Chapter 9
Applied Spectral Analysis

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9.1 Frequency, Spectral Analysis, Phase and Phasor

A signal is a representation of a physical phenomenon that evolves in time. There are two methods to describe the signal: time analysis and frequency analysis. The Fourier transform (FT) is the basic tool to pass from time to frequency analysis.

Definition 9.1 (Frequency) The *frequency* ν is the number of occurrences of a repeating event or oscillation per unit of time. Therefore if an event is periodic of period T, then

$$\nu = \frac{1}{T}.$$

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Frequency is measured in Hertz (Hz) (i.e. in cycles per second, cps). Alternatively $\omega = 2\pi\nu$ denotes the angular frequency, which measures the number of radians swept per unit of time in a circular motion.

A signal can contain many frequencies, i.e. it can be the sum of various components with different periods.

Definition 9.2 (Spectrum) The frequency content of a signal, i.e. the frequencies of the different components of the signal, is denoted as the *spectrum* of the signal.

The term *spectrum* (and its plural *spectra*), which etymologically refers to “images” or “apparitions” of persons not present physically (like ghosts), was introduced by Sir Isaac Newton to refer to the range of colours observed when light is dispersed through a prism. This term was rapidly adopted by the scientific community to refer to the fundamental components of any wave (like sound waves, seismic waves, electric signals, etc.).

Definition 9.3 (Spectral Analysis) The analysis of a signal in terms of a spectrum of frequencies or related quantities such as energies, eigenvalues, etc. is called *spectral analysis*.

Definition 9.4 (Phase) Let $x(t)$ be a time series or a periodic signal and T be its period,

$$x(t + T) = x(t) \quad \forall t .$$

Then the phase of $x(t)$ with respect to the initial time t_0 is

$$\varphi(t) = 2\pi \left[\!\left[\frac{t - t_0}{T} \right]\!\right] ,$$

where $[\![\cdot]\!]$ denotes the fractional part of a real number. Clearly, if t_0 is shifted by T, $\varphi(t)$ does not change. Therefore, the phase depends on $t_0 \mod T$.

A sinusoid can be represented mathematically by the Euler’s formula, i.e. as the sum of two complex-valued functions:

$$A \cdot \cos(\omega t + \theta) = A \cdot \frac{e^{i(\omega t+\theta)} + e^{-i(\omega t+\theta)}}{2} ,$$

where i is the imaginary unit, A the *amplitude* (i.e. the maximum absolute height of the curve), ω the *angular frequency* (i.e. how rapidly the function oscillates) and θ the *phase* (i.e. the starting point for the cosine wave). The frequency of the wave measured in Hertz is $\omega/2\pi$.

A sinusoid can also be written as

$$A \cdot \cos(\omega t + \theta) = \mathrm{Re}\{A \cdot e^{i(\omega t+\theta)}\},$$

where Re{·} is the real part. In fact

$$\begin{aligned}\mathrm{Re}\{A \cdot e^{i(\omega t+\theta)}\} &= \mathrm{Re}\{A \cdot \cos(\omega t + \theta) + iA \cdot \sin(\omega t + \theta)\}\\ &= A \cdot \cos(\omega t + \theta)\,.\end{aligned}$$

Definition 9.5 (Phasor) Given a sinusoidal signal represented in the time-domain form as

$$v(t) = A \cdot \cos(\omega t + \theta),$$

the *phasor* is the corresponding representation in the frequency-domain form

$$V(i\omega) = A \cdot e^{i\theta} = A\angle\theta.$$

Therefore a phasor is a "complex number, expressed in polar form, consisting of a magnitude equal to the peak amplitude of the sinusoidal signal and a phase angle equal to the phase shift of the sinusoidal signal referenced to a cosine signal" [20].

9.1.1 Fourier Series and Transform

Definition 9.6 (Fourier Series) Let us consider a complex-valued function $S(x)$ periodic with period T, which is integrable on any interval of length T. A *Fourier series* for $S(x)$ is

$$S_N(x) = \sum_{n=-N}^{N} c_n \cdot e^{i\frac{2\pi n x}{T}}\,,$$

where the coefficients c_n are given by

$$c_n = \frac{1}{T}\int_T S(x) e^{-i\frac{2\pi n x}{T}}\,dx\,.$$

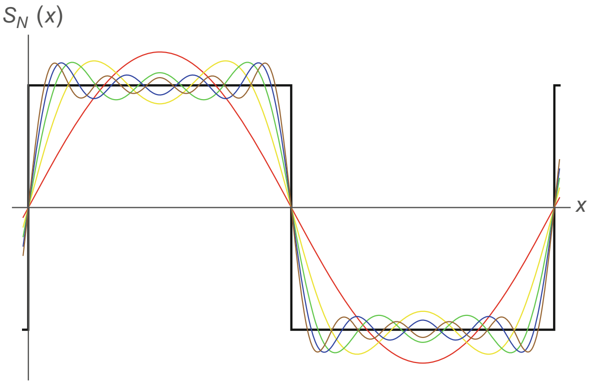
If $S(x)$ is real, then $\bar{c}_n = c_{-n}$.

Under suitable condition, is

$$S(x) = \lim_{N\to\infty} S_N(x).$$

Fourier series allows to represent periodic signals in terms of sums of simple sinusoidal functions (complex exponentials or sine and cosine functions), usually denoted as *harmonics*. The weight of each harmonic $e^{i\frac{2\pi n x}{T}}$ is given by the Fourier

Fig. 9.1 Gibbs phenomenon for a square wave (in black) for $N = 1, 3, 5, 7$ and 9



coefficient c_n, providing a representation of the periodic signal $S(x)$ in terms of the coefficients $\{c_n\}_{n\in\mathbb{Z}}$.

Care should be taken when approximating a periodic signal $S(x)$ by its truncated Fourier series $S_N(x)$, specially in the case where $S(x)$ is not continuous at some point x_0, since in this case $\lim_{N\to\infty} S_N(x) = \tilde{S}(x)$ with

$$\tilde{S}(x_0) = \frac{1}{2}\left(\lim_{x\to x_0^+} S(x) + \lim_{x\to x_0^-} S(x)\right).$$

In addition, $S_N(x)$ presents oscillations around x_0 that do not decrease in magnitude when N grows, known as Gibbs phenomenon (see Fig. 9.1).

Definition 9.7 (Fourier transform (FT)) The *Fourier transform* of a Lebesgue integrable function $f : \mathbb{R} \to \mathbb{C}$ is

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x\xi}\, dx\ ,$$

for any real number ξ.

Remark 9.1 (Fourier Transform) The Fourier transform is a representation of the function in terms of frequency instead of time; thus, it is a frequency-domain representation. It is invertible in the sense that $\hat{f}(\xi)$ can be taken back to $f(x)$. Therefore, linear operations that could be performed in the time domain have counterparts that can often be performed more easily in the frequency domain. Frequency analysis also simplifies the understanding and interpretation of the effects of various time-domain operations, both linear and non-linear. For instance, only non-linear or time-variant operations can create new frequencies in the frequency spectrum.

Remark 9.2 (Fourier Transform) The Fourier transform derives from the study of Fourier series in which complicated but periodic functions are reduced to the sum of simple waves represented by sines and cosines. The Fourier transform extends the Fourier series such that the period of the represented function goes to infinity. Thus, "the Fourier transform converts an infinitely long time-domain signal into a continuous spectrum of an infinite number of sinusoidal curves" [10].

Remark 9.3 (Fourier Transform) The Fourier transform, applied to a given complex function defined over the real line, returns a frequency spectrum containing all information of the original signal. For this reason the original function can be completely reconstructed through the inverse Fourier transform. However, in order to do so, the preservation of both the amplitude and phase of each frequency component is required.

Definition 9.8 (Discrete Fourier Transform (DFT)) Let us consider the complex numbers $x_0, \ldots, x_{N-1}$. The *discrete Fourier transform (DFT)* is defined as

$$X_k = \sum_{n=0}^{N-1} x_n e^{-i2\pi kn/N} \qquad k = 0, \ldots, N-1, \tag{9.1}$$

with $e^{i2\pi/N}$ a primitive N^{th} root of 1.

Remark 9.4 (Fast Fourier Transform (FFT)) Given the Fourier series defined in Definition 9.6 and the discretization in Definition 9.8, the problem of calculating (9.1) requires $O(N^2)$ operations (where operation means complex multiplication followed by complex addition). The reason is that there are N calculations X_k, and each calculation requires a sum of N terms. Cooley and Tukey [3] proposed an efficient method called *fast Fourier transform* (FFT) to compute the discrete Fourier transform, requiring only $O(2N \log_2 N)$ operations.

9.1.2 Spectral Density, Power Spectrum and Periodogram

Definition 9.9 (Energy Spectral Density) The *energy spectral density* of a continuous-time signal $x(t)$ describes how the energy of a signal or a time series is distributed with frequency, and it is denoted as $\mathcal{E}_x$ (unit^2 · second^2)

$$\mathcal{E}_x(f) = |\hat{x}(f)|^2$$

where

$$\hat{x}(f) = \int_{-\infty}^{\infty} e^{-2\pi ift} x(t)\, dt$$

is the Fourier transform of the signal and f is the frequency. The total energy of the signal is

$$\mathcal{E} = \int_{-\infty}^{\infty} \left|x(t)\right|^2 dt = \int_{-\infty}^{\infty} \left|\hat{x}(f)\right|^2 df.$$

If the signal is discrete, the total energy is defined as

$$\mathcal{E} = \sum_{n=-\infty}^{\infty} \left|x(n)\right|^2.$$

Definition 9.10 (Average Power) Given a signal $x(t)$, the average power P over all time is

$$P = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left|x(t)\right|^2 dt\ .$$

Remark 9.5 A stationary process may have a finite power but an infinite energy. This happens because energy is the integral of power, and the stationary signal continues over an infinite time. For this reason in such cases we cannot use the energy spectral density in Definition 9.9, but we need to introduce the concept of power spectral density.

Definition 9.11 (Amplitude Spectral Density) In analyzing the frequency content of the signal $x(t)$, one might like to compute the Fourier transform. However, for many signals of interest the Fourier transform does not formally exist. In such a case one can use a truncated Fourier transform where the signal is integrated only over a finite interval $[0, T]$ called as *amplitude spectral density*

$$\hat{x}_T(\omega) = \frac{1}{\sqrt{T}} \int_0^T x(t)\, e^{-i\omega t}\, dt\ .$$

Definition 9.12 (Power Spectral Density [19]) The *power spectral density* is

$$S_{xx}(\omega) = \lim_{T\to\infty} \left|\hat{x}_T(\omega)\right|^2 .$$

Remark 9.6 (Spectral Density) The *spectral density* describes how the energy of a continuous-time signal is distributed with frequency. The union of the various spectral densities is called *power spectrum* of the signal.

The spectral density is usually estimated using Fourier transform methods (such as the Welch method). Let us consider a sampling x_n, $n = 1, \ldots, \lfloor T/\Delta \rfloor$ of the signal $x(t)$ in the time window $[0, T]$, with sampling period Δ. Obviously, from this sampling we can evaluate only the spectral densities that are in $[2\pi/T, 2\pi/\Delta]$, since smaller frequencies generate waveforms with period longer than the time window we are considering, while higher frequencies cannot be captured by the used sampling time.

Definition 9.13 (DFT Periodogram) For the regularly sampled signal x_n, $n = 1, \ldots, N$ with sampling time Δ of the signal $x(t)$ ($x(n\Delta) = x_n$), the *periodogram* P is the function

$$P\left(\frac{k}{N\Delta}\right) = |X_k| = \left|\sum_{j=0}^{N-1} e^{-ik\frac{2\pi j}{N}} x_{j+1}\right| .$$

The periodogram can be used to estimate the spectral density of the signal[1] and is a first approximation of the signal power spectrum. For longer signals, it is possible to refine the power spectrum by averaging (even online) the different periodograms one obtains in each time window of length T. In other words, to estimate the power spectrum of a signal we need first to understand which windows of frequencies we are interested in. This define our sampling time and the length T of the time window. Then, in each time window $[kT, (k+1)T]$, compute the periodogram from the sampled signal and obtain the power spectrum of the signal by averaging the obtained periodograms.

Remark 9.7 The power spectrum of a signal is a fundamental instrument to identify the regularity of the signal. In fact, it answers the question "How much of the signal is at a certain frequency?" Signals generated by a system that exhibits a limit cycle will peak at the frequency related to the period of the limit cycle. Signals generated by a system that behave quasi-periodically give peaks at each of the different frequencies. Signals generated by a chaotic system give broad band components to the spectrum. Indeed this later can be used as a criterion for identifying that the system dynamics is chaotic.

The "most" chaotic signal is the one that exhibits a flat power spectrum, i.e. that has the same power at any frequency. This signal is called *white noise* and is defined as follows.

Definition 9.14 (White Noise) We say that the time series ϵ_t is a *white noise process* if

$$\begin{aligned} E(\epsilon_t) &= 0 \qquad \forall t \\ \mathrm{Var}(\epsilon_t) &= \sigma^2 \qquad \forall t \\ \mathrm{Covar}(\epsilon_t, \epsilon_s) &= 0 \qquad \forall s \neq t . \end{aligned} \tag{9.2}$$

[1]Other more sophisticated and more efficient methods can be used to estimate the power spectrum in case of non-equally sampled signals [4, 23].

If the random variable ϵ_t is normally distributed for all t, the white noise process is called *Gaussian white noise process*.

Remark 9.8 The covariances in the third line of Eq. (9.2) are called *autocovariances* of the time series ϵ_t.

9.1.3 Time–Frequency Representations of Signals: Gabor and Wavelet Transform

The Fourier transform of a non-periodic signal provides an accurate representation of the different frequencies (harmonics) of the signal, but with no information about the time at which each frequency is present. To avoid this problem other more sophisticated transforms are necessary.

Definition 9.15 (Gabor Transform) The Gabor transform of a signal $x(t)$ is given by Gabor [7] and Feichtinger and Strohmer [6]

$$G(x)(\tau, \omega) = \int_{-\infty}^{\infty} x(t) e^{-\pi(t-\tau)^2} e^{-i\omega t} dt.$$

Note that the Gabor transform of $x(t)$ is the Fourier transform of $x(t) \cdot e^{-\pi(t-\tau)^2}$, and therefore Gabor transform inherits all nice properties from Fourier transform. In fact it behaves even better since, due to the Gaussian, ill behaved signals $x(t)$ that do not have a Fourier transform can have a Gabor transform. The Gaussian factor $e^{-\pi(t-\tau)^2}$ can be seen as a window $e^{-\pi t^2}$ that it is shifted in the time domain to perform a *local* Fourier transform of the signal $x(t)$ at different times.

Depending on the application, better results can be obtained by substituting the Gaussian by a different window function (suitably normalized) $w(t)$ with compact support or with fast decay at infinity, in formula

$$G(x)(\tau, \omega) = \int_{-\infty}^{\infty} x(t) w(t - \tau) e^{-i\omega t} dt.$$

This more general type of transform is also denoted as Gabor transform, *short time Fourier transform* or *windowed Fourier transform*.

Definition 9.16 (Inverse Gabor Transform) The inverse Gabor transform is given by

$$x(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x)(\tau, \omega) w(t - \tau) e^{i\omega t} d\omega d\tau.$$

Gabor analysis is similar to Fourier analysis, but each harmonic is multiplied by a window function displaced in time. Thus Gabor analysis is a time–frequency representation of a signal. A 3D plot representation (or a contour plot) of $|G(x)(\tau, \omega)|$ is known as *spectrogram*. The redundancy introduced in this two-dimensional representation of a one-dimensional signal $x(t)$ traduces into a more accurate time representation of the different frequency components of the signal.

However, it is not possible to achieve an infinite precision for the localization in the time–frequency plane. In fact, the localization properties of the Gabor transform are restricted by the *uncertainty principle* (yes!, the same Heisenberg uncertainty principle of quantum mechanics, since the mathematics underlying these two fields are the same). This principle states that if the length of the time window grows, we obtain an accurate localization of frequencies but a poor description in time. On the contrary, if the length of the time window is decreased, we obtain better resolution in time, but poorer resolution in frequency.

Depending on the signal, one should select one of the two possibilities, or search for a compromise between both, and this is obtained for the original Gabor transform with a Gaussian window function, providing in addition the maximum theoretical localization in the time–frequency plane (for window functions of the same length). In practice other window functions $w(t)$ can be used, like the rectangular Hann/Hanning or Ham/Hamming windows [5, 15], to reduce computational cost or to enhance other properties of the Gabor transform. See Fig. 9.2 for a representation of the *uncertainty principle*.

We can increase the degree of localization (although at different rates for different frequencies) using a so-called time-scale representation instead of the time–frequency representation. This can be achieved with the following transform.

Definition 9.17 (Wavelet Transform) The wavelet transform of a signal $x(t)$ is given by Holschneider [8] and Mallat [11]

$$W(x)(a, \tau) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} x(t)\bar{\gamma}\left(\frac{t-\tau}{a}\right) dt\,, \qquad \tau \in \mathbb{R}, a > 0\,,$$

where $\bar{\gamma}(t)$ denotes the complex conjugate of $\gamma(t)$, which is a (generally complex) window function satisfying the admissibility condition:

$$0 < \int_{-\infty}^{\infty} \frac{|\hat{\gamma}(\omega)|^2}{|\omega|} d\omega < \infty.$$

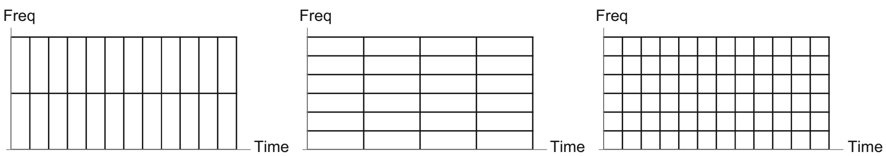


Fig. 9.2 Uncertainty principle in the time–frequency plane for a short time window (left), a wide time window (centre) and Gabor's Gaussian window

The admissibility condition implies, in particular, that $\hat{\gamma}(0) = 0$, i.e. $\gamma(t)$ has zero mean and must have oscillations, that is the reason for the name *wavelet* (or *ondelette* in French, since it is a wave with a short duration). The function $\gamma(t)$ is usually denoted as the *mother wavelet*, and the functions

$$\gamma_{a,\tau}(t) = \frac{1}{\sqrt{a}}\gamma\left(\frac{t-\tau}{a}\right)$$

are called *daughter wavelets*, representing scaled and displaced versions of the mother wavelet. The factor $1/\sqrt{a}$ is to preserve the total energy.

Definition 9.18 (Inverse Wavelet Transform) The inverse wavelet transform is given by

$$x(t) = \int_0^{\infty} \frac{da}{a^2} \int_{-\infty}^{\infty} d\tau W(x)(a,\tau)\gamma_{a,\tau}(t)\,.$$

In Fig. 9.3 we plot the Haar wavelet [18], a real wavelet that is one period of a square wave, and its daughter wavelets for $a = 2$ and $a = 1/2$.

For large values of a, $W(x)(a,\tau)$ will provide information on the long range behaviour of the signal $x(t)$ (where lower frequencies are important, and the effect of higher frequencies cancels out due to the oscillatory character of the wavelets). On the contrary, for small a, $W(x)(a,\tau)$ scrutinizes the short range

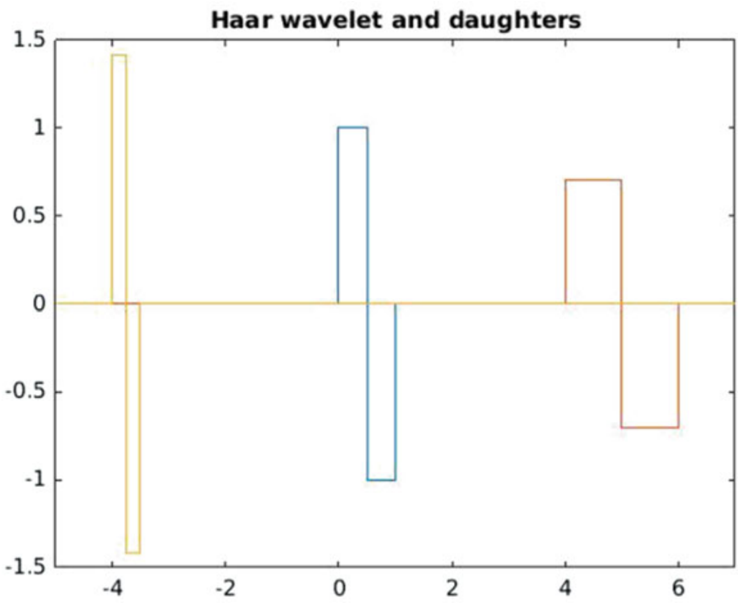


Fig. 9.3 Haar wavelet (centre) and its daughter wavelets for $a = 2$ (right) and $a = 1/2$ (left)

behaviour of $x(t)$ (where higher frequencies contribute and the lower frequencies cancel out). This indicates that although scales and frequencies are not the same, there is a reciprocal relation between them. In practise, the scales are represented logarithmically (in base 2), in the form $a = 2^{1/\lambda}$, in such a way that frequency $\propto \lambda$. Thus, although scale and frequency are not the same, it is possible to associate an approximate frequency, known as *pseudo-frequency*, to each scale (although this depends on the particular mother wavelet and the sampling time used).

Also, it is clear that for large values of a (lower frequencies) there is a poor resolution in time (since the daughter wavelet has a long duration), whereas for small values of a (higher frequencies) there is a good temporal resolution (since the daughter wavelet has a short duration). Therefore, the resolution of the wavelet transform is not uniform in the time-scale plane. Wavelets are commonly used in the analysis of signals with frequency varying in time (chirps [13, 21]), signals with discontinuities (like edges in images), fractals, etc., where the multiresolution properties of the wavelets provide more information than just the Fourier and Gabor analysis.

A 3D plot (or contour plot) of $|W(x)(a, \tau)|$ is denoted as *scalogram*, since the wavelet transform provides a scale-time representation of the signal.

In summary, Gabor and wavelet transforms, in contrast to the Fourier transform, provide information about a signal simultaneously in both time and frequency domains. They are widely applied tools in several fields where signal processing is required.

9.2 Applications

9.2.1 Power Spectrum of the Logistic Map

As in H.W. Lorenz [9] let us assume that a time series x_j, $j = 1, \ldots, n$ of a single variable has been observed at equidistant points in time. The Fourier transform of the series x_i is defined as

$$\overline{x}_k = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} x_j \exp(-2\pi i jk/n) , \qquad k = 1, \ldots, n .$$

It can be shown that the *autocorrelation function*, defined by

$$\psi_m = \frac{1}{n} \sum_{j=1}^{n} x_j x_{j+m} ,$$

can be written in terms of the Fourier transform:

$$\psi_m = \frac{1}{n} \sum_{k=1}^{n} |\overline{x}_k|^2 \cos\left(\frac{2\pi mk}{n}\right). \tag{9.3}$$

Inverting Eq. (9.3), we get

$$|\overline{x}_k|^2 = \frac{1}{n} \sum_{m=1}^{n} \psi_m \cos\left(\frac{2\pi mk}{n}\right).$$

The graph obtained by plotting $|\overline{x}_k|^2$ as a function of the frequency $2\pi k/n$ is the *power spectrum.*

Remark 9.9 (Power Spectrum Interpretation) A power spectrum displaying several distinguishable peaks is a sign of *quasiperiodic* behaviour. Dominating "peaks represent the basic frequencies of the motion, while minor peaks can be explained as linear combinations of the basic frequencies. If the underlying system is discrete, a single peak corresponds to a period-2 cycle, the emergence of two additional peaks to the left and to the right sides of the first peak, respectively, correspond to a period-4 cycle, 7 peaks correspond to a period-8 cycle, etc." [9].

If peaks emerge in a continuum the time series is either random or chaotic. In Fig. 9.4 it is shown the power spectrum of the logistic map for two different values of the bifurcation parameter μ.

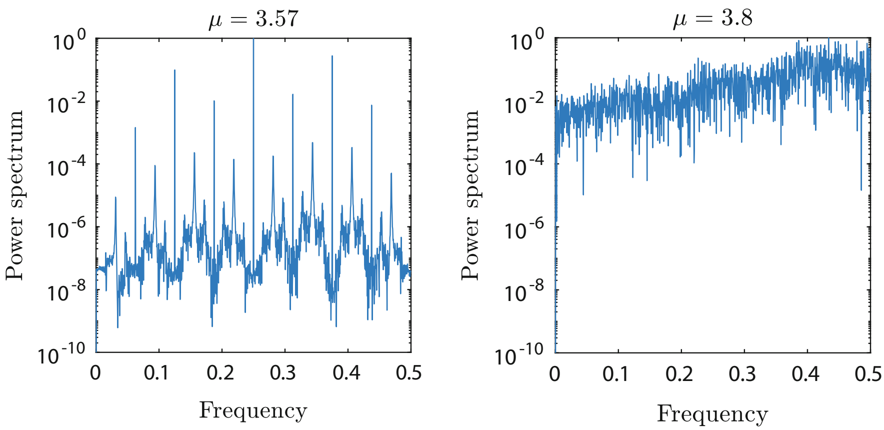


Fig. 9.4 Power spectrum of the logistic map obtained from a simulation starting at $x_0 = 0.5$ of 2000 samples. Figure on the left is obtained for $\mu = 3.57$, when the unique stable orbit having period 557120 ($2^6 \cdot 5 \cdot 1741$) is present. The power spectrum (obtained for a time series that is shorter than the attractor period) displays regular peaks. Figure on the right shows the power spectrum in the chaotic region where it is not possible to isolate dominating frequencies

9.2.2 State Space and Power Spectrum of a Periodic, Quasiperiodic and Chaotic Signals

In order to show how the power spectrum changes with the signal, let us consider the well-studied prototype of a generic two-dimensional map: the Hénon's two-dimensional dissipative map [1] defined by

$$\begin{cases} x_{n+1} = 1 + \beta x_n - \alpha y_n^2 \\ y_{n+1} = x_n \, . \end{cases} \tag{9.4}$$

The system in Eq. (9.4) is periodic, quasiperiodic or chaotic depending on the parameters α and β. For example, Fig. 9.5 shows for a periodic signal a discrete peak at the harmonic; Fig. 9.6 shows for a quasiperiodic signal discrete peaks at the harmonics and subharmonics; Fig. 9.7 shows for a chaotic signal a broadband component in its power spectrum. The latter decomposes the signal such that ones can detect whether the source is random/chaotic and its dominating frequencies.

9.2.3 Fourier Methods for Finding Frequency Components of a Given Signal

Matlab provides convenient libraries for spectral analysis. In this section we use Fourier transforms to find the frequency components of a signal buried in noise as retrieved in [12].

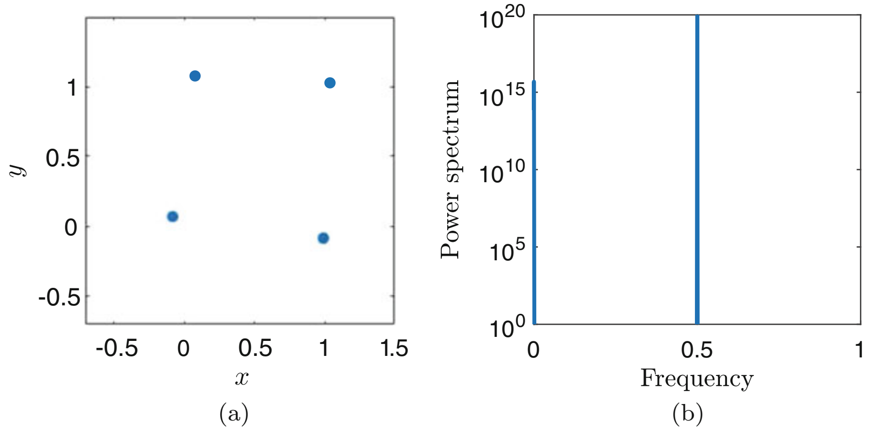


Fig. 9.5 (**a**) State space and (**b**) power spectrum for Eq. (9.4) with parameters $\alpha = 1$ and $\beta = 0.05$ (periodic signal)

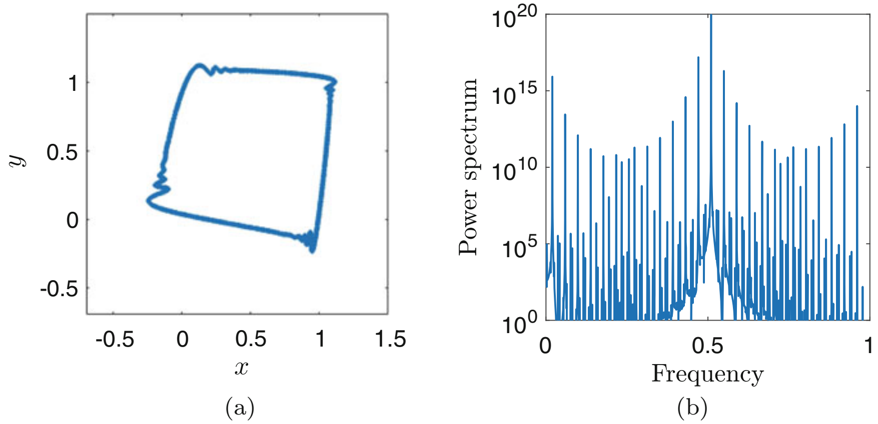


Fig. 9.6 (**a**) State space and (**b**) power spectrum for Eq. (9.4) with parameters $\alpha = 1$ and $\beta = 0.12$ (quasiperiodic signal)

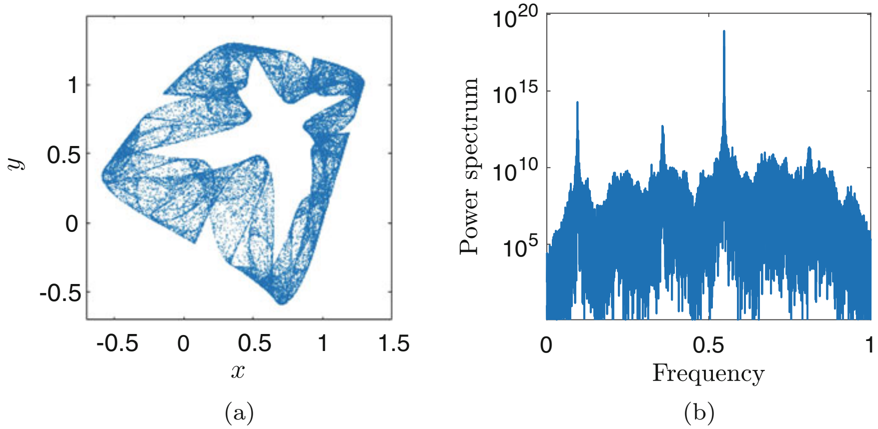


Fig. 9.7 (**a**) State space and (**b**) power spectrum for Eq. (9.4) with parameters $\alpha = 1$ and $\beta = 0.3$ (chaotic signal)

Let us consider a signal containing a 50 Hz sinusoid of amplitude 0.7 and 120 Hz sinusoid of amplitude 1:

$$S = 0.7 \sin(2\pi\ 50t) + \sin(2\pi\ 120t)\,, \tag{9.5}$$

and let us corrupt the signal with zero-mean white noise as defined in Definition 9.14 with a variance of 4:

$$X = S + 2\epsilon_t\,. \tag{9.6}$$

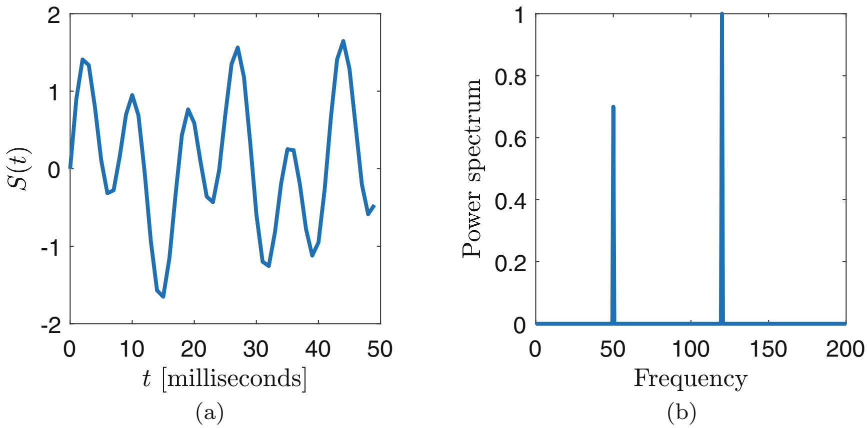


Fig. 9.8 Power spectrum for Eq. (9.4). Data sampled at 1 Hz. (periodic signal). (**a**) Signal in Eq. (9.5) in the time domain. (**b**) Phase spectrum for Eq. (9.5)

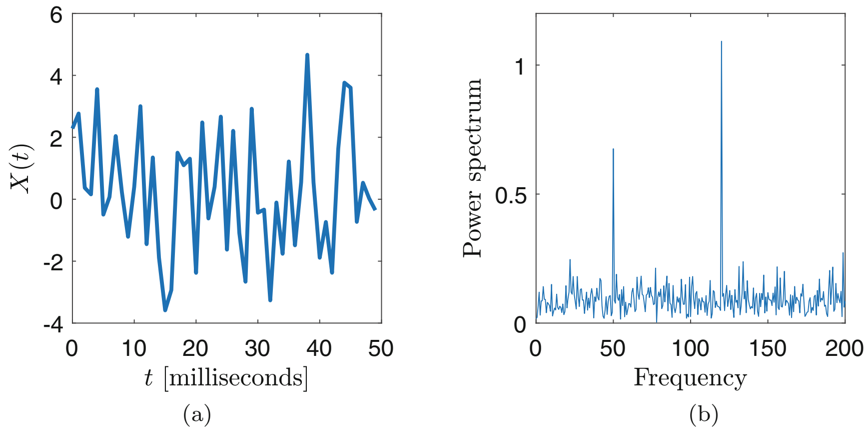


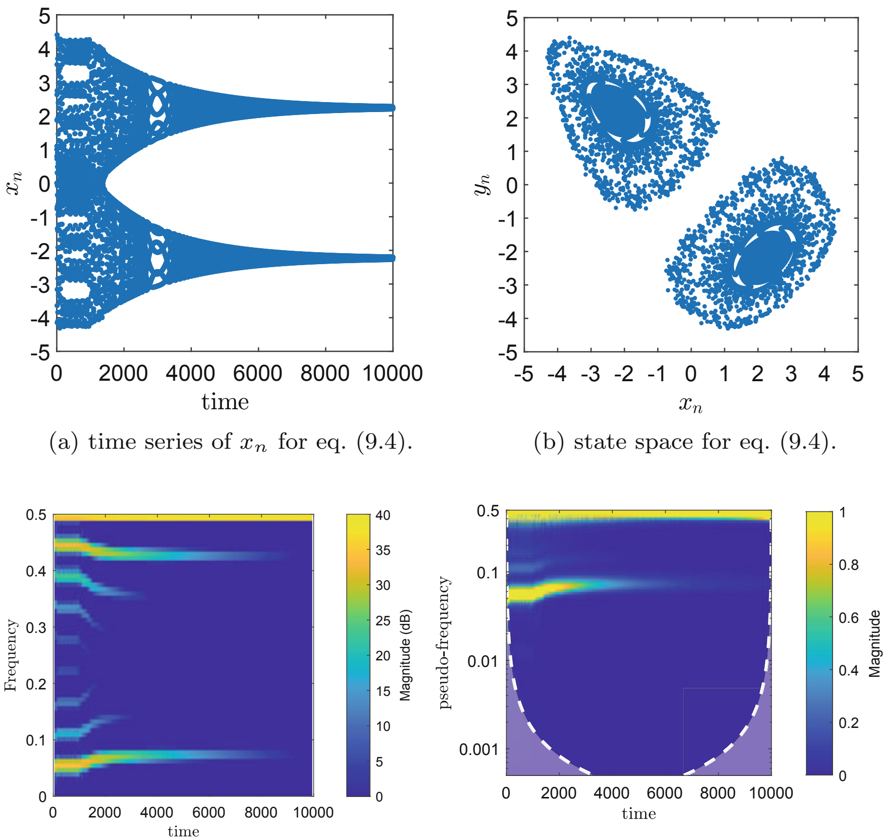
Fig. 9.9 Power spectrum for Eq. (9.6). Data sampled at 1 Hz. (periodic signal mixed with noise). (**a**) Signal in Eq. (9.6) in the time domain. (**b**) Power spectrum for Eq. (9.6)

Figure 9.8 shows the signal in the time domain as well as its power spectrum with just two spikes at 50 and 120 Hz, i.e. the dominating frequencies. Figure 9.9 shows the signal in the time domain as well as its power spectrum with a broadband component in its power spectrum. However, the two spikes at 50 and 120 Hz are clearly distinguishable along with other frequencies due to noise. For applications to economics see Chap. 16 and [16, 17].

9.2.4 Gabor and Wavelet Analysis of Hénon's Map

Let us consider Hénon's map (9.4) with $\alpha = 0.2$ and $\beta = 0.9991$, with state space shown in Fig. 9.10b. Applying the Gabor transform to the time series x_n, we obtain the spectrogram shown in Fig. 9.10c, where a few main frequencies can be observed, but with values that change with time. In particular, it is possible to see that at the beginning of the simulation more frequencies are present, which disappear while the system achieves convergence onto the map attractor, that is a period-2 cycle. The same can also be observed in the scalogram, when the wavelet transform is applied, in Fig. 9.10d, where the fact that different scales are used for different frequencies allowed us to better highlight this phenomenon at smaller frequencies.

See [22] for applications of Gabor analysis in economics, and see [2, 14] for applications of wavelets in economics.



(a) time series of x_n for eq. (9.4). (b) state space for eq. (9.4).

(c) Spectrogram of x_n in eq. (9.4). (d) scalogram of x_n in eq. (9.4).

Fig. 9.10 (a) Time series, (b) state space, (c) spectrogram and (d) scalogram for Eq. (9.4) with parameters $\alpha = 0.2$ and $\beta = 0.9991$

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