Chapter 18
An Empirical Test of Harrod's Model

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While Kaldor's theory strongly influenced the academic debate on business cycles, Harrod's theory inspired Solow's seminal paper "A Contribution to the Theory of Economic Growth" (1956) [36], that set the basis for modern growth theory. However, a recent re-evaluation of Harrod's theory [4, 14] challenges Solow's interpretation "which ultimately dominated the profession's view of Harrod" [14]. According to Solow, the Harrod model "implied a tendency toward progressive collapse of the economy". However this has "little to do with the problem of long-run growth as Solow understood it, but instead addressed medium-run fluctuations, the inherent instability" of economies" [14].

There are several reasons why in this chapter we focus on the Harrod's model. First of all, it is because of the abovementioned influence on the foundation of modern growth theory. Secondly, the Harrod model provides a dynamic framework and some guidelines to policy-makers, in terms of supply-side policies. In fact, they should consider the combination of investment, technological change, population

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growth, unemployment and aggregate demand. Another reason is that, in his framework, the warranted rate of growth is not a single (moving) equilibrium, but a "highly unstable" one. This takes the name of Harrod's knife-edge instability or the *instability principle*.

Similarly, but from a different starting point (i.e. static analysis and microeconomic foundations of macroeconomic dynamics), Leijonhufvud defines the notion of a stability corridor as a time-path in which economic activities "are reasonably well coordinated" [22]. Moreover the system is likely to behave differently for large than for moderate displacements from the "full coordination" time-path. Within some range from the path (referred to as "the corridor" for brevity), the system's homeostatic mechanisms work well, and deviation-counteracting tendencies increase in strength. Outside that range, these tendencies become weaker as the system becomes increasingly subject to "effective demand failures". If the system is displaced sufficiently "far out", the forces tending to bring it back may be so weak and sluggish that for all practical purposes the Keynesian "unemployment equilibrium" model is a sensible representation of its state. Inside the corridor, multiplier-repercussions are weak and dominated by neoclassical market adjustments. Outside the corridor, they should be strong enough for effects of shocks to the prevailing state to be endogenously amplified. Up to a point, multiplier-coefficients are expected to increase with distance from the ideal path. Within the corridor, the presumption is in favour of "monetarist" policy prescriptions, outside of it in favour of "fiscalist". Finally, although within the corridor market forces will be acting in the direction of clearing markets, institutional obstacles of the type familiar from the conventional Keynesian literature may, of course, intervene to make them ineffective at some point. Thus, a combination of monopolistic wage-setting in unionized occupations and legal minimum-wage restrictions could obviously cut the automatic adjustment process short before "equilibrium employment" is reached [22].

Both views, macroeconomic and dynamic (by Harrod) and static and micro-founded (by Leijonhufvud) converge to the "existence of thresholds at the start of the mechanisms that are at work" [21]. Therefore, the idea of dynamically unstable multiple equilibria or the alternative Harrod's suggestion of a Leijonhufvud's "corridor stability" is worth exploration in our opinion. In particular, whereas in the 1970s and the 1980s unemployment and stagflation discarded those theories, in the twentieth century "in the leading Western economies there have been prolonged periods when more saving would have been beneficial, and others with every appearance of inadequate effective demand" [12]. As the Harrod's model is one of the few able to predict that, "it still deserves serious attention" [12].

18.1 Background and Literature

The renewed interest in Harrod's also due to an epistemological work that has Besomi among its main contributors: "plunging into the original texts soon made it obvious that the subject of Harrod's dynamics was more intricate than the portrait given in textbook rendition" [4] and that many interpretations were erroneous. Baumol [2], for instance, asserts that "the main achievement of his [Harrod's] model lies in the ideas it inspired in those who did not fully understand it". In fact Harrod himself "claimed that his dynamics was essentially different from, and indeed more fundamental than, the mainstream interpretation of it (an interpretation which, of course, reflected the notion of dynamics which gained almost universal acceptance after the war)" [4].

The so-called Harrod's Dynamics is the result of a number of works resumed in "The Making of Harrod's Dynamics" by Besomi [4] and the most significant of which are "Towards a Dynamic Economics" (1948) [16] and "Economic Essays" (1972) [17]. This happened because Harrod "returned several times on the topic of his essays in correspondence with Keynes, who sometimes managed to force him to re-formulate his propositions" [4].

Harrod identified two stages in explaining economic dynamics: the first was the determination of the rate of growth at the equilibrium (given a certain ratio of saving over income and investment per unit increase of output), the second was related to the changes of those ratios (changes that would lead to different equilibria and would be responsible for cycles).

Because of the different formulations, Harrod's theory led to several distinct interpretations. For example, according to Tinbergen [39], the model was a combination of multiplier and accelerator that could not give rise to cyclical behaviour, but could only lead to an explosive growth or to an equilibrium. Samuelson [31, 32], in a different formulation with lagged variables, found that for a range of the multiplier and accelerator coefficients, there would be a cyclical behaviour.

Apart from that, the so-called Harrod–Domar model was extensively used to explain growth as the result of the optimal combination of saving and investment. This led to a debate on other factors determining the growth as well as around the multiplicity of equilibria and their instability. For example, according to Solow [36], relaxing Harrod's assumption of a constant capital/output ratio, the system would have drifted towards full employment. Moreover, while Harrod stressed the mentioned "principle of instability" to describe the adjustments between effective accumulation of capital and warranted accumulation, Solow's interpretation solved the puzzle by assuming that the warranted rate of growth (G_w) was constant and that technology was flexible (even though Harrod insisted on the fact that the G_w depended on time and cycle). Therefore, when Axel Leijonhufvud [22] sketched the idea of a corridor, in Economic Dynamics [15] Harrod confessed that it was an appropriate approach to what he was thinking about failures of effective demand.

Robinson [30] summarized a long debate on the post-Keynesian front and showed how multiple equilibria could be attained if different propensities to save across social classes were considered. Last but not least Kalecki *(1933–1939)* [20] and Kaldor [19] focused on the technical progress, on the non-linearities of the investment and savings functions and on the determinants of investment decisions. This inspired a number of works: from multiple attractors and global bifurcations [6] to homoclinic tangles [1], from the global existence of periodic solutions [18] to the existence of chaotic behaviour, not for a single specific value, but within a reasonable interval for each parameter [25, 26]. Shaikh [34] explains key differences between Harrodian and Keynesian theories and policies, proves the stability of the Harrodian warranted path and shows that the Keynesian paradox of thrift is transient. Moudud [23] shows how to combine taxation with public investments in order to raise the warranted growth rate (which, according to Harrod, is otherwise reduced by an increase in the budgeted deficit/GDP ratio). Serrano et al. [33] claim that Harrod's instability is an instance of what Hicks calls "static instability" and they show that the Sraffian Supermultiplier [24] model overcomes the Harrodian instability. Skott [35] contends that there is no need to introduce autonomous demand as the "driver of long-run economic growth and as a stabilizing force", but it would suffice to model "the supply side (the labour market) and/or economic policy" to obtain those results.

Finally Yoshida [41] and Sportelli [37, 38] offered, from the 1990s Harrod's Dynamics, a theoretical framework to explain jointly economic growth and business cycles through the Harrod's "instability principle". However for Yoshida the instability derives from a putty-clay technology in conjunction with flexibility of prices, while for Sportelli the instability derives from the gaps between Harrod's rates of growth: actual, warranted and natural. Further, in that framework [37], it has been shown that opening to foreign trade can lead to reducing cyclical instability of the economy as was suggested by Harrod.

18.2 Material and Methods

18.2.1 Cycles

Cyclical fluctuations in economy, Fig. 18.1, correspond to the duration or the amplitude between a high/peak and the succeeding low/trough [8]. The so-called peak–trough–peak (PTP) cycle affects the whole economy (e.g., wages, demand, prices, credit, etc.). Seasonal swings are typically short-term, but cyclical fluctuations could last for years. A depression is a prolonged and deep recession. As mentioned by Eckstein [11] "financial distress produces sharp discontinuities in flows of funds and spending and when the financial strains include tight monetary policy, much lessened availability of money and credit, sharp rises of interest rates, and deteriorating balance sheets for households, businesses, and financial institutions".

18.2.2 US Recessions

In the following Fig. 18.2 we display the path of US investments alongside recessions as reckoned by FRED (Table 18.1).

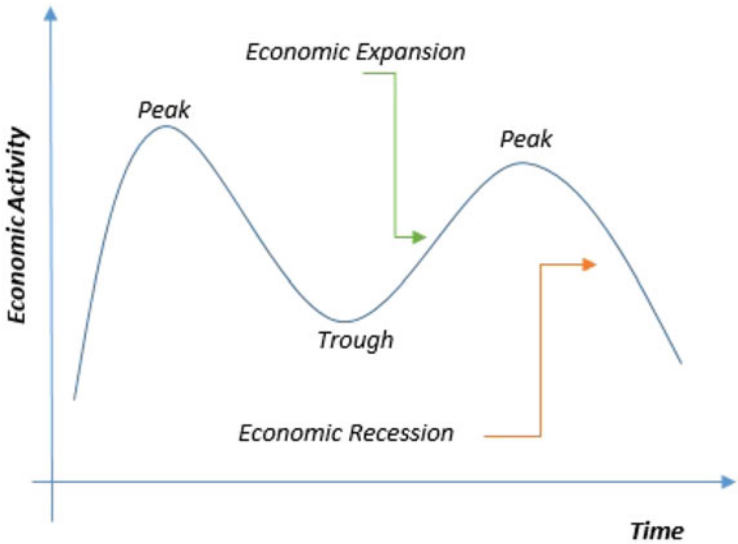

Fig. 18.1 The business cycle can be classified into four stages: (1) expansion when economic activity grows steadily; (2) boom when the aggregate demand grows more than the aggregate output which overheats the economy; (3) recession phase when the aggregate output cools down after a peak; (4) recovery after a *trough*. The so-called specific cycle amplitude corresponds to the vertical distance between the peak and the trough

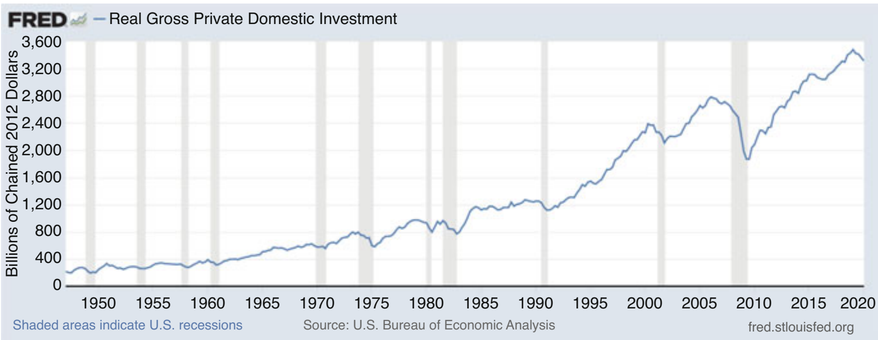

Fig. 18.2 US real gross private domestic investment (GPDIC1), billions of chained 2012 dollars, seasonally adjusted annual rate. Source: FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/GPDIC1, 25 May 2020. Greyed areas correspond to periods of economic recessions (Table 18.1)

Table 18.1 US recessions

Recessions			
From		To	
Quarter	Year	Quarter	Year
Q4	1948	Q4	1949
Q3	1953	Q1	1954
Q4	1957	Q1	1958
Q3	1960	Q1	1961
Q1	1970	Q4	1970
Q1	1974	Q2	1975
Q1	1980	Q2	1980
Q3	1981	Q4	1982
Q3	1990	Q1	1991
Q2	2001	Q4	2001
Q1	2008	Q3	2009

US. Bureau of Economic Analysis [3]

18.2.3 Empirical Data

In order to perform our test, we have retrieved data from several sources such as the Maddison Project, the World Bank, *IMF* and *BEA*. Annual world GDP estimate has been retrieved from the Maddison–Penn world table [7, 13], (from 1946 to 1961). This has been linked up with *World Bank*[1] and *IMF*[2] data (available from 1961 to 2018). Annual data has been changed into quarterly via the compounding law. Time series are retrieved from their original dataset or from FRED as detailed in the Appendix A Sect. A.1.

18.3 Calibration of Harrod's Model

To test empirically the Harrod model we evaluated the average distance between the historical data series reported in Appendix A.1 and the orbit produced by Eq. (13.23) starting at time 0. Mathematically, we want to compute the quantity:

$$D = \frac{1}{286} \sum_{t=0}^{286} \left(d(t) - \frac{1}{\hat{\tau}} \int_{t}^{t+1} \hat{\phi} \left(\hat{\tau} \tilde{t}, P \right) d\tilde{t} \right)^2, \tag{18.1}$$

[1] https://data.worldbank.org/indicator/NY.GDP.MKTP.KD.ZG.

[2] https://www.imf.org/external/datamapper/NGDP_RPCH@WEO/OEMDC/ADVEC/WEOWORLD.

Table 18.2 Harrod model parameters

#	Given model		Calibration		
			Cal. #1	Cal. #2	Cal. #3
	Parameter	Given value/range	Calibrated value		
1	α	0.5	0.28	0.29	1.09
2	ϵ	[0.2, 1.31]	0.13	0.58	0.52
3	σ	[2, 4)	1.42	1.67	2.45
4	G_f	0.03	0.03	0.00	0.54
5	C^*	4	4.00	4.00	3.18
6	β	2.5	2.50	2.50	2.20
7	m	0.07	0.04	0.04	1.23
8	φ	15	15.00	15.00	14.89
9	ξ	0.18	0.18	0.18	0.20
10	μ	1.4	0.78	0.90	2.06
11	γ	1	0.56	0.57	0.36
12	δ	6.2	6.20	6.20	5.94
13	ζ	1.9	1.06	1.09	2.25
Value of D			0.38	0.71	0.55

Original data as provided in [37] with related calibrations. $\bar{G} = max\ G_n$

where $d(t)$ is the vector that stacks the data of the rate of growth of domestic income, the expected rate of growth of aggregate demand, the share of income saved and the net export rate for the quarter t ($t = 0$ is the first quarter of 1947, $t = 286$ is the second quarter of 2018). Similarly, P is the vector of the 13 parameters of the model (reported in Table 18.2), and $\hat{\phi}$ stacks the four variables that solve the differential equation (13.23) with parameters set in P starting at $\hat{\phi}(0, P) = d(0)$: the integral between t and $t+1$ allows us to compute the average value of the (continuous) signal over the quarter of interest, to be compared with the data. Note that an additional dummy parameter $\hat{\tau}$ has been added. This parameter permits us to rescale the time of the signal produced by the model, in order to best fit with the time-scale of the data. The optimization variables are the $13 + 1$ parameters of Eq. (13.23), since they have physical meaning only when positive, this adds a set of constraints to be satisfied. Formally speaking, in order to find the best fitting solution, we solve the following constrained optimization problem:

$$\begin{aligned} &\min_{P,\hat{\tau}} \frac{1}{286} \sum_{t=0}^{286} \left(d(t) - \frac{1}{\hat{\tau}} \int_t^{t+1} \hat{\phi}\left(\hat{\tau}\tilde{t}, P\right) d\tilde{t} \right)^2 \\ &s.t.\ \hat{\phi}\left(\hat{\tau}t, P\right) \text{ is a solution of Eq. (13.23) with parameters set as } P \\ &\qquad \hat{\phi}\,(0, P) = d(0) \\ &\qquad P \geq 0,\ \hat{\tau} > 0. \end{aligned} \tag{18.2}$$

This problem is solved using the interior point method [5, 9] implemented in the Matlab `fmincon` routine. Since the problem is not convex, the optimization algorithm may converge to a local optimal solution. To better explore the space of the optimal solutions, we introduce a multi-start algorithm: the optimization is then run several times starting from a randomly perturbed sample drawn from a distribution centred in the parameter setting provided in Sportelli and Celi [37].

To evaluate the abovementioned version the Harrod model, Table 18.2 reports the parameters for both the original model and three calibrations we have obtained that present qualitatively different behaviours (together with the value of their distance D for calibration #2, D is computed with $t \leq 6$). The model, calibrated with real data, may display convergence to a long-run equilibrium (calibration #1, Fig. 18.3), divergence (calibration #2, Fig. 18.4) as well as a lightly damped oscillatory behaviour (calibration #3, Fig. 18.5). It is worth saying that the global optimum is obtained with calibration #1. However, with calibrations #2 and #3, we displayed sub-optimal results to provide a context for our results. In fact, qualitatively, we obtain similar values to the ones in [37]. Moreover we agree with this conclusion "when the value of ϵ is large enough, the long period dynamics of the saving rate is such that it can generate an irregular cycle in the system only if the net export rate is very low. On the contrary, starting from positive and meaningful values of the net export rate, the system may simply generate a limit cycle (or at most a double cycle) if a higher ϵ works together with adequate competitiveness on the foreign markets. This is the only formal result consistent with Harrod's intuition that a more moderate cyclical instability can emerge in an open economy compared to a closed one" [37].

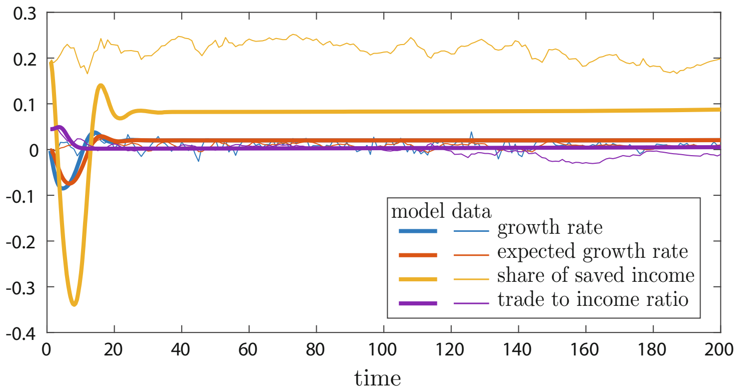

Fig. 18.3 Time series obtained with parameters of calibration #1 that displays convergence to the long-run equilibrium

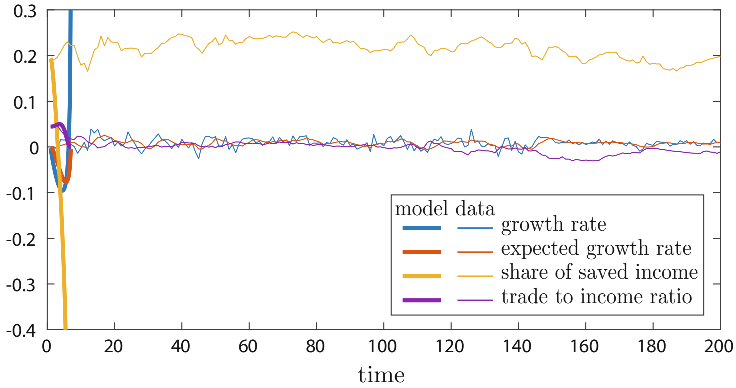

Fig. 18.4 Time series obtained with parameters of calibration #2 that displays divergence from the long-run equilibrium

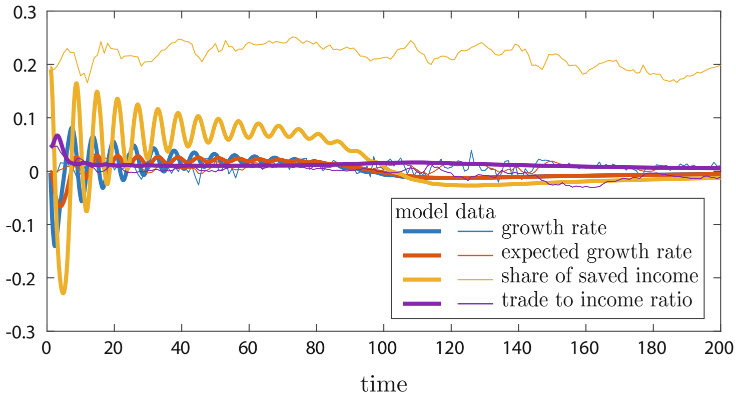

Fig. 18.5 Time series obtained with parameters of calibration #3 that displays lightly damped oscillatory behaviour around the long-run equilibrium

18.4 Conclusions

The Harrod's model [16] has the merit of rearranging Keynes's ideas into a dynamic framework with some additional specification on the supply side. In fact "where the warranted growth rate represents an economy's growth path on which aggregate demand and supply remain in balance, the model's natural growth rate reflects the supply of productive resources and the level of technology, the long-run limit to real output growth. The interaction between the warranted and natural growth rates provides a useful perspective for policymaking in today's environmentally constrained global economy. Also, since the growth of the labour force is built into the natural growth path, the model also helps to clarify policy choices in an economy impacted by immigration” [40]. Therefore “supply-side policies must be developed along with the standard Keynesian demand side policies, and the interactions between the two require disaggregated policies to address specific types of investment, technological change, and demand. That is, it is not generally possible to solve the unemployment problem by simply expanding aggregate demand” [40]. Harrod’s theory, and thereof modelization built on that [36, 37, 40, 41], may thus be seen as the link between classical economy (that stressed the importance of investment for growth) and the Keynesian approach “primarily concerned with the demand and income generating effect of investment” [10]. In real life, this theory was put into practice in India. In fact, the Indian fifth five year plan for the years 1974–1979 was based on a mix of a Harrod macroeconomic model and a Leontief inter-industry model, and it was aimed at achieving both self-reliance and growth. Main priorities on the industrial sectors were the development of: (1) core industry, (2) industry for export and diversification, (3) mass consumption production, (4) small industry and ancillary industry feeders of large industries. The target growth rate was 4.4% and, as a result, the actual growth rate was 4.8% [10].

Having said that, to recall the importance of the model, this test shows (for a specific set of parameters) that it is possible to find a match between Harrod’s suggestions and reality. This is relevant because in the long-standing debate about chaos and non-linear dynamics in economy, even the general usefulness of those concepts was questioned. “Stochastic modelling has proven to be able to simulate reality fairly well. However, a stochastic behaviour implies that reality is about exogenous randomness, while a chaotic behaviour means that reality is deterministic and non-linearities are endogenous” [28]. The ability of chaotic deterministic models to replicate reality is the common thread throughout this book [29].

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