + \int_0^t \int_{\mathbb{R}^n} (-\partial_t u(s, x) \psi_t(s, x) + s^{2\ell} \nabla u(s, x) \cdot \nabla \psi(s, x)) \, dx \, ds \\ &= \varepsilon \int_{\mathbb{R}^n} u_1(x) \psi(0, x) \, dx + \int_0^t \int_{\mathbb{R}^n} (|\partial_t u(s, x)|^p + |u(s, x)|^q) \psi(s, x) \, dx \, ds \end{aligned} \tag{6}$$

for any test function $\psi \in \mathcal{C}_0^\infty([0, T] \times \mathbb{R}^n)$ and any $t \in (0, T)$.

We point out that performing a further step of integration by parts in (6), we obtain the integral relation

$$\begin{aligned} & \int_{\mathbb{R}^n} (\partial_t u(t, x) \psi(t, x) - u(t, x) \psi_t(t, x)) \, dx + \int_0^t \int_{\mathbb{R}^n} u(s, x) (\psi_{tt}(s, x) - s^{2\ell} \Delta \psi(s, x)) \, dx \, ds \\ &= \varepsilon \int_{\mathbb{R}^n} (u_1(x) \psi(0, x) - u_0(x) \psi_t(0, x)) \, dx + \int_0^t \int_{\mathbb{R}^n} (|\partial_t u(s, x)|^p + |u(s, x)|^q) \psi(s, x) \, dx \, ds \end{aligned} \tag{7}$$

for any $\psi \in \mathcal{C}_0^\infty([0, T] \times \mathbb{R}^n)$ and any $t \in (0, T)$.

Let us state the main result of this paper.

Theorem 1.2. *Let $p, q > 1$ satisfy*

$$(((\ell + 1)n - 1)p - 2\ell(p - 1) - 2)(q - 1) < 4. \tag{8}$$

Let us assume that $u_0, u_1 \in L^1_{\text{loc}}(\mathbb{R}^n)$ are nonnegative, not both trivial and compactly supported functions with supports contained in B_R for some $R > 0$. Let

$$u \in \mathcal{C}([0, T], W^{1,1}_{\text{loc}}(\mathbb{R}^n)) \cap \mathcal{C}^1([0, T], L^1_{\text{loc}}(\mathbb{R}^n)) \cap L^q_{\text{loc}}((0, T) \times \mathbb{R}^n) \text{ such that } \partial_t u \in L^p_{\text{loc}}((0, T) \times \mathbb{R}^n)$$

be a weak solution to (1) according to Definition 1.1 with lifespan $T = T(\varepsilon)$.

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(n, \ell, p, q, u_0, u_1, R)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the weak solution u blows up in finite time. Furthermore, the upper bound estimate for the lifespan

$$T(\varepsilon) \leq C\varepsilon^{-\frac{p(q-1)}{\theta(n, \ell, p, q)}}$$

holds, where the positive constant C is independent of ε and

$$\theta(n, \ell, p, q) \doteq 2 - \frac{1}{2} [((\ell + 1)n - 1)p - 2\ell(p - 1) - 2](q - 1). \tag{9}$$

The paper is organized as follows: in Section 2 we provide the proof of Theorem 1.2. In Section 3 we recall some results already known in the case of the semilinear generalized Tricomi equation with power nonlinearity that follow straightforwardly by slightly modifying the proof in Section 2. Finally, in Section 4 we explain in detail the obtained blow-up range and we compare it with corresponding blow-up results for (2) and (4), respectively; in particular, we put a special emphasis on the analysis of the one dimensional case.

1.1. Notations

Throughout the paper we employ the following notations: the ball in \mathbb{R}^n with radius R around the origin is denoted B_R ; the notations $f \lesssim g$ means that there exists a positive constant C such that $f \leq Cg$ and, similarly, for $f \gtrsim g$; by writing $\psi \in \mathcal{C}_0^\infty([0, T] \times \mathbb{R}^n)$ we mean the existence of a compact set $\mathcal{K} \subset \mathbb{R}^{1+n}$ such that the support of the \mathcal{C}^∞ function ψ satisfies $\text{supp} \psi \subset \mathcal{K} \subset [0, T] \times \mathbb{R}^n$; K_ν denotes the modified Bessel function of second kind of order ν . Finally, as in the introduction, $p_0(n; \ell)$ is the positive solution to (3), $p_{\text{Str}}(n) = p_0(n; 0)$ denotes the Strauss exponent, and $p_{\text{Gla}}(n) = p_1(n; 0)$ denotes the Glassey exponent. When $1 + 2/(n - 1)$ refers to the nonlinear term $|u|^q$ we call it *Kato exponent* (denoted by $p_{\text{Kat}}(n)$), as it is customary in the related literature, although it coincides with the Glassey exponent.

2. Proof of the main result

Let u be a weak solution to (1) according to Definition 1.1 that fulfills the support condition (5). As time-dependent function, whose dynamic will provide the blow-up result, we consider the space average of u (following an approach introduced for the first time in [15]), namely,

$$U(t) \doteq \int_{\mathbb{R}^n} u(t, x) \, dx.$$

We will derive the iteration frame for this functional. Furthermore, we will employ also two auxiliary functionals, whose definitions will be provided in the next subsection, see (18) and (19), to determine a first lower bound estimate for U .

2.1. The test function

In this section, we recall from [1] the definition of a function $\Psi = \Psi(t, x)$ with separate variables that solves the linear generalized Tricomi equation, namely

$$\Psi_{tt} - t^{2\ell} \Delta \Psi = 0. \tag{10}$$

As x -dependent function we consider the eigenfunction for the Laplacian introduced in [16]

$$\varphi(x) \doteq \begin{cases} e^x + e^{-x} & \text{if } n = 1, \\ \int_{\mathbb{S}^{n-1}} e^{x \cdot \omega} \, d\sigma_\omega & \text{if } n \geq 2. \end{cases} \tag{11}$$

This function is radially symmetric, belongs to the class $\mathcal{C}^\infty(\mathbb{R}^n)$ and satisfies the following notable properties:

$$\Delta \varphi = \varphi, \tag{12}$$

$$\varphi(x) \sim |x|^{-\frac{n-1}{2}} e^{|x|} \quad \text{as } |x| \rightarrow \infty. \tag{13}$$

In the next lines we employ some well-known properties of the modified Bessel function of the second kind. Nevertheless, for the sake of readability we address the reader to the Appendix for a short recap of these properties. Let us recall the time-dependent function $\lambda = \lambda(t; \ell)$ introduced in [1] such that

$$\lambda(t; \ell) \doteq C_\ell t^{\frac{1}{2}} K_{\frac{1}{2\ell+2}} \left(\frac{1}{\ell+1} t^{\ell+1} \right), \tag{14}$$

with the positive constant C_ℓ allowing $\lambda(0; \ell) = 1$, where K_ν denotes the modified Bessel function of the second kind of order ν . Note that we use (A.3) to get a finite, positive value of λ as $t \rightarrow 0$. For the sake of brevity, hereafter, we will write simply $\lambda = \lambda(t)$ skipping the dependence on ℓ . Due to the fact that K_ν fulfills the second order ordinary differential equation (A.1) by straightforward computations it follows that λ satisfies the following problem:

$$\begin{cases} \lambda''(t) - t^{2\ell} \lambda(t) = 0, & t > 0, \\ \lambda(0) = 1, \lambda(\infty) = 0. \end{cases} \tag{15}$$

Therefore, combining (12) and (15), it follows that the positive function

$$\Psi(t, x) \doteq \lambda(t) \varphi(x)$$

solves the desired linear equation (10).

Let us point out that, using the recursive relations for the derivatives of modified Bessel functions of the second kind (cf. (A.2) in the Appendix) we have

$$\begin{aligned} \lambda'(t) &= \frac{C_\ell}{2} t^{-\frac{1}{2}} K_{\frac{1}{2\ell+2}} \left(\frac{1}{\ell+1} t^{\ell+1} \right) + C_\ell t^{\frac{1}{2}} \left(K_{\frac{1}{2\ell+2}} \left(\frac{1}{\ell+1} t^{\ell+1} \right) \right)' \\ &= -C_\ell t^{\frac{1}{2}+\ell} K_{\frac{1}{2\ell+2}-1} \left(\frac{1}{\ell+1} t^{\ell+1} \right) < 0, \end{aligned} \tag{16}$$

where in the last line we used that $K_\nu(\tau)$ is a positive function for $\nu \in \mathbb{R}$ and $\tau > 0$. Furthermore, there exists a constant $c_0 > 1$ such that

$$\frac{|\lambda'(t)|}{\lambda(t)} \geq \frac{t^\ell}{c_0} \text{ for any } t > 0 \text{ and } \frac{|\lambda'(t)|}{\lambda(t)} \leq c_0 t^\ell \text{ for any } t \geq 1, \tag{17}$$

as it has been shown in [17, Lemma 2.1]. Note that in (17) we used that λ is positive function (λ is strictly decreasing and $\lambda(\infty) = 0$).

Finally, we can introduce the two following auxiliary functionals:

$$U_0(t) \doteq \int_{\mathbb{R}^n} u(t, x) \Psi(t, x) \, dx, \tag{18}$$

$$U_1(t) \doteq \int_{\mathbb{R}^n} \partial_t u(t, x) \Psi(t, x) \, dx. \tag{19}$$

Our strategy in the next subsection is to determine lower bound estimates for U_0, U_1 .

2.2. Lower bound estimates for the auxiliary functionals

Let us start this subsection by deriving a lower bound estimate for U_0 .

Lemma 2.1. *Let u_0, u_1 be functions satisfying the same assumptions as in the statement of Theorem 1.2. Let U_0 be the functional associated with a local (in time) weak solution u of (1) and defined by (18). Then, there exists $T_0 > 0$ such that*

$$U_0(t) \gtrsim \varepsilon I_\ell[u_0, u_1] t^{-\ell} \text{ for any } t \in [2T_0, T], \tag{20}$$

where

$$I_\ell[u_0, u_1] \doteq \int_{\mathbb{R}^n} (u_1(x) - \lambda'(0; \ell) u_0(x)) \varphi(x) \, dx. \tag{21}$$

Proof. Due to the property of finite speed of propagation, we have that u fulfills (5), where $R > 0$ is chosen so that $\text{supp } u_0, \text{supp } u_1 \subset B_R$. In particular, thanks to this support condition for the solution u we may consider not compactly supported test functions in (6). Therefore, choosing $\Psi = \lambda\varphi \in C^\infty([0, T] \times \mathbb{R}^n)$ as test function in (7), for any $t \in (0, T)$ we get

$$\begin{aligned} &\int_{\mathbb{R}^n} \partial_t u(t, x) \Psi(t, x) \, dx - \int_{\mathbb{R}^n} u(t, x) \Psi_t(t, x) \, dx \\ &= \varepsilon \int_{\mathbb{R}^n} (u_1(x) - \lambda'(0) u_0(x)) \varphi(x) \, dx + \int_0^t \int_{\mathbb{R}^n} (|\partial_t u(s, x)|^p + |u(s, x)|^q) \Psi(s, x) \, dx \, ds \\ &= \varepsilon I_\ell[u_0, u_1] + \int_0^t \int_{\mathbb{R}^n} (|\partial_t u(s, x)|^p + |u(s, x)|^q) \Psi(s, x) \, dx \, ds, \end{aligned}$$

where we employed $\lambda(0) = 1$ and (10). Hence, by using (18), we may rewrite the previous identity as follows:

$$U'_0(t) - 2\frac{\lambda'(t)}{\lambda(t)}U_0(t) = \varepsilon I_\ell[u_0, u_1] + \int_0^t \int_{\mathbb{R}^n} (|\partial_t u(s, x)|^p + |u(s, x)|^q) \Psi(s, x) \, dx \, ds \tag{22}$$

for any $t \in (0, T)$. Since the nonlinearity is nonnegative, from (22) we have

$$U'_0(t) - 2\frac{\lambda'(t)}{\lambda(t)}U_0(t) \geq \varepsilon I_\ell[u_0, u_1] \tag{23}$$

for any $t \in (0, T)$. Multiplying both sides of the previous inequality by $(\lambda(t))^{-2}$ and integrating over $[0, t]$, it results

$$\frac{U_0(t)}{\lambda^2(t)} - \frac{U_0(0)}{\lambda^2(0)} \geq \varepsilon I_\ell[u_0, u_1] \int_0^t \frac{ds}{\lambda^2(s)}. \tag{24}$$

Thus, employing the sign assumption on the first initial data, we find

$$U_0(t) \geq \varepsilon I_\ell[u_0, u_1] \lambda^2(t) \int_0^t \frac{ds}{\lambda^2(s)}.$$

By using the asymptotic behavior for the modified Bessel function of the second kind (see (A.4) in the Appendix), we have

$$\lambda(t) = \sqrt{\frac{\pi(\ell + 1)}{2}} C_\ell t^{-\frac{\ell}{2}} e^{-\phi_\ell(t)} \left(1 + O(t^{-(\ell+1)})\right) \text{ as } t \rightarrow \infty. \tag{25}$$

Consequently, for large t so that $0 < T_0 \leq t < T$, we can estimate

$$U_0(t) \gtrsim \varepsilon I_\ell[u_0, u_1] t^{-\ell} e^{-2\phi_\ell(t)} \int_{T_0}^t s^\ell e^{2\phi_\ell(s)} \, ds.$$

Therefore, for $t \in [2T_0, T)$

$$\begin{aligned} U_0(t) &\gtrsim \varepsilon I_\ell[u_0, u_1] t^{-\ell} e^{-2\phi_\ell(t)} \int_{t/2}^t s^\ell e^{2\phi_\ell(s)} \, ds \\ &\gtrsim \varepsilon I_\ell[u_0, u_1] t^{-\ell} \left(1 - e^{2\phi_\ell(t/2) - 2\phi_\ell(t)}\right) = \varepsilon I_\ell[u_0, u_1] t^{-\ell} \left(1 - e^{2(2^{-(\ell+1)} - 1)\phi_\ell(t)}\right) \\ &\gtrsim \varepsilon I_\ell[u_0, u_1] t^{-\ell}. \end{aligned}$$

This completes the proof. \square

Next we derive a lower bound estimate for the functional U_1 . The proof of the next result is inspired by [13, Lemma 3.3].

Lemma 2.2. *Let u_0, u_1 be functions satisfying the same assumptions as in the statement of Theorem 1.2. Let U_1 be the functional associated with a local (in time) weak solution u of (1) and defined by (19). Then, there exists $T_0 > 0$ such that*

$$U_1(t) \gtrsim \varepsilon I_\ell[u_0, u_1] \text{ for any } t \in [2T_0, T), \tag{26}$$

where $I_\ell[u_0, u_1]$ was defined in (21).

Proof. Let us get started by showing that $U_1(t) \geq 0$ for any $t \in [0, T)$. According to this purpose, we introduce the further auxiliary functional

$$F_1(t) \doteq \int_{\mathbb{R}^n} \partial_t u(t, x) \varphi(x) \, dx \text{ for } t \in [0, T).$$

Since $U_1 = \lambda F_1$ and λ is a positive function, if we show that F_1 is nonnegative, then, it follows the nonnegativity of U_1 as well. Choosing φ as test function in (7) (this is possible, due to the support condition for u as explained in the proof of Lemma 2.1), we arrive at

$$\begin{aligned} \int_{\mathbb{R}^n} \partial_t u(t, x) \varphi(x) \, dx &= \int_0^t \int_{\mathbb{R}^n} s^{2\ell} u(s, x) \Delta \varphi(x) \, dx \, ds + \varepsilon \int_{\mathbb{R}^n} u_1(x) \varphi(x) \, dx \\ &\quad + \int_0^t \int_{\mathbb{R}^n} (|\partial_t u(s, x)|^p + |u(s, x)|^q) \varphi(x) \, dx \, ds \\ &= \int_0^t \frac{s^{2\ell}}{\lambda(s)} U_0(s) \, ds + \varepsilon \int_{\mathbb{R}^n} u_1(x) \varphi(x) \, dx + \int_0^t \int_{\mathbb{R}^n} (|\partial_t u(s, x)|^p + |u(s, x)|^q) \varphi(x) \, dx \, ds. \end{aligned}$$

We remark that (24) implies that $U_0(t)$ is nonnegative for $t \in [0, T)$. Therefore, due to the fact that the second data and the nonlinearity are nonnegative, from the previous identity, it follows that F_1 is nonnegative. We prove now the lower bound estimate (26). From (18) and (19), it follows the relations

$$U_1(t) = U_0'(t) - \frac{\lambda'(t)}{\lambda(t)} U_0(t). \tag{27}$$

Therefore, from (22) we obtain

$$U_1(t) - \frac{\lambda'(t)}{\lambda(t)} U_0(t) = \varepsilon I_\ell[u_0, u_1] + \int_0^t \int_{\mathbb{R}^n} (|\partial_t u(s, x)|^p + |u(s, x)|^q) \Psi(s, x) \, dx \, ds \tag{28}$$

for any $t \in (0, T)$. Differentiating the last identity with respect to t , we have

$$U_1'(t) - \frac{\lambda'(t)}{\lambda(t)} U_0'(t) + \left(-\frac{\lambda''(t)}{\lambda(t)} + \left(\frac{\lambda'(t)}{\lambda(t)} \right)^2 \right) U_0(t) = \int_{\mathbb{R}^n} (|\partial_t u(t, x)|^p + |u(t, x)|^q) \Psi(t, x) \, dx$$

for any $t \in (0, T)$. The employment of (27) and (15) in the last equation yields

$$\begin{aligned} \int_{\mathbb{R}^n} (|\partial_t u(t, x)|^p + |u(t, x)|^q) \Psi(t, x) \, dx &= U_1'(t) - \frac{\lambda'(t)}{\lambda(t)} U_1(t) - \frac{\lambda''(t)}{\lambda(t)} U_0(t) \\ &= U_1'(t) - \frac{\lambda'(t)}{\lambda(t)} U_1(t) - t^{2\ell} U_0(t). \end{aligned}$$

Using again the nonnegativity of the nonlinearity, for a fixed parameter ω we may rewrite

$$\begin{aligned} 0 &\leq U_1'(t) - \frac{\lambda'(t)}{\lambda(t)} U_1(t) - t^{2\ell} U_0(t) \\ &= U_1'(t) - 2\omega \frac{\lambda'(t)}{\lambda(t)} U_1(t) - h_1(t) \left(U_1(t) - \frac{\lambda'(t)}{\lambda(t)} U_0(t) \right) - h_2(t) U_0(t), \end{aligned} \tag{29}$$

where the functions h_1, h_2 are defined as follows:

$$h_1(t) \doteq \frac{\lambda'(t)}{\lambda(t)} (1 - 2\omega), \quad h_2(t) \doteq t^{2\ell} + \left(\frac{\lambda'(t)}{\lambda(t)} \right)^2 (1 - 2\omega).$$

By using (16) and (17) and the choosing $\omega \in \left(\frac{1}{2}, \frac{1}{2} + \frac{1}{2c_0^2} \right)$, we obtain that

$$h_1(t) \geq \frac{2\omega - 1}{c_0} t^\ell \geq 0 \quad \text{and} \quad h_2(t) \geq (1 + c_0^2(1 - 2\omega)) t^{2\ell} \geq 0 \tag{30}$$

for any $t \geq 1$. Therefore, combining (28), (29) and (30), we get

$$\begin{aligned} U_1'(t) - 2\omega \frac{\lambda'(t)}{\lambda(t)} U_1(t) &\geq h_1(t) \left(U_1(t) - \frac{\lambda'(t)}{\lambda(t)} U_0(t) \right) + h_2(t) U_0(t) \\ &\geq h_1(t) \left(U_1(t) - \frac{\lambda'(t)}{\lambda(t)} U_0(t) \right) \\ &\gtrsim \varepsilon I_\ell[u_0, u_1] t^\ell \end{aligned}$$

for $t \in (1, T)$. We remark that in the second step of the previous chain of inequalities the nonnegativity of the functional U_0 is employed. Multiplying the last estimate by $(\lambda(t))^{-2\omega}$ and integrating over $[1, t]$, we arrive at

$$\frac{U_1(t)}{\lambda^{2\omega}(t)} - \frac{U_1(1)}{\lambda^{2\omega}(1)} \gtrsim \varepsilon I_\ell[u_0, u_1] \int_1^t \frac{s^\ell}{\lambda^{2\omega}(s)} ds. \tag{31}$$

Employing the nonnegativity of the functional U_1 , that we proved at the really beginning of this proof, we may neglect the second term on the left-hand side of (31). Next, using again the asymptotic behavior of λ given in (25) as in the proof of Lemma 2.1, we derive the lower bound estimate for U_1 . From (31) it follows

$$\begin{aligned} U_1(t) &\gtrsim \varepsilon I_\ell[u_0, u_1] \lambda^{2\omega}(t) \int_1^t \frac{s^\ell}{\lambda^{2\omega}(s)} ds \\ &\gtrsim \varepsilon I_\ell[u_0, u_1] t^{-\ell\omega} e^{-2\omega\phi_\ell(t)} \int_{T_0}^t s^{\ell+\ell\omega} e^{2\omega\phi_\ell(s)} ds \\ &\gtrsim \varepsilon I_\ell[u_0, u_1] e^{-2\omega\phi_\ell(t)} \int_{t/2}^t s^\ell e^{2\omega\phi_\ell(s)} ds \\ &\gtrsim \varepsilon I_\ell[u_0, u_1] \left(1 - e^{-2\omega(1-2^{-(\ell+1)})\phi_\ell(t)} \right) \\ &\gtrsim \varepsilon I_\ell[u_0, u_1] \end{aligned}$$

for any $t \in [2T_0, T)$. Note that in the previous steps we may assume $T_0 > 1$ without loss of generality. The proof is complete. \square

2.3. Derivation of the iteration frame

Let us choose a bump function which is equal to 1 on the light-cone $\{(s, x) \in [0, t] \times \mathbb{R}^n : |x| \leq R + \phi_\ell(s)\}$. Then, from (6) we obtain

$$\int_{\mathbb{R}^n} \partial_t u(t, x) dx = \varepsilon \int_{\mathbb{R}^n} u_1(x) dx + \int_0^t \int_{\mathbb{R}^n} (|\partial_t u(s, x)|^p + |u(s, x)|^q) dx ds,$$

which leads to

$$U(t) = U(0) + U'(0)t + \int_0^t \int_0^s \int_{\mathbb{R}^n} (|\partial_t u(\tau, x)|^p + |u(\tau, x)|^q) dx d\tau ds \tag{32}$$

$$\geq \int_0^t \int_0^s \int_{\mathbb{R}^n} (|\partial_t u(\tau, x)|^p + |u(\tau, x)|^q) dx d\tau ds. \tag{33}$$

Additionally, from nonnegativity of u_0, u_1 it follows that $U(t) \geq 0$ for any $t \in (0, T)$.

From the paper [1], we know that

$$\left(\int_{B_{R+\phi_\ell(\tau)}} \Psi^{\frac{p}{p-1}}(\tau, x) dx \right)^{p-1} \lesssim \tau^{-\frac{\ell p}{2}} (R + \phi_\ell(\tau))^{(n-1)(p-1) - \frac{n-1}{2} p},$$

where hereafter the unexpressed multiplicative constants may depend on n, p, R, ℓ, u_0, u_1 but are independent of ε . By using Hölder’s inequality and Lemma 2.2, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |\partial_t u(\tau, x)|^p dx &\geq |U_1(\tau)|^p \left(\int_{B_{R+\phi_\ell(\tau)}} \Psi^{\frac{p}{p-1}}(\tau, x) dx \right)^{-(p-1)} \\ &\gtrsim \varepsilon^p \tau^{\frac{\ell p}{2}} (R + \phi_\ell(\tau))^{(n-1)(1-\frac{p}{2})} \end{aligned} \tag{34}$$

for any $t \in (2T_0, T)$. Therefore, if we combine (33) and (34), we derive a first lower bound for U , namely,

$$\begin{aligned} U(t) &\gtrsim \varepsilon^p (R + \phi_\ell(t))^{-(n-1)\frac{p}{2}} \int_{2T_0}^t \int_{2T_0}^s \tau^{\frac{\ell p}{2}} (R + \phi_\ell(\tau))^{n-1} d\tau ds \\ &\gtrsim \varepsilon^p (1+t)^{-(\ell+1)(n-1)\frac{p}{2}} \int_{2T_0}^t \int_{2T_0}^s (\tau - 2T_0)^{\frac{\ell p}{2} + (\ell+1)(n-1)} d\tau ds \\ &\geq K \varepsilon^p (1+t)^{-(\ell+1)(n-1)\frac{p}{2}} (t - 2T_0)^{\frac{\ell p}{2} + (\ell+1)(n-1)+2} \end{aligned} \tag{35}$$

for any $t \in (2T_0, T)$, where K is a suitable positive constant.

Eventually, in order to construct an iteration frame, we apply Hölder’s inequality together with the property of finite speed of propagation. More precisely, we find

$$\begin{aligned} U(t) &\gtrsim \int_0^t \int_0^s (R + \phi_\ell(\tau))^{-n(q-1)} (U(\tau))^q d\tau ds \\ &\geq C \int_0^t \int_0^s (1 + \tau)^{-(\ell+1)n(q-1)} (U(\tau))^q d\tau ds \end{aligned} \tag{36}$$

for any $t \in (0, T)$ and for a suitable positive constant C , where we neglected the influence of the nonlinearity of derivative type in (33). In the next section, we will employ (36) to get iteratively a sequence of lower bound estimates for the functional $U(t)$ starting from (35).

Remark 1. Repeating similar computations to those we made to get Eq. (35) (in particular, by using the lower bound estimate for U_0 from Lemma 2.1 instead of the lower bound estimate for U_1 from Lemma 2.2), we obtain

$$\begin{aligned} U(t) &\gtrsim \varepsilon^q (R + \phi_\ell(t))^{-\frac{(n-1)q}{2}} \int_{2T_0}^t \int_{2T_0}^s \tau^{-\frac{\ell q}{2}} (R + \phi_\ell(\tau))^{n-1} d\tau ds \\ &\geq \tilde{K} \varepsilon^q (1+t)^{-(n-1)(\ell+1)\frac{q}{2} - \frac{\ell q}{2}} (t - 2T_0)^{(n-1)(\ell+1)+2} \end{aligned} \tag{37}$$

for any $t \in (2T_0, T)$, where \tilde{K} is a suitable positive constant. By using either (35) or (37), we employed lower bound estimates for the nonlinearity of derivative type or for the power nonlinearity to get a first estimate from below for the functional U . A further lower bound estimate for U can be obtained by assuming u_1 nontrivial (so that, $\int_{\mathbb{R}^n} u_1(x) dx > 0$). Under this additional assumption, from (32) we derive

$$U(t) \geq \tilde{K}_0 \varepsilon t, \tag{38}$$

for any $t \in (0, T)$, where the positive multiplicative constant \tilde{K}_0 depends on u_1 .

2.4. Iteration argument

In this section, we are going to prove a sequence of lower bound estimates for the time-dependent functional U . In fact, we will prove that

$$U(t) \geq C_j (1+t)^{-\alpha_j} (t - 2T_0)^{\beta_j} \tag{39}$$

for any $t \in (2T_0, T)$ and any $j \in \mathbb{N}$, where $\{\alpha_j\}_{j \in \mathbb{N}}$, $\{\beta_j\}_{j \in \mathbb{N}}$ and $\{C_j\}_{j \in \mathbb{N}}$ are sequences of nonnegative real numbers that will be determined recursively in the iteration step.

Clearly, (39) holds true for $j = 0$ thanks to (35), provided that

$$C_0 \doteq K\varepsilon^p, \quad \alpha_0 \doteq (\ell + 1)(n - 1)\frac{p}{2}, \quad \beta_0 = \frac{\ell p}{2} + (\ell + 1)(n - 1) + 2.$$

In order to use an inductive argument, it remains to show the validity of the inductive step: we assume the validity of (39) for some $j \in \mathbb{N}$ and we have to prove its validity for $j + 1$ too. Plugging (39) in (36), we get

$$\begin{aligned} U(t) &\geq CC_j^q \int_{2T_0}^t \int_{2T_0}^s (1 + \tau)^{-(\ell+1)n(q-1)-\alpha_j q} (\tau - 2T_0)^{\beta_j q} d\tau ds \\ &\geq CC_j^q (1 + t)^{-(\ell+1)n(q-1)-\alpha_j q} \int_{2T_0}^t \int_{2T_0}^s (\tau - 2T_0)^{\beta_j q} d\tau ds \\ &\geq CC_j^q (\beta_j q + 1)^{-1} (\beta_j q + 2)^{-1} (1 + t)^{-(\ell+1)n(q-1)-\alpha_j q} (t - 2T_0)^{\beta_j q + 2}. \end{aligned}$$

Thus, we proved (39) for $j + 1$ provided that

$$C_{j+1} \doteq CC_j^q (\beta_j q + 1)^{-1} (\beta_j q + 2)^{-1}, \tag{40}$$

$$\alpha_{j+1} \doteq \underbrace{(\ell + 1)n(q - 1)}_{\doteq a} + \alpha_j q, \quad \beta_{j+1} \doteq \beta_j q + 2. \tag{41}$$

In particular, from (41) we have

$$\alpha_j = a + q\alpha_{j-1} = \dots = a \sum_{k=0}^{j-1} q^k + \alpha_0 q^j = \left(\frac{a}{q-1} + \alpha_0\right) q^j - \frac{a}{q-1}, \tag{42}$$

$$\beta_j = 2 + q\beta_{j-1} = \dots = 2 \sum_{k=0}^{j-1} q^k + \beta_0 q^j = \left(\frac{2}{q-1} + \beta_0\right) q^j - \frac{2}{q-1}. \tag{43}$$

Therefore, combining (41) and (43), it results

$$\beta_j = 2 + q\beta_{j-1} < \left(\frac{2}{q-1} + \beta_0\right) q^j.$$

Consequently, from (40) it follows

$$C_j \geq CC_{j-1}^q (\beta_{j-1} q + 2)^{-2} \geq \underbrace{C \left(\frac{2}{q-1} + \beta_0\right)^{-2}}_{\doteq \tilde{C}} C_{j-1}^q q^{-2j}.$$

Applying the logarithmic function to both sides of the last inequality and using in an iterative way the resulting relation, we obtain

$$\begin{aligned} \log C_j &\geq q \log C_{j-1} - 2j \log q + \log \tilde{C} \\ &\geq q^2 \log C_{j-2} - 2(j + (j - 1)q) \log q + (1 + q) \log \tilde{C} \\ &\geq \dots \geq q^j \log C_0 - 2 \log q \sum_{k=0}^{j-1} (j - k)q^k + \log \tilde{C} \sum_{k=0}^{j-1} q^k. \end{aligned}$$

Using the formula

$$\sum_{k=0}^{j-1} (j - k)q^k = \frac{1}{q - 1} \left(\frac{q^{j+1} - q}{q - 1} - j \right)$$

(whose validity can be proved through an inductive argument) we find

$$\begin{aligned} \log C_j &\geq q^j \log C_0 - \frac{2 \log q}{q - 1} \left(\frac{q^{j+1} - q}{q - 1} - j \right) + \frac{q^j - 1}{q - 1} \log \tilde{C} \\ &= q^j \left(\log C_0 - \frac{2q \log q}{(q - 1)^2} + \frac{\log \tilde{C}}{q - 1} \right) + \frac{2j \log q}{q - 1} + \frac{2q \log q}{(q - 1)^2} - \frac{\log \tilde{C}}{q - 1}. \end{aligned}$$

If we denote by $j_0 = j_0(n, p, q, \ell) \in \mathbb{N}$ the smallest integer greater than $\frac{\log \tilde{C}}{2 \log q} - \frac{q}{q-1}$, then, for any $j \geq j_0$ it holds

$$\begin{aligned} \log C_j &\geq q^j \left(\log C_0 - \frac{2q \log q}{(q - 1)^2} + \frac{\log \tilde{C}}{q - 1} \right) \\ &= q^j \log \left(K q^{-(2q)/(q-1)^2} \tilde{C}^{1/(q-1)} \varepsilon^p \right) = q^j \log (D \varepsilon^p), \end{aligned} \tag{44}$$

where $D \doteq K q^{-(2q)/(q-1)^2} \tilde{C}^{1/(q-1)}$. Combining (39), (42), (43) and (44), for $j \geq j_0$ and $t \geq 2T_0$ we arrive at

$$\begin{aligned} U(t) &\geq \exp \left(q^j \log(D \varepsilon^p) \right) (1 + t)^{-\alpha_j} (t - 2T_0)^{\beta_j} \\ &= \exp \left(q^j \left(\log(D \varepsilon^p) - \left(\frac{\alpha}{q-1} + \alpha_0 \right) \log(1 + t) + \left(\frac{\beta}{q-1} + \beta_0 \right) \log(t - 2T_0) \right) \right) (1 + t)^{\frac{\alpha}{q-1}} (t - 2T_0)^{-\frac{\beta}{q-1}}. \end{aligned}$$

For $t \geq T_1 \doteq \max\{1, 4T_0\}$ it holds $\log(1 + t) \leq \log(2t)$ and $\log(t - 2T_0) \geq \log(t/2)$, so, for $j \geq j_0$ it results

$$\begin{aligned} U(t) &\geq \exp \left(q^j \left(\log(D \varepsilon^p) + \left(\frac{2-\alpha}{q-1} + \beta_0 - \alpha_0 \right) \log t - \left(\frac{\alpha+2}{q-1} + \alpha_0 + \beta_0 \right) \log 2 \right) \right) (1 + t)^{\frac{\alpha}{q-1}} (t - 2T_0)^{-\frac{\beta}{q-1}} \\ &= \exp \left(q^j \left(\log \left(2^{-(\alpha_0+\beta_0) - \frac{\alpha+2}{q-1}} D \varepsilon^p t^{\frac{\theta(n,\ell,p,q)}{q-1}} \right) \right) \right) (1 + t)^{\frac{\alpha}{q-1}} (t - 2T_0)^{-\frac{\beta}{q-1}}, \end{aligned} \tag{45}$$

where for the exponent of t in the last equality we used

$$\begin{aligned} \frac{2-\alpha}{q-1} + \beta_0 - \alpha_0 &= \frac{2}{q-1} - (\ell + 1)n + (\ell + 1)(n - 1) + \frac{\ell p}{2} + 2 - (\ell + 1)(n - 1)\frac{p}{2} \\ &= \frac{2}{q-1} - ((\ell + 1)n - 1 - 2\ell)\frac{p}{2} - \ell + 1 \\ &= \frac{2}{q-1} - \frac{1}{2} [((\ell + 1)n - 1)p - 2\ell(p - 1) - 2] = \frac{\theta(n,\ell,p,q)}{q-1}, \end{aligned} \tag{46}$$

where $\theta(n, \ell, p, q)$ is defined in (9). Note that the blow-up condition on p, q in (8) is equivalent to require $\theta(n, \ell, p, q) > 0$. Let us choose $\varepsilon_0 > 0$ satisfying

$$\varepsilon_0^{-\frac{p(q-1)}{\theta(n, \ell, p, q)}} \geq 2^{-\frac{(\alpha_0 + \beta_0)(q-1) + a + 2}{\theta(n, \ell, p, q)}} D^{\frac{q-1}{\theta(n, \ell, p, q)}} T_1.$$

So, for any $\varepsilon \in (0, \varepsilon_0]$ and for $t \geq 2^{-\frac{(\alpha_0 + \beta_0)(q-1) + a + 2}{\theta(n, \ell, p, q)}} D^{-\frac{q-1}{\theta(n, \ell, p, q)}} \varepsilon^{-\frac{p(q-1)}{\theta(n, \ell, p, q)}}$ it results

$$t \geq T_1 \quad \text{and} \quad 2^{-(\alpha_0 + \beta_0) - \frac{a+2}{q-1}} D \varepsilon^p t^{\frac{\theta(n, \ell, p, q)}{q-1}} > 1,$$

hence, letting $j \rightarrow \infty$ in (45) it turns out that $U(t)$ blows up in finite time. Therefore, we proved the blowing-up of U for any $\varepsilon \in (0, \varepsilon_0]$ whenever (8) holds and, besides, as byproduct we established the upper bound estimate for the lifespan $T(\varepsilon) \lesssim \varepsilon^{-\frac{p(q-1)}{\theta(n, \ell, p, q)}}$.

3. Lifespan estimates for the power nonlinearity

Let us consider the Cauchy problem

$$\begin{cases} \partial_t^2 u - t^{2\ell} \Delta u = a|\partial_t u|^p + b|u|^q, & x \in \mathbb{R}^n, t > 0, \\ u(0, x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ \partial_t u(0, x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{47}$$

for $a, b \in \{0, 1\}$.

In Section 2, we proved Theorem 1.2 by using an iteration argument, whose first lower bound estimate for U is given by (35). In particular, we combined the influence of the nonlinearity of derivative type $|\partial_t u|^p$, provided by the lower bound estimate (35), with the iteration frame in (36), which has been derived thanks to the presence of the power nonlinearity $|u|^q$ on the right-hand side of our semilinear model.

On the other hand, as we pointed out in Remark 1, we know also other two lower bound estimates for U , given by (37) and (38), respectively (let us recall that, in order to derive (38), we need to require u_1 nonnegative and nontrivial). However, if we do employ one among these first lower bounds for U , somehow we neglect the effect $|\partial_t u|^p$ on the equation, obtaining the same blow-up result for the Tricomi equation with power nonlinearity $|u|^q$ from [1,18] in the subcritical case.

By using (37) in place of (35) and applying the machinery from Section 2.4, we find the following result which coincides with Theorem 1.2 in [18] for (47) when $a = 0, b = 1$.

Corollary 3.1. *Let us consider $n \geq 2$ and $p, q > 1$ satisfying*

$$1 < q < p_0(n; \ell),$$

where $p_0(n; \ell)$ is the positive root of the quadratic equation in (3). Let us assume that $u_0, u_1 \in L^1_{\text{loc}}(\mathbb{R}^n)$ are nonnegative, not both trivial and compactly supported functions with supports contained in B_R for some $R > 0$. Let

$$u \in \mathcal{C}([0, T], W^{1,1}_{\text{loc}}(\mathbb{R}^n)) \cap \mathcal{C}^1([0, T], L^1_{\text{loc}}(\mathbb{R}^n)) \cap L^q_{\text{loc}}((0, T) \times \mathbb{R}^n) \quad \text{such that} \quad \partial_t u \in L^p_{\text{loc}}((0, T) \times \mathbb{R}^n)$$

be a weak solution to (1) according to Definition 1.1 with lifespan $T = T(\varepsilon)$.

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(n, \ell, q, u_0, u_1, R)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the weak solution u blows up in finite time. Furthermore, the upper bound estimate for the lifespan

$$T(\varepsilon) \leq C \varepsilon^{-\frac{q(q-1)}{\gamma(n, \ell, q)}} \tag{48}$$

holds, where the positive constant C is independent of ε and

$$\gamma(n, \ell, q) \doteq (\ell + 1) + \frac{1}{2}((\ell + 1)n + 1 - 2\ell)q - \frac{1}{2}((\ell + 1)n - 1)q^2. \tag{49}$$

Using (38) rather than (37), we can prove a blow-up result for (1) for $1 < q < p_{\text{Kat}}((\ell + 1)n)$, where $p_{\text{Kat}}(n) = \frac{n+1}{n-1}$. This result provides actually an improvement of the upper bound for q in the blow-up range only in the one dimensional case. More precisely, we get the following result, which is already known in the literature (see [4, Remark 1.6]) for (47) when $a = 0, b = 1$.

Corollary 3.2. *Let us consider $n = 1$ and $p, q > 1$ satisfying*

$$1 < q < 1 + \frac{2}{\ell} = p_{\text{Kat}}(\ell + 1).$$

Let us assume that $u_0, u_1 \in L^1_{\text{loc}}(\mathbb{R})$ are nonnegative and compactly supported functions with supports contained in $[-R, R]$ for some $R > 0$. Additionally, we require that

$$\int_{\mathbb{R}} u_1(x) \, dx > 0.$$

Let $u \in \mathcal{C}([0, T), W^{1,1}_{\text{loc}}(\mathbb{R})) \cap \mathcal{C}^1([0, T), L^1_{\text{loc}}(\mathbb{R})) \cap L^q_{\text{loc}}((0, T) \times \mathbb{R})$ such that $\partial_t u \in L^p_{\text{loc}}((0, T) \times \mathbb{R})$ be a weak solution to (1) according to Definition 1.1 with lifespan $T = T(\varepsilon)$. Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(\ell, q, u_0, u_1, R)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the weak solution u blows up in finite time. Furthermore, the upper bound estimate for the lifespan

$$T(\varepsilon) \leq C\varepsilon^{-\left(\frac{q+1}{q-1} - (\ell+1)\right)^{-1}} \tag{50}$$

holds, where the positive constant C is independent of ε .

Besides, the upper bound for the lifespan in (48) can be improved for $n = 2$ for certain exponents q , provided that the integral of u_1 over the whole space is a positive quantity.

Corollary 3.3. *Let $n = 2$ and let us consider $p, q > 1$ satisfying*

$$1 < q < \frac{2(\ell + 1)}{2\ell + 1}.$$

Let us assume that $u_0, u_1 \in L^1_{\text{loc}}(\mathbb{R}^2)$ are nonnegative and compactly supported functions with supports contained in B_R for some $R > 0$. Additionally, we require that

$$\int_{\mathbb{R}^2} u_1(x) \, dx > 0.$$

Let $u \in \mathcal{C}([0, T), W^{1,1}_{\text{loc}}(\mathbb{R}^2)) \cap \mathcal{C}^1([0, T), L^1_{\text{loc}}(\mathbb{R}^2)) \cap L^q_{\text{loc}}((0, T) \times \mathbb{R}^2)$ such that $\partial_t u \in L^p_{\text{loc}}((0, T) \times \mathbb{R}^2)$ be a weak solution to (1) according to Definition 1.1 with lifespan $T = T(\varepsilon)$.

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(\ell, q, u_0, u_1, R)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the weak solution u blows up in finite time. Furthermore, the upper bound estimate for the lifespan

$$T(\varepsilon) \leq C\varepsilon^{-\left(\frac{q+1}{q-1} - 2(\ell+1)\right)^{-1}} \tag{51}$$

holds, where the positive constant C is independent of ε .

The improvement of the upper bound for the lifespan in the low dimensional case $n = 1$ and $n = 2$ from Corollaries 3.2–3.3 has already been observed for the case of the semilinear wave equation (see [19] and [20,21] for the semilinear damped wave equation in the scattering producing case).

Remark 2. For $n \geq 3$ we cannot improve the upper bound estimate in (48). As we have already pointed out, employing (38) rather than (37), we get a blow-up result for (1) provided that $1 < q < p_{\text{Kat}}((\ell + 1)n)$. Moreover, the next upper bound estimate for the lifespan can be proved:

$$T(\varepsilon) \leq C\varepsilon^{-\left(\frac{q+1}{q-1} - (\ell+1)n\right)^{-1}}.$$

It turns out that this upper bound for $T(\varepsilon)$ improves the one in (48) for

$$q < \frac{2(\ell + 1)}{(\ell + 1)n - 1}.$$

However, $2(\ell + 1)/((\ell + 1)n - 1) < 1$ for $n \geq 3$ and any $\ell > 0$, therefore, it makes sense to talk about an improvement of the lifespan estimate in (48) just in space dimensions $n = 1, 2$.

Remark 3. In this section, even though we formally got results for $a = b = 1$ in (47), we followed the approach applied to study the case with power nonlinearity, that is, for $a = 0, b = 1$. In particular, in this last case it is known that the results from Corollaries 3.1 and 3.2 are sharp (see [2] for $n \geq 3$ and [4] for $n = 1$ in the special case $\ell = \frac{1}{2}$).

4. Analysis of the obtained results

Let us define the following region:

$$\Gamma(n, \ell) \doteq \left\{ (p, q) \in (1, \infty)^2 : \left[((\ell + 1)n - 1)p - 2\ell(p - 1) - 2 \right] (q - 1) < 4 \right\}.$$

According to our main result, we proved the blow-up of local in time weak solutions to (1) under suitable sign assumptions for the Cauchy data provided that $(p, q) \in \Gamma(n, \ell)$. We observe that, given $n \geq 2$, $\{\Gamma(n, \ell)\}_{\ell \geq 0}$ is a family of sets decreasing by inclusion, so the smaller ℓ the wider $\Gamma(n, \ell)$.

We begin by pointing out that for $\ell = 0$, namely, for the case of the classical semilinear wave equation with combined nonlinearity, our result coincides exactly with the sharp result obtained by [7,8].

We focus now on the case $n \geq 2$ (we will investigate separately later the case $n = 1$). In this case, we remark that the range $\Gamma(n, \ell)$ is not fully contained in the strips

$$\{(p, q) \in (1, \infty)^2 : p < p_1(n; \ell) \text{ or } q < p_0(n; \ell)\},$$

delimited by $p = p_1(n; \ell)$, $q = p_0(n; \ell)$, which intersect in $S \doteq (p_1(n; \ell), p_0(n; \ell))$, see Fig. 1. In other words, thanks to the combined presence of a power nonlinearity and of a nonlinearity of derivative type on the right-hand side of (1), we find an enlargement of the blow-up range in comparison to the ranges for the corresponding semilinear models with either a nonlinearity of power type or a nonlinearity of derivative type. Let us consider for example the special case $p = q$. The intersection of $\partial\Gamma(n, \ell)$ with the diagonal yields the point $(p_{\text{diag}}(n; \ell), p_{\text{diag}}(n; \ell))$, where $p_{\text{diag}}(n; \ell)$ is the positive solution to the quadratic equation

$$((\ell + 1)n - 1 - 2\ell)p^2 - ((\ell + 1)n + 1 - 4\ell)p - 2(\ell + 1) = 0. \tag{52}$$

By straightforward computations it follows that $p_1(n; \ell) < p_0(n; \ell) < p_{\text{diag}}(n; \ell)$ for any $\ell > 0$ and $n \geq 2$. Therefore, already on the diagonal we are able to prove a blow-up result for (1) for pairs (p, q) satisfying $p > p_1(n; \ell)$ and $q > p_0(n; \ell)$. Since for $\ell = 0$ it holds $p_0(n; 0) = p_{\text{diag}}(n; 0) = p_{\text{Str}}(n)$, we notice that this is a completely new phenomenon in comparison to the case of the classical semilinear wave equation with the same kind of nonlinear term.

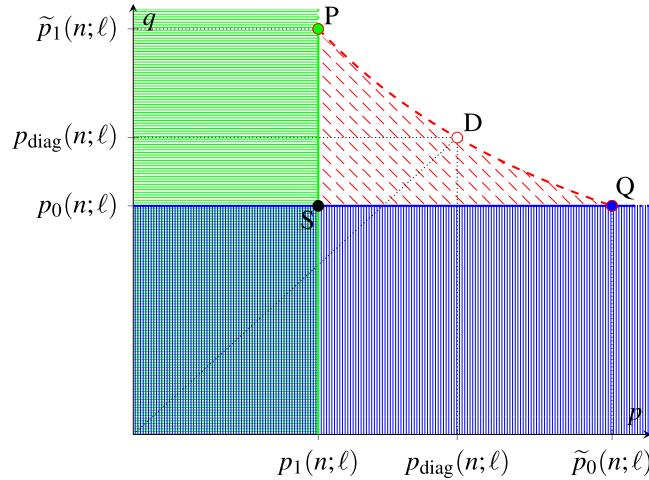


Fig. 1. Blow-up range for (1): case $n \geq 2$.

Remark 4. Let us remark that $p_{\text{diag}}(n; \ell)$ belongs somehow to the family of *Strauss-type* exponents. The Strauss exponent $p_{\text{Str}}(n)$ is the greatest root of the quadratic equation

$$(n - 1)(p - 1)^2 + (n - 3)(p - 1) - 4 = 0, \tag{53}$$

and it is the critical exponent for the semilinear wave equation with power nonlinearity. We are interested here in quadratic equations of the following kind:

$$\alpha(p - 1)^2 + \beta(p - 1) - 4 = 0 \tag{54}$$

or, equivalently,

$$\alpha(p - 1) = \alpha - \beta + \frac{4 - (\alpha - \beta)}{p}.$$

The previous relation emphasizes the importance of the quantities α and $\alpha - \beta$. Over the last years, several papers have been devoted to the treatment of semilinear wave models with critical exponents that are translation shifts of $p_{\text{Str}}(n)$ in the dimensional parameter (see [22–31] for models with time-dependent coefficients and [32,33] for models with space-dependent coefficients). Typically, these models present lower order terms with critical decay rates, beyond the principal part of the operator, which is the wave operator $\partial_t^2 - \Delta$, and in all cases the relation $\alpha - \beta = 2$ is satisfied as in (54). If we consider a generalized Tricomi operator $\partial_t^2 - t^{2\ell} \Delta$ instead of the wave operator, the quadratic equation that reveals $p_0(n; \ell)$ is

$$((\ell + 1)n - 1)(p - 1)^2 + ((\ell + 1)n - 3 + 2\ell)(p - 1) - 4 = 0. \tag{55}$$

In the last equation, we notice a rescaling in α due to the shape of the light-cone. Moreover, $\alpha - \beta = 2(1 - \ell)$. If we rewrite (52) in the following way:

$$((\ell + 1)n - 1 - 2\ell)(p - 1)^2 + ((\ell + 1)n - 3)(p - 1) - 4 = 0, \tag{56}$$

we observe that $\alpha - \beta = 2(1 - \ell)$ also in this case, so that, $p_{\text{diag}}(n; \ell)$ is obtained by $p_0(n; \ell)$ through a shift of magnitude -2ℓ in the coefficients α and β . In this sense, $p_{\text{diag}}(n; \ell)$ belongs to the family of Strauss-type exponents.

In Table 1, we summarize the values of parameters α , β , $\alpha - \beta$ for (54) related to the previously discussed semilinear hyperbolic equations.

On the other hand, taking the values of α in Table 1, by the relation $\frac{\alpha}{2}(p - 1) = 1$, we find the Glassey-type exponents $p_{\text{Gla}}(n)$, $p_{\text{Gla}}((\ell + 1)n)$ and $p_{\text{Gla}}((\ell + 1)n - 2\ell)$ for wave and generalized Tricomi operators (see Remark 5).

Table 1
Values of p , α , β and $\alpha - \beta$ in the quadratic equation (54) for different PDEs.

Semilinear PDE	Critical p	α	β	$\alpha - \beta$
$(\partial_t^2 - \Delta)u = u ^p$	$p_{\text{Str}}(n)$	$n - 1$	$n - 3$	2
$(\partial_t^2 - t^{2\ell}\Delta)u = u ^p$	$p_0(n; \ell)$	$(\ell + 1)n - 1$	$(\ell + 1)n - 3 + 2\ell$	$2(1 - \ell)$
$(\partial_t^2 - t^{2\ell}\Delta)u = \partial_t u ^p + u ^p$	$p_{\text{diag}}(n; \ell)$	$(\ell + 1)n - 1 - 2\ell$	$(\ell + 1)n - 3$	$2(1 - \ell)$

Let us determine now the intersection of the boundary of the blow-up region, i.e. $\partial\Gamma(n, \ell)$, with the straight lines with equations $q = p_0(n; \ell)$ and $p = p_1(n; \ell)$. We get started with the intersection of $\partial\Gamma(n, \ell)$ with the horizontal line. Denoting $Q = (\tilde{p}_0(n; \ell), p_0(n; \ell))$ the point such that

$$\partial\Gamma(n, \ell) \cap \{q = p_0(n; \ell)\} = \{Q\}$$

in the (p, q) -plane, if we combine $[((\ell + 1)n - 1)p - 2\ell(p - 1) - 2](q - 1) = 4$ and (55), then, we obtain that $\tilde{p}_0(n; \ell)$ is the greatest root of the quadratic equation

$$((\ell + 1)n - 1 - 2\ell)(p - 1)^2 + ((\ell + 1)n - 3 - 2\ell)(p - 1) - 2(\ell + 2) - \frac{4\ell(\ell + 1)}{(\ell + 1)n - 1 - 2\ell} = 0.$$

We underline explicitly that

$$\tilde{p}_0(n; 0) = p_{\text{Str}}(n) = p_{\text{diag}}(n; 0) = p_0(n; 0).$$

So, denoting $D \doteq (p_{\text{diag}}(n; \ell), p_{\text{diag}}(n; \ell))$, we find that $Q, D \rightarrow (p_{\text{Str}}(n), p_{\text{Str}}(n))$ as $\ell \rightarrow 0$. The fact that Q, D collapse on the same point as $\ell \rightarrow 0$ allows us to understand better the above pointed out phenomenon of the enlargement for $\ell > 0$ of the blow-up range on the diagonal. Indeed, for the semilinear Cauchy problem

$$\begin{cases} \partial_t^2 u - t^{2\ell}\Delta u = |\partial_t u|^p + |u|^p, & x \in \mathbb{R}^n, t > 0, \\ u(0, x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ \partial_t u(0, x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases}$$

the blow-up range is extended up to the wider range $1 < p < p_{\text{diag}}(n; \ell)$, in comparison to the corresponding semilinear Cauchy problems with either power nonlinearity $|u|^p$ or nonlinearity of derivative type $|\partial_t u|^p$. We recall that for $\ell = 0$ this circumstance does not occur.

We may rewrite $\partial\Gamma(n, \ell) = \{p > 1 : q = f(p; n, \ell)\}$ with

$$f(p; n, \ell) \doteq 1 + \frac{4}{((\ell + 1)n - 1 - 2\ell)p + 2\ell - 2}. \tag{57}$$

Using that the function $f = f(p; n, \ell)$ in (57) is strictly decreasing with respect to p , the relation $p_0(n; \ell) < p_{\text{diag}}(n; \ell)$ and that the point D is given by the intersection of $\partial\Gamma(n, \ell)$ with the diagonal $\{p = q\}$, we get $p_{\text{diag}}(n; \ell) < \tilde{p}_0(n; \ell)$.

Next we study the intersection of $\partial\Gamma(n, \ell)$ with the vertical line $p = p_1(n; \ell)$. Neglecting the influence of the power nonlinearity $|u|^q$ and following [5], a blow-up result for $1 < p \leq p_1(n; \ell)$ can be shown in the case with combined nonlinearity too.

For $p = p_1(n; \ell)$ on $\partial\Gamma(n, \ell)$ we find

$$q = \tilde{p}_1(n; \ell) \doteq \frac{((\ell + 1)n + 3)((\ell + 1)n - 1) - 4\ell}{((\ell + 1)n - 1)^2 - 4\ell}.$$

Since the function $f = f(p; n, \ell)$ in (57) is strictly decreasing with respect to p , D is given by the intersection of $\partial\Gamma(n, \ell)$ with the diagonal $\{p = q\}$ and $p_1(n; \ell) < p_{\text{diag}}(n; \ell)$, it follows $p_{\text{diag}}(n; \ell) < \tilde{p}_1(n; \ell)$.

Hence, if we denote by P the point in the (p, q) -plane such that

$$\partial\Gamma(n, \ell) \cap \{p = p_1(n; \ell)\} = \{P\},$$

then, from left to right we have the sorting P, D, Q on the branch of the hyperbola given by $\partial\Gamma(n, \ell)$. In other words, we obtained

$$p_1(n; \ell) < p_{\text{diag}}(n; \ell) < \tilde{p}_0(n; \ell),$$

$$p_0(n; \ell) < p_{\text{diag}}(n; \ell) < \tilde{p}_1(n; \ell).$$

Summarizing, we extend the blow-up region for (1) in the (p, q) -plane to the curvilinear triangle PQS. In Fig. 1 we collect all properties that we discussed on the blow-up region for (1) in this section.

Remark 5. Let us provide a wider overview on blow-up results for the semilinear generalized Tricomi equation with nonlinearity of derivative type. In [6] the authors found $p_{\text{Gla}}((\ell + 1)n - 2\ell)$ as upper bound for p in the blow-up range for $n \geq 2$. For combined nonlinearity, neglecting $|u|^q$ one deduces a blow-up result assuming $1 < p \leq p_{\text{Gla}}((\ell + 1)n - 2\ell)$. Plugging $p = p_{\text{Gla}}((\ell + 1)n - 2\ell)$ into the equation for $\partial\Gamma(n, \ell)$, we get $q = p_{\text{conf}}((\ell + 1)n)$, where $p_{\text{conf}}(n) \doteq \frac{n+3}{n-1}$ is the *conformal exponent* for the classical semilinear wave equation (cf. [34]).

Moreover, collecting the upper bound estimates from Theorem 1.2, Section 3 and [5,18] we have

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\frac{p(q-1)}{\theta(n,\ell,p,q)}} & \text{if } (p, q) \in \Gamma(n, \ell); \\ C\varepsilon^{-\frac{q(q-1)}{\gamma(n,\ell,q)}} & \text{if } 1 < q < p_0(n; \ell) \text{ and } n \geq 2; \\ C\varepsilon^{-\left(\frac{q+1}{q-1} - (\ell+1)n\right)^{-1}} & \text{if } 1 < q < p_{\text{Kat}}((\ell + 1)n) \text{ and } n = 1, 2; \\ C\varepsilon^{-\left(\frac{1}{p-1} - \frac{(\ell+1)n-1}{2}\right)^{-1}} & \text{if } 1 < p < p_{\text{Gla}}((\ell + 1)n); \\ \exp(C\varepsilon^{-q(q-1)}) & \text{if } q = p_0(n; \ell); \\ \exp(C\varepsilon^{-(p-1)}) & \text{if } p = p_{\text{Gla}}((\ell + 1)n). \end{cases}$$

For the lifespan estimate in the critical case $q = p_0(n; \ell)$ we refer to [18, Theorem 1.3], while for the upper bound estimates in the case $1 < p \leq p_{\text{Gla}}((\ell + 1)n)$ we address the interested reader to [5, Theorem 1.1].

Finally, let us consider more in detail the case $n = 1$. In the one dimensional case the blow-up condition (8) is

$$(-\ell p + 2\ell - 2)(q - 1) < 4. \tag{58}$$

The main difference in comparison to the higher dimensional cases is that the constant that multiplies p in (58) is negative, modifying substantially the shape of the blow-up region. We will consider separately the cases $\ell \in (0, 2]$, $\ell \in (2, 4]$ and $\ell \in (4, \infty)$.

For the semilinear wave equations ($\ell = 0$) with power nonlinearity $|u|^q$ and with nonlinearity of derivative type $|\partial_t u|^p$, respectively, it is well-known in the literature (cf. [15,35]) that for $n = 1$ blow-up results for local in time solutions hold for any $q > 1$ and any $p > 1$, respectively. Therefore, for the Cauchy problem associated with semilinear wave equation with combined nonlinearity in the 1-d case $\partial_t^2 u - \partial_x^2 u = |\partial_t u|^p + |u|^q$ for any $p, q > 1$ it is possible to prove the blow-up in finite time of the corresponding local in time solution, provided that suitable sign assumptions are required for the Cauchy data. In the first two subcases, namely for $\ell \in (0, 4]$, also for (1) the same situation occurs as for the corresponding wave equation ($\ell = 0$), that is, the blow-up range coincides with the entire set $\{p, q > 1\}$. However, in order to prove this fact we need to distinguish the case $\ell \in (0, 2]$ from the case $\ell \in (2, 4]$.

For $\ell \in (0, 2]$ it holds

$$\{(p, q) \in \mathbb{R}^2 : p, q > 1\} \subset \{(p, q) \in \mathbb{R}^2 : (-\ell p + 2\ell - 2)(q - 1) < 4\},$$

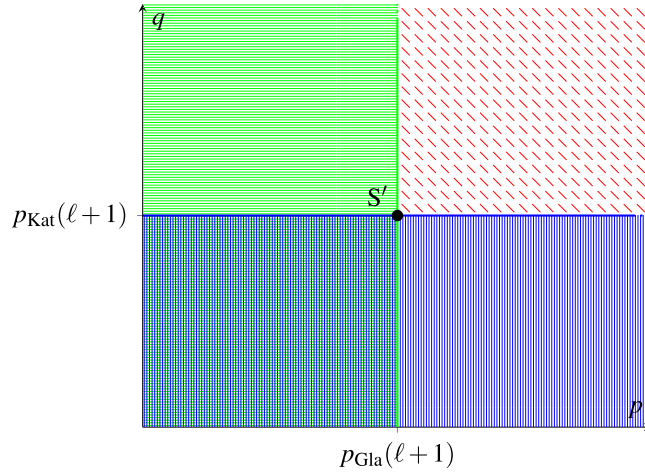


Fig. 2. Blow-up range for (1): case $n = 1, \ell \in (0, 2]$.

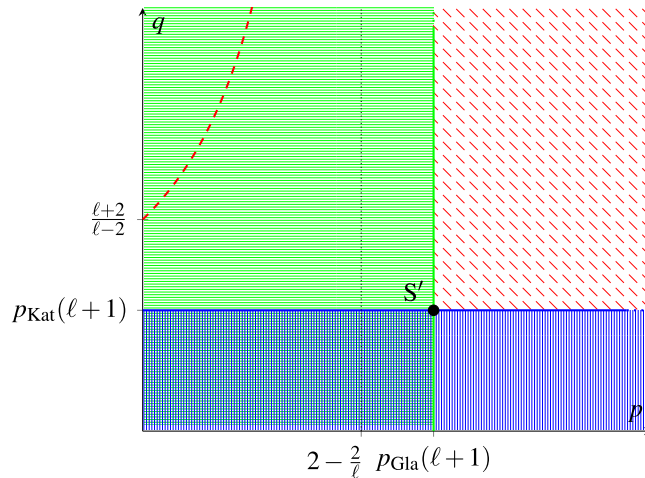


Fig. 3. Blow-up range for (1): case $n = 1, \ell \in (2, 4]$.

so it follows immediately that the blow-up range from [Theorem 1.2](#) is the full set $\{p, q > 1\}$, covering the case $p > p_{\text{Gla}}(\ell + 1)$ and $q > p_{\text{Kat}}(\ell + 1)$ as well (see [Fig. 2](#)). We point out that, as in the one dimensional case the critical exponent for the semilinear generalized Tricomi equation with power nonlinearity is $p_{\text{Kat}}(\ell + 1)$, we consider the point $S' \doteq (p_{\text{Gla}}(\ell + 1), p_{\text{Kat}}(\ell + 1))$ in place of S in the figures of the blow-up regions for $n = 1$.

On the other hand, for $\ell \in (2, 4]$ the hyperbola

$$\{(p, q) \in \mathbb{R}^2 : (-\ell p + 2\ell - 2)(q - 1) = 4\}$$

has asymptotes $q = 1$ and $p = 2 - \frac{2}{\ell} > 1$. Hence, the condition in [\(58\)](#) does not cover the entire set $\{p, q > 1\}$. Nevertheless, $2 - \frac{2}{\ell} < 1 + \frac{2}{\ell} = p_{\text{Gla}}(\ell + 1)$, so, using the blow-up result for $1 < p \leq p_{\text{Gla}}(\ell + 1)$ that can be proved by working just with the nonlinearity of derivative type (see [\[5, Theorem 1.1\]](#)), we close the gap left by [Theorem 1.2](#) (cf. [Fig. 3](#)).

Finally, for $\ell \in (4, \infty)$, even combining the results of [Theorem 1.2](#) and [\[5, Theorem 1.1\]](#), we do not obtain $\{p, q > 1\}$ as blow-up range for (1). More precisely, we have to exclude the region

$$\{(p, q) \in (1, \infty)^2 : p > p_{\text{Gla}}(\ell + 1) \text{ and } (-\ell p + 2\ell - 2)(q - 1) \geq 4\},$$

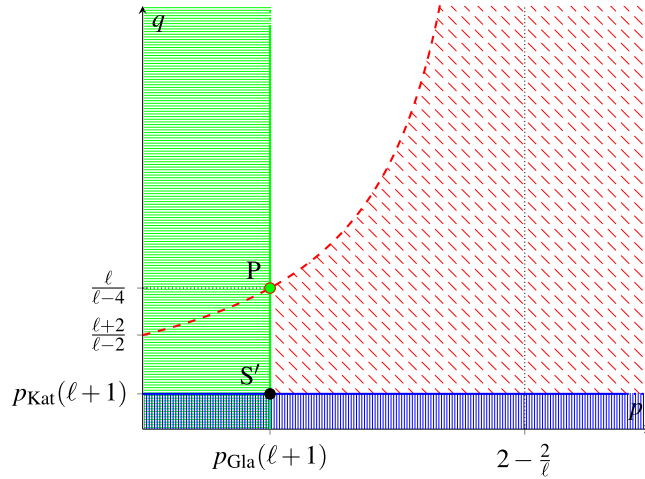


Fig. 4. Blow-up range for (1): case $n = 1$, $\ell \in (4, \infty)$.

contained in the strip $\{p_{\text{Gla}}(\ell + 1) < p < 2 - \frac{2}{\ell}\}$, from the blow-up range, since in this case the asymptote $p = 2 - \frac{2}{\ell}$ lies to the right of the critical value $p = p_{\text{Gla}}(\ell + 1)$ (see Fig. 4).

Final note. After the preparation of the final version of the present manuscript, the authors found out the existence of the preprint [36], where the same blow-up result for (1) has been obtained independently.

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Appendix. Properties of the modified Bessel function of the second kind

In this appendix, we recall the main properties of modified Bessel functions of the second kind that we employ throughout the paper. For further details we refer to [37, Chapter 10].

The modified Bessel function of the second kind of order ν satisfies the second-order linear differential equation with polynomial coefficients

$$\tau^2 K''_{\nu}(\tau) + \tau K'_{\nu}(\tau) - (\nu^2 + \tau^2) K_{\nu}(\tau) = 0. \tag{A.1}$$

The derivative of the modified Bessel function of the second kind can be expressed through the following recursive relations:

$$\begin{aligned} K'_{\nu}(\tau) &= -K_{\nu-1}(\tau) - \frac{\nu}{\tau} K_{\nu}(\tau), \\ K'_{\nu}(\tau) &= -K_{\nu+1}(\tau) + \frac{\nu}{\tau} K_{\nu}(\tau), \\ K'_{\nu}(\tau) &= -\frac{1}{2} (K_{\nu-1}(\tau) + K_{\nu+1}(\tau)). \end{aligned} \tag{A.2}$$

Finally, the following asymptotic estimates for K_ν hold for small and large argument, respectively,

$$K_\nu(\tau) \sim 2^{\nu-1} \Gamma(\nu) \tau^{-\nu} \quad \text{as } \tau \rightarrow 0^+, \text{ for } \Re\nu > 0, \quad (\text{A.3})$$

$$K_\nu(\tau) = \sqrt{\frac{\pi}{2\tau}} e^{-\tau} (1 + O(\tau^{-1})) \quad \text{as } \tau \rightarrow \infty. \quad (\text{A.4})$$

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