



On the Kuzmin model in fractional Newtonian gravity

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Abstract Fractional Newtonian gravity, based on the fractional generalization of Poisson's equation for Newtonian gravity, is a novel approach to Galactic dynamics aimed at providing an alternative to the dark matter paradigm through a non-local modification of Newton's theory. We provide an in-depth discussion of the gravitational potential for the Kuzmin disk within this new approach. Specifically, we derive an integral and a series representation for the potential, we verify its asymptotic behavior at large scales, and we provide illuminating plots of the resulting equipotential surfaces.

1 Introduction

Galaxy rotation curves and the formation of large-scale structure in the universe are among the most compelling indications that general relativity and the standard model of particle physics cannot account for all natural phenomena. The situation is even more severe than that, indeed it turns out that the theoretical tools which are currently available in physics can only resolve about 5% of the content of the universe. In more detail, in order to explain the current accelerating expansion of the universe it is customary to postulate the existence of an exotic *dark energy* [1,2] fluid, with positive energy and negative pressure, affecting the universe on its largest scale. Similarly, in order to account for structure formation after the Big Bang, as well as deviations from the expected Newtonian predictions for galaxy rotation curves, it seems to be necessary to include an additional dark component of the universe, featuring no direct coupling with electromagnetic radiation and an (almost) imperceptible pressure, which is dubbed as *dark matter* [3–5]. In the picture discussed above, dark matter and dark energy are treated as exotic forms of matter evading the Standard Model of particle physics. This exotic matter content finds its way in the so-called standard model of cosmology, also known as the Λ —cold dark matter model or Λ CDM for short, according to which the energy content of the universe splits into a 5% of ordinary (luminous) matter, 25% of dark matter, and about 70% is accounted for by dark energy. It is worth stressing that cold dark matter, namely dark matter moving with non-relativistic velocity, seems to be favored with

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respect to “warm” and “hot” models since it yields predictions for the cosmological large-scale structure that generally agree with current astronomical observations [6].

An alternative approach to dark matter and dark energy, which are typically added *ad hoc* in Einstein’s theory to reproduce the astronomical and cosmological observations, requires to rethink gravitational physics at a more fundamental level and include large-scale modifications of gravity aimed at reconciling theory and experiments. Notably, extensive efforts have been devoted toward the study of alternative theories of gravity that could replace, at least in part, dark matter and dark energy with the phenomenology of some additional gravitational degrees of freedom, see, e.g., [7–17]. However, lacking a direct detection of new particles signaling the emergence of physics beyond the Standard Model, and any definitive experimental proof of significant deviations from General Relativity, one can only conclude the jury is still out on what is really responsible for the odd phenomena observed at galactic and cosmological scales.

One of the most successful proposals of modification of gravity theory aimed at explaining the phenomenology typically traced back to dark matter is known as modified Newtonian dynamics (MOND), originally introduced by Milgrom in [18–21]. The idea behind this approach relies on the assumption that there exists a critical acceleration scale a_0 , whose value is empirically determined, such that Newton’s gravity dramatically changes when the magnitude of the acceleration of a test particle falls below this threshold. Specifically, under the simple assumption of spherical symmetry and considering a test particle on a stable orbit around a core mass M , denoting by $a = a(r)$ the acceleration of the test body MOND predicts that for $a \gg a_0$ one recovers standard Newtonian gravity, i.e.,

$$a \simeq \frac{G_N M}{r^2}, \tag{1}$$

whereas when $a \ll a_0$ the dynamics of the test particle is modified according to

$$\frac{a^2}{a_0} \simeq \frac{G_N M}{r^2} \tag{2}$$

with r denoting the distance from the center of the system. In other words, MOND recovers the standard Newtonian scaling of the acceleration $a(r) \sim 1/r^2$ at short scales, while the model yields the asymptotic behavior $a(r) \sim 1/r$ at large (Galactic) scales. This implies that the rotational velocity of a test body around a Galaxy center behaves as $v^2(r) \sim G_N m(r)/r$ in the innermost part of the Galaxy, with $m(r)$ denoting the total mass contained within a circular orbit of radius r , while $v^4(r) \sim G_N M a_0$ as one moves away from the Galaxy center. On other words, galaxy rotation curves flatten out as one moves asymptotically far from the Galaxy center, in full agreement with various astronomical observations [4, 22–24]. In [21] Bekenstein and Milgrom proposed a non-relativistic potential theory reproducing the MOND scenario based on a nonlinear modification of the Poisson equation of Newtonian gravity. The first robust relativistic MOND inspired model, known as tensor–vector–scalar gravity or TeVeS, was then proposed by Bekenstein [25]. Clearly, this last proposal is not exempt from problems; however, in the broader scheme of things it served as the seminal work for the study of dark matter phenomenology as an emergent effect of alternative theories of gravity.

Fractional calculus [26–28] offers a reliable set of tools for describing several physical phenomena which are not typically accounted for by model based on ordinary calculus (see, e.g., [28–30]). In recent years, this mathematical scheme has also been applied, in various forms, to gravity and fundamental physics, see, e.g., [31–34]. Focusing on the problem of dark matter phenomenology, the first *fractional* MOND-like non-relativistic potential theory was proposed by Giusti [35]. This approach is based on a fractional modification of the

Poisson equation of Newtonian gravity, where the ordinary Laplacian $-\Delta$ is replaced by the so-called fractional Laplacian $(-\Delta)^s$ with $s \in [1, 3/2)$. Notably, another model for a MOND-like non-relativistic potential theory, somehow related to fractional calculus, was proposed by Varieschi [36,37]. Varieschi’s approach is very similar to the one in [35] though the two are not identical as discussed in [36]. The key difference lays in the fact that Varieschi’s model is *not a fractional theory*. Indeed, Varieschi’s model relies on the use of a generalized gravitational Gauss’s law where the standard integration over \mathbb{R}^3 is replaced with an Hausdorff measure of \mathbb{R}^3 related to Weyl’s fractional integral. This procedure turns the model into a generalization of Newtonian gravity on a *fractal* space, involving a measure inspired by fractional calculus, for which *the field equation remains of integer order* (and thus *local*). This specific caveat, however, does not make Varieschi’s model any less interesting or less deserving of further investigation.

This work is organized as follows: first, we recall the basics of Giusti’s *fractional Newtonian gravity*, introduced in [35], and its implications for Galactic dynamics; second, we review the preliminary results discussed in [35] for the Kuzmin disk; third, we complete the analysis for the Kuzmin disk in fractional Newtonian gravity providing a discussion of the asymptotic behavior of the corresponding potential, a series representation for the full potential outside the plane of the disk, and a numerical study of the equipotential surfaces as one varies the fractional parameter $s \in [1, 3/2)$.

2 Fractional Newtonian gravity

Fractional Newtonian gravity [35] is an alternative to standard Newtonian gravity based on a modification of the Poisson equation for the gravitational potential. Specifically, the key ingredient of this model consists in the so-called *fractional Laplacian*.

Let $f(\mathbf{x})$ be a sufficiently well-behaved function on \mathbb{R}^3 ; one defines the Fourier transform of $f(\mathbf{x})$ as

$$\widehat{f}(\mathbf{k}) \equiv \mathcal{F}[f(\mathbf{x}); \mathbf{k}] = \int_{\mathbb{R}^3} e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) d^3x, \tag{3}$$

with \cdot denoting the standard Euclidean scalar product on \mathbb{R}^3 . Hence, if

$$\Delta f(\mathbf{x}) := \text{div}[\nabla f(\mathbf{x})]$$

denotes the Laplacian of $f(\mathbf{x})$, then it is easy to see that

$$\mathcal{F}[(-\Delta)f(\mathbf{x}); \mathbf{k}] = |\mathbf{k}|^2 \widehat{f}(\mathbf{k}), \tag{4}$$

with $|\mathbf{k}|^2 \equiv \mathbf{k}\cdot\mathbf{k}$. The *fractional Laplacian* [38,39] is therefore defined as the operator $(-\Delta)^s$ such that

$$\mathcal{F}[(-\Delta)^s f(\mathbf{x}); \mathbf{k}] = |\mathbf{k}|^{2s} \widehat{f}(\mathbf{k}). \tag{5}$$

Ten equivalent representations of this operator are discussed in [39]. Further details on the fractional Laplacian are analyzed and reviewed in [40].

Fractional Newtonian gravity [35] is therefore based on the fractional Poisson equation

$$(-\Delta)^s \Phi(\mathbf{x}) = -4\pi G_N \ell^{2-2s} \rho(\mathbf{x}), \tag{6}$$

with G_N denoting the Newtonian constant of gravitation, ℓ being a constant such that $[\ell] = \text{length}$, $\rho(\mathbf{x})$ is the mass density of the system, and $1 \leq s < 3/2$ denotes the fractional

parameter. It is often useful to deal with the fractional Poisson equation in the momentum space, hence taking the Fourier transform of both sides of Eq. (6) yields

$$\widehat{\Phi}(\mathbf{k}) = -\frac{4\pi G_N \ell^{2-2s}}{|\mathbf{k}|^{2s}} \widehat{\rho}(\mathbf{k}). \tag{7}$$

Remark 2.1 Note that Eq. (7) allows one to justify the condition $s < 3/2$ on \mathbb{R}^3 , see, e.g., [41] for details.

If one considers the case of a point-like source of mass density $\rho(\mathbf{x}) = \delta^3(\mathbf{x})$ then one finds

$$\Phi_s(\mathbf{x}) = -\frac{\Gamma(\frac{3}{2} - s)}{4^{s-1} \sqrt{\pi} \Gamma(s)} \left(\frac{\ell}{|\mathbf{x}|}\right)^{2-2s} \frac{G_N M}{|\mathbf{x}|}, \quad \text{for } 1 \leq s < \frac{3}{2}. \tag{8}$$

Clearly, this expression is not well behaved as $s \rightarrow (3/2)^-$, as expected. However, focusing on the momentum space representation of the fractional Poisson Eq. (7) for $s = 3/2$, i.e.,

$$\widehat{\Phi}(\mathbf{k}) = -\frac{4\pi G_N M}{\ell |\mathbf{k}|^3}, \tag{9}$$

the inverse Fourier transform of which can be regularized (see [35]) and yields

$$\Phi_{3/2}(\mathbf{x}) \stackrel{\text{reg}}{=} \frac{2 G_N M}{\pi \ell} \log(|\mathbf{x}|/\ell). \tag{10}$$

From $\mathbf{a} = -\nabla \Phi_s(\mathbf{x})$, and after recalling that

$$a(r) = \frac{v(r)^2}{r} = |\nabla \Phi_s(r)|, \tag{11}$$

with $r = |\mathbf{x}|$, one finds the expression of the orbital speed of a test particle around the center as a function of r and s , i.e.,

$$v_s(r) = \begin{cases} \frac{2^{\frac{3}{2}-s}}{\sqrt[4]{\pi}} \sqrt{\frac{\Gamma(\frac{5}{2} - s)}{\Gamma(s)}} \left(\frac{\ell}{r}\right)^{1-s} \sqrt{\frac{G_N M}{r}}, & \text{for } 1 \leq s < 3/2, \\ \sqrt{\frac{2 G_N M}{\pi \ell}}, & \text{for } s = 3/2. \end{cases} \tag{12}$$

This suggests that, in order to smoothly reproduce the flattening of Galaxy rotation curves in fractional Newtonian gravity one needs to turn the theory into a variable order. This is achieved by replacing s with a scale-dependent fractional parameter $s(r/\ell)$ such that $s(r/\ell) \rightarrow 1$ for $r < \ell$, whereas $s(r/\ell) \rightarrow (3/2)^-$ as $r \gtrsim \ell$.

Note that, differently from pure MOND, this approach is equipped with a critical length scale ℓ rather than an acceleration scale a_0 . Furthermore, even a variable-order version of Eq. (6) yields a linear theory, whereas MOND is inherently nonlinear in nature [21]. However, one can reconcile the phenomenology of the two theories at Galactic scales by means of the (empirical) *Tully–Fisher relation* [42]

$$v^4 = G_N M a_0,$$

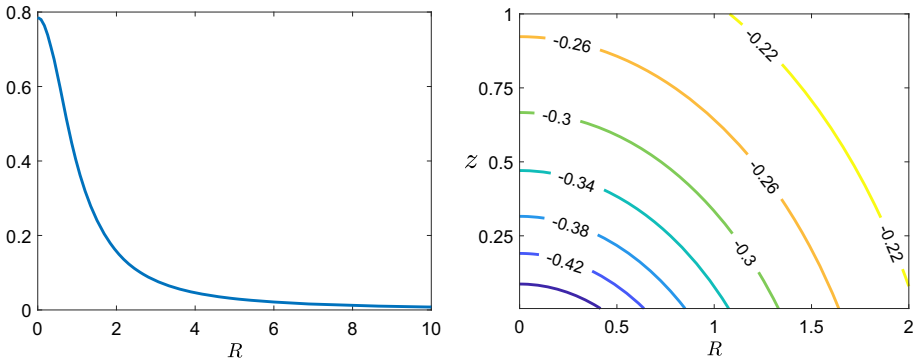


Fig. 1 Left: mass density of the Kuzmin disk on the plane of the disk ($z = 0$) as a function of R . Right: section of the three-dimensional equipotential surfaces ($\Phi_N(R, z) = \text{const.}$) for the Kuzmin disk. Assumptions: $G_N = 1, M = 0.5, R_0 = 1$

which leads to

$$\ell = \frac{2}{\pi} \sqrt{\frac{G_N M}{a_0}}, \tag{13}$$

with a_0 denoting the critical acceleration scale of MOND.

Note that in [35] the scheme of fractional Newtonian gravity and the corresponding MOND-like scenario have been generally put in connection with bootstrapped Newtonian and corpuscular gravity (see, e.g., [43–49] for details and [50] for a review on the topic).

3 The Kuzmin disk in fractional Newtonian gravity

The Kuzmin mass density that in cylindrical coordinates reads

$$\rho(R, z) = \frac{R_0 M}{2\pi (R^2 + R_0^2)^{3/2}} \delta(z), \tag{14}$$

with $R_0 > 0$ and $[R_0] = \text{length}$, is a widely used axisymmetric model for thin-disk Galaxies [51]. The classical Newtonian solution of Eq. (6) for the Kuzmin disk corresponds to the case $s = 1$ and yields

$$\Phi_N(R, z) \equiv \Phi_{s=1}(R, z) = -\frac{G_N M}{\sqrt{R^2 + (R_0 + |z|)^2}}, \tag{15}$$

see Fig. 1.

Since the Fourier transform of the Kuzmin density reads

$$\widehat{\rho}(\mathbf{k}) = \widehat{\rho}(\kappa) = M e^{-\kappa R_0},$$

with $\kappa = \sqrt{k_x^2 + k_y^2}$, then Eq. (6) for the Kuzmin disk, in the momentum space, reduces to

$$\widehat{\Phi}(\mathbf{k}) = \widehat{\Phi}(\kappa, k_z) = -4\pi G_N \ell^{2-2s} \frac{\widehat{\rho}(\mathbf{k})}{|\mathbf{k}|^{2s}} = -4\pi G_N M \ell^{2-2s} \frac{e^{-\kappa R_0}}{(\kappa^2 + k_z^2)^s}. \tag{16}$$

Inverting $\widehat{\Phi}(\mathbf{k})$ back to position space yields

$$\Phi_s(R, z) = -\frac{G_N M \ell^{2-2s}}{\pi} \int_0^\infty d\kappa \kappa e^{-\kappa R_0} J_0(\kappa R) \int_{\mathbb{R}} dk_z \frac{e^{ik_z z}}{(\kappa^2 + k_z^2)^s}. \tag{17}$$

In order to derive an expression for $\Phi_s(R, z)$ that is more easily treatable from a numerical perspective, one first has to consider the integral

$$I_1(s; \kappa, z) = \int_{\mathbb{R}} dk_z \frac{e^{ik_z z}}{(\kappa^2 + k_z^2)^s}. \tag{18}$$

Taking advantage of Euler’s formula $e^{ix} = \cos(x) + i \sin(x)$ and of the known symmetry properties of trigonometric functions, one can easily conclude that

$$I_1(s; \kappa, z) = 2 \int_0^\infty \frac{\cos(k_z z)}{(\kappa^2 + k_z^2)^s} dk_z. \tag{19}$$

Then, recalling that [52]

$$K_\nu(xz) = \frac{\Gamma(\nu + \frac{1}{2})(2z)^\nu}{\sqrt{\pi}x^\nu} \int_0^\infty \frac{\cos(xt)}{(t^2 + z^2)^{\nu + \frac{1}{2}}} dt, \tag{20}$$

with $K_\nu(z)$ denoting the modified Bessel function of the second kind, $\text{Re}(\nu) > -1/2, x > 0$, and $\arg(z) < \pi/2$, one infers that

$$I_1(s; \kappa, z) = \frac{2^{\frac{3}{2}-s} \sqrt{\pi}}{\Gamma(s)} \left(\frac{|z|}{\kappa}\right)^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(\kappa|z|). \tag{21}$$

Hence, for $z \neq 0$ one can rewrite Eq. (17) as

$$\Phi_s(R, z) = -\frac{2^{\frac{3}{2}-s} G_N M \ell^{2-2s} |z|^{s-\frac{1}{2}}}{\sqrt{\pi} \Gamma(s)} I_2(s; R, z),$$

with

$$I_2(s; R, z) = \int_0^\infty d\kappa \kappa^{\frac{3}{2}-s} e^{-\kappa R_0} J_0(\kappa R) K_{s-\frac{1}{2}}(\kappa|z|). \tag{22}$$

Remark 3.1 The case $z = 0$ has already been analyzed in [35] and yields

$$\begin{aligned} \Phi_s(R, 0) &= -\frac{G_N M \ell^{2-2s}}{\sqrt{\pi} R_0^{3-2s}} \frac{\Gamma(s - 1/2) \Gamma(3 - 2s)}{\Gamma(s)} \\ &\quad \times {}_2F_1\left(\frac{3}{2} - s, 2 - s; 1; -\frac{R^2}{R_0^2}\right), \end{aligned} \tag{23}$$

with ${}_2F_1(a, b; c; z)$ the Gaussian hypergeometric function (see [52] for details) and $1 \leq s < 3/2$. Moreover, the *regularized* potential on the plane of the disk for $s = 3/2$ reads

$$\Phi_{3/2}(R, 0) \stackrel{\text{reg}}{=} \frac{2}{\pi} \frac{G_N M}{\ell} \log \left[1 + \sqrt{1 + \left(\frac{R}{R_0}\right)^2} \right].$$

The behavior of the potential $\Phi_s(R, z)$ in Eq. (22) can be more easily understood through the plot of the corresponding equipotential surfaces. Thus, a numerical evaluation of the integral in Eq. (22), based on an adaptive Gauss–Kronrod quadrature, yields the illuminating plots and contours reported in Figs. 2 and 3.

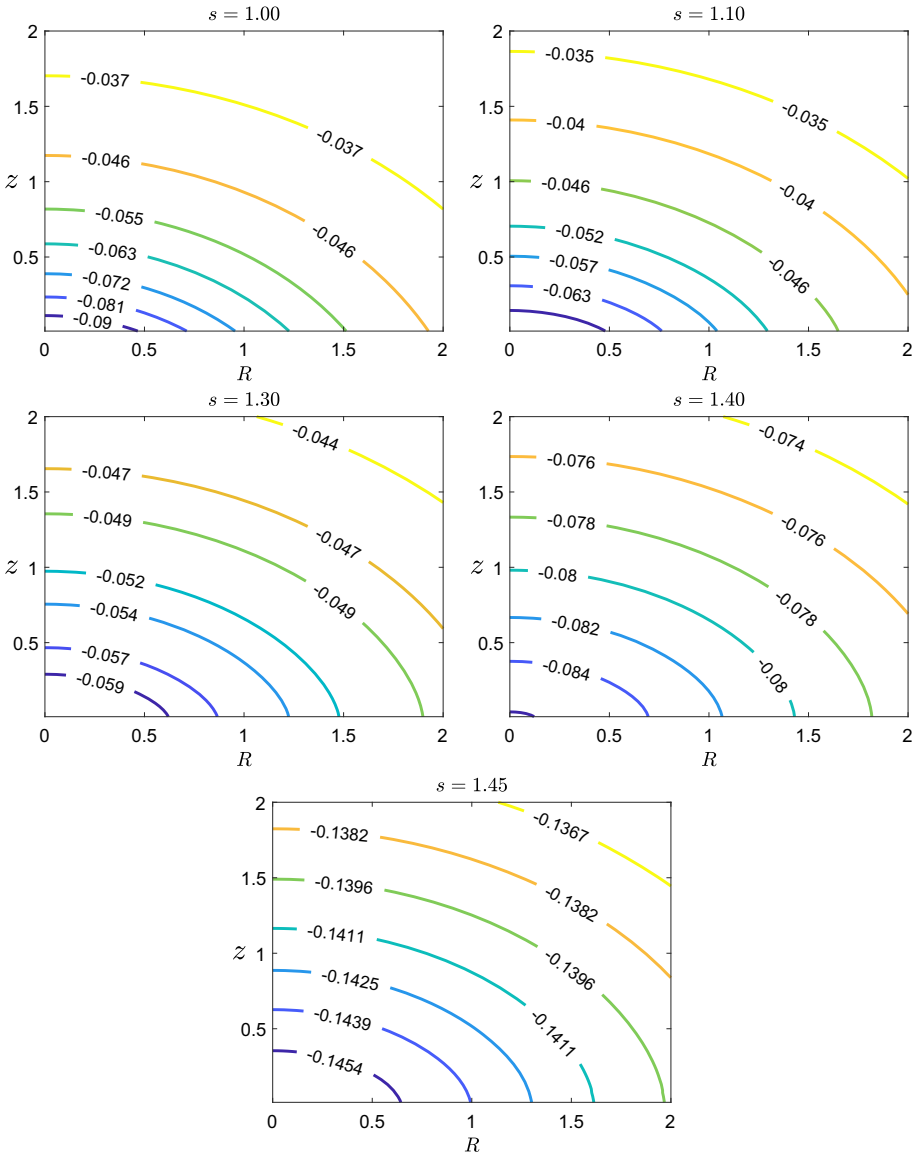
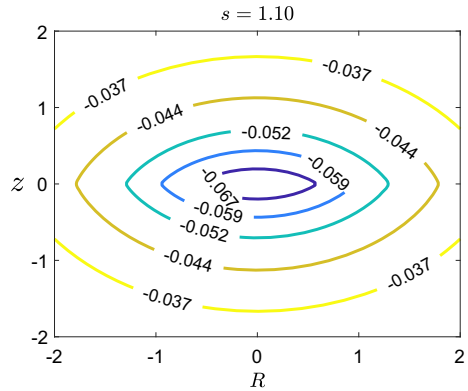


Fig. 2 Sections of $\Phi_s(R, z) = \text{const.}$ for positive R and z , assuming $G_N M = 0.1$, $R_0 = 1$, $\ell = 5$

3.1 Asymptotic behavior

In [35] it was shown that for a point particle the solution of Eq. (6) is given by Eqs. (8) and (10), where the latter corresponds to $s = 3/2$ and it is understood in the regularized sense. Thus, moving away from the Galaxy center one would expect to find a similar behavior from Eq. (22) for $r := \sqrt{R^2 + z^2} \gg R_0$.

Fig. 3 Cross section of an equipotential surface with $s = 1.1$, assuming $G_N M = 0.1$, $R_0 = 1$, $\ell = 5$



From [52] one recalls that

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right), \quad K_\nu(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}, \tag{24}$$

to the lowest order, when $x \rightarrow \infty$. This suggests that when $R, z \gg R_0$

$$I_2(s; R, z) \sim \frac{1}{\sqrt{R|z|}} \int_0^\infty \kappa^{\frac{1}{2}-s} e^{-\kappa(R_0+|z|)} \cos(\kappa R) d\kappa \\ \sim \frac{\Gamma\left(\frac{3}{2}-s\right)}{\sqrt{R|z|} [R^2 + (R_0 + |z|)^2]^{\frac{3-2s}{4}}} \cos\left[\frac{3-2s}{2} \arctan\left(\frac{R}{R_0 + |z|}\right)\right], \tag{25}$$

that yields $I_2 \sim r^{s-\frac{5}{2}}$ when $R, z \gg R_0$, i.e., $r = \sqrt{R^2 + z^2} \gg R_0$. From Eq. (22) one concludes that

$$\Phi_s(R, z) \simeq |z|^{s-\frac{1}{2}} I_2(s; R, z) \sim |z|^{s-\frac{1}{2}} r^{s-\frac{5}{2}} \sim r^{2s-3}, \tag{26}$$

assuming for simplicity $\mathcal{O}(R) = \mathcal{O}(|z|)$ as $r \rightarrow \infty$, which coincides with the asymptotic behavior of the potential for the point particle Eq. (8) for $1 \leq s < 3/2$. Furthermore, it is easy to show using the same procedure discussed above to the *Hadamard partie finie* of (22) for $s = 3/2$ (see, e.g., [53,54]) that $\Phi_{3/2}(R, z) \sim \log(r)$ as $r \rightarrow \infty$ with $\mathcal{O}(R) = \mathcal{O}(|z|)$.

3.2 Full potential outside the Galactic plane: a series representation

From [52] one recalls that

$$J_0(x) = \sum_{n=0}^\infty \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}. \tag{27}$$

Taking advantage of Lebesgue’s dominated convergence theorem, one can expand J_0 in Eq. (22) and interchange the summation and integral. This leads to

$$I_2(s; R, z) = \sum_{n=0}^\infty \frac{(-1)^n}{(n!)^2} \left(\frac{R}{2}\right)^{2n} \int_0^\infty \kappa^{\frac{3}{2}+2n-s} e^{-\kappa R_0} K_{s-\frac{1}{2}}(\kappa|z|) d\kappa \\ \equiv \sum_{n=0}^\infty \frac{(-1)^n}{(n!)^2} \left(\frac{R}{2}\right)^{2n} I_3(s, n; R, z), \tag{28}$$

with

$$I_3(s, n; R, z) := \int_0^\infty \kappa^{\frac{3}{2}+2n-s} e^{-\kappa R_0} K_{s-\frac{1}{2}}(\kappa|z|) d\kappa. \tag{29}$$

If one recalls the definitions of Kummer’s (confluent hypergeometric) functions [52]

$$M(a, b, z) = \sum_{n=0}^\infty \frac{(a)_n z^n}{(b)_n n!} \equiv {}_1F_1(a; b; z), \tag{30}$$

and

$$U(a, b, z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} M(a, b, z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} M(a-b+1, 2-b, z), \tag{31}$$

it is not hard to see that

$$K_\nu(z) = \sqrt{\pi}(2z)^\nu e^{-z} U\left(\nu + \frac{1}{2}, 2\nu + 1, 2z\right). \tag{32}$$

The last expression for the modified Bessel function of the second kind then implies that

$$K_{s-\frac{1}{2}}(\kappa|z|) = \sqrt{\pi} (2\kappa|z|)^{s-\frac{1}{2}} e^{-\kappa|z|} U(s, 2s, 2\kappa|z|), \tag{33}$$

that once inserted in Eq. (29) allows one to rewrite I_3 as

$$I_3(s, n; R, z) = \sqrt{\pi} (2|z|)^{s-\frac{1}{2}} \int_0^\infty \kappa^{2n+1} e^{-\kappa(R_0+|z|)} U(s, 2s, 2\kappa|z|) d\kappa. \tag{34}$$

If one recalls the known special integral (see, e.g., [55, §13.10(ii), Eq. 13.10.7])

$$\begin{aligned} & \int_0^\infty e^{-zt} t^{b-1} U(a, c, t) dt \\ &= \frac{\Gamma(b)\Gamma(b-c+1)}{\Gamma(a+b-c+1)z^b} {}_2F_1\left(a, b; a+b-c+1; \frac{z-1}{z}\right), \end{aligned} \tag{35}$$

with $\text{Re}(b) > \max\{\text{Re}(c) - 1, 0\}$ and $\text{Re}(z) > 0$, then Eq. (34) reduces to

$$\begin{aligned} I_3(s, n; R, z) &= \frac{\sqrt{\pi}(2|z|)^{s-\frac{1}{2}}}{(R_0 + |z|)^{2n+2}} \frac{\Gamma(2n+2)\Gamma(2n-2s+3)}{\Gamma(2n+3-s)} \\ &\quad \times {}_2F_1\left(s, 2n+2; 2n+3-s; \frac{R_0 - |z|}{R_0 + |z|}\right). \end{aligned} \tag{36}$$

Therefore, combining Eqs. (22), (28), and (36) one finds

$$\begin{aligned} \Phi_s(R, z) &= -\frac{2^{\frac{3}{2}-s} G_N M \ell^{2-2s} |z|^{s-\frac{1}{2}}}{\sqrt{\pi} \Gamma(s)} I_2(s; R, z) \\ &= -\frac{2 G_N M \ell^{2-2s} |z|^{2s-1}}{\Gamma(s) (R_0 + |z|)^2} \times \\ &\quad \times \sum_{n=0}^\infty \frac{(-1)^n}{(n!)^2} \left[\frac{R}{2(R_0 + |z|)} \right]^{2n} \frac{\Gamma(2n+2)\Gamma(2n-2s+3)}{\Gamma(2n+3-s)} \times \\ &\quad \times {}_2F_1\left(s, 2n+2; 2n+3-s; \frac{R_0 - |z|}{R_0 + |z|}\right), \end{aligned} \tag{37}$$

that ultimately provides an explicit expression of the potential, outside of the Galactic plane, in terms of a series of known special functions. However, such an expression can hardly be useful when dealing with observations; thus, the numerical evaluation of Eqs. (22) and (22) turns out to be a more practical path to follow.

4 Conclusions and outlook

Fractional Newtonian gravity [35], based on the fractional extension of Poisson's field equation for the gravitational potential obtained through the replacement of the Laplacian with the fractional Laplacian, represents a novel application of fractional calculus to astrophysics. Most notably, this theory naturally comprises both Newtonian gravity and MOND's asymptotic behavior as limiting scenario, respectively, obtained setting $s = 1$ and $s = 3/2$. This particular feature surprisingly allows one to naturally connect observations of Galaxy rotation curves with the more abstract theory of weakly singular integro-differential operators, and hence to non-local theories of gravity.

In this work, we have completed the analysis for an important toy model for the mass distribution of very thin-disk galaxies, known as the Kuzmin disk. First, in Eq. (22) we have provided an explicit integral representation of the potential generated by the disk outside the plane of the disk. Second, we have computed numerically the form of the equipotential surfaces for different values of the fractional parameter s and we provided some illuminating cross sections of these surfaces in Figs. 2 and 3. Third, in Sect. 3.1 we verified the asymptotic behavior of the potential in Eq. (22) when $r \rightarrow \infty$. Finally, in Eq. (37) we have provided an explicit series representation for the potential generated by the Kuzmin disk, outside the plane of the disk $z = 0$, thus filling a gap in the literature.

The program of fractional Newtonian gravity surely looks promising and deserving of further investigation. First, in order to properly reproduce Galaxy rotation curves one needs to turn the theory into a *variable-order one*, with $s = s(x/\ell)$ being a function reducing to 1 at short scale and approaching $3/2$ as one moves asymptotically far away from the center of the Galaxy. However, promoting this model to a variable-order theory leads to complications (see, e.g. [56]), both mathematical and numerical. These more serious topics will be discussed in detail in future studies.

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References

1. L. Amendola, S. Tsujikawa, *Dark Energy, Theory and Observations* (Cambridge University Press, Cambridge, 2010)
2. P. Brax, What makes the Universe accelerate? A review on what dark energy could be and how to test it. *Rep. Prog. Phys.* **81**(1), 016902 (2018)
3. G. Bertone, D. Hooper, J. Silk, Particle dark matter: evidence, candidates and constraints. *Phys. Rep.* **405**, 279–390 (2005). [arXiv:hep-ph/0404175](https://arxiv.org/abs/hep-ph/0404175) [hep-ph]
4. K. Garrett, G. Duda, Dark matter: a primer. *Adv. Astron.* **2011**, 968283 (2011). [arXiv:1006.2483](https://arxiv.org/abs/1006.2483) [hep-ph]
5. G. Bertone, D. Hooper, History of dark matter. *Rev. Mod. Phys.* **90**(4), 045002 (2018). [arXiv:1605.04909](https://arxiv.org/abs/1605.04909) [astro-ph.CO]

6. G.R. Blumenthal, S.M. Faber, J.R. Primack, M.J. Rees, Formation of galaxies and large scale structure with cold dark matter. *Nature* **311**, 517–525 (1984)
7. S. Capozziello, S. Carloni, A. Troisi, Quintessence without scalar fields. *Recent Res. Dev. Astron. Astrophys.* **1**, 625 (2003). [arXiv:astro-ph/0303041](#)
8. S.M. Carroll, V. Duvvuri, M. Trodden, M.S. Turner, Is cosmic speed-up due to new gravitational physics? *Phys. Rev. D* **70**, 043528 (2004). [arXiv:astro-ph/0306438](#)
9. T.P. Sotiriou, V. Faraoni, f(R) theories of gravity. *Rev. Mod. Phys.* **82**, 451 (2010). [arXiv:0805.1726](#) [gr-qc]
10. A. De Felice, S. Tsujikawa, f(R) theories. *Living Rev. Rel.* **13**, 3 (2010). [arXiv:1002.4928](#) [gr-qc]
11. S. Nojiri, S.D. Odintsov, Unified cosmic history in modified gravity: from F(R) theory to Lorentz non-invariant models. *Phys. Rep.* **505**, 59 (2011). [arXiv:1011.0544](#) [gr-qc]
12. S. Capozziello, M. De Laurentis, Extended theories of gravity. *Phys. Rep.* **509**, 167 (2011). [arXiv:1108.6266](#) [gr-qc]
13. S. Capozziello, M. De Laurentis, V. Faraoni, A Bird's eye view of f(R)-gravity. *Open Astron. J.* **3**, 49 (2010). [arXiv:0909.4672](#) [gr-qc]
14. S. Nojiri, S.D. Odintsov, V.K. Oikonomou, Modified gravity theories on a nutshell: inflation, bounce and late-time evolution. *Phys. Rep.* **692**, 1 (2017). [arXiv:1705.11098](#) [gr-qc]
15. S. Nojiri, S.D. Odintsov, Dark energy, inflation and dark matter from modified F(R) gravity. *TSPU Bull.* **N8(110)**, 7–19 (2011). [arXiv:0807.0685](#) [hep-th]
16. S. Nojiri, S.D. Odintsov, D. Saez-Gomez, Cosmological reconstruction of realistic modified F(R) gravities. *Phys. Lett. B* **681**, 74–80 (2009). [arXiv:0908.1269](#) [hep-th]
17. L. Heisenberg, A systematic approach to generalisations of general relativity and their cosmological implications. *Phys. Rep.* **796**, 1–113 (2019). [arXiv:1807.01725](#) [gr-qc]
18. M. Milgrom, A modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis. *Astrophys. J.* **270**, 365 (1983)
19. M. Milgrom, A modification of the Newtonian dynamics: implications for galaxies. *Astrophys. J.* **270**, 371 (1983)
20. M. Milgrom, A modification of the Newtonian dynamics: implications for galaxy systems. *Astrophys. J.* **270**, 384 (1983)
21. J. Bekenstein, M. Milgrom, Does the missing mass problem signal the breakdown of Newtonian gravity? *Astrophys. J.* **286**, 7 (1984)
22. K.G. Begeman, H I rotation curves of spiral galaxies. I - NGC 3198. *Astron. Astrophys.* **223**, 47 (1989)
23. F. Zwicky, On the masses of nebulae and of clusters of nebulae. *Astrophys. J.* **86**, 217 (1937)
24. E. Corbelli, P. Salucci, The extended rotation curve and the dark matter halo of M33. *Mon. Not. R. Astron. Soc.* **311**, 441 (2000). [arXiv:astro-ph/9909252](#)
25. J.D. Bekenstein, Relativistic gravitation theory for the MOND paradigm. *Phys. Rev. D* **70**, 083509 (2004). [arXiv:astro-ph/0403694](#) [astro-ph]
26. F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models* (World Scientific, Singapore, 2010)
27. S. Samko, A. Kilbas, O. Marichev, *Fractional Integrals and Derivatives* (Gordon and Breach Science Publishers, Yverdon, 1993)
28. A. Giusti et al., A practical guide to Prabhakar fractional calculus. *Fract. Calc. Appl. Anal.* **23**(1), 9–54 (2020). [arXiv:2002.10978](#) [math.CA]
29. R. Garrappa, F. Mainardi, G. Maione, Models of dielectric relaxation based on completely monotone functions. *Fract. Calc. Appl. Anal.* **19**(5), 1105–1160 (2016). [arXiv:1611.04028](#) [math-ph]
30. R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys. Rep.* **339**, 1–77 (2000). [arXiv:1611.04028](#) [math-ph]
31. G. Calcagni, Geometry of fractional spaces. *Adv. Theor. Math. Phys.* **16**(2), 549–644 (2012). [arXiv:1106.5787](#) [hep-th]
32. V.E. Tarasov, Fractional derivative regularization in QFT. *Adv. High Energy Phys.* **2018**, 7612490 (2018). [arXiv:1805.08566](#) [hep-th]
33. A.O. Barvinsky, P.I. Pronin, W. Wachowski, Heat kernel for higher-order differential operators and generalized exponential functions. *Phys. Rev. D* **100**(10), 105004 (2019). [arXiv:1908.02161](#) [hep-th]
34. A.M. Frassino, O. Panella, Quantization of nonlocal fractional field theories via the extension problem. *Phys. Rev. D* **100**(11), 116008 (2019). [arXiv:1907.00733](#) [hep-th]
35. A. Giusti, MOND-like fractional Laplacian theory. *Phys. Rev. D* **101**(12), 124029 (2020). [arXiv:2002.07133](#) [gr-qc]
36. G. U. Varieschi, Fractional Gravity and Modified Newtonian Dynamics, [arXiv:2003.05784](#) [gr-qc]
37. G. U. Varieschi, Newtonian Fractional Gravity and Disk Galaxies, [arXiv:2008.04737](#) [gr-qc]

38. P. R. Stinga, User's guide to the fractional Laplacian and the method of semigroups, in: A. Kochubei, Yu. Luchko, *Handbook of Fractional Calculus with Applications*, Vol. 2 Fractional Differential Equations (De Gruyter, 2019)
39. M. Kwaśnicki, Ten equivalent definitions of the fractional Laplace operator. *Fract. Calc. Appl. Anal.* **20**(1), 7–51 (2017). [arXiv:1507.07356](https://arxiv.org/abs/1507.07356) [math.AP]
40. A. Lischke et al., What is the fractional Laplacian? A comparative review with new results. *J. Comput. Phys.* **404**, 109009 (2020)
41. L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator. *Commun. Pure Appl. Math.* **60**, 67–112 (2007)
42. R.B. Tully, J.R. Fisher, A new method of determining distances to galaxies. *Astron. Astrophys.* **54**(3), 661–673 (1977)
43. R. Casadio, A. Giugno, A. Giusti, Matter and gravitons in the gravitational collapse. *Phys. Lett. B* **763**, 337–340 (2016). [arXiv:1606.04744](https://arxiv.org/abs/1606.04744) [hep-th]
44. R. Casadio, A. Giugno, A. Giusti, M. Lenzi, Quantum corpuscular corrections to the Newtonian potential. *Phys. Rev. D* **96**(4), 044010 (2017). [arXiv:1702.05918](https://arxiv.org/abs/1702.05918) [gr-qc]
45. R. Casadio, M. Lenzi, O. Micu, Bootstrapping Newtonian gravity. *Phys. Rev. D* **98**(10), 104016 (2018). [arXiv:1806.07639](https://arxiv.org/abs/1806.07639) [gr-qc]
46. R. Casadio, I. Kuntz, Bootstrapped Newtonian quantum gravity. *Eur. Phys. J. C* **80**(6), 581 (2020). [arXiv:2003.03579](https://arxiv.org/abs/2003.03579) [gr-qc]
47. M. Cadoni, R. Casadio, A. Giusti, W. Mück, M. Tuveri, Effective fluid description of the dark universe. *Phys. Lett. B* **776**, 242 (2018). [arXiv:1707.09945](https://arxiv.org/abs/1707.09945) [gr-qc]
48. M. Cadoni, R. Casadio, A. Giusti, M. Tuveri, Emergence of a dark force in corpuscular gravity. *Phys. Rev. D* **97**(4), 044047 (2018). [arXiv:1801.10374](https://arxiv.org/abs/1801.10374) [gr-qc]
49. M. Tuveri, M. Cadoni, Galactic dynamics and long-range quantum gravity. *Phys. Rev. D* **100**(2), 024029 (2019). [arXiv:1904.11835](https://arxiv.org/abs/1904.11835) [gr-qc]
50. A. Giusti, On the corpuscular theory of gravity. *Int. J. Geom. Methods Mod. Phys.* **16**(03), 1930001 (2019)
51. J. Binney, S. Tremaine, *Galactic Dynamics* (Princeton University Press, Princeton, 2011)
52. M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (US Department of Commerce. National Bureau of Standards Applied Mathematics series 55, 1965)
53. M. Riesz, L'intégrale de Riemann–Liouville et le problème de Cauchy. *Acta Math.* **81**, 1–222 (1949)
54. R. Estrada, R.P. Kanwal, Regularization, pseudofunction, and Hadamard finite part. *J. Math. Anal. Appl.* **141**, 195–207 (1989)
55. NIST Digital Library of Mathematical Functions. <http://dlmf.nist.gov/>, Release 1.0.27 of 2020-06-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds
56. S. Samko, Fractional integration and differentiation of variable order: an overview. *Nonlinear Dyn.* **71**, 653–662 (2013)