Uniform distribution on fractals

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Abstract

In this paper we introduce a general algorithm to produce u.d. sequences of partitions and of points on fractals generated by an IFS consisting of similarities which have the same ratio and which satisfy the open set condition (OSC). Moreover we provide an estimate for the elementary discrepancy of van der Corput type sequences constructed on this class of fractals.

Introduction

In this paper we will extend to certain fractals the concept of uniformly distributed (u.d.) sequences of partitions introduced by Kakutani in 1976 for the interval $[0,1]$ (cfr. [9]). Moreover, we will study in this setting the relation which has been recently established in [15] for $[0,1]$ between u.d. sequences of partitions and the classical theory of u.d. sequences of points, a theory which goes back to Weyl, [16].

The classical concept of u.d. sequences of points is more natural when we deal with the interval $[0,1]$ and with manifolds. On the other hand, when we work on fractals, in particular with fractals generated by Iterated Function Systems (IFS), partitions become a convenient tool for introducing a uniform distribution theory.

The advantage of considering partitions was implicitely used by Grabner and Tichy in [6] and by Cristea and Tichy in [3] even if they treated u.d. sequences of points. In these papers various concepts of discrepancies were introduced on the planar Sierpiński gasket and on the multidimensional Sierpiński carpet respectively, by using different kinds of partitions on these two fractals. In [6] an analogon of the classical van der Corput sequence has been constructed on the planar Sierpiński gasket. Similarly, in a succesive paper of Cristea, Pillichshammer, Pirsic and Scheicher [2] a sequence of van der Corput type has been defined on the s-dimensional Sierpiński carpet by exploiting the IFS-addresses of the carpet points. In all these papers the authors gave estimates for the *elementary* discrepancy of van der Corput type sequences, finding that is of the order $\mathcal{O}\left(\frac{1}{N}\right)$.

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The idea to study this special kind of sequences in relation to uniform distribution on IFS fractals is the starting point for this paper, too. In fact, we will introduce a procedure to define u.d. sequences of partitions and of points on the class of fractals generated by a system of similarities on $\mathbb{R}^d$ having the same ratio c and verifying the open set condition. Moreover, our estimate for the elementary discrepancy in this wider class of fractals is of the order $\mathcal{O}\left(\frac{1}{N}\right)$, too. Our results include those due to Cristea, Pillichshammer, Pirsic and Scheicher in [2], giving a more trasparent proof and taking in consideration the whole class of fractals described above.

The choice of the elementary discrepancy is convenient because the family of elementary sets is obtained in the most natural way by the construction of the fractal and because the elementary sets can be constructed for every IFS fractal regardless of the complexity of its geometric structure.

Let us give a brief outline of the paper.

In Section 1 we introduce some basic definitions and some preliminaries on uniform distribution in compact Hausdorff spaces. The books [10] and [4] are excellent general references.

In Section 2 we recall some notions about IFS fractals and we present the main results of this paper. In fact, we prove in Theorem 2.5 and in Theorem 2.6, that it is possible to construct explicitly u.d. sequences of partitions and of points on selfsimilar fractals generated by similarities with the same ratio and which satisfy the open set condition. This approach also allows to produce a different construction of the known van der Corput sequences on $[0, 1]$, using an observation which goes back to Mandelbrot.

Finally, in Section 3 we provide an estimate for the elementary discrepancy of the sequences of van der Corput type generated by our explicit algorithm and this result is presented in Theorem 3.1 .

1 Preliminaries

Let X be a compact Hausdorff space and let us denote by $\mathcal{B}$ the σ-algebra of Borel subsets of X. Suppose μ is a regular probability on $\mathcal{B}$. By $\mathcal{C}(X)$ we will denote the continuous real valued functions defined on X.

Definition 1.1.
A sequence $\{x_i\}$ of elements in X is said to be uniformly distributed (u.d.) with respect to μ, *if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(x_i) = \int_X f(t)\ d\mu(t)$$

for all $f \in \mathcal{C}(X)$.

Definition 1.2.
*A Borel set $M \subset X$ is called a μ-*continuity set *if $\mu(\partial M) = 0$, where ∂M denotes the boundary of M with respect to the topology on X.*

Definition 1.3.
Let $\{\pi_n\}$ be a sequence of partitions of X, where $\pi_n = \left\{A_1^n, A_2^n, \ldots, A_{k(n)}^n\right\}$ and the A_i^n's are μ-continuity sets. The sequence $\{\pi_n\}$ is said to be μ-uniformly distributed if for any $f \in \mathcal{C}(X)$, and any choice $t_i^n \in A_i^n$ we have

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} f(t_i^n) = \int_X f(t)\ d\mu(t).$$

The existence of u.d. sequences of partitions in separable metric spaces has been addressed, but not completely solved, in [1]. However, this will not be an issue in this paper, since known results on fractals will assure us the existence of all the partitions we need.

Given a class of real valued integrable functions $\mathcal{F}$, we say that it is *determining* for the uniform distribution of sequences of points if for any sequence $\{x_i\}$ in X the validity of the relation

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(x_i) = \int_X f(t)\ d\mu(t)\,, \tag{1}$$

for all $f \in \mathcal{F}$ already implies that $\{x_i\}$ is u.d..

Similarly, we say that $\mathcal{F}$ is *determining* for the uniform distribution of sequences of partitions if for any sequence $\{\pi_n\}$, where $\pi_n = \left\{A_1^n, A_2^n, \ldots, A_{k(n)}^n\right\}$, the validity of the relation

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} f(t_i^n) = \int_X f(t)\ dt$$

for all $f \in \mathcal{F}$ and for any choice $t_i^n \in A_i^n$ already implies that $\{\pi_n\}$ is u.d..

Observe the determining classes for the sequences of points play the same role for the sequences of partitions and viceversa.

For a family of real valued functions $\mathcal{F}$, we will denote by *span*$(\mathcal{F})$ the linear space generated by $\mathcal{F}$ and by $\overline{span(\mathcal{F})}$ its closure. The construction of many important determining classes is based on the following theorem.

Theorem 1.4. *If $\mathcal{F}$ is a class of real valued functions defined on X such that (1) holds for all $f \in \mathcal{F}$ and $\overline{\mathrm{span}(\mathcal{F})} \supset \mathcal{C}(X)$, then $\mathcal{F}$ is a determining class.*

It is also convenient to define determining classes of sets: if $\mathcal{G}$ is a class of sets, we say that it is determining if the corresponding class of characteristic functions $\{\chi_G :\ G \in \mathcal{G}\}$ is determining.

One more important definition is needed.

Definition 1.5.
Let $\mathcal{G}$ be a determining class of μ-continuity sets and $\omega = (x_1, \dots, x_N)$ a finite set of points in X. Then the discrepancy with respect to $\mathcal{G}$ (or $\mathcal{G}-$discrepancy) is defined by

$$D_N^{\mathcal{G}}(\omega) = \sup_{A \in \mathcal{G}} \left| \frac{1}{N} \sum_{i=1}^{N} \chi_A(x_i) - \mu(A) \right|.$$

In particular situations traditional terms are used instead of $\mathcal{G}$-discrepancy. For instance, if $X = [0,1]$ and $\mathcal{G}$ is the class of all subintervals of $[0,1]$, it is simply called *discrepancy*, while the term *star-discrepancy* is used if $\mathcal{G}$ is reduced to the class of all intervals of the type $[0, a[$.

If $\{y_n\}$ is a sequence of points, we associate to it the sequence of positive real numbers $D_N^{\mathcal{G}}(Y_N)$, where $Y_N = \{y_1, y_2, \dots y_N\}$. It follows from the definition that $\{y_n\}$ is u.d. if and only if $D_N^{\mathcal{G}}(Y_N)$ tends to zero when N tends to infinity.

In the next section we will introduce the van der Corput sequences on $[0,1]$. They play an important role because they are the "best distributed" sequences in the sense that their discrepancy is the smallest possible. In fact, Schmidt proved that in general the order $\mathcal{O}\left(\frac{\log N}{N}\right)$ cannot be improved, [13].

2 Fractals and Van der Corput sequences

From now on we will be concerned with uniform distribution on a special class of fractals, namely those which are generated by an Iterated Function System (IFS) of similarities having the same ratio $c \in]0,1[$ and satisfying the Open Set Condition (OSC).

Let us denote by $\mathcal{K}(\mathbb{R}^d)$ the space of all the non-empty compact subsets of $\mathbb{R}^d$ endowed with the Hausdorff distance, which makes it complete.

Let $\psi_1, \dots, \psi_m$ be similarities on $\mathbb{R}^d$ with $\|\psi_i(x) - \psi_i(y)\| = c\|x - y\|$, for all $x, y \in \mathbb{R}^d$, with $0 < c < 1$.

The unique fixed point F of the contraction $\psi(E) \mapsto \bigcup_{i=1}^{m} \psi_i(E)$ is called the *attractor* of the IFS, [8]. The set F is called a *self-similar* set and we have

$$F = \bigcup_{i=1}^{m} \psi_i(F).$$

Moreover, if $F_0 \in \mathcal{K}(\mathbb{R}^d)$ is such that $\psi_i(F_0) \subset F_0$ for $1 \le i \le m$, then the sequence of iterates $\{\psi^n(F_0)\}$ is decreasing and convergent to F in the Hausdorff metric as $n \to \infty$, with

$$F = \bigcap_{n=0}^{\infty} \psi^n(F_0)$$

(where $\psi^0(F_0) = F_0$ and $\psi^{n+1}(F_0) = \psi(\psi^n(F_0))$ for $n \ge 0$).
The set $F_0 = \psi^0(F_0)$ is called *initial set* and the iterates $\psi^n(F_0)$ are called *pre-fractals* for F.

Definition 2.1 (OSC).
A class of similarities $\psi_1, \ldots, \psi_m \colon \mathbb{R}^d \to \mathbb{R}^d$ satisfies the open set condition *if there exists a non-empty bounded open set V such that*

$$V \supset \bigcup_{i=1}^{m} \psi_i(V)$$

where $\psi_i(V)$ are pairwise disjoint.

If OSC holds the Hausdorff dimension of the attractor F is $s = -\frac{\log m}{\log c}$ and its s-dimensional Hausdorff measure $\mathcal{H}^s$ is positive and finite (cfr. [12], [5]).

The OSC ensures that the components $\psi_i(F)$ of the invariant set F cannot overlap too much. In fact, the following result holds (cfr. [8]).

Theorem 2.2. *Let $\psi_1, \ldots, \psi_m$ be similarities on $\mathbb{R}^d$ with ratio $0 < c < 1$ and let F be the attractor. If the OSC holds, then $\mathcal{H}^s(\psi_i(F) \cap \psi_j(F)) = 0$ for $i \neq j$.*

Our class of fractals includes the most popular kind of fractals as for instance the Cantor set, the Sierpiński triangle, the Sierpiński carpet, the von Koch curve and so on. But also $[0, 1]$ can be seen as the attractor of an IFS, in fact of infinitely many IFS's.

Fix a positive integer m and consider the mappings $\varphi_1, \ldots, \varphi_m$, from $\mathbb{R}$ to $\mathbb{R}$, where

$$\varphi_k(x) = \frac{k-1}{m} + \frac{x}{m}\ , \text{ for } 1 \leq k \leq m. \tag{2}$$

Then $[0, 1]$ is the attractor of this IFS. This observation goes back to Mandelbrot (cfr. [11]).

Now, we will see how this observation can be used to define on the kind of fractals we are considering (and also on $[0, 1]$) the van der Corput sequences of base $m > 1$.

Let $\psi = \{\psi_1, \ldots, \psi_m\}$ be our IFS and F its attractor. Assume that F_0 is the initial set such that

$$\psi_i(F_0) \subset F_0 \text{ for } i = 1, \ldots, m.$$

Fix a point $x_0 \in F$ and apply $\psi_1, \ldots, \psi_m$ in that order to x_0 getting so the points $x_1, \ldots, x_m$. At the second step, we apply the m mappings first to x_1, then to x_2 and so on, getting finally m^2 points ordered in a precise manner. Now we keep going, applying the functions of the IFS first to x_1 and continue so until we reach the point x_{m^2}, getting so m^3 points in the order determined by the construction. Iterating this procedure we get a sequence of points $\{x_n\}$ on F which will be called the van der Corput sequence.

Observe that if $[0, 1]$ is seen as the attractor of the IFS described in (2), and if $x_0 = 0$, this is the classical van der Corput sequence (cfr. [14], [7], [10]).

Let us come back to the general situation, showing now how a similar construction produces sequences of uniformly distributed partitions.

Observe that if we apply the ψ_i's to F, in the same order of before, we construct a sequence $\{\pi_k\}$ of partitions of F

$$\pi_k = \big\{\psi_{j_k}\psi_{j_{k-1}}\cdots\ \psi_{j_1}(F) : j_1, \ldots, j_k \in \{1, \ldots, m\}\big\}.$$

Each of the m^k sets E_j^k of the partition π_k contains exactly one point of the van der Corput sequence $\{x_n\}$ constructed above for $n = m^k$. We order the sets E_j^k accordingly.

Let us denote by $\mathscr{E}_k$ the collection of the m^k sets in π_k and by $\mathscr{E}$ the union of the families $\mathscr{E}_k$, for $k \in \mathbb{N}$. The sets of the class $\mathscr{E}$ are called *elementary sets.* Consider on F the normalized s-dimensional Hausdorff measure P, i.e.

$$P(A) = \frac{\mathcal{H}^s(A)}{\mathcal{H}^s(F)} \text{ for any Borel set } A \subset F$$

which is a regular probability.

Lemma 2.3. *The elementary sets are P-continuity sets.*

Proof.
Consider an elementary set $E_i = \psi_i(F) \in \mathscr{E}_1$. Let $x \in \partial E_i$. By definition, every neighbourhood U of x in the relative topology is such that $U \cap E_j \neq \emptyset$ for some $j \in \{1, 2, \ldots, m\}$ and $j \neq i$. But each E_j is closed, therefore $x \in E_j$. Hence ∂E_i is contained in $\bigcup\limits_{\substack{j=1\\ j\neq i}}^{m} (E_i \cap E_j)$. By Theorem 2.2, we have

$$\begin{aligned} 0 &\leq \mathcal{H}^s(\partial E_i) \leq \mathcal{H}^s\left(\bigcup_{\substack{j=1\\ j\neq i}}^{m} (E_i \cap E_j)\right) = \mathcal{H}^s\left(\bigcup_{\substack{j=1\\ j\neq i}}^{m} \big(\psi_i(F) \cap \psi_j(F)\big)\right) \\ &\leq \sum_{\substack{j=1\\ j\neq i}}^{m} \mathcal{H}^s\big(\psi_i(F) \cap \psi_j(F)\big) = 0. \end{aligned}$$

Now, a generic elementary set $A \in \mathscr{E}_k$ with $k \geq 2$ is a homothetic image of an elementary set in $\mathscr{E}_1$ and therefore $\mathcal{H}^s(\partial A) = 0$, too.

□

Lemma 2.4. *The class consisting of all elementary sets is determining.*

Proof.
Let $\mathcal{M}$ be the class consisting of all characteristic functions of sets $E \in \mathscr{E}$ and $f \in \mathcal{C}(F)$. By uniform continuity, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x') - f(x'')| < \varepsilon$ whenever $\|x' - x''\| < \delta$. Chose $n \in \mathbb{N}$ such that every $E_k^n \in \mathscr{E}_n$ has diameter smaller than δ. Take for any $E_k^n \in \mathscr{E}_n$ a point t_k and consider the function

$$g(y) = \sum_{k=1}^{m^n} f(t_k)\chi_{E_k^n}(y), \quad y \in F.$$

If $y \in E_k^n$, then $|g(y) - f(y)| = |f(t_k) - f(y)| < \varepsilon$.
Hence, $\mathrm{span}(\mathcal{M})$ is uniformly dense in $\mathcal{C}(F)$ and the conclusion follows by Theorem 1.4.

□

Theorem 2.5. *The sequence $\{\pi_n\}$ of partitions of F generated by the algorithm is u.d. with respect to the probability P.*

Proof.
Let us fix $E_h^k \in \pi_k$. By the previous lemma, we have to show that

$$\lim_{n\to\infty} \frac{1}{m^n} \sum_{j=1}^{m^n} \chi_{E_h^k}(t_j^n) = \int_F \chi_{E_h^k}(t)\, dP(t)$$

for every $E_j^n \in \pi_n$ and for every choice of $t_j^n \in E_j^n$. This is equivalent to

$$\lim_{n\to\infty} \frac{1}{m^n} \sum_{j=1}^{m^n} \chi_{E_h^k}(t_j^n) = \frac{1}{m^k}$$

because

$$\int_F \chi_{E_h^k}(t)\, dP(t) = P(E_h^k) = c^{sk} P(F) = c^{sk} = \frac{1}{m^k}.$$

Now, observe that for $n > k$, among the m^k sets generated by the algorithm, exactly one set of π_n is contained in the fixed set E_h^k. Since there are m^{n-k} sets of π_n which are contained in E_h^k, then

$$\lim_{n\to\infty} \frac{1}{m^n} \sum_{j=1}^{m^n} \chi_{E_h^k}(t_j^n) = \frac{m^{n-k}}{m^n} = \frac{1}{m^k}.$$

□

Theorem 2.6. *The sequence $\{x_i\}$ of points of F generated by the algorithm is u.d. with respect to P.*

Proof. By Lemma 2.4, the class $\mathscr{E}$ is determining. Hence, for a fixed set $E \in \mathscr{E}_k$, we have to prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \chi_E(x_i) = \int_E \chi_E\, dP = \frac{1}{m^k}. \tag{3}$$

Let $m^t \leq N < m^{t+1}$, then

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N}\chi_E(x_i) &= \frac{1}{N}\sum_{i=1}^{m+m^2+\ldots+m^{t-1}}\chi_E(x_i) + \frac{1}{N}\sum_{i=\frac{m^t-m}{m-1}}^{N}\chi_E(x_i) \\
&= \frac{\left(\frac{m^t-m}{m-1}\right)}{N}\cdot\frac{1}{\left(\frac{m^t-m}{m-1}\right)}\sum_{i=1}^{m+m^2+\ldots+m^{t-1}}\chi_E(x_i) \\
&+ \frac{N-\left(\frac{m^t-m}{m-1}\right)}{N}\cdot\frac{1}{N-\left(\frac{m^t-m}{m-1}\right)}\sum_{i=\frac{m^t-m}{m-1}}^{N}\chi_E(x_i) \qquad (4)
\end{aligned}$$

since $1+m+m^2+\ldots+m^{t-1} = \frac{m^t-1}{m-1}$.
Observe that for $i > \frac{m^t-1}{m-1}$, because of the order of the points x_i, among the first m^k points exactly one point of the sequence $\{x_i\}$ is contained in the fixed set E. Hence, for $t \to \infty$ we have

$$\frac{1}{\left(\frac{m^t-m}{m-1}\right)}\sum_{i=1}^{m+m^2+\ldots+m^{t-1}}\chi_E(x_i) \to \frac{1}{m^k}. \qquad (5)$$

Writing N as $N = \left(\frac{m^t-m}{m-1}\right) + Mm^k + r$ with $0 \leq r < m^k$, we have

$$\begin{aligned}
\frac{1}{N-\left(\frac{m^t-m}{m-1}\right)}\sum_{i=\frac{m^t-m}{m-1}}^{N}\chi_E(x_i) &= \frac{Mm^k}{N-\left(\frac{m^t-m}{m-1}\right)}\cdot\frac{1}{Mm^k}\sum_{i=\frac{m^t-m}{m-1}}^{N-r}\chi_E(x_i) \\
&+ \frac{r}{N-\left(\frac{m^t-m}{m-1}\right)}\cdot\frac{1}{r}\sum_{i=N-r+1}^{N}\chi_E(x_i). \qquad (6)
\end{aligned}$$

By the previous remarks we obtain that

$$\frac{1}{Mm^k}\sum_{i=\frac{m^t-m}{m-1}}^{N-r}\chi_E(x_i) = \frac{1}{m^k},$$

while for $N \to \infty$ and hence for $t \to \infty$ we have

$$\frac{r}{N-\left(\frac{m^t-m}{m-1}\right)}\sum_{i=N-r+1}^{N}\chi_E(x_i) \to 0$$

because $0 \leq \frac{r}{N} < \frac{m^k}{m^t}$.
Using the last two relations in (6) and taking the limit for $N \to \infty$ (and hence for $t \to \infty$) we have

$$\frac{1}{N-\left(\frac{m^t-m}{m-1}\right)}\sum_{i=1}^{N}\chi_E(x_i) \to \frac{1}{m^k}. \qquad (7)$$

Finally, (4) is a convex combination of two terms which both tend to $\frac{1}{m^k}$ for $N \to \infty$ by (5) and (7) and therefore the conclusion (3) holds.

$\square$

3 Order of convergence of the elementary discrepancy

In this section, F and P are as defined previously. The next theorem evaluates the elementary discrepancy.

Theorem 3.1. *Let $\{x_i\}$ be the sequence of points generated on F by the algorithm described in the previous section and let $N \geq 1$. Then the elementary discrepancy of the sequence $\omega = (x_1, \dots, x_N)$ is*

$$D_N^{\mathscr{E}}(\omega) = \frac{1}{N}.$$

Proof.
The lower bound is trivial. In fact, if we fix $k \in \mathbb{N}$, then

$$D_N^{\mathscr{E}_k}(\omega) \geq \frac{1}{N} - \frac{1}{m^k}.$$

In order to find an upper bound for $D_N^{\mathscr{E}}(\omega)$ let us consider $D_N^{\mathscr{E}_k}(\omega)$ for any $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$ and let $E \in \mathscr{E}_k$.

$$D_N^{\mathscr{E}_k}(\omega) = \sup_{E \in \mathscr{E}_k} \left| \frac{1}{N} \sum_{i=1}^{N} \chi_E(x_i) - \frac{1}{m^k} \right|.$$

Among the first m^k points of the sequence $\{x_i\}$ exactly one point is contained in the fixed set E because of the special order induced by the algoritm.

Let us distinguish two different cases:

1. For $N \leq m^k$, the set E contains at most one point of ω. Hence

$$\left| \frac{1}{N} \sum_{i=1}^{N} \chi_E(x_i) - \frac{1}{m^k} \right| = \max \left\{ \left| \frac{1}{N} - \frac{1}{m^k} \right|, \left| 0 - \frac{1}{m^k} \right| \right\} \leq \frac{1}{N}.$$

2. If $N > m^k$, we can write N as follows

$$N = Q \cdot m^k + r \text{ with } \ 0 \leq r < m^k \text{ and } Q \geq 1.$$

   Therefore, every $E \in \mathscr{E}_k$ contains either Q points or (r of them) $Q + 1$ points and hence

$$\left| \frac{1}{N} \sum_{i=1}^{N} \chi_E(x_i) - \frac{1}{m^k} \right| \leq \max \left\{ \left| \frac{Q}{N} - \frac{1}{m^k} \right|, \left| \frac{Q+1}{N} - \frac{1}{m^k} \right| \right\}.$$

   Note that

$$\left| \frac{Q}{N} - \frac{1}{m^k} \right| = \left| \frac{Qm^k - N}{Nm^k} \right| = \left| \frac{-r}{Nm^k} \right| < \frac{m^k}{Nm^k} = \frac{1}{N},$$

while

$$\left|\frac{Q+1}{N}-\frac{1}{m^k}\right| = \left|\frac{Qm^k+m^k-N}{Nm^k}\right| = \left|\frac{m^k-r}{Nm^k}\right| = \left|\frac{1}{N}-\frac{r}{Nm^k}\right| < \frac{1}{N}.$$

So we have that

$$\left|\frac{1}{N}\sum_{i=1}^{N}\chi_E(x_i)-\frac{1}{m^k}\right| < \frac{1}{N}.$$

Hence, it follows that for any $k \in \mathbb{N}$ we have $D_N^{\mathscr{E}_k}(\omega) < \frac{1}{N}$. This implies that $D_N^{\mathscr{E}}(\omega) \leq \frac{1}{N}$, as we wanted to prove.

□

4 Conclusions

This paper leaves open several questions.

The most relevant concerns a unified approach to the discrepancy on a wider class of fractals. This appears to be difficult due to different geometrical features of the various fractals.

Another problem concerns the extension of our results to IFS with OSC consisting of similarities which do not have the same ratio, and of course to more general fractals.

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