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ON FILIPPOV SOLUTIONS OF DISCONTINUOUS DAES OF INDEX 1 -

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ABSTRACT. We study discontinuous differential-algebraic equations (DDAEs) with a co-dimension 1 discontinuity manifold Σ. Our main objectives are to give sufficient conditions that allow to extend the DAE along Σ and, when this is possible, to define sliding motion (the sliding DAE) on Σ, extending Filippov construction to this DAE case. Our approach is to consider discontinuous ODEs associated to the DDAE and apply Filippov theory to the discontinuous ODEs, defining sliding/crossing solutions of the DDAE to be those inherited by the sliding/crossing solutions of the associated discontinuous ODEs. We will see that, in general, the sliding DAE on Σ is not defined unambiguously. When possible, we will consider in greater details two different methods based on Filippov's methodology to arrive at the sliding DAE. We will call these the direct approach and the singular perturbation approach and we will explore advantages and disadvantages of each of them. We illustrate our development with numerical examples.

Key words and phrases: Discontinuous differential equations, algebraic differential equations, Filippov, sliding solutions, singular perturbation theory, discontinuous slow-fast systems.

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1. INTRODUCTION

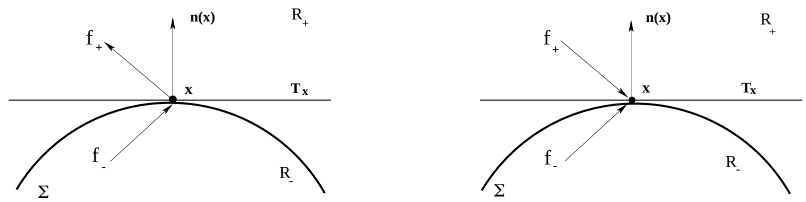
In this paper we study discontinuous DAEs (differential algebraic equations) with a co-dimension 1 discontinuity manifold Σ. By discontinuous DAE, DDAEs for short, we mean a DAE system where either or both the differential system and the algebraic constraint change discontinuously as solution trajectories reach a given discontinuity surface Σ.

For us, Σ will always be a surface of co-dimension 1, that is Σ will be always defined as 0-set of a smooth function:

$$\Sigma := \{x \in \mathbb{R}^n \ : h(x) = 0 \ , \ \ h : \ \mathbb{R}^n \to \mathbb{R}\} \,. \tag{1}$$

Geometrically, Σ divides $\mathbb{R}^n$ in two regions, R^+ and R^-, according to whether $h(x) > 0$ or $h(x) < 0$. We note that with this form of Σ, differential equations (ODEs not DAEs) with discontinuous right-hand side have been studied for a very long time,

FIGURE 1. Crossing and Sliding Point.



and that Filippov construction provides a powerful and well established theory to provide some meaning to these systems; see [9]. Next, we briefly recall the key points of Filippov theory.

Suppose one has the piecewise smooth ODE

$$R^- : \quad \dot{x} = f^-(x) \ , \ \ h(x) < 0 \, , \qquad R^+ : \quad \dot{x} = f^+(x) \ , \ \ h(x) > 0 \, , \tag{2}$$

with $x \in \mathbb{R}^n$, $f^\pm : \mathbb{R}^n \to \mathbb{R}^n$, $h : \ \mathbb{R}^n \to \mathbb{R}$, and Σ as in (1). Here, $f^\pm$ are assumed to be $\mathcal{C}^1$ (at least), and h is at least $\mathcal{C}^2$ in a neighborhood of Σ. Now, let $x \in \Sigma$ and compute the quantities

$$w^\pm(x) \ = \ \nabla h^T(x) f^\pm(x) \ . \tag{3}$$

Then, $x \in \Sigma$ is a *crossing, respectively sliding, point* of (2) if

$$\text{crossing:} \quad w^-(x) \cdot w^+(x) > 0, \ \ \text{sliding:} \quad w^-(x) \cdot w^+(x) < 0 \, ,$$

and further an *attractive sliding point* if

$$w^-(x) > 0 \ , \quad \text{and} \quad w^+(x) < 0 \ . \tag{4}$$

See Figure 1 for an illustration. (The case of $w^-(x) < 0$ and $w^+(x) > 0$ correspond to a so-called *repulsive sliding point*, which is an ill-posed configuration, and not further considered). Whereas, in the generic case, at crossing points of Σ the vector field is naturally defined as f^- or f^+, the vector field at attractive sliding points is not uniquely defined. Filippov theory postulates that, at an attractive sliding point, the solution must remain on Σ and will obey a differential system with vector field chosen as that convex combination of f^- and f^+ which lies on the tangent plane at Σ. That is, the sliding vector field is defined as:

$$\dot{x} \ = \ (1-\alpha) f^-(x) + \alpha f^+(x) \ , \quad \text{where} \quad \alpha = \frac{w^-(x)}{w^-(x) - w^+(x)} \, . \tag{5}$$

To reiterate, upon classifying points of Σ (and, thus, parts of Σ) as crossing or attractive, then one defines sliding motion according to Filippov theory on the portion of Σ which is attractive. Further, Filippov theory postulates that, generically, one will leave Σ with f^- or f^+ precisely when $\alpha = 0$ or $\alpha = 1$.

It should be appreciated that Filippov theory is not the only possible way to make sense of a discontinuous ODE like (2), and alternatives to Filippov theory exist; e.g., for control problems, see [21]. Filippov himself in [9, pp.61-63], gives a clear exposition of when the sliding vector field of a physical system can be defined as in equation (5) or whether a different definition should instead be considered. We point out (e.g., see [8, 21]) that the Filippov sliding vector field –see above– is also the vector field selected by a limiting process of a numerical discretization about the discontinuity surface; that is, the limit of the chattering behavior of a numerical trajectory that switches between the two given vector fields about the discontinuity surface (hysteresis). In the present paper, we will limit our consideration to Filippov construction when considering DDAEs.

Remark 1. *The paper* [3] *contains an overview of models, theory and numerics of* hybrid systems*, i.e. systems of ODEs or DAEs with instantaneous transitions triggered by zero sets of discontinuity functions. In* [18] *efficient numerical methods for discontinuous DAEs are presented. In the papers just cited, only crossing solutions (transitions) are taken into account (also in* [17] *transitions triggered by time dependent events for linear switched DAEs are considered). However, for DDAEs, there is no analog to Filippov sliding mode theory, and our main scope in this work is to understand how to do this in the present DDAE case.*

It must be appreciated that models of discontinuous DAEs have been proposed in the chemical engineering literature, see [1, 4], where the authors also proposed Filippov-like sliding solutions. Likewise, some effort on solvability and index reduction techniques for discontinuous DAEs is in [15, 19] who also put forward some Filippov-like solution techniques. However, no justification for these techniques has thus far been provided, and to fill this gap is one of our scopes in the present work.

Our goal is to study the solutions of DDAEs, and classify and define attractivity of Σ and crossing/sliding solutions, by adapting the known Filippov theory for discontinuous ODEs to the DDAEs. We will then follow this approach:

(i) we will consider discontinuous ODEs that have same solutions as the DDAE in R^- and R^+;
(ii) we will apply Filippov theory to the discontinuous ODEs;
(iii) we will define sliding/crossing solutions of the DDAE to be the same as the sliding/crossing solutions of the resulting discontinuous ODEs.

We will restrict to index 1 DAEs, but even this "simpler" case presents considerable challenges when trying to define attractivity of Σ and ensuing sliding motion on it. We will investigate the occurrence of sliding motion and give sufficient conditions that allow one to define Filippov sliding solutions along Σ. Some restrictions on Σ will be necessary to establish whether crossing or sliding should take place. Moreover, unless we make further restrictive assumptions on the algebraic constraints of the DAE under study, multiple sliding solutions of Filippov type can be defined on Σ and this in turn might give rise to different qualitative behaviors and hence to ambiguous sliding dynamics. In a nutshell, we can sum up our insight as saying that in general sliding solutions are not unambiguously defined. We will elaborate on these aspects in this paper.

In Section 2, we recall some basic results for smooth DAEs which are useful for our purposes. In Section 3, we discuss restrictions on Σ, appropriate forms for the algebraic constraints, and explore three possible approaches to define Filippov sliding solutions on Σ. In the standard DAE case, these three approaches are in fact equivalent to one another, but for DDAEs they can be quite different. In Section 4, we further restrict the form of the constraint defining the discontinuity surface Σ and provide a comparison in greater detail of the direct substition and singular perturbation approaches. In Section 5 we will present results for a DDAE arising in chemical engineering, and finally in Section 6, we give conclusions and briefly discuss the case of higher index.

2. Smooth DAEs

In the smooth case, the standard form of DAE we consider is one with differential variables $y \in \mathbb{R}^d$ and algebraic variables $z \in \mathbb{R}^a$, where $d + a = n$ the dimension of the ambient space, and can be given as

$$\begin{cases} \dot{y} = f(y,z) \\ g(y,z) = 0 \end{cases}, \quad t \geq 0 \;, \tag{6}$$

where the functions $f: \mathbb{R}^d \times \mathbb{R}^a \to \mathbb{R}^d$ and $g: \mathbb{R}^d \times \mathbb{R}^a \to \mathbb{R}^a$ are assumed to be sufficiently smooth: we will require $f \in \mathcal{C}^p(\mathbb{R}^d \times \mathbb{R}^a)$ and $g \in \mathcal{C}^p(\mathbb{R}^d \times \mathbb{R}^a)$, with $p \geq 1$. The DAE (6) is subject to initial conditions (y_0, z_0) satisfying the algebraic constraint: $g(y_0, z_0) = 0$, and we further will assume that the matrix

$$\frac{\partial g}{\partial z} \quad \text{is invertible along solution trajectories} \,. \tag{7}$$

In DAE terminology, the requirement (7) states that the DAE (6) is of index 1. We note that the class of problems (6)-(7) is also called a *ODE with constraints*, or a *semi-explicit DAE of index 1*, or a *Hessenberg index 1 DAE*; see [2]. Of course, the assumption (7) makes it possible, using the implicit function theorem, to locally solve the algebraic constraint for z in function of y : $z = k(y)$, where $k: \mathbb{R}^d \to \mathbb{R}^a$ is $\mathcal{C}^p$. Substituting this in the differential equation for y one is left with a standard initial value problem of ODEs. As simple as this consideration is, let us hence rewrite (6) with this acquired knowledge:

$$\begin{cases} \dot{y} = f(y,z) \\ z = k(y) \end{cases}, \quad t \geq 0 \;, \quad y(0) = y_0 \;, \;\; z_0 = k(y_0) \;. \tag{8}$$

Of course, now one just has to solve the DAE (8), but for later use let us nonetheless express three (mathematically equivalent) ways in which one may proceed.

(1) *Direct Substitution Approach* or *state-space form*. This is the approach given by integrating the ODE

$$\dot{y} = f(y, k(y)) \ , \quad t \geq 0 \ , \quad y(0) = y_0 \ . \tag{9}$$

The solution in this case is $y(t)$ and then one recovers $z(t) = k(y(t))$.

(2) *Singular Perturbation Approach*. Here one considers the enlarged system

$$\begin{cases} \dot{y} = f(y, z) \\ \epsilon \dot{z} = k(y) - z \end{cases} , \quad t \geq 0 \ , \quad y(0) = y_0 \ , \ \ z(0) = z_0 \ , \tag{10}$$

and then let $\epsilon \to 0$. If we call $(y_\epsilon(t), z_\epsilon(t))$ the solution of the ODE (10), then (using [20], and see Theorem 21 below) one has that $\lim_{\epsilon \to 0} (y_\epsilon(t), z_\epsilon(t)) = (y(t), z(t))$, the same solution of (9).

(3) *Weak Formulation Approach*. Here one differentiates the algebraic constraint in order to define a differential equation for z. Doing so in (8) gives

$$\begin{cases} \dot{y} = f(y, z) \\ \dot{z} = k_y(y) f(y, z) \end{cases} , \quad t \geq 0 \ , \quad y(0) = y_0 \ , \ \ z(0) = z_0 \ . \tag{11}$$

Again, the exact solution of (11) is the same as the solution of (9).

As we will elaborate in this work, these three formulations above are usually not equivalent when dealing with nonsmooth DAEs.

Remark 2. *It is worth remarking that the three formulations recalled above have all been used in the literature on numerical methods for DAEs; e.g., see* [11, 12].

3. General Index 1 DDAE

There are many ways to generalize index 1 DAEs to nonsmooth index 1 DAEs (or DDAEs). For example, consider the general model below, where both vector fields and algebraic constraints change (discontinuously) as the differential and algebraic variables cross a hypersurface Σ:

$$R^- : \quad \begin{cases} \dot{y} = f^-(y, z) \\ z = k^-(y) \end{cases} , \ \ h(y, z) < 0 \,, \qquad R^+ : \quad \begin{cases} \dot{y} = f^+(y, z) \\ z = k^+(y) \end{cases} , \ \ h(y, z) > 0 \,, \tag{12}$$

with $(y, z) \in \mathbb{R}^d \times \mathbb{R}^a$, $f^\pm : \mathbb{R}^d \times \mathbb{R}^a \to \mathbb{R}^d$, $k^\pm : \mathbb{R}^d \to \mathbb{R}^a$, $h : \ \mathbb{R}^d \times \mathbb{R}^a \to \mathbb{R}$, and we are letting $\Sigma := \{(y, z) \in \mathbb{R}^d \times \mathbb{R}^a \ : \ h(y, z) = 0\}$.

Remark 3. *We have assumed to have the same algebraic variables in R^+ and R^-. Even when this is seemingly not the same, we substituted directly in the respective differential system those that are not in common in the two regions. This case, for example, arises in Example 5 below.*

In the model (12), the fact that the function h depends on both y and z means that to define points on Σ as attractive or crossing, we must be able to access the derivative of z. In other words, in $R^{\pm}$ we must view the respective DAEs according to the "weak formulation approach," since we must be able to access the derivative of the algebraic variables, in order to obtain a well defined derivative $\frac{d}{dt}h(y(t),z(t))$ for trajectories that approach Σ from R^- or R^+; viz., $\frac{d}{dt}h(y(t),z(t)) = h_y\dot{y} + h_z\dot{z}$.

In short, we effectively need to consider the enlarged piecewise smooth system of ODEs for $x = \begin{bmatrix} y \\ z \end{bmatrix} \in \mathbb{R}^n$, given by

$$R^- \ : \ \dot{x} = \begin{bmatrix} f^-(y,z) \\ k_y^-(y)(f^-(x)) \end{bmatrix} \ , \ h(x) < 0 \, , \quad R^+ \ : \ \dot{x} = \begin{bmatrix} f^+(y,z) \\ k_y^+(y)(f^+(x)) \end{bmatrix} \ , \ h(x) > 0 \, , \tag{13}$$

with given initial conditions x_0. However, we must be careful that the initial conditions x_0 be **compatible** with the original (12); that is, if we have $x_0 = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} \in R^-$, then it must be that $z_0 = k^-(y_0)$, and if $x_0 \in R^+$, then it must be $z_0 = k^+(y_0)$. As a consequence, for $x_0 \in \Sigma$, we require that $k^+(y_0) = k^-(y_0)$ and this same condition must be satisfied along a suitable sliding solution. Hence, in order to apply Filippov's approach to (13), we require that:

$$\text{If} \quad x = \begin{bmatrix} y \\ z \end{bmatrix} \in \Sigma \, , \quad \text{then} \quad k^-(y) \ = \ k^+(y) \ . \tag{14}$$

Without (14), the problem (13) is not well posed. So, for (13), we assume that (14) holds and call $k(y)$ the common value of $k^{\pm}$ on Σ.

With this in mind, the case of the DDAE (12) reduces to the discontinuous ODE problem (13), which we can treat according to Filippov theory for discontinuous ODEs. Then, for example, at a point $x = (y,z) \in \Sigma$ the values of $w^{\pm}$ are as in (3): $w^{\pm}(x) = h_y(x)f^{\pm}(x) + h_z(x)\left(k_y^{\pm}(y)f^{\pm}(x)\right)$. And, $x \in \Sigma$ is, say, an attractive sliding point if $w^-(x) > 0$ and $w^+(x) < 0$. Sliding motion, therefore, will take place according to

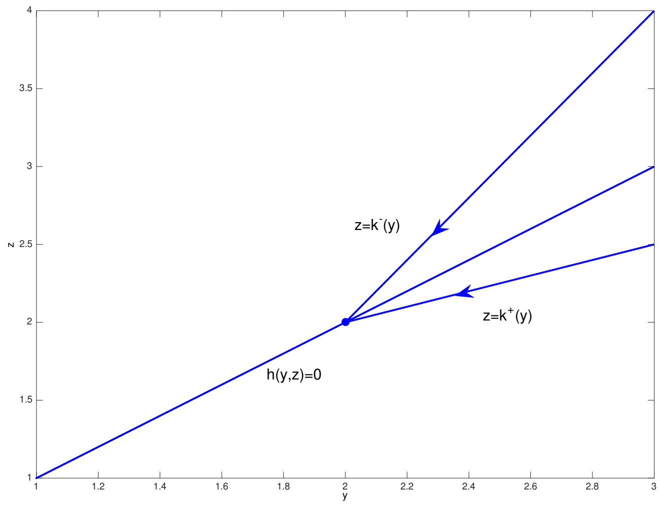
$$\dot{x} = \begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = (1-\alpha) \begin{bmatrix} f^-(y,z) \\ k_y^-(f^-) \end{bmatrix} + \alpha \begin{bmatrix} f^+(y,z) \\ k_y^+(f^+) \end{bmatrix} \ , \ \alpha(y,z) \ = \ \frac{w^-(y,z)}{w^-(y,z) - w^+(y,z)} \, , \tag{15}$$

with $x_0 \in \Sigma$, $x_0 = (y_0, k(y_0))^T$. In what follows, for ease of notation, we will drop the dependence of α on (y,z). In order to ensure that during sliding motion on Σ the solution of (15) remains on the manifold $z = k(y)$ as well, we must require that also the derivatives of k^+ and k^- coincide on Σ. Hence, we must have the following.

Compatibility Condition 1. *For the DDAE* (12) *to be well defined, we require that*

$$\textit{If} \quad (y,z) \in \Sigma, \qquad \textit{then} \qquad k^+(y) = k^-(y), \ \ k_y^+(y) = k_y^-(y) \ . \tag{16}$$

FIGURE 2. Example 4.



Example 4. *The following problem is an instance of* (12)*.*

$$R^- : \quad \begin{cases} \dot{y} = -1 \\ z = 2y - 2 \end{cases}, \quad y - z < 0\,, \qquad R^+ : \quad \begin{cases} \dot{y} = -1 \\ z = \frac{1}{2}y + 1 \end{cases}, \quad y - z > 0\,.$$

Here the discontinuity function is $h(y,z) = y - z$, and the functions defining the algebraic variables are $k^-(y) = 2y - 2$ and $k^+(y) = \frac{1}{2}y + 1$. There is only the point $(2,2)$ on Σ satisfying the compatibility condition, and thus this is the only possible initial condition on Σ. For $(y,z) \in \Sigma$ then $k^+(y) = k^-(y) = k(y) = 2$. In ./ 2 the solid line is $h(y,z) = 0$, while the dashed lines above and below the solid one are respectively $z = k^-(y)$ and $z = k^+(y)$. The point in bold is the point $(2,2)$ on the intersection of the three lines.The arrows indicate the direction of motion of the solution of the DAE in the two subregions R^- and R^+.

After rewriting the discontinuous DAE as in (13)*, we have*

$$w^- \;=\; 1 \quad \textit{and} \quad w^+ \;=\; -1/2 \quad \textit{thus} \quad \alpha = 2/3\;.$$

As a consequence, we would slide on Σ with differential equation $\dot{y} = \dot{z} = -1$, remaining on Σ but not on $z = k(y)$. Indeed, while (14) *is satisfied, the relation* (16) *is not. The vector field $\dot{y} = \dot{z} = -1$ obtained applying Filippov sliding vector field to* (13) *fails in describing a dynamic compatible with the original discontinuous DAE and hence can not be used in this context. See Example 5 below to see what happens instead when* (16) *is satisfied.*

Example 5. *This is an example of a DDAE* (12) *that satisfies* (16)*. As we will see, the Filippov sliding vector filed obtained by rewriting the original DDAE as a* *discontinuous ODE as in (13), is compatible with the original problem had the sliding solution remains on the intersection of* Σ *with the algebraic constraint* $z = k(y)$*. The problem is the following:*

$$R^- : \quad \begin{cases} \dot y = -z - 7y \\ z = 2y^2 - y + 1 \end{cases}, \ y - z + 1 < 0\,, \qquad R^+ : \quad \begin{cases} \dot y = z + 1 - y \\ z = y^2 + y \end{cases}, \ y - z + 1 > 0\,.$$

For this problem we have $k^-(y) = 2y^2 - y + 1$*,* $k^+(y) = y^2 + y$ *and they both intersect the discontinuity line* $z = y + 1$ *at the point* $(1, 2)$*. Hence on* Σ*,* $z = k(y) = 2$*. We rewrite the DAE as in (13) and obtain*

$$R^- : \quad \begin{cases} \dot y = -z - 7y \\ \dot z = (z + 7y)(1 - 4y) \end{cases}, \qquad R^+ : \quad \begin{cases} \dot y = z + 1 - y \\ \dot z = (2y + 1)(z + 1 - y) \end{cases}. \tag{17}$$

The components of the vector fields normal to Σ *are* $w^-(y, z) = (8y + 1)(4y - 2)$ *and* $w^+(y, z) = -4y$ *and* Σ *is attractive for* $y > 1/2$ *and* $z = y + 1$*. The point* $(1, 2)$ *satisfies this condition so that it is an attractive sliding point. If we compute Filippov sliding vector field for (17) at* $\{(1, 2)\}$*, we obtain* $\dot y = \dot z = 0$ *and the sliding solution remains at the intersection point of* Σ *with* $z = k(y)$*.*

To sum up, for the model (12), we propose the interpretation based on (13) as the appropriate interpretation to use for deriving a Filippov theory, for as long as the compatibility condition expressed by (16) is satisfied on Σ. In the next section, we consider the cases in which the discontinuity surface depends only on the differential variables y. We will see that different -non equivalent- interpretations are possible when $k^+ \neq k^-$ on Σ.

4. Discontinuity function independent from the algebraic variables

In this section we deal with Hessenberg Index 1 DDAEs with discontinuity function that does not depend on the algebraic variables, i.e. $h = h(y)$. Therefore, our model DDAE is the following.

$$R^- : \quad \begin{cases} \dot y = f^-(y, z) \\ z = k^-(y) \end{cases}, \ h(y) < 0\,, \qquad R^+ : \quad \begin{cases} \dot y = f^+(y, z) \\ z = k^+(y) \end{cases}, \ h(y) > 0\,, \tag{18}$$

with $(y, z) \in \mathbb{R}^d \times \mathbb{R}^a$, $f^\pm : \mathbb{R}^d \times \mathbb{R}^a \to \mathbb{R}^d$, $k^\pm : \mathbb{R}^d \to \mathbb{R}^a$. Above, the function $h : \mathbb{R}^d \to \mathbb{R}$, is assumed to be sufficiently smooth, we will let $\Sigma := \{(y, z) \in \mathbb{R}^d \times \mathbb{R}^a \ : \ h(y) = 0\}$, and $\Sigma_y := \{y \in \mathbb{R}^d \ : \ h(y) = 0\}$, and will further assume that h_y is full rank for all $y \in \Sigma_y$. As a consequence, Σ_y is a smooth $(d-1)$-dimensional manifold embedded in $\mathbb{R}^d$.

Admittedly, the model under study might seem reductive. For one thing, it may be hard to rewrite an Index 1 DAE in Hessenberg form; moreover, in the literature (see [1, 4, 15, 19]) several authors have put forward Hessenberg DDAEs with a different number of differentiable and algebraic variables in each subregion. However, we restrict our analysis to equation (18) where in each subregion there are the same differentiable and algebraic variables, and do so in order to derive a Filippov theory. In particular, the restriction on the differentiable variables is essential to apply the original Filippov theory; the restriction on the algebraic variables, on the other hand, could be dropped in certain cases, see Remark 20 for an example. Regardless, we will see that unexpected difficulties arise already in our simplified scenario of (18). As a matter of fact, unless we further restrict the class of DDAEs we consider, it is not possible to give a non ambiguous definition of crossing and sliding points and hence crossing/sliding solutions for Filippov type DDAEs.

In the quest for a non ambiguous Filippov type DDAE on Σ, we first restrict our attention to DDAEs with smooth algebraic constraints. Under this restriction, we can define sliding/crossing points in a non ambiguous way and, in case of sliding, we can define a unique Filippov sliding solution regardless of the equivalent discontinuous ODE we consider. This restriction and the results associated with it might seem at first trivial, however the need to put on firm ground these fairly intuitive ideas will become apparent after Section 4.2. There, we consider DDAEs with discontinuous algebraic constraints and show how a Filippov based approach does not even allow a non ambiguous classification of sliding/crossing points on Σ.

4.1. **Smooth algebraic constraint.** The first restriction we put on (18) is that the function(s) $k^{\pm}(y)$ expressing the algebraic constraint be the same (in principle, it would be sufficient for them to be the same in a neighborhood of Σ). So, we will take $k^-(y) = k^+(y) = k(y)$ and k to be at least a $\mathcal{C}^2$ function for all $y \in \mathbb{R}^d$; Example 3.1 in [4] is of this type. This will turn out to be the easy case, when all three classic approaches for DAEs give the same sliding vector field of Filippov type. In spite of the fact that this is indeed a simpler case, we discuss it for completeness here below.

Hence, the model under study can be written as

$$\begin{cases} \dot{y} = f^{\pm}(y,z)\ , & h(y) \gtrless 0\ , \\ z = k(y)\ , & \text{for all } \ y\ . \end{cases} \tag{19}$$

Presently, we can rewrite $R^{\pm} = \{(y,z) \in \mathbb{R}^d \times \mathbb{R}^a \mid h(y) \gtrless 0\}$, and since solutions of (18) must satisfy $z = k(y)$, the study of crossing/sliding regions can be restricted to

$$S \ = \ \{(y,z) \in \Sigma \ \mid \ z = k(y)\}\ .$$

Without loss of generality let $(y_0, z_0) \in R^-$, $z_0 = k(y_0)$ and assume that the solution of (19) with initial condition (y_0, z_0) reaches S at time $\bar{t}$ at the point $(\bar{y}, k(\bar{y}))$.

Next, we will show that all three possible reformulations (see Section 2) of the DDAE as a discontinuous ODE will give the same sliding vector field for the DDAE.

(1) *Direct substitution*, cfr. (9). We rewrite (19) as

$$\dot{y} = f^{\pm}(y, k(y)), \;\; h(y) \gtrless 0. \tag{20}$$

We classify sliding/crossing regions of Σ_y via Filippov theory. Let $y \in \Sigma_y$ and consider the components of $f^{\pm}$ normal to Σ_y:

$$w^{\pm}(y) = h_y^T(y) f^{\pm}(y, k(y)) \; . \tag{21}$$

Then, a point $\bar{y} \in \Sigma_y$ is a *crossing, respectively sliding, point* of (20) if

$$\text{crossing:} \quad w^-(\bar{y}) \cdot w^+(\bar{y}) > 0, \;\; \text{sliding:} \quad w^-(\bar{y}) \cdot w^+(\bar{y}) < 0\,.$$

Moreover $\bar{y} \in \Sigma_y$ is an *attractive sliding point*, respectively *repulsive sliding point*, if

$$w^-(\bar{y}) > 0 \;, \quad \text{and} \quad w^+(\bar{y}) < 0 \;, \;\; (w^-(\bar{y}) < 0 \;, \quad \text{and} \quad w^+(\bar{y}) > 0) \; . \tag{22}$$

The point $\bar{y} \in \Sigma_y$ is a tangential exit point into R^-, respectively into R^+, if

$$\begin{aligned} &\text{into } \; R^- : \quad w^-(\bar{y}) = 0, \;\; w^+(\bar{y}) < 0, \;\; w_y^-(\bar{y}) \neq 0 \;, \\ &\text{into } \; R^+ : \quad w^+(\bar{y}) = 0, \;\; w^-(\bar{y}) > 0, \;\; w_y^+(\bar{y}) \neq 0 \;. \end{aligned}$$

Repulsive sliding points will not be considered in the sequel, since trajectories through these (i.e., starting at these) are not uniquely defined. Crossing and attractive sliding points, instead, give rise to crossing or sliding Filippov solutions.

At a crossing point, solution trajectories simply leave Σ_y and pass from a region to the other, changing the vector field, hence producing a continuous but not smooth solution. At an attractive sliding point, instead, solutions trajectories will be forced to remain on Σ_y (and hence on Σ). According to Filippov theory, motion will proceed by solving the following differential equation

$$\dot{y} = (1-\alpha) f^-(y, k(y)) + \alpha f^+(y, k(y)) = f_F(y, k(y)), \;\; y(\bar{t}) = \bar{y} \;, \tag{23}$$

where $\alpha = \alpha(y)$ is such that $\dot{y}$ is in the tangent plane to Σ_y:

$$h_y^T(y) f_F(y, k(y)) \;=\; 0 \;,$$

and thus

$$\alpha(y) \;=\; \frac{w^-(y)}{w^-(y) - w^+(y)} \;, \tag{24}$$

so that $\alpha(y) \in (0,1)$ because of (22). The solution trajectory will continue solving (23) until it may reach an exit point, that is a point where it will leave Σ_y, a fact that generically will occur at tangential exit points. For later comparison, let us call $(y(t), z(t)) = (y_{\rm ds}(t), k(y_{\rm ds}(t)))$ the solution obtained with the present "direct substitution" approach.

(2) *Singular perturbation formulation*, cfr. (10). We replace the DDAE in each of the subregions $R^{\pm}$ with a singularly perturbed discontinuous ODE as follows:

$$\begin{cases} \dot{y} = f^{\pm}(y,z), & h(y) \gtrless 0 \\ \dot{z} = \frac{1}{\epsilon}(k(y)-z) \end{cases} . \tag{25}$$

It is well known that solutions of (25) converge uniformly (in a time interval that does not contain the initial time) to solutions of (19) in each subregion $R^{\pm}$ as the parameter ϵ goes to zero (e.g., see Theorem 21 below). For as long as it remains in R^{-}, call $(y^{\epsilon}_{\mathrm{sp}}(t), z^{\epsilon}_{\mathrm{sp}}(t))$ the solution of (25) with initial condition (y_0, z_0), so that

$$\lim_{\epsilon\to 0}(y^{\epsilon}_{sp}(t), z^{\epsilon}_{sp}(t)) = (y(t), z(t)), \tag{26}$$

uniformly in time in $[0, \bar{t}]$. It follows that for ϵ sufficiently small, there exists $t = \bar{t}_{\epsilon}$ such that $(y^{\epsilon}_{\mathrm{sp}}(\bar{t}_{\epsilon}), z^{\epsilon}_{\mathrm{sp}}(\bar{t}_{\epsilon}))$ reaches Σ at the point $(\bar{y}_{\epsilon}, \bar{z}_{\epsilon})$. Notice that, by continuity, we must have

$$\bar{t}_{\epsilon} \to \bar{t}, \;\; \bar{y}_{\epsilon} \to \bar{y}, \;\; \bar{z}_{\epsilon} \to \bar{z},$$

but we cannot expect $\bar{z}_{\epsilon} = k(\bar{y}_{\epsilon})$ for $\epsilon \neq 0$. In order to classify a crossing/sliding point of (25), we first write the components of the vector field normal to Σ:

$$w^{\pm}(y,z) = h_y(y) f^{\pm}(y,z).$$

A point $(y,z) \in \Sigma$ is a crossing (sliding) point for (25) if

$$\text{crossing: } w^{+}(y,z)w^{-}(y,z) > 0, \qquad \text{sliding: } w^{+}(y,z)w^{-}(y,z) < 0.$$

The conditions above do not depend on ϵ, hence a point $(y,z) \in \Sigma$ is a crossing/sliding point for (25) for all ϵ. From this and (26) it follows that for ϵ sufficiently small $(\bar{y}_{\epsilon}, \bar{z}_{\epsilon})$ is a crossing/sliding point if $(\bar{y}, k(\bar{y}))$ is a crossing/sliding point. If $(\bar{y}_{\epsilon}, \bar{z}_{\epsilon})$ is a crossing point, then $(y^{\epsilon}_{\mathrm{sp}}(t), z^{\epsilon}_{\mathrm{sp}}(t))$ crosses Σ to enter R^{+}. If, instead, $(\bar{y}_{\epsilon}, \bar{z}_{\epsilon})$ is an attractive sliding point, then $(y^{\epsilon}_{\mathrm{sp}}(t), z^{\epsilon}_{\mathrm{sp}}(t))$ remains on Σ, as solution of the following ODE

$$\begin{cases} \dot{y} = (1-\alpha^{\epsilon})f^{-}(y,z) + \alpha^{\epsilon} f^{+}(y,z), \\ \epsilon\dot{z} = (k(y)-z), \end{cases} \tag{27}$$

with α^{ϵ} chosen so that $h_y\dot{y} = 0$, i.e. $\alpha^{\epsilon} = \alpha^{\epsilon}(y,z) = \frac{h_y(y)f^{-}(y,z)}{h_y(y)(f^{-}-f^{+})(y,z)}$. Singular perturbation theory (again, see Theorem 21) ensures that in the limit for $\epsilon \to 0$, solutions of (27) converge to solutions of the DAE

$$\dot{y} = (1-\alpha^{0})f^{-}(y,z) + \alpha^{0} f^{+}(y,z), \;\; z = k(y),$$

where $\alpha^{0} = \frac{w^{-}(y)}{w^{-}(y)-w^{+}(y)}$ the same value as in (24). The theory guarantees uniform convergence in $[\bar{t}+\delta, \bar{t}+\Delta]$, with $\delta > 0$, and with $\Delta > 0$ chosen so that the solution of (27) remains on Σ in the time interval $[\bar{t}, \bar{t}+\Delta]$. Since

$\bar{z}_\epsilon \to k(\bar{y})$, in this case the convergence is uniform in $[\bar{t}, \bar{t}+\Delta]$. Therefore, taking the limit as $\epsilon \to 0$ of $(y^\epsilon_{\rm sp}(t), z^\epsilon_{\rm sp}(t)$, we have that this coincides with $(y(t), z(t))$ for $t \le \bar{t}$ and further with $(y_{\rm da}(t), z_{\rm da}(t))$ while sliding.

Remark 6. *System (25) is a discontinuous slow fast system with a smooth slow manifold. In [5] the authors study the qualitative behaviour of these discontinuous systems and they compare it with the qualitative behaviour of the reduced system, the same as (20) in this paper. They show that if the reduced system has an equilibrium or a periodic orbit on the discontinuity manifold, then, for ϵ sufficiently small, also the slow-fast system does. The proof of their result is based on results of singular perturbation theory as well.*

(3) *Weak formulation*, cfr. (11). Here we consider the weak formulation of the DDAE (19):

$$\begin{cases} \dot{y} = f^+(y,z) \\ \dot{z} = k_y(y) f^+(y,z) \end{cases}, \quad h(y) > 0 \ , \quad \begin{cases} \dot{y} = f^-(y,z) \\ \dot{z} = k_y(y) f^-(y,z) \end{cases}, \quad h(y) < 0 \ , \tag{28}$$

subject to consistent initial conditions (y_0, z_0) in R^-. Denote with $(y_{\rm w}(t), z_{\rm w}(t))$ the solution of (28), so that at $\bar{t}$ the solution reaches S at the point $(\bar{y}, k(\bar{y}))$.

Following Filippov, the conditions for $(\bar{y}, k(\bar{y}))$ to be a sliding/crossing point are the same as in the "Direct Approach" of point (1) above, and if $(\bar{y}, k(\bar{y}))$ is a crossing point, then $(y_{\rm w}(t), z_{\rm w}(t))$ crosses S to enter R^+. If instead $(\bar{y}, k(\bar{y}))$ is an attractive sliding point, $(y_{\rm w}(t), z_{\rm w}(t))$ must remain on Σ, as solution of the differential equation

$$\begin{cases} \dot{y} = (1-\alpha) f^-(y,z) + \alpha f^+(y,z), \\ \dot{z} = (1-\alpha) k_y f^-(y,z) + \alpha k_y f^+(y,z), \\ y(\bar{t}) = \bar{y}, \ \ z(\bar{t}) = k(\bar{y}), \end{cases} \tag{29}$$

with α so that $h_y^T \dot{y} = 0$, i.e.,

$$\alpha(y) = \frac{h_y(y) f^-(y,z)}{h_y(y)\left(f^-(y,z) - f^+(y,z)\right)}. \tag{30}$$

Since $z(t) = k(y(t))$ satisfies the differential equation for z in (29), then $z_{\rm w}(t) = k(y_{\rm w}(t))$ while sliding so that α in (30) is the same as in (24). It follows that $y_{\rm w}(t) = y_{\rm ds}(t)$.

We reiterate here that, under the assumption of a smooth algebraic constraint, and for initial condition $y_0, z_0 = k(y_0)$, the functions $(y_{\rm ds}(t), k(y_{\rm ds}))$, $\lim_{\epsilon\to 0}(y^\epsilon_{\rm sp}(t), z^\epsilon_{\rm sp}(t))$, and $(y_{\rm w}(t), z_{\rm w}(t))$, are all the same for all t. Therefore, in this case of smooth algebraic constraint, any of these can be called the **Filippov solution** of (19).

4.2. **Discontinuous algebraic constraint.** In this section we study the DDAE (18) where the algebraic constraints are different, in particular on Σ, $k^-(y) \neq k^+(y)$. In light of the results in Section 3, in particular see (14), the *Weak formulation* approach will not be considered, since we would need $k^+(y) = k^-(y)$ on Σ, which we do not have. For this reason, in this section we consider (see Section 2) only the *Direct Substitution Approach* and the *Singular Perturbation Approach*.

Our goals are the following: (i) highlight that the definition of sliding/crossing points depends on the specific approach we choose, and therefore that the sliding solution is in general not the same for the two approaches; (ii) show that, under certain assumptions, the singular perturbation approach leads to the use of a convex combination of the algebraic constraints during sliding motion; (iii) show that the sliding DAE on Σ obtained as limit of the singular perturbation approach is not necessarily of index 1.

4.2.1. *Direct Substitution Approach.* Much like in (9), we consider the following discontinuous ODE in $\mathbb{R}^d$

$$\dot{y} = \begin{cases} f^+(y, k^+(y)) \ , & h(y) > 0 \ , \\ f^-(y, k^-(y)) \ , & h(y) < 0 \ , \end{cases} \tag{31}$$

with discontinuity surface $\Sigma_y = \{y \in \mathbb{R}^d | h(y) = 0\}$, and $\Sigma = \Sigma_y \times \mathbb{R}^a$. System (31) is the same as [1, eq. (29)].

When adopting this approach, we say that a point $\bar{y} \in \Sigma_y$ is a *crossing/sliding* point for (18) if it is a crossing/sliding point for (31). Namely:

$$\begin{aligned} \text{crossing:} \quad & h_y(\bar{y}) f^-(\bar{y}, k^-(\bar{y}))\ h_y(\bar{y}) f^+(\bar{y}, k^+(\bar{y})) > 0 \ , \\ \text{sliding:} \quad & h_y(\bar{y}) f^-(\bar{y}, k^-(\bar{y}))\ h_y(\bar{y}) f^+(\bar{y}, k^+(\bar{y})) < 0 \ , \end{aligned}$$

and further an *attractive sliding point* if

$$h_y(\bar{y}) f^-(\bar{y}, k^-(\bar{y})) > 0 \quad \text{and} \quad h_y(\bar{y}) f^+(\bar{y}, k^+(\bar{y})) < 0 \, .$$

Sliding motion –according to Filippov theory– will take place as in (5), namely

$$\dot{y} \;=\; (1-\alpha) f^-(y, k^-(y)) + \alpha f^+(y, k^+(y)) \ , \quad \text{where} \quad \alpha = \frac{h_y f^-}{h_y \left(f^- - f^+\right)} \, , \tag{32}$$

and indeed the Filippov theory known for ODE systems applies to the discontinuous DAE in this case.

Remark 7. *We must notice that in the rewriting* (31)*, there is no longer any explicit dependence on the algebraic variables z. As a consequence, z is* **not specified** *on Σ by using the formulation* (31)*-*(32)*.*

4.2.2. *Singular Perturbation Approach.* In this section, we consider a certain singularly perturbed problem (see (33) below) as a possible mean to define solutions of the DDAE, similarly to what is done for DAEs (see the ϵ-Embedding method in [11] and [12]). We extended this approach to DDAEs.

At the same time, the singular perturbation problem we put forward is quite interesting in its own right, and it directly relates to singular perturbation theory of discontinuous slow-fast systems . This topic has seen several contributions in the past years, see for example [10], for discontinuous singularly perturbed systems with the discontinuity depending only on one slow variable, [13] for planar slow fast systems with discontinuity manifold depending also on the parameter ϵ and [5] for slow-fast systems with a smooth slow manifold). Our model DDAE is the reduced slow system of the following discontinuous slow fast system

$$\begin{cases} \dot{y} = f^-(y,z) \\ \epsilon\dot{z} = k^-(y) - z \end{cases}, \; h(y) < 0 \; ; \quad \begin{cases} \dot{y} = f^+(y,z) \\ \epsilon\dot{z} = k^+(y) - z \end{cases}, \; h(y) > 0 \; . \tag{33}$$

System (33) has a discontinuous slow manifold (if $k^- \neq k^+$) and nonsmooth slow variable (if $f^+ \neq f^-$) . When $\epsilon = 0$ in (33), the reduced system is the same as (18). Now, in $R^{\pm}$, singular perturbation theory ensures that, as $\epsilon \to 0$, solutions of (33) converge uniformly in time to solutions of (18). The convergence is uniform in a closed interval that does not contain 0 if the initial condition is not consistent with the algebraic constraint.

The conditions for crossing or sliding points on Σ for (33) are the same for all ϵ since $h = h(y)$ does not depend on z. Following our characterization given in the Introduction, see (4) and Figure 1, a point $(y,z) \in \Sigma$ is a *crossing* (resp. *sliding*) point for (33) if the following is verified

$$(h_y(y)f^-(y,z))(h_y(y)f^+(y,z)) > 0 \,(\text{resp. } (h_y(y)f^-(y,z))(h_y(y)f^+(y,z)) < 0).$$

Let $(y_0, k^-(y_0)) \in R^-$ and denote with $(y(t), z(t))$ the solution of (18) with initial condition $(y_0, k^-(y_0))$. Assume that $(y(t), z(t))$ reaches Σ transversally at time $t = \bar{t}$ at the point $(\bar{y}, k^-(\bar{y})) = (\bar{y}, \bar{z})$; then, it must be $h_y(\bar{y})f^-(\bar{y}, \bar{z}) > 0$ and, by continuity, $h_y(y)f^-(y,z) > 0$ in a neighborhood of $(\bar{y}, \bar{z})$ on Σ. Let $(y_\epsilon(t), z_\epsilon(t))$ be the solution of (33) with initial condition $(y_0, k^-(y_0))$. Tikhonov's Theorem (again, see Theorem 21) guarantees that

$$\lim_{\epsilon \to 0}(y_\epsilon(t), z_\epsilon(t)) = (y(t), z(t)),$$

in a suitable time interval $[0, T]$. Uniform convergence of $(y_\epsilon(t), z_\epsilon(t))$ to $(y(t), z(t))$ and $h_y(\bar{y})f^-(\bar{y}, \bar{z}) > 0$ imply that there is a time $t = \bar{t}_\epsilon$ such that $(y_\epsilon(\bar{t}_\epsilon), z_\epsilon(\bar{t}_\epsilon))$ reaches Σ at the point $(\bar{y}_\epsilon, \bar{z}_\epsilon)$ and

$$\lim_{\epsilon \to 0} \bar{t}_\epsilon = \bar{t}, \;\; \lim_{\epsilon \to 0} \bar{y}_\epsilon = \bar{y}, \;\; \lim_{\epsilon \to 0} \bar{z}_\epsilon = \bar{z}.$$

Note that, in general, for $\epsilon \neq 0$, $z_\epsilon \neq k^-(y_\epsilon)$.

Notice that if $\bar{t}_\epsilon < \bar{t}$, for ϵ sufficiently small we can continue the solution of

$$\begin{cases} \dot{y} = f^-(y,z) \\ \epsilon\dot{z} = k^-(y) - z \end{cases},$$

up to $\bar{t}$ as long as f^- and k^- are defined in a neighborhood of Σ. Then the uniform convergence is in $[0, \bar{t}]$.

We study (33) with the aid of Filippov theory. Then we will define solutions of (18) on Σ as the limit for $\epsilon \to 0$ of Filippov solutions of (33).

Remark 8. *When following the singular perturbation approach, we decide whether solutions of (18) slide on Σ or cross it at a given point by looking at the limiting behaviour of Filippov solutions of (33). There is indeed no criterium to classify $(\bar{y}, \bar{z})$ as a crossing/sliding point for (18).*

We proceed as follows. Assume that $(\bar{y}, \bar{z}) \in \Sigma$ is an attractive sliding point for (33), i.e. it satisfies the following:

$$h_y(y)f^+(y,z) < 0, \;\; h_y(y)f^-(y,z) > 0. \tag{34}$$

Then, for ϵ sufficiently small the point $(\bar{y}_\epsilon, \bar{z}_\epsilon)$ is an attractive sliding point for (33) as well. On Σ we consider sliding motion defined by the classic Filippov construction relative to the system (33), that is from:

$$\begin{cases} \dot{y} = (1-\alpha)f^-(y,z) + \alpha f^+(y,z), \\ \epsilon\dot{z} = (1-\alpha)k^-(y) + \alpha k^+(y) - z. \end{cases} \tag{35}$$

Due to continuity of solutions with respect to initial conditions, since $(\bar{y}_\epsilon, \bar{z}_\epsilon) \to (\bar{y}, \bar{z})$, we consider the following initial conditions for (35): $y_\epsilon(0) = \bar{y}$, $z_\epsilon(0) = \bar{z}$. We denote the solution of the Cauchy problem as $(y_\epsilon(t), z_\epsilon(t))$. In (35) α is determined in order to insure sliding on Σ: $\alpha = \alpha(y,z) = \frac{h_y^T f^-(y_\epsilon, z_\epsilon)}{h_y^T (f^- - f^+)(y_\epsilon, z_\epsilon)}$. Setting $\epsilon = 0$ in (35) we obtain the *reduced* differential algebraic equation

$$\begin{cases} \dot{y} = (1-\alpha)f^-(y,z) + \alpha f^+(y,z), \\ z = (1-\alpha)k^-(y) + \alpha k^+(y), \end{cases} \tag{36}$$

with $\alpha = \alpha(y,z) = \frac{h_y(y)f^-(y,z)}{h_y(y)(f^- - f^+)(y,z)}$ and $y(0) = \bar{y}$. The solution of (36) must satisfy the algebraic constraint $z = (1-\alpha)k^-(y) + \alpha k^+(y)$. Beware that the existence of a solution is not always guaranteed. However, in Theorem 13 we show that under certain assumptions there exists a function $z = \varphi(y)$ that satisfies the algebraic constraint. We denote with $(y_0(t), z_0(t))$ the solution of (36). Convergence of solutions of (35) to solutions of (36) as $\epsilon \to 0$ is not always guaranteed, even when the algebraic constraint can be explicitly resolved. Below, we give sufficient conditions for convergence following arguments used in singular perturbation theory (see [20, 14]).

Once convergence is verified (in a time interval that does not contain 0, since in general $z_0(0) \neq z_\epsilon(0)$), then we can use (36) as sliding equation for (18).

Remark 9. *Equation (36) for sliding solutions on Σ is used in [4, Equations (14)-(15)] without justification. However, the convergence of solutions of (33) to solutions of (36) as ϵ goes to 0 is not always guaranteed since (35) is not continuous at $\epsilon = 0$, and use of (36) requires justification. Moreover, it is not clear how to classify a point (y, z) with $z \neq k^{\pm}(y)$, as a sliding/crossing point for (18).*

Remark 10. *It is important to stress that there is a key difference between sliding motion defined by the direct approach, see (31)-(32), and sliding motion defined by the singular perturbation approach. The key difference is that in the latter approach the algebraic variables z are defined, whereas in the former they are not.*

Let

$$G(y, z) \;=\; (1 - \alpha(y, z))k^-(y) + \alpha(y, z)k^+(y) - z \;= 0, \tag{37}$$

with $\alpha(y, z) = \frac{h_y(y)f^-(y,z)}{h_y(y)(f^- - f^+)(y,z)}$, characterize the slow manifold of (35). Equation (36) blows up the dynamics of (35) along the slow manifold. Consider the time stretching $\tau = \frac{t}{\epsilon}$ and rewrite (35) using the derivative with respect to τ

$$\begin{cases} y' = \epsilon\left((1 - \alpha)f^-(y, z) + \alpha f^+(y, z)\right), \\ \quad z' = (1 - \alpha)k^-(y) + \alpha k^+(y) - z, \end{cases} \tag{38}$$

with initial conditions $y(0) = \bar{y}$, and $z(0) = \bar{z} = k^-(\bar{y})$. If $(y_\epsilon(t), z_\epsilon(t))$ is the solution of (35) then $(y_\epsilon(\epsilon\tau), z_\epsilon(\epsilon\tau))$ is the solution of (38). Setting $\epsilon = 0$, we obtain the *fast system*

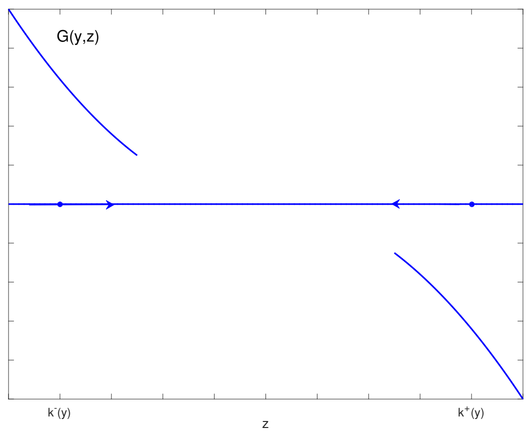
$$\begin{cases} y' = 0 \\ z' = (1 - \alpha(y, z))k^-(y) + \alpha(y, z)k^+(y) - z, \end{cases} \tag{39}$$

with $y(0) = \bar{y}$, $z(0) = \bar{z}$. Notice that, as long as the point (y, z) is an attractive sliding point of (33), α is in $(0, 1)$ and it is a smooth function of (y, z) inheriting the same regularity of f^- and f^+. Hence the vector field in (39) is smooth. In the following we will consider (39) as a differential equation in $\mathbb{R}^a$ depending on the parameter y. We denote with $(\tilde{y}(\tau), \tilde{z}(\tau)) = (\bar{y}, \tilde{z}(\tau))$ its solution. Continuity with respect to the parameter ϵ insures that the solution of (38) converges to the solution of (39), i.e.,

$$\lim_{\epsilon \to 0}(y_\epsilon(\epsilon\tau), z_\epsilon(\epsilon\tau)) = (\bar{y}, \tilde{z}(\tau)),$$

in a time interval $[0, \tau_0]$ with τ_0 so that the solution of (39) exists in $[0, \tau_0]$. In Proposition 11 below, we show that under certain assumptions solutions of (39) exist for all $\tau \geq 0$ and hence τ_0 can be taken sufficiently large. Denote with $K(y)$

FIGURE 3. Plot of the function $G = G(y, z)$ for y fixed in function of z



the convex hull of $k^+(y)$ and $k^-(y)$ in $\mathbb{R}^a$: $K(y) = \text{conv}\{k^+(y), k^-(y)\}$, and define the following set

$$\Sigma_K = \{(y, z) \in \Sigma \mid (y, z) \text{ satisfies } (34)\ \forall z \in K(y)\}.$$

We will show convergence of solutions of the full problem (35) to solutions of the reduced problem (36) in this set. Hence in what follows we will consider initial conditions in the set Σ_K. For $(y, z) \in \Sigma_K$, observe that α is in $(0, 1)$. We fix y and consider the map $G(y, \cdot) : K(y) \to \mathbb{R}^a$, $G(y, z) = 1 - \alpha(y, z))k^-(y) + \alpha(y, z)k^+(y)$. Since $\alpha \in (0, 1)$, $G(y, \cdot)$ maps $K(y)$ into itself and Brouwer Fixed Point Theorem implies that $G(y, \cdot)$ has a fixed point in $K(y)$. It follows that the fast equation (39) has at least one equilibrium in $K(y)$. We would like to give sufficient conditions so that (39) has a manifold of asymptotically stable equilibria depending on the parameter y, $z = \varphi(y)$. In Proposition 11 we restrict our attention to the case of one algebraic variable, i.e. $a = 1$, and in Theorem 13 we consider $a = 1$ together with two differential variables, i.e. $d = 2$, and h linear.

Proposition 11. *Let $a = 1$ in (18), $G : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, $k^\pm : \mathbb{R}^d \to \mathbb{R}$. Denote with $K(y)$ the convex combination of $k^+(y)$ and $k^-(y)$ in $\mathbb{R}$. Then for all y such that $(y, z) \in \Sigma_K$, there exists at least one asymptotically stable equilibrium of (39).*

Proof. We study the sign of G in the set $K(y)$ for $(y, z) \in \Sigma_K$. Without loss of generality, let $k^-(y) < k^+(y)$. Then

$$G(y, k^-(y)) = \alpha^0(y, k^-(y))(k^+(y) - k^-(y)) > 0,$$
$$G(y, k^+(y)) = (1 - \alpha^0(y, k^+(y))(k^-(y) - k^+(y)) < 0.$$

Therefore, $G(y, z)$ has a zero in $K(y)$, see Figure 3. Given the sign of $G(y, \cdot)$

at the extrema $k^-(y)$ and $k^+(y)$, there must be at least one asymptotically stable equilibrium of (39). □

From the proof of Proposition 11, see Figure 3, we see also that, for $(y, z) \in \Sigma_K$, solutions of $z' = G(y, z)$ with initial condition in $K(y)$ remain inside $K(y)$ and hence solutions of (39) are defined for all $\tau \geq 0$. It follows that $\lim_{\epsilon\to 0}(y_\epsilon(\epsilon\tau), z_\epsilon(\epsilon\tau)) = (\tilde{y}(\tau), \tilde{z}(\tau))$, in $[0, \tau_0]$ with τ_0 arbitrarily large. We now consider the case of one algebraic variable and two differential variables, i.e., $a = 1$ and $d = 2$, together with h linear. Without loss of generality let $h(y) = y_1$.

Remark 12. *When (18) has two differential variables, the vector field in (39) is on Σ and hence it depends on just one parameter. It follows that the conditions in item (i) and (ii) in Theorem 13 below are generic. In particular: we do no consider canard points ($\dot{y}_2 = 0$ in (ii), see [14]) and triple equilibria (in (ii) the additional condition on equilibria) since they are not generic for $d = 2$ and $a = 1$.*

Theorem 13. *Let $a = 1$, $d = 2$ and $h(y) = y_1$. Assume $(\bar{y}, k^-(\bar{y})) \in \Sigma_K$. Denote with $(\bar{y}, z(\bar{y}))$ the equilibrium of (39) closest to $(\bar{y}, k^-(\bar{y}))$. Assume that one of the two assumptions (i)-(ii) below is satisfied:*

(i) $\left[\frac{\partial}{\partial z}G(\bar{y}, z)\right]_{z=z(\bar{y})} \neq 0$;

(ii) $\left[\frac{\partial}{\partial z}G(\bar{y}, z)\right]_{z=z(\bar{y})} = 0$ *together with*

$$\left[\frac{\partial^2}{\partial z^2}G(\bar{y}, z)\right]_{z=z(\bar{y})} \neq 0, \quad \left[\frac{\partial}{\partial y}G(y, z)\right]_{(\bar{y},z(\bar{y}))} \neq 0 \ ,$$
$$\dot{y}_2 = e_2^T[(1-\alpha)f^- + \alpha f^+](\bar{y}, z(\bar{y})) \neq 0 \ .$$

Moreover, for any other possible equilibrium $(\bar{y}, \bar{z})$: $G(\bar{y}, \bar{z}) = 0$, we assume that

$$\textit{If} \quad \left[\frac{\partial}{\partial z}G(\bar{y}, z)\right]_{z=\bar{z}} = 0, \quad \textit{then} \quad \left[\frac{\partial^2}{\partial z^2}G(\bar{y}, z)\right]_{z=\bar{z}} \neq 0.$$

Then, there exists $\varphi : \mathbb{R} \to \mathbb{R}$ defined in a neighborhood of $\bar{y}_2$, such that $G(y, \varphi(y_2)) = 0$ and moreover there exists $T > 0$ such that

$$\lim_{\epsilon\to 0}(y_\epsilon(t), z_\epsilon(t)) = (y_0(t), \varphi(e_2^T y_0(t))), \ \ \textit{uniformly in } [\delta, T], \ \ \delta > 0.$$

Case (i) implies that $\varphi(\bar{y}_2) = z(\bar{y})$. The same equality is not guaranteed under case (ii).

Proof. The proof follows by using results of singular perturbation theory. We use Tikhonov's Theorem (see Theorem 21, [20]) for the proof of *(i)*, while for *(ii)* we also have to consider what happens at fold points of $z' = G(\bar{y}, z)$. This case is not contemplated in Theorem 21 but it can be studied by looking at the behavior of the reduced system (36) in a neighborhood of $(\bar{y}, z(\bar{y}))$ (see [14]).

(i) From the definition of $z(\bar{y})$, we have $G(\bar{y}, z(\bar{y})) = 0$. The assumption $\frac{\partial}{\partial z} G(\bar{y}, z(\bar{y})) \neq 0$, implies that $z(\bar{y})$ is a simple zero of $G(\bar{y}, z)$. Then, by the Implicit Function Theorem, there exists a neighborhood of $\bar{y}_2$, $I_{\bar{y}_2}$, a neighborhood of $z(\bar{y})$, $I_{z(\bar{y})}$ and a function $\varphi : I_{\bar{y}_2} \to I_{z(\bar{y})}$, $\varphi(\bar{y}_2) = z(\bar{y})$, such that $G(y, \varphi(y_2)) = 0$. Moreover, since $G(\bar{y}, z)$ is positive in a left neighborhood of $z(\bar{y})$, $z(\bar{y})$ must be an asymptotically stable equilibrium of $z' = G(\bar{y}, z)$ and for y_2 in a neighborhood of $\bar{y}_2$, $\varphi(y_2)$ must be an asymptotically stable equilibrium of the fast system (39). The convergence of $(y_\epsilon(t), z_\epsilon(t))$ to $(y_0(t), \varphi(e_2^T y_0(t)))$ follows easily from Tikhonov's Theorem (see Theorem 21 and [20, Section 39]).

(ii) The assumptions guarantee that $(\bar{y}, z(\bar{y}))$ is a generic fold of (39), see [16, 6]. Proposition 11 guarantees the existence of at least one asymptotically stable equilibrium of (39). Denote with $z_{as}(\bar{y})$ the asymptotically stable equilibrium closest to $z(\bar{y})$. Without loss of generality, assume (as in Proposition 11) that $k^-(y) < k^+(y)$, so that in particular it must be $z_{as}(\bar{y}) > z(\bar{y})$. Under the given assumptions $z_{as}(\bar{y})$ is a simple zero of $G(\bar{y}, z)$ since otherwise (because of the change in sign through it), we would have

$$G(\bar{y}, z_{as}(\bar{y})) = 0, \ \left[\frac{\partial}{\partial z} G(\bar{y}, z)\right]_{z_{as}(\bar{y})} = 0, \ \left[\frac{\partial^2}{\partial z^2} G(\bar{y}, z)\right]_{z_{as}(\bar{y})} = 0,$$

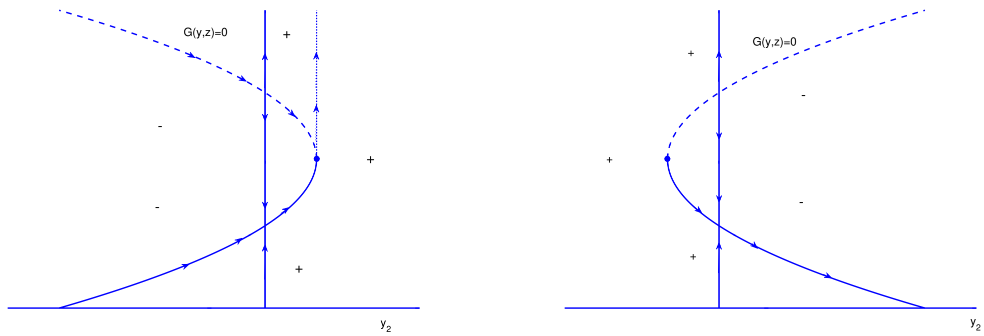
and this contradicts the assumptions in point *(ii)* of the theorem. Below we consider four possibilities.

$(ii)_1$ $\left[\frac{\partial}{\partial y} G(y, z)\right]_{\bar{y}, z(\bar{y})} > 0$ together with $\dot{y}_2 > 0$ in (36). This is illustrated in the left plot in Figure 4. In the figure we plot the set $G(y, z) = 0$ in the (y_2, z) plane (i.e. on Σ) in a neighborhood of $(\bar{y}, z(\bar{y}))$, marked as a full circle in the plot. The curve $G(y, z) = 0$ on Σ is a curve of equilibria of (39). In particular the solid line is the branch of asymptotically stable equilibria, while the dashed line indicates the unstable equilibria. $\dot{y}_2 > 0$ implies that the flow of the reduced problem (36) along the stable and unstable branch of the slow manifold is directed towards the fold. Then by Tikhonov's Theorem, solutions of (35) in the limit for $\epsilon \to 0$, follow the slow manifold along the stable branch. At $(\bar{y}, z(\bar{y}))$ then the limiting solution must continue along the dotted line in Figure 4. This is the solution of (39) and it satisfies $\lim_{\tau\to\infty} \tilde{z}(\tau) = z_{as}(\bar{y})$. Then

$$\lim_{\epsilon\to 0} (y_\epsilon(\epsilon\tau), z_\epsilon(\epsilon\tau)) = (\bar{y}, \tilde{z}(\tau)), \ \tau \in [0, \tau_0],$$

with τ_0 arbitrarily large. For τ_0 large enough $(y_\epsilon(\epsilon\tau), z_\epsilon(\epsilon\tau))$ enters a small neighborhood of $(\bar{y}, z_{as}(\bar{y}))$. From now on, the proof follows the same reasonings of Case *(i)*. The main difference consists in the fact that the function φ is defined in a neighborhood of $\bar{y}_2$ as in Case *(i)* but

FIGURE 4.



it has values in a neighborhood of $z_{as}(\bar{y})$. The point $(\bar{y}, z(\bar{y}))$ is then a jump point for the limiting solution of (35).

$(ii)_2$ $\left[\frac{\partial}{\partial y}G(y,z)\right]_{\bar{y},z(\bar{y})} < 0$ together with $\dot{y}_2 > 0$ in (36). This is depicted in the right plot of Figure 4. As in Case *(i)* we plot the stable branch of equilibria of (39) as a solid line and the unstable branch as a dashed line. The flow of the reduced problem (36) is directed away from the fold along the stable branch. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be such that $\varphi(\bar{y}_2) = z(\bar{y})$ and $z = \varphi(y_2)$ is the stable branch of equilibria of (39) in the figure. Then, by Tikhonov's Theorem, there exists $T > 0$ such that

$$\lim_{\epsilon \to 0}(y_\epsilon(t), z_\epsilon(t)) = (y_0(t), \varphi(e_2^T y_0(t))),$$

in $[0, T]$. The time instant T is chosen such that the solution of (36) remains on $z = \varphi(y_2)$.

$(ii)_3$ $\left[\frac{\partial}{\partial y}G(y,z)\right]_{\bar{y},z(\bar{y})} > 0$ together with $\dot{y}_2 < 0$, this can be treated similarly to case $(ii)_2$.

$(ii)_4$ $\left[\frac{\partial}{\partial y}G(y,z)\right]_{\bar{y},z(\bar{y})} < 0$ together with $\dot{y}_2 < 0$, this can be treated similarly to case $(ii)_1$.

□

As highlighted in Theorem 13, Case *(ii)*, the DAE (36) on Σ might not retain index 1 at some points. Further in the subcases $(ii)_1$ and $(ii)_4$ of the proof, the algebraic variable might jump at generic folds while the solution is sliding. See Example 19 below for an illustration of this phenomenon. At jump points $y_0(t)$ is continuous and not smooth. See also Remark 15 for another instance of jump points when the assumptions of Theorem 13 are not satisfied.

Definition 1. Let $(\bar{y}, k^-(\bar{y})) \in \Sigma_K$. Then the *singularly perturbed* sliding solution of (18) with initial condition $y(0) = \bar{y}$, $z(0) = k^-(\bar{y})$ is the function $(y^0(t), \varphi(e_2^T y_0(t)))$ in Theorem 13.

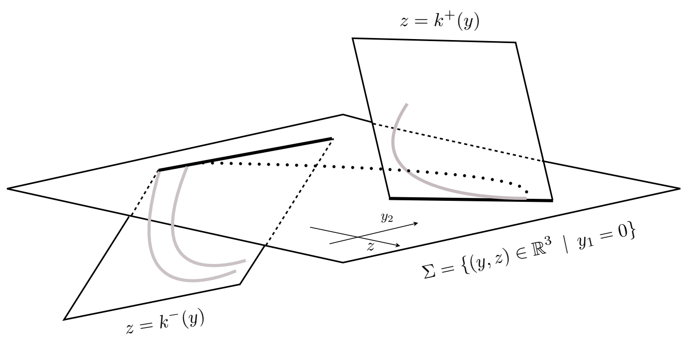
Remark 14. *For $(y, z) \in \Sigma_K$, the fast system (39) might have more than one asymptotically stable equilibrium in $K(y)$. For $y = \bar{y}$, denote with $z(\bar{y})^-$ (resp. $z(\bar{y})^+$) the asymptotically stable equilibrium of (39) closer to $k^-(\bar{y})$ (resp. $k^+(\bar{y})$). If we consider only two differential variables, $d = 2$, then generically these equilibria are simple. Then the Implicit Function Theorem implies that there is a stable branch of equilibria of (39) through $z(\bar{y})^-$, $z = \varphi^-(z)$, and one stable branch through $z(\bar{y})^+$, $z = \varphi^+(y)$. It follows that singularly perturbed solutions of (18) that enter Σ from R^- will slide along the branch $z = \varphi^-(y)$, while solutions that enter Σ from R^+ will follow the branch $z = \varphi^+(y)$. This is a main difference with the Filippov theory for discontinuous differential equations were sliding solutions on co-dimension 1 discontinuity manifold only depend on the entry point and not on whether the solution enters Σ from R^- or R^+.*

Remark 15. *If the singularly perturbed sliding solution leaves the region Σ_K at a point $(\tilde{y}, \varphi(\tilde{y}))$ while following a stable branch of equilibria of the fast system (39) then the assumptions of Theorem 13 do not hold anymore. But, we are still able to say what happens to the singularly perturbed sliding solution after the point $(\tilde{y}, \varphi(\tilde{y}))$. The solution remains on Σ until it reaches either a point at which $\alpha = 0, 1$, or a fold of $z = \varphi(y)$. This last exit point in general is not a tangential exit point and $z = z(t)$ will jump at this exit point, while $y(t)$ will be continuous at the exit point. See Example 17 for an illustration of this phenomenon. These fold points are generic exits and we see another main difference with the theory of Filippov systems for discontinuous ODEs, where generic exits from a co-dimension 1 discontinuity manifold can only take place at tangential exit points and the solution is differentiable at these points.*

A main advantage of Definition 1 is that it allows the characterization of sliding solutions (and sliding points) of a discontinuous DAE via the classic Filippov theory. Instead, had we defined sliding solutions on Σ directly as the solution of the DAE (36), we would have had to extend Filippov theory to the discontinuous DAE without any justification for such an extension. Moreover, defining the sliding solution of (18) directly as the solution of equation (36) has other shortcomings.

i) The dynamics of (18) is in general not defined in a neighborhood of the set $G(y, z) = 0$, see Figure 5. As a consequence, it is not clear how to establish whether a point is an attractive sliding point for (18) by looking at the dynamics in a neighborhood of Σ. For the same reason, it is not clear how to characterize when a sliding solution must leave Σ. See Remark 15 to see how to characterize exits of singularly perturbed sliding solutions instead, and Example 19 for an illustration.

FIGURE 5. DDAE. The dotted curve is the set $G(y,z) = 0$. The lighter curves are solution trajectories in R^- and R^+.



ii) The equation $G(y,z) = 0$ is in general nonlinear in z and for each y there could be more than one z that satisfies it. Is there a mechanism that chooses the appropriate solution? See Remark 14 for a mechanism in the case of singularly perturbed sliding solutions instead.

Before looking at a few examples, we emphasize that sliding motion for (18) cannot be characterized in a non ambiguous way when $k^-(y) \neq k^+(y)$. To witness, the "Direct Approach" (with sliding based on (32)) and the "Singular Perturbation Approach" (with sliding based on (36)) typically lead to different sliding solutions. This is clear also upon noticing that the equation for α is linear in (32), but nonlinear (in general) in (36). We illustrate in Example 17.

Remark 16. *In [19, (16 a-b)], the authors put forward a sliding equation on Σ that is a combination of (32) and (36). Namely, in order to compute sliding solutions, the authors of [19] use (32), but also update $z = (1-\alpha(y))k^-(y) + \alpha(y)k^+(y)$ on Σ. We failed to justify this hybrid approach via singular perturbation and the classic Filippov theories.*

Example 17. *In this example we illustrate the following: i) sliding solutions obtained via the "Direct Approach" and the "Singular Perturbation Approach" are different; ii) exits from Σ while sliding according to (36) might occur also at points* *that are not tangential exit points. Consider the following DDAE:*

$$\begin{cases} \dot{y}_1 = z - 2y_2 + 1 \\ \dot{y}_2 = 1 \\ z = y_2 \end{cases}, \quad y_1 < 0, \qquad \begin{cases} \dot{y}_1 = -\frac{1}{2}z + y_2 - 1 \\ \dot{y}_2 = 1 \\ z = 2y_2 + 1 \end{cases}, \quad y_1 > 0.$$

i) *Direct Substitution. The discontinuous system* (31) *is*

$$\begin{cases} \dot{y}_1 = -y_2 + 1 \\ \dot{y}_2 = 1 \end{cases}, \quad y_1 < 0, \qquad \begin{cases} \dot{y}_1 = -\frac{3}{2} \\ \dot{y}_2 = 1 \end{cases}, \quad y_1 > 0. \tag{40}$$

This is a discontinuous ODE in $\mathbb{R}^2$ *and the discontinuity line* $\Sigma_y = \{y_1 = 0\}$*. It is easy to see that* Σ_y *is attractive for* $y_2 < 1$*. On* Σ_y*, the value* $y_2 = 1$ *is a tangential exit point into* R^- *so that solutions of* (40) *with initial conditions* $(0, y_2)$*,* $y_2 < 1$*, slide on* Σ_y *until they reach* $y_2 = 1$ *and then they leave* Σ_y *to enter* R^-*.*

ii) *Singular Perturbation Approach. The singularly perturbed discontinuous ODE* (33) *is*

$$\begin{cases} \dot{y}_1 = z - 2y_2 + 1 \\ \dot{y}_2 = 1 \\ \epsilon\dot{z} = y_2 - z \end{cases}, \quad y_1 < 0, \qquad \begin{cases} \dot{y}_1 = -\frac{1}{2}z + y_2 - 1 \\ \dot{y}_2 = 1 \\ \epsilon\dot{z} = 2y_2 + 1 - z \end{cases}, \quad y_1 > 0. \tag{41}$$

The set Σ *is the* (y_2, z)*-plane and a point on* Σ *is an attractive sliding point if* $h_y^T f^-(y, z) = z - 2y_2 + 1 > 0$ *and* $h_y^T f^+(y, z) = -\frac{1}{2}z + y_2 - 1 < 0$*. Then* Σ *is attractive for* (41) *if* $z > 2y_2 - 1$*. In Figure 6 the sliding attractive points are the ones in the shaded region. The dotted line is* $z = 2y_2 - 1$ *and it is the line of tangential exit points in* R^- *for* (41)*. Notice that the intersection between* $z = k^-(y)$ *and* $z = 2y_2 - 1$ *is the point* $(0, 1, 1)$ *and the sliding solutions of* (40) *exit* Σ *to enter* R^- *at* $(0, 1)$ *for* $z = k^-(y)$*. The set* Σ_K *defined in Theorem 13 is*

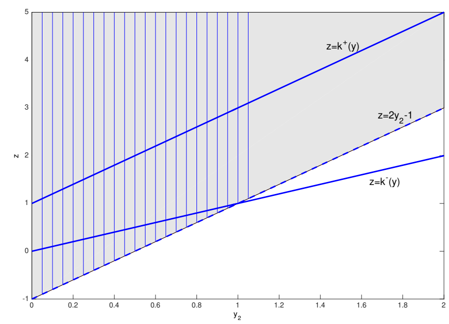
$$\Sigma_K = \{(y, z) \in \Sigma \mid z > 2y_2 - 1, \; y_2 < 1\},$$

and it is the striped region in Figure 6. We define the sliding vector field on Σ *via Filippov's convexification*

$$\begin{cases} \dot{y}_1 = (1-\alpha)(z - 2y_2 + 1) + \alpha(-\frac{1}{2}z + y_2 - 1) \\ \dot{y}_2 = 1 \\ \epsilon\dot{z} = (1-\alpha)y_2 + \alpha(2y_2 + 1) - z, \end{cases} \tag{42}$$

with α *such that* $(1-\alpha)(z - 2y_2 + 1) + \alpha(-\frac{1}{2}z + y_2 - 1) = 0$ *From the last equation we obtain* $\alpha = \alpha(y_2, z) = \frac{z - 2y_2 + 1}{\frac{3}{2}z - 3y_2 + 2}$*. In order to study solutions as* ϵ

FIGURE 6. Example 17. Attractive region for (41) (the shaded area) and set Σ_K (the striped area).



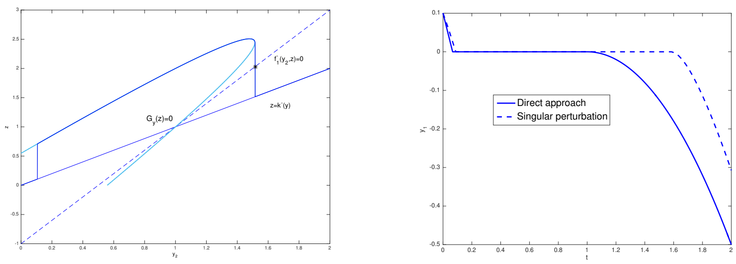
goes to 0*, we consider the reduced fast system*

$$\begin{cases} y_1' = 0 \\ y_2' = 0 \\ z' = \frac{z-2y_2+1}{\frac{3}{2}z-3y_2+2}(y_2+1) + y_2 - z \end{cases} . \tag{43}$$

The equilibria of (43) *must satisfy* $\frac{z-2y_2+1}{\frac{3}{2}z-3y_2+2}(y_2+1)+y_2-z=0$*. The curve of equilibria* $G(y_2,z)=0$ *is shown on the left in Figure 7 as a light solid line. The point* $(1,1)$ *is a tangential exit point for* (41)*, but it is an unstable equilibrium of* (43)*. It follows that the limit for* ϵ *that goes to* 0 *of solutions of* (42) *will never approach it. There is instead a fold of* (43) *at* $y_2 = 10-\sqrt{72} \simeq 1.5147$ *and* $z \simeq 2.4437$*, see also the plot of* $G(y,z)=0$*. The limiting solution of the singular perturbation system must leaves* Σ *at* $y_2 \simeq 1.5174$ *to enter* R^-*. In Figure 7 in the left plot the solid line is the solution of the singularly perturbed discontinuous system for* $\epsilon = 10^{-3}$*. The computed solution is obtained applying an event location method to* (41) *with a 4-th order variable stepsize Runge Kutta method and relative and absolute tolerance* `tol` $= 10^{-8}$*. The method computes the exit point from* Σ *into* R^- *for* $y_2 \simeq 1.5158$ *and* $z \simeq 2.0668$*. This exit point is marked with an asterisk, and it lies on the intersection between the solution and the dashed line. This is the line in the* (y_2,z)*-plane of tangential exit points into* R^-*. In the right plot, the first component of the solutions obtained with the Direct Approach and with the Singular Perturbation Approach are plotted in function of time. Clearly, the two approaches give two different sliding solutions of* (42)*.*

Remark 18. *The existence of more than one Filippov-like sliding solution is not necessarily a drawback. Indeed in the literature of discontinuous dynamical systems ambiguous solutions are well known (see [9, 21] for example).*

Figure 7.



We believe that there is no correct sliding solution but that each possible solution will have a different applicability. For example if we wish to define the algebraic variable while sliding, then the direct approach is not able to give us this information.

Example 19. *In this example we show that (36) might lead to a discontinuous z, while sliding. Consider this DDAE*

$$\left\{\begin{array}{c}\dot{y}_1 = \frac{1}{2}\\ \dot{y}_2 = 1\\ z = -y_2 + \frac{1}{2}\end{array}\right., \quad y_1 < 0, \qquad \left\{\begin{array}{c}\dot{y}_1 = -2z^2\\ \dot{y}_2 = 1\\ z = -y_2 + \frac{3}{2}\end{array}\right., \quad y_1 > 0. \tag{44}$$

The discontinuity set Σ is the (y_2, z)-plane. Now (33) is

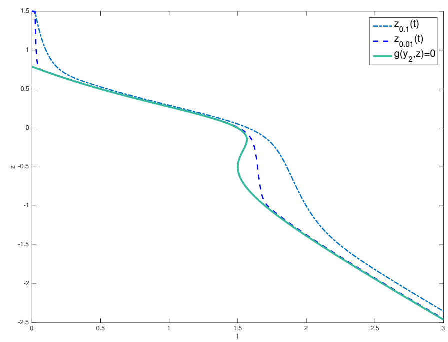
$$\left\{\begin{array}{c}\dot{y}_1 = \frac{1}{2}\\ \dot{y}_2 = 1\\ \epsilon\dot{z} = (-y_2 + \frac{1}{2} - z)\end{array}\right., \quad y_1 < 0, \qquad \left\{\begin{array}{c}\dot{y}_1 = -2z^2\\ \dot{y}_2 = 1\\ \epsilon\dot{z} = (-y_2 + \frac{3}{2} - z)\end{array}\right., \quad y_1 > 0. \tag{45}$$

All the points on Σ (except $z = 0$) are attractive sliding points for (45) and the corresponding Filippov sliding equation is the following

$$\left\{\begin{array}{c}\dot{y}_1 = 0\\ \dot{y}_2 = 1\\ \epsilon\dot{z} = -y_2 + \frac{1}{2} + \alpha(z) - z\end{array}\right., \tag{46}$$

with $\alpha(z) = \frac{1}{1+4z^2}$. Here, $G(y,z) = -y_2 + \frac{1}{2} + \frac{1}{4+z^2} - z$. The curve $G(y,z) = 0$ has two folds on Σ that can be computed imposing $\frac{d}{dz}G(y,z) = 0$: $(\frac{3}{2}, -\frac{1}{2})$ and $(1.567, -0.148)$. In Figure 8 we show the solutions of (45) with initial condition $(0.1, 0, \frac{3}{2})$ and $\epsilon = 0.1$, dotted line, and $\epsilon = 0.01$, dashed line. The solid line is the curve $G(y,z) = 0$. The numerical solutions are computed with event location method applied to (45), (see [8]). As ϵ approaches 0, the solution follows the set $z = k^+(y)$ in R^+, then it reaches Σ and starts sliding on it along the set $G(y,z) = 0$. When it reaches the fold $(0, 1.567, -0.148)$ it jumps and keeps sliding on the branch of attractive equilibria of the reduced fast system $y_1' = 0$, $y_2' = 0$, $z' = -y_2 + \frac{1}{2} + \alpha(z) - z$.

Figure 8.

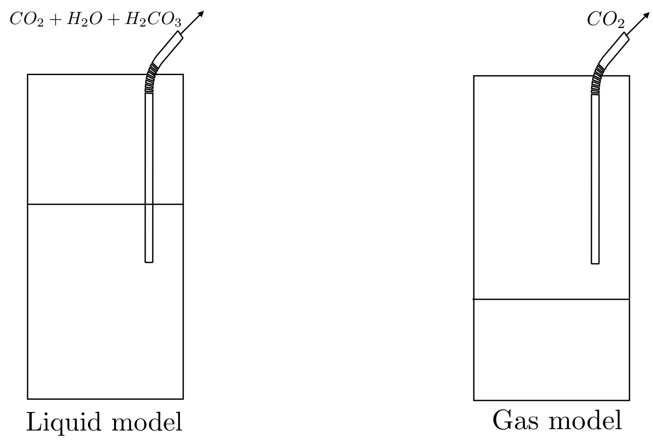


5. An ideal gas liquid interaction model

In this section we study an ideal gas liquid model (see [1, 4]) used in the chemical engineering literature for soft drink production. In the tank a mixture of CO_2, H_2O and H_2CO_3 is present. The mixture leaves the tank through the outlet tube until its level drops below the tube. When this happens, the CO_2 is the only substance that leaves the tank through the tube, see Figure 9. The first instance is described by the liquid model, the second by the gas model. For a thorough description of the model we refer the reader to the papers [1, 4]. Below we just give the equations that model the gas-liquid interaction and solve the nonsmooth DAEs using the Direct Approach and the Singular Perturbation Approach.

The variables y_1, y_2 and y_3 are the molar hold ups for CO_2, H_2O and H_2CO_3. The algebraic variable z instead is the molar flow rate of CO_2 through the valve. In [1, 4] two additional algebraic variables are considered in the liquid model, namely the molar flow rate of H_2O and of H_2CO_3. See Remark 20 below. The discontinuity surface is $h(y_2, y_3) = \frac{y_2}{\rho_l} + \frac{y_3}{\rho_a} - V_d$. The model equations we consider are written below. The liquid phase is defined for $h(y_2, y_3) > 0$ while the gas model is defined for $h(y_2, y_3) < 0$.

FIGURE 9. Ideal gas liquid model



$$
\begin{array}{ll}
\begin{array}{c}\text{Liquid model:}\\ h(y_2,y_3)>0\end{array} & \left\{\begin{array}{rc}
\frac{dy_1}{dt} = & F_1 - z - k_c\frac{y_1y_2}{V}\\[2ex]
\frac{dy_2}{dt} = & F_2 - \frac{y_2}{M_l+y_2+y_3}k_lX(P-P_{out}) - k_c\frac{y_1y_2}{V}\\[2ex]
\frac{dy_3}{dt} = & -\frac{y_3}{M_l+y_2+y_3}k_lX(P-P_{out}) + k_c\frac{y_1y_2}{V}\\[2ex]
z = & k_lX(P-P_{out}) - \frac{y_2+y_3}{M_l+y_2+y_3}k_lX(P-P_{out}),
\end{array}\right. ,\\[8ex]
\begin{array}{c}\text{Gas model:}\\ h(y_2,y_3)<0\end{array} & \left\{\begin{array}{rc}
\frac{dy_1}{dt} = & F_1 - z - k_c\frac{y_1y_2}{V}\\[2ex]
\frac{dy_2}{dt} = & F_2 - k_c\frac{y_1y_2}{V}\\[2ex]
\frac{dy_3}{dt} = & k_c\frac{y_1y_2}{V}\\[2ex]
z = & k_gX(P-P_{out})
\end{array}\right. .
\end{array}
\tag{47}
$$

with

$$
P = \frac{y_1RT}{V - y_2/\rho_l - y_3/\rho_a}, \qquad M_l = \frac{y_2+y_3}{1640-P}P, \qquad V_d = 2.25,
$$

and

$$F_1 = 0.5, \qquad F_2 = 7.5, \quad k_c = \tfrac{0.4333}{4000}, \quad V = 10$$

$$k_l = 2.5, \qquad k_g = 3, \qquad X = 1, \qquad P_{out} = 1$$

$$R = 0.0820574587, \quad T = 293, \quad \rho_a = 16, \quad \rho_l = 50.$$

Remark 20. *In [1, 4] the liquid model and the gas model do not have the same number of algebraic variables. More specifically, the liquid model equations have two additional algebraic variables: z_2 and z_3, defined as $z_i = -\frac{y_i}{M_l+y_2+y_3} k_l X(P - P_{out})$, $i = 1, 2$. The differential equations for y_2 and y_3 are then written in function of z_2 and z_3 as: $\frac{dy_2}{dt} = F_2 - z_2 - k_c \frac{y_1 y_2}{V}$ and $\frac{dy_3}{dt} = -z_3 + k_c \frac{y_1 y_2}{V}$. We substituted the expressions of z_2 and z_3 in the differential equations for y_2 and y_3. Thus, even if in the original DDAE the algebraic variables in the two subregions are seemingly not the same, the direct substitution of the algebraic variables that are not in common allows to rewrite the model equations as in (18).*

In this model the derivative of y_1 is the only one that depends on the algebraic variable z. Since $h^T f^{\pm}(y, z) = \frac{1}{\rho_l}\dot{y}_2 + \frac{1}{\rho_a}\dot{y}_3$, it follows that α does not depend on $\dot{y}_1$ and hence does not depend on z. It follows that the attractivity of Σ is independent from the algebraic variable. This together with the linear dependence of $\dot{y}_1$ on z implies that the Direct Approach and the Singular Perturbation approach give the same sliding equations.

In Figure 10 we plot the numerical approximations of $(y_1(t), y_2(t), y_3(t))$ computed for the Singularly Perturbed System with $\epsilon = 0.01$ and for the Direct Approach system. The initial conditions are respectively $(0.72\ \ 95\ \ 0\ \ 3.4)$ and $(0.72\ \ 95\ \ 0)$. The time interval for the numerical computations is $[0, 10]$. Both solutions are computed via an event location integrator. For the integration we used an adaptive Runge Kutta Fehlberg method with absolute and relative tolerance 10^{-9}. The two numerical solutions are indistinguishable in the plot.

In Figure 11 we plot $z(t)$ computed with the Singular Perturbation Approach. The solution starts in the gas phase, then enters the liquid phase, finally it starts sliding. The crossing and sliding points are marked with circles in the figure. The value of α at the entry point is $\alpha \simeq 0.5338$, at the last computed point is $\alpha \simeq 0.9496$. The Direct Approach does not allow to compute $z(t)$ during sliding motion.

6. Conclusions and Difficulties

In this paper we adopted a Filippov based approach to study discontinuous DAEs (DDAEs) of Index 1, in Hessenberg form, with a co-dimension 1 discontinuity manifold Σ. Already in this seemingly simple case, in our work we showed that restrictions are needed in order to guarantee the existence of a unique solution. The key

FIGURE 10.

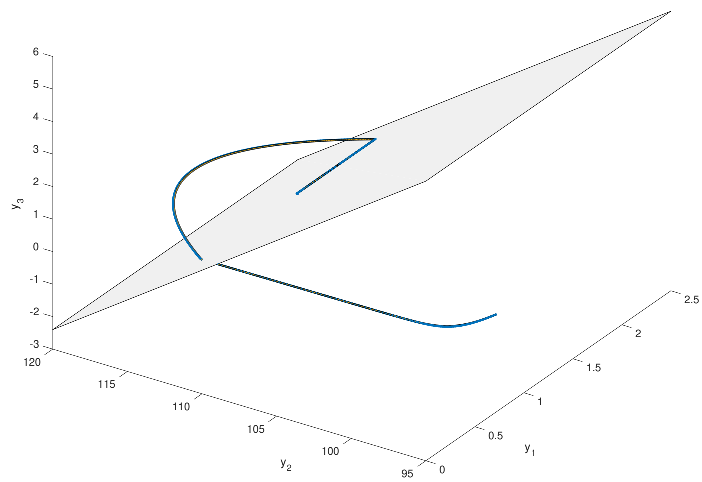
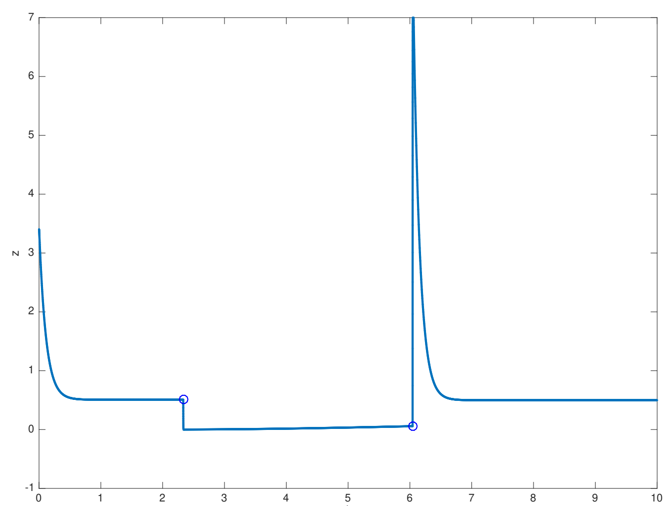


FIGURE 11.



is to account for the interplay between the manifolds Σ and that expressed by the algebraic constraints.

Since Filippov theory was developed (see [9]) for discontinuous ODEs, then we proposed the following approach in order to extend it to DDAEs.

- Rewrite the DDAE under study as a discontinuous ODE with same solution.
- Study the resulting discontinuous ODE via Filippov methodology.
- Obtain a characterization of crossing/sliding points and sliding solutions in the present DDAE case.

We summarize below our main results.

i) In the most general case of Σ being the 0-set of a function $h(y,z)$, we need that the algebraic constraint is continuous and differentiable in a neighborhood of Σ in order to extend the results of Filippov theory.
ii) In case Σ is the zero set of a function $h = h(y)$, we consider three different formulations to rewrite the DAE as an ODE (the Direct Substitutions Approach, a Singular Perturbation Approach and one based on a Weak Formulation of the problem).
   (a) When the algebraic constraint is smooth, the classification of crossing/sliding points and sliding solutions does not depend on the particular approach chosen. We conclude that in this case the DDAE is well defined.
   (b) If the algebraic constraint is discontinuous, then the Weak Formulation is not properly defined. Moreover, the classification of crossing/sliding points and sliding solutions is in general not equivalent for the Direct and the Singular Perturbation approaches.

The extension of our results to the case of discontinuity surface of higher codimension (e.g., see [7]) and the case of index 2, remain to be done.

For example, extension to the results to Index 2 DAEs in Hessenberg form is in principle simple, but will require further restrictions. Indeed, consider the index 2 DDAE:

$$(48)\quad R^- : \quad \begin{cases} \dot y = f^-(y,z) \\ 0 = g^-(y) \end{cases}, \quad h(y,z) < 0\,, \qquad R^+ : \quad \begin{cases} \dot y = f^+(y,z) \\ 0 = g^+(y) \end{cases}, \quad h(y,z) > 0\,,$$

As usual, we differentiate the algebraic constraint with respect to y:

$$(49)\quad \begin{cases} \dot y = f^-(y,z) \\ 0 = g^-_y(y) f^-(y,z) \end{cases}, \quad h(y,z) < 0\,, \qquad \begin{cases} \dot y = f^+(y,z) \\ 0 = g^+_y(y) f^+(y,z) \end{cases}, \quad h(y,z) > 0\,.$$

Then, if $f_z^{\pm} \neq 0$ we can rewrite the two algebraic constraints locally as $z = k^{\pm}(y)$, and the results of Sections 3 and 4 apply. But, we note that now we have to impose more restrictive conditions on the constraints than we have in the case of index 1.

7. Appendix. Tikhonov's Theorem.

The purpose of this Appendix is to give details on Tikhonov Theorem, as presented in [20, Setion 39].

Consider the following singularly perturbed problem

$$\begin{cases} \dot{y} = & f(y,z), \\ \dot{z} = & \frac{1}{\epsilon} G(y,z), \end{cases} \tag{50}$$

with $f(y,z) : \mathbb{R}^d \times \mathbb{R}^a \to \mathbb{R}^d$, $G(y,z) : \mathbb{R}^d \times \mathbb{R}^a \to \mathbb{R}^a$, $a, d \geq 1$, f and G are C^1 functions in E, with E open subset of $\mathbb{R}^d \times \mathbb{R}^a$. Let $y(0) = \bar{y}$, $z(0) = \bar{z}$ and denote with $(y_\epsilon(t), z_\epsilon(t))$ the solution of the corresponding Cauchy problem. If we set $\epsilon = 0$ in (50) we obtain the *reduced* problem

$$\begin{cases} \dot{y} = & f(y,z), \\ 0 = & G(y,z), \end{cases} \tag{51}$$

with initial condition $y(0) = \bar{y}$. The solution of (51) must belong to the set $G(y,z) = 0$. When $\epsilon \to 0$ the solutions of (50) have almost zero derivative with respect to y unless the solution trajectory is in a $O(\epsilon)$ neighborhood of the set $G(y,z) = 0$. It means that solutions of (50) either diverge of they converge to solutions of (51). Tikhonov's Theorem explores the behavior of limiting solutions of (50) as $\epsilon \to 0$, under the following assumptions:

a) There exists a compact convex set $A \subset \mathbb{R}^d$ and a function $\varphi : A \to \mathbb{R}^a$, $\varphi \in C^1(A)$, such that $(y, \varphi(y)) \in E$ and

$$G(y, \varphi(y)) = 0.$$

b) There exists $\eta > 0$, η independent of y, such that

$$\text{if} \quad \|z - \varphi(y)\| < \eta, \ \ z \neq \varphi(y), \ \ y \in A,$$

$$\text{then} \quad G(y,z) \neq 0, \ \ y \in A.$$

This means that the root $z = \varphi(y)$ is an isolated root of $G(y,z) = 0$.

c) The root $z = \varphi(y)$ is an asymptotically stable equilibrium of the boundary layer equation

$$z' = G(y,z),$$

for y fixed, $y \in A$.

d) The asymptotic stability of $z = \varphi(y)$ is uniform for $y \in A$.

Theorem 21. *Suppose assumptions (a)-(d) are satisfied and moreover assume that* $\bar{z}$ *is in the basin of attraction of* $\varphi(\bar{y})$ *for the equation* $z' = G(\bar{y}, z)$. *Denote with* $(y_0(t), z_0(t))$ *the solution of the reduced problem* (51) *with initial conditions* $y(0) = \bar{y}$, $z(0) = \varphi(\bar{y})$. *Then the solution of the full problem* (50) *converges to* $(y_0(t), z_0(t))$ *as* ϵ *goes to* 0, *i.e.,*

$$\lim_{\epsilon\to 0} y_\epsilon(t) = y_0(t), \qquad 0 \le t < T,$$

$$\lim_{\epsilon\to 0} z_\epsilon(t) = \varphi(y_0(t)), \quad 0 < t \le T,$$

where $T > 0$ *is such that* $z = \varphi(y_0(t))$ *is an isolated asymptotically stable equilibrium of* $z' = G(y_0(t), z)$, *for* $0 \le t \le T$. *The convergence is uniform in* $[0, T]$ *for* $y_\epsilon(t)$ *and is uniform in* $[t_1, T]$ *with* $0 < t_1$, *for* $z_\epsilon(t)$. $\square$

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