EDITORIAL SURVEY

WHY FRACTIONAL DERIVATIVES WITH NONSINGULAR KERNELS SHOULD NOT BE USED

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Abstract

In recent years, many papers discuss the theory and applications of new fractional-order derivatives that are constructed by replacing the singular kernel of the Caputo or Riemann-Liouville derivative by a non-singular (i.e., bounded) kernel. It will be shown here, through rigorous mathematical reasoning, that these non-singular kernel derivatives suffer from several drawbacks which should forbid their use. They fail to satisfy the fundamental theorem of fractional calculus since they do not admit the existence of a corresponding convolution integral of which the derivative is the left-inverse; and the value of the derivative at the initial time $t = 0$ is always zero, which imposes an unnatural restriction on the differential equations and models where these derivatives can be used. For the particular cases of the so-called Caputo-Fabrizio and Atangana-Baleanu derivatives, it is shown that when this restriction holds the derivative can be simply expressed in terms of integer derivatives and standard Caputo fractional derivatives, thus demonstrating that these derivatives contain nothing new.

1. **Introduction**

Fractional calculus, namely the study of the generalization of the standard theory of calculus to derivatives and integrals of non-integer orders, has attracted much attention in recent years from different disciplines. It is not only of interest to mathematicians; its success derives from its proven effectiveness in accurately describing innumerable physical phenomena, ranging from biophysics to astrophysics.

Throughout the history of this theory, several definitions for non-integer order operators have been proposed; each one is an attempt to extend the classical notions of integral and derivative. Among these proposals, two particular ones have stood the test of time and are now universally accepted: the celebrated works of Bernhard Riemann and Joseph Liouville, and its modification suggested by Mkhitar Dzhrbashyan. Most notably, the latter turned out to be equivalent to the operator independently inferred by Michele Caputo as a direct result of the generalization of the standard Laplace transform of ordinary derivatives to the fractional regime. The proposal of Riemann and Liouville, which started the entire field of fractional calculus, was based on performing an analytic continuation of Cauchy's formula for repeated integration. This operator is now known as the Riemann-Liouville (RL) fractional integral. Similarly, starting from Cauchy's integral formula for the nth derivative of an analytic function, one can then provide a definition of fractional derivative. This definition of derivative is however well posed only when applied to some very well-behaved functions, so it is naturally desirable to extend the class of permissible functions.

This extension can be achieved by defining a fractional derivative via an integro-differential operator with a locally absolutely integrable kernel, in the form of the well-known Riemann-Liouville (RL) fractional derivative; see (20) in Appendix A for details. But this definition of fractional derivative is not the only possible extension, and an alternative definition was formulated much later by Dzhrbashyan and Caputo (who worked independently). This new idea is known in the literature as the Dzhrbashyan-Caputo (or, simply, Caputo for shortness) fractional derivative; for its definition see (21) in Appendix A.

Each of the RL and Caputo derivatives of real order $\alpha > 0$ is a *left-inverse* operator for the RL fractional integral. They are each represented as Volterra-like convolution integro-differential operators with integral kernel $k(t) = t^{m-\alpha-1}/\Gamma(m - \alpha)$, where $m = \lceil \alpha \rceil$ is the smallest integer greater than or equal to α. If α is not an integer, this kernel is weakly singular at $t = 0$ and it is locally absolutely integrable on the positive real axis. For further discussion we refer the reader to [1, 9, 15, 19, 24, 26].

The weakly singular nature of the kernel has some important consequences. For example, as shown in [28], solutions of time-dependent fractional differential equations (FDEs) with RL or Caputo derivatives typically exhibit weak singularities at the initial time $t = 0$. This phenomenon presents challenging difficulties for the theoretical and numerical analysis of FDEs. One can foresee that numerical difficulties are to be expected because standard numerical methods for solving differential equations are usually based on *polynomial* approximations of the unknown solution — but polynomials do not provide accurate approximations in the neighbourhood of singularities.

In an attempt to avoid the difficulties caused by singularities, some authors have proposed modifications of the RL and Caputo derivatives that are based on the replacement of their weakly singular kernel by some *non-singular* function that is continuous on the *closed* interval $[0, T]$ with $T > 0$. For instance, the exponential function was employed as a replacement for the standard kernel in the Dzhrbashyan-Caputo derivative (21) of order $0 < \alpha < 1$ to obtain the following so-called Caputo–Fabrizio (CF) derivative:

$$^{\mathrm{CF}}D_0^\alpha f(t) = \frac{M(\alpha)}{1 - \alpha} \int_0^t \exp\left(-\frac{\alpha}{1 - \alpha}(t - \tau)\right) f'(\tau) \, \mathrm{d}\tau, \qquad (1)$$

where $M(\alpha)$ is a normalization factor such that $M(0) = M(1) = 1$. Similarly, the Mittag-Leffler function

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + 1)}$$

is used instead of the exponential function to define the so-called Atangana-Baleanu (AB) derivative

$$^{\mathrm{ABC}}D_0^\alpha f(t) = \frac{B(\alpha)}{1 - \alpha} \int_0^t E_\alpha\left(-\frac{\alpha}{1 - \alpha}(t - \tau)^\alpha\right) f'(\tau) \, \mathrm{d}\tau, \qquad (2)$$

where $B(\alpha)$ has the same role and properties as $M(\alpha)$. We mention for completeness that RL-type versions of the CF and AB operators have also appeared in the literature.

NOTE: For simplicity, in our integral operators we always take the initial time to be $t = 0$, since a choice of a different initial time makes no essential difference to the arguments that we shall present.

REMARK 1.1. The CF derivative (1) has an analytic kernel, while the ABC derivative (2) has a kernel that is continuous but not differentiable at $t = 0$. Continuity of the kernel suffices in our discussion, but later in the paper we use its derivative on some open interval $(0, T)$. Anyway, the essence of the integral operators discussed in this paper is that their

kernel is bounded in $[0, T]$. Both kernels of CF and ABC derivatives are sufficiently well behaved to satisfy our arguments.

At first sight, fractional derivatives defined using non-singular kernels may appear very attractive since they avoid several difficulties that are caused by the singular nature of the RL and Dzhrbashyan-Caputo kernel. Thus, it is unsurprising that these simpler operators have become quite popular since their appearance about five years ago. *But these operators with non-singular kernels have serious shortcomings that strongly discourage their use.* Some previous papers [6, 11, 12, 16, 27] have already mentioned some of these; here we attempt to give a comprehensive and coherent account of the drawbacks.

STRUCTURE OF THE PAPER: In Section 2 we prove first that Caputo-type derivatives that are defined using non-singular kernels must fail to satisfy the fundamental theorem of fractional calculus. In other words, they do not allow the existence of a corresponding convolution integral for which the derivative is the left-inverse. Although one can find definitions of "CF and AB integrals" $^{\mathrm{CF}}J_0^\alpha$ and $^{\mathrm{AB}}J_0^\alpha$ in the literature, nevertheless (as we shall see in Section 3) the corresponding derivatives do not act on them as a left-inverse, since $^{\mathrm{CF}}D_0^\alpha\left[^{\mathrm{CF}}J_0^\alpha f(t)\right] \neq f(t)$ and $^{\mathrm{ABC}}D_0^\alpha\left[^{\mathrm{AB}}J_0^\alpha f(t)\right] \neq f(t)$ unless the unnatural and restrictive condition $f(0) = 0$ is imposed on the space of function where these operators act. That is, if these integrals are used to solve differential equations involving CF or ABC derivatives, they can provide a correct solution only when the vector field vanishes at the origin, which is clearly an unreasonable assumption for most problems. In Section 3 we also show that this issue is shared by any derivative with non-singular kernel since they satisfy a *zero-zero property*: the derivative at 0 is always 0. For time-fractional initial-boundary value problems of parabolic type, one finds that a related restriction is automatically imposed on the initial data — see Section 4. Moreover, in Section 5 we show that, if one makes the unnatural assumption that the vector field vanishes at the origin, in order to ensure that CF and ABC derivatives are left-inverse of CF and AB integrals, one then obtains differential equations that are equivalent to standard differential equations of integer or fractional order; in other words, the introduction of these derivatives does not add anything new to the standard (RL and Caputo) theory of fractional calculus. In Section 6 we show that derivative-type operators defined by a non-singular kernel fail to satisfy various proposed extensions of the classical notion of derivative, so whether one should use the term "derivative" to describe these operators is doubtful. Some final remarks are given in Section 7.

NOTATION. We write $C[0,T]$ for the space of continuous functions on the interval $[0,T]$ of the real line, and $L^1[0,T]$ denotes the space of Lebesgue-integrable functions on this interval. The space of absolutely continuous functions on $[0,T]$ is denoted by $AC[0,T]$; recall that $f \in AC[0,T]$ if and only if $f(t) = f(0) + \int_0^t f'(s)\,ds$ for $0 \le t \le T$ with some $f' \in L^1[0,T]$.

2. The fundamental theorem of fractional calculus

For the Caputo derivative (21), by [1, Theorem 3.7] one has

$$^CD_0^\alpha \left[J_0^\alpha f(t) \right] = f(t)$$

for all $f \in C[0,T]$ and $0 < t \le T$. This is a fractional equivalent of (part of) the fundamental theorem of classical calculus. In this section we investigate what conditions a fundamental theorem of fractional calculus imposes on the kernels of the differential and integral operators. This analysis is based on ideas previously presented in [6, 11, 12, 16].

First, we state the following well-known technical result, which will be used more than once in this paper.

LEMMA 2.1. *Let $g \in L^1[0,T]$. Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that $\left| \int_E g(x)\,dx \right| < \varepsilon$ for every measurable set $E \subset [0,T]$ with measure less than δ.*

P r o o f. See for example [17, p. 300, Theorem 6].   □

Suppose that, imitating (21) for $0 < \alpha < 1$, we define for a function $f \in AC[0,T]$ a Caputo-type derivative D_ϕ by

$$D_\phi f(t) := \int_0^t \phi(t-\tau)f'(\tau)\,d\tau, \quad 0 < t \le T, \tag{3}$$

where the kernel function ϕ is as yet unspecified, except that we require $\phi \in L^1[0,T]$ to ensure that $D_\phi f(t)$ is defined almost everywhere (it is well known that the convolution of two functions in $L^1[0,T]$ also lies in $L^1[0,T]$; cf., e.g., [1, proof of Theorem 2.1]).

Such operators based on non-singular kernels usually have a normalization factor — see (1) and (2) for example — that multiplies the integral, depends on α, and ensures that D_ϕ approaches the classical first-order derivative when $\alpha \to 1$. For brevity we do not write this factor explicitly in (3); instead it is absorbed into the kernel ϕ.

In order to have a fundamental theorem of fractional calculus for our derivative D_ϕ, we need to define a corresponding integral operator J_ψ, defined by

$$J_\psi g(t) := \int_0^t \psi(t-\tau)g(\tau)\,d\tau \quad \text{for} \ \ 0 < t \le T,$$

where $\psi \in L^1[0, T]$ is yet to be chosen in such a way that $D_\phi[J_\psi f(t)] = f(t)$ for all $f \in AC[0, T]$ and $0 < t \leq T$. Writing out this identity in detail, we have

$$f(t) = \int_0^t \phi(t - \tau)(J_\psi f)'(\tau) \, d\tau = \int_0^t \phi(\tau)(J_\psi f)'(t - \tau) \, d\tau$$

$$= \frac{d}{dt} \left\{ \int_0^t \phi(\tau)(J_\psi f)(t - \tau) \, d\tau \right\},$$

here the second equation follows from a simple change of variable, while the third is a consequence of Leibniz's Rule for differentiating integrals, combined with $\lim_{t \to 0} J_\psi f(t) = 0$ (which follows from Lemma 2.1 since $\psi \in L^1[0, T]$ and f bounded implies that the integrand of $J_\psi f$ lies in $L^1[0, T]$). Now make another change of variable, then recall the definition of J_ψ to get

$$f(t) = \frac{d}{dt} \left\{ \int_0^t \phi(t - \tau)(J_\psi f)(\tau) \, d\tau \right\}$$

$$= \frac{d}{dt} \left\{ \int_{\tau=0}^t \phi(t - \tau) \left[\int_{s=0}^\tau \psi(\tau - s) f(s) \, ds \right] d\tau \right\}.$$

Next, apply Fubini's theorem to swap the order of integration, then apply Leibniz's Rule again:

$$f(t) = \frac{d}{dt} \left\{ \int_{s=0}^t f(s) \left[\int_{\tau=s}^t \phi(t - \tau)\psi(\tau - s) \, d\tau \right] ds \right\}$$

$$= f(t) \lim_{s \to t} \left[\int_{\tau=s}^t \phi(t - \tau)\psi(\tau - s) \, d\tau \right] +$$

$$+ \int_{s=0}^t f(s) \frac{d}{dt} \left[\int_{\tau=s}^t \phi(t - \tau)\psi(\tau - s) \, d\tau \right] ds.$$

We want this equation to hold true for all $f \in AC[0, T]$ and $0 < t \leq T$. This is possible only if

$$\lim_{s \to t} \int_{\tau=s}^t \phi(t - \tau)\psi(\tau - s) \, d\tau = 1 \quad \text{and} \quad \frac{d}{dt} \int_{\tau=s}^t \phi(t - \tau)\psi(\tau - s) \, d\tau = 0.$$

The change of variables $r = \tau - s$ shows that each integral here equals $\int_{r=0}^{t-s} \phi(t - s - r)\psi(r) \, dr$. Thus, the value of the integral depends on the length $t - s$ of the interval of integration but not separately on t and s. Consequently one can rewrite $\lim_{s \to t}$ in the first condition as $\lim_{t \to s}$. But the second condition says that $\int_{\tau=s}^t \phi(t - \tau)\psi(\tau - s) \, d\tau$ is a constant as t varies; then the first condition forces

$$\int_{\tau=s}^t \phi(t - \tau)\psi(\tau - s) \, d\tau = 1 \quad \text{for} \quad 0 \leq s < t \leq T. \tag{4}$$

Equations of the form (4) are known as Sonine equations. They impose a certain requirement on the functions ϕ and ψ (which, up to now, were merely required to lie in $L^1[0,T]$) and their interaction. Suppose that one of these functions is bounded on $[0,T]$; say, $|\phi(t)| \leq M$ for $0 \leq t \leq T$. Then

$$\left| \int_{\tau=s}^t \phi(t-\tau)\psi(\tau-s)\,\mathrm{d}\tau \right| \leq M \int_{\tau=s}^t |\psi(\tau-s)|\,\mathrm{d}\tau,$$

and by Lemma 2.1 the right-hand side will go to zero if $s \to t$. But this implies that the Sonine equation (4) cannot be satisfied when s is close to t. Thus we cannot have ϕ bounded on $[0,T]$ (and likewise for ψ).

The above argument can be summarised as follows:

THEOREM 2.1. *Given a Caputo-type fractional derivative of the form (3) whose kernel $\phi : [0,T] \to \mathbb{R}$ is bounded, one cannot define a corresponding integral operator such that the fundamental theorem of fractional calculus is valid.*

In particular, this theorem applies to kernels that are continuous functions on $[0,T]$.

A similar result for fractional derivatives of RL-type is derived in [11, 12].

3. Derivatives with non-singular kernel impose restrictive and unnatural assumptions

In this section we show that differential equations involving derivatives with a non-singular kernel impose severe (and unnatural) constraints on the initial conditions.

We do this by first considering differential equations involving the CF and ABC derivatives, and then moving on to the general case of non-singular kernels, which we analyse using Laplace transforms. The discussion in this section is for initial-value problems posed on $[0,T]$; a related restriction for initial-boundary value problems will be presented in Section 4.

3.1. **CF and ABC derivatives are not the left-inverse of the corresponding integrals.** A so-called CF integral $^{\mathrm{CF}}J_0^\alpha$ has been proposed in the literature, defined by

$$^{\mathrm{CF}}J_0^\alpha f(t) = \frac{1-\alpha}{M(\alpha)} f(t) + \frac{\alpha}{M(\alpha)} \int_0^t f(\tau)\,\mathrm{d}\tau, \quad t \geq 0. \qquad (5)$$

It has the property that $^{\mathrm{CF}}J_0^\alpha\big[^{\mathrm{CF}}D_0^\alpha f(t)\big] = f(t) - f(0)$; that is, the differential operator is the *right*-inverse of the integral operator on the space of functions $\{f \in AC[0,T] : f(0) = 0\}$. This property is similar to the

identity $\int_0^t f'(s)\,\mathrm{d}s = f(t) - f(0)$ enjoyed by classical first-order derivatives and the standard integral operator. But first-order derivatives also have the *left*-inverse property $\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(s)\,\mathrm{d}s = f(t)$, whereas for the CF integral and derivative we have the following result.

PROPOSITION 3.1. *Let* $f \in AC[0,T]$. *The CF derivative and the CF integral satisfy the relation*

$$^{CF}D_0^\alpha \left[{}^{CF}J_0^\alpha f(t) \right] = f(t) - \exp\left(-\frac{\alpha}{1-\alpha}t \right) f(0). \tag{6}$$

P r o o f. See Appendix B. □

This unfavourable result says that the CF derivative $^{CF}D_0^\alpha$ is the left-inverse of $^{CF}J_0^\alpha$ only on the restricted space $\{f \in AC[0,T] : f(0) = 0\}$ and not on the full space $AC[0,T]$, as one would expect (and as is the case for the Caputo derivative [1, Theorem 3.7]).

The constraint $f(0) = 0$ on functions for which $^{CF}D_0^\alpha$ is the left-inverse of $^{CF}J_0^\alpha$ has serious consequences if $^{CF}J_0^\alpha$ is employed to solve an initial-value problem such as

$$\begin{cases} {}^{CF}D_0^\alpha y(t) = g(t, y(t)), \\ y(0) = y_0. \end{cases} \tag{7}$$

For applying $^{CF}J_0^\alpha$ to both sides of this differential equation, one obtains

$$y(t) = y_0 + \frac{1-\alpha}{M(\alpha)} g(t, y(t)) + \frac{\alpha}{M(\alpha)} \int_0^t g(\tau, y(\tau))\,\mathrm{d}\tau. \tag{8}$$

But, replacing $f(t)$ in (6) by $g(t, y(t))$, we see immediately that

$$^{CF}D_0^\alpha y(t) = g(t, y(t)) - \exp\left(-\frac{\alpha}{1-\alpha}t \right) g(0, y_0).$$

Hence $^{CF}D_0^\alpha y(t) \neq g(t, y(t))$ if $g(0, y_0) \neq 0$. That is, although one might believe erroneously that $y(t)$ in (8) is the solution of (7), this is not true unless $g(0, y_0) = 0$.

The situation is similar for the so-called AB integral

$$^{AB}J_0^\alpha f(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \quad t \geq 0. \tag{9}$$

Here again $^{AB}J_0^\alpha \left[{}^{ABC}D_0^\alpha f(t) \right] = f(t) - f(0)$, but $^{ABC}D_0^\alpha$ is not the left-inverse of $^{AB}J_0^\alpha$ since the following analog of Theorem 3.1 holds (see Appendix B for the proof):

PROPOSITION 3.2.   *Let $f \in AC[0,T]$. The ABC derivative and the AB integral satisfy the relation*

$$^{\mathrm{ABC}}D_0^\alpha \left[^{\mathrm{AB}}J_0^\alpha f(t) \right] = f(t) - E_\alpha \left(-\frac{\alpha}{1-\alpha} t^\alpha \right) f(0). \qquad (10)$$

Like the CF derivative, the ABC derivative is the left-inverse of the AB integral only on the restricted space $\{ f \in AC[0,T] : f(0) = 0 \}$.

The use of $^{\mathrm{AB}}J_0^\alpha$ to solve a differential equation with the ABC derivative of the same type of (7) will thus produce a function $y(t) = y_0 + ^{\mathrm{AB}}J_0^\alpha g(t, y(t))$ that is not a solution of the equation since

$$^{\mathrm{ABC}}D_0^\alpha y(t) = g(t, y(t)) - E_\alpha \left(-\frac{\alpha}{1-\alpha} t^\alpha \right) g(0, y_0).$$

In general the CF and AB integrals cannot be used to solve differential equations with the corresponding fractional derivatives, unless one imposes the additional and restrictive condition $g(0, y_0) = 0$ to have the identities $^{\mathrm{CF}}D_0^\alpha \left[^{\mathrm{CF}}J_0^\alpha g(t, y(t)) \right] = g(t, y(t))$ and $^{\mathrm{ABC}}D_0^\alpha \left[^{\mathrm{AB}}J_0^\alpha g(t, y(t)) \right] = g(t, y(t))$.

To appreciate how unnatural the condition $g(0, y_0) = 0$ is, consider the simple linear problem where $g(t, y(t)) = \lambda y(t)$ in (7) with a CF or ABC derivative. Imposing the condition $g(0, y_0) = 0$, so that the CF or AB integral solves the problem correctly, requires either $\lambda = 0$ or $y_0 = 0$; but then the problem has only the trivial constant solution $y(t) \equiv y_0$ for all $t \geq 0$. Introducing new operators only to describe constant solutions is not worthwhile!

Propositions 3.1 and 3.2 will be generalized in Theorem 3.2 of Section 3.3.

3.2. **Non-singular kernel derivatives are always zero at zero.** The restriction on the initial condition of differential equations with CF and ABC derivatives is consequence of the fact the these derivatives are zero at the origin. For instance, taking the power function $f(t) = t^\gamma$ for constant $\gamma > 0$, one can compute

$$^{\mathrm{ABC}}D_0^\alpha f(t) = \frac{B(\alpha)}{1-\alpha} \Gamma(\gamma + 1) t^\gamma E_{\alpha, \gamma+1} \left(-\frac{\alpha}{1-\alpha} t^\alpha \right),$$

and consequently $^{\mathrm{ABC}}D_0^\alpha f(t) \big|_{t=0} = 0$ (similarly for $^{\mathrm{CF}}D_0^\alpha f(t)$). We call this the *zero-zero property* (namely, the derivative at 0 is always 0). It holds true not only for CF and ABC derivatives, and not only for the function $f(t) = t^\gamma$, but much more generally, as we now show.

THEOREM 3.1 (Zero-zero property). *Let ϕ be bounded on $[0,T]$, D_ϕ the operator defined by (3) and $f \in AC[0,T]$. Then*

$$\lim_{t \to 0^+} D_\phi f(t) = 0.$$

P r o o f. Since ϕ is bounded on $[0,T]$, for any $t \in (0,T]$ one has

$$\left| D_\phi f(t) \right| = \left| \int_0^t \phi(t-\tau)f'(\tau)\,\mathrm{d}\tau \right| \leq \left(\sup_{t \in [0,T]} |\phi(t)| \right) \int_0^t |f'(\tau)|\,\mathrm{d}\tau.$$

But $f \in AC[0,T]$ means that $f' \in L^1[0,T]$, so Lemma 2.1 implies the desired result. $\qquad\square$

REMARK 3.1.   The argument used to prove Theorem 3.1 fails for the Caputo derivative (21), because then the kernel blows up as $t \to 0^+$ and consequently does not have a maximum value; the function $f(t) = t^\alpha$ is a counterexample.

Consider now a general differential equation, with a non-singular (i.e. bounded) kernel derivative D_ϕ, of the form

$$\begin{cases} D_\phi y(t) = g(t, y(t)) \\ y(0) = y_0 \end{cases}, \tag{11}$$

for which Theorem 3.1 gives $0 = D_\phi y(t)\big|_{t=0^+} = g(0, y_0)$. Hence (11) can have a solution only if $g(0, y_0) = 0$.

Thus in (11) one is forced to choose the initial data y_0 such that $g(0, y_0) = 0$. This is of course restrictive — and may even be impossible in some cases.

3.3. No inversion of non-singular kernel derivatives without restrictions.
Overlooking the *zero-zero property* of derivatives with bounded kernel (Theorem 3.1) can lead to the construction of integral operators, such as $^{\mathrm{CF}}J_0^\alpha$ and $^{\mathrm{AB}}J_0^\alpha$, that are sometimes misinterpreted as inverse operators for the corresponding derivatives. For when these integral operators are applied to solve the differential equation (11) they do not yield correct solutions unless one imposes restrictions on the data, as we shall show in Theorem 3.2, which generalizes Propositions 3.1 and 3.2.

We work in the following general framework. The bounded kernel $\phi(t)$ that defines the non-singular derivative D_ϕ is usually defined for all $t \geq 0$, but the problems that we consider are typically posed on a bounded interval $[0,T]$. Thus we regard ϕ as defined only on $[0,T]$, and for the purpose of taking its Laplace Transform (LT) we extend $\phi(t)$ to $(0,\infty)$ by

setting $\phi(t) = 0$ for all $t > T$. This extension (or any other extension of ϕ on (T, ∞)) does not affect the differential equations that we investigate.

Then we make the following assumptions on the kernel function ϕ :

H1: $\phi(t)$ is continuous on $[0, T]$;
H2: $\phi(t)$ is differentiable on $(0, T)$ and $\phi'(t)$ has at worst an integrable singularity at $t = 0$.

The assumptions **H1** and **H2** are not restrictive; for example, they are satisfied by the CF and AB kernels.

The LT is defined in the usual way: for all suitable functions g and $s > 0$, the LT of g is

$$\hat{g}(s) := \int_{t=0}^{\infty} e^{-st} g(t) \, \mathrm{d}t,$$

the assumptions **H1** –**H2** and our zero extension of $\phi(t)$ from $[0, T]$ to $[0, \infty)$ ensure that the LT $\hat{\phi}(s)$ of ϕ exists for all $s > 0$.

Set $u(t) = D_\phi f(t) = \int_0^t \phi(t - \tau) f'(\tau) \, \mathrm{d}\tau$ for $0 < t \leq T$. Then standard LT properties (see, e.g., [1, Section D.3]) give

$$\hat{u}(s) = \hat{\phi}(s)\big[s\hat{f}(s) - f(0)\big],$$

where $\hat{f}(s)$ and $\hat{u}(s)$ are the LT of $f(t)$ and $u(t)$ respectively. Hence

$$\hat{f}(s) = \frac{1}{s} f(0) + \hat{\psi}(s)\hat{u}(s), \quad \text{where} \quad \hat{\psi}(s) := \frac{1}{s\hat{\phi}(s)}. \tag{12}$$

But the final value theorem [1, Theorem D.13] for the LT and **H1** yields

$$\lim_{s \to \infty} s\hat{\phi}(s) = \lim_{t \to 0^+} \phi(t) = \phi(0). \tag{13}$$

Consequently $\lim_{s \to \infty} \hat{\psi}(s) = 1/\phi(0) \neq 0$. It then follows from [2, Theorem 23.2] that $\hat{\psi}(s)$ cannot be the LT of any function $\psi(t)$. Thus, one cannot invert the LT in (12) to obtain a solution of the form $f(t) = f(0) + \int_0^t \psi(t - \tau) u(\tau) \, \mathrm{d}\tau$.

One might try to circumvent this obstacle by the following device: set

$$\hat{\psi}^\star(s) = \hat{\psi}(s) - \frac{1}{\phi(0)}$$

and reformulate (12) as

$$\hat{f}(s) = \frac{1}{s} f(0) + \frac{1}{\phi(0)} \hat{u}(s) + \hat{\psi}^\star(s)\hat{u}(s). \tag{14}$$

Since $\lim_{\mathrm{Re}(s) \to \infty} \hat{\psi}^\star(s) = 0$, one cannot exclude *a priori* the existence of a function $\psi^\star(t)$ whose LT is $\hat{\psi}^\star(s)$. If such a function exists, one can transform (14) back to the time domain, obtaining $f(t) = f(0) + \tilde{J}_\psi u(t)$, where

$$\tilde{J}_\psi u(t) = \frac{1}{\phi(0)} u(t) + \int_0^t \psi^\star(t - \tau) u(\tau) \, \mathrm{d}\tau. \tag{15}$$

Thus we now have an operator $\tilde{J}_\psi$, analogous to ${}^{\mathrm{CF}}J_0^\alpha$ and ${}^{\mathrm{AB}}J_0^\alpha$ in Section 3.1, such that

$$\tilde{J}_\psi \big[D_\phi f(t) \big] = f(t) - f(0).$$

It turns out however that D_ϕ is not necessarily the left inverse of $\tilde{J}_\psi$, as we now show.

THEOREM 3.2. Let $\tilde{J}_\psi$ be the operator defined in (15) and $f \in AC[0, T]$. Then

$$D_\phi \big[\tilde{J}_\psi f(t) \big] = f(t) - \frac{\phi(t)}{\phi(0)} f(0) - \phi(t) \cdot \lim_{t \to 0^+} J_{\psi^\star} f(t),$$

where $J_{\psi^\star} f(t) = \int_0^t \psi^\star(t - \tau) f(\tau) \, \mathrm{d}\tau$.

P r o o f. It is straightforward to evaluate

$$D_\phi \big[\tilde{J}_\psi f(t) \big] = \int_0^t \phi(t - \tau) \frac{\mathrm{d}}{\mathrm{d}\tau} \left[\frac{1}{\phi(0)} f(\tau) + \int_0^\tau \psi^\star(\tau - u) f(u) \, \mathrm{d}u \right] \mathrm{d}\tau$$

$$= \frac{1}{\phi(0)} D_\phi f(t) + \int_0^t \phi(t - \tau) g(\tau) \, \mathrm{d}\tau,$$

where $g(t) := \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \psi^\star(t - \tau) f(\tau) \, \mathrm{d}\tau$. Then the LT of g is

$$\hat{g}(s) = s \hat{\psi}^\star(s) \hat{f}(s) - J_{\psi^\star} f(t) \big|_{t=0} = \frac{1}{\hat{\phi}(s)} \hat{f}(s) - \frac{s}{\phi(0)} \hat{f}(s) - \lim_{t \to 0^+} J_{\psi^\star} f(t).$$

Hence

$$\mathcal{L}\left(\int_0^t \phi(t - \tau) g(\tau) \, \mathrm{d}\tau \, ; \, s \right) = \hat{\phi}(s) \left[\frac{1}{\hat{\phi}(s)} \hat{f}(s) - \frac{s}{\phi(0)} \hat{f}(s) - \lim_{t \to 0^+} J_{\psi^\star} f(t) \right]$$

$$= \hat{f}(s) - \frac{\hat{\phi}(s) \big(s \hat{f}(s) - f(0) \big)}{\phi(0)} - \frac{\hat{\phi}(s)}{\phi(0)} f(0) - \hat{\phi}(s) \cdot \lim_{t \to 0^+} J_{\psi^\star} f(t).$$

Inverting the LT, we get

$$\int_0^t \phi(t - \tau) g(\tau) \, \mathrm{d}\tau = f(t) - \frac{1}{\phi(0)} D_\phi f(t) - \frac{\phi(t)}{\phi(0)} f(0) - \phi(t) \cdot \lim_{t \to 0^+} J_{\psi^\star} f(t)$$

from which the result follows. $\qquad \square$

For well behaved functions it is possible that $\lim_{t \to 0^+} J_{\psi^\star} f(t) = 0$; but then, whenever $f(0) \neq 0$, it is clear that $D_\phi \big[\tilde{J}_\psi f(t) \big] \neq f(t)$. That is, the operator $\tilde{J}_\psi u(t)$ does not give a solution of the differential equation (11)

unless, once again, the restriction $g(0, y_0) = 0$ is imposed on the vector field $g(t, y(t))$.

4. Parabolic time-fractional initial-boundary value problems

In this section we follow [27]. Let Ω be a bounded domain in $\mathbb{R}^n$ for some $n \geq 1$. Let $T > 0$ be fixed. We consider initial-boundary value problems posed on $\Omega \times [0, T]$.

Let $\alpha \in (0, 1)$. For any suitable function $g(x, t)$ defined on $\Omega \times [0, T]$, the Caputo fractional temporal derivative D_t^α of order α is (see (21)):

$$D_t^\alpha g(x, t) := \frac{1}{\Gamma(1 - \alpha)} \int_{\tau=0}^t (t - \tau)^{-\alpha} \frac{\partial g(x, \tau)}{\partial \tau} \, d\tau, \quad \text{for } x \in \Omega, \ 0 < t \leq T.$$

Consider the time-fractional initial-boundary value problem

$$D_t^\alpha u - \Delta u = f(x, t) \tag{16a}$$

for $(x, t) \in Q := \Omega \times (0, T]$, with

$$u(x, t) = g(x, t) \quad \text{for } (x, t) \in \partial\Omega \times (0, T], \tag{16b}$$

$$u(x, 0) = u_0(x) \quad \text{for } x \in \bar{\Omega}, \tag{16c}$$

where the given functions g and u_0 are continuous on the closures of their domains.

Suppose now that $D_t^\alpha g$ is replaced by

$$\tilde{D}_t^\alpha g(x, t) := \int_{\tau=0}^t K(t, \tau) \frac{\partial g(x, \tau)}{\partial \tau} \, d\tau \quad \text{for } x \in \Omega, \ 0 < t \leq T, \tag{17}$$

where *the kernel $K(t, \tau)$ is nonsingular,* i.e., K is continuous on $[0, T] \times [0, T]$. (Here, similarly to (3), we do not write down any explicit normalisation factor for D_t^α; this factor is absorbed into the kernel K.) Note that the kernel $K(t, \tau)$ includes kernels of the form $\phi(t - \tau)$ as a special case.

For this nonsingular kernel, one has the following remarkable result:

THEOREM 4.1. [27, Theorem 1] *Let $u(x, t)$ be a solution of the initial-boundary value problem (16), where a continuous-kernel fractional derivative $\tilde{D}_t^\alpha u$ is used in (16a). Suppose that for each $x \in \Omega$, the function $u(x, \cdot)$ lies in $AC[0, T]$. Then the initial data $u_0(x) = u(x, 0)$ must satisfy the equation $\Delta u_0(x) = f(x, 0)$ on Ω.*

P r o o f. Theorem 3.1 implies that $\lim_{t \to 0^+} \tilde{D}_t^\alpha u(x, t) = 0$ for each $x \in \Omega$. Hence, taking the limit of equation (16a) as $t \to 0^+$, we get $\Delta u_0(\cdot) = f(\cdot, 0)$ on Ω. □

REMARK 4.1.   The hypothesis of Theorem 4.1 that for each $x \in \Omega$, the function $u(x, \cdot)$ lies in $AC[0, T]$ is not restrictive. This condition is satisfied by almost every example in the literature on time-fractional initial-boundary value problems.

The next example shows the powerful consequences of Theorem 4.1.

EXAMPLE 4.1.   Consider the fractional heat equation
$$\tilde{D}_t^\alpha v - \partial^2 v / \partial x^2 = 0 \ \text{ for } (x, t) \in (0, 1) \times (0, T],$$
where $\tilde{D}_t^\alpha$ is a continuous-kernel fractional derivative, the boundary data are $v(0, t) = v(1, t) = 0$ and the initial data $v(x, 0) = v_0(x)$, where $v_0(x) \in C^2[0, 1]$ is unspecified except that it satisfies the initial-boundary compatibility condition $v_0(0) = v_0(1) = 0$, so that any solution u of (16) is continuous on $\bar{\Omega} \times [0, T]$.

Assume that for each x, the solution $v(x, \cdot)$ of this problem lies in $L^1[0, T]$. Then Theorem 4.1 and the above compatibility condition show that v_0 must satisfy the conditions
$$-v_0''(x) = 0 \ \text{ on } (0, 1), \quad v_0(0) = v_0(1) = 0.$$
But these conditions imply that $v_0 \equiv 0$. As all the data of this example are now zero, we get $v \equiv 0$.

Thus, using a continuous-kernel fractional derivative forces the problem to have as its solution $v \equiv 0$; the apparent freedom of choice that one has for v_0 is only an illusion.

In [27], the differential operator $-\Delta u$ of (16a) is replaced by a much more general spatial operator, and it is shown that under reasonable conditions, *the initial data* (16c) *is determined uniquely by the other data of the problem.* Example 4.1 is a particular case of this phenomenon. Such a restrictive condition is extremely unnatural and it is clearly caused by the use of a continuous-kernel fractional derivative.

5. **Are CF and ABC derivatives really new and necessary?**

As we saw in Section 3, the differential equation (11) with the CF and ABC derivatives (or any other non-singular kernel derivative) requires $g(0, y_0) = 0$ in order to have a solution, where $g(t, y(t))$ is the vector field of the differential equation.

Suppose that one limits the use of these derivatives to those problems that satisfy the condition $g(0, y_0) = 0$. We now show that in this special case, the CF and ABC derivatives serve no purpose since the problem can then be described by simpler operators.

Consider the CF initial-value problem (7). Assume that $g(0, y_0) = 0$. Then the solution of the problem is (8). Differentiating this equation gives

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) = \frac{1 - \alpha}{M(\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} g(t, y(t)) + \frac{\alpha}{M(\alpha)} g(t, y(t))$$

— so $y(t)$ is the solution of an integer-order differential equation! Thus there is no need to use a fractional derivative to find y; this function can be handled in the framework of classical calculus.

This observation that the CF derivative is not truly fractional but can be reformulated using integer-order derivatives is discussed in [31].

Similarly, for the ABC derivative, under the assumption that $g(0, y_0) = 0$ one gets

$$^{\mathrm{C}}D_0^\alpha y(t) = \frac{1 - \alpha}{B(\alpha)} {}^{\mathrm{C}}D_0^\alpha g(t, y(t)) + \frac{\alpha}{B(\alpha)} g(t, y(t)),$$

so y is the solution of a Caputo differential equation and introducing the ABC derivative is unnecessary.

A further connection between CF and ABC derivatives and some standard operators of integer and fractional-order operators was shown in [5].

REMARK 5.1. The CF and ABC derivatives are sometimes described as special cases of the fractional Prabhakar derivative, but this is not true. Introduced in [3] to provide a Caputo-like regularization of the operator previously introduced in [14], the Prabhakar derivative is defined as

$$^{\mathrm{C}}\mathcal{D}_{\alpha,\beta,\lambda;0}^\gamma f(t) = \int_0^t (t - u)^{m-\beta-1} E_{\alpha,m-\beta}^{-\gamma} \left(\lambda(t - u)^\alpha \right) f^{(m)}(u) \, \mathrm{d}u,$$

with $m = \lceil \beta \rceil$. It is not obtained by replacing the standard power law kernel of the Dzhrbashyan-Caputo derivative with a particular realization of the three-parameter ML function [25]

$$E_{\alpha,\beta}^\gamma(z) = \frac{1}{\Gamma(\gamma)} \sum_{k=0}^\infty \frac{\Gamma(\gamma + k)z^k}{k! \Gamma(\alpha k + \beta)}, \quad \alpha, \beta, \gamma \in \mathbb{C}, \quad \mathrm{Re}(\alpha) > 0, \ z \in \mathbb{C},$$

but rather it is defined as the left-inverse of the Prabhakar integral

$$\mathcal{J}_{\alpha,\beta,\lambda;0}^\gamma f(t) = \int_0^t (t - u)^{\beta-1} E_{\alpha,\beta}^\gamma \left(\lambda(t - u)^\alpha \right) f(u) \, \mathrm{d}u, \quad \alpha, \beta > 0.$$

Hence, unlike CF and ABC operators, the Prabhakar derivative $^{\mathrm{C}}\mathcal{D}_{\alpha,\beta,\lambda;0}^\gamma$ naturally satisfies the fundamental theorem of fractional calculus. Moreover, the kernel of $^{\mathrm{C}}\mathcal{D}_{\alpha,\beta,\lambda;0}^\gamma f(t)$ is always singular at $t = 0$ (except for the limit case $\beta \in \mathbb{N}$ discussed in details in [6]) and no zero-zero property holds

with the Prabhakar derivative. The standard Dzhrbashyan-Caputo derivative of order β is obtained when $\gamma = 0$ or $\lambda = 0$. We refer to [7] for a complete treatment of the Prabhakar fractional calculus.

6. Are non-singular kernel derivatives really derivatives?

One should also consider whether operators obtained by inserting a non-singular kernel in the RL and Caputo derivatives can really be described as derivatives. While several papers make systematic attempts to determine whether or not a new operator is fractional [13, 20, 21, 29, 30], to the best of our knowledge very few attempts have been made to discern whether or not an operator is a derivative. There are contributions by Ortigueira and Machado [22, 23] based on systems theory, but we wish to explore this question using only mathematical considerations.

In our Appendix A.1 we describe the *indirect process* for the derivation of the RL and Caputo derivatives. In this process one first generalises integer-order repeated integrals to any real positive order, then one defines derivatives as operators that are the inverse of the integral (by analogy with integer-order calculus, where the derivative can be viewed as the inverse operator of the integral). The RL (20) and Caputo (21) derivatives obtained in this way are formulated in terms of convolution integrals. Under assumptions that are reasonable and unrestrictive, they are equivalent to operators obtained by a more straightforward generalization of the usual definition of the integer order derivative (see the description of the *direct process* in Appendix A.2).

Defining derivatives by means of integrals may appear unnatural at first sight, but the property of acting as an inverse of the repeated integral, and the fact that the same operators can be obtained by generalizing the integer-order derivative, together provide a compelling justification for recognising the RL and Caputo derivative operators as bona fide derivatives.

The construction of fractional derivatives with non-singular kernels imitates, but only in a partial way, the indirect process described above. It might appear attractive to modify the RL and Caputo derivatives by replacing their singular kernels by a non-singular function, but then it is difficult to justify the statement that these new operators are really derivatives. Indeed:

- as we have shown in Sections 2 and 3, there is no integral of which non-singular kernel derivatives are the inverse operators; thus the *indirect process* leading to the construction of the RL and Caputo derivatives is partly imitated but is not fully replicated;
- there is no evidence that these new "derivatives" can be generated through a direct generalisation of the integer-order derivative,

as described in Appendix A.2 for the RL and Caputo fractional derivatives.

Consequently we think that it is truly questionable to describe as derivatives the operators discussed in this paper. Formulas such as (1) and (2), and more generally (3), are more akin to integral operators than to derivatives and calling them derivatives is misleading; to avoid confusion, the more general term *operator* rather than the specific term *derivative* should be used.

6.1. **A further observation.** Integration by parts of (1) and (2) yields

$$
{}^{\mathrm{CF}}D_0^\alpha f(t) = \frac{M(\alpha)}{1-\alpha} \left[f(t) - \exp\left(-\frac{\alpha}{1-\alpha}t\right) f(0) \right.
$$

$$
\left. - \frac{\alpha}{1-\alpha} \int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right) f(\tau)\,\mathrm{d}\tau \right]
$$

and

$$
{}^{\mathrm{ABC}}D_0^\alpha f(t) = \frac{B(\alpha)}{1-\alpha} \left[f(t) - E_\alpha\left(-\frac{\alpha}{1-\alpha}t^\alpha\right) f(0) \right.
$$

$$
\left. - \frac{\alpha}{1-\alpha} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right) f(\tau)\,\mathrm{d}\tau \right]
$$

(note that a similar calculation is impossible for derivatives with a singular kernel, such as the RL and Caputo derivatives). These identities surely cast doubt on any claim that ${}^{\mathrm{CF}}D_0^\alpha$ and ${}^{\mathrm{ABC}}D_0^\alpha$ represent derivatives, since they merely comprise evaluations of the function f and a weighted integral of f, i.e., no differentiation of f is involved.

7. **Concluding remarks**

In this paper we have discussed the properties and drawbacks of the operators, commonly called non-singular kernel derivatives, that are obtained by replacing the singular kernel of the Caputo derivative with a non-singular function. While the so-called CF and ABC derivatives are the best-known operators of this type, our analysis covers any derivative with non-singular kernel.

We have shown that non-singular kernel derivatives do not in general have an inverse that can be written as a convolution integral — unlike the RL and Caputo derivatives, which enjoy this property. One can construct an integral with the operator as its left-inverse only if the function is zero at the origin. This follows from a "zero-zero" property: non-singular kernel

derivatives are always zero when evaluated at the initial time $t = 0$. Consequently, it is possible to solve differential equations with non-singular kernel derivatives only when a very restrictive and unnatural assumption is made on the initial condition.

We then go on to show that if one accepts this restrictive condition, then the CF and ABC derivatives can be replaced by finite combinations of operators that are already known from classical calculus and the Caputo derivative calculus.

We also cast doubt on the belief that a non-singular kernel derivative can be regarded as a true form of derivative.

Our overwhelming conclusion from all this evidence is that derivatives with non-singular kernel should never be used.

Appendix A. **Background material on fractional calculus**

Standard fractional derivatives such as the RL and Dzhrbashyan-Caputo derivatives can be introduced by following a *direct process* that starts from the integer-order derivative and leads to a fractional generalisation of the difference quotient. Alternatively, by an *indirect process* one first obtains the RL integral as a generalisation of the usual integer-order repeated integral, then inverses of this integral, which are formulated in terms of a convolution integrals, are defined as fractional derivatives.

For completeness of exposition we briefly describe here the two processes and show that they lead to equivalent operators.

A.1. **Indirect process: generalization of integer-order integrals and inversion.** To introduce fractional derivatives, begin by considering the standard (integer-order) n-fold repeated integral

$$J_0^n f(t) := \frac{1}{(n-1)!} \int_0^t (t - \tau)^{n-1} f(\tau) \, \mathrm{d}\tau, \quad t > 0. \tag{18}$$

Then the Euler-Gamma function $\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t} \, \mathrm{d}t$, $\mathrm{Re}(x) > 0$, which satisfies $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$, allows us to extend (18) from integers n to any real positive number α by setting

$$^{\mathrm{RL}}J_0^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) \, \mathrm{d}\tau, \quad t > 0. \tag{19}$$

This is the fractional Riemann-Liouville integral.

A left-inverse of the operator $^{\mathrm{RL}}J_0^\alpha$ is an operator that when applied to $^{\mathrm{RL}}J_0^\alpha f(t)$ gives back the original function $f(t)$. It is possible to find more than one such operator. In fact, writing $m = \lceil \alpha \rceil$ for the smallest integer greater than or equal to α and D^m for the usual integer-order differentiation, both the fractional RL derivative

$$^{\mathrm{RL}}D_0^\alpha f(t) := \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_0^t (t-\tau)^{m-\alpha-1} f(\tau)\,\mathrm{d}\tau, \quad t > 0 \qquad (20)$$

and the Caputo derivative

$$^{\mathrm{C}}D_0^\alpha f(t) := \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)\,\mathrm{d}\tau, \quad t > 0, \qquad (21)$$

are left-inverses of $^{\mathrm{RL}}J_0^\alpha$ since $^{\mathrm{RL}}D_0^\alpha\big[^{\mathrm{RL}}J_0^\alpha f\big] = {}^{\mathrm{C}}D_{t_0}^\alpha\big[^{\mathrm{RL}}J_0^\alpha f\big] = f$ under reasonable hypotheses on f (see, e.g., [1, Theorems 2.14 and 3.7]). It is well known [1, Definition 3.2] that these two operators are related by

$$^{\mathrm{C}}D_0^\alpha f(t) = {}^{\mathrm{RL}}D_0^\alpha\Big[f(t) - T_{m-1}[f,0](t)\Big], \qquad (22)$$

where $T_{m-1}[f,0](t)$ is the Taylor polynomial of $f(t)$ expanded around 0, viz.,

$$T_{m-1}[f,0](t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} f^{(k)}(0).$$

A.2. Direct process: generalization of integer-order derivatives.
To describe a more *direct process* we first consider the usual definition of the first-order derivative

$$f'(t) = \lim_{h\to 0^+} \frac{f(t) - f(t-h)}{h}, \qquad (23)$$

which is easily extended to any $n \in \mathbb{N}$ by simple recursion to obtain

$$f^{(n)}(t) = \lim_{h\to 0^+} \frac{1}{h^n} \sum_{j=0}^n \omega_j^{(n)} f(t-jh), \quad \omega_j^{(n)} = (-1)^j \binom{n}{j}. \qquad (24)$$

(For ease of presentation we take into consideration only limits from the right). Once again appealing to the Euler-Gamma function, the binomial coefficients can be reformulated as

$$\binom{n}{j} = \frac{n!}{j!(n-j)!} = \begin{cases} \dfrac{\Gamma(n+1)}{j!\Gamma(n+1-j)} & j = 0, 1, \ldots, n, \\ 0 & j > n. \end{cases} \qquad (25)$$

Then the simple observation that $\omega_j^{(n)} = 0$ for any $j > n$ allows us to rewrite (24) as an infinite series

$$f^{(n)}(t) = \lim_{h\to 0^+} \frac{1}{h^n} \sum_{j=0}^\infty \omega_j^{(n)} f(t-jh),$$

and, since (25) permits coefficients $\omega_j^{(n)}$ with n replaced by a non-integer parameter $\alpha > 0$, a generalisation to fractional order of (24) is easily obtained:

$$^{\mathrm{GL}}D^{\alpha}f(t) = \lim_{h\to 0^{+}} \frac{1}{h^{\alpha}} \sum_{j=0}^{\infty} \omega_{j}^{(\alpha)} f(t-jh), \quad \omega_{j}^{(\alpha)} = (-1)^{j}\frac{\Gamma(\alpha+1)}{j!\Gamma(\alpha-j+1)}. \quad (26)$$

The operator $^{\mathrm{GL}}D^{\alpha}$ is generally known as the Grünwald-Letnikov (GL) fractional derivative because proposed independently by Grünwald [10] and Letnikov [18] in 1867 and 1868 respectively. Although it is perhaps the most straightforward generalization of the integer-order derivative to any fractional order, it has some drawbacks:

- $^{\mathrm{GL}}D^{\alpha}f(t)$ requires the knowledge of the whole history of the function f in $(-\infty, t]$: while this may not be a difficulty from a purely mathematical point of view when $f(t)$ is known analytically for all t, when $^{\mathrm{GL}}D^{\alpha}$ is applied in differential equations the solution $f(t)$ (usually the state of a system) is known only starting from a given initial time. For this reason the use of $^{\mathrm{GL}}D^{\alpha}$ is mainly confined to signals theory, where signals are often decomposed into sin and cos functions whose values are available for all t;
- the series in (26) converges only for a restricted class of functions (for instance, bounded functions when $0 < \alpha < 1$) and this limitation is too restrictive for general applications.

To overcome these drawbacks, a common strategy is to fix a starting point, say for convenience 0, and impose suitably chosen values for $f(t)$ on $(-\infty, 0)$. Usually the following functions are considered:

$$f^{\mathrm{R}}(t) = \begin{cases} 0 & t \in (-\infty, 0) \\ f(t) & t \geq 0 \end{cases} \quad , \quad f^{\mathrm{C}}(t) = \begin{cases} T_{m-1}[y,0](t) & t \in (-\infty, 0) \\ f(t) & t \geq 0. \end{cases}$$

The replacement of f by f^{R} or f^{C}, together with the property of the coefficients $\omega_{j}^{(\alpha)}$ (see, e.g., [4]) that $\sum_{j=0}^{\infty} \omega_{j}^{(\alpha)} j^{k} = 0$ for $k = 0, 1, \dots, m-1$, leads to the two distinct fractional derivatives

$$^{\mathrm{GL}}D^{\alpha}f^{\mathrm{R}}(t) = \lim_{h\to 0^{+}} \frac{1}{h^{\alpha}} \sum_{j=0}^{N} \omega_{j}^{(\alpha)} f(t - jh),$$

$$^{\mathrm{GL}}D^{\alpha}f^{\mathrm{C}}(t) = \lim_{h\to 0^{+}} \frac{1}{h^{\alpha}} \sum_{j=0}^{N} \omega_{j}^{(\alpha)} \left[f(t - jh) - T_{m-1}[y,0](t - jh) \right],$$

where $N = \lfloor t/h \rfloor$.

Interestingly, there is a link between the indirect and direct processes. A result in fractional calculus [1, Theorem 2.25] states that if $f \in C^{m}[0, T]$, then $^{\mathrm{GL}}D^{\alpha}f^{\mathrm{R}}(t) = {}^{\mathrm{RL}}D_{0}^{\alpha}f(t)$; consequently, in view of (22), one also has $^{\mathrm{GL}}D^{\alpha}f^{\mathrm{C}}(t) = {}^{\mathrm{C}}D_{0}^{\alpha}f(t)$.

Appendix B. **CF and ABC derivatives are not left-inverse of CF and AB integrals**

For completeness, in this section we give the elementary derivations showing that the CF derivative $^{\mathrm{CF}}D_0^\alpha$ is not the left-inverse of the so-called CF integral $^{\mathrm{CF}}J_0^\alpha$. For notational convenience, throughout this section we use the abbreviation

$$W_\alpha = \frac{\alpha}{1-\alpha}.$$

P r o o f   o f   P r o p o s i t i o n   3.1. Let $f \in AC[0,T]$. Observe that

$$^{\mathrm{CF}}D_0^\alpha\left[^{\mathrm{CF}}J_0^\alpha f(t)\right] = \underbrace{\frac{1-\alpha}{M(\alpha)}\,^{\mathrm{CF}}D_0^\alpha f(t)}_{(A)} + \underbrace{\frac{\alpha}{M(\alpha)}\,^{\mathrm{CF}}D_0^\alpha \int_0^t f(s)\,\mathrm{d}s}_{(B)}.$$

Integration by parts allows us to evaluate the first integral (A):

$$(A) = \int_0^t \exp\left(-W_\alpha(t-\tau)\right)f'(\tau)\,\mathrm{d}\tau$$

$$= f(t) - \exp\left(-W_\alpha t\right)f(0) - W_\alpha \int_0^t \exp\left(-W_\alpha(t-\tau)\right)f(\tau)\,\mathrm{d}\tau.$$

For the integral (B), one has immediately

$$(B) = W_\alpha \int_0^t \exp\left(-W_\alpha(t-\tau)\right)\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau f(s)\,\mathrm{d}s\,\mathrm{d}\tau$$

$$= W_\alpha \int_0^t \exp\left(-W_\alpha(t-\tau)\right)f(\tau)\,\mathrm{d}\tau.$$

Now we are done, since $^{\mathrm{CF}}D_0^\alpha\left[^{\mathrm{CF}}J_0^\alpha f(t)\right] = (A) + (B)$. □

In a similar way, one can show that the ABC derivative $^{\mathrm{ABC}}D_0^\alpha$ is not the left-inverse of the so-called AB integral $^{\mathrm{AB}}J_0^\alpha$.

P r o o f   o f   P r o p o s i t i o n   3.2. Let $f \in AC[0,T]$. Observe that

$$^{\mathrm{ABC}}D_0^\alpha\left[^{\mathrm{AB}}J_0^\alpha f(t)\right] = \underbrace{\frac{1-\alpha}{B(\alpha)}\,^{\mathrm{ABC}}D_0^\alpha f(t)}_{(C)} + \underbrace{\frac{\alpha}{B(\alpha)}\,^{\mathrm{ABC}}D_0^\alpha \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau)\,\mathrm{d}\tau}_{(D)}.$$

Integration by parts yields

$$(C) = \int_0^t E_\alpha\left(-W_\alpha(t-\tau)^\alpha\right)f'(\tau)\,\mathrm{d}\tau$$

$$= f(t) - E_\alpha\left(-W_\alpha t^\alpha\right)f(0) - W_\alpha \int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}\left(-W_\alpha\tau^\alpha\right)f(t-\tau)\,\mathrm{d}\tau.$$

To evaluate (D) we first observe that

$$(D) = W_\alpha \int_0^t E_\alpha\Big(-W_\alpha(t-\tau)^\alpha\Big)\frac{\mathrm{d}}{\mathrm{d}\tau}{}^{\mathrm{RL}}J_0^\alpha f(\tau)$$

and by means of the LT we can evaluate [19, Eq. (1.10)]

$$\mathcal{L}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}{}^{\mathrm{RL}}J_0^\alpha f(\tau)\,\mathrm{d}\tau\,;\,s\right) = s\frac{1}{s^\alpha}\hat{f}(s) - {}^{\mathrm{RL}}J_0^\alpha f(0^+) = s^{1-\alpha}\hat{f}(s),$$

where we used standard rules for the LT of the first-order derivative together with ${}^{\mathrm{RL}}J_0^\alpha f(0^+) = 0$ since $f \in AC[0,T]$. Therefore, since the LT of $t^{\beta-1}E_{\alpha,\beta}(-\lambda t^\alpha)$ is $s^{\alpha-\beta}/(s^\alpha + \lambda)$ [8, Eq. (4.10.1)], we get

$$\mathcal{L}\left(\int_0^t E_\alpha\Big(-W_\alpha(t-\tau)^\alpha\Big)\frac{\mathrm{d}}{\mathrm{d}\tau}{}^{\mathrm{RL}}J_0^\alpha f(\tau)\,;\,s\right) = \frac{1}{s^\alpha + W_\alpha}\hat{f}(s).$$

The inversion of the LT gives

$$(D) = W_\alpha \int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}\Big(-W_\alpha\tau^\alpha\Big)f(t-\tau)\,\mathrm{d}\tau$$

from which the result follows since ${}^{\mathrm{ABC}}D_0^\alpha\big[{}^{\mathrm{AB}}J_0^\alpha f(t)\big] = (C) + (D)$. $\qquad\square$

Acknowledgments

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