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ORIGINAL PAPER

Electrodynamics from the viewpoint of modern continuum theory—A review

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This paper wants to draw attention to several issues in electrodynamic field theory and to make way for a rational continuum approach to the subject. The starting point are the balances for magnetic flux and electric charge, both in a very general formulation for volumes and for open surfaces, all of which can deform and be immaterial or material. The spatial point-of-view for the description of fields is favored and its advantages in comparison to the concept of material particles is explained. A straightforward answer to the question of how to choose units for the electromagnetic fields most suitably is also presented. The transformation properties of the electromagnetic fields are addressed by rewriting the balances in space–time notation. Special attention is paid to the connection between the two sets of electromagnetic fields through the so-called Maxwell–Lorentz–æther relations. The paper ends with an outlook into constitutive theory of matter under the influence of electromagnetic fields and a discussion on curious developments in context with Maxwell's equations.

1 | INTRODUCTION AND OUTLINE TO THE PAPER

The question as to whether the world is discrete or continuous in nature dates back to the Greek philosophers and before. Whatever the answer may be, there is one thing for certain: The continuum viewpoint has a clear mathematical advantage because it can make use of the powerful tools of tensor analysis and calculus.

Typically, students of theoretical physics are introduced to the notion of continuous fields for the first time in a class on electrodynamics. On the other hand, mechanical engineering students encounter fields in lectures on fluid mechanics or, more general, in continuum mechanics. The latter covers all states of matter and materials. Here, they learn to distinguish between the general conservation laws of physics, which are expressed in terms of balances of certain physical quantities, independently of the considered material, and constitutive relations. In contrast to the former, constitutive equations are not universal. They are material-specific and complement the balances. Principles, such as material frame indifference or the second law of thermodynamics, are used to decrease the wealth of possible forms of the constitutive relations. In the end, one arrives at field equations, which can then be analyzed by using various techniques for solving partial differential equations.

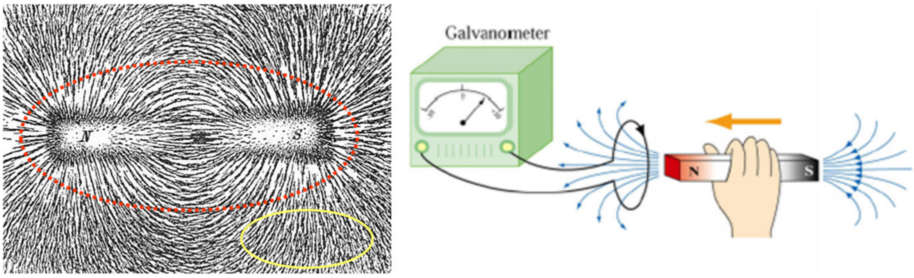

FIGURE 1 Experiments demonstrating the conservation of magnetic flux [13, 14]

It is fair to say that electrodynamics is still underway to make full use of the concepts and techniques established by modern continuum mechanics, although the foundations of rational electrodynamics were laid more than 60 years ago in the classic handbook text on continuum theory by Truesdell and Toupin [1], Chapter F]. Not many books were written in the same spirit. Noteworthy exceptions are, for example, [2–5]. Only in the latest edition of the famous textbook on electrodynamics by Jackson [6, Section 5.15], a footnote on general transport theorems for open deformable surfaces has been added in context with Faraday's law of induction. Other classical physics textbooks use the concept of balances for Maxwell's equations in a very rudimentary form (e.g., [7, Chapter 4], [8, Section 38, Section 45], [9, Chapter 4]), such that the considered volumes and open surfaces cannot deform at all or do not move. The same holds for modern textbooks of theoretical electrical engineering, for example, [10, Chapter 2].

The purpose of this paper is mostly didactic. Our intention is to create awareness of the rather abstract ideas of rational electrodynamics and to provide a bridge from theoretical physics to engineering. Therefore, in a way, we follow up on a quote in Truesdell and Toupin [1, pg. 668]: "We trust that our rather abstract postulation of the laws of conservation of charge and magnetic flux will not discourage the reader seeking more concrete and familiar results. Our main reason for this approach is to emphasize the independence of the conservation laws from any geometry of space–time. The ideas of conservation as formulated here have, in a certain sense, topological significance, transcending both the intuition and the mathematics of length, time, and angles." The paper is organized as follows:

- Maxwell's equation will be introduced as consequences of global conservation laws/balances, first, for the magnetic flux and, second, for electric charge, in an inertial frame.
- After that, the question of electromagnetic units can be addressed because the balances are unique up to certain constants.
- In the same context, it will be emphasized that the strengths of the four electromagnetic fields can directly be compared if Gaussian units are used.
- All fields are then re-introduced by rewriting the two sets on Maxwell's equations in space–time notation and by postulating a worldvector for charge and two worldtensors, the electromagnetic field tensor and the charge-current potential, with corresponding transformation properties in space–time.
- The Maxwell–Lorentz–æther relations will be introduced connecting the four electromagnetic fields with each other. It will be emphasized and shown that this relation becomes particularly simple for the class of Lorentz space-time transformations.
- Then interface conditions will be considered and it will be asked how the corresponding jump conditions look like. The question will be answered by deriving the local Maxwell equations in singular points.
- The paper ends with an first glance at constitutive equations and discusses with an open mind, albeit critically, "alternative views" on electrodynamics that were currently presented in the literature.

2 | THE LAW OF INDUCTION OR THE FIRST SET OF MAXWELL'S EQUATIONS

2.1 | The experimental evidence

Consider the situation shown in Figure 1 on the left: Due to a bar magnet under a sheet of paper on which iron filings were distributed, forces and torques are acting on these filings so that they orientate themselves along "lines of force." In fact, this is the term introduced by Faraday and Maxwell in the early works on electromagnetic phenomena, [11, pg. 295], [12, Sect. 541, pg. 175]. Today, we refer to these structures as magnetic field lines, $\boldsymbol{B}$. We are tempted to say that they "emerge" at one end of the magnet and "disappear" at the other end. Hence, if we choose first a control volume v^{s} with a closed[1] surface $\partial v^{\mathrm{s}}(t)$ outside the magnet (see the yellow contour in Figure 1, left inset), which may deform in time but otherwise is a mathematical object of imagination, it is straightforward to conclude that

$$\oint_{\partial v^{\mathrm{s}}(t)} \boldsymbol{B}(\boldsymbol{x}^{\mathrm{s}}, t) \cdot \boldsymbol{n}(\boldsymbol{x}^{\mathrm{s}}, t)\, \mathrm{d}a^{\mathrm{s}} = 0, \tag{1}$$

if $\boldsymbol{n}$ is the outward normal to that closed surface with surface element $\mathrm{d}a^{\mathrm{s}}$. The circle in the (surface) integral indicates that the boundary of the region of the region v^{s} is closed, which, in addition, is symbolically indicated by the "∂" symbol. A similar convention will be applied for the closed contour of open surfaces.

The functional dependence of the two fields, $\boldsymbol{B}$ and $\boldsymbol{n}$, require some explanation. As it will be explained in Appendix A in more detail, $\boldsymbol{x}^{s}$ refers to the position of a grid cell at a fixed observer clock time t, all chosen with respect to the laboratory frame. It is assumed that the cells of the chosen grid do not move nor deform in time.[2] This is what is known as the *spatial* viewpoint in continuum fluid mechanics. For that reason, the label "s" has been added to the position vector. It has also been added to the closed surface of the control volume, ∂v^{s}, because it encompasses a set of observational points, which all belong to the chosen spatial grid. Of course, the control volume (and its closed surface) can move and deform in time, $\partial v^{\mathrm{s}}(t)$. Therefore, the set of points encompassed by the control volume will change as time passes. Mathematically, we describe that temporal change by a bijective mapping χ^{s} as follows,

$$\boldsymbol{x}^{\mathrm{s}} = \chi^{\mathrm{s}}(\boldsymbol{X}^{\mathrm{s}}, t), \tag{2}$$

which, following solid continuum mechanics conventions, we shall call the *motion* of immaterial points belonging to the control volume $v^{\mathrm{s}}(t)$. $\boldsymbol{X}^{\mathrm{s}}$ is the position of an immaterial observational point at some initial time $t = 0$, which we may want to refer to as *reference placement*. Clearly, this allows us to assign a fictitious velocity to all the observational points within $v^{\mathrm{s}}(t)$ by

$$\boldsymbol{v}^{\mathrm{s}} = \left.\frac{\partial \chi^{\mathrm{s}}(\boldsymbol{X}^{\mathrm{s}}, t)}{\partial t}\right|_{\boldsymbol{X}^{\mathrm{s}}}, \tag{3}$$

which we may want to call velocity of an *observational point* within the control volume and, in particular, on its closed surface.

We now imagine the bar magnet to be within the control surface (see the dotted red contour in Figure 1, left inset). Then Equation (1) is no longer self-evident. It now turns into an axiom that this equation must still hold true. In empirical physics terms, we could say that there are no *magnetic monopoles*, which may act as sources to the integral on the left. The latter is also known as the total magnetic flux trespassing the closed surface. This is an assumption in classical electrodynamics, supported (so far) by missing experimental evidence regarding magnetic monopoles. In fact, we may now think that the $\boldsymbol{B}$-lines continue within the magnet and form a closed loop. Moreover, it should be pointed out that because the $\boldsymbol{B}$-field leads to a torque and orienting force on the iron filings, it must be considered as a force field and the reason why we observe some reorientation. In the spirit of cause and action of Aristotalian philosophy, we attribute it to be the cause and think of it as a primitive quantity that needs no further explanation. The same held true in context with Newton's law of motion: The change of momentum of a physical body (the effect) is due to the action of the notion force (the cause), which we do not explain but consider as given, that is, as being "primitive" without requiring further clarification.

Having said that and assuming continuity of the $\boldsymbol{B}$-field, Gauss' integral theorem can be applied and the first local Maxwell equation in regular points results,

$$\int_{v^{\mathrm{s}}(t)} \nabla^{\mathrm{s}} \cdot \boldsymbol{B}\, \mathrm{d}v^{\mathrm{s}} = 0 \qquad \Rightarrow \qquad \nabla^{\mathrm{s}} \cdot \boldsymbol{B} = 0, \tag{4}$$

where by ∇^{s}, we mean the gradient of the $\boldsymbol{B}$-field across the grid cells.

[1] Here "closed" means that it is the geometrical boundary to a three-dimensional object but not that it is impermeable to matter.

[2] For the observer, a coordinate system is chosen that does not move. Otherwise the coordinates of the position vector of the cells would be time-dependent.

Obviously, without the bar magnet, there would be no $\boldsymbol{B}$-field at all, neither outside, where it is visualized by the iron filings, nor inside, where we cannot see it, but still suppose it is there. But then the continuity assumption of $\boldsymbol{B}$ at the surface of the magnet sounds odd, because there is clearly a discontinuity of mass density from the surface of the magnet to the surrounding air or vacuum. There is no intuitive reason why the $\boldsymbol{B}$-field should behave differently, and by using the standard pillbox argument of continuum mechanics (see Section 7 for further details), we can only argue on purely mathematical grounds that from Equation (1), one obtains

$$[\![\boldsymbol{B}]\!] \cdot \boldsymbol{n} = 0, \tag{5}$$

if we postulate that Equation (1) holds true no matter whether the closed surface ∂v^{s} surrounds the magnet or not. In the last equation, $\boldsymbol{n}$ is the outward normal on the magnet's surface and the double bracket denotes the jump of $\boldsymbol{B}$ from the outside "'+" to the inside "−," $[\![\boldsymbol{B}]\!] = \boldsymbol{B}^{+} - \boldsymbol{B}^{-}$.

Also note that the singular interface is often a material one, as in the case of the surface of the bar magnet. However, it does not need to be, as in the case of electromagnetic shock waves traveling through vacuum. Moreover, the interface may have a speed of its own, which can be different from the velocity of the control volume shown in Equation (3). However, both of these velocities do not enter the jump condition.

All of this leads to such inevitable questions as: Are the $\boldsymbol{B}$-lines "attached" to matter? What if the matter starts moving, how will they "follow" this movement? For the time being, there is no reason to assume that the $\boldsymbol{B}$-field is firmly "connected" to matter. We leave it open and postulate that $\boldsymbol{B}$ "moves" on the basis of equations still to be developed. These equations will be the full set of Maxwell equations, and Equations (1), (4), and (5) are just the first set in global and local form. We shall see that the development of these equations is not unproblematic, and in the end, they are postulates or axioms that could only be falsified or their technical application could be limited based on experiments. However, at this point (and later), we should be puzzled by the fact that the $\boldsymbol{B}$-field also acts outside of the magnet, where there is no ponderable matter present. This is so different from what we are used to in (fluid) mechanics.

Moreover, we could continue asking: Suppose a "magnetizable" medium is now added outside of the magnet, which creates the $\boldsymbol{B}$-field. How will the field induce magnetism in the new material and how quickly will this be done? Will the "old" $\boldsymbol{B}$-field be altered because of the presence of the new material? And how are all results going to change if the new material is moved? We will continue to pose such questions over and over again in this paper. Nevertheless, in order to answer them, we need equations, the aforementioned Maxwell equations. In what follows, we will present a possible way of how to arrive at them. However, let us point out once more that the way of the derivation we favor is not free of dispute and neither is the final form of these equations in matter.

Consider now the situation shown in Figure 1 on the right. It is a sketch of the so-called induction experiment demonstrating the effect of the electromotoric force. A noose made of metallic wire is connected to a galvanometer.[3] Initially, it does not move and its diameter does not change. However, a bar magnet approaches or recedes, such that the magnetic flux through the surface encompassed by the noose changes in time. Then, a voltage is observed on the meter. We say that a change of the magnetic flux induces that voltage. Moreover, the direction of the voltage changes in the magnet is flipped and now approaches the noose with its other pole. The proper signs are defined in the rule of Lenz [15].

In order to describe this experiment mathematically, we first define the magnetic flux, $\Phi(t)$, through an open surface $a^{\mathrm{s}}(t)$ with outward normal $\boldsymbol{n}(\boldsymbol{x}^{\mathrm{s}}, t)$ (see also Figure A2, right), both in spatial description, as follows:

$$\Phi(t) = \int_{a^{\mathrm{s}}(t)} \boldsymbol{B}(\boldsymbol{x}^{\mathrm{s}}, t) \cdot \boldsymbol{n}(\boldsymbol{x}^{\mathrm{s}}, t)\, \mathrm{d}a^{\mathrm{s}}. \tag{6}$$

This definition is very general: The open surface is time-dependent, $a^{\mathrm{s}} = a^{\mathrm{s}}(t)$, this means it can move in space and change its shape, and it can be fictitious or material. However, in our experiment, the surface is open [4], time-independent,

[3] Here we do not explain the operating principle of this device. We simply say that it displays a "voltage," and we also do not explain that voltage can be expressed as a line integral of the electric field. In a way, we are stuck between the devil and the deep blue sea, since all we want to say is that the movement of the magnet leads to an effect, which we can display and make visible. Also, note that we avoid to use the notion "electric current," which we reserve for the principle of the conservation of electric charge in Section 3. But clearly, using so many words already indicates that there must be a connection between the law of induction or conservation of magnetic flux, which we discuss now, and this other principle. As always it is difficult to decide what was first, the hen or the egg.

[4] An "open" surface means that this is not the boundary to a three-dimensional object.

and not made of matter, that is, fictitious, with the exception of its closed circumference, the wire, so we denote it just by a^{s} without reference to time t. But the magnetic field changes at a fixed point $\boldsymbol{x}^{\mathrm{s}}$. It is explicitly time-dependent, as it becomes evident by the second entry in its functional dependence, $\boldsymbol{B}(\boldsymbol{x}^{\mathrm{s}},t)$. Again according to the principle of cause and action, there must be a reason for the occurrence of a voltage. This reason is another "force," the electric force, and we account for it by a second primitive field quantity known as the electric field, $\boldsymbol{E}(\boldsymbol{x}^{\mathrm{s}},t)$. It is somehow acting in the wire, that is, in a nonfictitious, material circumference ∂a^{s}, of our open surface, a^{s}, and we write

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a^{\mathrm{s}}} \boldsymbol{B}(\boldsymbol{x}^{\mathrm{s}},t)\cdot\boldsymbol{n}(\boldsymbol{x}^{\mathrm{s}})\,\mathrm{d}a^{\mathrm{s}} = -\oint_{\partial a^{\mathrm{s}}} \boldsymbol{E}(\boldsymbol{x}^{\mathrm{s}},t)\cdot\boldsymbol{\tau}(\boldsymbol{x}^{\mathrm{s}})\,\mathrm{d}l. \tag{7}$$

It is evident from the experiment that the normal, $\boldsymbol{n}$, of the surface is not explicitly time-dependent, just as the tangent to the circumference of the wire, $\boldsymbol{\tau}$, is not.

Now we modify the experiment: The magnet stays put and creates a time-independent but spatially varying field, $\boldsymbol{B}(\boldsymbol{x}^{\mathrm{s}})$. However, the noose is moved and/or the surface it encompasses may change in size over time by tightening or slackening the noose, $a^{\mathrm{s}}(t)$. Again, we observe a voltage the occurrence of which we attribute to a force, $\boldsymbol{v}\times\boldsymbol{B}$. We write

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a^{\mathrm{s}}(t)} \boldsymbol{B}(\boldsymbol{x}^{\mathrm{s}})\cdot\boldsymbol{n}(\boldsymbol{x}^{\mathrm{s}},t)\,\mathrm{d}a^{\mathrm{s}} = -\oint_{\partial a^{\mathrm{s}}(t)} \boldsymbol{v}(\boldsymbol{x}^{\mathrm{s}},t)\times\boldsymbol{B}(\boldsymbol{x}^{\mathrm{s}})\cdot\boldsymbol{\tau}(\boldsymbol{x}^{\mathrm{s}},t)\,\mathrm{d}l. \tag{8}$$

The arguments of the fields in these equations show clearly what varies and what not, and for an understanding as well as a mathematical description, it is essential that we use a spatial description, because these experiments also work if we pump off the surrounding air, so that $\boldsymbol{B}$ acts *in vacuo*. Note that the velocity $\boldsymbol{v}(\boldsymbol{x}^{\mathrm{s}},t)$ in this experiment is not the one from Equation (3), rather it is the velocity of the material points the metallic noose is made of.

2.2 | Mathematical generalization

We now combine the possibilities of both experiments. This results in the general form of *Faraday's law of induction*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a^{\mathrm{s}}(t)} \boldsymbol{B}(\boldsymbol{x}^{\mathrm{s}},t)\cdot\boldsymbol{n}(\boldsymbol{x}^{\mathrm{s}},t)\,\mathrm{d}a^{\mathrm{s}} = -\oint_{\partial a^{\mathrm{s}}(t)} [\boldsymbol{E}(\boldsymbol{x}^{\mathrm{s}},t)+\boldsymbol{v}(\boldsymbol{x}^{\mathrm{s}},t)\times\boldsymbol{B}(\boldsymbol{x}^{\mathrm{s}},t)]\cdot\boldsymbol{\tau}(\boldsymbol{x}^{\mathrm{s}},t)\,\mathrm{d}l, \tag{9}$$

where on the basis of experiments, we are inclined to think of the closed contour ∂a^{s} as a line made of matter, whereas the rest of the open surface a^{s} can be immaterial. Note that the electric and the magnetic field on the right hand side of Equation (9) are somehow present "inside" of the wire. Again, we do not answer the question how they are connected to the matter the wire is made of.

In the context of this relation, let us define the most general form of a balance for a vector flux $\boldsymbol{\gamma}$ through an open, not necessarily material surface $a(t)$ in spatial description:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a^{\mathrm{s}}(t)} \boldsymbol{\gamma}(\boldsymbol{x}^{\mathrm{s}},t)\cdot\boldsymbol{n}(\boldsymbol{x}^{\mathrm{s}},t)\,\mathrm{d}a^{\mathrm{s}} = -\oint_{\partial a^{\mathrm{s}}(t)} \boldsymbol{\varphi}(\boldsymbol{x}^{\mathrm{s}},t)\cdot\boldsymbol{\tau}(\boldsymbol{x}^{\mathrm{s}},t)\,\mathrm{d}l + \int_{a^{\mathrm{s}}(t)} [\boldsymbol{s}(\boldsymbol{x}^{\mathrm{s}},t)+\boldsymbol{p}(\boldsymbol{x}^{\mathrm{s}},t)]\cdot\boldsymbol{n}(\boldsymbol{x}^{\mathrm{s}},t)\,\mathrm{d}a^{\mathrm{s}}. \tag{10}$$

We refer to $\boldsymbol{\varphi}$ as the flux along the circumference of $a(t)$. It may contain convective as well as nonconvective parts. $\boldsymbol{s}$ is the supply of $\boldsymbol{\gamma}$ through the surface, and $\boldsymbol{p}$ is called its production. Note the analogy to balances for volumetric quantities within an (open) control volume v^{s}, which is not necessarily material, but may move freely through space, where all field quantities are written in spatial form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{v^{\mathrm{s}}(t)} \psi(\boldsymbol{x}^{\mathrm{s}},t)\,\mathrm{d}v^{\mathrm{s}} = -\oint_{\partial v^{\mathrm{s}}(t)} \boldsymbol{\phi}(\boldsymbol{x}^{\mathrm{s}},t)\cdot\boldsymbol{n}(\boldsymbol{x}^{\mathrm{s}},t)\mathrm{d}a^{\mathrm{s}} + \int_{v^{\mathrm{s}}(t)} [s(\boldsymbol{x}^{\mathrm{s}},t)+p(\boldsymbol{x}^{\mathrm{s}},t)]\,\mathrm{d}v^{\mathrm{s}}. \tag{11}$$

An example for ψ is the internal energy density ρu (ρ being the mass density). Then a part of $\boldsymbol{\phi}$ is the heat flux vector $\boldsymbol{q}$, the supply s is the radiation density ρr, and the production is the frictional loss $\boldsymbol{\sigma}^{\mathrm{d}} : (\nabla \bigotimes \boldsymbol{v})$ [5], $\boldsymbol{\sigma}^{\mathrm{d}}$ being the dissipative part of the Cauchy stress tensor.

The difference between supplies, $\boldsymbol{s}$, s, and productions, $\boldsymbol{p}$, p, is that the supplies can be controlled and "switched off," at least in principle, whereas the productions are inherent to the system and the physical process it undergoes. They develop on their own and elude our control. It is customary to say that a physical quantity is conserved if the production terms $\boldsymbol{p}$ or p vanish. In case of the latter, we refer to the corresponding balance as a "conservation law."

Several remarks are now in order:

- In context with this point-of-view, Equation (9) must be interpreted as a balance equation for the magnetic flux defined in Equation (6).
- In the same spirit, the quantity in brackets on the right-hand side of Equation (9) is a flux along the periphery, just as the surface integral over the heat flux vector is the (nonconvective) flux of internal energy across a closed surface of a 3D region in space. It contains a nonconvective and a convective part. It is a.k.a. the *electromotoric* or *the Lorentz force* density:

$$\mathfrak{E} = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}. \tag{12}$$

  The phrase "electromotoric force" is an hommage to von Siemens, who used this term frequently in his works. Because of this force, he could built his dynamos, which made his fortune [16].

  It is tempting to interpret this force as the agent for setting the electron cloud in the wire into motion. But this is dangerous: First, the free metal electrons are on a different scale than the macroscopic continuum. A micromodel and homogenization are required to analyze the connection in detail. This, however, is beyond the phenomenological line of thought we follow in this paper. Second, when thinking in terms of electric currents, the message of Faraday's law of induction will be clouded and it might even be confused with the Ampère–Øersted law that will be discussed in Section 3. The Faraday experiment addresses exclusively the change of the magnetic flux, expressed by the magnetic field $\boldsymbol{B}$. For obvious reasons, the latter quantity is, therefore, sometimes also referred to as "magnetic induction."
- There are no supply terms and, in particular, no production terms in Equation (9). Therefore, in summary, we may say that it expresses the *principle of conservation of magnetic flux*. It forms the *first axiom of electromagnetic field theory*.
- The magnetic and the electric field, $\boldsymbol{E}$, $\boldsymbol{B}$, are force related quantities, combined in the electromotoric force density $\mathfrak{E}$ of Equation (12). In this context, we explicitly cite Sommerfeld [17, pg. 11], even though the two other electromagnetic fields, $\boldsymbol{D}$ and $\boldsymbol{H}$, he mentions have not been introduced yet: "We may indicate finally a subdivision of physical entities into entities of intensity and entities of quantity. **E** and **B** belong to the first class, **D** and **H**, to the second. The entities of the first class are answers to the question 'how strong,' those of the second class, to the question 'how much.' In the theory of elasticity, for example, the stress is an entity of intensity, the corresponding strain, one of quantity; in the theory of gases pressure and volume form a corresponding pair of entities. In **D**, the quantity character is clearly evident as the quantity of electricity that has passed through; in **H**, the situation is slightly obscured by the fact that there are no isolated magnetic poles (see Section 3). We are in general inclined to regard the entities of intensity as cause, the corresponding entities of quantity as their effect."

  Based on the induction experiment, we fully agree with Sommerfeld that the two fields $\boldsymbol{E}$ and $\boldsymbol{B}$ answer to the question "how strong," that is, they are force related. However, our interpretation of cause and effect differs from his. Let us look at Equation (9): First of all, if we accept his classification only "cause" quantities are involved here, namely $\boldsymbol{E}$ and $\boldsymbol{B}$, which is absurd. Second, we could argue that a change of the magnetic flux through an open surface (the cause) will lead to an effect in its circumference, recorded by the galvanometer. Or we see the cause in the movement $\boldsymbol{v}$ of the open surface, specifically in its material circumference, which will lead to a change of flux through the area (the effect). Here, the situation is similar to Newton's Lex II [18, pg. 52] when stated in the form "change of momentum in time is equal to force:" We can consider the force as cause for the temporal change of momentum, the effect. Or we can lengthen a spring, the cause, and create a force in it, the effect. Indeed, this indicates already that the possibility to make a clear distinction between cause and effect—as Aristotle [19, pp. 284] wanted it to be in his causality principle—is wishful thinking.

[5] In index: $\sigma^{\mathrm{d}}_{ij} \frac{\partial v_i}{\partial x_j}$.

Let us repeat that by anticipating the notion of charge (see Section 3) we could say that if we interpret the right hand side of Equation (9) as an effect, namely the cloud of electrons in the metallic circumference is dragged by the force fields $\boldsymbol{E}$ and $\boldsymbol{B}$ to create a current, which becomes visible on the galvanometer. In this context it should, first, be mentioned that then cause and effect seem reversed in comparison with Newton's Second Law, where the *temporal change of momentum* is usually viewed as the effect, whereas here the *temporal change of magnetic flux* is obviously the cause. However, it was already mentioned that the opinion as to what is cause and what is effect can be questioned. Second, as we shall see in Section 3, electric charges and currents are the creators of the fields $\boldsymbol{D}$ and $\boldsymbol{H}$. And since we believe that charges and currents do not exist without matter, it seems reasonable to assume that $\boldsymbol{D}$ and $\boldsymbol{H}$ are "connected" to matter as well and will adjust suitably when matter is changing its position. On the other hand, on the basis of what was said above, it then also seems likely that the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are in some way related to $\boldsymbol{D}$ and $\boldsymbol{H}$, but the former, in Equation (9), were both thought to exist and function without precisely specifying how they are "attached" to matter. In short, the situation is complex, and some additional aspects will be considered in Sections 5 and 8. Moreover, we shall see over and over that cause and effect are "relative" and will also come back to a reevaluation of Sommerfeld's statement once the two fields $\boldsymbol{D}$ and $\boldsymbol{H}$ were introduced.

- We now argue purely on mathematical grounds and apply Equation (9) to a closed surface of a nonmaterial control volume, such that $a^{\mathrm{s}}(t) \to \partial v^{\mathrm{s}}(t)$ and $\partial a^{\mathrm{s}}(t) \to 0$. Then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\partial v^{\mathrm{s}}(t)} \boldsymbol{B}(\boldsymbol{x}^{\mathrm{s}}, t) \cdot \boldsymbol{n}(\boldsymbol{x}^{\mathrm{s}}, t)\, \mathrm{d}a^{\mathrm{s}} = 0 \quad \Rightarrow$$
$$\oint_{\partial v^{\mathrm{s}}(t)} \boldsymbol{B}(\boldsymbol{x}^{\mathrm{s}}, t) \cdot \boldsymbol{n}(\boldsymbol{x}^{\mathrm{s}}, t)\, \mathrm{d}a^{\mathrm{s}} = \oint_{\partial v^{\mathrm{s}}(0)} \boldsymbol{B}(\boldsymbol{x}^{\mathrm{s}}, 0) \cdot \boldsymbol{n}(\boldsymbol{x}^{\mathrm{s}}, 0)\, \mathrm{d}a^{\mathrm{s}}, \tag{13}$$

where an integration in time was carried out. Following an argument often presented in the continuum literature [20, pp. 305], [5, pg. 11] we assume that the control surface passed through a vanishing $\boldsymbol{B}$-field at time 0, $B(\boldsymbol{x}^{\mathrm{s}}, 0) = \boldsymbol{0}$, or if we say that the $\boldsymbol{B}$-field was not "switched on" at time 0 yet, then

$$\oint_{\partial v^{\mathrm{s}}(t)} \boldsymbol{B}(\boldsymbol{x}^{\mathrm{s}}, t) \cdot \boldsymbol{n}(\boldsymbol{x}^{\mathrm{s}}, t)\, \mathrm{d}a^{\mathrm{s}} = 0 \quad \Rightarrow \quad \nabla^{\mathrm{s}} \cdot \boldsymbol{B} = 0 \tag{14}$$

at all times, t.

This agrees formally with the result of the initial experiment, Equation (1). Hence, one could conclude that the physical principle "there are no magnetic monopoles" is already contained in the general form (9) and that a separate axiom is unnecessary. On the other hand, several, rather awkward arguments were required to obtain the final result. Nevertheless, these arguments can be found in the literature mentioned above. In summary, the authors of this paper believe that it is simpler to accept the axiom that there are no magnetic monopoles.

- Note that Equation (14) is sometimes interpreted as a volumetric balance of the type (11) if one assigns in the general relation $\boldsymbol{\phi} \to \boldsymbol{B}$. However, there is no volumetric quantity, that is, $\psi = 0$, no volumetric supply, $s = 0$, and no volumetric production, $p = 0$. This perception is not rewarding and we will get back to it when we discuss the work of Ivanova [21] at the end of this chapter.

We will now apply a transport theorem for open control surfaces. In the Appendix A such a mathematical rule is presented for a flux $\boldsymbol{\gamma}$ though an open, control surface $a^{\mathrm{s}}(t)$ that moves fictitiously with the velocity field $\boldsymbol{v}^{\mathrm{s}}$ shown in Equation (3). When applied to the case of the magnetic field, we must write

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{a^{\mathrm{s}}(t)} \boldsymbol{B}(\boldsymbol{x}^{\mathrm{s}}, t) \cdot \boldsymbol{n}(\boldsymbol{x}^{\mathrm{s}}, t)\, \mathrm{d}a^{\mathrm{s}} = \int_{a^{\mathrm{s}}(t)} \left(\frac{\partial \boldsymbol{B}(\boldsymbol{x}^{\mathrm{s}}, t)}{\partial t} + \boldsymbol{v}^{\mathrm{s}}(\boldsymbol{x}^{\mathrm{s}}, t) \nabla^{\mathrm{s}} \cdot \boldsymbol{B}(\boldsymbol{x}^{\mathrm{s}}, t) \right) \cdot \boldsymbol{n}(\boldsymbol{x}^{\mathrm{s}}, t)\, \mathrm{d}a^{\mathrm{s}} +$$
$$\oint_{\partial a^{\mathrm{s}}(t)} (\boldsymbol{B}(\boldsymbol{x}^{\mathrm{s}}, t) \times \boldsymbol{v}^{\mathrm{s}}(\boldsymbol{x}^{\mathrm{s}}, t)) \cdot \boldsymbol{\tau}(\boldsymbol{x}^{\mathrm{s}}, t)\, \mathrm{d}l. \tag{15}$$

It was noted that in case of the induction experiment shown in Figure 1 on the right that the circumference had to be made of wire moving with the material velocity $\boldsymbol{v}(\boldsymbol{x}^{\mathrm{s}}, t)$ expressed in spatial coordinates. Therefore, we now insist on replacing $\boldsymbol{v}^{\mathrm{s}}(\boldsymbol{x}^{\mathrm{s}}, t)$ by $\boldsymbol{v}(\boldsymbol{x}^{\mathrm{s}}, t)$ in the contour term of this equation[6] and by combining Equations (9) and (15) and after applying Stokes' integral theorem obtain

$$\int_{a^{\mathrm{s}}(t)} \left(\frac{\partial \boldsymbol{B}(\boldsymbol{x}^{\mathrm{s}}, t)}{\partial t} + \nabla^{\mathrm{s}} \times \boldsymbol{E}(\boldsymbol{x}^{\mathrm{s}}, t) \right) \cdot \boldsymbol{n}(\boldsymbol{x}^{\mathrm{s}}, t) \, \mathrm{d}a^{\mathrm{s}} = 0. \tag{16}$$

Because of the assumed continuity of all fields, we obtain the second local Maxwell equation, the induction law in regular points in spatial form:

$$\frac{\partial \boldsymbol{B}(\boldsymbol{x}^{\mathrm{s}}, t)}{\partial t} + \nabla^{\mathrm{s}} \times \boldsymbol{E}(\boldsymbol{x}^{\mathrm{s}}, t) = \boldsymbol{0}. \tag{17}$$

The case of (moving) discontinuities will be treated in Section 7. Note that in the last two equations spatial differentiation was performed, ∇^{s}, which refers to the gradient of properties between grid cells. If there is continuous matter distributed over the grid, and a material description of all fields is used, it can be replaced with the material differentiation in space, ∇:

$$\frac{\partial \boldsymbol{B}(\boldsymbol{x}, t)}{\partial t} + \nabla \times \boldsymbol{E}(\boldsymbol{x}, t) = \boldsymbol{0}. \tag{18}$$

2.3 | Comparison of the various presented forms of the law of induction with the literature

In the previous section, we put an emphasis on concepts known from continuum mechanics:

- All of our fields were expressed in spatial description. This is worth emphasizing, because the magnetic flux $\boldsymbol{B} \cdot \boldsymbol{n} \, \mathrm{d}a$ in a grid-point of space $\boldsymbol{x}^{\mathrm{s}}$ at some time t does not require that ponderable matter is present there.
- Faraday's law of induction and the law of nonexistence of magnetic monopoles are of the balance type.
- The magnetic flux is a conserved quantity.

Typically physics textbooks do not emphasize all of these points clearly. In our review of physics textbooks, we begin with Section 17 of the Feynman lectures on physics [22] where Feynman starts from the law of induction in regular points and integrates it over a "mathematical curve fixed in space" enclosing a "fixed surface." He then continues by explaining that "the flux rule applies whether the flux changes because the field changes or because the circuit moves (or both)." Then, he introduces the electromotoric force and emphasizes that in moving wires, the force stems from the second term $\boldsymbol{v} \times \boldsymbol{B}$. However, Feynman continues to make a very important point, which is in contrast to our beliefs from above: "The part of the emf that comes from the $\boldsymbol{E}$-field does not depend on the existence of a physical wire (as does the $\boldsymbol{v} \times \boldsymbol{B}$ part). The $\boldsymbol{E}$-field can exist in free space, and its line integral around any imaginary line fixed in space is the rate of change of the flux of $\boldsymbol{B}$ through that line." Note again that this is noteworthy, because the way we have introduced $\boldsymbol{E}$ in Equation (9) was w.r.t. a material line. We may, therefore, state an extended opinion and rewrite Faraday's law of induction as

$$\int_{a^{\mathrm{s}}(t)} \left(\frac{\partial \boldsymbol{B}(\boldsymbol{x}^{\mathrm{s}}, t)}{\partial t} + \boldsymbol{v}^{\mathrm{s}} \nabla^{\mathrm{s}} \cdot \boldsymbol{B}(\boldsymbol{x}^{\mathrm{s}}, t) \right) \cdot \boldsymbol{n}(\boldsymbol{x}^{\mathrm{s}}, t) \, \mathrm{d}a^{\mathrm{s}} = - \oint_{\partial a^{\mathrm{s}}(t)} \boldsymbol{E}(\boldsymbol{x}^{\mathrm{s}}, t) \cdot \boldsymbol{\tau}(\boldsymbol{x}^{\mathrm{s}}, t) \, \mathrm{d}l, \tag{19}$$

where the surface $a^{\mathrm{s}}(t)$ and its periphery $\partial a^{\mathrm{s}}(t)$ in this expression may now be completely imaginary. Note that we did not put $\nabla^{\mathrm{s}} \cdot \boldsymbol{B}(\boldsymbol{x}^{\mathrm{s}}, t) = 0$ yet, in order to emphasize that this derivative in space is not material at all. But, yes, this term

[6] This is possible as long as the fictitious velocity field $\boldsymbol{v}^{\mathrm{s}}(\boldsymbol{x}^{\mathrm{s}}, t)$ within the contour assumes continuously the material velocity $\boldsymbol{v}(\boldsymbol{x}^{\mathrm{s}}, t)$ of the wire, which is the contour.

vanishes. However, this form with or without $\nabla^{\mathrm{s}} \cdot \boldsymbol{B}(\boldsymbol{x}^{\mathrm{s}}, t)$ does not allow to see the balance character of Faraday's law of induction clearly.

In summary, following Feynman, the electric and the magnetic field, $\boldsymbol{E}$ and $\boldsymbol{B}$, do not need matter for their existence. They can "stimulate" each other, a temporal change of $\boldsymbol{B}$ will lead to $\boldsymbol{E}$ responding by a (negative) curl or rotor. This is why light can propagate through space without a supporting medium. Both fields are self-sufficient.

It is interesting to note that Feynman does not discuss the case of a closed surface in Section 17 of his presentation of the induction law. The issue of a magnetic monopole is not mentioned at all and in his summary of Maxwell equations on pg. 18-2, it is stated without a comment that the flux through a closed surface is zero.

In Chapter 4 of their field theory volume [9], Landau and Lifshitz discuss the Maxwell equations and in particular the law of induction on pp. 70. First, it should be noted that they work in Heaviside–Lorentz units (see Section 4 on units below). Hence, in a Lorentz system, no distinction between the fields $\boldsymbol{H}$ and $\boldsymbol{B}$ and the fields $\boldsymbol{E}$ and $\boldsymbol{D}$ is and must be made (see Section 5 and Equation (44)). They start with the local forms in regular points and integrate over closed and open surfaces and volumes, as well as over closed lines, but they do not say as to whether these geometric objects are fictitious or material. This is not an issue at all. Moreover, they refer to the "circulation of the electric field" alone, without the $\boldsymbol{v} \times \boldsymbol{B}$ term, as the electromotive force.

On pg. 210 of Chapter 5, the 3rd edition of Jackson's famous physics textbook on electromagnetic theory [6], an attempt is made in a footnote to follow up on the transport theorem for open surfaces (15) used in continuum theory. To put his ideas in context with ours, note that instead of Equation (15), we may write

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{a^{\mathrm{s}}(t)} \boldsymbol{B} \cdot \boldsymbol{n} \, \mathrm{d}a^{\mathrm{s}} = \int\limits_{a^{\mathrm{s}}(t)} \left(\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{B} \, \nabla^{\mathrm{s}} \cdot \boldsymbol{v}^{\mathrm{s}} + \boldsymbol{v}^{\mathrm{s}} \cdot \nabla^{\mathrm{s}} \boldsymbol{B} - \boldsymbol{B} \cdot \nabla^{\mathrm{s}} \boldsymbol{v}^{\mathrm{s}} \right) \cdot \boldsymbol{n} \, \mathrm{d}a^{\mathrm{s}} = \int\limits_{a^{\mathrm{s}}(t)} \left(\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} + \boldsymbol{B} \, \nabla^{\mathrm{s}} \cdot \boldsymbol{v}^{\mathrm{s}} - \boldsymbol{B} \cdot \nabla^{\mathrm{s}} \boldsymbol{v}^{\mathrm{s}} \right) \cdot \boldsymbol{n} \, \mathrm{d}a^{\mathrm{s}}. \tag{20}$$

In here, ${}^{\mathrm{d}\boldsymbol{B}}/_{\mathrm{d}t}$ is the total derivative, to which Jackson refers to as convective derivative, most likely because he calls the contour a "circuit," which is meant as a material. However, he also claims that the two other terms vanish by saying that "$\boldsymbol{v}$ is treated as a fixed vector in the differentiation," which makes no sense, neither for a material nor a fictitious velocity.

Magnetic monopoles are an extensive topic in Jackson's book, and a whole section is dedicated to their possible existence, see pp. 273. But they are not mentioned in Section 5.15 dedicated to the global form of the law of induction. However, on page 16, reference is made to the fact that the integral condition (1) represents their nonexistence. Hence, for Jackson, the law of induction seems to be independent of the law of no magnetic monopoles.

In Section 3 of his textbook [17], Sommerfeld gives an account of the Maxwell equations in integral form. On page 12, he discusses Faraday's law of induction and points out the importance to think of the electromotoric force as being abstract and not just related to "closed metallic circuits." However, it is fair to say that his mathematical terminology of open and closed surfaces and lines lack the rigor of modern continuum theory. For example, it is never completely clear in every equation as to whether surfaces and lines may change in time or not and how the time derivatives are treated properly. On the other hand, Sommerfeld (pg. 15) considers the law of non existence of magnetic monopoles in the form $(4)_2$ and (13) and assigns the status of a "supplementary axiom" to it.

Of course, also Becker discusses Faraday's law of induction in integral and local form in the various German and English editions of his textbooks on electrodynamics ([23, 24], §5.3, pp. 103, [8], coauthored and edited by Sauter). The importance of relative motion on the two parts in the electromotoric force (12) are discussed in detail. The question as to whether the law of magnetic monopoles is independent or not is not answered and Equation (4) drops more or less out of the sky, so-to-speak, and is "derived" verbally. Needless to say that the various time dependencies and their treatment does not reach the standards of modern continuum theory.

The books of Becker originated from the very early textbook on Maxwell's theory by Abraham ([25], coauthored by the professor of technical mechanics, August Föppl, who got interested in the new field theory). In this book, the formal mathematical connection to fluid mechanics concepts is emphasized, and time derivatives in various frames of reference are discussed in detail ([25, pp. 113]) and then applied to the magnetic fields, first without explicitly saying so: pp. 119. The time derivative of the magnetic flux (20) is then discussed in a surprisingly modern manner, [25, pg. 399],

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \boldsymbol{B} \cdot \boldsymbol{n} \, \mathrm{d}a = \int \left(\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times (\boldsymbol{B} \times \boldsymbol{v}) + \boldsymbol{v} \nabla \cdot \boldsymbol{B} \right) \cdot \boldsymbol{n} \, \mathrm{d}a, \tag{21}$$

which agrees with Equation (15). A reference to the work of Hertz is made on pg. 413 of Ref. [25] where a possible departure of the law of magnetic monopoles is discussed. Their discussion then culminates in Faraday's law of induction in the form

$$\int \nabla \times (\boldsymbol{E} - \boldsymbol{E}^e) \cdot \boldsymbol{n} \, \mathrm{d}a = -\int \left(\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times (\boldsymbol{B} \times \boldsymbol{v}) + \boldsymbol{v} \nabla \cdot \boldsymbol{B} \right) \cdot \boldsymbol{n} \, \mathrm{d}a. \tag{22}$$

This is in agreement with Equation (16) if we accept the nonexistence of magnetic monopoles (as finally done in Abraham and Föppl [25] by departing from Hertz) and identify what they call "impressed electromagnetic force" $\boldsymbol{E}^e$ as $-\boldsymbol{v} \times \boldsymbol{B}$. In summary, these early authors still profited considerably from their knowledge of hydrodynamics. However, the terminology for the electromagnetic force is still obscure. Things become even more confusing, when we look at the edited English translation of this book by Becker [26]. On page 142 of this book Faraday's law of induction is presented in local regular form when a "body, in which $\boldsymbol{E}$ is to be calculated, is moving with the velocity" $\boldsymbol{v}$ as

$$\nabla \times \boldsymbol{E} = -\underline{\dot{\boldsymbol{B}}}. \tag{23}$$

The time derivative on the right-hand side is referred to as a "special kind of time differentiation," namely

$$\underline{\dot{\boldsymbol{B}}} = \dot{\boldsymbol{B}} + \boldsymbol{v} \, \nabla \cdot \boldsymbol{B} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}). \tag{24}$$

By studying previous arguments in this book (pg. 39/136/141), one comes to the conclusion that $\dot{\boldsymbol{B}}$ should be $\partial \boldsymbol{B}/\partial t - \boldsymbol{v}\nabla \cdot \boldsymbol{B} \equiv \partial \boldsymbol{B}/\partial t$. We must then confirm Becker that the field $\boldsymbol{E}$ in Equation (23) cannot be the $\boldsymbol{E}$-field of Equation (17), which is w.r.t. an inertial frame and not w.r.t. a moving body. Such a babel of time derivatives in combination with imprecise definitions which field representation in which frame of reference has been chosen explains that until today, miscellaneous forms of the local Maxwell equations with curious time derivatives can be found in the literature. We will look more into these issues in Section 5 on transformation properties of electromagnetic fields and in Section 8.2, where time derivatives are an issue. At this point, it should also be mentioned that Morro [27] discusses the presentation of Faraday's induction law in the physics literature in detail on pg. 234 very critically and comes to similar conclusions as we do.

Let us now turn to the viewpoint of authors versed in modern continuum theory. We start by discussing Chapter F in the Handbook of Physics article of Truesdell and Toupin [1], where modern continuum terminology was used, probably for the first time, and which allegedly was written primarily by Toupin. This work embeds electromagnetic theory in the framework of a 4D space-time continuum from the very start. This will concern us in Section 5, when we talk about the transformation properties of the four primary electromagnetic fields. Moreover, the tensor calculus in which the chapter is written requires getting used to, because the text constantly jumps between primary tensorial quantities and their duals. These are introduced in a very general manner for arbitrary rank, rather than concentrating on the relevant objects of electromagnetism. However, a bridge to conventional 3D theory is also built and, in particular, Faraday's law of induction is discussed in terms of a conserved integral balance equation, see pg. 673:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{S}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} + \oint_{C} \boldsymbol{\mathfrak{E}} \cdot \mathrm{d}\boldsymbol{x} = 0, \tag{25}$$

where $\mathrm{d}\boldsymbol{a} = \boldsymbol{n}\mathrm{d}a$ and $\mathrm{d}\boldsymbol{x} = \boldsymbol{\tau}\mathrm{d}l$. $\boldsymbol{\mathfrak{E}}$ is referred to as "electromotive intensity," which is synonymous to the term "electromotoric force" of von Siemens. It is also explicitly mentioned that $\mathcal{S}$ is a "2-dimensional surface in space moving with the particles of the motion and C is its complete boundary." Hence, the surface is definitely completely a material one. This, however, seems unnecessary in view of the experiments shown in Figure 1 on the right. Nevertheless, on pg. 674, it is said that Equation (25) is the "traditional form of Faraday's law of induction for moving circuits." However, as it was pointed out above, here it is only necessary that the closed contour is material.

However, it should be pointed out that Toupin also obtains both, Equations (1) and (9), from a single abstract principle, namely by looking at a closed surface (or closed two-dimensional "circuit" in space-time as he calls it on pg. 674). Now, if the magnetic field worldtensor $\boldsymbol{\varphi}$ (see Section 5, Equation (63)) is integrated over the surface after scalar multiplication the result vanishes (pg. 667). Hence, for him, these relations are consequences of the conservation of magnetic flux in space–time, which leads us to conclude that a separate physical postulate based on the nonexistence of magnetic monopoles is unnecessary. Indeed, it is stressed that the worldvector form of "the law of conservation of magnetic flux" is a postulate (pg. 667), it is one of the axioms of electromagnetism.

Regarding the transport theorem for moving open surfaces of the A and in particular its application to the magnetic flux shown in Equation (15), we point out that Toupin devotes the whole Section 277 to this topic, albeit in a rather abstract manner. However, it is clearly mentioned that geometric entities, such as surfaces or curves, are all “moving with the particles of a motion.” Hence, the analysis is purely material. Toupin calls on pg. 676 the expression

$$\frac{\mathrm{d_c}\hat{\boldsymbol{F}}}{\mathrm{d}t} = \frac{\partial \hat{\boldsymbol{F}}}{\partial t} + \mathrm{dual}\big(\mathrm{div}\hat{\boldsymbol{F}} \times \boldsymbol{v}\big) + \mathrm{curl}\big(\hat{\boldsymbol{F}} \times \boldsymbol{v}\big) \tag{26}$$

the “convected time flux” of an “axial density” $\hat{\boldsymbol{F}}$, which is a contravariant k-vector density, $\hat{F}^{r_1 r_2 \ldots r_p}$, $p + k = 3$. Indeed, the various differential operations and the meaning of his tensorial notation become a little clearer if one turns completely to index notation:

$$\frac{\mathrm{d_c}\hat{F}^{r_1 r_2 \ldots r_p}}{\mathrm{d}t} = \frac{\partial \hat{F}^{r_1 r_2 \ldots r_p}}{\partial t} + p\frac{\partial \hat{F}^{[r_1 r_2 \ldots r_{p-1}|s|}}{\partial x^s} v^{r_p]} + (p+1)\frac{\partial}{\partial x^s}\Big(\hat{F}^{[r_1 r_2 \ldots r_p} v^{s]}\Big). \tag{27}$$

The brackets around multi-indices are nowhere explained in the text and neither is the meaning of an index in absolute signs. We can only make an intelligent guess. The multi-index brackets indicate antisymmetrization, which we know from tensors second rank:

$$A_{ij} = A_{(ij)} + A_{[ij]}\ , \quad A_{(ij)} = \frac{1}{2}\big(A_{ij} + A_{ji}\big)\ , \quad A_{[ij]} = \frac{1}{2}\big(A_{ij} - A_{ji}\big). \tag{28}$$

In the case of a sequence with p-indices, the factors of two must be replaced by $p!$. Moreover $|s|$ could mean that this index does not participate in the antisymmetrization process. Hence, we may now apply these relations to our case of interest, where the axial density is a simple axial vector $\boldsymbol{\gamma}$, for example, the magnetic field $\boldsymbol{B}$. Hence $p = 1$ and we obtain in simple Cartesian coordinates

$$\frac{\mathrm{d_c}\gamma_i}{\mathrm{d}t} = \frac{\partial \gamma_i}{\partial t} + \frac{\partial \gamma_s}{\partial x_s} v_i + 2\frac{\partial}{\partial x_s}\big(\gamma_{[i} v_{s]}\big) \tag{29}$$

or

$$\frac{\mathrm{d_c}\boldsymbol{\gamma}}{\mathrm{d}t} = \frac{\partial \boldsymbol{\gamma}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{\gamma} + \boldsymbol{\gamma}\nabla\cdot\boldsymbol{v} - \boldsymbol{\gamma}\cdot\nabla\boldsymbol{v} \equiv \frac{\partial \boldsymbol{\gamma}}{\partial t} - \nabla\times(\boldsymbol{v}\times\boldsymbol{\gamma}) + \boldsymbol{v}\nabla\cdot\boldsymbol{\gamma}, \tag{30}$$

the “convected time-flux” of $\boldsymbol{\gamma}$.

The sections on electromagnetism in the books by Müller ([28], Chapter V, and [20], Chapter 9) are very much inspired by the Handbook article: Faraday’s law of induction is considered as a conservation law for the magnetic flux $\boldsymbol{B}$. The law of nonexistence of magnetic monopoles is a consequence of the general flux balance for an open material surface. Even Toupin’s term “electromotoric intensity” for $\mathfrak{E}$ is used. Moreover, four-vector notation is applied. The same holds true for the books by Kovetz [3, 4].

Although not unaffected by the Handbook article the book by Hutter et al. [5] allows for a wider view on continuum electrodynamics by taking the works of other authors in account who do not follow the “rational viewpoint” in a strictly orthodox manner. Nevertheless, Faraday’s law of induction is viewed as a conservation law expressed by a balance over a *material* open surface (pg. 11) and the law of nonexistence of magnetic monopoles is also considered to be a corollary. His convective time derivative (pg. 12), which he denotes by $\overset{\star}{\boldsymbol{\gamma}}$, is formulated for a material particle and agrees with Equation (30). The same holds for the article by Morro [27], pg. 233, where it is denoted by $\overset{\diamond}{\boldsymbol{\gamma}}$.

Finally, we come to the books by Eringen and Maugin ([29, 30]). Before they get to the continuum level (Chapter 3), they consider the Maxwell equations on a discrete, “microscopic, atomistic” level first and then perform a homogenization (Chapter 2). In fact, on the microscopic level in an inertial frame, they require only the electric and magnetic field $\boldsymbol{E}$ and $\boldsymbol{B}$. The two other electric and magnetic fields $\boldsymbol{D}$ and $\boldsymbol{H}$ appear only on the macroscopic level. We will get back to this issue below, first, when we talk about the conservation of electric charge in Section 3 and, second, in the Section 5 on transformation properties. Note that Jackson [6] also follows this line of reasoning in his Chapter 4 and Section 6.6 when he introduces polarization and magnetization. However, his way of presentation is certainly not written in the language of continuum mechanics and micromechanics.

Moreover, one can tell that the roots of Eringen and Maugin are in the continuum mechanics of solids, since their presentation is fully based on the motion of a material point. For example, all of their transport theorems are formulated in terms of material open surfaces and material volumes, Section 1.13. It is interesting to note that, initially, they do not make use of the concept of a global conservation law for the magnetic flux (nor for charge): Section 3.3. This is because they start at the microscopic level and homogenization takes them directly to the local continuum point. Also, the notion and use of the convective time differentiation in the local induction law exists and agrees with Equation (30) (see $(1.12.12)_2$ on pg. 18, where it is denoted by a "$*$" above the field in question) however, it appears in context with the Galilean invariance principle (Section 3.4) applied to the local fields and equations and not from the transport theorem applied to the temporal change of magnetic flux (Equation (3.4.16) on pg. 54, in Heaviside-Lorentz units, which explains the factor $1/c$):

$$\nabla \times \mathfrak{E} + \frac{1}{c}\overset{*}{\boldsymbol{B}} = \mathbf{0}. \tag{31}$$

The global form of Faraday's induction law in the form (25) appears later in Section 3.9, Equation (3.9.6), where it is claimed that it holds for "nonrelativistically moving matter."

Maugin's monography [31] on continuum mechanics of electromagnetic solids focuses on the continuum description of solids subjected to elecromagnetic fields. Consequently, the general kinematic framework is based on the material particle. Maxwell's equations appear in local form for regular material points in Chapter 3 and are simply accepted and not further discussed.

A totally different point of view of Faraday's law of induction is presented in Section 6.6 of the edited book [32], which is based on articles by Zhilin. It is "derived" in local form based on mechanical equations of micropolar media. Further details regarding the derivation within the framework of this theory and a discussion of the attempts to understand electromagnetism mechanically can be found in Refs. [33–37], Section 2.5. It should be pointed out that the approach of Zhilin and disciples goes beyond the early attempts of understanding Maxwell's equations from a hydrodynamic analogy, which was for example presented by Sommerfeld [38], pp. 108. Indeed, a mechanical interpretation within the framework of the modern mechanics of micropolar media could be stimulating and useful. It is not surprising that the Zhilin school is very much convinced that by using the right mechanics electrodynamic phenomena can be described, modeled, and in the end better understood. In this context, for them, the reestablishment of the concept of the æther is important. This becomes evident in the Editor's footnote on pg. 353: "То, что в современной физике описывается уравнениями Максвелла, согласно трактовке П. А. Жилина, принятой в шестой главе, является возмущением в эфире (электромагнитном поле). Уравнения для возмущений, распространяющихся в электромагнитном поле, выведены в разделе 6.8. (Примеч. ред.)"[7] We will get back to that in Section 8. Moreover, it is fair to say that Zhilin does not give much credit to the attempts in physics to base Maxwell's equations on experimental evidence and to the logical extrapolation of these in terms of conservation laws, as it was done in the present section dedicated to the principle of conservation of magnetic flux. On pg. 353, we can hear his sentiments quite clearly: "В современной физике считается, что уравнения Максвелла являются чем-то вроде божественного откровения и потому просто постулируются."[8] It is our opinion that the situation is not quite as bad and will continue to prove it in the next Section 3 .

Finally, Ivanova raises an interesting issue in [21] regarding the law of magnetic monopoles: "However, if equation $\nabla \cdot \boldsymbol{B} = 0$ is a balance equation, then it is unclear why it cannot be changed when we turn from statics to dynamics. In dynamics, any balance equation contains at least one term with the time derivative." In this context, it should be said that, first, the "no magnetic monopole law" is dynamic. This becomes evident if it is introduced in the way it was done here: It results from a balance for a vectorial flux across an open surface (with an explicit time derivative in front of an integral), namely Equation (9). If specialized to the case of a closed surface, we arrive at Equations (13) and (14). Second, as observed correctly in Ivanova [21], Equation $(14)_2$ looks as if it came from a balance for a volumetric quantity, Equation (11), with the setting $\psi = 0$, $\boldsymbol{\phi} \rightarrow \boldsymbol{B}$, $s = 0$, $p = 0$. But this impression is deceiving and confusion does not arise if a balance for a flux is considered to begin with. Indeed, both types of balances are important, but all in good time, as we proceed to show in the next section.

[7] "What in modern physics is described by Maxwell's equations, according to the interpretation of P. A. Zhilin, adopted in the sixth chapter, is a perturbation in the ether (electromagnetic field). Equations for disturbances propagating in an electromagnetic field are derived in Section 6.8. (Editor's Note)"

[8] "In modern physics, it is believed that Maxwell's equations are something like a divine revelation and therefore are simply postulated."

3 | CONSERVATION OF CHARGE OR THE SECOND SET OF MAXWELL'S EQUATIONS

3.1 | The experimental evidence

According to the current scientific doctrine, there is no electric charge without matter. And matter has ponderable mass. Hence mass and electric charge are intimately connected to each other. In fact, a moving electric charge can even contribute to its own inertia, the so-called electromagnetic mass, as outlined in Chapter 28 of the Feynman physics course [39]. However, in continuum physics, we strictly distinguish between mass and electric charge and define a total volumetric electric charge density $q = \overline{q}(\boldsymbol{x}^{\mathrm{s}}, t) = {}^{\mathrm{d}Q}/_{\mathrm{d}v^{\mathrm{s}}}$, where $\mathrm{d}Q$ is the total [9] amount of charge within the spatial volume element $\mathrm{d}v^{\mathrm{s}}$. Hence, two primitive volumetric properties of matter located at the observational point $\boldsymbol{x}^{\mathrm{s}}$ exist *independently* of each other, the mass density ρ and the total charge density q. It is for that reason that we do not introduce a specific total charge density per unit mass as we so often do, for example, the specific linear momentum $\boldsymbol{v}$, or the specific body force $\boldsymbol{f}$, and so forth.

It is well known that mass can only be transported convectively.[10] Hence the mass flux through the surface element of a control volume is $\rho(\boldsymbol{v}^{\mathrm{s}} - \boldsymbol{v})$. Charge can also be transported that way, $q(\boldsymbol{v}^{\mathrm{s}} - \boldsymbol{v})$, but, similar to the heat flux $\boldsymbol{q}$, in addition, and only on the continuum scale, there is an electric flux density $\boldsymbol{j} = \overline{\boldsymbol{j}}(\boldsymbol{x}^{\mathrm{s}}, t)$ not associated with convective transport. We call that the density of the total nonconvective electric current. In fact, in a sloppy manner of speech, both are "electric currents" and the adjectives "convective" and "nonconvective" are very often not mentioned at all. Therefore, further remarks are required to explain the difference: Consider a car battery hooked up to the starter of the engine. Before loading the battery consists of (at least) two plates of lead (Pb) surrounded by sulfuric acid (H_2SO_4), the so-called electrolyte. A chemical reaction occurs and both plates are initially coated with $PbSO_4$. If the battery is charged for the first time, one plate returns to a state of pure lead, more precisely a "lead foam," and the other one is coated with lead oxide (PbO_2). In other words, there was an electric current through the electrolyte based on ion migration connected with visible mass transport. Other more exotic examples of convective electric currents are the flow of the liquid, electrically charged outer iron–nickel core of the Earth, or charged plasma eruptions on the Sun. Now consider a wire made of copper and put it into the wall socket: The wire will become very hot (unless the fuse blows earlier), but there is no visible mass transport nor chemical reaction. On a microscopic scale, we would argue that the "electron cloud" surrounding the copper atoms was set in motion, which led to friction hence heat.

In this spirit, we postulate the second basic law of continuum electrodynamics, the law of the conservation of electric charge:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{v^{\mathrm{s}}(t)} q \, \mathrm{d}v^{\mathrm{s}} = \oint_{\partial v^{\mathrm{s}}(t)} [q \, (\boldsymbol{v}^{\mathrm{s}} - \boldsymbol{v}) - \boldsymbol{j}] \cdot \boldsymbol{n}^{\mathrm{s}} \mathrm{d}a^{\mathrm{s}}. \tag{32}$$

The minus sign in context with the total electric current density is convention: If $\boldsymbol{j}$ points into the volume $v^{\mathrm{s}}(t)$, the amount of electric charge shall increase. In terms of the general volumetric balance (11), we conclude that the negative flux $-\boldsymbol{\phi}$ is given by $q \, (\boldsymbol{v}^{\mathrm{s}} - \boldsymbol{v}) - \boldsymbol{j}$, which means it is partly convective and partly nonconvective in nature. There are no volumetric supplies and productions of charge, s and p, respectively, and because of the latter, charge is a truly conserved quantity. If fact, it is even possible to seal off, and therefore, actively control, the volume v^{s} completely from outside, by using a membrane $\partial v^{\mathrm{s}}(t)$ that is impermeable to matter and an electric insulator.

We can now apply the transport theorem (A8) for volumetric quantities of Appendix A to find the local form of the conservation of charge in regular points in spatial description:

$$\frac{\partial q}{\partial t} + \nabla^{\mathrm{s}} \cdot (q\boldsymbol{v} + \boldsymbol{j}) = 0 \quad \Leftrightarrow \quad \frac{\delta q}{\delta t} + q \nabla^{\mathrm{s}} \cdot \boldsymbol{v} + \nabla^{\mathrm{s}} \cdot \boldsymbol{j} = 0, \tag{33}$$

where the material derivative in spatial description from Equation (A5) has been used.

[9] The adjective "total" will become clearer in Section 6, where we shall decompose the total electric charge density into free electric charge and polarization charge. Similarly we will treat the total electric current density.

[10] This is the belief of classical mechanics. Indeed, radioactive materials can also lose mass by radiation. However, the conversion and equivalence of mass and energy will not be analyzed here.

If we wish to consider a material volume $v(t)$ and use a Eulerian description of fields, that is, $q = \check{q}(\boldsymbol{x}, t)$, $\boldsymbol{j} = \check{\boldsymbol{j}}(\boldsymbol{x}, t)$[11] we must write instead of (32)

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{v(t)} q \, \mathrm{d}v = - \oint_{\partial v(t)} \boldsymbol{j} \cdot \boldsymbol{n} \, \mathrm{d}a. \tag{34}$$

After application of the transport theorem for material volumes (A11), this leads to the local form

$$\frac{\partial q}{\partial t} + \nabla \cdot (q\boldsymbol{v} + \boldsymbol{j}) = 0 \quad \Leftrightarrow \quad \frac{\mathrm{d}q}{\mathrm{d}t} + q\nabla \cdot \boldsymbol{v} + \nabla \cdot \boldsymbol{j} = 0, \tag{35}$$

where ∇ is the gradient between neighboring material points and ${}^{\mathrm{d}}/_{\mathrm{d}t}$ the material derivative of a field in Eulerian description. Note that Equations (33) and (35) look rather similar but that the meaning of the fields in terms of their arguments is completely different. It is worth mentioning that Equation $(35)_1$ is often referred to as the continuity equation in the literature and that the total current density $q\boldsymbol{v} + \boldsymbol{j}$ is sometimes combined in one symbol, $\boldsymbol{J}$.

We will now propose a formal solution to the global and local equations for the conservation of electric charge by introducing two new fields, which for obvious reasons are then referred to as "potentials." We start with the global version, which has the form of a vectorial flux balance:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{a^{\mathrm{s}}(t)} \boldsymbol{D} \cdot \boldsymbol{n} \, \mathrm{d}a^{\mathrm{s}} = \oint_{\partial a^{\mathrm{s}}(t)} [\boldsymbol{H} + \boldsymbol{D} \times \boldsymbol{v}^{\mathrm{s}}] \cdot \boldsymbol{\tau} \, \mathrm{d}l + \int_{a^{\mathrm{s}}(t)} [q(\boldsymbol{v}^{\mathrm{s}} - \boldsymbol{v}) - \boldsymbol{j}] \cdot \boldsymbol{n} \, \mathrm{d}a^{\mathrm{s}}. \tag{36}$$

This equation is also known as the Ampère–Øersted law in the literature. $\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{x}^{\mathrm{s}}, t)$ and $\boldsymbol{H} = \boldsymbol{H}(\boldsymbol{x}^{\mathrm{s}}, t)$ will be referred to as the potentials for total charge and total current, respectively. Closing the surface, $a^{\mathrm{s}}(t) \to \partial v^{\mathrm{s}}(t)$ (36) yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\partial v^{\mathrm{s}}(t)} \boldsymbol{D} \cdot \boldsymbol{n} \, \mathrm{d}a^{\mathrm{s}} = \oint_{\partial v^{\mathrm{s}}(t)} [q(\boldsymbol{v}^{\mathrm{s}} - \boldsymbol{v}) - \boldsymbol{j}] \cdot \boldsymbol{n} \, \mathrm{d}a^{\mathrm{s}}. \tag{37}$$

Hence, in comparison with Equation (32), we conclude that if we put

$$\oint_{\partial v^{\mathrm{s}}(t)} \boldsymbol{D} \cdot \boldsymbol{n} \, \mathrm{d}a^{\mathrm{s}} \equiv \int_{v^{\mathrm{s}}(t)} q \, \mathrm{d}v^{\mathrm{s}} \tag{38}$$

the global law of conservation of charge can be fulfilled. Assuming continuity of all fields, Gauss' theorem can be applied to Equation (38) and we arrive at the following local form in regular spatial points:

$$\nabla^{\mathrm{s}} \cdot \boldsymbol{D} = q. \tag{39}$$

This equation is often called *Gauss' law of electrostatics*, although it does not hold only for static conditions. We have explained this at the end of Section 2 for the law of magnetic monopoles: Now this equation is also the result of a flux balance across an open surface with a full time derivative, namely Equation (36), which was then specialized to the case of a closed surface. If we switch to the perspective of a description for material points we must write

$$\nabla \cdot \boldsymbol{D} = q, \tag{40}$$

and all fields are in Eulerian form, for example, $\boldsymbol{D} = \check{\boldsymbol{D}}(\boldsymbol{x}, t)$. On first glance, this should always be possible, because unlike the magnetic and electric fields, the charge potential was initially introduced in context with electrically charged matter. However, as we shall see shortly, the charge potential can be extended into empty space, which is relevant, if concrete initial-boundary value problems are solved. Hence the possibility of always using the material points perspective is slightly

[11] See Appendix A for further explanations of the nomenclature.

deceiving unless we attribute some materialistic quality to the vacuum. This has been done in terms of the so-called æther, which led to ongoing extensive scientific controversies since its first appearance.

Furthermore, we can apply the transport theorem (A18) to (36), observe Equation (39) and after some algebraic manipulations obtain

$$\int_{a^{\mathrm{s}}(t)} \left(\frac{\partial \boldsymbol{D}}{\partial t} + q\boldsymbol{v} \right) \cdot \boldsymbol{n} \, \mathrm{d}a^{\mathrm{s}} = \oint_{\partial a^{\mathrm{s}}(t)} \boldsymbol{H} \cdot \boldsymbol{\tau} \, \mathrm{d}l - \int_{a^{\mathrm{s}}(t)} \boldsymbol{j} \cdot \boldsymbol{n} \, \mathrm{d}a^{\mathrm{s}} \equiv \int_{a^{\mathrm{s}}(t)} (\nabla^{\mathrm{s}} \times \boldsymbol{H} - \boldsymbol{j}) \cdot \boldsymbol{n} \, \mathrm{d}a^{\mathrm{s}}, \tag{41}$$

if we assume continuity of the total current potential $\boldsymbol{H}$ and apply Stokes' theorem. So, finally the local form in regular spatial points of the open surface results

$$-\frac{\partial \boldsymbol{D}}{\partial t} + \nabla^{\mathrm{s}} \times \boldsymbol{H} = q\boldsymbol{v} + \boldsymbol{j}, \tag{42}$$

where all fields depend of the variables $\boldsymbol{x}^{\mathrm{s}}$ and t of spatial notation.

From the perspective of a formulation in terms of material points, we must write

$$-\frac{\partial \boldsymbol{D}}{\partial t} + \nabla \times \boldsymbol{H} = q\boldsymbol{v} + \boldsymbol{j}, \tag{43}$$

where all fields are thought to be written in Eulerian form.

It should be pointed out that Equation (36) is of the form of a balance for the vector flux $\boldsymbol{D}$ through an open nonmaterial surface with a nonmaterial circumference according to Equation (10): The circumferential flux $-\boldsymbol{\varphi}$ is given by $\boldsymbol{H} + \boldsymbol{D} \times \boldsymbol{v}^{\mathrm{s}}$, there is a supply, $\boldsymbol{s} \to q(\boldsymbol{v}^{\mathrm{s}} - \boldsymbol{v}) - \boldsymbol{j}$, but no production, $\boldsymbol{p} = \boldsymbol{0}$. Hence, $\boldsymbol{D}$ is a conserved quantity as it should be, since it is the potential to the total electric charge, which is conserved. Moreover, Equation (37) looks like a volumetric balance of the type (11), if we assign $\psi \to \nabla^{\mathrm{s}} \cdot \boldsymbol{D}$. Then $-\boldsymbol{\phi} \to q(\boldsymbol{v}^{\mathrm{s}} - \boldsymbol{v}) - \boldsymbol{j}$. There is no volumetric supply nor production, $s = 0$, $p = 0$. However, just as in the case of the magnetic monopole law (1), this interpretation is very dangerous.

The first term in the last two equations (including the minus) is known as *Maxwell's displacement current* in the literature and in this context some remarks are in order to put the last two Maxwell equations Equations (39) and (42) into the proper historic context and link them to experiments:

- The "law of electrostatics" (39) was established indirectly in (static) experiments by Coulomb [42], pp. 107–115] and Cavendish [43, pp. 104–113], who studied the inverse square law of repulsion and attraction between electrically charged spheres. This relation is mathematically identical to the cannibalized form of Newton's law of gravitational attraction between two masses m and M at center-to-center distance r, $F = G^{mM}/_{r^2}$. In fact, Cavendish also attempted to measure the gravitational constant G in a similar manner as he investigated experimentally the law of electrostatics. It is for that reason that frequently Lagrange (Sur l'attraction des sphéroïdes elliptiques in Lagrange [44, pp. 619-649) and Gauss (Theoria attractionis corporum sphaeroidicorum ellipticorum homogeneorum methodo nova tractata, [45], pp. 3-22) are given credit to have studied Coulomb's electrostatic attraction and repulsion as well. They did not, or at least they did not care much about the underlying physics! For them, it was a pure mathematical exercise, and if it was related to reality at all, they had gravitation in mind as their texts clearly indicate. One of the first who put their work in context with electromagnetism was Duhem ([46], for example, pp. 1–56). He emphasizes the analogy to gravitation and uses potential theory for finding solutions to Equation (39), albeit purely for electrostatics.
- Equation (39) is visualized in Figure 2 (left): A person touches the charged head of a van der Graaff generator, and as a result, the hair stands on edge. We may interpret this in two ways. First, as the result from the presence of the $\boldsymbol{D}$-field emerging from electric charges. This explains why $\boldsymbol{D}$ is a.k.a. *electric excitation* [47]: Electric charges "excite" the surrounding space even if it is vacuum. Alternatively, one could say that the presence of $\boldsymbol{D}$ "polarizes" the hair and due to that there is repulsion, that is, a force reaction. This is why $\boldsymbol{D}$ is a.k.a. the electric displacement field and it also explains Maxwell's terminology for $-{}^{\partial \boldsymbol{D}}/_{\partial t}$. On the other hand, we have already established the electric force field $\boldsymbol{E}$ and the question arises why and how $\boldsymbol{D}$ and $\boldsymbol{E}$ are different. At this point, it may suffice to say that $\boldsymbol{D}$ was introduced in context with electric charge as a measure of its quantity, as indicated in the Sommerfeld citation from above.
- The Ampère–Øersted law (42) is the mathematical abstraction of results of experiments with live wires by Øersted (Expériences sur l'effet du conflict électrique sur l'aiguille aimant, [48], pp. 1–6) and Ampère (Mémoire sur l'action

FIGURE 2 Experiments demonstrating the conservation of electric charge [40, 41]

mutuelle de deux courants électriques, sur celle qui existe entre un courant électrique et un aimant ou le globe terrestre, et celle de deux aimants l'un sur l'autre, [48], pp. 7–55). They observed that wires carrying an electric current create a magnetic response around them (see Figure 2 (right)), which is made visible by the alignment of iron filings. Moreover, two wires attract and repel each other depending on the direction of the electric current in each wire. Therefore, analogously to the case of $\boldsymbol{D}$, one may say that an electric current excites the surrounding space, which may be vacuum, by creating a "magnetic excitation" $\boldsymbol{H}$, where the latter is a measure of the quantity of the electric current. On the other hand, we have already established the magnetic field $\boldsymbol{B}$ as a force field and surely $\boldsymbol{H}$ and $\boldsymbol{B}$ should be related to each other. We will give a possible answer to this puzzle later in Section 5.

- It is interesting to note that the closed surface ∂v^{s} in Equation (32) related to $\boldsymbol{D}$ can be material or immaterial. In case of the latter, the application of Gauss' theorem might look problematic, because at some radius, we have to move from a region void of matter to a matter containing one, where the charge q begins. We will attempt to clarify this issue in Section 7. Also, in contrast to the experiment leading to Equation (9) the circumference ∂a^{s} in Equation (36) does not have to be material. The iron filings of Figure 2 (right) are just to show that $\boldsymbol{H}$ is present at that position.

3.2 | Comparison of the various presented forms of charge conservation with the literature

As in Section 2.3, we start our discussion with some pertinent physics textbooks.

Feynman considers the principle of the conservation of electric charge in Section 13-2 of Ref. [39]. Not surprisingly, it is not presented in the spirit of rational continuum theory. Rather it is some mix of global and local statements similar looking to Equations (34) and (35). In view of his equation (13.3), it must also be pointed out that in this section of his book, Feynman sees the flux of electric current to be exclusively convective, namely as $q\boldsymbol{v}$, $\boldsymbol{v}$ being some "mean velocity" of "individual charges, say electrons." No difference is made between the microscopic and the continuum scale, and the idea of homogenization does not exist. Moreover, there is no connection between the conservation of charge and the laws of Gauss and Ampère–Øersted. These appear independently in Sections 4.3 and 13-4, in local and global form as variations of Equations (38), (40), (36), and (43). It is interesting to note that Feynman uses in them the electric and the magnetic field, $\boldsymbol{E}$ and $\boldsymbol{B}$, only. As we shall see in Section 5, this is perfectly legitimate if they are based in an inertial frame. However, Feynman does not mention that, and $\boldsymbol{D}$ is for him just a "new vector" that needs to be defined if polarization is to be considered and the total charge density is additively decomposed into a part of free and polarization charge densities (see his Section 10-4). Similarly, $\boldsymbol{H}$ appears as a "new vector field" in context with the decomposition of the electric current (which now no longer is just a convective transport of electric charge density) in Section 36-2, and its fundamental relevance is not further discussed. Rather, it is viewed as problematic only in context with dimensions and units (see his discussion at the end of Section 36-2).

Landau and Lifshitz [9] derive in §29 surrogates of Equations (34) and (35) very similarly to Feynman. In particular, they explicitly say that the electric current density $\boldsymbol{J}$ is given exclusively by the convective flux $q\boldsymbol{v}$. The local equation of continuity is then rewritten by combining q and $\boldsymbol{j}$ in a worldvector (see Section 5 for this concept). In §30, this worldvector is connected to the gradient of the antisymmetric worldtensor consisting of $\boldsymbol{D}$ and $\boldsymbol{H}$, or rather $\boldsymbol{E}$ and $\boldsymbol{H}$, because their 4D formalism seems restricted to inertial frames. We will clarify this issue in Section 5. For the time being, it may suffice to say that the potential character of $\boldsymbol{D}$ and $\boldsymbol{H}$ is addressed by this formalism.

On page 3, Jackson [6] sees the continuity equation (35) as "implicit in the [local] Maxwell equations" (40) and (43), which shows that he puts the Maxwell equations first and the principle of conservation of charge second. Note that a distinction is made between $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{D}$, $\boldsymbol{H}$. Instead of decomposing the total current density, $q\boldsymbol{v} + \boldsymbol{j}$, the electric current is simply referred to as $\boldsymbol{J}$, and it never becomes immediately clear what is contained in that symbol. For example, on pg. 174, it is referred to as "charges in motion" without explaining as to whether this refers to the atomic or to the continuum scale. On pg. 554, the continuity equation reappears in context with converting it into a worldvector equation. Now, it is clearly mentioned that "the microscopic Maxwell equations" are meant, although this is unnecessary as we shall see in Section 5. $\boldsymbol{D}$ is referred to as "electric displacement," $\boldsymbol{B}$ is called "magnetic induction," and $\boldsymbol{H}$ the "magnetic field" (pg. 13). The reason for this is most likely that $\boldsymbol{B}$ is the relevant field in the law of induction and $\boldsymbol{H}$ created a magnetic effect around live electric wires in Øersted's experiments.

It is interesting to note that Sommerfeld [17] emphasizes on pg. 72 of his book the need to determine the two fields $\boldsymbol{E}$ and $\boldsymbol{D}$ by "an imaginary experiment, that is, an observational method, even if it cannot be carried out in practice." He stresses the fact that $\boldsymbol{E}$ is a force field, whilst $\boldsymbol{D}$ is described "as the quantity of electricity which, at a given point, has passed through an area F during the excitation of the field, divided by the magnitude of F." This is essentially the verbal form of the mathematical statement (38). In fact this equation is provided on pg. 15 of his book. However, it is claimed that it holds for a "nonconductor." In the same context, Equation (37) appears (pg. 14), albeit specialized to a closed material surface and without clearly distinguishing between a convective and nonconvective currents. Equation (34) describing the conservation of charge results on pg. 15, and it is clearly not the starting point for introducing the fields $\boldsymbol{D}$ and $\boldsymbol{H}$.

On the other hand, regarding the formal mathematical correspondence of the equation of the nonexistence of magnetic monopoles (14) (which means there are no magnetic charges) and (38), which tells us that the source of the $\boldsymbol{D}$ field are the electric charges, he quotes Hertz and says: "The first Equation (6 a) [(38)] may be expressed, with Hertz, in the form: There is no true magnetism. In this statement, one proceeds from the assumption, formerly regarded as obvious, that **B** is the magnetic analog of **D**. From our standpoint, however, this analog is **H**, and not **B**. We shall hence have to relate the definition of magnetism, in particular of the pole strength P (see Section 7), not to **B** but to **H**." It is for such reasons that Sommerfeld calls $\boldsymbol{H}$ the magnetic excitation. In fact a discussion regarding the difference between $\boldsymbol{B}$ and $\boldsymbol{H}$ starts again on page 89. Clearly for him, $\boldsymbol{B}$ can be determined from the force, so measured, acting on a test body. Interestingly, regarding $\boldsymbol{H}$, he anticipates the Maxwell–Lorentz–æther relations (which we will discuss in Section 5) and says that it is simply proportional to $\boldsymbol{B}$. With respect to the name of the field $\boldsymbol{D}$, he says quite correctly: "We shall call this preferably electric "excitation," but shall also frequently employ, particularly in the first part of these Lectures, the customary term "dielectric displacement" (Maxwell's designation)." For Sommerfeld, it is important to make this distinction because: "With regard to Maxwell's designation "dielectric displacement," we note that it fits strictly not the vector **D** itself, but only that fraction of **D** which arises from the presence of ponderable matter and which wiIl later (see Section llC) be designated as the polarization **P**."

Finally, in view of the fact that we introduced $\boldsymbol{D}$ and $\boldsymbol{H}$ as the potentials to charges and currents, we can also come back to the quote by Sommerfeld on cause and effect in Section 2.2. We may now definitely say that electric charges and currents are the causes and $\boldsymbol{D}$ and $\boldsymbol{H}$ are the corresponding effects. In this sense, Sommerfeld is correct. Moreover, the charge and current are clearly entities of quality and so are their offsprings, $\boldsymbol{D}$ and $\boldsymbol{H}$, as Sommerfeld claims.

Becker refers to the conservation of charge in global and local form when he introduces the concept of an electric current ([23], pp. 102, [24], pg. 83, and [8], section 38) It is a consequence of the Maxwell equations and not the primary principle. $\boldsymbol{D}$ is referred to as electric displacement. Just like Jackson, he calls $\boldsymbol{H}$ the magnetic field and $\boldsymbol{B}$ the magnetic induction. Gauss' law in global and local form, (38) and (40), respectively, is written in terms of $\boldsymbol{E}$ and not $\boldsymbol{D}$ ([23], pp. 50, [24], pg. 15, and [8], Section 19), which is correct, if stated in an inertial frame, but this fact is not explained clearly.

Being the predecessor to Becker's books Abraham argues exactly the same ([25], Section 36, pp. 129) as far as Gauss' law is concerned. $\boldsymbol{D}$ is called electric displacement and $\boldsymbol{H}$ is called magnetic field strength (pg. 217). Obviously following Hertz (see Sommerfeld's remark above) Abraham sees $\boldsymbol{E}$ as the electric analog to $\boldsymbol{H}$ and $\boldsymbol{B}$ the magnetic analog to $\boldsymbol{D}$ and $\boldsymbol{E}$ (much more explicit than Becker acknowledges the need for four fields by saying on pg. 217: "Dabei wird der elektrischen Feldstärke $\mathfrak{E}$ die "magnetische Feldstärke" $\mathfrak{H}$ gegenübergestellt, während die "magnetische Induktion" $\mathfrak{B}$ der mit 4π multiplizierten elektrischen Verschiebung $\mathfrak{D}$ gegenübergestellt wird."[12]) Hence his opinion is contrary to Sommerfeld's

[12] "Thereby the electric field strength $\mathfrak{E}$ is juxtaposed to the "magnetic field strength" $\mathfrak{H}$, while the "magnetic induction" $\mathfrak{B}$ is juxtaposed to the electric displacement $\mathfrak{D}$ multiplied by 4π."

(see the citation above) and the one expressed in this article. Not too surprisingly this sentence does not appear at all in Becker's supervised English translation of Abraham's book [26] at the beginning of the corresponding Chapter VII. A conservation law for charge is not discussed at all by Abraham.

We will now compare the formulation presented in Section 3.1 with various monographs on continuum electrodynamics. For Toupin, conservation of charge is a fundamental principle of electrodynamics. It is available in global as well as in local form ([1], pg. 673 and 677) for material volumes and particles, that is, Equations (34) and $(35)_1$, respectively. The idea of $\boldsymbol{D}$ and $\boldsymbol{H}$ being the potentials of charge and current is elaborated in detail in Section 276. The various designations of these two fields, which have already been mentioned, are discussed in a footnote on pg. 674. It is fair to say that our formulation in Section 3.1 was inspired by Toupin's Handbook article. This is also true for the books by Müller ([28], Section 9.2.2, and [20], Chapter 9), Kovetz ([3], pp. 1, and [4], pp. 1), and Hutter et al. ([5], pp. 12), or the articles by Steigmann [49] and Morro [27], which, however, all focus on material volumes and material particles. On the other hand, they make a distinction between convective and nonconvective electric current.

In the work of Eringen and Maugin, the principle of conservation of electric charge is expressed in global and local form on the continuum level ([29], pg. 73 and 74). However, it is not put at the top in the sense that charge and current potentials $\boldsymbol{D}$ and $\boldsymbol{H}$ are introduced to solve it formally. This may be due to the fact that they start on the atomistic level and move up to the continuum scale by homogenization. Also, $\boldsymbol{D}$ and $\boldsymbol{H}$ are both referred to as electric displacement and magnetic field, respectively (pg. 91 and pg. 100). They result after homogenization of their atomic counterparts, called $\boldsymbol{d}$ and $\boldsymbol{h}$, which are introduced formally by relating them to atomic electric polarization and atomic electric field or atomic magnetic polarization and atomic magnetic field, pg. 39. In fact, sometimes both $\boldsymbol{H}$ and $\boldsymbol{B}$ are referred to as magnetic fields. Note that it is perfectly legitimate to make no distinction as long as the equations are considered in an inertial frame and Heaviside–Lorentz units are used. Then $\boldsymbol{E} = \boldsymbol{D}$ and $\boldsymbol{H} = \boldsymbol{B}$, as we shall see in Sections 4 and 5. However, for didactic and conceptional reasons, this is not an advisable way to go. But, a distinction is made between convective and nonconvective electric current, pg. 51.

Finally, it should be mentioned that in the book by Zhilin [32] the Ampère-Øersted law is also "derived" in Section 6.6 in local form based on micropolar theory. Moreover, Zhilin puts Maxwell's equations (40) and (43) first (written in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$ and without convective currents) and considers the principle of conservation of charge as secondary and not as fundamental as we do (pg. 354): "Физики предпочитают называть уравнение (7.55) [i.e., (35)] законом сохранения заряда и считать его законом Природы."[13] and "Таким образом, уравнение (7.55) ни в коем случае нельзя трактовать как закон Природы — это именно необходимое условие разрешимости классических уравнений Максвелла."[14] Spoken like a true mechanics scholar and mathematician who is appalled by the physicists' desire for new laws of nature! As we shall explain later, there are two types of balances. Balances for conserved as well as nonconserved physical quantities. In particular, according to Truesdell and Toupin [1], Maxwell's equations are the result of the principles of conservation of magnetic flux and electric charge, both of which can be expressed in terms of a balance. In context with mechanical quantities, this issue was also addressed recently in Müller et al. [37].

4 | DIMENSIONS AND UNITS OF THE ELECTROMAGNETIC FIELDS

4.1 | The Maxwell–Lorentz–æther relations

Recall the experimental findings in Sections 2.1 and 3.1:

- The electric field $\boldsymbol{E}$ exerts a force on a charge at rest w.r.t. an inertial frame of reference. By definition the force acts in the direction of the vector $\boldsymbol{E}$ for positive charges and vice versa.
- A magnetic field $\boldsymbol{B}$ exerts a force on a charge moving with the velocity $\boldsymbol{v}$ w.r.t. an inertial frame of reference. For a positive charge the force follows the direction $\boldsymbol{v} \times \boldsymbol{B}$ and vice versa.

[13] "Physicists prefer to call equation (7.55) the law of conservation of charge and consider it a law of nature. From the point of view of mechanics, in the general case, there are no conservation laws, but there are equations for the balance of certain quantities."
[14] "Thus, the equation (7.55) can in no case be interpreted as a law of nature—it is precisely the necessary condition for the solvability of the classical Maxwell equations."

- Charges at rest w.r.t. an inertial frame of reference excite the surrounding space by producing the electric excitation field (or charge potential) $\boldsymbol{D}$, which is felt in terms of a force by a test charge at rest w.r.t. an inertial frame placed at a certain distance from the exciting charge.
- Electric currents, which on the continuum scale are decomposed into a convective and a nonconvective part, and on the atomic scale may simply be interpreted as charged matter moving w.r.t. an inertial frame of reference, excite the surrounding space by producing a magnetic excitation field (or current potential) $\boldsymbol{H}$. The presence of this field can be felt in terms of a force by magnetizable matter (iron filings) or another live wire placed parallel to the original one. The force on both wires is repulsive if their electric currents run in the opposite directions and vice versa.
- The aforementioned inertial frame of reference is a primitive concept and not explained any further. It is assumed as given and existing.

Judging by these facts, one must conclude that the pairs of electric and magnetic fields, $(\boldsymbol{E}, \boldsymbol{D})$ and $(\boldsymbol{B}, \boldsymbol{H})$, respectively, are related to each other. The simplest relationship is to postulate proportionality in an inertial frame, and that we shall do. This postulate is known as the Maxwell–Lorentz–æther relations. It is another axiom of electrodynamics. This strange name has a historic origin, because it was believed that in an inertial frame, there is a medium at rest, called the æther, which transports the electromagnetic fields. However, following Newton by saying *hypotheses non fingo,* we will not attempt to explain how electromagnetic fields come into being, nor how they are transported through empty space, nor what electric charge actually is. But we shall explore the factors of proportionality in more detail and denote them temporarily by α and β, such that

$$\boldsymbol{D} = \alpha \boldsymbol{E}\ , \quad \boldsymbol{B} = \beta \boldsymbol{H}. \tag{44}$$

In order to be more specific, we must now define physical units for the electric charge. In this context, note that the property "electric charge" is a dimension and not a unit, just like mass or length or time are dimensions that exist independently of a physical unit. Units are just used to quantify a dimension and make it accessible to engineering experiments.

Before we get to the units of electric charge and of the electromagnetic fields, we must conclude from the above that

$$\dim(\boldsymbol{E}) = \frac{\text{force}}{\text{charge}},\ \dim(\boldsymbol{D}) = \frac{\text{charge}}{\text{length}^2},\ \dim(\boldsymbol{B}) = \frac{\text{force}}{\text{charge} \times {}^{\text{length}}/_{\text{time}}},\ \dim(\boldsymbol{H}) = \frac{{}^{\text{charge}}/_{\text{time}}}{\text{length}}. \tag{45}$$

Hence, the dimensions of the electromagnetic fields are very different, and this also shows that the two electric and the two magnetic fields are very different in their nature. Now it will be even more surprising that it is possible to choose the unit of electric charge such, that they all four fields have the same units. Unfortunately, this unit for electric charge is not the choice of the modern engineer. We proceed to discuss this point.

4.2 | Coulomb's and Biot–Savart's laws revisited

In order to discuss possible units for the electric charge and current, we start from Maxwell's equations in global form. Specifically, we evaluate Equation (38) for an electric (point) charge e at rest and Equation (36) for the stationary case of a constant electric current I flowing through a wire in $\boldsymbol{e}_z$ direction. The first problem has then full spherical and the second full cylindrical symmetry. This implies that we obtain at a radial distance r:

$$\boldsymbol{D} = D(r)\boldsymbol{e}_r = \frac{e}{4\pi r^2}\boldsymbol{e}_r\ , \quad \boldsymbol{H} = H(r)\boldsymbol{e}_\theta = \frac{I}{2\pi r}\boldsymbol{e}_\theta. \tag{46}$$

Obviously, the charge potential $\boldsymbol{D}$ is purely radial (in the direction of the unit vector $\boldsymbol{e}_r$), r being the radial distance from the location of the charge. The current potential $\boldsymbol{H}$ is pointing in circumferential direction $\boldsymbol{e}_\theta$ of a circular area of radius r, such that the wire penetrates this area perpendicularly in its center.

Also, recall that an electric field $\boldsymbol{E}$ exerts a force $\boldsymbol{F}_\text{e}$ on a point test charge e' at rest. Moreover, a magnetic field $\boldsymbol{B}$ results in a force contribution $\text{d}\boldsymbol{F}_\text{m}$ on a continuous charge distribution moving with the velocity $\boldsymbol{v}$ in the direction of an infinitely long straight wire oriented in $\boldsymbol{e}_z$-direction. This corresponds to an electric current $\boldsymbol{I}' = I'\boldsymbol{e}_z$, which is equivalent to a charge flow per unit time, $I'\text{d}z\ \boldsymbol{e}_z$, that traveled the distance $\text{d}z$. Of course, $\text{d}z$ denotes a small piece of the wire along which the current is flowing. Clearly for an infinitely large wire $\boldsymbol{F}_\mathrm{m}$ would become infinite. This is why a force per unit length, $\mathrm{d}\boldsymbol{F}_\mathrm{m}/\mathrm{d}z$, must be considered in the second case. Both forces are then given by

$$\boldsymbol{F}_\mathrm{e} = a e' \boldsymbol{E} \ , \quad \frac{\mathrm{d}\boldsymbol{F}_\mathrm{m}}{\mathrm{d}z} = b \boldsymbol{I}' \times \boldsymbol{B}. \tag{47}$$

Note that it was necessary to introduce coefficients a and b because the units of electric charge and electric current have not been defined yet, so that they can be related to a force, the unit of which has not been specified either.

By combining Equations (44) and (47), we obtain

$$\boldsymbol{F}_\mathrm{e} = \frac{a}{4\pi\alpha}\frac{e\,e'}{r^2}\boldsymbol{e}_r \ , \quad \frac{\mathrm{d}\boldsymbol{F}_\mathrm{m}}{\mathrm{d}z} = -\frac{b\beta}{2\pi}\frac{I\,I'}{r}\boldsymbol{e}_r. \tag{48}$$

The first relation is Coulomb's law, which was mentioned in Section 3.1. Now it explicitly shows that it has the form of Newton's law of gravitation in its cannibalized version, that is, not in field formulation in terms of the Poisson equation for gravitation. The second relation is known as Biot–Savart's law, which quantifies the forces arising in Ampère–Øersted magnetic experiments with wires.

We are now in a position to define units of electric charge. We start with the so-called cgs-system, which stands for centimeter, gram, seconds, all of which are units of mechanics. In fact, this was all that was known when electromagnetism appeared in physics. Therefore, it was only natural to "include" the new quality electric charge in a mechanics framework. Indeed, the unit of electric charge, called 1 esu = electro-static unit, was defined by using Equation $(48)_1$ as a corner stone: Two point charges of the strength 1 esu placed at a distance of 1 cm result in a force of 1 dyn=$1\mathrm{g\,cm}/\mathrm{s}^2$. Hence there is no real need for the factor $\frac{a}{4\pi\alpha}$ any more and we can deal with it at our leisure:[15] In the "pure" cgs-system (a.k.a. Gaussian system), it was decided to put it simply equal to one. In the Heaviside–Lorentz system (a.k.a. Rationalized Gaussian system, which is favored by Eringen and Maugin, [29]) it was decided to keep the factor 4π, so that $\frac{a}{\alpha} = 1$. We conclude by looking at Equation $(48)_1$ that in both these unit systems, 1 esu can be expressed in pure mechanical form by 1 $\mathrm{g}^{1/2}\mathrm{cm}^{3/2}\mathrm{s}^{-1}$. Consequently, the charge (volume) density q has the unit $\mathrm{g}^{1/2}\mathrm{cm}^{-3/2}\mathrm{s}^{-1}$.

It should be pointed out that this affects the way how the Maxwell equations must be written. It will be necessary to include additional constants, and the mathematical factor 4π is just one of them. Much more critical is the advent of the physical quantity known as the speed of light, c. It always appears when a velocity related quantity is concerned, for example, the electric current density, $\boldsymbol{j}$, or the convective electric current density $q\boldsymbol{v}$. In fact, we must insist that the unit of electric current corresponds to its dimension, which is electric charge per time. So the unit of an electric current must be $\mathrm{g}^{1/2}\mathrm{cm}^{3/2}\mathrm{s}^{-2}$ and the current densities $q\boldsymbol{v}$ or $\boldsymbol{j}$, which are electric currents distributed over a surface, have $\mathrm{g}^{1/2}\mathrm{cm}^{-1/2}\mathrm{s}^{-2}$. Then a simple unit check of Equation $(48)_2$ reveals that the units of the factor $b\beta$ must be $\mathrm{s}^2\mathrm{cm}^{-2}$, which is the inverse of the units of the square of a velocity. Therefore, in the Gauss system, the choice was $b\beta = 4\pi/c^2$ and in the Heaviside–Lorentz system $b\beta = 1/c^2$. In summary, Equation (48) now reads

$$\begin{aligned} &\boldsymbol{F}_\mathrm{e} = \frac{e\,e'}{r^2}\boldsymbol{e}_r \ , \quad \frac{\mathrm{d}\boldsymbol{F}_\mathrm{m}}{\mathrm{d}z} = -\frac{2 I/c\ I'/c}{r}\boldsymbol{e}_r \qquad \text{(Gauss system)} \\ &\boldsymbol{F}_\mathrm{e} = \frac{1}{4\pi}\frac{e\,e'}{r^2}\boldsymbol{e}_r \ , \quad \frac{\mathrm{d}\boldsymbol{F}_\mathrm{m}}{\mathrm{d}z} = -\frac{1}{2\pi}\frac{I/c\ I'/c}{r}\boldsymbol{e}_r \qquad \text{(Heaviside} - \text{Lorentz system).} \end{aligned} \tag{49}$$

However, we are not finished with the Gaussian or the Heaviside–Lorentz system yet. If we take a look at the dimensions of each field shown in Equation (45) and compute the units, we find

$$\begin{aligned} &\mathrm{units}(\boldsymbol{E}) = \mathrm{g}^{1/2}\mathrm{cm}^{-1/2}\mathrm{s}^{-1}, \ \mathrm{units}(\boldsymbol{D}) = \mathrm{g}^{1/2}\mathrm{cm}^{-1/2}\mathrm{s}^{-1}, \\ &\mathrm{units}(\boldsymbol{B}) = \mathrm{g}^{1/2}\mathrm{cm}^{-3/2}, \ \mathrm{units}(\boldsymbol{H}) = \mathrm{g}^{1/2}\mathrm{cm}^{1/2}\mathrm{s}^{-2}. \end{aligned} \tag{50}$$

Obviously the units of the electric field and the charge potential are the same, so that they can be simply added and compared in the Gauss and in the Heaviside–Lorentz system. This is a considerable advantage if one wishes to assess their strength. However, the units of the magnetic field and of the current potential are still different. In this form, they

[15] Also see Carron [50] for a detailed discussion.

cannot directly be compared. But the ratio between the units of $\boldsymbol{H}$ and $\boldsymbol{B}$ is ${}^{\mathrm{cm}^2}/_{\mathrm{s}^2}$, so we can easily rescale them by using the speed of light as a scaling factor. And therefore the following assignments in the Gauss and in the Heaviside–Lorentz system are made:

$$\boldsymbol{E} \to \boldsymbol{E}^{\mathrm{G}},\ \boldsymbol{D} \to \boldsymbol{D}^{\mathrm{G}},\ \boldsymbol{B} \to c\boldsymbol{B} \equiv \boldsymbol{B}^{\mathrm{G}},\ \boldsymbol{H} \to \frac{\boldsymbol{H}}{c} \equiv \boldsymbol{H}^{\mathrm{G}}. \tag{51}$$

The index "G" is a reminder that these are fields in the Gauss system and the same assignment holds for the Heaviside–Lorentz case. It is unfortunate that textbooks rarely make use of such accurate denominations and omit indices "G" or "HL" (Heaviside–Lorentz) completely, because "everybody knows that." In any case, now the strength of all four fields can directly be compared, which is good for practical purposes but is also the root of much misunderstanding.

For the charge and current density, we must pay attention because factors of 4π must be assigned or not depending on whether we are working in the Gauss or in the Heaviside–Lorentz system so that the Maxwell equations will lead to Equation (49):

$$q \to 4\pi q^{\mathrm{G}},\ \boldsymbol{j} \to 4\pi \boldsymbol{j}^{\mathrm{G}},\ q \to q^{\mathrm{HL}},\ \boldsymbol{j} \to \boldsymbol{j}^{\mathrm{HL}}. \tag{52}$$

However, so far our choices of units concerned only the quotients or products ${}^{a}/_{\alpha}$ and $b\beta$ in Equation (48) leading to Equation (49). But what about each of the four coefficients? In order to determine them uniquely, we demand that the Maxwell–Lorentz æther relations (44) shall be simple and look like this in the Gaussian as well as in the Heaviside–Lorentz system:

$$\boldsymbol{D} = \boldsymbol{E}\ ,\quad \boldsymbol{B} = \boldsymbol{H}, \tag{53}$$

where we have already omitted indices at the field symbols.

Hence, $\alpha = 1$ and $\beta = 1$. However, this makes the two magnetic and the two electric fields in an inertial frame indistinguishable from each other, if the Gaussian or the Heaviside–Lorentz system is used. Moreover, $a = 4\pi$ and $b = {}^{4\pi}/_{c^2}$ for Gaussian units or without the 4π for Heaviside–Lorentz.

It was emphasized already but cannot be mentioned often enough: On the one hand side, all of this brings a certain benefit of simplicity, but on the other hand, it opens up room for endless discussions and confusion what the differences between the two electric fields $(\boldsymbol{E}, \boldsymbol{D})$ and the two magnetic fields $(\boldsymbol{B}, \boldsymbol{H})$ are and as to whether we really need two of them each. The answer is a clear "yes" for two reasons: (a) they are fundamentally different in concept, as becomes evident if we look at the dimensions in Equation (45); (b) the Maxwell–Lorentz–æther relations do not keep the simple form (53) (or the simple proportionality (44)) if we switch to a noninertial frame, as we shall see in Section 5 .

After these careful deliberations, we are now in a position to state Maxwell's equations for the Gaussian system in global form and in spatial notation, based on Equations (14), (9), (38), and (36):[16]

$$\begin{aligned}
&\oint\limits_{\partial v^{\mathrm{s}}(t)} \boldsymbol{B}\cdot\boldsymbol{n}\,\mathrm{d}a^{\mathrm{s}} = 0\ ,\quad \frac{\mathrm{d}}{\mathrm{d}ct}\int\limits_{a^{\mathrm{s}}(t)} \boldsymbol{B}\cdot\boldsymbol{n}\,\mathrm{d}a^{\mathrm{s}} = -\oint\limits_{\partial a^{\mathrm{s}}(t)} \left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right)\cdot\boldsymbol{\tau}\,\mathrm{d}l,\\
&\oint\limits_{\partial v^{\mathrm{s}}(t)} \boldsymbol{D}\cdot\boldsymbol{n}\,\mathrm{d}a^{\mathrm{s}} \equiv 4\pi\int\limits_{v^{\mathrm{s}}(t)} q\,\mathrm{d}v^{\mathrm{s}}\ ,\\
&\frac{\mathrm{d}}{\mathrm{d}ct}\int\limits_{a^{\mathrm{s}}(t)} \boldsymbol{D}\cdot\boldsymbol{n}\,\mathrm{d}a^{\mathrm{s}} = \oint\limits_{\partial a^{\mathrm{s}}(t)} \left(\boldsymbol{H} + \boldsymbol{D}\times\frac{\boldsymbol{v}^{\mathrm{s}}}{c}\right)\cdot\boldsymbol{\tau}\,\mathrm{d}l + \frac{4\pi}{c}\int\limits_{a^{\mathrm{s}}(t)} [q(\boldsymbol{v}^{\mathrm{s}} - \boldsymbol{v}) - \boldsymbol{j}]\cdot\boldsymbol{n}\,\mathrm{d}a^{\mathrm{s}}.
\end{aligned} \tag{54}$$

[16] In the Heaviside–Lorentz system, the factor 4π must be omitted.

In local form, this reads[17]

$$\nabla^s \cdot \boldsymbol{B} = 0\,, \qquad \frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} + \nabla^s \times \boldsymbol{E} = \boldsymbol{0}\,,$$
$$\nabla^s \cdot \boldsymbol{D} = 4\pi q\,, \quad -\frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} + \nabla^s \times \boldsymbol{H} = \frac{4\pi}{c}(q\boldsymbol{v} + \boldsymbol{j}). \tag{55}$$

The equations describing the conservation of charge keep their forms (e.g., (32) or (33)) in both unit systems, whereas for the Lorentz force density $\boldsymbol{f}$ on (moving) charge densities or electric currents, we must write (no factors 4π)

$$\boldsymbol{f} = q\boldsymbol{E} + \frac{1}{c}(q\boldsymbol{v} + \boldsymbol{j}) \times \boldsymbol{B}. \tag{56}$$

By looking at these equations, another advantage of the Gauss or the Heaviside–Lorentz system becomes visible: Velocity-related quantities are normalized by the speed of light, and this allows us to identify them as “small” or “large” depending on whether these velocities are close to the speed of light or not. This is why the Gauss and the Heaviside–Lorentz system are popular whenever fundamental considerations are of interest.

However, in (electrical) engineering, where materials do not move in any way close to the speed of light, these systems are rarely used any more and have been replaced by SI units, a.k.a. MKSA system.[18] Here the property charge is introduced indirectly through the electric current, which by dimension is charge per unit time. Until 2019, the unit of the electric current, 1 A(mpere), was based on the statement that “The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed one meter apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newtons per meter of length” [51]. Therefore, we conclude from Equation $(48)_2$ that $b\beta = 4\pi \times 10^{-7}\,\frac{\mathrm{N}}{\mathrm{A}^2}$. This combination is denoted by the symbol μ_0 and known as the *magnetic permeability of the vacuum*. This interesting terminology will become clearer when we will take a look into simple constitutive equations in Section 6.

This in mind we now take a second look at Equation $(48)_1$, which we want to make ready for MKSA units. Because here we insist that charges are measured in units of Coulomb (= As) we conclude that the unit of $\frac{a}{\alpha}$ must be $\frac{\mathrm{Nm}^2}{\mathrm{A}^2}$. These are the units of μ_0 multiplied by the square of a velocity, hence in the MKSA system, we choose $\frac{a}{\alpha} = \mu_0 c^2$. This factor is also denoted by $\frac{1}{\epsilon_0}$, where ϵ_0 is called the *permittivity of the vacuum*, a term that will also become clearer in Section 6. Consequently, Coulomb’s and Biot-Savart’s laws (48) read in MKSA units:

$$\boldsymbol{F}_{\mathrm{e}} = \frac{1}{4\pi\epsilon_0}\frac{e\,e'}{r^2}\boldsymbol{e}_r\,, \quad \frac{\mathrm{d}\boldsymbol{F}_{\mathrm{m}}}{\mathrm{d}z} = -\frac{\mu_0}{2\pi}\frac{I\,I'}{r}\boldsymbol{e}_r. \tag{57}$$

Note that Maxwell’s equations and the equations describing conservation of charge keep the form shown in Sections 2.1 and 3.1. No factors whatsoever need to be added. Moreover, according to the dimensions shown in Equation (45), the MKSA units of the various fields are as follows: the electric field $\boldsymbol{E}$ in $\frac{\mathrm{N}}{\mathrm{As}}$, the charge potential $\boldsymbol{D}$ in $\frac{\mathrm{As}}{\mathrm{m}^2}$, the magnetic field $\boldsymbol{B}$ in $\frac{\mathrm{N}}{\mathrm{Am}}$, and the current potential $\boldsymbol{H}$ in $\frac{\mathrm{A}}{\mathrm{m}}$. They have all different units and cannot directly be compared.

Furthermore, the form of the Maxwell–Lorentz æther relations (44) needs to be clarified. In MKSA, the choices are $a = 1$, $b = 1$ and because of $\frac{a}{\alpha} = \frac{1}{\epsilon_0}$ and $b\beta = mu_0$, we then find

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E}\,, \quad \boldsymbol{B} = \mu_0 \boldsymbol{H}\,, \quad \epsilon_0\mu_0 = \frac{1}{c^2}. \tag{58}$$

Finally, we find for the Lorentz force density of an electric charge density q (in units of $\frac{\mathrm{As}}{\mathrm{m}^3}$) and the electric current $\boldsymbol{j}$ (in units of $\frac{\mathrm{A}}{\mathrm{m}^2}$):

$$\boldsymbol{f} = q\boldsymbol{E} + (q\boldsymbol{v} + \boldsymbol{j}) \times \boldsymbol{B}. \tag{59}$$

[17] If a description for material points is desired, ∇^s must simply be replaced by ∇. Also note that if the divergence operation $\nabla^s\cdot$ is applied to the Faraday induction law in local regular points $(55)_2$, and the law of magnetic monopoles $(55)_1$ is observed, we have an identity. If this operation is applied to the Ampère–Øersted law $(55)_4$, and Gauss’ law $(55)_3$ is observed, the conservation law of charge in local regular form $(33)_1$ (in Heaviside Lorentz units) results.

[18] Système Internationale and Meter, Kilogram, Second, Ampere.

At this point, the following statement is appropriate to make: If one wishes to rewrite electromagnetic relations from SI to the Gauss system, the necessary substitutions read

$$\boldsymbol{E}^{\mathrm{SI}} \to c\sqrt{\frac{\mu_0}{4\pi}}\boldsymbol{E}^{\mathrm{G}},\ \boldsymbol{D}^{\mathrm{SI}} \to \frac{1}{c\sqrt{4\pi\mu_0}}\boldsymbol{D}^{\mathrm{G}},\ \boldsymbol{B}^{\mathrm{SI}} \to \sqrt{\frac{\mu_0}{4\pi}}\boldsymbol{B}^{\mathrm{G}},\ \boldsymbol{H}^{\mathrm{SI}} \to \frac{1}{\sqrt{4\pi\mu_0}}\boldsymbol{H}^{\mathrm{G}}. \tag{60}$$

For the Heaviside–Lorentz system, the factor 4π must be omitted.

4.3 | Comparison with the literature

All classic physics textbooks discuss the issue of dimensions and units in electromagnetism or mention it at least.

Feynman [22] introduces units for all electromagnetic quantities as need arises. The focus is on the MKSA system, the issue of the Gaussian or the Heaviside–Lorentz system is not addressed at all.

Landau and Lifshitz [9] favor the Gaussian system of units but they mention Heaviside–Lorentz in a footnote (pg. 73) saying that in this case, the "field equations have a more convenient form (4π does not appear)… ." Deeper issues are not investigated and the MKSA system does not appear at all.

Jackson [6] devotes a long appendix to the issue of dimensions and units. It is fair to say that he is very much aware of the explosive nature of this topic. As in the present article, his presentation centers around Coulomb's and Biot-Savart's law. What can be considered as a disadvantage is that he does not consistently distinguish between the need for two electric and two magnetic fields that are fundamentally different in nature. In fact, all equations are presented for the case of an inertial system, where the quantities in the two sets are proportional to each other. Being no specialist in continuum theory (see the discussion on the continuum mechanics related literature in this section), the aspect of the Maxwell–Lorentz æther relations is also not addressed. The fact that $\boldsymbol{E}$ and $\boldsymbol{B}$ have the same units in the Gaussian system is mentioned (pg. 780) but its relevance is left unexplained. However, it is mentioned (pg. 780) that MKSA is geared toward engineering practice while the Gaussian system is "more suitable for microscopic problems" and for "the relativistic electrodynamics of the latter part of the book, we retain Gaussian units as a matter of convenience." In summary, the appendix can be used for easy reference and conversion but it does not really reveal the issues behind.

Sommerfeld [17] is quite aware of the issues behind dimensions and unit of electromagnetic quantities and starts the discussion already in his preface (pg. VI). The plot thickens in §7 C/D and in §8, all of which deal with this topic. Quite revealing of his profound knowledge of the history and differences between the Gaussian and the Heaviside–Lorentz point-of-view are the sentences on pg. 42/43: "Historically the forms (8) and (12) [our Equation $(49)_1$] of Coulomb's law result from an effort to approach as closely as possible the customary form of Newton's law. We shall denote the suppression of the numerical factor 4π in Coulomb's law as conventional, our retention of it as rational. It is in fact evident that in a problem with spherical symmetry, such as the Coulomb problem, the factor 4π is appropriate." and "Heaviside makes the following striking comparison: In passing from the measurement of distance to the measurement of area, one might define as unit of area the area of a circle of radius 1. This would be logically possible. It would however lead to the strange result that a square with the side 1 would have the area $1/\pi$. Everyone would then say that π was at the wrong place."

Sommerfeld also emphasizes the need to distinguish between two electric and two magnetic fields by using different units. In this context, he says on pg. 45 "It is to be welcomed, from our point of view, that, by international agreement, separate designations gauss and oersted have been introduced for the two magnetic vectors **B** and **H**. Historically, the name gauss also seems proper for **B**, since Gauss' methods of determining magnetic moment rest on measurements of force and hence refer to **B** and not to **H**. The unhappy term 'magnetic field' for **H** should be avoided as far as possible. It seems to us that this term has led into error none less than Maxwell himself, who, in art. 625 of the Treatise puts the force exerted by the field on a magnetic pole m equal to $m\mathbf{H}$." He alludes to the fact that in the Gaussian system, all fields have the same units and that this might cause problems (pg. 50): "Furthermore the dimensions of the two pairs become the same, since now … ε_0 and μ_0 in (14a), become pure numbers, … The Gaussian system obscures the dimensional character of the four fundamental vectors **E**, **D**, **B**, **H** completely… ."

The 1957 edition of the book by Becker [23] favors Gaussian units. However, he emphasizes already in his preface its importance for fundamental investigations (in context with relativity theory) and the usefulness of the MKSA system for the engineering practice. The latter is then more or less exclusively used in the later edition [24] and in the translation [8]. Heaviside–Lorentz units are also mentioned ([23], pg. 51) but considered as not established neither in pure nor in applied physics. Becker's mindset becomes very clear from the following citation in the Foreword of [8]: "The transition from the four basic units of the MKSA system—units which at first sight seem to appear naturally—to the three basic units of the Gaussian CGS system, requires that Coulomb's law be not only considered as an experimental law, but that at the same time it be regarded as the defining equation for the unit of charge in the Gaussian system. This is because the resulting constant of proportionality, having the dimensions of force times length squared, divided by charge squared, is arbitrarily assumed to be dimensionless and equal to 1. This procedure, violently criticized by certain advocates of the MKSA system, corresponds exactly with today's custom in high-energy physics of combining the length and time dimensions with one another through the arbitrary assumption that the velocity of light in a vacuum is dimensionless and equal to 1." It should also be pointed that the differences between the two electric and two magnetic fields, their proportionality in terms of the Maxwell–Lorentz–æther relations, as well as the restriction to an inertial system are not explained in a crystal-clear manner. In fact, we read on pg. 49 of [24]: "Die Einführung des Verschiebungsvektors $\boldsymbol{D}$ und seines Gegenstückes $\boldsymbol{D}^*$ ist daher im Grunde erst für die Behandlung der Elektrodynamik in der Materie erforderlich."[19] This shows a certain laxness in view of the criticality of the underlying issues.

The book by Abraham and Föppl [25] takes us back in time where the definition of units and the understanding of phenomena was still in its infancy. This becomes evident by cute formulations, such as: "Man hat ganz willkürlich der Elektrizität des mit dem geriebenen Katzenfell berührten Kügelchens das positive Vorzeichen gegeben und infolgedessen der Elektrizität der geriebenen Siegellackstange das negative."[20]. It must be mentioned that the 1932 English translation of this book [26] is rather different from the German original and shows the clear handwriting of Becker who among other things added drawings and an appendix on Gaussian units. However, as it was already indicated above, the clear distinction between the two electric and two magnetic fields gets lost. For Abraham, $\boldsymbol{D}$ is the aforementioned excitation of the vacuum by the presence of charge, which he describes on pg. 145 in a rather baroque way by mentioning the æther: "Sie legt die Annahme nahe, daß auch der leere Raum elektrische Wirkungen vermittelt, daß er der Sitz eines elektrischen Feldes sein kann. Da man früher dem Raum nur geometrische Eigenschaften beizulegen pflegte, so hat man für den mit elektromagnetischen Eigenschaften behafteten Raum ein besonderes Wort "Äther" eingeführt."[21] and "Wir verbinden heute mit dem Worte "Äther" keineswegs die Vorstellung einer hypothetischen Substanz; vielmehr gebrauchen wir dieses historisch überlieferte Wort heute als Abkürzung, wenn wir ohne Weitschweifigkeiten von dem Raume als Träger eines elektromagnetischen Feldes sprechen."[22]

Not surprisingly, the wording is more "politically correct" in the Becker translation (pg. 70): "Instead of the word vacuum we also occasionally use the word "æther", not connecting it in any way with the idea of a hypothetical substance, but merely using the word when we are speaking of space as the carrier of an electromagnetic field." Moreover, dimensions and units of electromagnetism are discussed in both books: Section 67 of Abraham's book discusses several unit system incl. the Gaussian one. It is emphasized that the latter has the property of providing the same units for all fields ("… werden in dem Gaußschen Maßsysteme elektrische und magnetische Größen paritätisch behandelt."[23] and "Der von den neueren Weiterbildungen der Maxwellschen Theorie gemachten Annahme, daß das elektromagnetische Feld eigentlich als Feld im Äther zu betrachten ist, das nur durch die in der Materie enthaltene Elektrizität bzw. durch deren Bewegung modifiziert wird, paßt sich dieses Dimensionssystem am besten an. Denn $\mathfrak{E}$ und $\mathfrak{D}$, $\mathfrak{H}$ und $\mathfrak{B}$ werden hier wesensgleich …"[24]). In contrast to that the translation [26], concentrates exclusively on the Gaussian system (Chapter VII, Section 3).

Let us now discuss the continuum related literature. Dimensions and units are an important issue in Chapter F of the Handbook article [1] from the very beginning on. Unfortunately, the proposed nomenclature is rather unusual and first applied in very general terms to tensorial objects of arbitrary rank. For better understanding, reference is made to Ericksen's appendix on tensors in a footnote on pg. 660. Things become clearer when concrete objects are addressed. All four electromagnetic fields are strictly distinguished and the summary in Section 2.1 agrees completely with Equation (45).

[19] Ultimately the introduction of the displacement vector $\boldsymbol{D}$ and of its counterpart $\boldsymbol{D}^*$ is required only for the treatment of electrodynamics in matter.
[20] "The positive sign has been given, quite arbitrarily, to the electricity on the little ball rubbed by the catskin, and consequently the negative sign to the rubbed sealing wax." translation from [26], pg.54
[21] "It suggests that empty space also mediates electrical effects, that it can be the seat of an electric field. Because space used to be associated with only geometric properties, a special word 'ether' was introduced for space with electromagnetic properties."
[22] "Today we do not associate the word 'ether' with the idea of a hypothetical substance; rather, we use this historically handed-down word today as an abbreviation when we speak without rambling about space as a carrier of an electromagnetic field."
[23] "… electrical and magnetic quantities are treated equally in the Gaussian system."
[24] "The assumption made by the more recent developments of Maxwell's theory that the electromagnetic field is actually to be regarded as a field in the ether, which is only modified by the electricity contained in the matter or by its movement, this dimensional system fits best. Then $\mathfrak{E}$ and $\mathfrak{D}$, $\mathfrak{H}$ and $\mathfrak{B}$ become consubstantial here …"

The Maxwell–Lorentz æther relations are introduced in Section 279 in the form shown in Equation (58) and generalized to arbitrary frames in Section 280. We will get back to that in Section 5 of this paper. However, it is interesting to note that Toupin is very self-critical of his approach: “At the same time, we give warning that the aether relations represent an assumption not adopted in every existing theory of electromagnetism…, whereas the conservation laws of charge and magnetic flux are, to our knowledge, common to all.”

A less critical attitude is found in the books by Müller [20, 28], pg. 309, where the æther relations are simply accepted as a commodity. These books also focus exclusively on the MKSA system, whereas the Handbook article in principle allows for more but then also mentions only SI units explicitly, pg. 681.

The books by Kovetz [3, 4], Section 11, are clearly aware of the special status and criticality of the æther relations: “The third principle of electromagnetism states: a Euc1idean, inertial frame exists in which the relations (58) with ϵ_0 and μ_0 two positive, universal constants, hold everywhere and at all times inside material bodies as weIl as in empty space.” MKSA and Gaussian units are both mentioned and used. In the appendix, it is stressed that in Gaussian units, all four field have the same units and the question is asked “Since the Gaussian fields $\mathbf{D}'$, $\mathbf{H}'$, $\mathbf{E}'$, $\mathbf{B}'$, $\mathbf{P}'$, and $\mathbf{M}'$ all have the same dimensions, would it not be sensible to employ the same unit, for example the gauss, for all of them?” The answer is a Salomonic one: “It would not be tactful for an author who uses SI units to attempt an answer to this question.”

Hutter et al. [5] present the æther relations in MKSA notation in Section 2.5 on material objectivity. For them, they introduce, if used, invariance of the Maxwell equations under Lorentz transformations. In a footnote on pg. 56, they point out that “While we apply SI-units, Gaussian units are used in [249], and it is a well-known fact that a c^{-2}-term in one system of units is not necessarily a c^{-2}-term in the other system of units as well.” This is important to realize if nonrelativistic approximations to electrodynamic materials science problems are sought.

Steigmann [49] is pointing in a similar direction when he says: “It is well known that these relations [i.e., Equation (58)] are invariant under the Lorentz group of transformations rather than the Galilean transformations that preserve the equations of conventional non-relativistic mechanics. However, these transformations are asymptotically coincident if the material velocity relative to a Galilean frame is much smaller than c and if the diameter of the material body is bounded …”

Finally, in the work of Eringen and Maugin [29], the æther relations are not mentioned at all. The Maxwell equations are first looked at the microscopic level in an inertial frame, so there is no real need for two independent electric and magnetic fields: Equation (2.7.17), pg. 38. Then homogenization leads to the continuum scale and four fields appear formally: Equation (3.3.2), pg. 54. An attitude purely related to constitutive theory becomes visible when $\boldsymbol{D}$ is additively decomposed into $\boldsymbol{E}$ and the polarization $\boldsymbol{P}$, and similarly $\boldsymbol{B}$ into $\boldsymbol{H}$ plus the magnetization $\boldsymbol{M}$. Heaviside–Lorentz units are used throughout their monograph without comment.

5 | TRANSFORMATION PROPERTIES OF THE ELECTROMAGNETIC FIELDS

5.1 | Worldtensor notation of the Maxwell equations

In this section, it will be investigated how the mathematical form of the Maxwell equations will change when switching to a noninertial frame. Recall that so far these relations were only formulated in an inertial system. It is well known that the balance of linear momentum will contain additional terms, the so-called “fictitious forces,” if the frame of the observer is noninertial. Consequently, several questions arise: First, is there is an analogous phenomenon in electrodynamics, second, what is the specific transformation law of frames in which the Maxwell equations keep the form that was presented in Chapters 2 and 3, third, how do the Maxwell equations look in other frames and, fourth, how do the various fields of electrodynamics transform under arbitrary change of observer.

We will give answers to these questions based on the worldtensor formalism presented in Chapter F of the handbook article by Truesdell and Toupin [1] but there are several shortcomings. First, the worldtensor (or space-time) formalism will be presented completely in index form. An alternative might be using the exterior tensor calculus, which is described in context with Maxwell’s equations, for example, in Thorne et al. [52], Chapter 4. However, there no distinction is made between the four fields and the Maxwell–Lorentz æther relations are not considered. Second, it will remain unanswered as to whether this approach really addresses a change of observer or is “just” a coordinate transformation, even though it is one that changes space and time simultaneously. The difference between a change of observer and a pure coordinate transformation is explained for classical continuum mechanics for example in Ivanova et al. [53]. Its extension to electromagnetic phenomena is left to future work.

We start our discussion with the definition of a mixed worldtensor object $\boldsymbol{f}$ whose law of transformation under arbitrary space–time coordinate transformations

$$x^{A'} = x^{A'}(x^B) \tag{61}$$

is given by

$$f^{M'N'\dots}_{\dots R'} = \left|\frac{\partial \boldsymbol{x}'}{\partial \boldsymbol{x}}\right|^{-w} \operatorname{sgn}\left(\frac{\partial \boldsymbol{x}'}{\partial \boldsymbol{x}}\right)^{p} \frac{\partial x^{M'}}{\partial x^{M}} \frac{\partial x^{N'}}{\partial x^{N}} \dots \frac{\partial x^{R}}{\partial x^{R'}} f^{MN\dots}_{\dots R}. \tag{62}$$

Several comments are in order:

- In the Handbook, Equation (61) is initially a coordinate transformation of an n-dimensional space. It becomes a 4D space-time transformation in Section 270. In contrast to the Handbook, we assign in x^B the index $B = 0$ to time, that is, $x^0 = ct$, and the remaining three indices $b \in (1, 2, 3)$ to space. Moreover, the choice of the symbol $\boldsymbol{x}$ or x^B is an unfortunate coincidence and must, in general, not be confused with Cartesian orthogonal coordinates. Also, the representation (62) is not necessarily Cartesian, since otherwise the distinction between co- and contravariant indices would make no sense. On the other hand, if convenient, Cartesian coordinates will often be used in what follows.
- The positive or negative number w is referred to as the *weight* of the world tensor. The number p can assume the values 0 and 1. Certain combinations of both numbers lead to different denominations for the tensor. In this article, the following ones are useful to know: $p = 0$ and $w = 0$ is called an *absolute* tensor. The case $p = 0$, $w \neq 0$ is referred to as relative tensor of weight w. If $p = 1$, $w \neq 0$, the object $\boldsymbol{f}$ is referred to as an *axial* or pseudotensor of weight w. Further classifications will be made whenever necessary.

According to the nomenclature of Truesdell and Toupin [1], the electric and magnetic fields, $\boldsymbol{E}$ and $\boldsymbol{B}$ will be combined in an absolute, covariant, completely antisymmetric tensor of rank 2 or 2-vector $\boldsymbol{\varphi}$, the electromagnetic field

$$\varphi_{AB} = \left[\begin{array}{c|c} 0 & -E_b/c \\ \hline E_a/c & \varepsilon_{abc}B^c \end{array}\right], \tag{63}$$

which transforms according to ($p = 0$, $w = 0$):

$$\varphi_{A'B'} = \frac{\partial x^A}{\partial x^{A'}} \frac{\partial x^B}{\partial x^{B'}} \varphi_{AB}. \tag{64}$$

If we introduce the 4D Levi-Civita tensor by

$$\varepsilon^{ABCD} = \begin{cases} +1 & \text{if } A, B, C, D = 0, 1, 2, 3 \text{ and even permutations} \\ -1 & \text{if } A, B, C, D = 1, 0, 2, 3 \text{ and even permutations}, \\ 0 & \text{otherwise} \end{cases} \tag{65}$$

which is a relative contravariant tensor ($p = 0, w = 1$) of rank 4 or a contravariant four-vector density

$$\varepsilon^{A'B'C'D'} = \left|\frac{\partial \boldsymbol{x}'}{\partial \boldsymbol{x}}\right|^{-1} \frac{\partial x^{A'}}{\partial x^{A}} \frac{\partial x^{B'}}{\partial x^{B}} \frac{\partial x^{C'}}{\partial x^{C}} \frac{\partial x^{D'}}{\partial x^{D}} \varepsilon^{ABCD}, \tag{66}$$

we can write the Maxwell equations for the electromagnetic flux (17), (4), which were established in an inertial system, in the following form

$$\varepsilon^{ABCD} \frac{\partial \varphi_{CD}}{\partial x^B} = 0. \tag{67}$$

Several comments are due:

- It is difficult to decide as to whether the differentiation w.r.t. position in Equation (67) is of the spatial or of the material type when the space–time notation is used. It seems natural to assume that a spatial description should be used, because the two electromagnetic fields that are combined in $\boldsymbol{\varphi}$ can exist in vacuum without reference to matter.
- If Equation (67) is expanded, the Maxwell equations (17) and (4) appear in the form $2\frac{\partial B^b}{\partial x^b} = 0$ and $-\frac{1}{2c}(\frac{\partial B^a}{\partial t} + \varepsilon^{abd}\frac{\partial E_d}{\partial x^b}) = 0$, which does not matter, because the right-hand side is zero. There is a deeper issue behind the factor of 2, related to the use of tensors and their duals, introduced, for example, in Truesdell and Toupin [1], pp. 661. However, we will not investigate details here.
- When we assign contra- and covariant indices to the objects $\boldsymbol{B}$ and $\boldsymbol{E}$, we follow a suggestion in Kovetz [4], pg. 34. Indeed, formally the distinction between co- and contravariant indices of position (labeled by small Latin letters, a, b, etc.) is ultimately irrelevant, because Cartesian coordinates of position are meant. However, they are useful when they are used in context with space–time indices characterized by capital Latin letters, A, B, etc. Moreover, this way of writing creates some nice symmetry effects. A similar remark concerns the objects $\boldsymbol{D}$ and $\boldsymbol{H}$ further down.

Equations (64), (66), and (67) can now be used to show that the conservation of magnetic flux holds in all systems and the following form-invariance can be established:

$$\varepsilon^{A'B'C'D'}\frac{\partial \varphi_{C'D'}}{\partial x^{B'}} = 0. \tag{68}$$

It is important to note that in Equation (67) or (68), partial and not covariant derivatives occur (cf., [54], pp. 173, or [20], pg. 396). This is characteristic of a conservation law: The magnetic flux $\boldsymbol{\varphi}$ is a conserved quantity.

We now combine charge density q and the total electric current $q\boldsymbol{v} + \boldsymbol{j}$ in a worldobject. In the nomenclature of Truesdell and Toupin [1], the charge-current vector $\boldsymbol{\sigma}$ is a relative ($p = 0, w \neq 0$) contravariant tensor of rank 1 and weight +1, a.k.a. contravariant one-vector density, which in the inertial system reads in Cartesian coordinates:

$$\sigma^A = [cq, qv^a + j^a], \tag{69}$$

and which transforms according to

$$\sigma^{A'} = \left|\frac{\partial \boldsymbol{x}'}{\partial \boldsymbol{x}}\right|^{-1}\frac{\partial x^{A'}}{\partial x^A}\sigma^A. \tag{70}$$

Then the conservation of charge (33) in the inertial system can be expressed by

$$\frac{\partial \sigma^A}{\partial x^A} = 0. \tag{71}$$

However, by means of Equation (70) and the identity

$$\frac{\partial}{\partial x^A}\left(\left|\frac{\partial \boldsymbol{x}'}{\partial \boldsymbol{x}}\right|\frac{\partial x^A}{\partial x^{A'}}\right) = 0 \tag{72}$$

we can immediately transform this equation into an arbitrary other world frame (or better to arbitrary other worldcoordinates $x^{A'}$):

$$0 = \frac{\partial \sigma^A}{\partial x^A} = \frac{\partial}{\partial x^A}\left(\left|\frac{\partial \boldsymbol{x}'}{\partial \boldsymbol{x}}\right|\frac{\partial x^A}{\partial x^{A'}}\sigma^{A'}\right) = \cdots = \left|\frac{\partial \boldsymbol{x}'}{\partial \boldsymbol{x}}\right|\frac{\partial \sigma^{A'}}{\partial x^{A'}} \;\Rightarrow\; \frac{\partial \sigma^{A'}}{\partial x^{A'}} = 0. \tag{73}$$

This confirms form invariance of the conservation of charge when written in space–time notation. We may say that it is valid in every frame and its concrete form can be specified if the space–time transformation (61) between an inertial frame and another completely arbitrarily moving one is detailed. Note that it was important that in Equation (71), partial and not covariant differentiation is used. As mentioned above, this is an inherent property of space–time conservation laws.

Equation (71) is satisfied if the antisymmetric charge current potential $\boldsymbol{\eta}$ is used:

$$\frac{\partial \eta^{AB}}{\partial x^B} = \sigma^A \quad \text{with} \quad \eta^{AB} = -\eta^{BA}. \tag{74}$$

In view of Equation (70) and in order to fulfill the requirement that the tensorial properties on both sides of the tensor equation (74) are the same, it is required that η^{AB} transforms like a contravariant relative tensor of weight $w = +1$, that is, it is a contravariant two-vector density:

$$\eta^{A'B'} = \left|\frac{\partial \boldsymbol{x}'}{\partial \boldsymbol{x}}\right|^{-1} \frac{\partial x^{A'}}{\partial x^A}\frac{\partial x^{B'}}{\partial x^B}\eta^{AB}. \tag{75}$$

If we now put

$$\eta^{AB} = \left[\begin{array}{c|c} 0 & cD^b \\ \hline -cD^a & \varepsilon^{abc}H_c \end{array}\right], \tag{76}$$

the second set of Maxwell equations (40) and (42) results. Forminvariance under arbitrary transformations is easily shown if we observe Equations (70), (72), and (75):

$$\frac{\partial \eta^{A'B'}}{\partial x^{B'}} = \sigma^{A'}. \tag{77}$$

5.2 | Euclidean transformations and objective tensors of electromagnetism

When ignoring transformations to frames with nonrectangular axes and curvilinear coordinates, the most general transformation between two frames S and S'[25] in classical mechanics are the Euclidean transformations in Cartesian coordinates, which represent time-dependent rotations and translations. They are particularly relevant to nonrelativistic constitutive theory and, therefore, deserve our special attention:

$$\begin{aligned} x^{0'} &= x^0 \\ x^{a'} &= Q^{a'}_{.b} x^b + b^{a'} \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} x^0 &= x^{0'} \\ x^b &= Q^{.b}_{a'}\left(x^{a'} - b^{a'}\right). \end{aligned} \tag{78}$$

$Q^{a'}_b$ is an orthogonal[26] matrix ($Q^{a'}_{.c}Q^{.c}_{b'} = \delta^{a'}_{b'}$, $Q^{c'}_{.a}Q^{.b}_{c'} = \delta^b_a$, $\det \boldsymbol{Q} = \pm 1$) representing the relative rotation of the frames, and $b^{a'}$ is a vector representing the translational distance between their origins (from S' to S). Both may be time-dependent. Hence,

$$\begin{aligned} &\frac{\partial x^{A'}}{\partial x^B} = \left[\begin{array}{c|c} 1 & 0 \\ \hline \dot{Q}^{a'}_{.b}x^b + \dot{b}^{a'} & Q^{a'}_{.b} \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline \frac{w^{a'}}{c} & Q^{a'}_{.b} \end{array}\right], \quad \frac{\partial x^B}{\partial x^{A'}} = \left[\begin{array}{c|c} 1 & 0 \\ \hline \dot{Q}^{.b}_{c'}\left(x^{c'} - b^{c'}\right) - Q^{.b}_{c'}\dot{b}^{c'} & Q^{.b}_{a'} \end{array}\right] \\ &w^{a'} = c\left[\Omega^{a'}_{c'}\left(x^{c'} - b^{c'}\right) + \dot{b}^{a'}\right], \quad \Omega^{a'}_{c'} = \dot{Q}^{a'}_{.b}Q^{.b}_{c'}. \end{aligned} \tag{79}$$

The dots refer to differentiation with respect to $x^0 = ct$. Hence, $w^{a'}$ is the relative velocity (angular and translational) and $c\Omega^{a'}_{c'}$ is the antisymmetric (meaning $\dot{Q}^{a'}_{.b}Q^{.b}_{c'} = -\dot{Q}^{.b}_{c'}Q^{a'}_{.b}$) matrix of angular velocity between the frames S and S'.

[25] Often S is assumed to be "at rest," that is, as an inertial system, while S' translates and rotates with respect to it.
[26] This allows for rotations and mirror imaging.

Moreover, the velocity $\boldsymbol{v}$ and the acceleration $\boldsymbol{a}$ transform according to

$$\begin{aligned} v^{a'} &= Q^{a'}_{.b} v^b + w^{a'} \\ a^{a'} &= Q^{a'}_{.b} a^b + \underline{2c\Omega^{a'}_{c'}\left(v^{c'} - \dot{b}^{c'}\right) - c^2\Omega^{a'}_{b'}\Omega^{b'}_{c'}\left(x^{c'} - b^{c'}\right) + c^2\dot{\Omega}^{a'}_{c'}\left(x^{c'} - b^{c'}\right) + c^2\ddot{b}^{a'}}. \end{aligned} \tag{80}$$

Specifically $-w^{a'}$ is the velocity of frame S' relative to frame S. The terms with $\boldsymbol{\Omega}$ represent the Coriolis, centrifugal, and Euler accelerations. Assume that S is an inertial system. Then the underlined terms are responsible that the balance of momentum does not have the same form in the frame S'. When multiplied by the (negative) mass density, these inertial accelerations lead to “fictitious forces.” Hence, the balance of momentum is not frame-indifferent under Euclidean transformations.

If $\boldsymbol{Q}$ is independent of time ($\Rightarrow Q^{a'}_{.b}(\not t)$, $\Omega^{a'}_{c'} = 0$) and $\boldsymbol{b}$ is linear in time ($\Rightarrow c\dot{b}^{a'} = -V^{a'} = \text{const.}$), the Euclidean reduces to the Galilean transformation:

$$x^{a'} = Q^{a'}_{.b} x^b - V^{a'} t\ , \quad v^{a'} = Q^{a'}_{.b} v^b - V^{a'}\ , \quad a^{a'} = Q^{a'}_{.b} a^b. \tag{81}$$

Hence, the balance of momentum is frame-indifferent w.r.t. Galilean Transformations. Here “fictitious” forces do not appear and the mechanics is exactly the same as in the original inertial system. This is why before the advent of electrodynamics (the period of so-called “classical physics”) it was believed that the Galilean systems define the group of inertial systems.

We will now use Equation (79) to find out how the various fields of electrodynamics transform under Euclidean transformations and as to whether they contain inertial terms as well.[27] If the latter is not the case, we will follow the usual nomenclature of continuum mechanics and call them objective or Euclidean tensors of various rank (see, e.g., [5], pg. 21, or [29], pg. 16). We start with the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$. If we observe that the 3D spatial Levi–Civita symbol is an axial Euclidean tensor of third rank,

$$\varepsilon^{a'b'c'} = \det \boldsymbol{Q}\ Q^{a'}_{.a} Q^{b'}_{.b} Q^{c'}_{.c} \varepsilon^{abc}, \tag{82}$$

and apply the transformation $(79)_2$ to Equations (64) and (63), we find

$$\begin{aligned} B^{a'} &= \det \boldsymbol{Q}\ Q^{a'}_{.a} B^a \quad \text{(axial Euclidean vector)}, \\ E_{a'} &= Q^{.a}_{a'}\left(E_a - \varepsilon_{abc} Q^{.b}_{d'} w^{d'} B^c\right) \quad \text{(no Euclidean vector)}. \end{aligned} \tag{83}$$

If the transformation of velocities $(80)_1$ is observed, the last equation can be rewritten in the form

$$E_{a'} + \varepsilon_{a'b'c'} v^{b'} B^{c'} = Q^{.a}_{a'}\left(E_a + \varepsilon_{abc} v^b B^c\right) \quad \Rightarrow \quad \mathfrak{E}_{a'} = Q^{.a}_{a'} \mathfrak{E}_a \quad \text{(Euclidean vector)}. \tag{84}$$

The last result was to be expected, because specific forces should be Euclidean vectors.

We now turn to the charge-current vector (69) and find by observing the transformation law (70) in context with the Euclidean transformation $(79)_1$:

$$\begin{aligned} q' &= q \quad \text{(Euclidean scalar)}, \\ q' v^{a'} + j^{a'} &= Q^{a'}_{.a}(q v^a + j^a) + q w^{a'} \quad \text{(no Euclidean vector)}. \end{aligned} \tag{85}$$

By observing the transformation law for the velocity $(80)_1$, it is found that

$$j^{a'} = Q^{a'}_{.a} j^a \quad \text{(Euclidean vector)}. \tag{86}$$

[27] Intuitive arguments which fields of electromagnetism should be polar or axial in nature and Euclidean or not can be found in Section 13 of Müller [55].

Finally, we turn to the charge–current potential. By combining Equations (75), (76) with $(79)_1$, we find

$$
\begin{aligned}
D^{a'} &= Q^{a'}_{.a} D^{a} \quad \text{(Euclidean vector)},\\
H_{a'} &= \det \boldsymbol{Q}\; Q^{.a}_{a'}\Big(H_a + \varepsilon_{abc} Q^{.b}_{d'} w^{d'} D^{c}\Big) \quad \text{(no axial Euclidean vector)}.
\end{aligned} \tag{87}
$$

However, similar to Equation $(83)_2$ we can turn the last result into

$$
H_{a'} - \varepsilon_{a'b'c'} v^{b'} D^{c'} = \det \boldsymbol{Q}\; Q^{.a}_{a'}\big(H_a - \varepsilon_{abc} v^{b} D^{c}\big) \quad \text{(axial Euclidean vector)}. \tag{88}
$$

5.3 | The Maxwell–Lorentz æther relations and the Lorentz transformations

Section 5.1 taught us that the Maxwell equations (67) and (74) are forminvariant[28] under all space–time transformations incl. Euclidean and Galilean transformations. This comes as a surprise, because we often hear that the Maxwell equations are forminvariant only under Lorentz transformations. The solution to this puzzle is hidden in the fact that so far we have strictly distinguished between two sets of electric and magnetic fields, $(\boldsymbol{E}, \boldsymbol{D})$ and $(\boldsymbol{B}, \boldsymbol{H})$, and did not connect them yet. The connection is achieved by using the Maxwell–Lorentz æther relations (58). However, it was pointed out already that they hold only in an inertial system. If we look at the complicated transformation rules for the electric field $\boldsymbol{E}$ and for the current potential $\boldsymbol{H}$, which are both nonobjective under Euclidean transformation, it must be suspected that the simple proportionality of fields according to the Maxwell–Lorentz æther relations does no longer hold if we switch to a moving frame. We proceed to investigate this in detail. For this purpose, we rewrite the Maxwell–Lorentz æther relations (58) in space–time notation. In what follows, we will essentially follow Sections 280– 2.2 of the Handbook [1] with slight variations in the nomenclature.

In order to express the aether relations with the worldtensors $\boldsymbol{\eta}$ and $\boldsymbol{\varphi}$, which were defined in Equations (63) and (76), we write

$$
\eta^{AB} = \sqrt{\frac{\epsilon_0}{\mu_0}}\,\Gamma^{AC}\Gamma^{BD}\varphi_{CD}, \tag{89}
$$

where the choice

$$
\Gamma^{AB} = \sqrt{c}\begin{bmatrix} -1/c^2 & 0 \\ 0 & \delta^{ab} \end{bmatrix} \tag{90}
$$

satisfies the aether relations (58) in the initial inertial laboratory frame. $\boldsymbol{\Gamma}$ is a relative contravariant worldtensor of weight $w = +1/2$ and obeys the following transformation rule to another frame:

$$
\Gamma^{A'B'} = \left|\frac{\partial \boldsymbol{x}'}{\partial \boldsymbol{x}}\right|^{-\frac{1}{2}} \frac{\partial x^{A'}}{\partial x^{A}}\frac{\partial x^{B'}}{\partial x^{B}}\Gamma^{AB}. \tag{91}
$$

By means of this choice, we have guaranteed that the relation (89) is indeed a tensor equation, since $\boldsymbol{\eta}$ was introduced as a contravariant two-vector density of weight $w = +1$, while $\boldsymbol{\varphi}$ was just a two-vector.

Now it becomes possible to find the transformation $x^{A'} = x^{A'}(x^B)$ to all other *inertial* frames, for which by definition, the Maxwell–Lorentz æther relations keep the simple form (58):

$$
\begin{bmatrix} -1/c^2 & 0 \\ 0 & \delta^{a'b'} \end{bmatrix} = \left|\frac{\partial \boldsymbol{x}'}{\partial \boldsymbol{x}}\right|^{-\frac{1}{2}} \frac{\partial x^{A'}}{\partial x^{A}}\frac{\partial x^{B'}}{\partial x^{B}} \begin{bmatrix} -1/c^2 & 0 \\ 0 & \delta^{ab} \end{bmatrix}. \tag{92}
$$

[28] The notions “forminvariance” or “general covariance” replace the ideas of “frame-indifference” and “objectivity” known from rational continuum theory. Further details regarding their meaning can be found in Refs. [56] and [57].

In Truesdell and Toupin [1], pg. 682, it is now argued that these relations $x^{A'} = \bar{x}^{A'}(x^B)$ between inertial frames must be linear. Then, their general form reads

$$x^{0'} = \gamma\left(x^0 - \frac{V_r}{c}x^r\right), \quad x^{a'} = Q^{a'}_{.b}\left[\left(\delta^b_c + (\gamma - 1)\frac{V^b V_c}{\boldsymbol{V}^2}\right)x^c - \gamma\frac{V^b}{c}x^0\right], \tag{93}$$

where $\gamma = 1/\sqrt{1 - V^2/c^2}$. As in Equation (81), $\boldsymbol{V}$ is the constant velocity by which the origin of S' removes from the origin of S. It is known as the Lorentz boost velocity.

Equation (93) are the Lorentz transformations. They ensure that the Maxwell–Lorentz æther relations stay simple. Now, this means that if we wish to replace $\boldsymbol{D}$ by $\epsilon_0\boldsymbol{E}$ and $\boldsymbol{H}$ by $\boldsymbol{B}/\mu_0$ in Equations (39) and (42), we are allowed to do so, but a price must be paid: The resulting Maxwell equations are then valid only in Lorentz frames. Note that for $|\boldsymbol{V}| << c$, the Galilean transformation results from Equation (93). We may conclude by saying that the class of Lorentz systems are the inertial frames of "modern" physics and replace the Galilean transformations of "classical physics."

5.4 | Comparison with the literature

The physics literature is very enthusiastic when it comes to the worldtensor formalism of electrodynamics even though most books are less stringent and precise than the Handbook. In fact for physicists, this description is linked very much to the (general) theory of relativity and Lorentz transformations. They do not realize the aspect that, for example, worldtensors also comprise changes between Galilean and Euclidean frames. For them, such an association would simply be faulty. In Chapter 25, pg. 25-7 of Ref. [22], Feynman presents the balance of charge in "four-vector notation," that is, essentially Equation (71). It is mentioned that this combination "is an invariant scalar, if it is zero in one frame it is zero in all frames." We could give him the benefit of the doubt that he means what he says and it is not thinking of Lorentz frames only. However, a look at how Gauss' law (39) and the Ampère–Øersted relation (42) are treated on pg. 25-9 clearly shows that he has Lorentz transformations in mind, because the factor ϵ_0 appears. On the other hand, the polar and axial nature of $\boldsymbol{E}$ and $\boldsymbol{B}$ is explained without making use of four-vectors, pg. 13-12.

Landau and Lifshitz [9] start the four-vector formalism from the very beginning on, always referring to Lorentz transformations. There is only one electromagnetic field tensor (pg. 65) and the distinction between η^{AB} and φ_{AB} is not made. The same symbols appear in the co- and in the contravariant form of this tensor and no ϵ_0, μ_0 or c appear because Gaussian units are used. This makes it impossible to understand clearly in which frames the 4D Maxwell laws really hold. Obviously in Lorentz frames, which is true if validity of the simple Maxwell–Lorentz æther relations is assumed. The conservation law of magnetic flux (67) appears on pg. 71, the conservation of charge (71) on pg. 77, and the potential relation (74), when evaluated with the Maxwell–Lorentz æther relations in a Lorentz system, on pg. 79.

Jackson [6] discusses the worldtensor notation in Chapter 11 entitled "Special Theory of Relativity." Not surprisingly, the same symbol is used for η^{AB} and φ_{AB}. Just like Landau and Lifshitz, the presentation is in Gaussian units. The transformation of the electromagnetic fields are limited to Lorentz transformations, Section 10.10.

The 1957 edition of the book by Becker [23] introduce the electrodynamic field equations in worldtensor form in Section 81. A four-vector equation for the conservation of charge can be found on pg. 240, interestingly only for convective transport of charge. This may be due to the desire to use the Lorentz transformation for velocity and time, which is explained on pg. 241. Moreover, the imaginary constant is used in order to avoid the concept of the 4D metric tensor. On pg. 241, we find an analog to the definition of the electromagnetic field tensor $\boldsymbol{\varphi}$ from Equation (63). However, it becomes obvious that no distinction between the two electric and the two magnetic fields is made when the two sets of Maxwell equations (67) and (74) are presented on pg. 242, where the same worldtensor is used for both. It should be noted that Equation (67) is presented in the alternative form

$$\frac{\partial\varphi_{AB}}{\partial x^C} + \frac{\partial\varphi_{CA}}{\partial x^B} + \frac{\partial\varphi_{BC}}{\partial x^A} = 0. \tag{94}$$

The 1972 edition of that book is good for a few surprises: The conservation of the four-vector current is presented without the imaginary unit on pg. 236 with a current that does not specify the convective and nonconvective part. The difference between these two types of current is explored in more detail in 11.2. It is noteworthy that Becker starts his discussion in the rest frame of matter, so that the convective part of the current vanishes. This is problematic in the sense that a "rest frame" is strictly speaking not necessarily an inertial frame. An instantaneous local Lorentz rest frame would be the proper terminology. Moreover, two worldtensors are introduced, one analog to $\boldsymbol{\varphi}$ (63) containing $\boldsymbol{E}$ and $\boldsymbol{B}$ on pg. 236, and an analog to $\boldsymbol{\eta}$ (76) containing $\boldsymbol{D}$ and $\boldsymbol{H}$ on pg. 237. The two sets of Maxwell equations (94) and (74) follow correctly in worldvector form on the same page. Then, immediately after that it is said: "Sie [the Maxwell equations] vereinfachen sich noch weiter für den Fall des Vakuums."[29] followed by the Maxwell–Lorentz æther relations (essentially in worldtensor form (89)) without calling them so. Let us interpret all of this as follows: The "vacuum" is synonymous to the "æther" and "vacuum frames" result from each other by means of a Lorentz transformation and, finally, in a Lorentz system the æther is at rest.[30]

As expected electrodynamics and its invariance in space–time is also an important topic in all books on (general) relativity. Weinberg does not distinguish between the electromagnetic field and the potential world tensors, $\boldsymbol{\varphi}$ and $\boldsymbol{\eta}$, respectively ([58], Chapter 7, pg. 41 and pg. 127). Instead, the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are combined in one worldtensor, which is used to formulate the conservation of magnetic flux and the conservation of charge in four-vector notation similar to Equations (67) and (74). The Maxwell–Lorentz æther relations appear indirectly when on pg. 42, where the Lorentz metric (not introduced here but related to the æther relations) is used to switch from co- to contravariant indices, in other words to switch between φ_{AB} and η^{AB}.

Misner et al. ([52], pp. 80) share this point-of-view. It should be mentioned that they present electromagnetism also alternatively in terms of differential forms and external calculus. Indeed, the latter way of writing is very popular in the modern relativistic and continuum literature ([59, 60] (offering various ways of writing the Maxwell equations at different times in history in the table on pg. 29), [61], pp. 179, [62]). However, the use of this notation is not totally autotelic. It seems to lead to growing awareness that the Maxwell equations and the æther relations have a different status and two sets of fields must be used. For example, in Hehl and Obukhov [63] or Hehl and Obukhov [64], we read "In this sense, the Maxwell equations are "more universal" than the Maxwell–Lorentz spacetime relations [= the aether relations]. The latter ones are not completely untouchable. We may consider (6.1) [= the aether relations] as constitutive relations for spacetime itself." A similar attitude is shared in [65], Chapter IV, section 4, where the two worldtensors $\boldsymbol{\varphi}$ and $\boldsymbol{\eta}$ are related to each other by "electromagnetic constitutive equations" and are equal only "in the vacuum" (pg. 390). These viewpoints will be reconsidered in Section 6, when we briefly discuss the issue of constitutive equations in electromagnetism.

Let us now turn to the continuum mechanics literature. In the opinion of the authors Toupin's article, Chapter F [1] is the authoritative reference when it comes to the worldvector formalism of electrodynamics. It should be mentioned that the question which quantities of electrodynamics can be considered as Euclidean or "objective" tensors is touched but not answered on pg. 684[31] in the Handbook, but not addressed at all when it comes to the constitutive equations of electrodynamics in Section 308. It could be argued that when the article was written, this part of the "principle of material objectivity" ([5], pg. 21) was still in the development stage.

Regarding this issue, the books by Müller go far beyond the original reference ([28], pp. 118 and [20], Section 9.6) and give a full account of all objective and nonobjective quantities of electromagnetism under Euclidean transformations. The books by Kovetz do not go that far, even though they are written in the spirit of rational electrodynamics. They restrict themselves to Galilean transformations without giving reasons and despite the fact that they embed Galilean transformations into the context of spacetime transformations ([3], pg. 5, [4], pg. 22).

Hutter et al. [5] are refreshingly honest when they present objective Euclidean electromagnetic vectors on pg. 23 and say on pg. 24 "Based on the properties (2.5.12) [=Euclidean vectors] we may then request as is done classically, that the material response be invariant under the Euclidian group. This is an approximation, because the Maxwell equations can never be rendered Lorentz invariant this way." This is very true, because in a constitutive theory based on Euclidean

[29] "They [the Maxwell equations] further simplify for the case of the vacuum."

[30] In view of the fact that the class of Lorentz systems preserve the constancy of the speed of light (also see [55], pp. 376, where this approach is compared with the idea that the invariance of the æther relations defines the class of Lorentz systems) the following citation from Sommerfeld [17], pg. 235 points in the same direction: "We may say: The constancy of the velocity of light is today the only valid *remnant of the ether concept*. If at present we should speak of an ether, we would have to assign a separate ether to every frame of reference, i.e. speak e.g. of a primed and an unprimed ether. … Thus in parts I and II, we have almost never spoken of the "ether", but used instead the not readily misinterpreted word 'vacuum'."

[31] "The three-dimensional form of the law of conservation of charge and the charge and current equations taking into account a surface distribution of charge and current can be obtained from the world invariant equations (283.2) and (283.6) by introducing a Euclidean frame. The resulting equations have an interesting but complicated structure; Since we do not make use of these equations here, we leave to the reader the task of deriving them."

tensors, we would have to use in the Euclidean system æther relations of a more complicated form:[32]

$$\boldsymbol{D} = \epsilon_0(\boldsymbol{E} + \boldsymbol{w} \times \boldsymbol{B}) \ , \quad \boldsymbol{B} = \mu_0(\boldsymbol{H} - \boldsymbol{w} \times \boldsymbol{D}). \tag{95}$$

It is interesting to note that the right-hand side consists of expressions similar to the objective quantities from Equations (84) and (88), if we replace in there the material velocity $\boldsymbol{v}$ by the "frame velocity" $\boldsymbol{w} = \boldsymbol{\Omega} \cdot (\boldsymbol{x}' - \boldsymbol{b}) + \dot{\boldsymbol{b}}$.

Writing these relations in SI units has the disadvantage that it remains unclear as to whether the differences to Equation (58) are small or large. All it shows is that there are differences, which, if the relations are inserted, destroy the frame independence of the second set of Maxwell equations (39) and (42). These will suddenly contain frame-dependent terms. Another option is to use Gaussian or Heaviside–Lorentz units. If we observe Equation (60), then Equation (95) reads

$$\boldsymbol{D} = \boldsymbol{E} + \frac{\boldsymbol{w}}{c} \times \boldsymbol{B} \ , \quad \boldsymbol{B} = \boldsymbol{H} - \frac{\boldsymbol{w}}{c} \times \boldsymbol{D}, \tag{96}$$

which gives a better feel for the size of the corrections. In this context also recall that in Gaussian or Heaviside–Lorentz units, all electromagnetic fields are commensurable.

Let us now consider the work of Eringen and Maugin, in particular, the first volume [29]. Here the notion Maxwell–Lorentz æther relations is not used at all. Similar to Kovetz, the invariance of the Maxwell equations is investigated for Galilean transformation in Section 3.4. The authors emphasize on pg. 54 that all their transformation laws for the electromagnetic fields are nonrelativistic and this *caveat* is understandable since the four-vector formalism is not mentioned at all in the first volume. Interestingly, when a phenomenological constitutive theory is developed later, the principle of material frame indifference is based on Euclidean transformations (see pp. 136). On pg. 137, we read: "Presently, no proof of contradiction to the applicability of (5.4.7) [= the Euclidean transformation] exists." In summary, it seems only fair to follow Hutter et al. [5] pg. 24, and classify this work under nonrelativistic theories.

In Chapter 15 of the second volume [30], the worldtensor formalism is presented. In Section 15.4, pg. 729, the two world tensors $\boldsymbol{\varphi}$ and $\boldsymbol{\eta}$ seemingly appear, all in index formulation. However, the presentation of the two authors is rather difficult to follow, because notions like "Galilean formulation," "inertial frame (laboratory frame)," "co-moving frame," and "Lorentz transformations" are used at a rapid pace and not clearly distinguished. In short, it lacks the clarity of the Handbook, which interestingly is only mentioned as reference to the kinematics part Section 15.3 of this chapter.

Finally, in Zhilin [66] and in the book [32] by Zhilin, some comments regarding the transformation properties of the electromagnetic fields can be found. For example, on pg. 351 of the book, an axiom regarding the nature of the magnetic and the electric field is presented stating that $\boldsymbol{B}$ is of axial and $\boldsymbol{E}$ of polar nature, which is in agreement with Equations $(83)_1$ and (84). Zhilin associates the axiom with the name of Hertz. It is fair to say that, in his complete works, in particular, the papers dealing with electrodynamics, [67], there is no direct reference to this issue. In fact, Hertz does not really make use of the concept of vectors even though he refers to Heaviside's work in the introduction, where the vector concept of electromagnetism is used to perfection [68].

However, in the book by Abraham [25], we find some hints that Maxwell may have thought about this already: "… doch hat die Entwicklung der Wissenschaft die Vermutung Maxwells bestätigt, daß die magnetischen Vektoren axialer, die elektrischen polarer Art sind."[33] ([25], pg. 24) and "Da der Vektor $\mathfrak{K}$ [=force vector] polar ist, so sind über die Natur des Skalars e [= electric charge] und des Vektors $\mathfrak{E}$ [=electric field vector] nur zwei Annahmen möglich. Entweder e ist ein eigentlicher Skalar und $\mathfrak{E}$ ein polarer Vektor, oder e ist ein Pseudoskalar und $\mathfrak{E}$ ein axialer Vektor. Wir wollen uns jetzt schon für die erstere Annahme entscheiden, indem wir die elektrische Ladung als Skalar im eigentlichen Sinne betrachten."[34] ([25], pg. 128). In fact, the latter statement is in contradiction to the Handbook [1] where on pg. 666, it is said in a footnote: "Our reasons for assuming that the charge transforms as an axial scalar instead of an absolute scalar will be explained in Section 283." However, no explanation can be found in that section. On the other hand, it was not possible for us to confirm Abraham's statement with the published work of Maxwell directly. We may only say this: In Section 23 of Ref. [69], Maxwell considers "Right-handed and left-handed relations in space." Among other things, he talks about

[32] Unlike in Section 5.2, three-vector notion has been used and no distinction between co- and contravariant components is made to allow for an easy comparison with Equation (58).

[33] "… but the development of science has confirmed Maxwell's supposition that the magnetic vectors are of the axial, the electric ones of the polar type."

[34] "Since the vector $\mathfrak{K}$ [=force vector] is polar, only two assumptions regarding the nature of the scalar e [=electric charge] and the vector $\mathfrak{E}$ [=electric field vector] are possible. Either e is a proper scalar and $\mathfrak{E}$ a polar vector, or e is a pseudo scalar and $\mathfrak{E}$ an axial vector. Already now we want to choose the former assumption, by considering the electric charge being a scalar in the proper sense."

sign reversal for oriented surface and volume elements. Following Ichiguchi [70], we are tempted to conclude: "He [= Maxwell] considered $\boldsymbol{E}$ and $\boldsymbol{H}$ as quantities defined with respect to a line, and $\boldsymbol{D}$ and $\boldsymbol{B}$ as quantities defined with respect to a plane … This suggests that Maxwell himself thought of $\boldsymbol{E}$ and $\boldsymbol{H}$ as polar vectors and $\boldsymbol{D}$ and $\boldsymbol{B}$ as axial vectors."[35]

Then, the editors of Zhilin's book mention in a footnote on pg. 351: "Напомним, что в шестой главе предложена модель электромагнитного поля, согласно которой вектор электрического поля следует трактовать как аксиальный вектор, а вектор магнитного поля — как полярный вектор. (Примеч. ред.)"[36] However, it is not explained, which choice is more appropriate. It must also be pointed out that according to Equations (84) and (83), the electromotoric force $\mathfrak{E}$ and not the electric field $\boldsymbol{E}$ is the objective quantity. Nevertheless, after proposing the axiom, Zhilin uses objectivity for $\boldsymbol{E}$ for transformation of Maxwell equations (pg. 349 and pg. 352), in which a total time derivative is used. Indeed, we have seen in Equation (30) for the convective time derivative that it might be possible to extract the rotor term $\boldsymbol{v} \times \boldsymbol{B}$, which is required in Equation (84). In Section 8, a possible solution to this conundrum is suggested. Further information and references on Zhilin's work on the invariance properties of the Maxwell equations can be found in Altenbach et al. [72].

6 | POLARIZATION AND MAGNETIZATION

6.1 | Additive decomposition of charge and current densities

A full exposé on constitutive equations of electromagnetism, in particular when coupled to thermo-mechanics, is beyond the purpose of this review paper. May it therefore suffice to introduce polarization and magnetization in a rational manner. We will use arguments from the Handbook [1], Section 2.3. We start with an important quote: "… charge and current are the fundamental entities while polarization and magnetization are simply auxiliary fields introduced as mathematical devices providing a convenient description of special distributions of charge and current in special types of materials. This may be called the principle of Ampère and Lorentz …" In this spirit, we decompose the charge and current within a material volume $v(t)$ and across an open material surface $a(t)$[37] additively as follows:

$$\begin{aligned} \int_{v(t)} q \,\mathrm{d}v &= \int_{v(t)} q^{\mathrm{f}} \,\mathrm{d}v - \oint_{\partial v(t)} \boldsymbol{P} \cdot \boldsymbol{n} \,\mathrm{d}a, \\ \int_{a(t)} \boldsymbol{j} \cdot \boldsymbol{n} \,\mathrm{d}a &= \int_{a(t)} \boldsymbol{j}^{\mathrm{f}} \cdot \boldsymbol{n} \,\mathrm{d}a + \frac{\mathrm{d}}{\mathrm{d}t} \int_{a(t)} \boldsymbol{P} \cdot \boldsymbol{n} \,\mathrm{d}a + \oint_{\partial a(t)} \boldsymbol{M} \cdot \boldsymbol{\tau} \,\mathrm{d}l. \end{aligned} \tag{97}$$

We refer to q^{f} as the density of free (a.k.a. real or true) charges. $\boldsymbol{P}$ is the polarization vector and by application of Gauss theorem, we may immediately conclude that

$$q^{\mathrm{p}} = -\nabla \cdot \boldsymbol{P} \tag{98}$$

is the polarization (a.k.a. induced) charge density. The name stems from the fact that by an external $\boldsymbol{E}$ field, being a force field, a displacement of charges is induced within the previously electrically neutral molecules of the material within $v(t)$. After homogenization, an apparent surface charge remains on the surface $\partial v(t)$. Indeed, the dimension of the polarization vector is charge per unit surface. If we assume that within $v(t)$, there are no discontinuities, $\boldsymbol{P}$ can be extended throughout the body, resulting in the polarization charge density q^{p}, whose source (observe the divergence) is the polarization. The minus sign is pure convention.

[35] Maxwell's struggle to find the right equations has recently been described beautifully in Longair [71], where also the origin of the word "curl" is explained.

[36] "Recall that in the sixth chapter, a model of the electromagnetic field was proposed, according to which the electric field vector should be interpreted as an axial vector, and the magnetic field vector as a polar vector. (Editor's note)"

[37] It is possible to extend all arguments to open control volumes v^{s} and open control surfaces a^{s}. However, because this section of the paper is concerned with the response of matter to the external fields $\boldsymbol{E}$ and $\boldsymbol{B}$, it seems adequate to focus on material volumes and material open surfaces.

From Equation (38), we conclude that

$$\oint_{\partial v(t)} (\boldsymbol{D} + \boldsymbol{P}) \cdot \boldsymbol{n} \, \mathrm{d}a \equiv \int_{v(t)} q^{\mathrm{f}} \, \mathrm{d}v. \tag{99}$$

The quantity in parentheses, $\boldsymbol{\mathfrak{D}} = \boldsymbol{D} + \boldsymbol{P}$, is known as the charge potential in polarizable matter. Hence, we conclude that in regular points of the material, the following local equation holds:

$$\nabla \cdot \boldsymbol{\mathfrak{D}} = q^{\mathrm{f}}. \tag{100}$$

Analogously $\boldsymbol{j}^{\mathrm{f}}$ from Equation $(97)_2$ is known as the free electric current density, and $\boldsymbol{M}$ is the magnetization. If we apply the transport theorem (A19) to the second term in there, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{a(t)} \boldsymbol{P} \cdot \boldsymbol{n} \, \mathrm{d}a = \int_{a(t)} \left(\frac{\partial \boldsymbol{P}}{\partial t} - q^{\mathrm{p}} \boldsymbol{v} \right) \cdot \boldsymbol{n} \, \mathrm{d}a + \oint_{\partial a(t)} (\boldsymbol{P} \times \boldsymbol{v}) \cdot \boldsymbol{\tau} \, \mathrm{d}l, \tag{101}$$

where Equation (98) was used.

Moreover, if the material version of the integral form (41) of the Ampère–Øersted law and the definition of the charge potential in matter is used, it follows:

$$\int_{a(t)} \left(\frac{\partial \boldsymbol{\mathfrak{D}}}{\partial t} + q^{\mathrm{f}} \boldsymbol{v} \right) \cdot \boldsymbol{n} \, \mathrm{d}a^{\mathrm{s}} = \oint_{\partial a(t)} (\boldsymbol{H} - \boldsymbol{P} \times \boldsymbol{v} - \boldsymbol{M}) \cdot \boldsymbol{\tau} \, \mathrm{d}l - \int_{a(t)} \boldsymbol{j}^{\mathrm{f}} \cdot \boldsymbol{n} \, \mathrm{d}a. \tag{102}$$

The quantity in parentheses under the line integral, $\boldsymbol{\mathfrak{H}} = \boldsymbol{H} - \boldsymbol{P} \times \boldsymbol{v} - \boldsymbol{M}$, is known as the current potential in matter. If we assume continuity of all fields throughout $a(t)$, we obtain an alternative version to the local Ampère–Øersted law in regular points, Equation (43):

$$-\frac{\partial \boldsymbol{\mathfrak{D}}}{\partial t} + \nabla \times \boldsymbol{\mathfrak{H}} = q^{\mathrm{f}} \boldsymbol{v} + \boldsymbol{j}^{\mathrm{f}}. \tag{103}$$

In this context, it is worthwhile repeating the *caveat* from pp. 688 of the Handbook: "In most elementary and even advanced texts on electromagnetic theory, a clear distinction is not made between the partial potentials $\boldsymbol{\mathfrak{D}}$ and $\boldsymbol{\mathfrak{H}}$ and the resultant potentials $\boldsymbol{D}$ and $\boldsymbol{H}$ in discussion of polarizable and magnetizable media. However, it is only when one attempts to treat the problem of dielectrics and magnets set in motion that serious difficulties arise from confusing these fields."

Indeed, it is now time to talk about concrete forms for the constitutive equations of the polarization and the magnetization. However, this can easily become an extremely long story. Hence we will only slightly touch upon the subject. Let us imagine that the dielectric material is at rest in an inertial frame, $\boldsymbol{v} = \boldsymbol{0}$, and isotropic. Externally an electric and a magnetic field, $\boldsymbol{E}$ and $\boldsymbol{B}$, are applied. Then by definition of a nonmagnetizable ($\boldsymbol{M} = \boldsymbol{0}$), dielectric material only the electric field will have an effect. Because of its force character, it tends to polarize the molecules. Moreover, if this effect is assumed to be linear, we have

$$\boldsymbol{P} = \epsilon_0 \chi \boldsymbol{E}\,, \quad \boldsymbol{M} = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{\mathfrak{D}} = \boldsymbol{D} + \boldsymbol{P} \equiv \epsilon_0 \boldsymbol{E} + \boldsymbol{P} = \epsilon_0 (1 + \chi) \boldsymbol{E}\,, \quad \boldsymbol{\mathfrak{H}} = \boldsymbol{H} \equiv \frac{1}{\mu_0} \boldsymbol{B}. \tag{104}$$

Note that the æther relations were used. The dimensionless factor χ is known as the dielectric susceptibility. Now, if the material moves w.r.t. the inertial frame ($\boldsymbol{v} \neq \boldsymbol{0}$), this simple constitutive law looks immediately more complicated, because the $\boldsymbol{B}$-field can also induce polarization:

$$\begin{gathered} \boldsymbol{P} = \epsilon_0 \chi \boldsymbol{\mathfrak{E}} \quad \Rightarrow \\ \boldsymbol{\mathfrak{D}} = \epsilon_0 [(1 + \chi) \boldsymbol{E} + \chi \boldsymbol{v} \times \boldsymbol{B}]\,, \quad \boldsymbol{\mathfrak{H}} = \boldsymbol{H} + \boldsymbol{v} \times \boldsymbol{P} = \frac{1}{\mu_0} \left(\boldsymbol{B} + \frac{\chi}{c^2} \boldsymbol{v} \times \boldsymbol{\mathfrak{E}} \right), \quad \boldsymbol{\mathfrak{E}} = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}. \end{gathered} \tag{105}$$

The rational approach presented in this paper makes changes like that immediately obvious without the need for talking about the theory of relativity. However, because it is not used throughout the scientific world considerable confusion may arise to understand experiments like the rotating Wilson cylinder (cf., [4], pp. 119, [73]). Note that if one operates in Gaussian units, the last equation reads

$$\mathfrak{D} = (1+\chi)\boldsymbol{E} + \chi\frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\,,\quad \mathfrak{H} = \boldsymbol{B} + \chi\frac{\boldsymbol{v}}{c}\times\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right), \tag{106}$$

which allows immediately to see the order of the additional terms. If one so wishes, the second-order correction could be interpreted as a "relativistic effect."

6.2 | Comparison with the literature

Physicists often do not stress the importance of the additive decomposition of electric charges and currents enough. Moreover, typically no difference is made between the full potentials, $\boldsymbol{D}$ and $\boldsymbol{H}$, and the potentials of the free charges and currents, $\mathfrak{D}$ and $\mathfrak{H}$. Consequently, considerable confusion may arise what certain symbols exactly mean.

Feynman [22] introduces the concept of polarization in Chapter 10 of his lectures. He splits the electric charges additively. But then the partial charge potential $\mathfrak{D}$ appears in disguise in Section 10-4 and is immediately evaluated in an inertial system, that is, the æther relation for $\boldsymbol{D}$ is inserted. Magnetization is introduced conceptually in Chapter 35 and related to the magnetization current density in Chapter 36 by additive decomposition of the currents. Feynman chooses "to define a new field vector $\boldsymbol{H}$" and relates it to magnetization in Section 36-2. Note that this is not quite the partial potential $\mathfrak{H}$. Rather this quantity would be related by $\mathfrak{H} = {}^{\boldsymbol{H}}/_{\mu_0}$ in our nomenclature, which means it carries more the meaning of a "magnetic field in matter." The contribution of polarization is not mentioned.[38]

Because matter is concerned, Landau and Lifshitz study polarization and magnetization rightfully not in Volume 2 but in Volume 8 of their course on theoretical physics [74]. Chapter II is devoted to the electrostatics of dielectrics. As in the case of Feynman, the defining equation for $\mathfrak{D}$ is immediately evaluated in an inertial system, Section 6, in Gaussian units where $\boldsymbol{P}$ is multiplied by 4π. Free charges are referred to as "extraneous charges" (pg. 35).[39] The magnetization is introduced in Chapter IV, §29 where it also carries 4π when it is related to $\mathfrak{H}$ and $\boldsymbol{H}$. In fact, $\mathfrak{H}$ is replaced by $\boldsymbol{B}$, called the "magnetic induction," which is clearly the magnetic field vector. The origin of this problematic association is easy to trace, because in the Gaussian system, all electromagnetic field, including $\boldsymbol{P}$ and $\boldsymbol{M}$, have the same units.

Sommerfeld [17] uses SI units. In Sections §11C and §48, polarization and magnetization is introduced and related to the various electromagnetic fields of our nomenclature by $\boldsymbol{D} \to \mathfrak{D}$, $\epsilon_0\boldsymbol{E} \to \boldsymbol{D}$, $\boldsymbol{H} \to \mathfrak{H}$, and $\frac{\boldsymbol{B}}{\mu_0} \to \boldsymbol{H}$. Hence, agreement is reached if the æther relations (58) are observed.

The books by Becker show a similar pattern. In the early edition [23], polarization is introduced empirically in §26/27 in Gaussian units. A distinction between total and polarization charges is made, the partial charge potential $\mathfrak{D}$ is labeled $\boldsymbol{D}$, (confusingly) called "elektrische Verschiebung,"[40] and related to $\boldsymbol{P}$ by making use of the aether relations, that is, the total charge potential is set equal to $\boldsymbol{E}$, because of the use of Gaussian units. In Section §48, the magnetization is introduced and related to "magnetic fields," pg. 132, such that the partial current density $\mathfrak{H}$ is denoted by the letter $\boldsymbol{H}$, whereas for the total current potential, the letter $\boldsymbol{B}$ is used, which means that the æther relation in Gaussian units $(53)_2$ was applied. Hence, agreement with our nomenclature is obtained. In the more recent edition [24], SI units are used instead. The English edition [8] follows the older German one and uses Gaussian units.

Abraham [25] distinguishes in §43 between "wahrer" and "freier"[41] electricity and introduces the polarization just like in the older edition of Becker in §47. The magnetization follows analogously in §63, pg. 233.

In contrast to the other physics textbooks, Jackson [6] explains polarization and magnetization based on atomistic models including a homogenization procedure to move from the microscopic to the continuum scale in Chapters 4 and

[38] Note that the combination $\boldsymbol{M} - \boldsymbol{v}\times\boldsymbol{P}$ is also sometimes referred to as the Minkowski magnetization ([4], pp. 76) and subsumed in one symbol "$\boldsymbol{M}$," which gives additional rise to confusion. Feynman does not care, because he studies electrostatics of dielectrics without explicitly saying so.

[39] The meaning of this strange translation becomes a little clearer from the Russian term that is used, namely "сторонние" = "third party," which judging from the context is supposed to indicate that this charge does not need to be first induced in the material. Recall that we refer to it as the immediately available "free charge."

[40] Electric displacement

[41] "true" and "free"

5. His multipole expansions show that there are higher-order terms present, such as quadrupoles (pg. 146), which are not mentioned in the Handbook approach to polarization. On pg. 152 of this book, it is hinted that the dipole contribution is the dominant mode, but after so much mathematical effort on the microscopic level, the corrections on the continuum scale are surprisingly not detailed.[42] The introduction of $\boldsymbol{P}$ and $\boldsymbol{M}$ and relating them to the electromagnetic fields is as uncritical as in the other books, that is, the æther relations are always assumed when connecting them to the other electromagnetic fields.

Regarding the continuum related literature the following can be said:

In addition to what we presented, the Handbook article [1] also discusses transformation properties of magnetization and polarization and their embedding in a worldtensor formalism (pp. 686). However, rewriting the Maxwell equations (100) and (103) in worldtensor form is not discussed.

Not surprisingly, the books by Müller [20, 28] follow the Handbook, as far as the introduction of polarization and magnetization is concerned. However, an intuitive, atomistic interpretation of $\boldsymbol{P}$ in terms of dipoles sticking out at the surface of a material polarizable body and elementary Amperian currents circling around the periphery of a material open surface are given, [20], §9.4.1/2. But it is fair to say that the presented homogenization procedure and the transition from the microscopic to the continuum stage is far from being strict. Finally, we note the objective nature of polarization and magnetization (pg. 323):

$$P_{a'} = Q^{\cdot a}_{a'} P_a \quad \text{(Euclidean vector)}\,, \quad M_{a'} = \det \boldsymbol{Q}\, Q^{\cdot a}_{a'} M_a \quad \text{(axial Euclidean vector)}. \tag{107}$$

Kovetz distinguishes in his books between the Minkowski and the Lorentz magnetization: [3], pg. 79, [4], Chapter 7. In other words, the term $\boldsymbol{v} \times \boldsymbol{P}$ is made an issue. Applications of this term are shown in Section 22 (dragging on light by a dielectric) and 34 (Wilson's experiment). The Minkowski and Lorentz interaction models are discussed in more detail by Hutter el al. [5]. We will not discuss this intensively here.

Eringen and Maugin [29] introduce polarization and magnetization first at the atomic level (Sections 2.3/4). Then a homogenization procedure follows and the corresponding continuum are presented, Section 3.3. The æther relations for connecting these fields with the four electromagnetic fields are implicitly assumed, pg. 51, but never explicitly mentioned. A distinction between full and partial charge and current potentials is not made.

7 | THE ISSUE OF DISCONTINUITIES

7.1 | Balances for volumes and open surfaces crossed by singular surfaces and singular lines

The purpose of this section is to establish local balances for the electromagnetic fields in singular points. We start our discussion for the case of control volumes that are traversed by a discontinuity surface Σ as shown in Figure A2 (left), leading to an intersection $I(t)$, the so-called singular interface. The volumetric balance (11) needs to be generalized to cope with this situation. Indeed, a very general form for such a balance is the following one:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{v^{\mathrm{s}}_{+}\cup v^{\mathrm{s}}_{-}} \psi \,\mathrm{d}v^{\mathrm{s}} + \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{I(t)} \psi^{\mathrm{I}} \,\mathrm{d}a^{\mathrm{s}} = &-\int\limits_{a^{\mathrm{s}}_{+}\cup a^{\mathrm{s}}_{-}} \boldsymbol{\varphi}\cdot\boldsymbol{n}\,\mathrm{d}a^{\mathrm{s}} - \oint\limits_{\partial I} \boldsymbol{\varphi}^{\mathrm{I}}\cdot\boldsymbol{\nu}\,\mathrm{d}l^{\mathrm{s}} \\ &+ \int\limits_{v^{\mathrm{s}}_{+}\cup v^{\mathrm{s}}_{-}} (s+p)\,\mathrm{d}v^{\mathrm{s}} + \int\limits_{I} (s^{\mathrm{I}}+p^{\mathrm{I}})\,\mathrm{d}a^{\mathrm{s}},\end{aligned} \tag{108}$$

where $\boldsymbol{\nu}$ is the outward normal of the periphery of the intersection between the volume and the singular surface (see Figure A2, left).

Note that the to-be-balanced physical quantity, ψ or ψ^{I}, is defined for the volumes $v^{\mathrm{s}}_{\pm}(t)$ as well as for the singular interface $I(t)$. It can change because of fluxes across the periphery, $\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}^{\mathrm{I}}$, as well as supplies and productions within the volume or on the interface, s, p and s^{I}, p^{I}, respectively.

[42] Note that Ivanova and Kolpakov consider the impact of quadrupoles on the continuum level in a recent paper [75].

If now the transport theorem is used and the total flux is decomposed into a nonconvective and convective part, $\boldsymbol{\varphi} = \boldsymbol{\phi} - \psi(\boldsymbol{v}^{\mathrm{s}} - \boldsymbol{v}))$, we obtain:

$$
\begin{aligned}
&\int_{v_+^{\mathrm{s}} \cup v_-^{\mathrm{s}}} \frac{\partial \psi}{\partial t} \mathrm{d}v^{\mathrm{s}} + \int_{a_+^{\mathrm{s}} \cup a_-^{\mathrm{s}}} \psi\, \boldsymbol{v} \cdot \boldsymbol{n}\, \mathrm{d}a^{\mathrm{s}} - \int_{I(t)} [\![\psi]\!] \boldsymbol{w}^{\mathrm{I}} \cdot \boldsymbol{e}\, \mathrm{d}a^{\mathrm{s}} + \frac{\mathrm{d}}{\mathrm{d}t} \int_{I(t)} \psi^{\mathrm{I}}\, \mathrm{d}a^{\mathrm{s}} \\
&= - \int_{a_+^{\mathrm{s}} \cup a_-^{\mathrm{s}}} \boldsymbol{\phi} \cdot \boldsymbol{n}\, \mathrm{d}a^{\mathrm{s}} - \oint_{\partial I} \boldsymbol{\varphi}^{\mathrm{I}} \cdot \boldsymbol{\nu}\, \mathrm{d}l^{\mathrm{s}} + \int_{v_+^{\mathrm{s}} \cup v_-^{\mathrm{s}}} (s + p)\, \mathrm{d}v^{\mathrm{s}} + \int_{I} (s^{\mathrm{I}} + p^{\mathrm{I}})\, \mathrm{d}a^{\mathrm{s}},
\end{aligned}
\tag{109}
$$

where $\boldsymbol{e}$ is the normal on the intersection between the volume and the singular surface pointing from the "−" into the "+" region (see Figure. A2, left).

Of course, the singular interface is a generic model for many different physical situations. For example, it can be the contact or connecting zone between two materials of different type. Or it can be a shock wave traveling through space. Thus, it can be material or immaterial, but in all cases, it is idealized as a mathematical object with no thickness. Nevertheless, in general, the physical entity it is supposed to describe will have properties, and for this reason, interfacial fields are introduced in the general balance, such as the interface areal density, ψ_{I}, the total interface line flux, $\boldsymbol{\varphi}^{\mathrm{I}}$, with the outward normal $\boldsymbol{\nu}$, and the areal supply and production densities, s^{I} and p^{I}, respectively.

Recall that Equation (109) can be used to derive expressions in regular and singular points of the continuum, the so-called local balances. First, we concentrate on a regular point within the regions $v_{\pm}^{\mathrm{s}}$. Then, only volume-related terms are relevant and we obtain after application of the extended divergence theorem of Gauss (A12) to the flux term $\boldsymbol{\phi}$ the general local balance in regular points:

$$
\frac{\partial \psi}{\partial t} + \nabla^{\mathrm{s}} \cdot (\psi \boldsymbol{v} + \boldsymbol{\phi}) = s + p. \tag{110}
$$

It should be noted that this equation is written in spatial description. If one wishes to obtain the corresponding balance in material description, all fields must be expressed in Eulerian description and the index "s" on the nabla operator must be removed. The balance looks then essentially the same but its meaning is not obvious without further explanation. Also note that Equation (110) does not contain the mapping velocity $\boldsymbol{v}^{\mathrm{s}}$ any more. This is reasonable, because at a local level this quantity serves no purpose.

In electrodynamics, there are several examples for this type of balance: First, the nonexistence of magnetic monopoles (4), where it is wise to recall the *caveat* at the end of Section 2.3: This balance is "extremely mutilated" so-to-speak, since there is no volumetric quantity, no supply, and no production, but just a nonconvective flux, the magnetic field $\boldsymbol{B}$. Second, Gauss' law (39) with a nonconvective flux, the charge potential $\boldsymbol{D}$, and a supply, the charge density q. The comments after (39) apply, because it also just looks as if it came from a volumetric balance. Third, the conservation of charge (33), where there is a to-be-balanced volumetric quantity, the charge density q, convective and nonconvective fluxes, $q\boldsymbol{v}$ and the electric current density $\boldsymbol{j}$, respectively. It is interestingly to see that physical quantities can change their role in these equations. For example, q is sometimes the to-be-balanced volumetric quantity and sometimes a supply. Moreover, note that there are no productions, all quantities are conserved.

If we now wish to obtain a counterpart of the equation (110) for a point on the singular surface, the famous "Truesdellian pillbox argument" must be applied. The general idea of this procedure is concisely explained in Liu [76], pg. 35. Basically in Equation (109), now only surface-related terms are relevant. The treatment of the time derivative of the interface density ψ_{I} and the conversion of the line integral in Equation (A14) to a surface integral is explained in Müller [20] on pg. 52 in index form. However, for conciseness, we will neglect these terms in our presentation of electrodynamics. Then, the local balance in singular points reads

$$
[\![\boldsymbol{\phi} + \psi(\boldsymbol{v} - \boldsymbol{w}^{\mathrm{I}})]\!] \cdot \boldsymbol{e} = s^{\mathrm{I}} + p^{\mathrm{I}}. \tag{111}
$$

Similar remarks as in context with the balance in regular points (110) applies: This equation is meant as written in spatial form. However, in material description, it looks exactly the same and this time not even a nabla operator allows to suspect a difference. Note again that the mapping velocity $\boldsymbol{v}^{\mathrm{s}}$ does not appear. It should also be mentioned that the purpose of "jump conditions," such as Equation (111), is to enable the correct evaluation of boundary and transition conditions in initial-boundary-value problems. Moreover, the presence of the production terms in the jump condition (111) is a matter of debate: For example, if we allow volumetric productions to be present, the limit process of the pillbox argument can lead to nonvanishing contributions from the volume integrals over p. In fact, these can even be singular. However, this does not concern us, because it will turn out that the jump conditions of electrodynamics considered in this paper result from balances of conserved quantities.

An example for this type of equation is given by Equation (5): In view, the global law for the nonexistence of magnetic monopoles (14) we put $\psi \to 0, \boldsymbol{\phi} \to \boldsymbol{B}, s^{\mathrm{I}} = 0$, and $p^{\mathrm{I}} = 0$, the latter because we believe that the magnetic flux is a conserved quantity without a source, also on singular surfaces. Moreover, by looking at Gauss' law in global form (38) for the regular case—a conservation law—we conclude that there are no interface densities $\psi^{\mathrm{I}} = 0$, and no production $p^{\mathrm{I}} = 0$. However, the singular surface might carry a surface charge density, $s^{\mathrm{I}} \to q^{\mathrm{I}}$. Hence,

$$[\![\boldsymbol{D}]\!] \cdot \boldsymbol{e} = q^{\mathrm{I}}. \tag{112}$$

Finally, we take a look at the global equation of charge in regular points. We assign $\boldsymbol{\psi} \to q$ and $\boldsymbol{\phi} \to \boldsymbol{j}$ to write

$$[\![\boldsymbol{j} + q(\boldsymbol{v} - \boldsymbol{w}^{\mathrm{I}})]\!] \cdot \boldsymbol{e} = 0. \tag{113}$$

Note that this relation needs to be extended if the time dependence and movement of surface charges q^{I} and currents $\boldsymbol{\varphi} \to \boldsymbol{j}^{\mathrm{I}}$ running through the closed loop $\partial I(t)$ [43] were considered. They are neglected here.

Let us now consider a sufficiently general balance for a vector flux quantity $\boldsymbol{\gamma}$ through an open surface traversed by a singular surface Σ, so that a singular line $I(t)$ results as shown in Figure A2 (right), all in spatial description:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{a_+^{\mathrm{s}} \cup a_-^{\mathrm{s}}} \boldsymbol{\gamma} \cdot \boldsymbol{n} \, \mathrm{d}a^{\mathrm{s}} = - \int_{L_+^{\mathrm{s}} \cup L_-^{\mathrm{s}}} \boldsymbol{\varphi} \cdot \boldsymbol{\tau} \mathrm{d}l + \int_{a_+^{\mathrm{s}} \cup a_-^{\mathrm{s}}} (\boldsymbol{s} + \boldsymbol{p}) \cdot \boldsymbol{n} \, \mathrm{d}a^{\mathrm{s}} + \oint_{I(t)} \left(\boldsymbol{s}^{\mathrm{I}} + \boldsymbol{p}^{\mathrm{I}}\right) \cdot \boldsymbol{\nu} \, \mathrm{d}l^{\mathrm{s}}. \tag{114}$$

The symbols $\boldsymbol{s}$, $\boldsymbol{p}$ and $\boldsymbol{s}^{\mathrm{I}}$, $\boldsymbol{p}^{\mathrm{I}}$ refer to supplies and productions of the surfaces $a_{\pm}^{\mathrm{s}}(t)$ and the singular line $I(t)$, respectively. The total flux can be additively decomposed into a nonconvective and a convective part $\boldsymbol{\varphi} = \boldsymbol{\phi} - \boldsymbol{\gamma} \times (\boldsymbol{v} - \boldsymbol{v}^{\mathrm{s}})$. In combination with the transport theorem (A21), this allows us to rewrite Equation (114) as follows:

$$\begin{aligned} &\int_{a_+^{\mathrm{s}} \cup a_-^{\mathrm{s}}} \left(\frac{\partial \boldsymbol{\gamma}}{\partial t} + \boldsymbol{v} \nabla^{\mathrm{s}} \cdot \boldsymbol{\gamma}\right) \cdot \boldsymbol{n} \, \mathrm{d}a^{\mathrm{s}} + \int_{L_+^{\mathrm{s}} \cup L_-^{\mathrm{s}}} (\boldsymbol{\gamma} \times \boldsymbol{v}) \cdot \boldsymbol{\tau} \, \mathrm{d}l^{\mathrm{s}} - \int_{I(t)} \left([\![\boldsymbol{\gamma}]\!] \times \boldsymbol{w}^{\mathrm{I}}\right) \cdot \boldsymbol{\tau}^{\mathrm{I}} \, \mathrm{d}l^{\mathrm{s}} \\ &= - \int_{L_+^{\mathrm{s}} \cup L_-^{\mathrm{s}}} \boldsymbol{\phi} \cdot \boldsymbol{\tau} \mathrm{d}l + \int_{a_+^{\mathrm{s}} \cup a_-^{\mathrm{s}}} [\boldsymbol{s} + \boldsymbol{p}] \cdot \boldsymbol{n} \, \mathrm{d}a^{\mathrm{s}} + \int_{\partial I(t)} \left[\boldsymbol{s}^{\mathrm{I}} + \boldsymbol{p}^{\mathrm{I}}\right] \cdot \boldsymbol{\nu} \, \mathrm{d}l^{\mathrm{s}}. \end{aligned} \tag{115}$$

Initially, we focus of regular surface points on $a_{\pm}^{\mathrm{s}}$. Then only surface-related terms will survive in Equation (115). And if the generalized Stokes theorem (A20) is taken into account, we obtain

$$\frac{\partial \boldsymbol{\gamma}}{\partial t} + \boldsymbol{v} \, \nabla^{\mathrm{s}} \cdot \boldsymbol{\gamma} + \nabla^{\mathrm{s}} \times (\boldsymbol{\gamma} \times \boldsymbol{v} + \boldsymbol{\phi}) = \boldsymbol{s} + \boldsymbol{p}. \tag{116}$$

The following examples for this type of relation can be given: First, Faraday's law of induction (17), where we assign $\boldsymbol{\gamma} \to \boldsymbol{B}, \boldsymbol{\phi} \to \boldsymbol{\mathfrak{E}} \equiv \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}, \boldsymbol{s} \to \boldsymbol{0}$, and $\boldsymbol{p} \to \boldsymbol{0}$. Moreover, the nonexistence of magnetic monopoles (4) must be observed. Second, in the case of the Ampère–Øersted law (42), we put $\boldsymbol{\gamma} \to -\boldsymbol{D}, \boldsymbol{\phi} \to \boldsymbol{H} + \boldsymbol{D} \times \boldsymbol{v}, \boldsymbol{s} \to \boldsymbol{j}, \boldsymbol{p} \to \boldsymbol{0}$, and observe Gauss' law (39).

In order to obtain the local balance in a singular point on the line of intersection $I(t)$, the pillbox that led to Equation (111) is replaced be a closed noose encircling the interface point from $a_{\pm}^{\mathrm{I}}$. The procedure is illustrated in Jackson [6], pp. 17, where it is referred to as the "Stokesian loop." Obviously in Equation (115), only line integrals will survive and the final result

[43] Here the adjective "closed," indicated mathematically by ∂, means that it is the boundary to a two-dimensional object. However, it can be permeable to matter.

reads (also see [77])

$$\boldsymbol{n} \times [\![\boldsymbol{\phi}]\!] + \boldsymbol{n} \cdot [\![\boldsymbol{\gamma} \otimes \boldsymbol{w}^{\mathrm{I}} - \boldsymbol{w}^{\mathrm{I}} \otimes \boldsymbol{\gamma}]\!] = \boldsymbol{s}^{\mathrm{I}} + \boldsymbol{p}^{\mathrm{I}}. \tag{117}$$

The induction law (9) leads to the following assignments: $\boldsymbol{\phi} \to \boldsymbol{E}$, $\boldsymbol{\gamma} \to \boldsymbol{B}$, $\boldsymbol{s}^{\mathrm{I}} = \boldsymbol{0}$, $\boldsymbol{p}^{\mathrm{I}} = \boldsymbol{0}$. If the law of nonexistence of magnetic monopoles in the form (5) is taken into account one obtains

$$\boldsymbol{n} \times [\![\boldsymbol{E}]\!] - \boldsymbol{n} \cdot \boldsymbol{w}^{\mathrm{I}} [\![\boldsymbol{B}]\!] = \boldsymbol{0}. \tag{118}$$

In the case of the Ampère–Øersted law, we assign $\boldsymbol{\phi} \to \boldsymbol{H}$, $\boldsymbol{\gamma} \to -\boldsymbol{D}$, $\boldsymbol{s}^{\mathrm{I}} \to \boldsymbol{j}^{\mathrm{I}}$, $\boldsymbol{p}^{\mathrm{I}} = \boldsymbol{0}$ and observe (112)

$$\boldsymbol{n} \times [\![\boldsymbol{H}]\!] + \boldsymbol{n} \cdot \boldsymbol{w}^{\mathrm{I}} [\![\boldsymbol{D}]\!] = \boldsymbol{j}^{\mathrm{I}} + q^{\mathrm{I}} \boldsymbol{w}^{\mathrm{I}}. \tag{119}$$

7.2 | Comparison with the literature

It is fair to say that in the physics literature, jump conditions for the electromagnetic fields are barely explained: Feynman [39] considers jump conditions not systematically but only if there is need. In Section 33.3, he derives Equation (5) and special cases of Equations (118) and (119) in order to study the transmission and reflection of electromagnetic waves. In Landau and Lifshitz [74], Chapter II, §6/15 Chapter III, §29 some (specialized) jump conditions are simply applied without deriving them from first principles. It was already indicated that Jackson [6] in 1.5/6 puts more emphasis on the derivation and use of jump conditions for the electromagnetic fields. However, his approach is far from the general treatment of this paper. In the books by Becker jump condition are also mentioned on an "as there is need" basis. They are not consistently derived and far from generality: [23], §27, §30, §47, §59, [24], 2.3, 2.4, 5.5, 8.4 [8], §25, §27, §28, §47, §59. The predecessor of Becker's books by Abraham do not clearly allow to understand the jump between the field components because the sum and not the difference of fields on both sides of the discontinuous surface is considered, for example, [26], pg. 56. This is explained verbally and with a reference to a confusing picture, "Each normal component is taken in the direction *from* the field to the surface, as in fig. 7. p. 29." The Stokesian loop argument appears on pg. 191, leading to specialized versions of Equations (118) and (119), which are used to explain the reflective power of metals. All of this can be traced back to the German original by Abraham [25], pg. 73, pg. 149, pg. 231, all of which written in an extremely archaic form.

The continuum theory-based literature is very different from that. Hutter and Jöhnk [78] devote a whole chapter on the concept of jump conditions and their usefulness in continuum physics. Their presentation is based in material description and the Eulerian as well as the Lagrangian way of formulation is shown. In the Handbook [1], the jump relations (5), (118) for the electric and for the magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are summarized in Section 278 together with the jump for the electric charge (113). Because the latter is presented, Toupin may have decided not to show the jump relations for the charge and current potentials, $\boldsymbol{D}$ and $\boldsymbol{H}$-fields, respectively, even though the corresponding relations in regular points (42) and (39) are listed. From the context, it must be concluded that the presentation is in material description. Of course, the form of these jump conditions is the same as if they were meant in spatial notation.

Although the book by Müller [20] is an offspring of the handbook article it should be pointed out that a lot of effort goes first into explaining the jump relations from general principles, Sections 3.1.1.6 and 3.1.2.5. These are exploited later for the electromagnetic fields in Sections 9.2.1.2 and 9.2.2.2, all in material description. Kovetz [3] also highlights jump conditions, in 3.7 for the charge and current potentials and in 4.10 for the electric and magnetic fields. The jump condition for the charge (113) can be found in 3.5 for the special case of the matter being at rest, $\boldsymbol{v} = \boldsymbol{0}$. Hutter et al. [5] present the jump condition for the electromagnetic fields in Section 2.8. They write "The jump conditions obtained thereby will depend on whether one is dealing with the material or the spatial description," which on first glance is contrary to the opinion of this paper. However, when they use the word "spatial," they actually mean the Eulerian way of presenting a material description and "material" refers to the Lagrangian formulation when all equations are mapped fully onto the reference configuration.

Eringen and Maugin always present all possible sets of the Maxwell equations, and in particular also the ones relevant at discontinuous surfaces (in material description and in Heaviside–Lorentz units), for example, in Section 3.9 or 3.14. The transport theorems, the localization including pillbox and Stokesian loop arguments, and general balances for volumetric and flux quantities are outlined in great detail in the preparatory Sections 1.13 and 3.8.

8 | THE AETHER, VARIOUS TYPES OF TIME DIFFERENTIATION, AND MANY A QUAINT AND CURIOUS VOLUME OF FORGOTTEN LORE

8.1 | The æther

In the German physics literature, the terms "Fernwirkung" and "Nahwirkung" are frequently used. They can only inadequately be translated into English as "action at a distance" or "close-range interaction." *Fernwirkung* means that material substances separated over long distances by a vacuum (which is space free of ponderable matter) may still interact with each other, not necessarily instantaneous, but without the help of an intermediate "agent." Mathematically speaking, we would expect that equations describing close-range interaction contain some "material parameters" characteristic of the agent material.

In this sense, Newton's law of gravity seems to be a theory of the *Fernwirkungs*-type: In empty space, there is no visible agent to transfer the force in between two gravitating celestial bodies. Moreover, the interaction is instantaneous in the sense that the relevant equations do not contain a characteristic speed of interaction. This becomes evident in the Poisson version of Newton's law of gravity, $\Delta\varphi = -4\pi G\rho$, where φ is the gravitational potential, such that the gravitational body force on a mass element is given by $\boldsymbol{f} = -\nabla\varphi$, ρ is the mass density (which may be explicitly time-dependent in terms of spatial variables) and G is the universal gravitational constant.

If we now express the electric and magnetic fields in terms of a scalar and a vector potential, φ and $\boldsymbol{A}$, respectively, [44]

$$\boldsymbol{E} = -\nabla\varphi - \frac{\partial \boldsymbol{A}}{\partial t}\ , \quad \boldsymbol{B} = \nabla \times \boldsymbol{A}, \tag{120}$$

insert this in the local Maxwell equations in regular points, use the æther relations and the Lorenz gauge condition,[45] we obtain wave equations for both potentials in a Lorentz system:

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \Delta\varphi = \frac{1}{\epsilon_0}q\ , \quad \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \Delta\boldsymbol{A} = \mu_0(q\boldsymbol{v} + \boldsymbol{j}). \tag{121}$$

These could also be considered as equations of the *Fernwirkungs*-type, but with a finite speed of interaction, dictated by the speed of light, c. Indeed, in orthodox physics, an agent material is not required and the fields propagate through empty space with no material transmitter whatsoever.

However, one could see things differently and say: The gravitational constant G, the speed of light c, and the vacuum constants ϵ_0 and μ_0 are "material characteristics" of an agent, namely the æther "resting" in a Lorentz frame.

Indeed, such different opinions exist even in the recent literature, where it is insisted that the *Fernwirkungs* concept should be replaced with a *Nahwirkungs* theory, in which the æther exists and explains the transmission through the vacuum. In fact, this is an attempt at answering the perfectly legitimate question how light manages to travel through empty space. Moreover, it is fostered by Maxwell's attempts to describe the propagation of electricity through vacuum by the concept of rolling vortices (see [80, 81], pp. 311). The answer of orthodox physics to this is silence or disapproving head shaking. In private conversations between scientists sitting at a fireplace with a glass of whisky in their hands, we sometimes hear that "well, there is the particle-wave dualism of light and now here light is a phonon, which is like a bullet traveling through empty space, and a bullet needs no medium to travel."[46]

In Ivanova [83], pg. 115, we can read the following strong conviction: "We are convinced that the ether exists and it is some kind of substance consisting of particles that move and interact with each other." However, a few lines later we hear: "Is the ether a material medium, which is similar to ponderable matter and differs from it only by the material parameters, as scientists of the past believed? Certainly, it is not." A very true statement, indeed!

A nonponderable material needs a complex constitutive theory. One idea is to use a micropolar medium to characterize the material æther. This constitutive theory emphasizes rotational degrees of freedom in matter and has successfully been used in many real-life applications, such as blood flow or liquid crystal theory (see, for example, the work of Eringen [84–86]). In fact, this approach can be used to derive equations that are similar to Maxwell's equations in regular

[44] For simplicity, we use material notation here. Further details can be found, for example, in Müller [20], Section 9.3.

[45] Alternatively, Coulomb gauging could be used. However, then the hyperbolic character of the resulting equations gets lost. Issues like this are discussed for example in Brill and Goodman [79].

[46] The aspect of field versus particle interaction is also analyzed in the philosophical article by Pietsch [82].

points. This is illustrated with great care in the work of Ivanova ([21, 36, 83, 87]), where we can also find much interesting information and carefully collected details on the history of the æther. The review article by Aifantis [88], pp. 3 points in the same direction. Other constitutive approaches to the æther can be found in Refs. [89–96] (with comments on magnetic monopoles), [97]. Attempts for reconciliation of the æther model with the description of "wave-like particles" by the Schrödinger equation can be found in Refs. [98–101].[47]

The desire to understand electrodynamic phenomena, for example light, in mechanical terms as well as the anticipation of the futility to do so becomes particularly clear from the following quote by Lord Kelvin (handwritten addendum to Thomson [102], see [103]): "I can never satisfy myself until I can make a mechanical model of a thing. If I can make a mechanical model I can understand it. As long as I cannot make a mechanical model all the way through I cannot understand, and that is why I cannot get the electromagnetic theory. I firmly believe in an electromagnetic theory of light, and that when we understand electricity and magnetism and light we shall see them all together as parts of the whole. But I want to understand light as well as I can, without introducing things that we understand even less of. That is why I take plain dynamics. I can get a model in plain dynamics; I cannot in electromagnetics. But so soon as we have rotators to take the part of magnets, and something imponderable to take the part of magnetism, and realize by experiment Maxwell's beautiful ideas of electric displacements and so on, then we shall see electricity, magnetism, and light closely united and grounded in the same system."

Sommerfeld [38], pg. 111, has the following to say about the mechanical models of the æther and attempts to "derive" Maxwell's equations from equations of fluid mechanics models: "It is by no means our intention to assign any physical reality to this 'ether model'. Physicists had convinced themselves by the turn of the century that all attempts at a mechanical explanation of Maxwell's equations were doomed to failure. What we mean here is not a mechanical explanation but, at best, a mechanical analogy." And on pg. 113, he says even more harshly: "We shall have no reason to come back in Vol. III to the model of the ether discussed here. The electric charge and the structure of the electromagnetic field must be accepted as entities that transcend mechanics."

Shall we, in view of these words from an authority of physics, vilify all of the mentioned recent authors as obstinate, stubborn crackpots? It is the opinion of this paper that they do indeed have points to make and we should critically read and comment their work. Indeed, an objective comparison between Sommerfeld's approach to a mechanical interpretation of Maxwell's equations and the modern alternative based on micropolar continua has already been presented in Müller et al. [37]. We, therefore, turn directly to an another curious aspect of the Maxwell equations.

8.2 | Different types of differentiation in time

It is sometimes said that already Maxwell used total or substantial time derivatives in "his" equations. Indeed, on first glance, it looks like that when we see a ${}^{d}/_{dt}$ in his equations (A) and (H*) for the Ampère–Ørsted law on pp. 233 of Maxwell [12]. But then we must also realize that $\nabla \times \boldsymbol{H}$ in his equation (A) is expressed in "total" differentials of space coordinates, for example, ${}^{d}/_{dy}$. What is really meant in both cases are partial derivatives w.r.t. space and time. We proceed to prove it. First, note that in the famous Treatise on Natural Philosophy by Thomson and Tait [104], we can see in their discussion of the material time derivative on pg. 147 a feeble attempt to characterize partial derivatives in space and time by an additional index at the normal d's, for example, ${}^{d_t}/_{dt}$. Then they comment on it by: "Omitting again the suffixes, according to the usual imperfect notation for partial differential co-efficients, which on our new understanding can cause no embarrassment …" No embarrassment, indeed! We are not the only ones who fell into this trap. Second, in Torrance [105], pg. xii, we find: "He [= a Dr. A.D. Gilbert] also draws attention to the fact that while for functions of position and time modern notation uses, for example, the partial derivative notation $\frac{\partial}{\partial t}$, Clerk Maxwell uses the total derivative notation $\frac{d}{dt}$ throughout. To be accurate by today's standards, it would be necessary to use partials at all appropriate places, …" Third, in the same context, it is also interesting to take a look at Boltzmann's book on Maxwell's theory [106], who uses both ways of writing. For example, on pg. 81, he expands the divergence in the law of no monopoles in terms of partial derivatives w.r.t. Cartesian coordinates, whereas in his summary of Maxwell's equations on pg. 84, he uses "total" derivatives in the same equation.

Moreover, the textbooks on experimental physics by Pohl [107, 108] are often used to corroborate that scientists had realized early on that Maxwell's equations need to contain a material time derivative. It is claimed that in the case of that

[47] Also note the remarks in Feynman [22], Section 28-5, or Landau-Lifshitz [9] §75, which allude to modifications of Maxwell's theory required at the elementary particle level.

book, a dot was used for the material time derivative in the local forms of Faraday's and the Ampère–Øersted's law on pg. 72 and 78, respectively. Indeed, such a dot is sometimes sloppily applied in continuum mechanics textbooks for the material derivative (e.g., [78], pg. 150). However, maybe in the case of Pohl's book, this is not meant: On pg. 68 and 77 of the German original [107], Pohl clearly states that the dot means a partial derivative for what he calls fields $\boldsymbol{B}$ and $\boldsymbol{D}$. But then, in the recent English translation [108], edited under the auspices of his son, we see straight d's on the corresponding pg. 124. A salomonic way out of this dilemma is to say that the flux of fields through surfaces considered in these textbooks on experimental physics is homogeneous as, for example, figs. 6.1/2 on pg. 113/114 as the presented experiments suggest. Then there is no difference between the various time derivatives.

Christov states in Christov [109] quite apodictically that "Hertz (1900) [=[110]] realized that the cause of non-invariance was the use of partial time derivatives." and in Christov [95] "Quite ironically, ten years prior to the advent of RP [=Relativity Principle], the sacred equations had already been changed in the correct direction by Hertz [14] [= [110]], who proposed to use the convective derivatives in the terms where Maxwell had merely partial time derivative (see, also the survey in [15] [=[111]])." Pinheiro [111] points out the approximate location in Hertz' book, where this allegedly has been done, namely on pg. 244. However, in order to make sure that we know what was really said by Hertz, we now consult the German original [67].

First, it is justified to assume that by moving bodies, Hertz means bodies filled with matter. Indeed, right at the beginning of his article on the fundamental equations of electrodynamics of moving bodies (pp. 256) he says: "Wir beachten zunächst, dass wenn von bewegten Körpern schlechthin die Rede ist, wir stets nur an die Bewegung der ponderabelen Materie denken."[48] Then on pg. 259, we can read the following: "Wir behaupten nun, es sei der Einfluss der Bewegung derart, dass, wenn er allein wirksam wäre, er die magnetischen Kraftlinien mit der Materie fortfahren würde."[49] Granted, this somewhat nebulous phrase might indicate the idea of a material transport. A little further down the text, Hertz begins to clarify in mathematical terms what he means ("Um zunächst unsere Behauptung in Zeichen zu kleiden … "[50]) and studies the transport of magnetic lines of forces trespassing a small surface element with the movement of that element, which he does not call material though. Indeed, Hertz' discussion reminds slightly of Maxwell's alleged believe that the $\boldsymbol{B}$-vector is axial in nature (see the end of Section 5.3). Finally, on pg. 261, a set of equations ("… das folgende System der Grundgleichungen für bewegte Körper …"[51]) appears, which does very much resemble the one we shall derive now, and which, in a rather complex way, as we shall discuss, confirms the statements of Christov and Pinheiro.

To this end, recall Toupin's convective time derivative in the form (30) and apply it to the magnetic field $\boldsymbol{B}$ and the total charge potential $\boldsymbol{D}$:

$$\begin{aligned}\frac{\mathrm{d_c}\boldsymbol{B}}{\mathrm{d}t} &= \frac{\partial \boldsymbol{B}}{\partial t} - \nabla\times(\boldsymbol{v}\times\boldsymbol{B}) + \boldsymbol{v}\nabla\cdot\boldsymbol{B} \equiv \frac{\partial \boldsymbol{B}}{\partial t} - \nabla\times(\boldsymbol{v}\times\boldsymbol{B}),\\ \frac{\mathrm{d_c}\boldsymbol{D}}{\mathrm{d}t} &= \frac{\partial \boldsymbol{D}}{\partial t} - \nabla\times(\boldsymbol{v}\times\boldsymbol{D}) + \boldsymbol{v}\nabla\cdot\boldsymbol{D} \equiv \frac{\partial \boldsymbol{D}}{\partial t} - \nabla\times(\boldsymbol{v}\times\boldsymbol{D}) + q\boldsymbol{v},\end{aligned} \tag{122}$$

where the nonexistence of magnetic monopoles and Gauss' law, Equations (4) and (40), respectively, were used. A similar rationale can be found in Refs. [109, 112], where this time derivative is referred to as the "Oldroyd upper-convected derivative," and in Pinheiro [111]. In these papers, similar conclusions to the following ones are drawn.

These results are now inserted into the induction and the Ampère–Øersted law, Equations (18) and (43), respectively. We find

$$\frac{\mathrm{d}_c\boldsymbol{B}}{\mathrm{d}t} + \nabla\times(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}) = \boldsymbol{0}\ ,\quad -\frac{\mathrm{d}_c\boldsymbol{D}}{\mathrm{d}t} + \nabla\times(\boldsymbol{H}+\boldsymbol{D}\times\boldsymbol{v}) = \boldsymbol{j}. \tag{123}$$

Note by comparison with the results from Section 5.2, only (axial) objective quantities appear[52] in these equations. It is almost miraculous that the convective part $q\boldsymbol{v}$ of the electric current no longer exists. In this form, Faraday's law has axial and the Ampère–Øersted law polar Euclidean character, as expected.

[48] "We remark, in the first place, that when, in general, we speak of bodies in motion, we always think of the motion of ponderable matter only."
[49] "We now claim that the influence of the motion is such that, if it were acting alone, it would carry the magnetic lines of force with the matter."
[50] "In order to conceptualize our claim in symbols … "
[51] "… the following system of fundamental equations for moving bodies …"
[52] A good way to show that the convective time derivative of polar or axial Euclidean flux vectors is objective, is to start with Equation (29) and then make use of the Euclidean transformation in index notation, Equations (78) and $(80)_1$, respectively.

Indeed, Equations (123) look like Hertz' fundamental equations of moving bodies on pg. 261 of Hertz [67] if (a) we identify the symbol $\frac{d}{dt}$ in them with $\frac{\mathrm{d_c}}{\mathrm{d}t}$ despite the fact that $\frac{d}{dt}$ is clearly a partial derivative in his version of the Faraday and Ampère–Ørsted laws for bodies at rest on pg. 225; and (b) if we apply the law of magnetic monopoles and put the term $\boldsymbol{v}\nabla \cdot \boldsymbol{B}$ in these equations equal to zero. Hertz does not do that. In fact Abraham comments on this fact and says on pg. 413 of Abraham and Föppl [25]: "Hertz operiert dort mit der Annahme von wahrem Magnetismus. Wir wollen, der hier vertretenen Auffassung getreu, solchen ausschließen und daher div $\mathfrak{B}$ durchweg gleich Null setzen." [53] It remains unclear what "true magnetism" really means in physics terms. Then (c) in context with Hertz' term $\boldsymbol{v}\ \nabla \cdot \boldsymbol{D}$ in the equations on pg. 261 it needs to be clarified as to whether what he calls on pg. 224 "elektrische Strömung" [54] now contains convective currents $q\boldsymbol{v}$ or not. This is unclear, because the definition on pg. 224 was given in his article on fundamental equations of electromagnetism in bodies *at rest*. In fact, not only for the sake of history of science but also in view of the ongoing debates, each equation of Hertz' two articles deserves a detailed investigation in view of a modern continuum theory of electrodynamics. We leave this to future work. In summary, we may say that it is not surprising that until today people are uneasy when asked how the equations for the "electrodynamics of moving bodies" read correctly.

At this point, we would also like to compare the result (123) with the literature based on the following quote from Zhilin's book [32], pg. 348: [55]

"Далее рассматриваются уравнения Максвелла в пустоте, ибо включение токов требует обсуждения вопросов, не имеющих прямого отношения к рассматриваемой теме. В современной физике уравнения Максвелла в пустоте записываются в виде

$$\nabla \times \mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t}, \quad \nabla \cdot \mathcal{E} = 0, \quad \nabla \times \mathcal{B} = \frac{1}{c^2}\frac{\partial \mathcal{E}}{\partial t}, \quad \nabla \cdot \mathcal{B} = 0. \qquad (7.45)$$

В механике эти уравнения записывались бы несколько по-иному

$$\nabla \times \mathcal{E} = -\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}t}, \quad \nabla \cdot \mathcal{E} = 0, \quad \nabla \times \mathcal{B} = \frac{1}{c^2}\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}, \quad \nabla \cdot \mathcal{B} = 0. \qquad (7.46)$$

Так же, как и в случае с волновыми уравнениями, различие заключается в том, что в (7.45) входят частные производные по времени, а в (7.46) — полные производные по времени."[56]

There is much truth in these words, but also possible misunderstanding. Indeed, if one so wishes, it is possible to use something similar to total derivatives, namely the convective time derivative of a flux-like quantity, as it was shown in Equation (123). But then in the rotors, additional terms will appear since $\boldsymbol{E}$ and $\boldsymbol{H}$ are not objective. The crux is that Zhilin inserted the æther relations without saying and they hold only in inertial and not in Euclidean systems. However, note that Zhilin assumes only inner rotational but no translational degrees of freedom, hence $\boldsymbol{v} = \boldsymbol{0}$ and agreement is achieved. Moreover, if we use the convective time derivative in a system co-moving with the matter, then $\boldsymbol{v} = \boldsymbol{0}$ and we have also have perfect agreement with Zhilin's proposed equations, if in addition, the æther relations for Euclidean systems are observed, (95), which can also (formally) be simplified to become the æther relations of the inertial system. To this end, the co-movement must be such that $\boldsymbol{w} = \boldsymbol{0}$ at all times, which means it is necessary to constantly switch the co-moving frame. Doing all this is possible, but unnecessarily restrictive and can be avoided, if one accepts the worldvector formulation.

9 CONCLUSIONS

In this paper, the following was achieved:

[53] "There Hertz operates under the assumption of true magnetism. We want in accordance with the opinion held here exclude such a one and therefore throughout put div $\mathfrak{B}$ equal to zero."

[54] "electric flow [not current, mind!]"

[55] For a correct citation we have used Zhilin's letters. They correspond to our nomenclature as follows: $\mathcal{E} \to \boldsymbol{E}$, $\mathcal{B} \to \boldsymbol{B}$.

[56] "In what follows, Maxwell's equations in vacuum are considered, because the inclusion of currents requires discussion of issues that are not directly related to the topic under consideration. In modern physics, Maxwell's equations in vacuum are written as … In mechanics, these equations would be written somewhat differently … Just as in the case of the wave equations, the difference lies in the fact that (7.45) includes partial time derivatives, and (7.46) includes total time derivatives."

- Two independent sets of electromagnetic fields were introduced. The necessity for this was explained in details. The first set, $\boldsymbol{E}$ and $\boldsymbol{B}$, is related to the axiom of the conservation of magnetic flux, the second, $\boldsymbol{D}$ and $\boldsymbol{H}$, to the axiom of the conservation of charge.
- It was explained that both sets are connected in a simple linear fashion in inertial systems by the so-called Maxwell–Lorentz æther relations. Inertial systems are related to each other by Lorentz transformations. These guarantee that the simple linear proportionality of the Maxwell–Lorentz æther relations remains.
- In this context, the worldtensor formalism of electrodynamics was explained and specialized to the requirements of rational continuum theory, namely Euclidean transformations. Euclidean objective and nonobjective fields of continuum electrodynamics were identified.
- A synopsis on dimensions and units of the electromagnetic fields was presented. Advantages of the SI and of the Gaussian system of units were explained.
- A detailed analysis of all these items in view of the old as well as the recent physics and continuum literature was performed. The different viewpoints were presented and discussed, the pros and cons as well as unresolved issues were considered.
- The last chapter of this article was devoted to two curious aspects of electrodynamics. Both have an old history and yet they are a topic of ongoing interest in the modern literature: The notion of æther under the viewpoint of a material with a constitutive behavior and the issue of objective material and other time derivatives. The corresponding discussion is far from being finished and requires to be continued.

Finally, in his book [113], Truesdell laments over “The Tragicomedy of Classical Thermodynamics.” Had he (and not Toupin) been written Chapter F of the Handbook he would have surely concluded that there is an equally tragic comedy going on in electrodynamics.

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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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APPENDIX A: TRANSPORT THEOREMS

A.1 | Volumetric quantities

In order to introduce the spatial description of continua favored in this article, we argue as follows: An inertial observer is introduced who describes space in a continuous manner by introducing a three-dimensional grid as indicated in Figure A1. The observer also carries a clock to measure his current time, t. The space points are continuously distributed, fully immaterial, and identified by vectors $\boldsymbol{x}^{\mathrm{s}}$. The space point addressed by this vector is sometimes called an *observational point* [114].

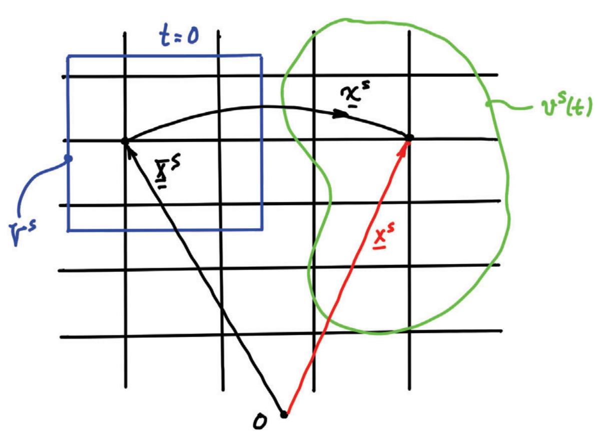


FIGURE A1 The concept of spatial description and open control volumes

At the current time, t, the observer considers a nonmaterial, arbitrarily moving volume, $v^{\mathrm{s}}(t)$, that encompasses a certain set of spatial points $\boldsymbol{x}^{\mathrm{s}}$. This volume and the enclosed observational points belong to the so-called *current placement*. If one so wishes, this volume is called "open" in the sense that matter may stream through it unaffectedly. It is also referred to as a *control volume*.

Its time dependency stems from the fact that the observer considers different sets of spatial points at different times, and this is why it is called "arbitrarily moving." Specifically, at time $t = 0$, a set of points $\boldsymbol{X}^{\mathrm{s}}$ within the space region V^{s} is considered. In the spirit of continuum mechanics, we refer to this volume as the *reference volume* and to the spatial points $\boldsymbol{X}^{\mathrm{s}}$ as the observational points of the *reference placement*. It should be pointed out that by definition V^{s} is not time-dependent.

We now require that the observational points of the current and of the reference placement are related to each other by a bijective mapping,

$$\boldsymbol{x}^{\mathrm{s}} = \boldsymbol{\chi}^{\mathrm{s}}(\boldsymbol{X}^{\mathrm{s}}, t). \tag{A1}$$

However, it must be pointed out that $\boldsymbol{X}^{\mathrm{s}}$ is not physically "transported" onto $\boldsymbol{x}^{\mathrm{s}}$, the points are only bijectively related in an abstract manner. In this sense, the motion $\boldsymbol{\chi}^{\mathrm{s}}$ is a figment of imagination, albeit a useful one as we shall demonstrate now.

We can assign a *mapping velocity* [115] to this motion, namely

$$\boldsymbol{v}^{\mathrm{s}} = \left.\frac{\partial \boldsymbol{\chi}^{\mathrm{s}}(\boldsymbol{X}^{\mathrm{s}}, t)}{\partial t}\right|_{\boldsymbol{X}^{\mathrm{s}}}, \tag{A2}$$

which is completely immaterial. Moreover, if we wish, we can also apply the chain rule to $\boldsymbol{\chi}^{\mathrm{s}}$ and find a total time differential:

$$\left.\frac{\partial \boldsymbol{\chi}^{\mathrm{s}}(\boldsymbol{X}^{\mathrm{s}}, t)}{\partial \boldsymbol{X}^{\mathrm{s}}}\right|_{t} \cdot \frac{\mathrm{d}\boldsymbol{X}^{\mathrm{s}}}{\mathrm{d}t} + \left.\frac{\partial \boldsymbol{\chi}^{\mathrm{s}}(\boldsymbol{X}^{\mathrm{s}}, t)}{\partial t}\right|_{\boldsymbol{X}^{\mathrm{s}}} \equiv \boldsymbol{v}^{\mathrm{s}}. \tag{A3}$$

This holds because the time differentiation of $\boldsymbol{X}^{\mathrm{s}}$ must vanish (in the first term), because space and time are independent. Note that the mapping velocity is a current property at time t and assigned to the point $\boldsymbol{x}^{\mathrm{s}}$. Various ways of writing are possible to express this fact, where functional values and functions are carefully distinguished by bars and hats,

$$\boldsymbol{v}^{\mathrm{s}} = \overline{\boldsymbol{v}}^{\mathrm{s}}(\boldsymbol{x}^{\mathrm{s}}, t) \equiv \overline{\boldsymbol{v}}^{\mathrm{s}}(\boldsymbol{\chi}^{\mathrm{s}}(\boldsymbol{X}^{\mathrm{s}}, t), t) = \hat{\boldsymbol{v}}^{\mathrm{s}}(\boldsymbol{X}^{\mathrm{s}}, t), \tag{A4}$$

and, clearly, the last variant is potentially most misleading in context with the fact that the mapping velocity is a current property, because of its dominance of the argument $\boldsymbol{X}^{\mathrm{s}}$.

The mapping velocity $\boldsymbol{v}^{\mathrm{s}}$ must not been confused with the velocity $\boldsymbol{v}$ of material particles. In this context, it should be mentioned that the material derivative of some field quantity in spatial coordinates is given by [114]:

$$\frac{\delta(\cdot)}{\delta t} = \frac{\mathrm{d}(\cdot)}{\mathrm{d}t} + (\boldsymbol{v} - \boldsymbol{v}^{\mathrm{s}}) \cdot \nabla^{\mathrm{s}}(\cdot)\,, \quad \nabla^{\mathrm{s}} = \frac{\partial}{\partial \boldsymbol{x}^{\mathrm{s}}}, \tag{A5}$$

where $\mathrm{d}/\mathrm{d}t$ is the total derivative of the field.

Fields, material as well as immaterial ones, characterizing physical properties exist in space and time and they are observed in the inertial frame of reference. Our discussion starts with volumetric fields ψ of dimension per length^3, which can be scalar, vectorial, or tensorial in nature. We write $\psi = \overline{\psi}(\boldsymbol{\chi}^{\mathrm{s}}(\boldsymbol{X}^{\mathrm{s}}, t), t)$, or $\psi = \overline{\psi}(\boldsymbol{x}^{\mathrm{s}}, t)$ for short.[57] By doing so, we indicate that the physical property ψ is considered now at time t at the spatial grid point $\boldsymbol{x}^{\mathrm{s}}$. However, at time $t = 0$, it was present at the grid point $\boldsymbol{X}^{\mathrm{s}}$. We call this the *spatial description* of the field property.

We now consider again the open control volume $v^{\mathrm{s}}(t)$, which at time t encloses a certain continuous set of points $\boldsymbol{x}^{\mathrm{s}}$, and which will alter, if time changes. On this set of points, the *additive* field property ψ is defined. Consequently, we can consider volume integrals of the form $\int_{v^{\mathrm{s}}(t)} \overline{\psi}(\boldsymbol{x}^{\mathrm{s}}, t)\mathrm{d}v^{\mathrm{s}}$ and ask how the time derivative of this quantity can be exchanged with the integration. This operation is two-fold because of the changing boundaries of the integral and of the

[57] Recall again that $\partial \boldsymbol{x}^{\mathrm{s}}/\partial t = \mathbf{0}$ because space and time are independent.

integrand being time-dependent. Based on Equation (A1), we follow the concepts that are usually applied to the movement of material particles and define a deformation gradient:

$$\boldsymbol{F}^{\mathrm{s}} \equiv \overline{\boldsymbol{F}}^{\mathrm{s}}(\boldsymbol{x}^{\mathrm{s}}, t) = \frac{\partial \chi^{\mathrm{s}}(\boldsymbol{X}^{\mathrm{s}}, t)}{\partial \boldsymbol{X}^{\mathrm{s}}}, \quad J^{\mathrm{s}} \equiv \overline{J}^{\mathrm{s}}(\boldsymbol{x}^{\mathrm{s}}, t) = \det \boldsymbol{F}^{\mathrm{s}} \quad \Rightarrow \quad \mathrm{d}v^{\mathrm{s}} = J^{\mathrm{s}} \mathrm{d}V^{\mathrm{s}}, \tag{A6}$$

where $\mathrm{d}V^{\mathrm{s}}$ is the volume element in the reference placement. Thus, we arrive at the analog to Nanson's first formula known from the continuum mechanics of material points:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}v^{\mathrm{s}}) = \nabla^{\mathrm{s}} \cdot \overline{\boldsymbol{v}}^{\mathrm{s}}(\boldsymbol{x}^{\mathrm{s}}, t)\mathrm{d}v^{\mathrm{s}}. \tag{A7}$$

Then, the time differentiation of the volume integral becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{v^{\mathrm{s}}(t)} \overline{\psi}(\boldsymbol{x}^{\mathrm{s}}, t)\, \mathrm{d}v^{\mathrm{s}} = \int_{v^{\mathrm{s}}(t)} \left(\frac{\partial \psi}{\partial t} + \boldsymbol{v}^{\mathrm{s}} \cdot \nabla^{\mathrm{s}} \psi + \psi \nabla^{\mathrm{s}} \cdot \boldsymbol{v}^{\mathrm{s}} \right) \mathrm{d}v^{\mathrm{s}} \equiv \int_{v^{\mathrm{s}}(t)} \left(\frac{\partial \psi}{\partial t} + \nabla^{\mathrm{s}} \cdot (\psi \boldsymbol{v}^{\mathrm{s}}) \right) \mathrm{d}v^{\mathrm{s}}. \tag{A8}$$

This is the transport theorem for volumetric quantities in spatial description. Note that all fields under the integrals are written in terms of spatial variables $\boldsymbol{x}^{\mathrm{s}}$ and t. We did not put bars on top of all symbols plus the arguments in parentheses in order to make the result look not too unwieldy, but we expect the reader to remember whenever the transport is applied. If there are no discontinuities within $v^{\mathrm{s}}(t)$, Gauss' theorem can be applied,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{v^{\mathrm{s}}(t)} \overline{\psi}(\boldsymbol{x}^{\mathrm{s}}, t)\, \mathrm{d}v^{\mathrm{s}} = \int_{v^{\mathrm{s}}(t)} \frac{\partial \psi}{\partial t} \mathrm{d}v^{\mathrm{s}} + \oint_{\partial v^{\mathrm{s}}(t)} \psi\, \boldsymbol{v}^{\mathrm{s}} \cdot \boldsymbol{n}\, \mathrm{d}a^{\mathrm{s}}, \tag{A9}$$

where $\boldsymbol{n}$ is the outward normal to the surface element da^{s}, all in spatial notation.

It should be noted that if the control volume is moving with the matter and if the material points of the matter carry a physical property ψ, the standard transport equation in material description can be formally obtained from the relations above. We assign a material volume by $v^{\mathrm{s}}(t) \to v(t)$. The fictitious mapping (A1) is replaced by the mapping of material points, χ, leading to the velocity $\boldsymbol{v}$ of material particles, both of which are not fictive,

$$\boldsymbol{x} = \chi(\boldsymbol{X}, t) \quad \Rightarrow \quad \boldsymbol{v} = \left. \frac{\partial \chi(\boldsymbol{X}, t)}{\partial t} \right|_{\boldsymbol{X}}. \tag{A10}$$

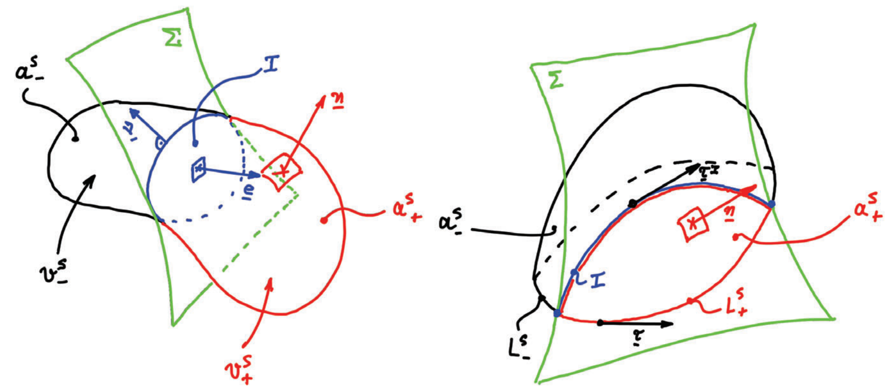
The physical property carried by the material particles can now mathematically be characterized by functional dependencies as follows, $\psi = \tilde{\psi}(\boldsymbol{X}, t)$, which is known as *Lagrangian description*, or $\psi = \breve{\psi}(\chi(\boldsymbol{X}, t), t)$, and $\psi = \breve{\psi}(\boldsymbol{x}, t)$ for short, a.k.a. *Eulerian description*. Especially, the latter must not be confused with the spatial description of fields, $\psi = \overline{\psi}(\boldsymbol{x}^{\mathrm{s}}, t)$. Then, Equation (A8) is replaced by

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{v(t)} \breve{\psi}(\boldsymbol{x}, t)\, \mathrm{d}v = \int_{v(t)} \left(\frac{\partial \psi}{\partial t} + \nabla \cdot (\psi \boldsymbol{v}) \right) \mathrm{d}v \equiv \int_{v(t)} \frac{\partial \psi}{\partial t} \mathrm{d}v + \oint_{\partial v(t)} \psi\, \boldsymbol{v} \cdot \boldsymbol{n}\, \mathrm{d}a, \tag{A11}$$

where the nabla operator ∇ refers to gradients between material points, and where the equivalence only holds if there are no discontinuities within $v(t)$. Analogously to Equations (A8) and (A9), we point out that all fields under the integrals are written in terms of Eulerian variables $\boldsymbol{x}(\boldsymbol{X}, t)$ and t. We just refrain from putting a breve on top of all symbols plus the dependency on variables in order not to make the equations look too convoluted.

Consider now the situation shown on the left of Figure A2: A "shock front" Σ passes through the volume $v^{\mathrm{s}}(t)$ and cuts it into two pieces, denoted by $v^{\mathrm{s}}_{\pm}(t)$. The front moves with its own velocity field $\boldsymbol{w}^{\mathrm{I}} = \boldsymbol{w}^{\mathrm{I}}(\boldsymbol{x}^{\mathrm{s}}, t)$, which is also described in spatial coordinates w.r.t. an inertial observer. The corresponding open surfaces are given by $a^{\mathrm{s}}_{\pm}(t)$. In order to close the surface of each subvolume, the surface of intersection $I(t)$ must be added: $\partial v^{\mathrm{s}}_{\pm} = a^{\mathrm{s}}_{\pm} \cup I$. The outward normal of $I(t)$ for $\partial v^{\mathrm{s}}_{\pm}$ is $\mp \boldsymbol{e}^{\mathrm{s}}$, which is indicated in the figure. Before we adjust Equation (A9) to this new situation, we note that Gauss divergence theorem must be extended in case of a discontinuity. For an arbitrary vector field $\boldsymbol{a}$, we have

FIGURE A2 Singular surface



$$\int_{v^{\mathrm{s}}_{\pm}(t)} \nabla^{\mathrm{s}} \cdot \boldsymbol{a} \, \mathrm{d}v^{\mathrm{s}} = \oint_{\partial v^{\mathrm{s}}_{\pm}(t)} \boldsymbol{a} \cdot \boldsymbol{n}^{\mathrm{s}} \, \mathrm{d}a^{\mathrm{s}} = \int_{a^{\mathrm{s}}_{\pm}(t)} \boldsymbol{a} \cdot \boldsymbol{n}^{\mathrm{s}} \, \mathrm{d}a^{\mathrm{s}} \mp \int_{I(t)} \boldsymbol{a} \cdot \boldsymbol{e}^{\mathrm{s}} \, \mathrm{d}a^{\mathrm{s}}$$

$$\Rightarrow \int_{v^{\mathrm{s}}_{+} \cup v^{\mathrm{s}}_{-}} \nabla^{\mathrm{s}} \cdot \boldsymbol{a} \, \mathrm{d}v^{\mathrm{s}} = \int_{a^{\mathrm{s}}_{+} \cup a^{\mathrm{s}}_{-}} \boldsymbol{a} \cdot \boldsymbol{n}^{\mathrm{s}} \, \mathrm{d}a^{\mathrm{s}} - \int_{I(t)} [\![\boldsymbol{a}]\!] \cdot \boldsymbol{e}^{\mathrm{s}} \, \mathrm{d}a^{\mathrm{s}}. \tag{A12}$$

Let us (temporarily) emphasize velocities on the boundary of control volumes by denoting them by the letter $\boldsymbol{w}$. Then, Equation (A9) can be used in the two regions $v^{\mathrm{s}}_{\pm}$ as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{v^{\mathrm{s}}_{\pm}(t)} \psi \, \mathrm{d}v^{\mathrm{s}} = \int_{v^{\mathrm{s}}_{\pm}(t)} \frac{\partial \psi}{\partial t} \mathrm{d}v^{\mathrm{s}} + \oint_{\partial v^{\mathrm{s}}_{\pm}(t)} \psi \, \boldsymbol{v}^{\mathrm{s}} \cdot \boldsymbol{n} \, \mathrm{d}a^{\mathrm{s}}$$

$$= \int_{v^{\mathrm{s}}_{\pm}(t)} \frac{\partial \psi}{\partial t} \mathrm{d}v^{\mathrm{s}} + \int_{a^{\mathrm{s}}_{\pm}(t)} \psi \, \boldsymbol{v}^{\mathrm{s}} \cdot \boldsymbol{n} \, \mathrm{d}a^{\mathrm{s}} \mp \int_{I(t)} \psi^{\pm} \, \boldsymbol{w}^{\mathrm{I}} \cdot \boldsymbol{e} \, \mathrm{d}a^{\mathrm{s}}. \tag{A13}$$

Now, we apply Equation (A12) by choosing $\boldsymbol{a} \rightarrow \psi \boldsymbol{v}^{\mathrm{s}}$ and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{v^{\mathrm{s}}_{+} \cup v^{\mathrm{s}}_{-}} \psi \, \mathrm{d}v^{\mathrm{s}} = \int_{v^{\mathrm{s}}_{+} \cup v^{\mathrm{s}}_{-}} \frac{\partial \psi}{\partial t} \mathrm{d}v^{\mathrm{s}} + \int_{a^{\mathrm{s}}_{+} \cup a^{\mathrm{s}}_{-}} \psi \, \boldsymbol{v}^{\mathrm{s}} \cdot \boldsymbol{n} \, \mathrm{d}a^{\mathrm{s}} - \int_{I(t)} [\![\psi]\!] \boldsymbol{w}^{\mathrm{I}} \cdot \boldsymbol{e} \, \mathrm{d}a^{\mathrm{s}}$$

$$= \int_{v^{\mathrm{s}}_{+} \cup v^{\mathrm{s}}_{-}} \left[\frac{\partial \psi}{\partial t} + \nabla^{\mathrm{s}} \cdot (\psi \boldsymbol{v}^{\mathrm{s}}) \right] \mathrm{d}v^{\mathrm{s}} + \int_{I(t)} [\![\psi \left(\boldsymbol{v}^{\mathrm{s}} - \boldsymbol{w}^{\mathrm{I}} \right)]\!] \cdot \boldsymbol{e} \, \mathrm{d}a^{\mathrm{s}}. \tag{A14}$$

Finally, two remarks are in order: The term $\boldsymbol{v}^{\mathrm{s}} - \boldsymbol{w}^{\mathrm{I}}$ is left within the jump brackets because if a material particle is considered $\boldsymbol{v}^{\mathrm{s}} \rightarrow \boldsymbol{v}$ and the material velocity $\boldsymbol{v}$ is definitely discontinuous on $I(t)$. Moreover, on first glance there seem to be no volumetric fields in physics that are not associated with matter, so that using a spatial description might be just an option and not be imperative. However, electromagnetic waves carry electromagnetic momentum and energy densities even *in vacuo*.

A.2 | Flux quantities

We now consider a flux density in spatial description $\boldsymbol{\gamma} = \overline{\boldsymbol{\gamma}}(\boldsymbol{x}^{\mathrm{s}}, t)$ and an open control surface $a^{\mathrm{s}}(t)$[58] covering a set of spatial points $\boldsymbol{x}^{\mathrm{s}}$, such that the flux through the surface is defined by $\int_{a^{\mathrm{s}}(t)} \overline{\boldsymbol{\gamma}}(\boldsymbol{x}^{\mathrm{s}}, t) \cdot \overline{\boldsymbol{n}}^{\mathrm{s}}(\boldsymbol{x}^{\mathrm{s}}, t) \mathrm{d}a^{\mathrm{s}}$. $\boldsymbol{n}^{\mathrm{s}}$ is the outward normal on this surface, also in spatial description. We ask for the spatial derivative of the flux, which is double time-dependent, namely w.r.t. the boundaries of the integral and of the integrands. In order to exchange the time differentiation with the

[58] Note that the surface $\partial v^{\mathrm{s}}(t)$ is the *closed* surface to an *open* control volume $v^{\mathrm{s}}(t)$. In other words, it is "open" for the passage of material particles. The adjective "open" for the surface $a^{\mathrm{s}}(t)$ refers to the fact that it is not a closed surface in the mathematical sense, and on top of that it is also "open" for the passage of material particles.

integration, we recall Nanson's second formula for material surface elements and related expressions, and rewrite them analogously in spatial description:

$$\boldsymbol{n}^{\mathrm{s}}\mathrm{d}a^{\mathrm{s}} = J^{\mathrm{s}}(\boldsymbol{F}^{\mathrm{s}})^{-\mathsf{T}} \cdot \boldsymbol{N}^{\mathrm{s}}\mathrm{d}A^{\mathrm{s}}\,, \quad \frac{\partial F^{\mathrm{s}}_{il}}{\partial t} = F^{\mathrm{s}}_{pl}\frac{\partial v^{\mathrm{s}}_i}{\partial x^{\mathrm{s}}_p}, \tag{A15}$$

where $\boldsymbol{N}^{\mathrm{s}}$ is the normal of the surface element $\mathrm{d}A^{\mathrm{s}}$ in the reference placement. Applying the mappings and carrying out all time differentiations yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{a^{\mathrm{s}}(t)} \boldsymbol{\gamma}\cdot\boldsymbol{n}^{\mathrm{s}}\mathrm{d}a^{\mathrm{s}} = \int\limits_{a^{\mathrm{s}}(t)} \left(\frac{\partial\boldsymbol{\gamma}}{\partial t} + \boldsymbol{v}^{\mathrm{s}}\cdot\nabla^{\mathrm{s}}\boldsymbol{\gamma} + \boldsymbol{\gamma}\nabla^{\mathrm{s}}\cdot\boldsymbol{v}^{\mathrm{s}} - \boldsymbol{\gamma}\cdot\nabla^{\mathrm{s}}\boldsymbol{v}^{\mathrm{s}}\right)\cdot\boldsymbol{n}^{\mathrm{s}}\mathrm{d}a^{\mathrm{s}}. \tag{A16}$$

The following identity holds:

$$\nabla^{\mathrm{s}}\times(\boldsymbol{\gamma}\times\boldsymbol{v}^{\mathrm{s}}) = \boldsymbol{v}^{\mathrm{s}}\cdot\nabla^{\mathrm{s}}\boldsymbol{\gamma} + \boldsymbol{\gamma}\nabla^{\mathrm{s}}\cdot\boldsymbol{v}^{\mathrm{s}} - \boldsymbol{\gamma}\cdot\nabla^{\mathrm{s}}\boldsymbol{v}^{\mathrm{s}} - \boldsymbol{v}^{\mathrm{s}}\nabla^{\mathrm{s}}\cdot\boldsymbol{\gamma}. \tag{A17}$$

If there are no discontinuities on the open surface, Stokes' integral theorem can be applied to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{a^{\mathrm{s}}(t)} \boldsymbol{\gamma}\cdot\boldsymbol{n}^{\mathrm{s}}\mathrm{d}a^{\mathrm{s}} = \int\limits_{a^{\mathrm{s}}(t)} \left(\frac{\partial\boldsymbol{\gamma}}{\partial t} + \boldsymbol{v}^{\mathrm{s}}\nabla^{\mathrm{s}}\cdot\boldsymbol{\gamma}\right)\cdot\boldsymbol{n}^{\mathrm{s}}\mathrm{d}a^{\mathrm{s}} + \oint\limits_{\partial a^{\mathrm{s}}(t)} (\boldsymbol{\gamma}\times\boldsymbol{v}^{\mathrm{s}})\cdot\boldsymbol{\tau}^{\mathrm{s}}\mathrm{d}l^{\mathrm{s}}, \tag{A18}$$

where $\tau^{\mathrm{s}} = \overline{\tau}^{\mathrm{s}}(\boldsymbol{x}^{\mathrm{s}}, t)$ is the tangent to the closed peripheral circuit $\partial a^{\mathrm{s}}(t)$ with line element $\mathrm{d}l^{\mathrm{s}}$ in spatial description. This is the transport theorem for open nonmaterial control surfaces in spatial description.

In complete analogy to Equation (A11), it is also possible to consider a material open surface, $a(t)$, and a material flux density $\boldsymbol{\gamma} = \check{\boldsymbol{\gamma}}(\boldsymbol{x}, t)$ assigned to the material points of the open surface, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{a(t)} \boldsymbol{\gamma}\cdot\boldsymbol{n}\,\mathrm{d}a = \int\limits_{a(t)} \left(\frac{\partial\boldsymbol{\gamma}}{\partial t} + \boldsymbol{v}\nabla\cdot\boldsymbol{\gamma}\right)\cdot\boldsymbol{n}\,\mathrm{d}a + \int\limits_{\partial a(t)} (\boldsymbol{\gamma}\times\boldsymbol{v})\cdot\boldsymbol{\tau}\,\mathrm{d}l. \tag{A19}$$

For an extension, consider now the situation shown on the right of Figure A2, where the "shock front" Σ passes through the open surface $a^{\mathrm{s}}(t)$ and cuts it into two pieces, denoted by $a^{\mathrm{s}}_{\pm}(t)$. The corresponding open lines are given by $L^{\mathrm{s}}_{\pm}(t)$. In order to close the lines of each subsurface, the line of intersection $I(t)$ must be added: $\partial a^{\mathrm{s}}_{\pm} = L^{\mathrm{s}}_{\pm} \cup I$. The tangent of $I(t)$ in $\partial a^{\mathrm{s}}_{\pm}$ is $\mp\boldsymbol{\tau}^{\mathrm{I}} = \pm\boldsymbol{\tau}$.

In order to generalize Equation (A19) to the new situation, we first consider a generalized Stokes theorem. For an arbitrary vector field $\boldsymbol{a}$, we write analogously to Equation (A12):

$$\int\limits_{a^{\mathrm{s}}_{+}\cup a^{\mathrm{s}}_{-}} (\nabla^{\mathrm{s}}\times\boldsymbol{a})\cdot\boldsymbol{n}^{\mathrm{s}}\,\mathrm{d}a^{\mathrm{s}} = \int\limits_{L^{\mathrm{s}}_{+}\cup L^{\mathrm{s}}_{-}} \boldsymbol{a}\cdot\boldsymbol{\tau}^{\mathrm{s}}\,\mathrm{d}l^{\mathrm{s}} - \int\limits_{I(t)} [\![\boldsymbol{a}]\!]\cdot\boldsymbol{\tau}\,\mathrm{d}l^{\mathrm{s}}. \tag{A20}$$

Moreover, similarly to Equation (A14), the surface transport theorem (A18) can be generalized as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{a^{\mathrm{s}}_{+}\cup a^{\mathrm{s}}_{-}} \boldsymbol{\gamma}\cdot\boldsymbol{n}\,\mathrm{d}a^{\mathrm{s}} &= \int\limits_{a^{\mathrm{s}}_{+}\cup a^{\mathrm{s}}_{-}} \left(\frac{\partial\boldsymbol{\gamma}}{\partial t} + \boldsymbol{v}^{\mathrm{s}}\nabla^{\mathrm{s}}\cdot\boldsymbol{\gamma}\right)\cdot\boldsymbol{n}\,\mathrm{d}a^{\mathrm{s}} \\ &\quad + \int\limits_{L^{\mathrm{s}}_{+}\cup L^{\mathrm{s}}_{-}} (\boldsymbol{\gamma}\times\boldsymbol{v}^{\mathrm{s}})\cdot\boldsymbol{\tau}\,\mathrm{d}l^{\mathrm{s}} - \int\limits_{I(t)} ([\![\boldsymbol{\gamma}]\!]\times\boldsymbol{w}^{\mathrm{I}})\cdot\boldsymbol{\tau}^{\mathrm{I}}\mathrm{d}l^{\mathrm{s}}.\end{aligned} \tag{A21}$$

Finally it should be pointed out that there exist flux quantities $\boldsymbol{\gamma}$ on open surfaces, the magnetic field $\boldsymbol{B}$ being the most prominent example, that need no "transport by matter" and require the use of a spatial description.