

MINIMAL SYMPLECTIC ATLASES OF HERMITIAN SYMMETRIC SPACES

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ABSTRACT. In this paper we estimate the minimal number of Darboux charts needed to cover a Hermitian symmetric space of compact type M in terms of the degree of their embeddings in $\mathbb{C}P^N$. The proof is based on the recent work of Y. B. Rudyak and F. Schlenk [20] and on the symplectic geometry tool developed by the first author in collaboration with A. Loi and F. Zuddas [14]. As application we compute this number for a large class of Hermitian symmetric spaces of compact type.

1. INTRODUCTION AND STATEMENTS OF THE MAIN RESULTS

Consider the open ball of radius r ,

$$B^{2n}(r) = \{(x, y) \in \mathbb{R}^{2n} \mid \sum_{j=1}^n x_j^2 + y_j^2 < r^2\}$$

in the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$, where $\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j$. In [20] Y. B. Rudyak and F. Schlenk introduced the invariant $S_B(M, \omega)$ for a closed symplectic manifold (M, ω) of dimension $2n$ defined by:

$$S_B(M, \omega) := \min\{k \mid M = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_k\},$$

where \mathcal{B}_j is the image of a Darboux chart $\varphi(B^{2n}(r_j)) \subset M$. This is the minimal number of symplectic charts needed to cover (M, ω) . The problem of estimating this number is closely related to two other problems, namely computing the Gromov width $c_G(M, \omega)$ and the Lusternik-Schnirelmann category $\text{cat}(M)$ of M . While the latter can be often computed or estimated very well, computing the former is an open and delicate matter. The Gromov width of a $2n$ -dimensional symplectic manifold (M, ω) , introduced in [7], is defined as

$$c_G(M, \omega) = \sup \{ \pi r^2 \mid \exists \varphi : (B^{2n}(r), \omega_0) \rightarrow (M, \omega) \}$$

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where φ is a symplectic embedding.

By Darboux's theorem $c_G(M, \omega)$ is a positive number or ∞ . Computations and estimates of the Gromov width for various examples can be found in [2, 3, 4, 5, 7, 8, 10, 11, 14, 15, 16, 17, 18, 19, 21, 24].

We adopt the following notation from [14].

Notation: *From now on we shall use the shortening HSSCT to denote a Hermitian symmetric space of compact type. Further, throughout the paper we shall denote by ω_{FS} the canonical symplectic (Kähler) form on an irreducible HSSCT normalized so that $\omega_{FS}(B) \in \{-\pi, \pi\}$ when B is a generator of $H_2(M, \mathbb{Z})$, and by A the generator for which $\omega_{FS}(A) = \pi$.*

The following theorem and its two corollaries are the main results of this paper.

Theorem 1. *Let (M, ω_{FS}) be a $2n$ -dimensional HSSCT and let $f : M \hookrightarrow \mathbb{C}P^N$ be any holomorphic isometric immersion of M in $\mathbb{C}P^N$ endowed with the Fubini–Study form ω . Then*

- (i) *If $\deg(f) \geq 2n$, then $S_B(M, \omega_{FS}) = \deg(f) + 1$*
- (ii) *If $\deg(f) < 2n$, then $\max\{n + 1, \deg(f) + 1\} \leq S_B(M, \omega_{FS}) \leq 2n + 1$.*

As holomorphic isometric immersion $f : M \hookrightarrow \mathbb{C}P^N$ we can take, for example, the coherent states map described in Section 1.1. In particular when M is the complex Grassmannian one can take f equal to the Plücker embedding. We recall the definition of degree of a holomorphic immersion in Section 2.1, while in Section 2.2 we compute it for all irreducible HSSCT.

The proof of Theorem 1 is based on the results obtained by Y. B. Rudyak and F. Schlenk in [20] about minimal atlases for compact symplectic manifolds together with the explicit computation of the Gromov width given by the first author in collaboration with A. Loi and F. Zuddas in [14] and the properties of the symplectic duality map introduced by A. J. Di Scala and A. Loi in [6] which, in particular, give us a symplectic embedding of the noncompact dual (Ω, ω_0) of (M, ω_{FS}) into (M, ω_{FS}) .

Using the explicit computation of the volume of a classical domain (Ω, ω_0) given by L. K. Hua in [9], we are able to prove the following corollary, which extends the computation of S_B for the Grassmannians given in [20] to any classical irreducible HSSCT. Before stating the corollary, we recall that a classical irreducible HSSCT is one of the following quotients of compact Lie groups:

$$\begin{aligned} \text{I}_{k,s} &= SU(s)/S(U(k) \times U(s-k)), \\ \text{II}_s &= SO(2s)/U(s), \\ \text{III}_s &= Sp(s)/U(s), \end{aligned}$$

$$IV_s = SO(s+2)/SO(s) \times SO(2).$$

Corollary 2. *Let (M, ω_{FS}) be a classical irreducible HSSCT of dimension $2n$. Then we have:*

$$S_B(I_{k,s}) = \deg(f) + 1, \quad \text{for } (k = 2 \text{ and } s \geq 7) \text{ or } k \geq 3 \quad (1)$$

$$S_B(II_s) = \deg(f) + 1, \quad \text{for } s \geq 6$$

$$S_B(III_s) = \deg(f) + 1, \quad \text{for } s \geq 5$$

$$n + 1 \leq S_B(IV_s) \leq 2n + 1, \quad \text{for } s \geq 2.$$

Otherwise, we have

$$\max\{n + 1, \deg(f) + 1\} \leq S_B(M, \omega_{FS}) \leq 2n + 1.$$

In the rank one case (i.e. $M = \mathbb{C}P^n$), we can set f equal to the identity map, so that $\deg(f) = 1$. On the other hand, [20, Corollary 5.8] tells us that

$$S_B(\mathbb{C}P^n, \omega_{FS}) = n + 1.$$

The second corollary is a straightforward consequence of Theorem 1:

Corollary 3. *Let $(M_1 \times M_2, \omega_{FS})$ be a product of HSSCT of dimension $2n$. If $M_1 \times M_2$ is different from $\mathbb{C}P^1 \times \mathbb{C}P^{n-1}$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$, then*

$$S_B(M_1 \times M_2, \omega_{FS}) = \deg(f) + 1,$$

where $f : M_1 \times M_2 \hookrightarrow \mathbb{C}P^N$ is any holomorphic isometric immersion. Otherwise, we have

$$\max\{n + 1, \deg(f) + 1\} \leq S_B(M, \omega_{FS}) \leq 2n + 1.$$

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1.1. The coherent states map. It is well known that an HSSCT M is a simply connected Kähler–Einstein manifold with strictly positive scalar curvature. Therefore the integrality of $\frac{\omega_{FS}}{\pi}$ implies the existence of a polarizing holomorphic hermitian line bundle (L, h) on M such that $c_1(L) = [\frac{\omega_{FS}}{\pi}]$ and the Ricci curvature of h satisfies $\text{Ric}(h) = \frac{\omega_{FS}}{\pi}$ (where $\text{Ric}(h) = -\frac{i}{2\pi} \partial\bar{\partial} \log(h(\sigma, \sigma))$ in a local trivialization $\sigma : U \subset M \rightarrow L$). Consider the space $H^0(L)$ consisting of global holomorphic sections s of L which are bounded with respect to

$$\langle s, s \rangle = \|s\|^2 = \int_M h(s(x), s(x)) \frac{\omega^n}{n!}.$$

As $H^0(L) \neq \{0\}$, given an orthonormal basis $\{s_0, \dots, s_N\} \subset H^0(L)$ (with respect $\langle \cdot, \cdot \rangle$), it is well defined the *coherent states map*, given by $f : M \rightarrow \mathbb{C}P^N$

$$f(x) = [s_0(x) : \dots : s_N(x)].$$

The Fubini–Study form ω of $\mathbb{C}P^N$ (normalized so that $\omega(B) \in \{-\pi, \pi\}$, when B is a generator of $H_2(\mathbb{C}P^N, \mathbb{Z})$) is given by

$$\omega = \frac{i}{2} \partial \bar{\partial} \log \left(\sum_{j=0}^N |Z_j|^2 \right),$$

it follows that

$$\begin{aligned} f^* \omega &= \frac{i}{2} \partial \bar{\partial} \log \left(\sum_{j=0}^N |s_j(x)|^2 \right) = \frac{i}{2} \partial \bar{\partial} \log \left(\frac{\sum_{j=0}^N h(s_j(x), s_j(x))}{h(\sigma(x), \sigma(x))} \right) \\ &= -\frac{i}{2} \partial \bar{\partial} \log (h(\sigma(x), \sigma(x))) + \frac{i}{2} \partial \bar{\partial} \log \left(\sum_{j=0}^N h(s_j(x), s_j(x)) \right) \\ &\quad \pi \operatorname{Ric}(h) + \frac{i}{2} \partial \bar{\partial} \log \epsilon(x) = \omega_{FS} + \frac{i}{2} \partial \bar{\partial} \log \epsilon(x), \end{aligned}$$

where $\epsilon : M \rightarrow \mathbb{R}$ is the so called ϵ -function defined by

$$\epsilon(x) = \sum_{j=0}^N h(s_j(x), s_j(x)),$$

one can prove that the ϵ -function (see e.g. [12, Theorem 4.3]) is invariant with respect the action of the group of holomorphic isometric transformation of (M, ω_{FS}) which act transitively on M . Therefore the ϵ -function is constant and we conclude that

$$f^* \omega = \omega_{FS}.$$

2. PROOFS OF THEOREM 1, COROLLARY 2 AND COROLLARY 3

Consider the following lower bound for $S_B(M, \omega)$ given by

$$\Gamma(M, \omega) := \left\lfloor \frac{\operatorname{Vol}(M, \omega) n!}{c_G(M, \omega)^n} \right\rfloor + 1,$$

where $\lfloor x \rfloor$ denote the maximal integer smaller than or equal to x . The following theorem summarizes the results about minimal atlases obtained in [20] that we need in the proof of Theorem 1.

Theorem A (Rudiyak–Schlenk [20]). *Let (M, ω) be a compact connected $2n$ -dimensional symplectic manifold.*

- i) If $\Gamma(M, \omega) \geq 2n + 1$, then $S_B(M, \omega) = \Gamma(M, \omega)$.*
- ii) If $\Gamma(M, \omega) < 2n + 1$ then $\max\{n + 1, \deg(f) + 1\} \leq S_B(M, \omega) \leq 2n + 1$.*

2.1. Proof of Theorem 1. We start recalling the definition of the degree of an holomorphic immersion $f : M \rightarrow \mathbb{C}P^N$. Suppose that $\dim(M) = 2n < 2N$, by Sard's Theorem there exists a point $q \notin f(M)$. Up to unitary transformation of $\mathbb{C}P^N$ we can suppose q to be the point of coordinates $[1, 0, \dots, 0]$. Consider the projection $p_k : \mathbb{C}P^k \setminus \{q\} \rightarrow \mathbb{C}P^{k-1}$, $p_k([Z_0, \dots, Z_k]) = [Z_1, \dots, Z_k]$ and define the map $F : M \rightarrow \mathbb{C}P^n$ by $F = \tilde{p} \circ f$, where $\tilde{p} = p_{n+1} \circ \dots \circ p_N$. The degree $\deg(f)$ of f is by definition the degree $\deg(F)$ of the map F , which is the integer number such that

$$F_*[M] = \deg(F)[\mathbb{C}P^n] \in H_{2n}(\mathbb{C}P^n, \mathbb{Z}). \quad (2)$$

What we need about $\deg(f)$ is summarized in the following Lemma:

Lemma 4. (W. Wirtinger [23], M. Barros, A. Ros, [1]) *The degree $\deg(f)$ is a positive integer such that*

$$\text{Vol}(M) = \deg(f)\text{Vol}(\mathbb{C}P^n), \quad (3)$$

where $\deg(f) = 1$ iff M is totally geodesic and $\deg(f) = 2$ iff f is congruent to the standard embedding of the quadric.

The proof follows from Theorem A once one observes that the volume of any n -dimensional projective variety X , with holomorphic embedding $f : X \hookrightarrow \mathbb{C}P^N$, is given by

$$\text{Vol}(X, \omega_{FS}) = \deg(f)\text{Vol}(\mathbb{C}P^n, \omega_{FS}), \quad (4)$$

$\text{Vol}(\mathbb{C}P^n) = \frac{\pi^n}{n!}$ and that the Gromov width of any HSSCT (see [14]) is given by $c_G(M, \omega_{FS}) = \pi$.

2.2. Proof of Corollary 2. Consider (Ω, ω_0) , the noncompact dual of (M, ω_{FS}) . In [6, Theorem 1.1] it is proved the existence of a global symplectomorphism

$$\Phi : (\Omega, \omega_0) \rightarrow (M \setminus \text{Cut}_p(M), \omega_{FS})$$

where $\text{Cut}_p(M)$ is the cut locus of (M, ω_{FS}) with respect to a fixed point $p \in M$ (see also [13]). Thus $\text{Vol}(M, \omega_{FS}) = \text{Vol}(\Omega, \omega_0)$. On the other hand the explicit expression of the volume $\text{Vol}(\Omega, \omega_0)$ can be found in L. K. Hua [9] and by (4) we are able to write the expression of $\deg(f)$ associated to any classical HSSCT, as follows.

Let $I_{k,s}$ be a HSSCT of type I, namely the Grassmannian of k -planes in \mathbb{C}^s . Notice that the dimension is $2n = 2(s-k)k$ and that $\text{rank}(I_{k,s}) = k$. We have that

$$\begin{aligned} \deg(f_{k,s}) &= \frac{\text{Vol}(I_{k,s}, \omega_{FS})}{\text{Vol}(\mathbb{C}P^{(s-k)k}, \omega_{FS})} = \\ &= \frac{1!2! \dots (s-k-1)!1!2! \dots (k-1)!((s-k)k)!}{1!2! \dots (s-1)!}. \end{aligned} \quad (5)$$

The case $I_{k,s}$ was already done by Rudyak–Schlenk [20] and we obtain (1) by [20, Corollary 5.10]. Moreover they prove that

$$S_B(I_{2,4}) \in \{5, 6\}$$

$$S_B(I_{2,5}) \in \{7, 8, 9, 10\}.$$

Let II_s be an irreducible HSSCT of the second type. The complex dimension is given by $n_s = \frac{(s-1)s}{2}$. We have,

$$\deg(f_{\text{II}_s}) = \frac{s(s-1)!}{2} \frac{2!4!\dots(2s-4)!}{(s-1)!s!\dots(2s-3)!}.$$

In order to apply Theorem 1 we need to study when

$$\frac{\deg(f_{\text{II}_s})}{n_s} \geq 2$$

One can see that the inequality is satisfied for $s = 6$ and that $\frac{\deg(f_{\text{II}_s})}{n_s} < \frac{\deg(f_{\text{II}_{s+1}})}{n_{s+1}}$ for any $s \geq 6$.

Let III_s be an irreducible HSSCT of the third type. The complex dimension is given by $n_s = \frac{(s+1)s}{2}$. We have,

$$\deg(f_{\text{III}_s}) = \frac{s(s+1)!}{2} \frac{2!4!\dots(2s-2)!}{s!(s+1)!(s+2)!\dots(2s-1)!}.$$

Arguing as before we see that $\frac{\deg(f_{\text{III}_s})}{n_s} \geq 2$ for any $s \geq 6$.

Let IV_s be an irreducible HSSCT of the fourth type (namely the complex quadric). Assume $s > 3$ (if $s = 1$ or $s = 2$ we have respectively $\text{IV}_1 = \mathbb{CP}^1$ or $\text{IV}_2 = \mathbb{CP}^1 \times \mathbb{CP}^1$). By Lemma 4, $\deg(f) = 2$. As $n = s \geq 3$, the result follows by (ii) of Theorem 1.

2.3. Proof of Corollary 3. Let ω_{FS}^1 and ω_{FS}^2 be the Fubini-Study forms associated to M_1 and M_2 . Since the associated volume form satisfies (with abuse of notation) $v_{\omega_{FS}} = v_{\omega_{FS}^1} \wedge v_{\omega_{FS}^2}$, we have $\text{Vol}(M_1 \times M_2) = \text{Vol}(M_1)\text{Vol}(M_2)$. By (4) we get:

$$\deg(f) = \frac{(n_1 + n_2)!}{n_1!n_2!} \deg(f_1) \deg(f_2),$$

where n_j is the complex dimension of M_j , $j = 1, 2$ and f, f_1 and f_2 are holomorphic isometric immersions of $M_1 \times M_2$, M_1 and M_2 . In order to apply (i) of Theorem 1, we have to check when

$$\deg(f_1) \deg(f_2) \frac{(n_1 + n_2 - 1)!}{n_1!n_2!} \geq 2. \quad (6)$$

First notice that when $\deg(f_1) \geq 2$ or $\deg(f_2) \geq 2$, since $\frac{(n_1 + n_2 - 1)!}{n_1!n_2!} \geq 1$, the inequality (6) is satisfied.

Assume now that $\deg(f_1) = \deg(f_2) = 1$. By Lemma 4, f_1, f_2 are totally geodesic, this force M_1 and M_2 to have rank 1, that is $M_1 = \mathbb{C}P^{n_1}$ and $M_2 = \mathbb{C}P^{n_2}$. Moreover it is easy to see that (6) is satisfied if and only if $n_1 \geq 3$ and $n_2 \geq 2$ or $n_1 \geq 2$ and $n_2 \geq 3$. The proof is complete.

Remark 5. When $M = \mathbb{C}P^1 \times \mathbb{C}P^{n-1}, \mathbb{C}P^2 \times \mathbb{C}P^2$ we are not able to compute $S_B(M, \omega_{FS})$. Even for the simple case of $\mathbb{C}P^1 \times \mathbb{C}P^1$ we know (private communication with F. Schlenk) that one can construct a covering by 4 symplectic balls but we still do not know if this number can be reduced to 3.

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