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Consistent Order Approximations in Extended Thermodynamics of Polyatomic Gases

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Abstract

In this article the known models are considered for relativistic polyatomic gases with an arbitrary number of moments, in the framework of Extended Thermodynamics. These models have the downside of being hyperbolic only in a narrow domain around equilibrium, called "hyperbolicity zone". Here it is shown how to overcome this drawback by presenting a new model which satisfies the hyperbolicity requirement for every value of the independent variables and without restrictions. The basic idea behind this new model is that hyperbolicity is limited in previous models by the approximations made there. It is here shown that hyperbolicity isn't limited also for an approximated model if terms of the same order are consistently considered, in a new way never used before in literature. To design and complete this new model, well accepted principles are used such as the "Entropy Principle" and the "Maximum Entropy Principle". Finally, new trends are analized and these considerations may require a modification of the results published so far; as a bonus, more manageable balance equations are obtained. This allows to obtain more stringent results than those so far known. For example, we will have a single quantity (the energy e) expressed by an integral and all the other constitutive functions will be expressed in terms of it and its derivatives with respect to temperature. Another useful consequence is its easier applicability to the case of diatomic and ultrarelativistic gases which are useful, at least for testing the model in simple cases.

1.Introduction

This article is placed in the context of Extended Thermodynamics. It was conceived with the aim of eliminating some negative aspects of Ordinary Thermodynamics (OT). For example, OT uses as field equations the conservation laws of mass and of momentum-energy closed through the Navier-Stokes and Fourier laws. In this way a parabolic set of equations is found which predicts infinite velocity of propagation of shock waves. This is in contradiction with the Einstenian Relativity Principle according to which nothing can propagate with a velocity greater than that of light.

Extended Thermodynamics was elaborated just to solve this problem, by substituting the parabolic set of equations with an hyperbolic one. Obviously, many other important mathematical and physical goals were realized. It beguns treating firstly the case of a monoatomic gas both in the classical case [1] and in the relativistic case [2] (See also [3]).

In a subsequent period an important upgrade of the theory was obtained by finding models also for polyatomic gases in the classical framework (See [4], [5] and in the relativistic framework (See [6], [7]).

The reason why this article is placed in the relativistic context is that the theory par excellence is the relativistic one and the classical one can be used only as an approximation only when the speeds involved, divided by that of light, aren't infinitesimal (to this end it isn't necessary that the particle speeds are close to the light speed); moreover, the classical theory is preferred (when this is possible) because it gives rise to simpler equations. But this is not the case in Continuum Thermodynamics because in the classical case its laws must respect the Galilean Equivalence Principle, while in the relativistic case we need not worry about it because the Einsteinian Equivalence Principle comes out automatically for the covariant form of the equations themselves. Finally, the polyatomic structure of molecules is not destroyed by the fact that their average velocity is high; at most, this can happen when their relative speed is high, i.e., when the temperature is high. To be more precise, the relativistic formulation becomes necessary when the term

$$\gamma = \frac{m\, c^2}{k_B\, T}$$

is large (here m is the particle mass, c the speed of light, k_B the Boltzmann constant and T the temperature). This happens for example when the temperature is not very high or when the particles haven't infinitesimal mass. In any case, the equations for the classical case can be easily obtained from the relativistic ones jut making their limit for c which tends to infinity. For this reason, articles on polyatomic gases in the context of relativity have already been published, such as [6], [8], [9], [10], [11].

Recently, in the article [12], the case has been considered of a model with an arbitrary but fixed number of moments to describe polyatomic gases in the relativistic context and the following system of balance equations has been found:

$$\partial_\alpha A^{\alpha\alpha_1\cdots\alpha_n} = A^{\alpha_1\cdots\alpha_n} \qquad \text{with n=0,1,...,N.} \tag{1}$$

where N is an arbitrary fixed number; for every choice of N a different model is obtained and we must choose that more fitting experimental results.

However, the explicit closure was not found in [12], nor the hyperbolicity requirement was investigated and we do it here. If we impose the MEP we find the following expression for the distribution function

$$f = \exp\left[-1 - \frac{\chi_\varepsilon + \overline{\chi}}{k_B}\right] \tag{2}$$

$$\text{with } \chi_\varepsilon = m\,\lambda + \lambda_\mu\, p^\mu \left(1 + \frac{I}{mc^2}\right),$$

$$\overline{\chi} = \sum_{n=2}^{N} \lambda_{\alpha_1\alpha_2\ldots\alpha_n}\, p^{\alpha_1}\, p^{\alpha_2} \ldots p^{\alpha_n}\, \frac{1}{m^{n-1}} \left(1 + \frac{n\, I}{m\, c^2}\right),$$

where I is a parameter which takes into account the internal energy of a molecule and p^α is the 4-momentum of a particle, satisfying the

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condition $p^\alpha\, p_\alpha \,=\, m^2\, c^2$. This expression for f implies the following expression for the 4-potential

$$h'^\alpha \,=\, -\,k_B\,c\int_{R^3} d\,\vec{P}\,\int_0^{+\infty} p^\alpha\, e^{-1-\frac{\chi_\varepsilon+\overline{\chi}}{k_B}}\,\varphi(I)dI \qquad (3)$$

where φ(I) measures "how much" the gas is polyatomic (A possibility is that φ(I) = I[a] and monoatomic gases are obtained in the limit for a going to -1; see [8]). After that, the tensors in the left hand sides of (1) becomes

$$A^{\alpha\alpha_1\dots\alpha_n} \,=\, \frac{\partial\, h'^\alpha}{\partial\,\lambda_{\alpha_1\dots\alpha_n}}, \qquad (4)$$

which are evidently simmetric with respect to every couple of indexes. Eq. (4) gives the closure we were looking for. But it is only symbolic, since we are not able to calculated the integral in (3) and, moreover, it is not integrable for every values of the independent variables, as we will see in Section 3. For this reason everyone is satisfied by using an approximated expression with a Taylor's expansion around equilibrium. This is defined as the state in which $\lambda_{\alpha_1\dots\alpha_n} = 0$ for n=2, ..., N, while λ and λ_μ remain independent variables; we will refer to this definition by adding the suffix ε. Subsequently, a change of variables is usually used from λ, λ_μ, $\lambda_{\alpha_1\dots\alpha_n}$ with 2 ≤ n ≤ N to the pressure p, the absolute temperature T, the 4-velocity U_α and $\lambda_{\alpha_1\dots\alpha_n}$ with 2 ≤ n ≤ N; we will refer to quantities depending only on p, T, U^α by adding the suffix E.

In particular, this change of variables is achieved by finding λ, λ_μ, from the relationships

$A^\alpha = \frac{m\,p}{k_B T}\,U^\alpha$, $U_\alpha U^\alpha = c^2$, $U_\alpha U_\beta A^{\alpha\beta}$ = e (p,T) c^2 ,

with e (p, T) the state function giving the energy density.

With this in mind, in the next section the closure will be found and discussed up to whatever order with respect to equilibrium; it will be proved that the new field equations here proposed are hyperbolic up to whatever order with respect to equilibrium, unlike those already known in literature . In sect. 3 the integrability of eq. (3) is studied, while in sect. 4, new trends of the relativistic model for polyatomic gases are proposed and these considerations may require a modification of the results published up to now but also lead to a simpler closure in the case of diatomic gases.

2. The hyperbolicity up to whatever order

Goal of this section is to demonstrate that the field equations above obtained are hyperbolic for any value of the independent variables. This result is completely opposite to that of other models in literature, as it can be seen from the articles [14], [15], [16], where hyperbolicity is valid only in a restricted set of the independent variables close to the equilibrium state. This narrow zone of hyperbolicity (in the models already known in the literature) is due to the approximations to which the balance equations are subjected. It is true that some approximation is still necessary, but if such approximations are made consistently, as indicated in this article, the hyperbolicity requirement is not limited.

In order to prove hyperbolicity up to whatever order, let us begin by using the well known serial development

$$e^{-x} \,=\, \sum_{r=0}^{+\infty}(-1)^r\,\frac{x^r}{r!}\;;$$

$$h'^\alpha \,=\, \sum_{r=0}^{+\infty}(-1)^r\,\frac{1}{r!}\,h'^{r,\alpha} \;\text{ with} \qquad (5)$$

$$h'^{r,\alpha} \,=\, -\,k_B\,c\int_{R^3} d\,\vec{P}\int_0^{+\infty} p^\alpha\, e^{-1-\frac{\chi_\varepsilon}{k_B}}\left(\frac{\overline{\chi}}{k_B}\right)^r\varphi(I)dI$$

The integrals here present can be easily calculated by using the results of Appendix A, and one finds

$$h'^{r,\beta_1} \,=\, -\,c\left(\frac{1}{k_B}\right)^{r-1}\sum\nolimits_{h_i\in S_r} H^{\beta_1\beta_2\dots\beta_\eta}_{h_2,h_3,\dots,h_N}\;\lambda_{\beta_2\beta_3}\dots\lambda_{\beta_{\eta-N+1}\dots\beta_\eta}, \qquad (6)$$

where η = 1+ 2 h_2 + 3h_3 + ... +N h_N , the factor $\lambda_{\beta_2\beta_3}\dots\lambda_{\beta_{\eta-N+1}\dots\beta_\eta}$ denotes the product of h_2 tensors $\lambda_{\beta\gamma}$, h_3 tensors $\lambda_{\beta\gamma\delta}$, and so on, of h_N tensors $\lambda_{\beta_1\dots\beta_N}$(Obviously, all with different indexes). Moreover, Sr is the set of integer numbers $S_r = \{0 \le h_i \le r\;,\; \sum_{i=2}^N h_i = r\}$ and

$$H^{\beta_1\dots\beta_\eta}_{h_2,h_3,\dots,h_N} = \int_{R^3} d\,\vec{P}\int_0^{+\infty} e^{-1-\frac{\chi_\varepsilon}{k_B}}\,p^{\beta_1}\dots p^{\beta_\eta}\,\left[\frac{1}{m}\left(1+\frac{2I}{mc^2}\right)\right]^{h_2}\cdot$$

$$\cdot\left[\frac{1}{m^2}\left(1+\frac{3I}{mc^2}\right)\right]^{h_3}\dots\left[\frac{1}{m^{N-1}}\left(1+\frac{NI}{mc^2}\right)\right]^{h_N}\varphi(I)\,d\,I = e^{-1-\frac{m\,\lambda}{k_B}}\,.$$

$$\cdot\sum_{p=0}^{\left[\frac{\eta}{2}\right]}\phi_{p,h_2,h_3,\dots,h_N}(\lambda,\gamma)\;h^{(\beta_1\beta_2}\dots h^{\beta_{2p-1}\beta_{2p}}\,l^{\beta_{2p+1}}\dots l^{\beta_\eta)}\,, \qquad (7)$$

$\gamma = \frac{mc}{k_B}\sqrt{\lambda^\alpha\lambda_\alpha}$. $l^\alpha = \frac{mc^2}{\gamma\,k_B}\lambda^\alpha$,

$$\phi_{p,h_2,h_3,\dots h_N=} = \binom{\eta}{2p}\frac{4\,\pi}{2p+1}m^{\eta+2}c^{2p+2}\int_0^{+\infty} J_{2p+2,\eta-2p}\,(\gamma^{\cdot})\psi_{p,h_2,h_3,\dots h_N}(I)\,d\,I\,,$$

and

$$\psi_{p,h_2,h_3,\dots h_N=} = \left[\frac{1}{m}\left(1+\frac{2I}{mc^2}\right)\right]^{h_2}\left[\frac{1}{m^2}\left(1+\frac{3I}{mc^2}\right)\right]^{h_3}\dots\left[\frac{1}{m^{N-1}}\left(1+\frac{NI}{mc^2}\right)\right]^{h_N}\varphi(I)$$

The integrability of this last expression is an immediate consequence of the theorems proved in [13]. By substituting (6) and (7) in (4) we find the requested expression for the fields $A^{\alpha\alpha_1\dots\alpha_n}$.

We can now prove hyperbolicity of the resulting field equations for every value of the Lagrange multipliers.

2.1. The convexity of $\xi_\alpha\, h'^\alpha$

Let us use the compact notation X_A to denote λ, λα, and the notation Y_C n to denote $\lambda_{\alpha_1\dots\alpha_n}$ for 2 ≤ n ≤ N. After that, for every constant time-like congruence ξ_α, we consider the quadratic form

$$K = \xi_\alpha\left(\frac{\partial^2\,h'^\alpha}{\partial\,X_A\,\partial X_B}\delta\,X_A\,\delta X_B + 2\,\frac{\partial^2\,h'^\alpha}{\partial\,X_A\,\partial Y_C^n}\delta\,X_A\,\delta Y_C^n + \frac{\partial^2\,h'^\alpha}{\partial Y_C^n\,\partial Y_D^q}\,\delta Y_C^n\,\delta\,Y_D^q\right) =$$

$$= -\,\frac{c}{k_B}\,\xi_\alpha\int_{R^3} d\,\vec{P}\int_0^{+\infty} e^{-1-\frac{\chi_\varepsilon}{k_B}}\left[(\delta\,\chi_\varepsilon)^2\sum_{r=0}^{+\infty}(-1)^r\,\frac{1}{r!}\left(\frac{\overline{\chi}}{k_B}\right)^r - 2\,\delta\,\chi_\varepsilon\,\delta\,\overline{\chi}\sum_{r=1}^{+\infty}(-1)^r\,\frac{1}{(r-1)!}\left(\frac{\overline{\chi}}{k_B}\right)^{r-1} + (\delta\overline{\chi})^2\sum_{r=2}^{+\infty}(-1)^r\,\frac{1}{(r-2)!}\left(\frac{\overline{\chi}}{k_B}\right)^{r-2}\right]p^\alpha\,\varphi(I)\,d\,I\,.$$

But we have that

$$\sum_{r=1}^{+\infty}(-1)^r\,\frac{1}{(r-1)!}\left(\frac{\overline{\chi}}{k_B}\right)^{r-1} = -\sum_{R=0}^{+\infty}(-1)^R\,\frac{1}{R!}\left(\frac{\overline{\chi}}{k_B}\right)^R,$$

$$\sum_{r=2}^{+\infty}(-1)^r\,\frac{1}{(r-2)!}2 = \sum_{R=0}^{+\infty}(-1)^R\,\frac{1}{R!}\left(\frac{\overline{\chi}}{k_B}\right)^R,$$

where for the first line we have changed index with the law r=1+R, while for the second line we have changed index with the law r=2+R. The result allows us to rewrite K as

$$K = -\,\frac{c}{k_B}\,\xi_\alpha\int_{R^3} d\,\vec{P}\int_0^{+\infty} e^{-1-\frac{\chi_\varepsilon}{k_B}}\sum_{r=0}^{+\infty}(-1)^r\,\frac{1}{r!}\left(\frac{\overline{\chi}}{k_B}\right)^r(\delta\,\chi_\varepsilon + \delta\,\overline{\chi}\,)^2\;\;p^\alpha\,\varphi(I)\,d\,I\,.$$

Since K is negative defined, we have that $\xi_\alpha\, h'^\alpha$ is a convex function for every value of the Lagrange multipliers and for every value of ξ_α; this property assures us that the field equations are hyperbolic.

But the expression (5) is burdened by the presence of the serial development. To avoid this, one could think of replacing it with the serial truncated to order r=M, i.e.,

$$h'^{\alpha}_{M} = -k_B\, c \int_{R^3} d\vec{P} \int_0^{+\infty} p^{\alpha} e^{-1-\frac{\chi_\varepsilon}{k_B}} \sum_{r=0}^{M} (-1)^r \frac{1}{r!}\left(\frac{\overline{\chi}}{k_B}\right)^r \varphi(I)dI . \quad (9)$$

In this way we find

$$K_M == -\frac{c}{k_B}\xi_\alpha \int_{R^3} d\vec{P}\int_0^{+\infty} e^{-1-\frac{\chi_\varepsilon}{k_B}}\Bigg[(\delta\chi_\varepsilon+\delta\overline{\chi})^2 \sum_{r=0}^{M-2}(-1)^r\frac{1}{r!}\left(\frac{\overline{\chi}}{k_B}\right)^r + (\delta\chi_\varepsilon)^2\left((-1)^{M-1}\frac{1}{(M-1)!}\frac{\overline{\chi}^{M-1}}{k_B^{M-1}} + (-1)^M\frac{1}{M!}\frac{\overline{\chi}^M}{k_B^M}\right) + 2\,\delta\chi_\varepsilon\,\delta\overline{\chi}\,(-1)^M\frac{1}{(M-1)!}\frac{\overline{\chi}^{M-1}}{k_B^{M-1}}\Bigg] p^\alpha\varphi(I)\,dI .$$

Therefore $\xi_\alpha h'^\alpha$ is a convex function for any value of its variables, while the convexity of $\xi_\alpha h'^\alpha_M$ is ensured only up to the order M-2 with respect to equilibrium since in this case the second and third terms in the bracket [...] of (10) are neglected.

This is in perfect agreement with [14] which shows how the hyperbolic zone increases when the equations develop until the second order, rather than the first one (See also [15], [16]).

So there is now the problem on how to "save goat and cabbage", i.e. to have field equations which are symmetric and at the same time hyperbolic for any value of the independent variables. We will now deal with this aspect. Let us substitute eq. (4) in eqs. (1) and distinguish the values n=0,1 and 2 ≤ n ≤ N, i.e.

$$\frac{\partial^2 h'^\alpha}{\partial\lambda^2}\partial_\alpha\lambda + \frac{\partial^2 h'^\alpha}{\partial\lambda\,\partial\lambda_\gamma}\partial_\alpha\lambda_\gamma + \sum_{q=2}^{N}\frac{\partial^2 h'^\alpha}{\partial\lambda\,\partial\lambda_{\beta_1\dots\beta_q}}\partial_\alpha\lambda_{\beta_1\dots\beta_q} = 0 ,$$

$$\frac{\partial^2 h'^\alpha}{\partial\lambda_\beta\partial\lambda}\partial_\alpha\lambda + \frac{\partial^2 h'^\alpha}{\partial\lambda_\beta\partial\lambda_\gamma}\partial_\alpha\lambda_\gamma + \sum_{q=2}^{N}\frac{\partial^2 h'^\alpha}{\partial\lambda_\beta\partial\lambda_{\beta_q\dots\beta_q}}\partial_\alpha\lambda_{\beta_q\dots\beta_q} = 0$$

$$\frac{\partial^2 h'^\alpha}{\partial\lambda_{\alpha_1\dots\alpha_n}\partial\lambda}\partial_\alpha\lambda + \frac{\partial^2 h'^\alpha}{\partial\lambda_{\alpha_1\dots\alpha_n}\partial\lambda_\gamma}\partial_\alpha\lambda_\gamma + \sum_{q=2}^{N}\frac{\partial^2 h'^\alpha}{\partial\lambda_{\alpha_1\dots\alpha_n}\partial\lambda_{\beta_1\dots\beta_q}}\partial_\alpha\lambda_{\beta_1\dots\beta_q} = I^{\alpha_1\dots\alpha_n} ,$$

or, in compact notation,

$$\frac{\partial^2 h'^\alpha}{\partial X_A\partial X_B}\partial_\alpha X_B + \frac{\partial^2 h'^\alpha}{\partial X_A\partial Y^q_D}\partial_\alpha Y^q_D = 0 , \qquad \frac{\partial^2 h'^\alpha}{\partial Y^n_C\partial X_B}\partial_\alpha X_B + \frac{\partial^2 h'^\alpha}{\partial Y^n_C\partial Y^q_D}\partial_\alpha Y^q_D = I^C , \quad (11)$$

If we substitute h'$^\alpha$ from eq. (5), there is no problem except for the presence of the serial expansion. Instead of this, if we substitute here h'$^\alpha$ with the approximated expression (9), we see that the coefficients of $\partial\alpha\, X_B$ and of $\partial\alpha\, Y^D_q$ are calculated at different order with respect to equilibrium. A possibility is to think that the approximation is consistent only if these coefficients are calculated at the same order. This is the same thing that is done in the simple problems of Rational Mechanics of rigid bodies when one does the series development of the equations of motion around an equilibrium position: The coefficients of the second derivatives of the Lagrangian parameters are simply calculated at equilibrium. In this case it follows that the consistent order approximation of the field equations (11) is

$$\frac{\partial^2 h'^\alpha_M}{\partial X_A\partial X_B}\partial_\alpha X_B + \frac{\partial^2 h'^\alpha_{M+1}}{\partial X_A\partial Y^q_D}\partial_\alpha Y^q_D = 0 , \qquad \frac{\partial^2 h'^\alpha_{M+1}}{\partial Y^n_C\partial X_B}\partial_\alpha X_B + \frac{\partial^2 h'^\alpha_{M+2}}{\partial Y^n_C\partial Y^q_D}\partial_\alpha Y^q_D = I^C_{M+1}, \quad (12)$$

where the order M+1 of I^c has been chosen because it was tested positevely in the articles [9], (17).

It is easy to verify that this system is hyperbolic for every value of the independent variables. In fact, it is symmetric; so hyperbolicity holds if the matrix of coefficients of $\partial_\alpha X_B$ and of $\partial_\alpha Y^D_q$ is negative defined, i.e., if the quadratic form

$$\tilde{K}_M = \xi_\alpha\left(\frac{\partial^2 h'^\alpha_M}{\partial X_A\partial X_B}\delta X_A\,\delta X_B + 2\,\frac{\partial^2 h'^\alpha_{M+1}}{\partial X_A\partial Y^n_C}\delta X_A\,\delta Y^n_C + \frac{\partial^2 h'^\alpha_{M+2}}{\partial Y^n_C\,\partial Y^q_D}\delta Y^n_C\;\delta Y^q_D\right)$$

is negative defined for every value of the Lagrange multipliers. This property holds iff M is an even number. In fact, by using (9) we find

$$\tilde{K}_M = -\frac{c}{k_B}\xi_\alpha\int_{R^3} d\vec{P}\int_0^{+\infty} e^{-1-\frac{\chi_\varepsilon}{k_B}}\sum_{r=0}^{M}(-1)^r\frac{1}{r!}\left(\frac{\overline{\chi}}{k_B}\right)^r(\delta\chi_\varepsilon + \delta\overline{\chi})^2 p^\alpha\phi(I)\,dI .$$

Now, in Appendix B we prove that iff M is an even number we have that $\sum_{r=0}^{M}(-1)^r\frac{1}{r!}\left(\frac{\overline{\chi}}{k_B}\right)^r > 0$ for every value of $\overline{\chi}$ and this fact proves our statement.

It is interesting the similarity with the results of [18] related to the classical case according to which the hyperbolicity can be obtained only if the highest order of the moments N is even; this condition is no longer valid in the relativistic case (because in [12] it is shown that, by making the non-relativistic limit, we obtain always a classical model with higher order of moments which is an even number). Here we have obtained instead that the highest order with respect to the equilibrium in which we stop the approximation must be an even number M.

It must however be said that the advantage of having obtained a hyperbolic system for any value of the independent variables has a cost; in fact, the system (12) hasn't the divergence form. For example, $(12)_1$ can be written as $\partial_\alpha f^{\alpha A}=0$ if and only if

$$\frac{\partial f^{\alpha A}}{\partial X_B} = \frac{\partial^2 h'^\alpha_M}{\partial X_A\,\partial X_B},\ \frac{\partial f^{\alpha A}}{\partial Y^q_D} = \frac{\partial^2 h'^\alpha_{M+1}}{\partial X_A\,\partial Y^q_D}$$

whose integrability condition is $\frac{\partial^3(h'^\alpha_{M+1} - h'^\alpha_M)}{\partial X_A\,\partial X_B\,\partial Y^q_D} = 0$

and this is not satisfied. A similar consideration can be done for $(12)_2$. We conclude that with this choice of truncation with respect to equilibrium, we must renounce to the divergence form of the field equations.

Thus two choices are present: The system

$$\frac{\partial^2 h'^\alpha_M}{\partial X_A\partial X_B}\partial_\alpha X_B + \frac{\partial^2 h'^\alpha_M}{\partial X_A\partial Y^q_D}\partial_\alpha Y^q_D = 0 , \qquad \frac{\partial^2 h'^\alpha_M}{\partial Y^n_C\partial X_B}\partial_\alpha X_B + \frac{\partial^2 h'^\alpha_M}{\partial Y^n_C\partial Y^q_D}\partial_\alpha Y^q_D = I^C_{M-1}, \quad (13)$$

with $I_{-1}^C=0$, $I_0^C=0$, or the system (12). The first one is hyperbolic only up to a certain order with respect to equilibrium (with a zone of hyperbolicity increasing with the growth of M) and the second one which is hyperbolic for any order with respect to equilibrium but which doesn't have the divergence form. Here the idea is assumed to prefer the second one of the above choices; it is clear that this decision is unpopular because people are so used to the first choice to consider it natural and look at everything else with distrust. The first choice consists in developing h'$^\alpha$ up to a certain order and then taking what comes out of it, letting it to decide the terms of the expansion around equilibrium. Now, it is true that h'$^\alpha$ has a role of the utmost importance to obtain symmetric equations and it has also suggested us how to construct the second choice, but this doesn't mean that it has universal importance to lay down laws also on other aspects, such as choosing which terms to hold or not in the expansion around equilibrium. Therefore it is very important as a mathematical tool, but its physical meaning is doubtful, while the second choice is based on the need to have hyperbolic equations for any value of the independent variables and, therefore, on the physical principle of cause and effect and on the fact that wave propagation speeds must not exceed that of light. This consistency of order around equilibrium, which is the basis of the second choice, brings to mind the article [19] where, however, the consistency was based on the choice of moments to be withheld or refused. But the equations that came out were so complicated that no one had the courage to work on it, despite the subsequent simplified versions [20] , [21]. Since it has no relation to

the ideas here reported, we limit ourselves to remember with appreciation the existence of these articles.
In any case, the system (13) with M+2 instead of M can always be used but, after that, from the coefficients of $\partial_\alpha X_B$ and $\partial_\alpha Y_D^q$ all the terms of order greater than M can be dropped obtaining in this way the system (12).

We conclude this section with an interesting theorem which holds both for the system (12), than for the system (13) with M ≥ 2 (as in [6]):
THEOREM: "A necessary and sufficient condition for the independent variables to tend towards an equilibrium value is that at the initial instant we have also $\partial_\alpha \lambda=0$, $\partial_{(\alpha} \lambda_{\beta)}=0$, i.e., equilibrium in the sense of ordinary thermodynamics of viscous, heat conducting fluids as stated in [2] near eq. (7.21)".

In fact, by calculating the field equations after that instant, they become

$$\frac{\partial^2 h'^\alpha_E}{\partial \lambda^2}\, \partial_\alpha \lambda + \frac{\partial^2 h'^\alpha_E}{\partial \lambda\, \partial \lambda_\gamma}\, \partial_{(\alpha}\lambda_{\gamma)} = 0\,,$$
$$\frac{\partial^2 h'^\alpha_E}{\partial \lambda_\beta \partial \lambda}\, \partial_\alpha \lambda + \frac{\partial^2 h'^\alpha_E}{\partial \lambda_\beta\, \partial \lambda_\gamma}\, \partial_{(\alpha}\lambda_{\gamma)} = 0\,,$$
$$\frac{\partial^2 h'^\alpha_1}{\partial \lambda_{\alpha_1 \ldots \alpha_n} \partial \lambda}\, \partial_\alpha \lambda + \frac{\partial^2 h'^\alpha_1}{\partial \lambda_{\alpha_1 \ldots \alpha_n}\, \partial \lambda_\gamma}\, \partial_{(\alpha}\lambda_{\gamma)} = 0\,,$$

where we have replaced $\partial_\alpha \lambda_\gamma$ with its symmetric part $\partial_{(\alpha} \lambda_{\gamma)}$ and this was possible because they were contracted with symmetric tensors with respect to α and γ. This is a system in the 14 unknown components of $\partial_\alpha \lambda$, $\partial_{(\alpha} \lambda_{\gamma)}$. Obviously, $\partial_\alpha \lambda=0$, $\partial_{(\alpha} \lambda_{\gamma)}=0$ is a solution of this system. To prove that it is the unique solution, we can consider a part of these equations; in particular, we can consider the third equations only with n=2 and take only the traceless part of this equation. In this way we can use the results of [6] and express the system multiplied by $-k_B/m$ as

$$V^\alpha_E\, \partial_\alpha \lambda + T^{\alpha\gamma}_E\, \partial_{(\alpha}\lambda_{\gamma)} = 0\,, T^{\alpha\beta}_E\, \partial_\alpha \lambda + m A^{\alpha\beta\gamma}_{E11}\, \partial_{(\alpha}\lambda_{\gamma)} = 0\,,$$
$$A^{\alpha<\beta\delta>}_E\, \partial_\alpha \lambda + m A^{\alpha<\beta\delta>\gamma}_{E12}\, \partial_{(\alpha}\lambda_{\gamma)} = 0\,,$$

where $A_E^{\alpha\beta\gamma}$, $A_{E11}^{\alpha\beta\gamma}$ and $A_{E12}^{\alpha\beta\delta\gamma}$ are the functions called $A_E^{\alpha\beta\gamma}$, $A_{11}^{\alpha\beta\gamma}$ and $A_{12}^{\alpha\beta\delta\gamma}$ in eqs. (A.6)-(A.7) of [6]. We see that here the matrix of the coefficients of the unknows $\partial_\alpha \lambda$, $\partial_{(\alpha} \lambda_{\gamma)}$ is the transposite of that in the system $(54)_{1,2}$ of [6] even if with different unknowns. To be more sure, let us firstly consider eq. $(14)_1$ multiplied by c^2/m, eq. $(14)_2$ contracted with U_β/m and eq. $(14)_3$ contracted with $(4\, U_\beta\, U_\gamma - c^2\, g_{\beta\gamma})/(3\, m\, c^2)$; in this way we obtain a linear homogeneous system of 3 equations in the 3 unknowns

$U_\alpha \partial_\alpha \lambda$, $c^{-2}\, U^\alpha\, U^\gamma \partial_{(\alpha} \lambda_{\gamma)}$, $c^2\, h^{\alpha\gamma}\, \partial_{(\alpha} \lambda_{\gamma)}$ whose determinant of the coefficients is the transposite of $\tilde{D}^\pi_1$ reported in eq. $(A.11)_1$ of [6]. Since we had $\tilde{D}^\pi_1 \neq 0$, we now desume that $U_\alpha \partial_\alpha \lambda=0$, $U^\alpha\, U^\gamma \partial_{(\alpha} \lambda_{\gamma)}=0$, $h^{\alpha\gamma}\, \partial_{(\alpha} \lambda_{\gamma)} =0$.
We consider now eq. $(14)_2$ contracted with $-h_\beta{}^\delta/m$ and eq. $(14)_3$ contracted with $-2h_\beta{}^\delta U_\gamma /(mc^2)$; in this way we obtain a linear homogeneous system of 2 equations in the 2 unknowns $h^{\alpha\delta}\partial_\alpha\lambda$, $2\, c^{-2}\, h^{\delta\mu}\, U^\nu\, \partial_{(\mu} \lambda_{\nu)}$ whose determinant of the coefficients is the transposite of

$$\begin{vmatrix} \dfrac{p}{m} & \dfrac{2A^0_{11}}{m} \\ \dfrac{B_4 c^2}{3} & \dfrac{2B_2 c^2}{3} \end{vmatrix}$$

Now, on line 3 of page 444 of [6] a determinant D_3 appears that is not null and proportional to the present one. So also the present determinant is not null and we can desume that $h^{\alpha\delta}\, \partial_\alpha \lambda = 0$, $h^{\delta\mu}\, U^\nu\, \partial_{(\mu} \lambda_{\nu)}=0$.

Finally, $(14)_3$ contracted with $h_\beta{}^{<\delta}\, h^{\theta>}{}_\gamma$ gives $\partial_{<\mu} \lambda_{\nu>} =0$, thus completing the proof.

Obviously, $\partial_{<\mu} \lambda_{\nu>} =0$ is equivalent to $\lambda_\alpha= \lambda_{E\alpha} = U_\alpha/T= C_{\alpha\beta}\, x^\beta + C_\alpha$ with $C_{\alpha\beta}$ and C_α constants and $C_{\alpha\beta}$ skew-symmetric. From this result it follows $C^2\, T^{-2} = C_{\alpha\gamma}\, C^\gamma{}_\beta\, x^\alpha\, x^\beta + 2\, C_{\alpha\gamma}\, C^\gamma\, x^\alpha + C_\gamma\, C^\gamma$.

3. On the integrability of eq. (3)

Let us define $G_0 = \sqrt{\lambda^\mu \lambda_\mu}\,, l^\beta = \frac{c\,\lambda^\beta}{G_0}\,,\ h^{\alpha\beta} = -g^{\alpha\beta} + \frac{l^\alpha l^\beta}{c^2}$.
It follows that $\chi_\varepsilon + \overline{\chi} =$ (15)

$$= \sum_{n=0}^{N} \boldsymbol{\lambda_{\alpha_1 \ldots \alpha_n}}\, \frac{\mathbf{1}}{\boldsymbol{m^{n-1}}}\left(\mathbf{1} + \frac{\boldsymbol{nI}}{\boldsymbol{mc^2}}\right)\left(-\boldsymbol{h^{\beta_1}_{\alpha_1}} + \frac{\boldsymbol{l^{\beta_1} l_{\alpha_1}}}{\boldsymbol{c^2}}\right)\ldots\left(-\boldsymbol{h^{\beta_n}_{\alpha_n}} + \frac{\boldsymbol{l^{\beta_n} l_{\alpha_n}}}{\boldsymbol{c^2}}\right)\boldsymbol{p^{\beta_1}}\ldots\boldsymbol{p^{\beta_n}} =$$
$$= \boldsymbol{m\,\lambda} + \left(\mathbf{1} + \frac{\boldsymbol{I}}{\boldsymbol{mc^2}}\right)\frac{\boldsymbol{G_0}}{\boldsymbol{c}}\,\boldsymbol{l_\beta\, l^\beta} + \sum_{n=2}^{N} \boldsymbol{\lambda_{\alpha_1 \ldots \alpha_n}}\, \frac{\mathbf{1}}{\boldsymbol{m^{n-1}}}\left(\mathbf{1} + \frac{\boldsymbol{nI}}{\boldsymbol{mc^2}}\right)\sum_{r=0}^{n}(-\mathbf{1})^r \binom{\boldsymbol{n}}{\boldsymbol{r}}\ \boldsymbol{h^{\beta_1}_{\alpha_1}} \ldots \boldsymbol{h^{\beta_r}_{\alpha_r}}$$
$$\boldsymbol{l^{\alpha_{r+1}}}\ldots\boldsymbol{l^{\alpha_n}}\,\boldsymbol{l_{\beta_{r+1}}}\ldots\boldsymbol{l_{\beta_n}}\,\boldsymbol{c^{2r-2n}}\,\boldsymbol{p^{\beta_1}}\ldots\boldsymbol{p^{\beta_n}}\,.$$

We can valuate this expression in the reference frame where $l^\alpha \equiv (c\,,0\,,0\,,0)$, $p^\alpha \equiv m\,c\,(\cosh s, \sinh s\, q^i)$ with $q^i\, q_i = -1$ and find

$$\boldsymbol{\chi_\varepsilon + \overline{\chi}}$$
$$= \boldsymbol{m\,\lambda} + \boldsymbol{m}\left(\mathbf{1} + \frac{\boldsymbol{I}}{\boldsymbol{mc^2}}\right)\boldsymbol{G_0\, c \cosh s}$$
$$+ \sum_{n=2}^{N}\left(\mathbf{1} + \frac{\boldsymbol{nI}}{\boldsymbol{mc^2}}\right)\sum_{r=0}^{n}\binom{\boldsymbol{n}}{\boldsymbol{r}}\,(-\mathbf{1})^r\,\boldsymbol{L_{n,i_1 \ldots i_r}}\,\boldsymbol{q^{i_1}}\ldots\boldsymbol{q^{i_r}}\,\boldsymbol{\sinh^r s \cosh^{n-r} s}$$

where $L_{n,i_1 \ldots i_r} = \lambda_{\alpha_1 \ldots \alpha_n}\, h^{\alpha_1}_{i_1} \ldots h^{\alpha_r}_{i_r}\, l^{\alpha_{r+1}} \ldots l^{\alpha_n}\, c^n$. So it is evident that the integrability of (3) holds iff

$$\lim_{s\to\infty} \chi_\varepsilon + \overline{\chi} = +\infty. \qquad (16)$$

This result corresponds to eq. (29) of [18] which concerned the non relativistic monoatomic case. In particular, $(29)_1$ imposes that the number N must be even; in our case this condition is not present.
In fact, the non relativistic limit of the present field equations has been considered in [12] and is summarized by its eq. (11) where the tensor H_{SA} appears. From its definition in eq. $(12)_1$ we see that it is obtained by taking s traces of a tensor of order A + 2s. Since A ≤ N-s, it follows that A + 2 s ≤ N+s. Since s ≤ N, it follows that A + 2 s ≤ 2 N. This value is effectively present; in fact, s=N and A=0 is an admissible set of values and with them we have A + 2 s = 2 N. In other words, F_{N0} is obtained by taking N traces of a tensor of order 2N. So the highest order present in our independent variables is even (2N).
But eq. $(29)_2$ of [18] is another condition which has a counterpart also in the present model. In fact, from the above result, we can define $f_N\left(\vec{q}\right)$, from

$$\lim_{s\to\infty} \cosh^{-N} s\, (\chi_\varepsilon + \overline{\chi}) = m\left(1 + \frac{NI}{mc^2}\right)\sum_{r=0}^{N}(-1)^r\binom{N}{r} L_{N,i_1 \ldots i_r} q^{i_1} \ldots q^{i_r} = f_N\left(\vec{q}\right),$$

because $\lim_{s\to+\infty} \tanh s = 1$. So a necessary condition for (16) is that $f_N\left(\vec{q}\right) \geq 0$ for all unitary vectors $\vec{q}$, while a sufficient condition for (16) is that $f_N\left(\vec{q}\right) > 0$ for all unitary vectors $\vec{q}$. The doubtful case remains $f_N\left(\vec{q}\right) = 0$ for some unitary vector $\vec{q}$. Let us call m the minimum value of the function $f_N\left(\vec{q}\right)$ on the closed and limited set $q^i\, q_i =-1$; then the necessary condition becomes m ≥ 0 and the sufficient one becomes m > 0. (The only difference with respect to $(29)_2$ of [18] is due to the opposite sign of their χ and the present $\chi_\varepsilon + \overline{\chi}$).
Obviously, also the doubtful cases can be studied but the results are complicated and not relevant. The type of calculations involved can be understood by the following 3 subcases:

- The case N=1. We see that (15) becomes $\chi_\varepsilon + \overline{\chi}= m\ \lambda + m \left(1 + \frac{I}{mc^2}\right) G_0\, c \cosh s$ which surely satisfies the condition (16) and we are sure of integrability.
- The case N=2. We see that m ≥ 0 is a necessary and sufficient condition for integrability. In fact, the doubtful

case is $\sum_{r=0}^{2}(-1)^r\binom{2}{r}L_{2,i_1\ldots i_r}q^{i_1}\ldots q^{i_r}=0$ for some $\vec{q}$ and, in this case, we have that $\lim_{s\to\infty} e^{-S}(\chi_\varepsilon+\overline{\chi}) = m\left(1+\frac{I}{mc^2}\right)=\frac{G_0c}{2}>0$ so that also in this case (16) is satisfied.

- The case N=3. We see that (15) gives

$\chi_\varepsilon+\overline{\chi}=\frac{e^{3s}}{8}f_3\left(\vec{q}\right)+\frac{e^{2s}}{4}f_2\left(\vec{q}\right)+e^s\,m\,g_3\left(\vec{q}\right)$ plus a linear combination of e^0, e^{-s}, e^{-2s}, e^{-3s} and with

$$g_3\left(\vec{q}\right)=\left(1+\frac{I}{mc^2}\right)\frac{G_0c}{2}+\left(1+\frac{3I}{mc^2}\right)\frac{3\,L_3}{8}-L_{3,i}q^i-L_{3,ij}\,q^iq^j+L_{3,ijk}\,q^iq^jq^k.$$

It follows that, in the above mentioned doubtful case $f_3\left(\vec{q}\right)=0$ for $\vec{q}$ belonging to a set S_3 of unitary vectors, the necessary condition is $f_2\left(\vec{q}\right)\geq 0$ for all $\vec{q}\in S_3$ and the sufficient condition is $f_2\left(\vec{q}\right)>0$. So another doubtful case arises if $f_2\left(\vec{q}\right)=0$ for a subset S_2 of S_3. In this case the sufficient condition for integrability is $g_3\left(\vec{q}\right)>0$ for all $\vec{q}\in S_2$, while the necessary condition is $g_3\left(\vec{q}\right)\geq 0$, otherwise the condition (16) is certainly not satisfied.

It is not easy to study the conditions $f_N\left(\vec{q}\right)\geq 0$, $g_3\left(\vec{q}\right)\geq 0$ for all unitary vectors $\vec{q}$ and similar; for the sake of curiosity we consider now the cases N=2,3. In any case, it is a disappointment that hyperbolicity doesn't hold for every value of the independent variables. As we have seen, the loophole consists in replacing our field equations with their consistent approximation around the equilibrium.

3.1. The condition $f_2\left(\vec{q}\right)\geq 0$, for all unitary vectors

- If $L_{2,ij}=0$, $L_{2,i}=0$ this condition is easy and is expressed by $L_2\geq 0$.
- If $L_{2,ij}=0$, $L_{2,i}\neq 0$ it is easy too and we prove now that it is equivalent to the two conditions $L_2>0$, $(L_2)^2\geq 4\,L_{2,i}\,L_{2,j}\,\delta^{ij}$.

In fact, in this case we have

$f_2\left(\vec{q}\right)=\left(1+\frac{2I}{mc^2}\right)\left(L_2-2L_{2,i}\,q^i\right)$. In the particular value $q^i=\frac{L_{2,i}}{\sqrt{L_{2,j}L_{2,k}\delta^{jk}}}$ the condition $f_2\left(\vec{q}\right)\geq 0$, becomes $L_2\geq 2\sqrt{L_{2,j}L_{2,k}\,\delta^{j,k}}$ from which the above relations hold. Vice versa, if they are satisfied, we have $\left|L_{2,i}\,q^i\right|=\left|\vec{L}_2\cdot\vec{q}\right|\leq\left|\vec{L}_2\right|\left|\vec{q}\right|=\left|\vec{L}_2\right|\leq\frac{L_2}{2}$, where we have applied the Schwartz's theorem. It follows that $-\frac{1}{2}L_2\leq L_{2,i}\,q^i\leq\frac{1}{2}L_2$ whose right hand side gives $f_2\left(\vec{q}\right)\geq 0$.

- If L2,ij≠0, we have

$f_2\left(\vec{q}\right)=\mathrm{m}\left(1+\frac{2I}{mc^2}\right)\left(L_2-2L_{2,i}\,q^i+L_{2,ij}\,q^iq^j\right)$, so that $f_2\left(\vec{q}\right)=0$ is a quadratic surface S and we have the following cases:

I. If S consists of two coinciding planes and $L_{2,ij}\,\delta^{ij}<0$, then $f_2\left(\vec{q}\right)\geq 0$ has no solution.

II. If S consists of two coinciding planes and $L_{2,ij}\,\delta^{ij}>0$, then $f_2\left(\vec{q}\right)\geq 0$ is identically satisfied.

III. If S consists of two complex parrallel and distinct planes and $L_{2,ij}\,\delta^{ij}<0$, then $f_2\left(\vec{q}\right)\geq 0$ has no solution.

IV. If S consists of two complex parrallel and distinct planes and $L_{2,ij}\,\delta^{ij}>0$, then $f_2\left(\vec{q}\right)\geq 0$ is identically satisfied.

V. If S consists of two real parrallel and distinct planes, they divide the space in 3 regions; in one or two of these regions R_+ the condition $f_2\left(\vec{q}\right)\geq 0$ is satisfied while in the remaining part R_- it is not satisfied. Thus the sphere of center the origin and radius 1 must be contained all in one of the regions R_+; this is equivalent to say that also $\vec{q}=\vec{0}$ must be contained in one of the regions R_+, i.e., $L_2>0$ and that the surface $q^i\,q_i=-1$ doesn't intersect S or it is tangent to one or both the planes of S. Obviously, this last condition may be expressed in terms of $L_{2,ij}$, $L_{2,i}$, L_2.

VI. If S consists of two complex incident and distinct planes and the matrix $L_{2,ij}$ is negative semidefined, then $f_2\left(\vec{q}\right)\geq 0$ has no solution.

VII. If S consists of two complex incident and distinct planes and the matrix $L_{2,ij}$ is positive semidefined, then $f_2\left(\vec{q}\right)\geq 0$ is identically satisfied.

VIII. If S consists of two real incident and distinct planes, they divide the space in 4 regions; in two of these regions R_+ the condition $f_2\left(\vec{q}\right)\geq 0$ is satisfied while in the remaining two regions R_- it is not satisfied. Thus the sphere of center the origin and radius 1 must be contained all in one of the two regions R_+; this is equivalent to say that also $\vec{q}=\vec{0}$ must be contained in one of the regions R_+, i.e., $L_2>0$ and that the surface $q^i\,q_i=-1$ doesn't intersect S or it is tangent to one or both the planes of S.

IX. If S is a complex cylinder and the matrix $L_{2,ij}$ is negative semidefined, then $f_2\left(\vec{q}\right)\geq 0$ has no solution.

X. If S is a complex cylinder and the matrix $L_{2,ij}$ is positive semidefined, then $f_2\left(\vec{q}\right)\geq 0$ is identically satisfied.

XI. If S is a real cylinder, it divides the space in 2 regions (3 regions if it is an hyperbolic cylinder): In one or two of these regions R_+ the condition $f_2\left(\vec{q}\right)\geq 0$ is satisfied while in the remaining one R_- it is not satisfied. Thus the sphere of center the origin and radius 1 must be contained all in the region R_+; this is equivalent to say that also $\vec{q}=\vec{0}$ must be contained in this region R_+, i.e., $L_2>0$ and that the surface $q^i\,q_i=-1$ doesn't intersect the cylinder or it is tangent to it.

XII. If S is a complex cone and the matrix $L_{2,ij}$ is negative defined, then $f_2\left(\vec{q}\right)\geq 0$ has no solution.

XIII. If S is a complex cone and the matrix $L_{2,ij}$ is positive defined, then $f_2\left(\vec{q}\right)\geq 0$ is identically satisfied.

XIV. If S is a real cone, it divides the space in 2 regions: one of these is constitute by the two inner parts of the two semicones and the other is the external part. In one of these regions R_+ the condition $f_2\left(\vec{q}\right)\geq 0$ is satisfied while in the remaining one R_- it is not satisfied. Thus the sphere of center the origin and radius 1 must be contained all in the region R_+; this is equivalent to say that also $\vec{q}=\vec{0}$ must be contained in this region R_+, i.e., $L_2>0$ and that the surface $q^i\,q_i=-1$ doesn't intersect the cone or it is tangent to it.

XV. If S is a complex ellipsoid and the matrix $L=\begin{pmatrix}L_{2,ij} & L_{2,i}\\ L_{2,j} & L_2\end{pmatrix}$ is negative defined, then $f_2\left(\vec{q}\right)\geq 0$ has no solution.

XVI. If S is a complex ellipsoid and the matrix L is positive defined, then $f_2\left(\vec{q}\right)\geq 0$ is identically satisfied.

XVII. If S is a real ellipsoid, it divides the space in 2 regions: one of these is its inner part and the other is its external part. In of these regions R_+ the condition $f_2\left(\vec{q}\right)\geq 0$ is satisfied while in the remaining one R_- it is not satisfied. Thus the sphere of center the origin and radius 1 must be contained all in the region R_+; this is equivalent to say that also must be contained in this region R_+, i.e., $L_2>0$ and that the surface $q^i\,q_i=-1$ doesn't intersect the ellipsoid or it is tangent to it.

XVIII. If S is a paraboloid or an elliptic hyperboloid, it divides the space in 2 regions; in of these regions R_+ the condition $f_2\left(\vec{q}\right)\geq 0$ is satisfied while in the remaining one R_- it is not satisfied. Thus the sphere of center the origin and radius 1 must be contained all in the region R_+; this is equivalent to say that also $\vec{q}=\vec{0}$ must be contained in this region R_+, i.e., $L_2>0$ and that the surface $q^i\,q_i=-1$ doesn't intersect S or it is tangent to it.

XIX. If S is an hyperbolic hyperboloid, it divides the space in 3 regions: two of these are the internal parts of the two flaps of the hyperboloid and the other is the remaining one. In one or two of these regions the condition $f_2\left(\vec{q}\right)\geq 0$ is satisfied while in the remaining one it is not satisfied. Thus the sphere of center the origin and radius 1 must be contained all in one of the regions with $f_2\left(\vec{q}\right) \geq 0$; this is equivalent to say that also $\vec{q} = \vec{0}$ must be contained in this region R_+, i.e., $L_2 > 0$ and that the surface $q^i q_i = -1$ doesn't intersect the hyperboloid or it is tangent to it.

3.2. The condition $f_3\left(\vec{q}\right) \geq 0$, for all unitary vectors

The expression of $f_3\left(\vec{q}\right) = m\left(1 + \frac{3I}{mc^2}\right)\left(L_3 - 3L_{3,i}\, q^i + 3\, L_{3,ij}\, q^i q^j - L_{3,ijk} q^i q^j q^k\right)$.

Obviously, our condition is very complicated to study. So we are content to see that there are cases in which it is satisfied and cases in which it is not. For example, if $L_{3,ijk} = L_{3,(i}\, \delta_{jk)}$, then $f_3\left(\vec{q}\right)$ becomes $f_3\left(\vec{q}\right) = m\left(1 + \frac{3I}{mc^2}\right)\left(L_3 - 4L_{3,i}\, q^i + 3\, L_{3,ij}\, q^i q^j\right)$ which is similar to that of $f_2\left(\vec{q}\right)$ except for replacing 2I with 3I, L_2 with L_3, $L_{2,i}$ with 2 $L_{3,i}$ and $L_{2,ij}$ with 3 $L_{3,ij}$. So the considerations of the above subsection can be applied realizing our purpose in this way.

4. New trends of the relativistic model for polyatomic gases

Goal of this subsection is to present a completely new way to obtain balance equations for polyatomic gases. Instead of multiplying the Boltzmann-Chernikov equation by $\frac{c}{m^{j-1}}\, p^{\alpha_1} \dots p^{\alpha_j}\left(1 + \frac{jI}{mc^2}\right)\varphi(I)$, it is here multiplied by

$\frac{c}{m^{j-1}}\, p^{\alpha_1} \dots p^{\alpha_j}\left(1 + \frac{I}{mc^2}\right)^j \varphi(I)$. This is physically more significant because the 4-momentum p^α appears through $p^\alpha\left(1 + \frac{I}{mc^2}\right)$ so that its modulus is no more constant but it depends on the internal energy of the particle. The choice previously used in literature $\frac{c}{m^{j-1}}\, p^{\alpha_1} \dots p^{\alpha_j}\left(1 + \frac{jI}{mc^2}\right)\varphi(I)$ was motivated by the fact that with it the non-relativistic limit of the balance equations gives us those known in the literature for the classical case. But this is also true of the choice presented here.

The present choice has notable mathematical consequences; for example, a single quantity (the energy e) is expressed by an integral and all the other constitutive functions can be expressed in terms of it and its derivatives with respect to temperature. Another useful consequence is its easier applicability to the case of diatomic and ultra-relativistic gases which are useful, at least for testing the model in simple cases.

Therefore there is the possibility that all the articles already present in the literature, in this context, must be revisited following the directives of this article and citing it. Further investigations may then be possible, thanks to the more manageable form of the present balance equations.

To justify this new approach, we observe that, in the article (12), the authors started from the definition:

$$A^{\alpha\alpha_1 \dots \alpha_J} = \frac{c}{m^{j-1}} \int_{R^3} d\,\vec{P} \int_0^{+\infty} f\; p^\alpha p^{\alpha_1} \dots p^{\alpha_j}\left(1 + a_j\, \frac{I}{mc^2}\right)\varphi(I)\; d\, I\,,$$

and proved that the necessary and sufficient condition to obtain, at the non relativistic limit, the hierarchy of polyatomic gases is that $a_j = j$. Now it is interesting to observe that $\left(1 + \frac{jI}{mc^2}\right)$ is nothing more than the first two terms in the binomial power $\left(1 + \frac{I}{mc^2}\right)^j$. Therefore the same results are obtained, at the classical limit, if we replace the previous definition with

$$A^{\alpha\alpha_1 \dots \alpha_J} = \frac{c}{m^{j-1}} \int_{R^3} d\,\vec{P} \int_0^{+\infty} f\; p^\alpha p^{\alpha_1} \dots p^{\alpha_j}\left(1 + \frac{I}{mc^2}\right)^j \varphi(I)\; d\, I\,, \quad (18)$$

Obviously, the expression of the closure must be changed accordingly.

From the physical point of view, it seems that $\left(1 + \frac{jI}{mc^2}\right)$ doesn't have great significance, while $\left(1 + \frac{I}{mc^2}\right)^j$ (except for the exponent) is significant because $\left(1 + \frac{I}{mc^2}\right)$ is the total energy divided by mc^2, i.e., the normalized total energy. This fact suggests the following new approach:

Let us start from the Boltzmann-Chernikov equation

$$p^\alpha\, \partial_\alpha f = Q\,, \quad (19)$$

contract it by $\frac{c}{m^{j-1}}\, p^{\alpha_1} \dots p^{\alpha_j}\left(1 + \frac{I}{mc^2}\right)^j \varphi(I)$ and integrate it in $d\,\vec{P}\; d\, I$. So we obtain the field equations $\partial_\alpha A^{\alpha\alpha_1 \dots \alpha_J} = I^{\alpha_1 \dots \alpha_J}$ with $A^{\alpha\alpha_1 \dots \alpha_J}$ given by (18) for j=0,1,..., J.
By applying the "Maximum Entropy Principle" (MEP), we find that

$$f = e^{-1-\frac{\chi}{k_B}} \text{ with } \chi = \sum_{j=0}^{J} \frac{1}{m^{j-1}}\, \lambda_{\alpha_1 \dots \alpha_j}\, p^{\alpha_1} \dots p^{\alpha_j}\left(1 + \frac{I}{mc^2}\right)^j. \quad (20)$$

By defining the four-potential h'^α from

$$h'^\alpha = -\, k_B\, c \int_{R^3} d\,\vec{P} \int_0^{+\infty} f\; p^\alpha \varphi(I)\; d\, I\,, \quad (21)$$

we obtain that the above definition (18) becomes

$$A^{\alpha\alpha_1 \dots \alpha_j} = \frac{\partial\, h'^\alpha}{\partial\, \lambda_{\alpha_1 \dots \alpha_j}}\,. \quad (22)$$

This is important because ensure the symmetric hyperbolic form of the field equations

$$\partial_\alpha A^{\alpha\alpha_1 \dots \alpha_J} = I^{\alpha_1 \dots \alpha_J} \text{ i.e., } \frac{\partial^2\, h'^\alpha}{\partial\, \lambda_{\alpha_1 \dots \alpha_j} \partial\, \lambda_{\beta_1 \dots \beta_i}}\, \partial_\alpha \lambda_{\beta_1 \dots \beta_i} = I^{\alpha_1 \dots \alpha_j}\,, \quad (23)$$

provided the convexity of the function $\xi_\alpha h'^\alpha$ for any time-like unitary and constant congruence ξ_α.

4.1. The variables at equilibrium

Thermodynamical equilibrium is defined as the state where $\lambda_{a_1 \dots a_j} = 0$ for $j \geq 2$. In this case (18) becomes

$$A_E^{\alpha\alpha_1 \dots \alpha_j} = \frac{c}{m^{j-1}} \int_{R^3} d\,\vec{P} \int_0^{+\infty} e^{-1-\frac{1}{k_B}\left[m\,\lambda^E + \lambda^E_\mu\, p^\mu\left(1+\frac{I}{mc^2}\right)\right]}\; p^\alpha p^{\alpha_1} \dots p^{\alpha_j}\left(1 + \frac{I}{mc^2}\right)^j \varphi(I)\; d\, I\,. \quad (24)$$

It is interesting for the future, that from (24) it follows $\frac{\partial A_E^{\alpha\alpha_1 \dots \alpha_j}}{\partial\, \lambda_{\alpha_{j+1}}} = -\frac{m}{k_B}\, A_E^{\alpha\alpha_1 \dots \alpha_{j+1}}$, so that the tensor of order j+2 can be easily obtined from that of order j+1.

For j=0,1 q. (24) is the same of [6], so that we may copy the corresponding results and see that λ^E and λ^E_μ are given by

$$n = 4\,\pi\, m^3 c^3 e^{-1-\frac{m\,\lambda^E}{k_B}} \int_0^{+\infty} J^{\cdot}_{2,1}\; \varphi(I)\; dI\,, \lambda^E_\mu = \frac{U_\mu}{T}\,, \quad (25)$$

where T is the absolute temperature and we have used the functions

$$J_{m,n}(\gamma) = \int_0^{+\infty} e^{-\gamma \cosh s}\; sinh^m s\; cosh^n s\; d\, s\,, \quad (26)$$

$$\gamma = \frac{mc^2}{k_B T}\,, \gamma^{\cdot} = \gamma\left(1 + \frac{I}{mc^2}\right)\,,\; J^{\cdot}_{m,n} = J_{m,n}(\gamma^{\cdot})\,.$$

After that, we have

$$V^\alpha = m\, n\, U^\alpha\,, T_E^{\alpha\beta} = \frac{e}{c^2}\, U^\alpha U^\beta + p\, h^{\alpha\beta}\,, \quad (27)$$

Where $h^{\alpha\beta} = -\, g^{\alpha\beta} + \frac{U^\alpha U^\beta}{c^2}$, e is the energy, p is the pressure and they are given by

$$p = \frac{n\, m\, c^2}{\gamma}\,,\; e = n\, m\, c^2\, \frac{\int_0^{+\infty} J^{\cdot}_{2,2}\left(1+\frac{I}{m\, c^2}\right)\varphi(I)\; d\, I}{\int_0^{+\infty} J^{\cdot}_{2,1}\; \varphi(I)\; d\, I}\,. \quad (28)$$

The expression of $A_E^{\alpha\alpha_1\ldots\alpha_j}$ in (24) for j ≥ 2 is new but, by applying the Representation Theorems, we know that it must have the form

$$A_E^{\alpha_1\ldots\alpha_{j+1}} = \sum_{h=0}^{\left[\frac{j+1}{2}\right]} \phi_{h,j}\; h^{(\alpha_1\alpha_2}\ldots h^{\alpha_{2h-1}\alpha_{2h}}\, U^{\alpha_{2h+1}}\ldots U^{\alpha_{j+1})}. \quad (29)$$

(The second index in $\phi_{h,j}$ remember us that it belongs to the tensor $A_E^{\cdots}$ of order j+1). The right hand sides of this equation and of (24) must be equal; this gives an equation which we can contract by $h_{\alpha_1\alpha_2}\ldots h_{\alpha_{2k-1}\alpha_{2k}}\, U_{\alpha_{2k+1}}\ldots U_{\alpha_{j+1}}$ so obtaining

$$\phi_{k,j} = 4\pi \binom{j+1}{2k}\frac{1}{2k+1}c^{2k+3}\, m^4\, e^{-1-\frac{m\lambda^E}{k_B}} \int_0^{+\infty} J_{2k+2,+1-2k}\left(1+\frac{I}{mc^2}\right)^j \varphi(I)\; I\,, \quad (30)$$

where the integrations have been calculted with the same method of [6], from the line after eq. (24) up to eq. (27).

In particular, eqs. (29)-(30) with j=0 confirm eqs. $(27)_1$ and $(25)_1$ with $\phi_{0,0}$= m n.

Now we can use eq. $(25)_1$ to desume $e^{-1-\frac{m\lambda^E}{k_B}}$ and substitute it in (30) so that it becomes

$$\phi_{k,j} = m\, n\, c^{2k}\frac{1}{2k+1}\binom{j+1}{2k}\frac{\int_0^{+\infty} J_{2k+2,j+1-2k}\left(1+\frac{I}{mc^2}\right)^j \varphi(I)\, d\, I}{\int_0^{+\infty} J_{2,1}\,\varphi(I)\, d\, I}. \quad (31)$$

In particular, eqs. (29), (31) with j=1 confirm eqs. $(27)_2$ and (28) with $\phi_{0,1}$= e c-2 and with $\phi_{1,1}$= p (The identity γ $J_{4,0}$ (γ) = 3 $J_{2,1}$ (γ) has been used, from which it follows γ $\left(1+\frac{I}{m\,c^2}\right)$ $J_{4,0}$ (γ*) = 3 $J_{2,1}$ (γ^*)).

Finally, eqs. (29), (31) with j=2 give

$$A_E^{\alpha\alpha_1\alpha_2} = \phi_{0,2}\, U^\alpha U^{\alpha_1}\, U^{\alpha_2} + \phi_{1,2}\, U^{(\alpha}h^{\alpha_1\alpha_2)} \quad (32)$$

with

$$\phi_{0,2} = m\, n\,\frac{\int_0^{+\infty} J_{2,3}\left(1+\frac{I}{mc^2}\right)^2 \varphi(I)\, d\, I}{\int_0^{+\infty} J_{2,1}\,\varphi(I)\, d\, I},\quad \phi_{1,2} = m\, n\, c^2\,\frac{1}{3}\,\frac{\int_0^{+\infty} J_{4,1}\left(1+\frac{I}{mc^2}\right)\varphi(I)\, d\, I}{\int_0^{+\infty} J_{2,1}\,\varphi(I)\, d\, I}\,.$$

We see that $(32)_1$ is the same of eq. (48) of [6] with the identification $\phi_{0,2} = A_1^0$, $\phi_{1,2} = 3\,A_{11}^0$, and the expressions $(32)_{2,3}$ correspond to eqs. (49), (50) of [6] except that now we have $\left(1+\frac{I}{mc^2}\right)^2$ instead of $1+\frac{2I}{mc^2}$, as expected for the present different choice. We note that also with the new variables the tensor (4) of order j+2 can be obtained from that of order j+1. In particular, we have

$$\phi_{h,j+1} = \frac{j+2}{j+2-2h}\left(\frac{e}{nmc^2}\,\phi_{h,j} - \frac{\partial\,\phi_{h,j}}{\partial\,\gamma}\right) \quad for\ h = 0,\ldots,\left[\frac{j+1}{2}\right],$$

$$\phi_{\frac{j+2}{2},j+1} = \frac{c^2}{\gamma}\,\phi_{\frac{j}{2},j} \ \text{(present only if j is even).} \quad (32a)$$

To prove these relations, let us take the derivative of eq. (31) with respect to γ and take into account that $\frac{\partial}{\partial\gamma}\left(\int_0^{+\infty} J_{2,1}\,\phi(I)\, d\, I\right)^{-1} = \left(\int_0^{+\infty} J_{2,1}\,\phi(I)\, d\, I\right)^{-2}\left(\int_0^{+\infty} J_{2,2}\left(1+\frac{I}{mc^2}\right)\phi(I)\, d\, I\right) = \frac{e}{nmc^2}\left(\int_0^{+\infty} J_{2,1}\,\phi(I)\, d\, I\right)^{-1}$. So we obtain

$\frac{\partial\,\phi_{k,j}}{\partial\,\gamma} = -\,\phi_{k,j+1}\,\frac{j+2-2k}{j+2} + \phi_{k,j}\,\frac{e}{nmc^2}$, from which $(32a)_1$ follows. To obtain $(32a)_2$ let us write (31) with j+1 instead of j and use the identity of [2]:

$$\gamma\, J_{2k+2,j+2-2k} = -(j+2-2k)\, J_{2k,j+1-2k}\,(\gamma) + (j+3) J_{2k,j+3-2k}\,(\gamma)\,.$$

If j is even, this relation with k=(j+2)/2 gives $(32a)_2$ .For the other values of j we can furtherly use the identity $J_{2k,j+1-2k} = J_{2k,j+3-2k} - J_{2k+2,j+1-2k}$ and our equation gives $\phi_{k,j+1}$ =

$$\frac{j+2}{\gamma}\left(\phi_{k,j} + c^2\,\frac{j+3-2k}{2k}\,\phi_{k-1,j}\right) \ \text{for k= 1, ..., [(j+1)/2].} \quad (32b)$$

So, for k=0 and for k=(j+2)/2 and j even we can take (32a), while for k= 1, ..., [(j+1)/2], we can take $(32a)_1$ or (32b). Obviously, $for\ h = 1,\ldots,\left[\frac{j+1}{2}\right]$ this fact implies the identity

$$\frac{1}{\gamma}\left(\phi_{h,j} + c^2\,\frac{j+3-2h}{2h}\,\phi_{h-1,j}\right) = \frac{1}{j+2-2h}\left(\frac{e}{nmc^2}\,\phi_{h,j} - \frac{\partial\,\phi_{h,j}}{\partial\,\gamma}\right). \quad (32c)$$

A confirmation of this result can be obtained in the following way: From (25) we deduce the differentials

$$d\,\lambda^E = -\frac{k_B}{m}\left[d\,(\ln n) + \frac{e}{nmc^2}\, d\,\gamma\right],\ d\,\lambda_\mu^E = \frac{k_B}{mc^2}\left(U_\mu\, d\,\gamma + \gamma\, d\, U_\mu\right).$$

The differential of (24) in the old variables is

$$d\, A_E^{\alpha\alpha_1\ldots\alpha_j} = -\frac{m}{k_B}\left(A_E^{\alpha\alpha_1\ldots\alpha_j}\, d\,\lambda^E + A_E^{\alpha\alpha_1\ldots\alpha_{j+1}}\, d\,\lambda_{\alpha_{j+1}}^E\right) =$$

$$= A_E^{\alpha\alpha_1\ldots\alpha_j}\left[d\,(\ln n) + \frac{e}{nmc^2}\, d\,\gamma\right] - \frac{1}{c^2}A_E^{\alpha\alpha_1\ldots\alpha_{j+1}}\left(U_{\alpha_{j+1}}\, d\,\gamma + \gamma\, d\, U_{\alpha_{j+1}}\right).$$

This must be equal to the differential of (24) with respect to the new variables, i.e.,

$$d\, A_E^{\alpha\alpha_1\ldots\alpha_j} = \frac{\partial\, A_E^{\alpha\alpha_1\ldots\alpha_j}}{\partial\, n}\, dn + \frac{\partial\, A_E^{\alpha\alpha_1\ldots\alpha_j}}{\partial\,\gamma}\, d\gamma + \frac{\partial\, A_E^{\alpha\alpha_1\ldots\alpha_j}}{\partial\, U_{\alpha_{j+1}}}\, d\, U_{\alpha_{j+1}};$$

By equating the coefficients of dn, dγ, dU_β, we see that the first one gives an identity, while the other 2 can be used to desume

$$\frac{U_{\alpha_{j+1}}}{c^2}\, A_E^{\alpha\alpha_1\ldots\alpha_{j+1}} = \frac{e}{nmc^2}A_E^{\alpha\alpha_1\ldots\alpha_j} - \frac{\partial A_E^{\alpha\alpha_1\ldots\alpha_j}}{\partial\,\gamma},$$

$A_E^{\alpha\alpha_1\ldots\alpha_j\beta}\, h_\beta^{\alpha_{j+1}} = \frac{c^2}{\gamma}\,\frac{\partial\, A_E^{\alpha\alpha_1\ldots\alpha_j}}{\partial\, U_\beta}\, h_\beta^{\alpha_{j+1}}$. From these it follows

$$A_E^{\alpha\alpha_1\ldots\alpha_{j+1}} = \frac{c^2}{\gamma}\,\frac{\partial A_E^{\alpha\alpha_1\ldots\alpha_j}}{\partial\, U_\beta}\, h_\beta^{\alpha_{j+1}} + \left(\frac{e}{nmc^2}\, A_E^{\alpha\alpha_1\ldots\alpha_j} - \frac{\partial\, A_E^{\alpha\alpha_1\ldots\alpha_j}}{\partial\,\gamma}\right) U^{\alpha_{j+1}}.$$

By using eq. (29) and (32c) we find that $A_E^{\alpha_1\ldots\alpha_{j+2}}$ is symmetric (as it must be). After that, we have that (29) holds with j+1 instead of j and with $\phi_{h,j+1}$ given by (32a).

4.2. The first order parts of the closure with respect to equilibrium

From eq. (20) we obtain $f - f_E = -\frac{1}{k_B}f_E\,\Delta\chi$ with

$$\Delta\chi = m\,(\lambda-\lambda^E) + (\lambda_\alpha - \lambda_\alpha^E)p^\alpha\left(1+\frac{I}{mc^2}\right) + \sum_{j=2}^{J}\frac{1}{m^{j-1}}\lambda_{\alpha_1\ldots\alpha_j}\, p^{\alpha_1}\ldots p^{\alpha_j}\left(1+\frac{I}{mc^2}\right)^j.$$

This result can be used to desume from (18) the linear departure of $A^{\alpha\alpha_1\ldots\alpha_J}$ from equilibrium

$$A^{\alpha\alpha_1\ldots\alpha_j} - A_E^{\alpha\alpha_1\ldots\alpha_j} = -\frac{m}{k_B}\left[(\lambda-\lambda^E)A_E^{\alpha\alpha_1\ldots\alpha_j} + \left(\lambda_{\alpha_{j+1}} - \lambda_{\alpha_{j+1}}^E\right)A_E^{\alpha\alpha_1\ldots\alpha_{j+1}} + \sum_{i=2}^{J}\lambda_{\alpha_{j+2}\ldots\alpha_{j+i}}\; A_E^{\alpha\alpha_1\ldots\alpha_j\alpha_{j+1}\ldots\alpha_{j+i}}\right]. \quad (33)$$

In particular this relation, for j=0,1 is

$$0 = A^\alpha - A_E^\alpha = -\frac{m}{k_B}\left[(\lambda-\lambda^E)A_E^\alpha + \left(\lambda_{\alpha_1} - \lambda_{\alpha_1}^E\right)A_E^{\alpha\alpha_1} + \sum_{i=2}^{J}\lambda_{\alpha_1\ldots\alpha_i}\; A_E^{\alpha\alpha_1\ldots\alpha_i}\right],$$

$$A^{\alpha\alpha_1} - A_E^{\alpha\alpha_1} = -\frac{m}{k_B}\left[(\lambda-\lambda^E)A_E^{\alpha\alpha_1} + \left(\lambda_{\alpha_2} - \lambda_{\alpha_2}^E\right)A_E^{\alpha\alpha_1\alpha_2} + \sum_{i=2}^{J}\lambda_{\alpha_2\ldots\alpha_{2+i}}\; A_E^{\alpha\alpha_1\alpha_2\ldots\alpha_{1+i}}\right]. \quad (34)$$

In the first one of these equations we have used the physical property that there isn't a linear deviation of four-velocity from equilibrim; for the second one we can use the physical property that there isn't a non equilibrium energy, i.e., $U_\alpha U_\beta\,(A^{\alpha\beta}-A_E^{\alpha\beta})=0$. This last condition, jointly with $(34)_1$ contracted by c^{-2} U_α gives a system from which we desume the values

$$\lambda - \lambda^E = -\frac{1}{D}\sum_{i=2}^{J} U_\alpha \lambda_{\alpha_1\ldots\alpha_i} \left(\phi_{0,2}\, c^2 A_E^{\alpha\alpha_1\ldots\alpha_i} - \frac{e}{c^2} U_\beta A_E^{\alpha\beta\alpha_1\ldots\alpha_i}\right), \quad (35)$$

$$(\lambda_\mu - \lambda_\mu^E)U^\mu = -\frac{1}{D}\sum_{i=2}^{J} U_\alpha \lambda_{\alpha_1\ldots\alpha_i} \left(mn\, U_\beta A_E^{\alpha\beta\alpha_1\ldots\alpha_i} - e\, A_E^{\alpha\alpha_1\ldots\alpha_i}\right),$$

Where $D = \begin{vmatrix} mnc^2 & e \\ e & \phi_{0,2}\, c^2 \end{vmatrix}$.

We contract now eq. $(34)_1$ with $-h_{\alpha\beta}$ so obtaining

$$(\lambda_\mu - \lambda_\mu^E)h^{\mu\beta} = \frac{1}{p}\sum_{i=2}^{J} h_\alpha^\beta\, \lambda_{\alpha_1\ldots\alpha_i} A_E^{\alpha\alpha_1\ldots\alpha_i}.$$

This result, jointly with $(35)_2$ allows us to obtain

$$\lambda_\alpha - \lambda_\alpha^E = -\sum_{i=2}^{J} \lambda_{\alpha_2\ldots\alpha_{i+1}} \left[\frac{U_\alpha}{D}\left(mn\, U_{\alpha_1} U_{\alpha_{i+2}} A_E^{\alpha_1\alpha_2\ldots\alpha_{i+2}} - e\, U_{\alpha_1} A_E^{\alpha_1\alpha_2\ldots\alpha_{i+1}}\right) + \frac{1}{p} h_{\alpha\alpha_1} A_E^{\alpha_1\alpha_2\ldots\alpha_{i+1}}\right]. \quad (36)$$

By substituting $(35)_1$ and (36) in (33), the dependence on $\lambda - \lambda^E$ and $\lambda_\alpha - \lambda_\alpha^E$ is there eliminated. In particular, $(34)_1$ becomes simply $\partial_\alpha (mnU^\alpha)=0$.

In this way the independent variables remain n, γ, U^α and $\lambda_{\alpha_1\ldots\alpha_i}$ fro i≥2.

4.3. A possible change of integration variables

We note that in the above expression (31) of $\phi_{k,j}$ there are two integrations, one over d I and the other over ds which is implicit in the definition of $J_{m,n}$. To try to reduce one of integrations, we introduce the following change of variables from (I, s) belonging to [0, + ∞[X [0, + ∞ [to (r , q) belonging to the domain $D = \{ r \in [0, +\infty[, 0 \le q \le \sinh r\}$: $s = settcosh \frac{\cosh r}{\sqrt{\cosh^2 r - q^2}}$, $I = m\, c^2\left(\sqrt{\cosh^2 r - q^2} - 1\right)$, (37)

The Iacobian of this transformation has absolute value $|J| = m\, c^2 \frac{\sinh r}{\sqrt{\cosh^2 r - q^2}}$.

It can be seen that the above transformation is invertible and its inverse is

$$r = settcosh\left[\left(1+\frac{I}{mc^2}\right)\cosh s\right], \quad q = \sinh r\left(1+\frac{I}{mc^2}\right),$$

By using this transformation, we can transform the following expression appearing in (31):

$$\frac{\int_0^{+\infty} J_{2k+2,j+1-2k}\left(1+\frac{I}{mc^2}\right)^j \varphi(I)\, d\, I}{\int_0^{+\infty} J_{2,1}\, \varphi(I)\, d\, I} = \quad (38)$$

$$= \frac{\int_0^{+\infty} dr \int_0^{\sinh r} e^{-\gamma\cosh r} q^{2k+2} \cosh^{j+1-2k} r \sinh r\, \eta(r,q) d\, q}{\int_0^{+\infty} dr \int_0^{\sinh r} e^{-\gamma\cosh r} q^2 \cosh r \sinh r\, \eta(r,q) d\, q}, \text{ with}$$

$$\eta(r,q) = \frac{\varphi\left[mc^2\left(\sqrt{\cosh^2 r - q^2} - 1\right)\right]}{(\cosh^2 r - q^2)^2}.$$

Now, in eq. (19) of [6], it has been shown that the measure φ(I) can be whatever funtion whose limit for c going to infinity is I^a.

- A possibility is to take directly $\varphi(I) = I^a$. In this case, in eq. (38) we can substitute $\eta(r,q) = \frac{\left(\sqrt{\cosh^2 r - q^2} - 1\right)^a}{(\cosh^2 r - q^2)^2}$. (39)
- Another possibility is to take

$\varphi(I) = \left(1+\frac{I}{mc^2}\right)^4 I^a$. In this case, in eq.(38) we can substitute $\eta(r,q) = \left(\sqrt{\cosh^2 r - q^2} - 1\right)^a$. (40)

In the particular case of a **diatomic gas** we have a=0 and eqs. (39), (40) become $\eta(r,q) = (\cosh^2 r - q^2)^{-2}$ and $\eta(r,q) = 1$ respectively. In the second case the integrations in d q can be easily calculated and (38) becomes

$$\frac{\int_0^{+\infty} J_{2k+2,j+1-2k}\left(1+\frac{I}{mc^2}\right)^j \varphi(I)\, d\, I}{\int_0^{+\infty} J_{2,1}\, \varphi(I)\, d\, I} = \frac{3}{2k+3}\,\frac{J_{2k+4,j+1-2k}(\gamma)}{J_{4,1}(\gamma)}. \quad (43)$$

It is evident that this expression can be written in terms of the modified Bessel functions

$K_n(\gamma) = \int_0^{+\infty} e^{-\gamma\cosh s} \cosh(ns)\, d\, s$ which, in turns, can be written in terms of K_2 and K_3 by means of the recurrence relation (29) of [6].

Also with the choice (39) the integrations in d q can be calculated but the result isn't so much elegant as in (43) and it isn't clear how it can be expressed in terms of the Bessel functions. To do this integration we need to calculate first the integral $\int_0^{\sinh r} \frac{q^{2k+2}}{(q^2 - \cosh^2 r)^2}\, d\, q$. (44)

This is calculated in the Appendix C. By using it (38) with the first choice becomes

$$\left(J_{2,1}(\gamma) - \int_0^{+\infty} r\, e^{-\gamma\cosh r} \sinh r\, d\, r\right)^{-1} \left(J_{2k+2,j+1-2k}(\gamma) - (2k+1)\int_0^{+\infty} r\, e^{-\gamma\cosh r} \cosh^j r \sinh r\, d\, r + \sum_{\eta=0}^{k-1} \frac{2k+1}{2\eta+1} J_{2\eta+2,j-\eta-1}\right),$$

which obviously cannot be expressed in terms of the Bessel functions. These results suggest a more physically meaningful approach; this is shown in the next subsection.

4.4. A more physically meaningful approach

We note that (18) remains unchanged if we use, instead of the variables p^α the expressions $p^{*\alpha}=p^\alpha(1+I/mc^2)$. These are more more physically significative because they take into account the fact that the moment of the particle has also a contribution from internal energy. Obviously, the new variables are linked by the condition

$$P^{\cdot\alpha} p_{\cdot\alpha} = m^2 c^2 \left(1+\frac{I}{m\, c^2}\right)^2 \ge m^2 c^2. \quad (45)$$

which represents a cone C of the 4-dimensional space. So we can multiply the Boltzmann-Chernikov equation (19) by $\frac{c}{m^{j-1}} p^{\cdot\alpha_1}\ldots p^{\cdot\alpha_j}\left(1+\frac{I}{m\, c^2}\right) I^a$ and integrate it in $d\,\vec{p}^{\cdot}$ over the cone C. So we obtain again $(23)_1$ but with $A^{\alpha\alpha_1\ldots\alpha_J} = \frac{c}{m^{j-1}}\int_C f\, p^{*\alpha} p^{*\alpha_1}\ldots p^{*\alpha_j} I^a\, d\,\vec{p}^{\cdot}$, (46)

$$I^{\alpha_1\ldots\alpha_J} = \frac{c}{m^{j-1}}\int_C Q\, p^{*\alpha_1}\ldots p^{*\alpha_j} I^a\, d\,\vec{p}^{\cdot},$$

Moreover, we obtain again (20), (21) but with

$$\chi = \sum_{j=0}^{J} \frac{1}{m^{j-1}} \lambda_{a_1\ldots a_j} p^{*\alpha_1}\ldots p^{*\alpha_j}, \quad h'^\alpha = -k_B\, c\int_C f\, p^{*\alpha} I^a\, d\,\vec{P}^*,$$

We call now m c q the modulus of the spatial component of $P^{\cdot\alpha}$ and m c cosh r its time-like component. In this way (45) becomes $\cosh^2 r - q^2 \ge 1$\$, i.e., $\cosh^2 r - 1 \ge q^2$ or, equivalently, $\sinh r \ge q \ge 0$. After that, by introducing spherical coordinates for the spatial components of $P^{\cdot\alpha}$, we have the parametrization

$P^{\cdot\alpha}$=mc(cosh r, q sin ϑ cos φ, q sin ϑ sin φ, q cos ϑ) with r ≥ 0, 0 ≤ ϑ ≤ π, 0 ≤ φ ≤ 2π, 0 ≤ q ≤ sinh r.

The Jacobian of this transformation is J= m^4c^4 sinh r q^2 sin ϑ. After that, we can integrate $(46)_1$ at equilibrium. It becomes as (24), except that now the integration is in $d\,\vec{p}^{\cdot}$, over the 4-dimensional domain C. For j=0,1 we obtain again (24) but with

$$n = 4\pi\, m^5 c^5 (mc^2)^a\, e^{-1-\frac{m\lambda^E}{k_B}} \quad (47)$$

$$\int_0^{+\infty} d\, r \int_0^{\sinh r} e^{-\gamma\cosh r} \cosh r \sinh r \left(\sqrt{\cosh^2 r - q^2} - 1\right)^a q^2\, d\, q,$$

$$\frac{e}{nmc^2} = \frac{\int_0^{+\infty} d\, r \int_0^{\sinh r} e^{-\gamma\cosh r} \cosh^2 r \sinh r \left(\sqrt{\cosh^2 r - q^2} - 1\right)^a q^2\, d\, q}{\int_0^{+\infty} d\, r \int_0^{\sinh r} e^{-\gamma\cosh r} \cosh r \sinh r \left(\sqrt{\cosh^2 r - q^2} - 1\right)^a q^2\, d\, q},$$

$$\frac{3p}{nmc^2} = \frac{\int_0^{+\infty} d\,r \int_0^{\sinh r} e^{-\gamma\cosh r} \sinh r \left(\sqrt{\cosh^2 r - q^2} - 1\right)^a q^2\, d\,q}{\int_0^{+\infty} d\,r \int_0^{\sinh r} e^{-\gamma\cosh r} \cosh r \sinh r \left(\sqrt{\cosh^2 r - q^2} - 1\right)^a q^2\, d\,q}\,.$$

To modify the last one of these expressions, we start from the known identity $J_{m,0}(\gamma) = \frac{m-1}{\gamma} J_{m-2,1}(\gamma)$ from which it follows

$$\int_0^{+\infty} ds \int_0^{+\infty} e^{-\gamma\left(1+\frac{I}{mc^2}\right)\cosh s} \sinh^m s \left(1+\frac{I}{mc^2}\right) \varphi(I)\; dI =$$

$$\frac{m-1}{\gamma} \int_0^{+\infty} ds \int_0^{+\infty} e^{-\gamma\left(1+\frac{I}{mc^2}\right)\cosh s} \sinh^{m-2} s \cosh s \left(1+\frac{I}{mc^2}\right) \varphi(I)\; dI\,.$$

If we use the change of integration variables (37) and use $\varphi(I) = \left(1+\frac{I}{mc^2}\right)^4 I^a$, as in the choice before eq. (40), we transform this identity in another one; in the particular case m=4 it becomes

$$\int_0^{+\infty} dr \int_0^{\sinh r} e^{-\gamma\cosh r} \sinh r \left(\sqrt{\cosh^2 r - q^2} - 1\right)^a q^4\, d\,q = \frac{3}{\gamma}\int_0^{+\infty} dr$$

$$\int_0^{\sinh r} e^{-\gamma\cosh r} \sinh r\; \cosh r \left(\sqrt{\cosh^2 r - q^2} - 1\right)^a q^4\, d\,q\,.$$

By using this identity we see that $(47)_3$ becomes $p = \frac{n\,m\,c^2}{\gamma}$ as in eq. (28). Also $(47)_{1,2}$ are equal to (25) and $(28)_2$, as it can be seen by using the integration variables (37) and $\varphi(I) = \left(1+\frac{I}{mc^2}\right)^4 I^a$.

For the other values of j, eq. $(46)_1$ at equilibrium becomes again (29) but with $\phi_{k,j} =$

$$= 4\,\pi \binom{j+1}{2k} \frac{1}{2k+1} c^{2k+5}\, m^6\, e^{-1-\frac{m\,\lambda^E}{k_B}} \int_0^{+\infty} dr \int_0^{\sinh r} e^{-\gamma\cosh r} \cosh^{j+1-2k} r$$

$$\sinh r\; (m\,c^2)^a \left(\sqrt{\cosh^2 r - q^2} - 1\right)^a q^{2k+2}\, d\,q\;,$$

instead of (30). Now we can use eq. $(47)_1$ to desume $e^{-1-\frac{m\,\lambda^E}{k_B}}$ and substitute it in the previous equation so that it becomes

$$\phi_{k,j} = m\,n\,c^{2k} \frac{1}{2k+1}\binom{j+1}{2k}\;.$$

$$\frac{\int_0^{+\infty} d\,r \int_0^{\sinh r} e^{-\gamma\cosh r} \cosh^{j+1-2k} r \sinh r \left(\sqrt{\cosh^2 r - q^2} - 1\right)^a q^{2k+2}\, d\,q}{\int_0^{+\infty} d\,r \int_0^{\sinh r} e^{-\gamma\cosh r} \cosh r \sinh r \left(\sqrt{\cosh^2 r - q^2} - 1\right)^a q^2\, d\,q}\,.$$

instead of (31). Obviously, in the particular case of a **diatomic gas**, where a=0, in all these scalar functions it is easy to integrate in dq and the results become expressed in terms of $J_{m,n}(\gamma)$; consequently, they can be expressed easily in terms of the Bessel functions.

Conclusion

In this article the reason have been identified why the so far known relativistic models for polytomic gases suffer from the drawback of a narrow zone of hyperbolicity: The approximations used there. Since some kind of approximations around equilibrium are still necessary, a method has been identified to make them in a consistent way. This allowed us to find a new model that is hyperbolic for any value of its independent variables, thus satisfying the principles of cause and effect and that Einstein's relativity. Finally, a new way of constructing relativistic balance equations has been identified which is physically more significant and which is more manageable for applications to particular cases, such as that of diatomic and ultra-relativistic gases. It also has the undoubted advantage of having all its constitutive scalar functions expressed in terms of energy and its derivatives with respect to temperature.

Appendix

5. A - An useful set of integrals above used

In the main text of this article we faced integrals of the type

$$\int_{R^3} d\,\vec{P} \int_0^{+\infty} e^{-1-\frac{\chi\varepsilon}{k_B}}\, p^{\alpha_1}\ldots p^{\alpha_n}\, \psi(I)\, d\,I = \qquad (49)$$

$$= e^{-1-\frac{m\lambda}{k_B}} \int_{R^3} d\,\vec{P} \int_0^{+\infty} e^{-\frac{1}{k_B}\lambda_\mu p^\mu\left(1+\frac{I}{mc^2}\right)}\, p^{\alpha_1}\ldots p^{\alpha_n}\, \psi(I)\, d\,I =$$

$$= e^{-1-\frac{m\lambda}{k_B}} \sum_{r=0}^{\left[\frac{n}{2}\right]} \phi_{r,n}(\lambda,\gamma)\; h^{(\alpha_1\alpha_2} \ldots h^{\alpha_{2r-1}\alpha_{2r}}\, l^{\alpha_{2r+1}} \ldots l^{\alpha_n)}\;,$$

where the expression in the right hand side is suggested by the Representation Theorems and

$$\gamma = \frac{m\,c}{k_B}\sqrt{\lambda_\alpha\lambda^\alpha}\;,\; l^\alpha = \frac{mc^2}{\gamma\,k_B}\lambda^\alpha\,.$$

By contracting (49) with $h_{\alpha_1\alpha_2}\ldots h_{\alpha_{2R-1}\alpha_{2R}}\, l_{\alpha_{2R+1}}\ldots l_{\alpha_n}$ we find that only the term with r=R survives in the summation and, moreover,

$$\phi_{r,n} = \binom{n}{2r}\frac{4\pi}{2r+1} m^{n+2}\, c^{2r+2}$$

$$\int_0^{+\infty} ds \int_0^{+\infty} e^{-\left(1+\frac{I}{mc^2}\right)\gamma\cosh s}\; \sinh^{2r+2} s\; \cosh^{n-2r} s\, \psi(I)\, ds\, dI\,,$$

where calculations have been performed in the reference frame where l^α has only the zero component, i.e., $l^\alpha \equiv$ (c , 0, 0, 0) and the same procedure used in [6] has been followed. The result can be written also as

$$\phi_{r,n} = \binom{n}{2r}\frac{4\pi}{2r+1} m^{n+2}\, c^{2r+2} \int_0^{+\infty} J_{2r+2,n-2r}(\gamma^{\cdot})\, \psi(I)\, dI\,.$$

6. B - A property used above

In the main text of this article we have used the following

THEOREM: Let us consider the function

$f_M(x) = \sum_{r=0}^{M} (-1)^r \frac{1}{r!} x^r$; it satisfies the properties: 1) If M is even, then $f_M(x) > 0\ \forall\, x$ while 2) if M is odd, then $\exists\ x_M^\cdot : f_M(x) > 0 \Leftrightarrow x < x_M^\cdot$.

Let us prove it with the iterative procedure on M.

- It is true for M=0 because $f_0(x) = 1$,
- It is true for M=1 because $f_1(x) = 1-x>0 \Leftrightarrow x<1$, so that $x_1^* = 1$.
- Let us suppose that it holds up to the number M=2p and prove that it holds also for M=2p+1 and M=2p+2.

In fact, we have $f'_{2p+1}(x) = -f_{2p}(x) < 0$ so that $f_{2p+1}(x)$ is a decreasing function. Since $\lim_{x\to-\infty} f_{2p+1}(x) = +\infty$ and $\lim_{x\to+\infty} f_{2p+1}(x) = -\infty$, our property is satisfied for M=2p+1. Let us prove it also for M=2p+2: We have $f'_{2p+2}(x) = -f_{2p+1}(x)$ so that $f_{2p+2}(x)$ is a decreasing function for $x < x^*_{2p+1}$ and an increasing function for $x > x^*_{2p+1}$.

Moreover, we have $f_{2p+2}(x) = f_{2p+1}(x) + \frac{x^{2p+2}}{(2p+2)!}$ so that $f_{2p+2}(x^*_{2p+1}) = \frac{(x^\cdot_{2p+1})^{2p+2}}{(2p+2)!} > 0$. In other words, $f_{2p+2}(x)$ decreases from + ∞ to the minimum value $\frac{(x^\cdot_{2p+1})^{2p+2}}{(2p+2)!} > 0$ and, after that, increases up to + ∞. It follows that $f_{2p+2}(x) > 0$ for every value of x and this concludes our proof.

7. C - Calculation of the integral (44)

Let us perform firstly an integration by parts so that it becomes

$$\frac{1}{2}\int_0^{\sinh r} \left(\frac{-1}{q^2-\cosh^2 r}\right)' q^{2k+1}\, d\,q =$$

$$= \frac{1}{2}\sinh^{2k+1} r + \frac{2k+1}{2}\int_0^{\sinh r} \frac{q^{2k}}{q^2-\cosh^2 r}\, d\,q.$$

So there remains to calculate the integral $\int_0^{\sinh r} \frac{q^{2k}}{q^2-\cosh^2 r}\, d\,q$. To this end we start from the well known identity

$$q^{2k} - \cosh^{2k} r == (q^2 - \cosh^2 r) \sum_{\eta=0}^{k-1} q^{2\eta} \cosh^{2k-\eta-2} r \;\rightarrow$$

$$\rightarrow \frac{q^{2k}}{q^2 - \cosh^2 r} = \frac{\cosh^2 r}{q^2 - \cosh^2 r} + \sum_{\eta=0}^{k-1} q^{2\eta} \cosh^{2k-\eta-2} r \,.$$

So we can now integrate and find $\int_0^{\sinh r} \frac{q^{2k}}{q^2 - \cosh^2 r}\, d\, q =$

$$= \int_0^{\sinh r} \frac{\cosh^{2k-1} r}{2} \left(\frac{1}{q - \cosh r} - \frac{1}{q + \cosh r} \right) d\, q + \sum_{\eta=0}^{k-1} \left| \frac{q^{2\eta+1}}{2\eta+1} \right|_0^{\sinh r}$$

$$\cosh^{2k-\eta-2} r = \frac{\cosh^{2k-1} r}{2} \left| \ln \left| \frac{q - \cosh r}{q + \cosh r} \right| \right|_0^{\sinh r} +$$

$$+ \sum_{\eta=0}^{k-1} \frac{1}{2\eta+1} \sinh^{2\eta+1} r \cosh^{2k-\eta-2} r = \frac{\cosh^{2k-1} r}{2} \ln \left| \frac{\sinh r - \cosh r}{\sinh r + \cosh r} \right| +$$

$$+ \sum_{\eta=0}^{k-1} \frac{1}{2\eta+1} \sinh^{2\eta+1} r \cosh^{2k-\eta-2} r = - r \cosh^{2k-1} r +$$

$$+ \sum_{\eta=0}^{k-1} \frac{1}{2\eta+1} \sinh^{2\eta+1} r \cosh^{2k-\eta-2} r \,,$$

where we have used the property

$$\ln \left| \frac{\sinh r - \cosh r}{\sinh r + \cosh r} \right| = \ln \left| \frac{-e^{-r}}{e^r} \right| = \ln (e^{-2r}) = -2\, r.$$

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The author declares that there is no conflict of interest. He has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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