AN ANALYSIS OF SOLUTIONS TO FRACTIONAL NEUTRAL DIFFERENTIAL EQUATIONS WITH DELAY

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Abstract. This paper discusses some properties of solutions to fractional neutral delay differential equations. By combining a new weighted norm, the Banach fixed point theorem and an elegant technique for extending solutions, results on existence, uniqueness, and growth rate of global solutions under a mild Lipschitz continuous condition of the vector field are first established. Be means of the Laplace transform the solution of some delay fractional neutral differential equations are derived in terms of three-parameter Mittag-Leffler functions; their stability properties are hence studied by using use Rouché's theorem to describe the position of poles of the characteristic polynomials and the final value theorem to detect the asymptotic behavior. By means of numerical simulations the theoretical findings on the asymptotic behavior are verified.

1. Introduction

Delay differential equations play an important role in describing various phenomena in biosciences, chemistry, economics, engineer, and physics. Very ofter, indeed, transport and propagation (of material, energy, or information) in interconnected systems does not happen instantaneously but it is affected by some delay which suggests the use of delay differential equations for modeling these phenomena. For theory, applications and numerical methods of delay differential equations we refer to [1, 2, 3, 4] and references therein.

Recently, fractional-order systems with delay attracted considerable research attention because they allows to describe systems in which the rate of change depends not only on the present and delayed state but also on the whole past memory. In [5], by the final value theorem for

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Key words and phrases. Fractional derivative, Fractional neutral differential equation with delay, Existence and uniqueness of solutions, Exponential boundedness, Stability, Numerical simulation.

Accepted version of the paper published in [Commun. Nonlinear Sci. Numer. Simul., 2021, 105854] and available at https://doi.org/10.1016/j.cnsns.2021.105854. The work of HTT and HDT is supported by The International Center for Research and Postgraduate Training in Mathematics–Institute of Mathematics–Vietnam Academy of Science and Technology under the Grant ICRTM01-2020.09 and under the Grant ICRTM03-2021.02 respectively. The work of R.Garrappa is partially supported by a GNCS-INdAM 2020 Project.

Laplace transforms, the well-known method of steps, and the Argument principle, the authors have studied stability of delayed fractional-order differential equations of linear type.

In [6], the authors have obtained general results on the existence, uniqueness and growth rate of solutions to fractional-order systems with delays based on the Banach fixed point theorem and a weighted norm. In [7] necessary and sufficient condition for stability have been provided by studying the location of the eigenvalues of the system matrix. By the linearization method and generalized Mittag-Leffler functions, in [8, 9] the authors have proved the stability of nonlinear fractional-order delay systems. Furthermore, using Lyapunov applicant functional, in [10] the authors also obtained a sufficient condition for stability. In [11], the problem of the correct initialization of fractional delay differential equations has been investigated.

In delay differential equations of neutral type the derivative of the unknown function appears with and without delays and therefore their analysis is more involved than in the non-neutral counterpart. These systems find application in the description of a number of physical systems and/or phenomena, ranging from the motion of radiation electrons, to population growth, spread of epidemics, networks with loss-less transmission lines, and others (see, *e.g.*,[4, 12, 13, 14] and references therein). To the best of our knowledge, only few works concern with fractional neutral delay differential equation (FNDDEs) and below we briefly review the main contributions to this topic.

In [15], based on Krasnoselskii's fixed point theorem, the authors proved the existence of at least one solution to a class of FNDDEs with bounded delay. The existence of mild solutions for a class of abstract fractional neutral integro-differential equations with state-dependent delay is studied in [16] by the Leray-Schauder alternative fixed point theorem. Recently, in [17] a new inequality of Halanay type was derived to describe the behavior of solutions of FNDDEs of Hale type and conditions for contractivity and dissipativity were established as well. In [18] the investigation concerned the robust stability of a class of FNDDE with uncertainty and input saturation. Recently, an analysis of finite-time stability for some nonlinear FNDDEs has been proposed in [19] .

This paper is devoted to discussing some qualitative properties of solutions to FNDDEs. The paper is organized as follows. In Section 2, we briefly recall some basic notations concerning fractional derivatives and fractional delay differential equations. In Section 3 we give a result on the existence and uniqueness of global solutions to FNDDEs and in Section 4 we prove their exponential boundedness. In Section 5 we derive an explicit representation, based on generalized three-parameter Mittag-Leffler functions, of solutions of linear FNDDEs. In Section 6 we discuss in details the stability of two classes of linear FNDDEs and some numerical simulations are presented in Section 7 to illustrate the main theoretical results obtained in the paper.

2. Preliminaries

In this section we recall some definitions and a result on the integral representation of solutions of FNDDEs which will be used in the sequel. For $0 < \alpha < 1$, $[a, b] \subset \mathbb{R}$ and a measurable function $x : [a, b] \to \mathbb{R}$ such that $\int_a^b |x(\tau)| d\tau < \infty$, the *Riemann-Liouville (RL) integral* of order α is defined by

$$I_{a+}^{\alpha} x(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s) \mathrm{d}s, \quad t \in (a, b),$$

where $\Gamma(\cdot)$ is the Gamma function. The *Riemann–Liouville fractional derivative* ${}^{\mathrm{RL}}D_{a+}^{\alpha} x$ of a integrable function $x : [a, b] \to \mathbb{R}$ is defined by

$${}^{\mathrm{RL}}D_{a+}^{\alpha} x(t) = D I_{a+}^{1-\alpha} x(t) \text{ for almost } t \in (a, b],$$

with $D = \mathrm{d}/\mathrm{d}t$ the usual integer-order derivative. The *Caputo fractional derivative* ${}^{\mathrm{C}}D_{a+}^{\alpha} x$ of a continuous function $x : [a, b] \to \mathbb{R}$ is defined as

$${}^{\mathrm{C}}D_{a+}^{\alpha} x(t) := {}^{\mathrm{RL}}D_{a+}^{\alpha} (x(t) - x(a)) \text{ for almost } t \in (a, b].$$

For more details on fractional calculus, we would like to introduce the reader to the monographs [20, 21, 22] and to the interesting work by G. Vainikko [23]. Let τ and N be arbitrary real constants such that $\tau > 0, N \neq 0$, and $\phi \in C^1\big([-\tau, 0]; \mathbb{R}\big)$ be a given function. In this paper we consider the following FNDDE

$$ {}^{\mathrm{C}}D^{\alpha}_{0+}\big[x(t) + Nx(t-\tau)\big] = f(t, x(t), x(t-\tau)), \quad t \geq 0, \tag{1} $$

subject to the initial condition

$$ x(t) = \phi(t), \quad \forall t \in [-\tau, 0], \tag{2} $$

where $x : [0, \infty) \to \mathbb{R}$ is a unknown function and $f : [0, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. To prove the existence of solutions to problem (1)–(2), we need to reformulate it into an equivalent delay integral equation. This is stated in the following lemma.

Lemma 2.1. A function $x \in C\big([-\tau, \infty); \mathbb{R}\big)$ is solution of the problem (1)-(2) on $[-\tau, \infty)$ if and only if it is solution of the delay integral equation

$$ \begin{aligned} x(t) &= \phi(0) + N\phi(-\tau) - Nx(t-\tau) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x(s-\tau))\mathrm{d}s, \ \ \forall t \in [0, \infty), \end{aligned} \tag{3} $$

and satisfies $x(t) = \phi(t)$, $\forall t \in [-\tau, 0]$.

Proof. The proof of this lemma is similar to the one of [20, Lemma 6.2] (see also [24]) and therefore it is omitted. □

3. Existence and uniqueness of global solutions of FNDDEs

Let $T > 0$ be arbitrary and consider, on the finite interval $[-\tau, T]$, the initial value problem

$$ \begin{aligned} {}^{\mathrm{C}}D^{\alpha}_{0+}\big[x(t) + Nx(t-\tau)\big] &= f(t, x(t), x(t-\tau)), \qquad & t \in (0, T], \\ x(t) &= \phi(t), & t \in [-\tau, 0]. \end{aligned} \tag{4} $$

Here $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ it is assumed to satisfy the following assumptions:

- (A1) f is continuous on $[0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$;
- (A2) there exists a continuous function $L : [0, T] \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that for any $t \in [0, T]$, $x, \hat{x}, y \in R$, it is $|f(t, x, y) - f(t, \hat{x}, y)| \leq L(t, y)|x - \hat{x}|$.

By proposing a new weighted norm and modifying the approach in the proof of [6, Theorem 3.1], we are able to obtain the following result on the existence and uniqueness of a global solution to (4).

Theorem 3.1. Assume that conditions (A1) and (A2) hold. Then, the FNDDE (4) has a unique solution on the interval $[-\tau, T]$.

Proof. We first consider the case $0 < T \leq \tau$. In this case, the integral representation (3) of the solution becomes

$$ x(t) = \phi(0) + N\phi(-\tau) - N\phi(t-\tau) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), \phi(s-\tau))\mathrm{d}s $$

for $t \in [0, T]$. Let $\beta := \max_{t\in[0,T]} L(t, \phi(t-\tau))$ and λ be a large positive constant which will be chosen later. On the space $C([0, \tau]; \mathbb{R})$, we define the metric

$$ d_\lambda(\xi, \hat{\xi}) := \underset{t\in[0,r]}{sup} \frac{|\xi(t) - \hat{\xi}(t)|}{\mathrm{e}^{\lambda t}}, \quad \forall \xi, \hat{\xi} \in C([0, \tau]; \mathbb{R}). $$

It is obvious that $C([0,r];\mathbb{R})$ equipped with d_λ is complete. We now consider the operator $\mathcal{T}_\phi : C([0,\tau];\mathbb{R}) \to C([0,\tau];\mathbb{R})$ defined as

$$\begin{aligned}(\mathcal{T}_\phi\,\xi)(t) := \phi(0)+&N\phi(-\tau) - N\phi(t-\tau)\\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,\xi(s),\phi(s-\tau))\mathrm{d}s,\ \forall t\in[0,\tau].\end{aligned}$$

For any $\xi,\hat{\xi}\in C([0,r];\mathbb{R})$ and any $t\in[0,T]$ we have

$$\begin{aligned}|(\mathcal{T}_\phi\,\xi)(t)-(\mathcal{T}_\phi\,\hat{\xi})(t)| &\le \frac{\max_{s\in[0,t]} L(s,\phi(s-\tau))}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|\xi(s)-\hat{\xi}(s)|\mathrm{d}s\\ &\le \frac{\mathrm{e}^{\lambda t}\beta}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathrm{e}^{-\lambda(t-s)}\frac{|\xi(s)-\hat{\xi}(s)|}{\mathrm{e}^{\lambda s}}\mathrm{d}s\\ &\le \frac{\mathrm{e}^{\lambda t}\beta}{\lambda^\alpha} d_\lambda(\xi,\hat{\xi})\end{aligned}$$

from which we infer

$$\frac{|(\mathcal{T}_\phi\,\xi)(t)-(\mathcal{T}_\phi\,\hat{\xi})(t)|}{\mathrm{e}^{\lambda t}} \le \frac{\beta}{\lambda^\alpha} d_\lambda(\xi,\hat{\xi}),\ \ \forall t\in[0,T]$$

and hence

$$d_\lambda(\mathcal{T}_\phi\,\xi,\mathcal{T}_\phi\,\hat{\xi}) \le \frac{\beta}{\lambda^\alpha} d_\lambda(\xi,\hat{\xi}),\ \ \forall \xi,\hat{\xi}\in C([0,T];\mathbb{R}).$$

Take $\lambda > 0$ large enough, for example, $\lambda^\alpha > \beta$. Then, the operator $\mathcal{T}_\phi$ is contractive on $(C([0,T];\mathbb{R}), d_\lambda)$. By virtue of Banach fixed point theorem, there exists a unique fixed point $\xi^*_\tau(\cdot)$ of $\mathcal{T}_\phi$ in $C([0,T];\mathbb{R})$. The proof now follows by putting

$$\Phi_T(t,\phi) := \begin{cases}\phi(t) & \text{if } t\in[-\tau,0],\\ \xi^*_\tau(t) & \text{if } t\in[0,T].\end{cases}$$

which is clearly the unique solution of the problem (4) on $[-\tau,T]$.

For the case $T>\tau$, by exploiting an approach already proposed in [6], we split the interval $[0,T]$ into subintervals $[0,\tau]\cup\cdots\cup[(k-1)\tau,T]$, where $k\in\mathbb{N}$ satisfying $0\le T-k\tau<\tau$. The existence and uniqueness of solutions to (4) on $[-\tau,k\tau]$ will be showed by induction. Assume that (4) has a unique solution denoted by $\Phi_{\ell\tau}(\cdot)$ on $[-\tau,\ell\tau]$ with $\ell\in\mathbb{Z}_{\ge 0}$ and $0\le\ell<k$. On the space $C([\ell\tau,(\ell+1)\tau];\mathbb{R})$, let

$$\begin{aligned}\mathcal{T}_{(\ell+1)\tau}\xi(t) := \phi(0) &+ N\phi(-\tau) - N\Phi_{\ell\tau}(t-\tau)\\ &+\frac{1}{\Gamma(\alpha)}\int_0^{\ell\tau}(t-s)^{\alpha-1} f(s,\Phi_{\ell\tau}(s),\Phi_{\ell\tau}(s-\tau))\mathrm{d}s\\ &+\frac{1}{\Gamma(\alpha)}\int_{\ell\tau}^t (t-s)^{\alpha-1} f(s,\xi(s),\Phi_{\ell\tau}(s-\tau))\mathrm{d}s,\quad t\in[\ell\tau,(\ell+1)\tau].\end{aligned}$$

Consider $\beta_\ell := \max_{t\in[\ell\tau,(\ell+1)\tau]} L(t,\Phi_{\ell\tau}(t-\tau))$. Then,

$$\begin{aligned}|(\mathcal{T}_{(\ell+1)\tau}\xi)(t)-(\mathcal{T}_{(\ell+1)\tau}\hat{\xi})(t)| &\le \frac{\beta_\ell}{\Gamma(\alpha)}\int_{\ell\tau}^t (t-s)^{\alpha-1}|\xi(s)-\hat{\xi}(s)|\mathrm{d}s\\ &\le \frac{\mathrm{e}^{\lambda t}\beta_\ell}{\Gamma(\alpha)}\int_{\ell\tau}^t (t-s)^{\alpha-1}\mathrm{e}^{-\lambda(t-s)}\frac{|\xi(s)-\hat{\xi}(s)|}{\mathrm{e}^{\lambda s}}\mathrm{d}s\\ &\le \frac{\mathrm{e}^{\lambda t}\beta_\ell}{\lambda^\alpha} d_{\ell,\lambda}(\xi,\hat{\xi}),\ \ \forall t\in[\ell\tau,(\ell+1)\tau].\end{aligned}$$

Here, $d_{\ell,\lambda}(\xi,\hat{\xi}) := \max_{t\in[\ell\tau,(\ell+1)\tau]}\frac{|\xi(t)-\hat{\xi}(t)|}{\mathrm{e}^{\lambda t}}$ for any $\xi,\hat{\xi}\in C([\ell\tau,(\ell+1)\tau];\mathbb{R})$. If we choose $\lambda > \beta_\ell^{1/\alpha}$ the operator $\mathcal{T}_{(\ell+1)\tau}$ is contractive on the Banach space $(C([\ell\tau,(\ell+1)\tau];\mathbb{R}), d_{\ell,\lambda})$.

Hence, $\mathcal{T}_{(\ell+1)\tau}$ has a unique fixed point $\xi^*_{\ell\tau}$ in $C([\ell\tau,(\ell+1)\tau];\mathbb{R})$. Define now the function

$$\Phi_{(\ell+1)\tau}(t) := \begin{cases} \Phi_{\ell\tau}(t) & \text{if } t \in [-\tau, \ell\tau], \\ \xi^*_{\ell\tau}(t) & \text{if } t \in [\ell\tau, (\ell+1)\tau] \end{cases}$$

which is the unique solution of (4) on $[-\tau,(\ell+1)\tau]$. Finally, let $\Phi_{k\tau}(\cdot)$ be the unique solution to (4) on $[-\tau, k\tau]$. We construct an operator $\mathcal{T}_f$ on $C([k\tau, T];\mathbb{R})$ by

$$\begin{aligned} \mathcal{T}_f\xi(t) := {} & \phi(0) + N\phi(-\tau) - N\Phi_{k\tau}(t-\tau) \\ & + \frac{1}{\Gamma(\alpha)}\int_0^{k\tau}(t-s)^{\alpha-1} f(s,\Phi_{k\tau}(s),\Phi_{k\tau}(s-\tau))\mathrm{d}s \\ & + \frac{1}{\Gamma(\alpha)}\int_{k\tau}^{t}(t-s)^{\alpha-1} f(s,\xi(s),\Phi_{k\tau}(s-\tau))\mathrm{d}s, \quad t\in[k\tau,T]. \end{aligned}$$

Since the above estimates $\mathcal{T}_f$ has a unique fixed point ξ^*_f in $C([k\tau,T];\mathbb{R})$. Therefore, we can consider

$$\Phi(t,\phi) := \begin{cases} \Phi_{k\tau}(t) & \text{if } t \in [-\tau, k\tau], \\ \xi^*_f(t) & \text{if } t \in [k\tau, T] \end{cases}$$

which is the the unique solution of (4) on $[-\tau, T]$ and allows to conclude the proof. □

We observe that Theorem 3.1 provides a generalization of previous results (see, for instance, [25, Theorem 2.3], [26, Theorem 5.1] and [27, Theorem 3.2]) which applies to the special case $N = 0$. A different proof, for problems with a nonlinear neutral term, is instead presented in [28] under weaker assumptions which, however, ensure existence but not uniqueness.

Corollary 3.2. Consider the system (1)–(2) and assume that the function f satisfies assumptions (A1) and (A2) for $t \in [0,\infty)$. Then, the system has a unique global solution on $[-\tau,\infty)$.

Proof. We omit the proof since similar to the proof of [6, Corollary 3.2]. □

4. Exponential boundedness of FNDDEs

Let $\phi \in C^1([-\tau,0],\mathbb{R})$ be an arbitrary function. We consider the system

$$\begin{aligned} {}^{\mathrm{C}}D^{\alpha}_{0+}\big[x(t)+Nx(t-\tau)\big] &= f(t,x(t),x(t-\tau)), && t\in(0,\infty), \\ x(t) &= \phi(t), && t\in[-\tau,0]. \end{aligned} \tag{5}$$

for which f is assumed continuous and satisfying the following conditions:

(H1) there exits a positive constant L such that

$$|f(t,x,y)-f(t,\hat{x},\hat{y})| \le L\big(|x-\hat{x}|+|y-\hat{y}|\big), \quad \forall t\ge 0, \quad \forall x,y,\hat{x},\hat{y}\in\mathbb{R};$$

(H2) there exits a positive constant λ such that

$$\sup_{t\ge 0}\frac{\int_0^t (t-s)^{\alpha-1}|f(s,0,0)|\mathrm{d}s}{\mathrm{e}^{\lambda t}} < \infty.$$

The following bound for the growth rate of solutions of (5) holds.

Theorem 4.1. Assume that conditions (H1) and (H2) hold. Then, the global solution $\Phi(\cdot,\phi)$ on the interval $[-\tau,\infty)$ of (5) is exponentially bounded.

Proof. Let $\lambda > 0$ be the constant satisfying condition (H2). Denote by $C_\lambda([-\tau,\infty);\mathbb{R})$ the set of all continuous functions $\xi : [-\tau,\infty) \to \mathbb{R}$ such that

$$||\xi||_\lambda := \sup_{t\ge 0}\frac{\xi^*(t)}{\exp(\lambda t)} < \infty, \ \ \xi^*(t) := \sup_{-\tau\le\theta\le t}|\xi(\theta)|.$$

It is immediate that $(C_\lambda([-\tau,\infty);\mathbb{R});||\cdot||_\lambda)$ is a Banach space. On this space we construct the operator $\mathcal{T}_\phi$ as follows

$$\begin{aligned}(\mathcal{T}_\phi\xi)(t) &:= \phi(t), \quad t\in[-\tau,0],\\ (\mathcal{T}_\phi\xi)(t) &:= \phi(0)+N\phi(-\tau)-N\xi(t-\tau)\\ &\qquad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,\xi(s),\xi(s-\tau))\mathrm{d}s, \quad t\geq 0.\end{aligned}$$

It is easy to see that $\mathcal{T}_\phi\xi\in C([-\tau,\infty);\mathbb{R})$ for any $\xi\in C_\lambda([-\tau,\infty);\mathbb{R})$ but we will show that for any $\xi\in C_\lambda([-\tau,\infty);\mathbb{R})$ it is $\mathcal{T}_\phi\xi\in C_\lambda([-\tau,\infty);\mathbb{R})$ as well. Indeed, let $\xi\in C_\lambda([-\tau,\infty);\mathbb{R})$ be arbitrary, for any $t\geq\tau$, we have

$$\begin{aligned}|(\mathcal{T}_\phi\xi)(t)| &\leq |\phi(0)|+|N||\phi(-\tau)|+|N||\xi(t-\tau)|\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,\xi(s),\xi(s-\tau))-f(s,0,0)|\mathrm{d}s\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,0,0)|\mathrm{d}s\\ &\leq C_1+|N|\xi^*(t)+\frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(|\xi(s)|+|\xi(s-\tau)|)\mathrm{d}s\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,0,0)|\mathrm{d}s.\end{aligned}$$

An immediate consequence is that

$$\begin{aligned}|(\mathcal{T}_\phi\xi)(t)| &\leq C_1+|N|\mathrm{e}^{\lambda t}\frac{\xi^*(t)}{\mathrm{e}^{\lambda t}}+\frac{2L\mathrm{e}^{\lambda t}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathrm{e}^{-\lambda(t-s)}\frac{\xi^*(s)}{\mathrm{e}^{\lambda s}}\mathrm{d}s\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,0,0)|\mathrm{d}s\\ &\leq C_1+|N|\mathrm{e}^{\lambda t}||\xi||_\lambda+\frac{2L\mathrm{e}^{\lambda t}}{\lambda^\alpha}||\xi||_\lambda+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,0,0)|\mathrm{d}s, \quad \forall t\geq\tau.\end{aligned}$$

and hence for any $t\geq\tau$ it is

$$\frac{(\mathcal{T}_\phi\xi)^*(t)}{\mathrm{e}^{\lambda t}}\leq\frac{C_1}{\mathrm{e}^{\lambda t}}+|N|||\xi||_\lambda+\frac{2L}{\lambda^\alpha}||\xi||_\lambda+\frac{1}{\Gamma(\alpha)}\sup_{t\geq 0}\frac{\int_0^t (t-s)^{\alpha-1}|f(s,0,0)|\mathrm{d}s}{\mathrm{e}^{\lambda t}}$$

from which we infer

$$\sup_{t\geq 0}\frac{(\mathcal{T}_\phi\xi)^*(t)}{\mathrm{e}^{\lambda t}}<\infty.$$

The next step is to prove that $\mathcal{T}_\phi$ is a contractive operator on $(C_\lambda([-\tau,\infty);\mathbb{R});||\cdot||_\lambda)$. Let $\xi,\hat{\xi}\in(C_\lambda([-\tau,\infty);\mathbb{R});||\cdot||_\lambda)$ be arbitrary, we have the following estimates on the intervals $[-\tau,0]$, $[0,\tau]$ and $[\tau,\infty)$. When $t\in[-\tau,0]$ it is immediate to see that

$$|(\mathcal{T}_\phi\xi)(t)-(\mathcal{T}_\phi\hat{\xi})(t)|=0$$

whilst whenever $t\in[0,\tau]$ we have

$$\begin{aligned}|(\mathcal{T}_\phi\xi)(t)-(\mathcal{T}_\phi\hat{\xi})(t)| &\leq\frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(|\xi(s)-\hat{\xi}(s)|+|\xi(s-\tau)-\hat{\xi}(s-\tau)|)\mathrm{d}s\\ &\leq\frac{2L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\xi-\hat{\xi})^*(s)\mathrm{d}s\\ &\leq\frac{2L\mathrm{e}^{\lambda t}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathrm{e}^{-\lambda(t-s)}\frac{(\xi-\hat{\xi})^*(s)}{\mathrm{e}^{\lambda s}}\mathrm{d}s\\ &\leq\frac{2L\mathrm{e}^{\lambda t}}{\lambda^\alpha}||\xi-\hat{\xi}||_\lambda.\end{aligned}$$

Finally, for $t \in [\tau, \infty)$ it is

$$\begin{aligned}|(\mathcal{T}_\Phi \xi)(t) - (\mathcal{T}_\Phi \hat{\xi})(t)| &\leq |N||\xi(t-\tau) - \hat{\xi}(t-\tau)| \\ &\quad + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (|\xi(s) - \hat{\xi}(s)| + |\xi(s-\tau) - \hat{\xi}(s-\tau)|) \mathrm{d}s \\ &\leq |N| \mathrm{e}^{\lambda t} \frac{(\xi - \hat{\xi})^*(t-\tau)}{\mathrm{e}^{\lambda(t-\tau)} \mathrm{e}^{\lambda\tau}} + \frac{2L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\xi - \hat{\xi})^*(s) \mathrm{d}s \\ &\leq \mathrm{e}^{\lambda t} \frac{|N|}{\mathrm{e}^{\lambda\tau}} ||\xi - \hat{\xi}||_\lambda + \mathrm{e}^{\lambda t} \frac{2L}{\lambda^\alpha} ||\xi - \hat{\xi}||_\lambda.\end{aligned}$$

Therefore, on the whole interval $[-\tau, \infty)$ we obtain

$$|(\mathcal{T}_\phi \xi)(t) - (\mathcal{T}_\phi \hat{\xi})(t)| \leq \mathrm{e}^{\lambda t} \left(\frac{|N|}{\mathrm{e}^{\lambda\tau}} + \frac{2L}{\lambda^\alpha} \right) ||\xi - \hat{\xi}||_\lambda, \ \ \forall t \in [-\tau, \infty)$$

and, consequently,

$$(\mathcal{T}_\phi \xi - \mathcal{T}_\phi \hat{\xi})^*(t) \leq \mathrm{e}^{\lambda t} \left(\frac{|N|}{\mathrm{e}^{\lambda\tau}} + \frac{2L}{\lambda^\alpha} \right) ||\xi - \hat{\xi}||_\lambda, \ \ \forall t \geq 0.$$

for which it is

$$||(\mathcal{T}_\phi \xi - \mathcal{T}_\phi \hat{\xi})||_\lambda \leq \left(\frac{|N|}{\mathrm{e}^{\lambda\tau}} + \frac{2L}{\lambda^\alpha} \right) ||\xi - \hat{\xi}||_\lambda.$$

Choose λ large enough, i.e. such that

$$\frac{|N|}{\mathrm{e}^{\lambda\tau}} + \frac{2L}{\lambda^\alpha} < 1,$$

assures that $\mathcal{T}_\phi$ is contractive on $(C_\lambda([-\tau, \infty); \mathbb{R}); || \cdot ||_\lambda)$. The unique fixed point ξ^* of $\mathcal{T}_\phi$ is therefore the unique solution to (5) in $C_\lambda([-\tau, \infty); \mathbb{R})$. Moreover, this solution is exponentially bounded. □

5. Explicit representation of solutions of linear FNDDEs

For $a, b, N \in \mathbb{R}$ and an arbitrary continuous function $\phi(t) : [-\tau, 0] \to \mathbb{R}$, we now consider the special case of linear FNDDEs

$$\begin{aligned} {}^{\mathrm{C}}D^\alpha_{0+} \big[x(t) + N x(t-\tau) \big] &= a x(t) + b x(t-\tau), && t \in (0, T] \\ x(t) &= \phi(t), && t \in [-\tau, 0] \end{aligned} \tag{6}$$

for which we are interested in providing an explicit representation of the solution. Since assumptions (H1) and (H2) introduced in Section 4 are trivially verified, the solution $x(t)$ possesses the Laplace transform (LT), say $X(s)$, and from well-known results on the LT of the fractional Caputo derivative we have

$$\mathcal{L}\Big({}^{\mathrm{C}}D^\alpha_0 x(r) \, , s \Big) = s^\alpha X(s) - s^{\alpha-1} \phi(0).$$

Hence, by taking the LT to both sides of (6), we obtain

$$s^\alpha X(s) + N s^\alpha \mathcal{L}\Big(x(t-\tau) \, , s \Big) - s^{\alpha-1} \big[\phi(0) + N \phi(-\tau) \big] = a X(s) + b \mathcal{L}\Big(x(t-\tau) \, , s \Big). \tag{7}$$

We know (see, for instance, [7, Eq (3.2)] or [11, Proposition 4.2]) that

$$\mathcal{L}\Big(x(t-\tau) \, , s \Big) = \mathrm{e}^{-s\tau} X(s) + \mathrm{e}^{-s\tau} \hat{X}_\tau(s), \quad \hat{X}_\tau(s) = \int_{-\tau}^0 \mathrm{e}^{-st} \phi(t) \mathrm{d}t \tag{8}$$

and, therefore, one immediately obtains

$$\Big(1 - \frac{b - N s^\alpha}{s^\alpha - a} \mathrm{e}^{-s\tau} \Big) X(s) = \frac{s^{\alpha-1}}{s^\alpha - a} \big[\phi(0) + N \phi(-\tau) \big] + \frac{b - N s^\alpha}{s^\alpha - a} \mathrm{e}^{-s\tau} \hat{X}_\tau(s).$$

For sufficiently large $|s|$ the use of the series expansion

$$\Big(1 - \frac{b - Ns^{\alpha}}{s^{\alpha} - a} \mathrm{e}^{-s\tau}\Big)^{-1} = \sum_{k=0}^{\infty} \frac{(b - Ns^{\alpha})^k}{(s^{\alpha} - a)^k} \mathrm{e}^{-s\tau k}$$

leads to

$$X(s) = \frac{s^{\alpha-1}}{s^{\alpha} - a} \sum_{k=0}^{\infty} \frac{(b - Ns^{\alpha})^k}{(s^{\alpha} - a)^k} \mathrm{e}^{-s\tau k} \big[\phi(0) + N\phi(-\tau)\big] + \sum_{k=1}^{\infty} \frac{(b - Ns^{\alpha})^k}{(s^{\alpha} - a)^k} \mathrm{e}^{-s\tau k} \hat{X}_{\tau}(s)$$

and, hence, after exploiting standard rules for powers of binomials

$$(b - Ns^{\alpha})^k = \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell} N^{\ell} b^{k-\ell} s^{\alpha\ell}$$

we obtain the following representation of the LT of the solution of the linear FNDDE (6)

$$\begin{aligned} X(s) &= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell} N^{\ell} b^{k-\ell} \frac{s^{\alpha+\alpha\ell-1}}{(s^{\alpha} - a)^{k+1}} \mathrm{e}^{-s\tau k} \big[\phi(0) + N\phi(-\tau)\big] \\ &+ \sum_{k=1}^{\infty} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell} N^{\ell} b^{k-\ell} \frac{s^{\alpha\ell}}{(s^{\alpha} - a)^{k}} \mathrm{e}^{-s\tau k} \hat{X}_{\tau}(s) \end{aligned} \tag{9}$$

An explicit representation of the solution of (6) in the time domain can be obtained by inversion of the LT (9) only once the initial function $\phi(t)$ has been specified. The following preliminary results are however necessary.

Let $\alpha > 0$ and $\beta, \gamma \in \mathbb{R}$ be some parameters, and consider the three-parameter Mittag-Leffler function (also known as the Prabhakar function) [29, 30]

$$E^{\gamma}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\gamma)} \sum_{j=0}^{\infty} \frac{\Gamma(\gamma + j) z^j}{j! \Gamma(\alpha j + \beta)},$$

for which, when $t \geq 0$ and a is any real or complex value, we have the following result concerning the LT

$$\mathcal{L}\Big(e^{\gamma}_{\alpha,\beta}(t;a)\,, s\Big) = \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha} - a)^{\gamma}}, \quad e^{\gamma}_{\alpha,\beta}(t;a) := t^{\beta-1} E^{\gamma}_{\alpha,\beta}(a t^{\alpha}), \tag{10}$$

for $Re(s) > 0$ and $|s| > |a|^{\frac{1}{\alpha}}$. Furthermore, whenever $\tau \geq 0$ it is a basic fact in the theory of LT (see, for instance, [31, Theorem 1.31]) that

$$\mathcal{L}^{-1}\left(\frac{s^{\alpha\gamma-\beta}}{(s^{\alpha} - a)^{\gamma}} \mathrm{e}^{-sp}\,, s\right) = \begin{cases} e^{\gamma}_{\alpha,\beta}(t - \tau; a) & t \geq \tau \\ 0 & t < \tau \end{cases} \tag{11}$$

We are now able to provide an explicit representation of the solution of linear FNDDEs for some instances of the initial function $\phi(t)$. In the following, for any real value x, with $\lfloor x \rfloor$ we will denote the greatest integer less or equal to x.

Proposition 5.1. If $\phi(t) = x_0$, $\forall t \in [-\tau, 0]$, the exact solution of the linear FNDDE (6) is

$$\begin{aligned} x(t) &= \sum_{k=0}^{\lfloor t/\tau \rfloor} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell} N^{\ell} b^{k-\ell} e^{k+1}_{\alpha,\alpha(k-\ell)+1}(t - \tau k; a)(1 + N)x_0 \\ &- \sum_{k=1}^{\lfloor t/\tau \rfloor} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell} N^{\ell} b^{k-\ell} e^{k}_{\alpha,\alpha(k-\ell)+1}(t - \tau k; a)x_0 \\ &+ \sum_{k=1}^{\lfloor t/\tau \rfloor+1} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell} N^{\ell} b^{k-\ell} e^{k}_{\alpha,\alpha(k-\ell)+1}(t - \tau k + \tau; a)x_0. \end{aligned}$$

Proof. Since $\phi(t) = x_0$, $\forall t \in [-\tau, 0]$, it is immediate to compute

$$\hat{X}_\tau(s) = -\frac{1}{s}\big(1 - \mathrm{e}^{s\tau}\big)x_0$$

and, hence, the LT $X(s)$ obtained in (9) becomes

$$\begin{aligned} X(s) = &\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^\ell N^\ell b^{k-\ell}\frac{s^{\alpha+\alpha\ell-1}}{(s^\alpha - a)^{k+1}}\mathrm{e}^{-s\tau k}(1+N)x_0 \\ &-\sum_{k=1}^{\infty}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^\ell N^\ell b^{k-\ell}\frac{s^{\alpha\ell-1}}{(s^\alpha - a)^{k}}\mathrm{e}^{-s\tau k}x_0 \\ &+\sum_{k=1}^{\infty}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^\ell N^\ell b^{k-\ell}\frac{s^{\alpha\ell-1}}{(s^\alpha - a)^{k}}\mathrm{e}^{-s\tau (k-1)}x_0. \end{aligned}$$

The proof now follows after recognizing the presence, in each summation, of the LT (10) of the three-parameter Mittag-Leffler (ML) function, applying Eq. (11), and, for any t, truncating each summation at the maximum index k such that $t \geq \tau k$ (first and second summation) or $t \geq \tau(k-1)$ (third summation). □

Proposition 5.2. If $\phi(t) = x_0 + mt$, $\forall t \in [-\tau, 0]$, the exact solution of the linear FNDDE (6) is

$$\begin{aligned} x(t) = &\sum_{k=0}^{\lfloor t/\tau\rfloor}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^\ell N^\ell b^{k-\ell} e^{k+1}_{\alpha,\alpha(k-\ell)+1}(t-\tau k; a)\big[x_0 + N\phi(-\tau)\big] \\ &-\sum_{k=1}^{\lfloor t/\tau\rfloor}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^\ell N^\ell b^{k-\ell} e^{k}_{\alpha,\alpha(k-\ell)+1}(t-\tau k; a)x_0 \\ &+\sum_{k=1}^{\lfloor t/\tau\rfloor+1}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^\ell N^\ell b^{k-\ell} e^{k}_{\alpha,\alpha(k-\ell)+1}(t-\tau k+\tau; a)\phi(-\tau) \\ &-\sum_{k=1}^{\lfloor t/\tau\rfloor}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^\ell N^\ell b^{k-\ell} e^{k}_{\alpha,\alpha(k-\ell)+2}(t-\tau k; a)m \\ &+\sum_{k=1}^{\lfloor t/\tau\rfloor+1}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^\ell N^\ell b^{k-\ell} e^{k}_{\alpha,\alpha(k-\ell)+2}(t-\tau k+\tau; a)m. \end{aligned}$$

Proof. When $\phi(t) = x_0 + mt$, $t \in [-\tau, 0]$, a standard computation allows to evaluate

$$\begin{aligned} \hat{X}_\tau(s) = \int_{-\tau}^{0}\mathrm{e}^{-st}\phi(t)\mathrm{d}t &= -\frac{1}{s}\big(1-\mathrm{e}^{s\tau}\big)x_0 + m\Big[-\frac{1}{s^2} - \frac{\tau}{s}\mathrm{e}^{s\tau} + \frac{1}{s^2}\mathrm{e}^{s\tau}\Big] \\ &= -\frac{1}{s}x_0 + \frac{1}{s}\mathrm{e}^{s\tau}\phi(-\tau) - \frac{1}{s^2}m + \frac{1}{s^2}\mathrm{e}^{s\tau}m \end{aligned}$$

and, after inserting the above expression for $\hat{X}_\tau(s)$ in the formula (9) for the LT of the solution of (6), we obtain

$$\begin{aligned} X(s) &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^\ell N^\ell b^{k-\ell}\frac{s^{\alpha+\alpha\ell-1}}{(s^\alpha-a)^{k+1}}\mathrm{e}^{-s\tau k}\big(x_0+N\phi(-\tau)\big) \\ &-\sum_{k=1}^{\infty}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^\ell N^\ell b^{k-\ell}\frac{s^{\alpha\ell-1}}{(s^\alpha-a)^{k}}\mathrm{e}^{-s\tau k}x_0 \\ &+\sum_{k=1}^{\infty}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^\ell N^\ell b^{k-\ell}\frac{s^{\alpha\ell-1}}{(s^\alpha-a)^{k}}\mathrm{e}^{-s\tau(k-1)}\phi(-\tau) \\ &-\sum_{k=1}^{\infty}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^\ell N^\ell b^{k-\ell}\frac{s^{\alpha\ell-2}}{(s^\alpha-a)^{k}}\mathrm{e}^{-s\tau k}m \\ &+\sum_{k=1}^{\infty}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^\ell N^\ell b^{k-\ell}\frac{s^{\alpha\ell-2}}{(s^\alpha-a)^{k}}\mathrm{e}^{-s\tau(k-1)}m \end{aligned}$$

and the proof is concluded in the same way as the proof of Proposition 5.1. □

The above explicit representations of exact solutions is of interest since it allows to accurately evaluate the solutions of linear FNDDEs once a procedure for the computation of the three-parameter ML functions $e^k_{\alpha,\beta}(t;a)$ is available. To this purpose the method devised in [32] to compute k-th order derivatives $E^{(k)}_{\alpha,\beta}(z)$ of the two-parameter ML function $E_{\alpha,\beta}(z)$ can be exploited since three-parameter ML functions are related to derivatives of two-parameter ML functions by the relationship $E^k_{\alpha,\beta}(z)=E^{(k)}_{\alpha,\beta-\alpha k+\alpha}(z)/(k-1)!$.

Anyway, this approach does not seems suitable for computation on intervals of large size since it could require the evaluation of a considerable number of three-parameter ML functions. Moreover, a specific explicit representation of the exact solution must be derived in dependence of the selected initial function $\phi(t)$. For this reason, in the Section devoted to present numerical simulations we will derive a specific numerical scheme.

6. Asymptotic behavior of solutions of linear FNDDEs

This section is devoted to discuss the asymptotic behavior of solutions to linear FNDDEs (6). We will focus on two different cases, namely when $a<0, b=0$ and when $a<0, 0<|b|<|a|$.

6.1. Case (C1): $a<0$, $b=0$.

In this case, the linear FNDDE becomes

$$ {}^{\mathrm{C}}D^\alpha_{0+}\big[x(t)+Nx(t-\tau)\big]=ax(t),\quad t\ge 0, \tag{12}$$

and, thanks to (7) and (8), the LT $X(s)$ of the solution $x(t)$ is

$$X(s)=\frac{s^{\alpha-1}(\phi(0)+N\phi(-\tau))-Ns^\alpha\mathrm{e}^{-s\tau}\int_{-\tau}^{0}\mathrm{e}^{-su}\phi(u)\mathrm{d}u}{s^\alpha+Ns^\alpha\mathrm{e}^{-s\tau}-a}. \tag{13}$$

To investigate the asymptotic behavior of $x(t)$ it is necessary to locate possible poles of $X(s)$ in the complex plane. Denote the denominator of $X(s)$ by

$$Q(s):=s^\alpha+Ns^\alpha\mathrm{e}^{-s\tau}-a.$$

Due to the fact that $X(s)$ has just a single pole at the origin in addition to zeros of $Q(s)$, we can restrict ourselves to study the roots of the equation $Q(s)=0$.

Lemma 6.1. Let $a<0$. The following statements hold:

(i) if $|N|\le 1$, then $Q(s)=0$ has no roots in the closed right half plane $\{z\in\mathbb{C}:\Re(z)\ge 0\}$;

(ii) if $|N| > 1$, then $Q(s) = 0$ has at least one root in the open right half plane $\{z \in \mathbb{C} : \Re(z) > 0\}$.

Proof. (i) Since $Q(0) \neq 0$, the equation $Q(s) = 0$ is equivalent to

$$1 + N\mathrm{e}^{-\tau s} = as^{-\alpha},\ s \neq 0. \tag{14}$$

We will show that (14) has no root in $\{z \in \mathbb{C} : \Re(z) \geq 0\}$. Indeed, on the contrary, assume that (14) has a root $s_0 \neq 0$ with $\Re(s_0) \geq 0$. Note that $1 + N\mathrm{e}^{-\tau s_0} \in D_1 := \{z \in \mathbb{C} : |z - (-1)| \leq |N|\}$ and $as_0^{-\alpha} \in D_2 := \{z \in \mathbb{C} : |\arg(z)| \leq \frac{\alpha\pi}{2}\}$. Furthermore, for $|N| \leq 1$, two domains D_1 and D_2 intersect at most one point at the origin which implies a contradiction.

(ii) To prove this point, we only have to show that the Eq. (14) has at last one root in the open right half plane $\{z \in \mathbb{C} : \Re(z) > 0\}$. To this purpose consider the functions $f(s) := 1 + N\mathrm{e}^{-s\tau}$ and $g(s) := -as^{-\alpha}$ and $N > 1$ (the case $N < -1$ is proved in a similar way). It is easy to see that $s_k := \frac{\log N}{\tau} + \mathrm{i}\frac{(2k+1)\pi}{\tau}$, $k \in \mathbb{Z}$, are roots of the equation $f(s) = 0$. Let R be a positive constant and define $C := C_1 \cup C_2 \cup C_3 \cup C_4$, where

$$C_1 := \left\{z \in \mathbb{C} :\ z = s_1 + \mathrm{i}R,\ \frac{\log N}{2\tau} \leq s_1 \leq \frac{3\log N}{2\tau}\right\},$$

$$C_2 := \left\{z \in \mathbb{C} :\ z = \frac{3\log N}{2\tau} + \mathrm{i}s_2,\ R \leq s_2 \leq R + \frac{2\pi}{\tau}\right\},$$

$$C_3 := \left\{z \in \mathbb{C} :\ z = s_1 + \mathrm{i}(R + \frac{2\pi}{\tau}),\ \frac{\log N}{2\tau} \leq s_1 \leq \frac{3\log N}{2\tau}\right\},$$

$$C_4 := \left\{z \in \mathbb{C} :\ z = \frac{\log N}{2\tau} + \mathrm{i}s_2,\ R \leq s_2 \leq R + \frac{2\pi}{\tau}\right\}.$$

For $s \in C$ we we obtain the estimates

$$|f(s)| > 1 - \frac{N}{\mathrm{e}^{-\tau R}} > \frac{1}{2},\ \text{for } R \text{ large enough},$$

$$|g(s)| \leq \frac{|a|}{R^\alpha} \to 0 \text{ as } R \to \infty.$$

and, therefore, by choosing R sufficiently large, it is

$$|f(s)| > |g(s)|,\ \forall s \in C.$$

Since $f(s)$ has at least one zero in the domain D bounded by C, by Rouché's theorem (see, e.g., [2, Theorem 12.2, p. 398]) there is at least one zero point of $Q(s) = f(s) + g(s)$ in D and hence in $\{z \in \mathbb{C} : \Re(z) > 0\}$ which allows to conclude the proof. □

We are now in a position to state the main result for the case *(C1)*.

Theorem 6.2. Let $a < 0$ and consider the linear FNDDE (12). The following statements hold:

(i) if $|N| \leq 1$, then the solution of (12) is asymptotically stable;
(ii) if $|N| > 1$, then the solution of (12) is unstable.

Proof. (i) As shown above, the LT $X(s)$ of the solution $x(t)$ of (12) does not have any poles in the closed right half-plane $\{s \in \mathbb{C} : \Re(s) \geq 0\}$ except for a simple pole at the origin. Hence, since from (13) it is $\lim_{s\to 0} sX(s) = 0$, by the final value theorem for LT [20, Theorem D. 13, p. 232], we have

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = 0$$

which implies that (12) is asymptotically stable.
(ii) The proof of this part follows since the LT $X(s)$ has at least one pole in the open right half-plane of the complex domain. □

Remark 6.3. We have not considered the case $a = 0$ since it is trivial. Whenever $a > 0$, $Q(s)$ has at least one root in the open right half plane $\{z \in \mathbb{C} : \Re(z) > 0\}$ for any N and $\tau \geq 0$ but the location of possible other roots depends on a, N and τ and requires a more in-depth analysis; we think however that this analysis is not necessary since the presence of a root of $Q(s)$ in the open right half plane $\{z \in \mathbb{C} : \Re(z) > 0\}$ makes unstable the solution of the linear FNDDE (12).

6.2. **Case (C2):** $a < 0$, $0 < |b| < |a|$. The linear FNDDE is now

$$ {}^{\mathrm{C}}D^{\alpha}_{0+}\big[x(t) + Nx(t-\tau)\big] = ax(t) + bx(t-\tau), \quad t \geq 0, \tag{15}$$

and, by exploiting again (7) and (8), the LT $X(s)$ of the solution $x(t)$ is

$$X(s) = \frac{s^{\alpha-1}(\phi(0) + N\phi(-\tau)) + (b - Ns^{\alpha})\mathrm{e}^{-s\tau}\int_{-\tau}^{0} \mathrm{e}^{-su}\phi(u)\mathrm{d}u}{s^{\alpha} + Ns^{\alpha}\mathrm{e}^{-s\tau} - a - b\mathrm{e}^{-\tau s}}. \tag{16}$$

It is easy to see that $s = 0$ is only a simple pole of $X(s)$. We now put

$$P(s) := s^{\alpha} + Ns^{\alpha}\mathrm{e}^{-s\tau} - a - b\mathrm{e}^{-\tau s}$$

and by the following lemma we provide information about location of zero points of P.

Lemma 6.4. Assume that $a < 0$ and $0 < |b| < |a|$.

(i) If $|N| \leq 1$, then $P(s) = 0$ has no roots in the closed right half-plane $\{s \in \mathbb{C} : \Re(s) \geq 0\}$.
(ii) If $|N| > 1$, then $P(s) = 0$ has at least one root in the open right half-plane $\{s \in \mathbb{C} : \Re(s) > 0\}$.

Proof. (i) Denote $D_1 := \{z \in \mathbb{C} : \Re(z) \geq 0\}$. Since $P(0) \neq 0$, there exists a small enough $\varepsilon > 0$ such that $P(s) \neq 0$ in the ball $B := \{s \in \mathbb{C} : |s| \leq \varepsilon\}$. On the other hand, for $s \in D_1$ it is

$$|P(s)| \geq |s|^{\alpha}(1 - |N|) - (|a| + |b|) \to \infty, \quad \text{as } |s| \to \infty.$$

Thus, there is $R > 0$ such that $P(s) \neq 0$ for all $s \in D_1 \cap \{z \in \mathbb{C} : |z| \geq R\}$. Denote $C_1 := \{z \in \mathbb{C} : z = \varepsilon(\cos\varphi + \mathrm{i}\sin\varphi),\ -\pi/2 \leq \varphi \leq \pi/2\}$, $C_3 := \{z \in C : z = R(\cos\varphi + \mathrm{i}\sin\varphi),\ -\pi/2 \leq \varphi \leq \pi/2)\}$, $C_2 := \{z \in \mathbb{C} : z = r(\cos\pi/2 - \mathrm{i}\sin\pi/2\}$ and $C_4 := \{z \in \mathbb{C} : z = r(\cos\pi/2 + \mathrm{i}\sin\pi/2)\}$. Put $f(s) := s^{\alpha} - a$, $g(s) := Ns^{\alpha}\mathrm{e}^{-s\tau} - b\mathrm{e}^{-\tau s}$. On C_1 and C_3, let $s = s_1 + \mathrm{i}s_2 = r(\cos\varphi + \mathrm{i}\sin\varphi)$, where $s_1 > 0$, $r = \varepsilon$ or $r = R$ and $\varphi \in [-\pi/2, \pi/2]$. We have

$$\begin{aligned}
f(s) &= s^{\alpha} - a = r^{\alpha}\cos(\alpha\varphi) - a + \mathrm{i}r^{\alpha}\sin(\alpha\varphi),\\
g(s) &= Nr^{\alpha}\mathrm{e}^{\mathrm{i}\alpha\varphi}\mathrm{e}^{-\tau(s_1+is_2)} - b\mathrm{e}^{-\tau(s_1+is_2)}\\
&= Nr^{\alpha}\mathrm{e}^{-\tau s_1}\cos(\alpha\varphi - \tau s_2) - b\mathrm{e}^{-\tau s_1}\cos(\tau s_2) + \mathrm{i}\big[Nr^{\alpha}\mathrm{e}^{-\tau s_1}\sin(\alpha\varphi - \tau s_2) + b\mathrm{e}^{-\tau s_1}\sin(\tau s_2)\big]
\end{aligned}$$

and hence

$$|f(s)|^2 = r^{2\alpha} + a^2 - 2ar^{\alpha}\cos(\alpha\varphi), \tag{17}$$

$$|g(s)|^2 = N^2r^{2\alpha}\mathrm{e}^{-2\tau s_1} + b^2\mathrm{e}^{-2\tau s_1} - 2bNr^{\alpha}\mathrm{e}^{-2\tau s_1}\cos(\alpha\varphi). \tag{18}$$

From (17), (18) and the assumptions that $s_1 \geq 0$, $|N| \leq 1$ and $|b| < |a|$, we see that

$$|f(s)| > |g(s)| \text{ on } C_1 \text{ and } C_3. \tag{19}$$

Now, we will compare $|f|$ and $|g|$ on C_4. For any $s \in C_4$, we describe $s = \mathrm{i}r = r(\cos\pi/2 + \mathrm{i}\sin\pi/2)$, where $r \in [\varepsilon, R]$. By a simple computation, we obtain the estimates

$$\begin{aligned}
|f(s)|^2 &= r^{2\alpha} + a^2 - 2ar^{\alpha}\cos\frac{\alpha\pi}{2},\\
|g(s)|^2 &= N^2r^{2\alpha} + b^2 - 2Nr^{\alpha}b\cos\frac{\alpha\pi}{2} \leq N^2r^{2\alpha} + b^2 + 2|N|r^{\alpha}|b|\cos\frac{\alpha\pi}{2},
\end{aligned}$$

which imply that

$$|f(s)| > |g(s)| \text{ on } C_4. \tag{20}$$

Similarly, on C_2, we also have

$$|f(s)| > |g(s)|$$

and, together with (19), (20), we obtain

$$|f(s)| > |g(s)| \text{ on } C := C_1 \cup C_2 \cup C_3 \cup C_4. \tag{21}$$

From (21), by Rouché's theorem, P has no zero in the domain D bounded by the contour C defined as above. Thus, P has no zero point in the closed right half-plane of the complex plane. (ii) As in the proof of Lemma 6.1 (ii), we only need to show that the following equation has at least one root in the open right half-plane $\{z \in \mathbb{C} : \Re(z) > 0\}$:

$$1 + N\mathrm{e}^{-\tau s} - \frac{a}{s^\alpha} - \frac{b\mathrm{e}^{-\tau s}}{s^\alpha} = 0. \tag{22}$$

To do this, we set $f(s) := 1 + N\mathrm{e}^{-\tau s}$ and $g(s) := -\frac{a}{s^\alpha} - \frac{b\mathrm{e}^{-\tau s}}{s^\alpha}$. Take the contour C as in the proof of Lemma 6.1 (ii) with a large enough $R > 0$. It is known that f has one zero in the domain bounded by C and $f(s) \neq 0$ on this contour. On the other hand $|g(s)| \to 0$ as $|s| \to \infty$ with $s \in \{z \in \mathbb{C} : \Re(z) > 0\}$. Thus, for sufficiently large R we have

$$|g(s)| < \min_{s \in C} |f(s)| \leq |f(s)| \text{ for all } s \in C,$$

which together with Rouché's theorem imply that (22) has one root in the domain bounded by C, namely the equation $P(s) = 0$ has at least one root in $\{z \in \mathbb{C} : \Re(z) > 0\}$, thus allowing to complete the proof. □

Based on Lemma 6.4 and arguments as in the proof of Theorem 6.2, we obtain the following result.

Theorem 6.5. Let $a < 0$, $0 < |b| < |a|$ and consider the linear FNDDE (15). The following statements hold:

- (i) if $|N| \leq 1$, then the solution of (15) is asymptotically stable;
- (ii) if $|N| > 1$, then the solution of (15) is unstable.

Remark 6.6. Studying different combinations of the parameters a and b with respect to those considered in Lemma 6.4 and Theorem 6.5 appears more challenging and it would require a more involved analysis. We just observe that when $a + b > 0$ there is at least one root of $P(\cdot)$ in the open right half plane $\{z \in \mathbb{C} : \Re(z) > 0\}$ for any N and $\tau \geq 0$ and hence equation (15) is unstable for every N and $\tau \geq 0$. In the remaining cases, the stability property of this equation depends on all parameters a, b, N and τ.

More detailed information about the asymptotic behavior of solutions of the linear FNDDE (6), under conditions for which they turn out asymptotically stable, can be obtained thanks to the representations of the exact solutions in terms of Prabhakar functions given in Propositions 5.1 and 5.2. Indeed we know (see, for instance, [33, 34]) that when $0 < \alpha \leq 1$, $a < 0$ and $k \geq 1$ the asymptotic behaviour of the generalized Prbahakar function $e^k_{\alpha,\beta}(t;a)$ is given by

$$e^k_{\alpha,\beta}(t;a) = \frac{(-1)^k}{\Gamma(k)} \sum_{j=0}^{\infty} \frac{\Gamma(j+k)a^{-(j+k)}}{j!\Gamma(\beta - \alpha(j+k))} t^{\beta - \alpha(k+j) - 1}, \quad t \to \infty.$$

Therefore asymptotically stable solutions of the linear FNDDE (6) possesses algebraic expansions as $t \to \infty$, a common feature of stable fractional-order systems (see, for instance, [35, 36, 37, 38] for non-delayed equations or [39, 40, 9] for fractional-order delayed equations).

7. Numerical simulations

With the aim of verifying the theoretical findings on the asymptotic behavior of solutions of linear FNDDEs, we consider here a numerical scheme based on the application of a standard product-integration (PI) rule of rectangular rule to the integral representation (3). Methods of this kind are widely employed to solve fractional differential equations (see, for instance [41]) and they can be easily adapted to solve FNDDEs as well.

Let $h > 0$ and consider an equispaced grid $t_n = nh$, $n = 0, 1, \dots$, thanks to which the integral in (3) can be rewritten in a piece-wise way

$$\begin{aligned} x(t_n) &= \phi(0) + N\phi(-\tau) - Nx(t_n - \tau) \\ &\qquad + \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{\alpha-1} f(s, x(s), x(s-\tau))\mathrm{d}s. \end{aligned}$$

The vector field $f(s, x(s), x(s - \tau))$ is hence approximated, in each interval $[t_k, t_{k+1}]$, by the constant values assumed in one of the endpoints of $[t_k, t_{k+1}]$. For stability reasons, and avoid to introduce instability due to the numerical scheme, we prefer to device an implicit method and adopt the approximation $f(s, x(s), x(s-\tau)) \approx f(t_{k+1}, x(t_{k+1}), x(t_{k+1} - \tau))$, $s \in [t_k, t_{k+1}]$. After integrating in an exact way each integral we obtain the approximations $x_n \approx x(t_n)$ given by

$$x_n = \phi(0) + N\phi(-\tau) - Nx(t_n - \tau) + h^\alpha \sum_{k=1}^{n} b_{n-k}^{(\alpha)} f(t_k, x_k, x_{k-\tau/h}),$$

where convolution weights $b_n^{(\alpha)}$ are defined by $b_n^{(\alpha)} = ((n+1)^\alpha - n^\alpha)/\Gamma(\alpha+1)$. The approximation $x_{k-\tau/h}$ of $x(t_k - \tau)$ is obtained by interpolation of the two closest available approximations of the solution when $t_k - \tau$ is not a grid point or when it does not belong to $[-\tau, 0]$. First-degree polynomial interpolation is clearly sufficient to preserve the first-order convergence of the PI rule. Finally, Newton-Raphson iterations are used to determine x_n from the above implicit scheme when f is nonlinear.

We now apply the above scheme to present some numerical examples illustrating the main results proposed in this paper.

Example 7.1. Consider the equation

$$\begin{aligned} &{}^{\mathrm{C}}D_{0+}^{0.7}\big[x(t) + x(t-1)\big] = -5x(t),\ t > 0 \\ &x(\cdot) \in C([-1, 0]; \mathbb{R}) \end{aligned} \tag{23}$$

which from the theory of Subsection 6.1 is expected to present asymptotically stable solutions. In Figure 1, we show the numerical simulation of the trajectory $\Phi(\cdot, \phi)$ of the solution to (23) with the initial condition $\phi(t) = 0.2$ on $[-1, 0]$ which clearly show a stable behavior.

Example 7.2. Consider the equation

$$\begin{aligned} &{}^{\mathrm{C}}D_{0+}^{0.7}\big[x(t) - 1.5x(t-1)\big] = -5x(t),\ t > 0 \\ &x(\cdot) \in C([-1, 0]; \mathbb{R}). \end{aligned} \tag{24}$$

Since $|N| > 1$, we expect unstable solutions for Eq. (24). Indeed, as we can see from Figure 2, where it is depicted the trajectory of the solution $\Phi(\cdot, \phi)$ when $\phi(t) = 0.2$ on $[-1, 0]$, the numerical simulation confirms the theoretical expectation.

Example 7.3. Consider now the equation

$$\begin{aligned} &{}^{\mathrm{C}}D_{0+}^{0.7}\big[x(t) + 0.5x(t-1)\big] = -5x(t) + 0.5x(t-1),\ t > 0 \\ &x(\cdot) \in C([-1, 0]; \mathbb{R}) \end{aligned} \tag{25}$$

whose stability properties are studied in Subsection 6.2. As shown in Theorem 6.5 (i), this the solution of this equation is asymptotically stable. In Figure 3 it is presented the trajectory of

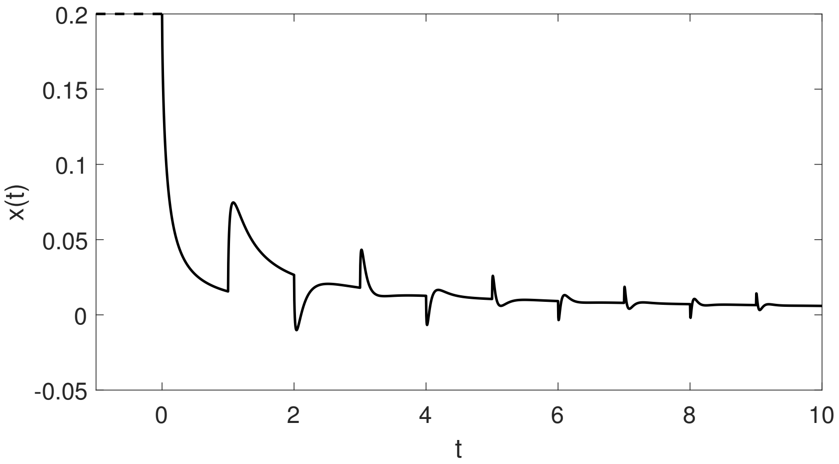

FIGURE 1. Trajectory of the solution $\Phi(\cdot,\phi)$ to system (23) when $\phi(t) = 0.2$ on $[-1, 0]$.

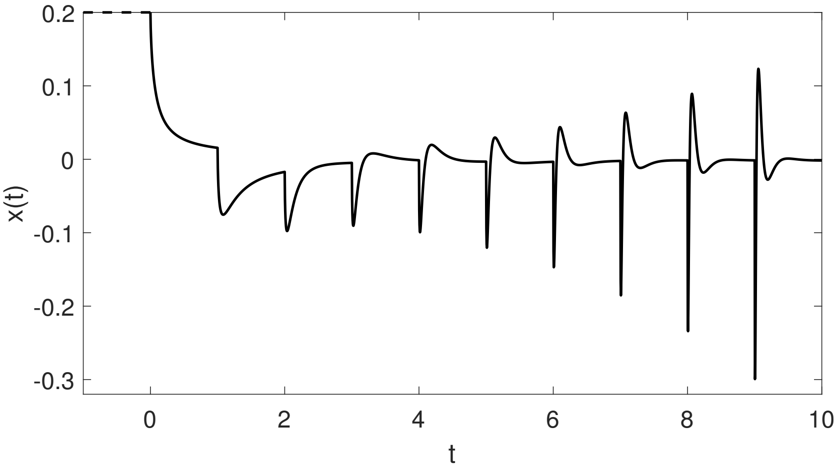

FIGURE 2. Trajectory of the solution $\Phi(\cdot,\phi)$ to system (24) when $\phi(t) = 0.2$ on $[-1, 0]$.

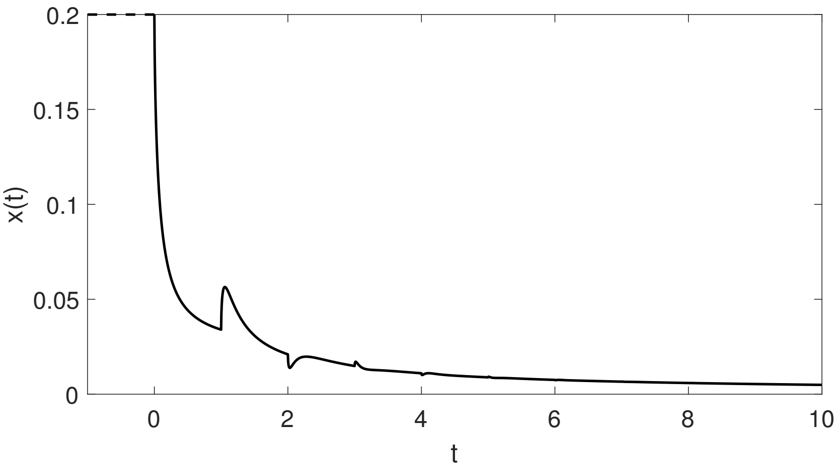

FIGURE 3. Trajectory of the solution $\Phi(\cdot,\phi)$ to system (25) when $\phi(t) = 0.2$ on $[-1, 0]$.

the solution $\Phi(\cdot,\phi)$ to (25) with the initial condition $\phi(t) = 0.2$ on $[-1, 0]$ which clearly appears to be stable as predicted from theory.

Example 7.4. We finally consider the equation

$$\begin{aligned} {}^{\mathrm{C}}D^{0.7}_{0+}\big[x(t) + 1.5x(t-1)\big] &= -5x(t) + 0.5x(t-1),\ t > 0 \\ x(\cdot) &\in C([-1, 0]; \mathbb{R}). \end{aligned} \tag{26}$$

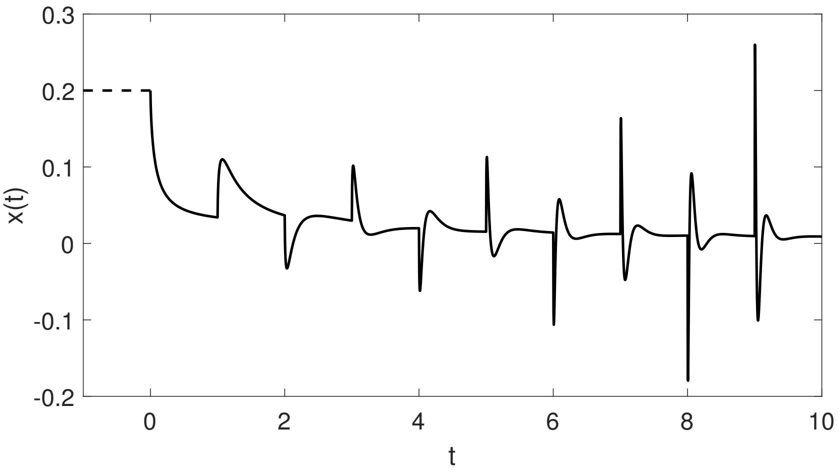

FIGURE 4. Trajectory of the solution $\Phi(\cdot, \phi)$ to system (26) when $\phi(t) = 0.2$ on $[-1, 0]$.

in which the presence of the neutral coefficient $N > 1.5$ suggests an unstable behavior as shown in Theorem 6.5 (ii). Indeed, also in this case, the simulation of the trajectory of the corresponding solution $\Phi(\cdot, \phi)$ to (26), with the same initial condition $\phi(t) = 0.2$ on $[-1, 0]$, confirms the theoretical findings (see Figure 4).

ACKNOWLEDGMENT

The authors are grateful to the reviewers for their valuable remarks which helped to improve the work.

The work of HTT is supported by The International Center for Research and Postgraduate Training in Mathematics–Institute of Mathematics–Vietnam Academy of Science and Technology under the Grant ICRTM01-2020.09. The work of HDT is supported by The International Center for Research and Postgraduate Training in Mathematics–Institute of Mathematics–Vietnam Academy of Science and Technology under the Grant ICRTM03-2021.02. A part of this paper was completed while the first author HTT was a postdoc at the Vietnam Institute for Advanced Study in Mathematics (VIASM). HTT would like to thank VIASM for support and hospitality. The work by RG is founded by INdAM under a 2020 GNCS project.

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