Published for SISSA by Springer

Received: October 22, 2021
Revised: March 18, 2022
Accepted: March 29, 2022
Published: April 21, 2022

Gravity from symmetry: duality and impulsive waves

Laurent Freidel[a] **and Daniele Pranzetti**[a,b]

[a] *Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario N2L 2Y5, Canada*
[b] *Università degli Studi di Udine, via Palladio 8, Udine I-33100, Italy*

E-mail: lfreidel@perimeterinstitute.ca, dpranzetti@perimeterinstitute.ca

Abstract: We show that we can derive the asymptotic Einstein's equations that arises at order $1/r$ in asymptotically flat gravity purely from symmetry considerations. This is achieved by studying the transformation properties of functionals of the metric and the stress-energy tensor under the action of the Weyl BMS group, a recently introduced asymptotic symmetry group that includes arbitrary diffeomorphisms and local conformal transformations of the metric on the 2-sphere. Our derivation, which encompasses the inclusion of matter sources, leads to the identification of covariant observables that provide a definition of conserved charges parametrizing the non-radiative corner phase space. These observables, related to the Weyl scalars, reveal a duality symmetry and a spin-2 generator which allow us to recast the asymptotic evolution equations in a simple and elegant form as conservation equations for a null fluid living at null infinity. Finally we identify non-linear gravitational impulse waves that describe transitions among gravitational vacua and are non-perturbative solutions of the asymptotic Einstein's equations. This provides a new picture of quantization of the asymptotic phase space, where gravitational vacua are representations of the asymptotic symmetry group and impulsive waves are encoded in their couplings.

Keywords: Classical Theories of Gravity, Models of Quantum Gravity, Space-Time Symmetries

ArXiv ePrint: 2109.06342

https://doi.org/10.1007/JHEP04(2022)125

Contents

1 Introduction

The program of *local holography* is grounded in the fundamental role played by symmetries. It aims to provide a new description of quantum geometry in terms of the representation theory of the gravitational symmetries associated to the codimension-2 surface bounding a general finite region in spacetime, the *corner* [1–10]. Since the seminal work of Emmy Noether [11], the notion of symmetry has represented a very helpful and effective tool to unravel the correct description of the fundamental forces of Nature, both at the classical and, in the case of the Standard Model and Condensed Matter, in the quantum regime. We believe that this invaluable tool will ultimately prove itself crucial also to guide us through the ultimate and most impervious stretch of this discovery journey, leading to the quantization of gravity.

From this perspective, it is fundamental to understand the pivotal role of symmetries in describing the properties of a gravitational system in a finite bounded region of spacetime. The full power of the Noether theorem for local symmetries implies that the symmetry charges are supported by codimension two surfaces lying at the corner of the spacetime region under consideration [1]. This symmetry group lying at the corner can naturally be split into 'kinematical' symmetries that carries no symplectic flux and are readily quantizable and 'dynamical' symmetries that include supertranslations along the null normals and which carry fluxes.[1] The study of the kinematical gravitational symmetries of a finite bounded region of spacetime has been performed originally in [1] for the Einstein-Hilbert formulation of gravity and then further extended to other first order formulations in [7, 8].

These analyses have led to the notion of *corner symmetry group*, which is the kinematical subgroup generated by internal gauge transformations and the residual diffeomorphisms which vanish at the corner. In the Einstein-Hilbert formulation, the corner symmetry algebra $\mathfrak{g}_S$ has been shown [1] to have the semi-direct sum structure

$$\mathfrak{g}_S = \mathrm{diff}(S) \oplus \mathfrak{sl}(2,\mathbb{R})^S \,, \tag{1.1}$$

where $\mathrm{diff}(S)$ corresponds to the Lie algebra generated by diffeomorphisms tangent to the corner S and $\mathfrak{sl}(2,\mathbb{R})^S$ the Lie algebra generated by the surface boosts that linearly transform the normal plane of S in a position-dependent way.

The inclusion of normal supertranslations that move the corner has led to the notion of *extended corner symmetry group* in [14, 15] and given by

$$\mathfrak{g}_S^{\mathrm{ext}} = \left(\mathrm{diff}(S) \oplus \mathfrak{sl}(2,\mathbb{R})^S\right) \oplus (\mathbb{R}^2)^S \,, \tag{1.2}$$

where the second semi-direct sum involves the two normal time translations.

At the same time, it has been shown in [16] that a similar semi-direct sum structure (1.2) captures the symmetries of a general non-stationary null surface at finite distance equipped with a thermal structure. This group was dubbed Weyl BMS, or BMSW

[1] Recent developments appeared shortly after the first version of this manuscript have allowed us to also include supertranslation as canonical transformation into the gravitational phase space by extending this with a dressing field [12, 13].

JHEP04(2022)125

for short, in [17] and shown two satisfy two key properties. On the one hand, it is a subgroup of the extended corner symmetry group: the subgroup that preserves, up to scale, the canonical null generator of the null surface. On the other hand, it is also the symmetry group of null infinity.

More precisely, the Lie algebra of BMSW possesses a semi-direct sum structure

$$\mathsf{bmsw} := \left(\mathrm{diff}(S) \oplus \mathbb{R}^S_W\right) \oplus \mathbb{R}^S_T\,, \tag{1.3}$$

where $\mathbb{R}^S_W$ denotes the Weyl transformations labeled by functions W on the sphere, while $\mathbb{R}^S_T$ denotes the super-translations labelled by weight $1/2$ densities T on the sphere. This algebra contains all the known extensions of the BMS algebra [18–20] that have been recently introduced as candidates for the gravitational symmetries of null infinity in [21–25]. Besides super-translation transformations, it includes arbitrary sphere diffeomorphisms and local Weyl rescalings of the 2D sphere metric at $\mathcal{I}$. Importantly, it has been shown in [15] that the extended corner symmetry algebra (1.2) reduces to the bmsw Lie algebra in the limit $r \to \infty$, with the Weyl rescaling corresponding to the $\mathfrak{sl}(2,\mathbb{R})^S$ generator preserving the null generator of $\mathcal{I}$, while the $\mathbb{R}^S_T$ contribution corresponds to the normal super-translation along $\mathcal{I}$. This result provides clear evidence that the local holography program can be equally well applied to null infinity and this is what we concentrate on in this manuscript.

From this perspective, it is fundamental to understand the full power of symmetries in describing the properties of a gravitational system, and investigate the role of the kinematical subgroup of BMSW as well as the role of supertranslations. Therefore, our goal is to understand how far the symmetry principle can take us in the description of a gravitational system and its asymptotic dynamics, and from there to the quantum realm of gravity. We aim to establish that the dominant asymptotic Einstein's equations can be recovered purely from a symmetry argument. A first indication that this is indeed possible, in the context of null infinity, comes from the analysis of [17], where a new charge bracket generalizing a previous proposal of Barnich and Troessaert [22], and derived from first principles in [15], was introduced to represent the bmsw Lie algebra in terms of the Noether charges associated to it. It was shown that the demand that the BMSW Lie algebra being represented at all times along $\mathcal{I}$ without any 2-cocycle extension is *equivalent* to imposing the asymptotic Einstein's equations at null infinity. On a similar vein, evidence that the asymptotic symmetry group is strong enough to reconstruct the MHV sector of S-matrix amplitudes has been given by Banerjee et al. [26–29]. Providing evidence that symmetry can be strong enough to significantly constraint S-matrix amplitudes is one of the cornerstones of the program of local holography [30–32]. Our work can be viewed as a classical and group-theoretical analog of this quest. It provides new evidence that symmetry might be strong enough to determine the dynamics. Let us point out that an approach similar in spirit has been applied in [33] to relate the study of dynamics in Carrollian geometries to the analysis of symmetries at null infinity; in this case though the group of interest is the boundary group of Carrollian diffeormorphism, not the corner symmetry group.

Here, we exploit the BMSW group structure in order to derive the asymptotic Einstein's equations at null infinity in a more direct way, making the symmetry argument even

JHEP04(2022)125

more explicit. More precisely, in section 3, after deriving the symmetry transformations of the asymptotic metric components, we first identify a set of semi-covariant observables $(\mathcal{N}^{AB}, \mathcal{J}^A, \mathcal{M}, \tilde{\mathcal{M}}, \mathcal{P}_A, \mathcal{T}_{AB})$. They are defined as Bondi metric functionals that do not possess quadratic anomalies, under the BMSW transformations. They also transform homogeneously (i.e. tensorially) under the non-extended BMSW group, namely when time super-translations are not included. In section 3.5 we show that these are in direct relation with the five asymptotic Weyl scalars [34–36] at null infinity. We then look for combinations containing time derivatives of the semi-covariant observables, which transform homogeneously, that is with no anomalies at all, under the BMSW group. This singles out five relations which express the asymptotic Einstein's evolution equations at leading order in the large-r expansion around null infinity in an elegant and simple form. As derived in section 4, these read

$$\dot{\mathcal{J}}^A = \frac{1}{2} D_B \mathcal{N}^{AB}\,, \tag{1.4a}$$

$$\dot{\mathcal{M}} = \frac{1}{2} D_A \mathcal{J}^A + \frac{1}{8} C_{AB} \mathcal{N}^{AB}\,, \tag{1.4b}$$

$$\dot{\tilde{\mathcal{M}}} = \frac{1}{2} D_A \tilde{\mathcal{J}}^A + \frac{1}{8} C_{AB} \tilde{\mathcal{N}}^{AB}\,, \tag{1.4c}$$

$$\dot{\mathcal{P}}_A = D_A \mathcal{M} + \tilde{D}_A \tilde{\mathcal{M}} + C_{AB} \mathcal{J}^B\,, \tag{1.4d}$$

$$\dot{\mathcal{T}}_{AB} = D_{\langle A} \mathcal{P}_{B\rangle} + \frac{3}{2} \left(C_{AB} \mathcal{M} + \tilde{C}_{AB} \tilde{\mathcal{M}} \right)\,. \tag{1.4e}$$

In the expressions above $\mathcal{N}^{AB} = \dot{N}^{AB}$ is the time derivative of the news tensor; the tilde denotes a notion of duality in the gravitational phase space at null infinity that we introduce in section 3 and it encompasses the notion of dual gravitational charges introduced in [37–39] and further studied in [40–44]. The Einstein's evolution equations recast as in (1.4) exhibit a manifest invariance under this duality transformation. Moreover, we show how the symmetry argument can be applied in the presence of matter as well, allowing us to derive the correct combination of stress-energy tensor components (and their derivatives) that source the Einstein's evolution equations.

While our derivation of the asymptotic Einstein's evolution equations is totally independent from the Newman-Penrose formalism, the final form of the equations agree with their central results [34–36]. This can be seen by exploiting the explicit relation between the semi-covariant observables and the asymptotic Weyl scalars, summarized in section 3.5, and showing that (1.4) agrees with the Newman-Penrose derivation of the time evolution of the asymptotic Weyl scalars. The advantage and novelty of our approach is the explicit derivation of the symmetry transformations for these observables from the Bondi gauge variables (see also [17, 45]) and the emphasis that invariance under the BMSW asymptotic symmetry group is enough to ensure the derivation of the equations of motion. This represents the first main result of the paper. It gives a posteriori a justification for the success of the Penrose-Newman formalism, by showing that it is naturally adapted to the concept of asymptotic symmetries. This sets the stage for the rest of our analysis and it opens the way towards a quantum analysis.

More precisely, in section 5 we focus our attention on the non-radiative phase space. The no radiation condition is defined by the vanishing of the time derivative of the news tensor, that is $\dot{N}^{AB} = 0$, which provides a more relaxed definition of non-radiative phase space than the usual condition $N^{AB} = 0$, and corresponds to the case where no outgoing radiation is registered at $\mathcal{I}$. We can, under the no radiation condition, integrate the evolution equations and we construct a new set of *conserved* charges $(j_A, m, \tilde{m}, p_A, t_{AB})$ defined in terms of the covariant ones. These charges parametrize the non-radiative corner phase space on $\mathcal{I}$ and their transformation properties are obtained to be

$$\delta_{(T,W,Y)} j^A = [\mathcal{L}_Y + 4W]\, j^A\,, \tag{1.5a}$$

$$\delta_{(T,W,Y)} m = [\mathcal{L}_Y + 3W]\, m + j^A \partial_A T + \frac{T}{2} D_A j^A\,, \tag{1.5b}$$

$$\delta_{(T,W,Y)} \tilde{m} = [\mathcal{L}_Y + 3W] \tilde{m} + \tilde{j}^A \partial_A T + \frac{T}{2} D_A \tilde{j}^A\,, \tag{1.5c}$$

$$\delta_{(T,W,Y)} p_A = [\mathcal{L}_Y + 2W]\, p_A + \frac{3}{2}(m\partial_A T + \tilde{m}\tilde{\partial}_A T) + \frac{T}{2}\left(\partial_A m + \tilde{\partial}_A \tilde{m} + c_{AB} j^B\right)\,, \tag{1.5d}$$

$$\delta_{(T,W,Y)} t_{AB} = [\mathcal{L}_Y + W]\, t_{AB} + \frac{8}{3} p_{\langle A}\partial_{B\rangle} T + T\left(\frac{2}{3} D_{\langle A} p_{B\rangle} + \frac{1}{2} c_{AB} m + \frac{1}{2}\tilde{c}_{AB}\tilde{m}\right)\,, \tag{1.5e}$$

where T, W, Y are transformation parameters which are functions of the coordinates on the celestial sphere and label respectively supertranslations, Weyl rescalings and tangent diffeomorphisms. This set of transformations represents the second main result of the paper as it generalizes[2] to the BMSW group the one identified by Barnich et al. [46–48] in the Penrose-Newman formalism for the extended BMS group [21, 22]. Our derivation provides a more direct and independent derivation of these transformation laws from the Bondi formalism. These transformations constitute the starting point for the construction in [49] of the moment map between the non-radiative corner phase space of null infinity and the dual Lie algebra of its full symmetry group. Indeed the charge conservation and the closure of symmetry transformations are two indicators that the charges can be understood as moment maps representing the action of an extended symmetry group on the asymptotic gravity phase space. Even if the explicit action of the new spin-2 charge t_{AB} on the gravity phase space has now been revealed in [50, 51],[3] establishing that these conserved charges, including the dual mass and the spin two charge aspect, define a moment map for a generalization of the BMS group still needs to be carried out.

In order to understand the relationship between the charge aspects we revealed and the radiation, we investigate in section 6 how an initial vacuum state of the non-radiative phase space is changed by an impulsive gravitational wave localized at $u = 0$, transitioning into a new vacuum. To do so, we find the non-linear impulsive solutions that describe this transition by integrating the evolution equations (1.4) in the case where the Weyl tensor component $\mathcal{N}_{AB}$ is proportional to a delta function $\delta(u)$ through an impulse news

[2] In [46–48] the metric is restricted to be conformally spherical and the diffeomorphisms are restricted to be local Killing vector fields. Our derivation relax this restriction and includes the full group of sphere diffeomorphism.

[3] These two references have also appeared on the arXiv shortly after the first version of this manuscript.

function on the 2-sphere. By demanding continuity of the induced metric, we are able to integrate all the evolution equations without encountering distributional singularities. Quite surprisingly, we find that all the Weyl scalars are activated by the gravitational impulse. This is in contrast with the usual solution for an impulsive gravitational wave where only one Weyl scalar is non-vanishing [52–60] — although other Weyl scalars can be activated from collisions —. More precisely, the remaining covariant charges are of the form

$$\mathcal{J}_A(u) = \mathcal{J}_A^{\mathrm{NR}} + \mathcal{J}_A^{\mathrm{R1}}\,, \tag{1.6a}$$

$$\mathcal{M}(u) = \mathcal{M}^{\mathrm{NR}} + \mathcal{M}^{\mathrm{R1}}\,, \tag{1.6b}$$

$$\tilde{\mathcal{M}}(u) = \tilde{\mathcal{M}}^{\mathrm{NR}} + \tilde{\mathcal{M}}^{\mathrm{R1}}\,, \tag{1.6c}$$

$$\mathcal{P}_A(u) = \mathcal{P}_A^{\mathrm{NR}} + \mathcal{P}_A^{\mathrm{R1}} + \mathcal{P}_A^{\mathrm{R2}}\,, \tag{1.6d}$$

$$\mathcal{T}_{AB}(u) = \mathcal{T}_{AB}^{\mathrm{NR}} + \mathcal{T}_{AB}^{\mathrm{R1}} + \mathcal{T}_{AB}^{\mathrm{R2}}\,, \tag{1.6e}$$

where the label NR denotes the non-radiative expressions given in (5.9), R1 a distributional radiative component linear in the impulse news and R2 a secular radiative component quadratic in the impulse news. The explicit expressions for the impulsive wave transition represent the third main result of the paper and they are given in section 6.1.

We present our conclusions in section 7 and many technical derivations of various relations used in the main text in a series of appendices A, B, C, D.

Notation. We use units in which $8\pi G = 1$ and $c = 1$. Greek letters are used for spacetime indices and uppercase Latin letters $\{A, B, C, \dots\}$ for coordinates over the 2D sphere. The symbol $\stackrel{\mathcal{I}}{=}$ is used when the right-hand side is evaluated at future null infinity $\mathcal{I}$. We denote the symmetric, trace-free part of a tensor T_{AB} with the brackets $\langle\cdot\rangle$, namely

$$T_{\langle AB\rangle} = \frac{1}{2}\left(T_{AB} + T_{BA} - q_{AB}q^{CD}T_{CD}\right)\,, \tag{1.7}$$

where q_{AB} is the asymptotic metric on the two-sphere.

2 Future null infinity

Let us introduce Bondi coordinates $x^\mu = (u, r, \sigma^A)$, where u labels null outgoing geodesic congruences which intersect infinity along 2d spheres, r is a parameter along these geodesics measuring the sphere's radius (r is the luminosity distance) and σ^A denotes coordinates on the celestial sphere. In these coordinates, the metric is given by [18, 19, 61]

$$\mathrm{d}s^2 = -2e^{2\beta}\mathrm{d}u\,(\mathrm{d}r + \Phi\mathrm{d}u) + r^2\gamma_{AB}\left(\mathrm{d}\sigma^A - \frac{\Upsilon^A}{r^2}\mathrm{d}u\right)\left(\mathrm{d}\sigma^B - \frac{\Upsilon^B}{r^2}\mathrm{d}u\right). \tag{2.1}$$

This metric satisfy the Bondi *gauge conditions* given by

$$g_{rr} = 0, \qquad g_{rA} = 0, \qquad \partial_r\sqrt{\gamma} = 0. \tag{2.2}$$

JHEP04(2022)125

In addition to the gauge condition we impose *extended*[4] Bondi *asymptotic boundary conditions* [21, 47, 62], which are given by

$$g_{ur} \overset{\mathcal{I}}{=} -1, \qquad \frac{1}{r} g_{uu} \overset{\mathcal{I}}{=} 0, \qquad \frac{1}{r} g_{uA} \overset{\mathcal{I}}{=} 0, \qquad \partial_u q_{AB} \overset{\mathcal{I}}{=} 0. \tag{2.3}$$

In this work we assume the usual Bondi like asymptotic boundary conditions which imply that the metric components $(\Phi, \beta, \gamma_{AB}, \Upsilon^A)$ have the following fall-off behavior[5]

$$\Phi = F(u,\sigma^A) - \frac{M(u,\sigma^A)}{r} + o(r^{-1})\,, \tag{2.4a}$$

$$\beta = \frac{b(u,\sigma^A)}{r^2} + o(r^{-2})\,, \tag{2.4b}$$

$$\Upsilon^A = U^A(u,\sigma^A) - \frac{2q^{AB}}{3r}(P_B + C_{BC}U^C + \partial_B b)(u,\sigma^A) + o(r^{-1})\,, \tag{2.4c}$$

$$\gamma_{AB} = q_{AB}(u,\sigma^A) + \frac{C_{AB}(u,\sigma^A)}{r} + \frac{1}{r^2}\left(D_{AB} + \frac{1}{4} q_{AB} C_{CD} C^{CD}\right)(u,\sigma^A) + \frac{E_{AB}(u,\sigma^A)}{r^3} + o(r^{-3})\,. \tag{2.4d}$$

The expansions of the different coefficients are needed to obtain the expansion of the metric $g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ to order[6] $O(r^{-1})$. Here M is the Bondi mass aspect, U^A is the asymptotic velocity, C_{AB} is twice the asymptotic shear. If one restricts q_{AB} to be the round sphere metric $\mathring{q}_{AB}$ with $R(\mathring{q}) = 2$, one recovers the restricted Bondi *boundary* conditions. We will avoid doing that in the following and keep the asymptotic conditions just stated. Because of the Bondi determinant gauge condition the symmetric tensors C_{AB}, D_{AB}, E_{AB} are all *traceless* when contracted with the inverse asymptotic metric q^{AB}. The $O(r^{-2})$ factor in the metric expansion is uniquely determined by the Bondi gauge condition; the demand that logarithmic anomalies vanish requires $D_{AB} = 0$ [63] and we assume this in the following. When evaluating the asymptotic, we use the metric q_{AB} to lower and raise the indices $\{A, B, \dots\}$ on the 2-sphere.

The leading asymptotic Einstein's equations (EEs) give a first relation

$$\partial_u q_{AB} = 0\,, \tag{2.5}$$

which can be understood as a boundary condition, implying the u-independence of the leading order of the metric component g_{AB}. While we derive from symmetry all the other EEs, this one is assumed and taken as a boundary condition from now on. There is then

[4] Note that we do not require $R(q) = 2$ which is why we call our boundary conditions *extended*.

[5] This is a restrcition of our analysis: one could be more general by allowing b to be order 1 and by allowing Φ to admit a term growing linearly in r. We follow here the original treatment [18–20] and all the subsequent extensions [21–25] of asymptotic symmetries of null infinity, we have chosen fall-off conditions that do not include these terms from the beginning in order to simplify the rest of the analysis. We come back to this point at the beginning of section 4.

[6] Since $\mathrm{d}r$ is of order $O(r)$, g_{ur} needs to be expanded to order $O(r^{-2})$, since $g_{AB} = r^2\gamma_{AB}$, γ_{AB} needs to be expanded to order $O(r^{-3})$ and since $g_{uA} = \gamma_{AB}\Upsilon^A$, Υ^A needs to be expanded to order $O(r^{-1})$, to achieved $O(r^{-1})$ for the expansion of the metric $g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$.

a second set of asymptotic vacuum Einstein's equations given by the relations[7]

$$\mathsf{E}_F := F - \frac{R(q)}{4} = 0\,, \tag{2.6}$$

$$\mathsf{E}_U^A := U^A + \frac{1}{2} D_B C^{AB} = 0\,, \tag{2.7}$$

$$\mathsf{E}_b := b + \frac{1}{32} C_{AB} C^{AB} = 0\,, \tag{2.8}$$

where D_A is the covariant derivative associated with q_{AB}. These can be understood as constraints between phase space data and we will show below how they can be obtained simply from the BMSW transformation properties and the requirement of covariance under it. The next two asymptotic equations are

$$\mathsf{E}^M := \dot{M} - \frac{1}{4} D_A D_B N^{AB} - \frac{1}{8}\Delta R + \frac{1}{8} N_{AB} N^{AB} = 0\,, \tag{2.9}$$

$$\begin{aligned}\mathsf{E}_A^P := \dot{P}_A - D_A M - \frac{1}{8} D_A \left(C^{BC} N_{CB}\right) - \frac{1}{4} C_{AB} \partial^B R \\ - \frac{1}{4} D_C \left(D_A D_B C^{BC} - D^C D^B C_{AB}\right) \\ - \frac{1}{4} D_B \left(N^{BC} C_{AC} - C^{BC} N_{AC}\right) + \frac{1}{4} N^{BC} D_A C_{BC} = 0\,,\end{aligned} \tag{2.10}$$

and they correspond to evolution equations for the energy aspect M and the momentum aspect P_A. At the order we are working in, the last asymptotic equation is an evolution equation for the spin-2 tensor E_{AB}. It was written explicitly in the gauge we are adopting here by Nichols in [64]

$$\begin{aligned}\mathsf{E}_{AB}^E := \dot{E}_{AB} - \frac{1}{2} M C_{AB} - \frac{1}{3} D_{\langle A} P_{B\rangle} - \frac{1}{96} D_{\langle A} D_{B\rangle} (C_{CD} C^{CD}) \\ - \frac{1}{4} C_{AB} N_{CD} C^{CD} + \frac{1}{8} \epsilon_A{}^C C_{CB} \epsilon_D{}^E D_E D_C C^{CD}\,.\end{aligned} \tag{2.11}$$

These three evolution equations are derived as well using our symmetry argument.

2.1 BMSW vector fields

The infinitesimal BMSW diffeomorphisms introduced in [17] are spacetime diffeomorphisms preserving the *boundary* conditions above. These are labelled by a vector field Y^A on S representing asymptotic diffeomorphisms of the celestial sphere, a super-translation parameter T and a Weyl transformation parameter W, which are all independent of u and r. The BMSW vector fields can be conveniently written, as $\xi_{(\tau,Y)}$, in terms of the parameter $\tau = \tau(T, W)$ given by

$$\tau := T + uW\,, \qquad \dot{\tau} = W\,, \qquad \ddot{\tau} = 0. \tag{2.12}$$

[7]The boundary condition (2.5) together with the Einstein's equation (2.6) clearly imply that the leading order of g_{uu} is time independent, namely $\partial_u F = 0$.

The BMSW vector fields $\xi_{(\tau,Y)}$ are characterized as the bulk vector fields preserving the Bondi gauge and asymptotic conditions (2.2), (2.3) which evaluate on $\mathcal{I}$ to $\xi_{(\tau,Y)} \stackrel{\mathcal{I}}{=} \bar{\xi}_{(\tau,Y)}$, where the asymptotic BMSW vector fields are

$$\begin{aligned}\bar{\xi}_{(\tau,Y)} &:= \tau\partial_u + Y^A\partial_A - \dot{\tau}\, r\partial_r, \\ &= T\partial_u + W(u\partial_u - r\partial_r) + Y^A\partial_A.\end{aligned} \tag{2.13}$$

We see that with the first parametrization τ labels time translations, while $\dot{\tau}$ labels conformal rescaling. In the second parametrization T labels time translations, while W labels asymptotic boosts.

To write down explicitly the bulk extension $\xi_{(\tau,Y)}$, it is convenient to define

$$I^{AB} := \left(\int_r^\infty \frac{\mathrm{d}r'}{r'^2} e^{2\beta}\gamma^{AB}\right), \tag{2.14}$$

which is such that $I^{AB} \stackrel{\mathcal{I}}{=} 0$ and $r\partial_r I^{AB} = -\frac{1}{r}e^{2\beta}\gamma^{AB}$. The corresponding vector fields are of the form[8]

$$\xi^u_{(\tau,Y)} = \tau\,, \tag{2.15a}$$

$$\xi^A_{(\tau,Y)} = Y^A - I^{AB}\partial_B\tau\,, \tag{2.15b}$$

$$\xi^r_{(\tau,Y)} = r\left(\frac{1}{2}D_A(I^{AB}\partial_B\tau) + \frac{1}{2r^2}\Upsilon^A\partial_A\tau - \dot{\tau}\right). \tag{2.15c}$$

We can check that $W(\sigma^A)$ induces a Weyl rescaling of the celestial sphere, as

$$\mathcal{L}_\xi\sqrt{q} = (D_AY^A - 2W)\sqrt{q}\,, \tag{2.16}$$

where $q := \det(q_{AB})$. The generalized BMS group proposed in [23, 25] is recovered by setting $W = \frac{1}{2}D_AY^A$, so that the condition $\delta\sqrt{q} = 0$ is preserved by the symmetry transformations.

2.2 BMSW symmetry transformations

The boundary BMSW symmetry group is asymptotically generated by the vector fields $\xi_{(\tau,Y)}$. Its Lie algebra is isomorphic to the double semi-direct sum [17]

$$\mathsf{bmsw} := \left(\mathrm{diff}(S) \oplus \mathbb{R}^S_W\right) \oplus \mathbb{R}^S_T. \tag{2.17}$$

The first factor $\mathbb{R}^S_W$ denotes the Weyl transformations labeled by functions W on the sphere, while the second factor $\mathbb{R}^S_T$ denotes the super-translations labelled by functions T on the sphere of density weight $1/2$. The commutators are given by

$$[\bar{\xi}_{(\tau_1,Y_1)}, \bar{\xi}_{(\tau_2,Y_2)}] = \bar{\xi}_{(\tau_{12},Y_{12})}. \tag{2.18}$$

[8]We have used that, thanks to the Bondi gauge, we have $D^\gamma_A Z^A = \frac{1}{\sqrt{\gamma}}\partial_A(\sqrt{\gamma}Z^A) = \frac{1}{\sqrt{q}}\partial_A(\sqrt{q}Z^A) = D_AZ^A$, for a generic vector Z on the sphere, with D^γ_A the covariant derivative associated to γ_{AB}.

Here we parametrized the $(\mathbb{R}^S_W \oplus \mathbb{R}^S_T)$ by functions on the sphere $\tau = T + uW$ which are linear in time and we have denoted

$$\tau_{12} = \tau_1\dot{\tau}_2 - \tau_2\dot{\tau}_1 + Y_1[\tau_2] - Y_2[\tau_1]\,, \quad Y^A_{12} = [Y_1, Y_2]^A_{\text{Lie}}\,, \tag{2.19}$$

where $Y[\tau] := Y^A\partial_A\tau$.

The quantities of physical interests such as $(q_{AB}, C_{AB}, F, M, b, U_A, P_A)$ are functionals $\Phi^i(g_{\mu\nu})$ of the metric. The transformations of these functionals are given by the chain rule $\delta_{(\tau,Y)}\Phi^i = \int \frac{\delta\Phi^i}{\delta g_{\mu\nu}}\mathcal{L}_{\xi_{(\tau,Y)}}g_{\mu\nu}$. In practice, to evaluate the variations $\delta_{(\tau,Y)}\Phi^i$ we use that $g_{\mu\nu}$ is determined by Φ_i and we evaluate the condition

$$\mathcal{L}_{\xi_{(\tau,Y)}}g_{\mu\nu}[\Phi^i] = \frac{\partial}{\partial\epsilon}g_{\mu\nu}[\Phi^i + \epsilon\delta_{(\tau,Y)}\Phi^i]\Big|_{\epsilon=0}\,. \tag{2.20}$$

The explicit derivations are given in appendix A.

By focusing on the metric component g_{AB}, one can easily derive the transformations

$$\delta_{(\tau,Y)}q_{AB} = [\mathcal{L}_Y - 2\dot{\tau}]\,q_{AB}\,, \tag{2.21a}$$

$$\delta_{(\tau,Y)}C_{AB} = [\tau\partial_u + \mathcal{L}_Y - \dot{\tau}]\,C_{AB} - 2D_{\langle A}D_{B\rangle}\tau\,. \tag{2.21b}$$

Taking the trace of q implies that

$$\delta_{(\tau,Y)}\sqrt{q} = \left[D_AY^A - 2\dot{\tau}\right]\sqrt{q}\,. \tag{2.21c}$$

Taking the time derivative $N_{AB} = \dot{C}_{AB}$ and the divergence of the shear $(D{\cdot}C)^B := D_AC^{AB}$ implies that

$$\delta_{(\tau,Y)}N_{AB} = [\tau\partial_u + \mathcal{L}_Y]\,N_{AB} - 2D_{\langle A}\partial_{B\rangle}\dot{\tau}\,, \tag{2.21d}$$

$$\delta_{(\tau,Y)}(D{\cdot}C)^B = [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}](D{\cdot}C)^B + (N^{BA}\partial_A\tau - C^{BA}\partial_A\dot{\tau}) - (R(q)\partial^B\tau + \partial^B\Delta\tau)\,. \tag{2.21e}$$

The second equality is established in appendix B. Next, focusing on the g_{uu} component, one can derive the two transformations

$$\delta_{(\tau,Y)}F = [\mathcal{L}_Y + 2\dot{\tau}]\,F + \frac{1}{2}\Delta\dot{\tau}, \tag{2.21f}$$

$$\begin{aligned}\delta_{(\tau,Y)}M = {}&[\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}]\,M + \left(\frac{1}{2}D_BN^{AB} + \partial^AF\right)\partial_A\tau\\ &+ \frac{1}{4}N^{AB}D_A\partial_B\tau + \frac{1}{4}C^{AB}D_A\partial_B\dot{\tau} + \frac{1}{2}(\mathsf{E}^A_U\partial_A\dot{\tau} - \dot{\mathsf{E}}^A_U\partial_A\tau)\,,\end{aligned} \tag{2.21g}$$

where E^A_U is the asymptotic Einstein's equation (2.7). The g_{ur} component gives the transformation

$$\delta_{(\tau,Y)}b = [\tau\partial_u + \mathcal{L}_Y + 2\dot{\tau}]\,b + \frac{1}{8}C^{AB}D_A\partial_B\tau + \frac{1}{4}\mathsf{E}^A_U\partial_A\tau\,. \tag{2.21h}$$

JHEP04(2022)125

From the Lie derivative of the g_{uA} component we can read off the transformation of the two functionals

$$\delta_{(\tau,Y)}U_A = [\tau\partial_u + \mathcal{L}_Y + \dot\tau]\,U_A + \frac{1}{2}(4F\partial_A\tau + \partial_A\Delta\tau) + \frac{1}{2}(C_A{}^B\partial_B\dot\tau - N_A{}^B\partial_B\tau)\,, \tag{2.21i}$$

$$\begin{aligned}\delta_{(\tau,Y)}P_A \mathrel{\hat=} {}& [\tau\partial_u + \mathcal{L}_Y + 2\dot\tau]P_A + 3M\partial_A\tau - \frac{1}{8}C_{BC}N^{BC}\partial_A\tau + \frac{1}{2}C_{AB}N^{BC}\partial_C\tau \\ & + \frac{3}{4}(D_AD_CC_B{}^C - D_BD_CC_A{}^C)\partial^B\tau + \frac{1}{4}\partial_A(C^{BC}D_BD_C\tau) \\ & + \frac{1}{2}D_{\langle A}D_{B\rangle}\tau D_CC^{BC} + C_{AB}\left(F\partial^B\tau + \frac{1}{4}\partial^B\Delta\tau\right) - 2\dot{\mathsf{E}}_b\partial_A\tau\,.\end{aligned} \tag{2.21j}$$

The hatted equality refers to the fact that we have used the asymptotic equations $\mathsf{E}_U^A \mathrel{\hat=} 0$ to simplify the r.h.s. of expression (2.21j).

Finally the most extensive calculation concerns the variation of the traceless component E_{AB}. One finds in appendix A.4.1 that

$$\begin{aligned}\delta_{(\tau,Y)}E_{AB} \mathrel{\hat=} {}& [\tau\partial_u + \mathcal{L}_Y + \dot\tau]E_{AB} \\ & + \frac{4}{3}\left(P_{\langle A} + \frac{1}{4}D_DC^{DC}C_{C\langle A} - 8\partial_{\langle A}b\right)\partial_{B\rangle}\tau \\ & + \frac{1}{2}\left(C^{CD}D_CC_{AB} - C_{AB}D_CC^{CD}\right)\partial_D\tau - D_{\langle A}C_{B\rangle C}C^{CD}\partial_D\tau \\ & + 4bD_{\langle A}\partial_{B\rangle}\tau - \frac{1}{4}C_{AB}C^{CD}D_C\partial_D\tau - \frac{16}{3}\mathsf{E}_bD_{\langle A}\partial_{B\rangle}\tau + \frac{32}{3}\partial_{\langle A}\mathsf{E}_b\partial_{B\rangle}\tau\,.\end{aligned} \tag{2.22}$$

2.3 Relation to the extended corner symmetry group

As shown in [1], the *corner symmetry* group is the gravitational symmetry associated to a generic codimension-2 surface called corner. This surface can be thought of as bordering a bounded region of space. The *corner symmetry algebra* is simply the subalgebra of diffeomorphisms that do not change the position of the corner surface. It is given by the semi-direct sum of the surface diffeomorphism and surface boosts. Explicitly, we have

$$\mathfrak{g}_S = \mathrm{diff}(S) \oplus \mathfrak{sl}(2,\mathbb{R})^S\,. \tag{2.23}$$

Its extension to include time translations normal to the surface yields the notion of the *extended corner symmetry algebra*, as revealed in [14, 15], which includes also two copies of $\mathbb{R}$ corresponding to the surface translations along the two normal directions. Its explicit structure is given by

$$\mathfrak{g}_S^{\mathrm{ext}} = \left(\mathrm{diff}(S) \oplus \mathfrak{sl}(2,\mathbb{R})^S\right) \oplus (\mathbb{R}^2)^S\,. \tag{2.24}$$

As shown in [15], the bmsw Lie algebra (2.17) corresponds to a subalgebra of the extended corner symmetry algebra $\mathfrak{g}_S^{\mathrm{ext}}$ in the bulk, where the $\mathbb{R}_W^S$ contribution is given by one of the $\mathfrak{sl}(2,\mathbb{R})$ generators (namely, the one preserving the null generator of $\mathcal{I}$), while the $\mathbb{R}_T^S$ contribution corresponds to one of the two normal super-translations (namely, the one along $\mathcal{I}$). We can thus understand the BMSW group as the $r\to\infty$ limit (a contraction) of the extended corner symmetry group.

JHEP04(2022)125

3 Anomalies

The anomaly operator Δ_τ associated with a functional $\mathcal{O}$ of conformal dimension s is given by

$$\Delta_\tau \mathcal{O} := \delta_{(\tau,Y)}\mathcal{O} - (\tau\partial_u + \mathcal{L}_Y + s\dot{\tau})\mathcal{O}. \tag{3.1}$$

This anomaly measures the difference between the natural action of the BMSW group on $\mathcal{O}$ and its field space action. By construction the anomaly only depends on τ. The transformation rules reported in the previous section have the general structure

$$\delta_{(\tau,Y)}\mathcal{O} = [\tau\partial_u + \mathcal{L}_Y + s\dot{\tau}]\mathcal{O} + L^A_{\mathcal{O}}\partial_A\tau + \bar{L}^A_{\mathcal{O}}\partial_A\dot{\tau} + Q^{AB}_{\mathcal{O}}D_A\partial_B\tau + \bar{Q}^{AB}_{\mathcal{O}}D_A\partial_B\dot{\tau}. \tag{3.2}$$

The first term is the homogeneous transformation that involves the scale weight[9] s of the functional $\mathcal{O}$. All scale weights of the different functionals can be found by assigning scale weight $s(\mathrm{d}s^2) = 0$, while $s(r) = s(\mathrm{d}r) = +1$ and $s(u) = s(\mathrm{d}u) = -1$ in the metric expansion, hence the scale weight of ∂_u is $+1$. Functionals that transform homogeneously are sections of the scale bundle P.[10] The inhomogeneous terms are of two types: $(L^A_{\mathcal{O}}, \bar{L}^A_{\mathcal{O}})$ which we call *linear anomalies* and terms $(Q^{AB}_{\mathcal{O}}, \bar{Q}^{AB}_{\mathcal{O}})$ which are the *quadratic anomalies*. An example of anomaly is

$$\Delta_\tau C_{AB} = -2D_{\langle A}\partial_{B\rangle}\tau\,. \tag{3.3}$$

The functional $\mathcal{O}$ is said to be tensorial when both linear and quadratic anomalies vanish. The first examples of tensorial combinations are the quantities $q_{AB}, \dot{N}_{AB}$ which satisfy

$$\Delta_\tau q_{AB} = 0, \qquad \Delta_\tau \dot{N}_{AB} = 0\,. \tag{3.4}$$

They are operators of scale weight $(-2, +1)$ respectively. The main theme of our paper is that we can recover the equations of motion by identifying the tensorial combinations. For instance, we can easily see from the previous expressions that the asymptotic equations of motion (2.5), (2.7), (2.8) all transform tensorially: $\mathsf{E}_F := R - \frac{1}{4}F$ as a section of weight 2, E_b as a section of weight 2, E^A_U as a vector of weight 3.

Another class of operators which will be of interest to us is the *pseudo-tensors* of weight s. These are characterized by the fact that the quadratic anomaly and the anomaly $\bar{L}^A_{\mathcal{O}}$ linear in $\dot{\tau}$ vanish while the linear anomaly $L^A_{\mathcal{O}}$ does not. The pseudo-tensors can be understood as tensorial for the subgroup of symmetry that does not include super-translations.

The next example we want to study involves the Liouville stress tensor [25]. Given a metric q_{AB} we can define its Liouville stress tensor[11] to be the symmetric traceless tensor $T_{AB}(q)$ such that

$$D_A T^{AB} + \frac{1}{2}\nabla^B R = 0. \tag{3.5}$$

[9] s can also be understood as a boost weight.

[10] We call *scale bundle* a line bundle $P \to \mathcal{I}$ over $\mathcal{I}$ whose automorphism group includes the asymptotic BMSW vector fields (2.13).

[11] The conserved energy momentum tensor of Liouville is $\tau_{AB} := T_{AB} + \frac{1}{2}q_{AB}R(q)$. Its trace is $q^{AB}\tau_{AB} = R(q)$. The tensor $T_{AB}(q)$ is also called the Geroch tensor when $q = e^{\varphi}\mathring{q}$ [65].

The fact that this tensor can be uniquely determined follows first from the fact that the equation $\mathring{D}_A T^{AB} = 0$ implies, when S is a sphere, that $T^{AB}(\mathring{q}) = 0$. Second, from the following covariance properties under Weyl transformation with parameter W and diffeomorphism $\varphi : S \to S$

$$T_{AB}(e^{2W}q) = T_{AB}(q) - 2(D_{\langle A}WD_{B\rangle}W + D_{\langle A}D_{B\rangle}W), \qquad \varphi^*(T_{AB}(q)) = T_{AB}(\varphi^*(q)). \tag{3.6}$$

Indeed, by the uniformization theorem, any metric on the sphere can be written as $q = e^W\varphi^*(\mathring{q})$. So the transformation properties (3.6) allow one to determine $T_{AB}(q)$. This means that the combination

$$\bar{N}_{AB} := N_{AB} - T_{AB}(q) \tag{3.7}$$

possesses no anomaly $\Delta_\tau \bar{N}_{AB} = 0$. It is a tensor operator of scale weight 0. From this we can construct a covariant current

$$\mathcal{J}^A := \frac{1}{2}D_B\bar{N}^{BA} = \frac{1}{2}D_B N^{AB} + \frac{1}{4}\partial^A R\,, \tag{3.8}$$

where the second equality follows from (3.5). The covariant current yields the Weyl scalar Ψ_3, it possesses no quadratic anomaly and it is a pseudo-tensor of dimension 4

$$\delta_{(\tau,Y)}\mathcal{J}^A = [\tau\partial_u + \mathcal{L}_Y + 4\dot{\tau}]\,\mathcal{J}^A + \frac{1}{2}\dot{N}^{AB}\partial_B\tau\,. \tag{3.9}$$

This can be seen by taking the time derivative of (2.21e) (an alternative derivation is given in appendix B). The explicit relation between all the covariant observables and the Weyl scalars is shown in section 3.5. Since $\dot{N}^{AB}$ is a tensor and $\mathcal{J}^A$ is a pseudo-tensors whose anomaly vanish when $\dot{N}^{AB} = 0$, we can define the non-radiative vacua to be such that $\mathcal{J}^A = 0 = \dot{N}_{AB}$. The non-radiative vacua are transformed into each other by the symmetry transformations.

3.1 Covariant mass

We are interested in combinations of the physical quantities parametrizing the Bondi metric (2.1) that transform as pseudo-tensors, with no quadratic anomaly and no Weyl linear anomaly. To this aim, we introduce the notion of *covariant mass*

$$\mathcal{M} := M + \frac{1}{8}C_{AB}N^{AB}\,. \tag{3.10}$$

The justification for this name comes from the fact that, by means of (2.21g), (2.21b), (2.21d), the quantity above transforms[12] as

$$\delta_{(\tau,Y)}\mathcal{M} \,\hat{=}\, [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}]\,\mathcal{M} + \mathcal{J}^A\partial_A\tau\,. \tag{3.11}$$

We thus see that only a linear anomaly term appears in the transformation of $\mathcal{M}$ and moreover that the linear anomaly depends only on τ, not $\dot{\tau}$. A nontrivial consistency

[12] We recall that the hatted equality refers to the fact that we use the asymptotic equations $\mathsf{E}_U^A = 0$.

check for this formula comes from the fact that the variation of $\mathcal{J}^A$ does not contain any quadratic anomaly terms, as shown by (3.9).

This indicates also that if the non-radiative structure $\mathcal{J}^A = 0 = \dot{N}_{AB}$ is satisfied, then the covariant mass aspect $\mathcal{M}$ transforms homogeneously. Moreover, flat vacua can be defined by the conditions $\mathcal{M} = 0 = \mathcal{J}^A = \dot{N}^{AB}$. These were parametrized and studied in [25, 66].

3.2 Duality and covariant mass

In this section we show that it is possible to construct from C_{AB}, N_{AB} and their derivative another scalar of dimension 3 that possesses no quadratic anomaly: the *dual covariant mass*. To describe its construction, let us introduce the volume form on S denoted ϵ_{AB} and given by $\frac{1}{2}\epsilon_{AB}\mathrm{d}\sigma^A \wedge \mathrm{d}\sigma^B = \sqrt{q}\mathrm{d}^2\sigma$. Raising one of its indices with the metric, one gets the complex structure

$$\epsilon_A{}^B := \epsilon_{AC}q^{CB}, \qquad \epsilon_A{}^B\epsilon_B{}^C = -\delta_A^C. \tag{3.12}$$

The complex structure is a tensor of weight 0

$$\delta_{(\tau,Y)}\epsilon_A{}^B = (\tau\partial_u + \mathcal{L}_Y)\epsilon_A{}^B. \tag{3.13}$$

We can use this complex structure to define a duality transform for the traceless tensors C, N and the derivatives. We introduce the notation

$$\tilde{C}_{AB} := \epsilon_A{}^C C_{CB} = \epsilon_B{}^C C_{AC}\,, \qquad \tilde{N}_{AB} := \epsilon_A{}^C N_{CB} = \epsilon_B{}^C N_{AC}\,, \qquad \tilde{V}_A := \epsilon_A{}^B V_B\,. \tag{3.14}$$

Note that the equality $\epsilon_A{}^C C_{CB} = \epsilon_B{}^C C_{AC}$ is only valid for symmetric traceless tensors.

The tilde operation is a duality $\tilde{\tilde{N}}_{AB} = -N_{AB}$, and we have the properties

$$\tilde{V}_A W^A = -V_A\tilde{W}^A, \qquad \tilde{N}_{AB}V^B = -N_{AB}\tilde{V}^B, \tag{3.15}$$

where we denoted $\tilde{V}^A = q^{AB}\tilde{V}_B$. In particular, this means that

$$\widetilde{(D\cdot N)}_A = \epsilon_A{}^B D^C N_{CB} = (D\cdot\tilde{N})_A = -(\tilde{D}\cdot N)_A\,. \tag{3.16}$$

The tensor ϵ_{AB} can be used to convert 2-forms on the sphere into (pseudo)-scalars. In particular, given $J_{AB} = J_{[AB]}$ a 2-form on S this can be written as

$$J_{AB} = \frac{1}{2}\tilde{J}\epsilon_{AB}, \qquad \tilde{J} = \epsilon^{AB}J_{AB}. \tag{3.17}$$

An identity that we will repeatedly use in the following derivations is the condition that

$$D_{[A}N_{B]}{}^C\partial_C\tau = D_C N_{[A}{}^C\partial_{B]}\tau, \tag{3.18}$$

which follows from the Fierz identity $D_{[A}N_B{}^C\partial_{C]}\tau = 0$ and the fact that N is traceless. It will also be useful to simplify some tensors using the identity

$$\epsilon_{AB}\epsilon^{CD} = \delta_A^C\delta_B^D - \delta_A^D\delta_B^C. \tag{3.19}$$

Given these preliminaries we can now present the construction of the dual covariant mass. From the transformation (2.21e) of $(D{\cdot}C)_B$ we conclude in appendix B that

$$\delta_{(\tau,Y)}(D_A(D{\cdot}\tilde{C})^A) = [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}](D_A(D{\cdot}\tilde{C})^A) + 4\tilde{\mathcal{J}}^A\partial_A\tau + (\tilde{N}^{BC}D_B\partial_C\tau + C^{BC}\tilde{D}_B\partial_C\dot{\tau}). \tag{3.20}$$

This means that the following combination

$$\tilde{\mathcal{M}} := \frac{1}{4}(D_A(D{\cdot}\tilde{C})^A) + \frac{1}{8}C_{AB}\tilde{N}^{AB}\,, \tag{3.21}$$

called the *covariant dual mass*, possesses no quadratic anomalies. The explicit transformation follows from the transformations (3.20) and (2.21b), (2.21d) and it is given by

$$\delta_{(\tau,Y)}\tilde{\mathcal{M}} = [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}]\tilde{\mathcal{M}} + \tilde{\mathcal{J}}^A\partial_A\tau\,. \tag{3.22}$$

This is in absolute parallel with the mass transformation formula (2.21g). Note that the role of the mass aspect M is played here by the "vorticity" of the fluid with velocity U_A:

$$\tilde{M} := -\frac{1}{2}\epsilon^{AB}D_AU_B = \frac{1}{4}(D_A(D{\cdot}\tilde{C})^A). \tag{3.23}$$

The covariant mass $\mathcal{M}$ and the dual covariant mass $\tilde{\mathcal{M}}$ determine respectively the real and the imaginary part of the Weyl scalar Ψ_2 at $\mathcal{I}$.

3.3 Covariant momentum

We now focus on the construction of the covariant momentum. The transformation of the momentum[13] is given in (2.21j). In order to analyze and simplify this equation, one can follow the same strategy as the one that led to the definition of the covariant mass and look for counter-terms that cancel all the quadratic anomaly terms proportional to $D_A\partial_B\tau$ and $\Delta\tau$. To do so one first establishes, using (2.21e) again, that

$$\begin{aligned}\delta_{(\tau,Y)}(D_CC^{CB}C_{BA}) = [\tau\partial_u + \mathcal{L}_Y + 2\dot{\tau}](D_CC^{CB}C_{BA}) + C_{AB}(N^{BC}\partial_C\tau - C^{BC}\partial_C\dot{\tau})\\ - C_{AB}(R\partial^B\tau + \partial^B\Delta\tau) - 2D_{\langle A}D_{B\rangle}\tau D_CC^{CB}\,.\end{aligned} \tag{3.24}$$

We also use that

$$\begin{aligned}\delta_{(\tau,Y)}\partial_A\left(-\frac{1}{32}C_{BC}C^{BC}\right) \hat{=} [\tau\partial_u + \mathcal{L}_Y + 2\dot{\tau}]\partial_A\left(-\frac{1}{32}C_{BC}C^{BC}\right) + \frac{1}{8}\partial_A(C^{BC}D_B\partial_C\tau)\\ - \frac{1}{32}\partial_u\left(C_{BC}C^{BC}\right)\partial_A\tau - \frac{1}{16}C_{BC}C^{BC}\partial_A\dot{\tau}\,.\end{aligned} \tag{3.25}$$

This means that the last three terms in the variation (2.21j) of P_A can therefore be cancelled by the modification

$$\mathcal{P}_A := P_A + \frac{1}{4}(D_CC^{CB})C_{BA} + \frac{1}{16}\partial_A(C_{BC}C^{BC})\,, \tag{3.26}$$

[13] To compare these transformations with the one of [22, 25] one needs to use that $N_A = P_A + \partial_Ab$.

which defines the *covariant momentum* and yields the Weyl scalar Ψ_1. To compute explicitly the variation of this covariant momentum one uses that

$$C_{AB}C^{BC} = \frac{1}{2}(C_{BD}C^{BD})\delta^C_A, \qquad C_{AB}N^{BC} = \frac{1}{2}(C_{BD}N^{BD})\delta^C_A + \frac{1}{2}(C_{BD}\tilde{N}^{BD})\epsilon_A{}^C\,. \tag{3.27}$$

This means that the variation of the covariant momentum drastically simplifies into

$$\delta_{(\tau,Y)}\mathcal{P}_A \mathrel{\hat{\hat{=}}} [\tau\partial_u + \mathcal{L}_Y + 2\dot{\tau}]\mathcal{P}_A + 3\mathcal{M}_{AB}\partial^B\tau\,, \tag{3.28}$$

where the tensor $\mathcal{M}_{AB}$ is given by

$$\mathcal{M}_{AB} := \mathcal{M}q_{AB} + \tilde{\mathcal{M}}\epsilon_{AB}\,, \tag{3.29}$$

and the double hatted equality refers to the fact that we have used both asymptotic equations $\mathsf{E}^A_U \mathrel{\hat{\hat{=}}} 0 \mathrel{\hat{\hat{=}}} \mathsf{E}_b$.

The momentum transformation involves the mass and dual mass on a symmetric level. It is a clear improvement from the cumbersome transformation (2.21j) and it exhibits, as anticipated, a self-dual symmetry of the transformation rules. This expression also provides a powerful and nontrivial consistency check of (2.21j). Indeed, since $\mathcal{P}_A$ does not contain quadratic anomaly, its variation should also be expressed only in terms of semi-covariant tensors that do not contain quadratic anomalies. This is indeed the case since both $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are semi-covariant. To summarize, the transformation property (3.28) of the covariant momentum is self-dual and given by

$$\delta_{(\tau,Y)}\mathcal{P}_A \mathrel{\hat{\hat{=}}} [\tau\partial_u + \mathcal{L}_Y + 2\dot{\tau}]\mathcal{P}_A + 3\left(\mathcal{M}\partial_A\tau + \tilde{\mathcal{M}}\tilde{\partial}_A\tau\right)\,. \tag{3.30}$$

3.4 Covariant stress

We finally focus on the construction of the covariant spin-2 observable. It is easy to see that the first two terms in the last line of (2.22) are cancelled in the following combination

$$\mathcal{T}_{AB} := 3\left(E_{AB} - \frac{1}{16}C_{AB}C_{CD}C^{CD}\right)\,, \tag{3.31}$$

which defines the *covariant stress* and yields the Weyl scalar Ψ_0. Using the definition of the covariant momentum, we can write its transformation as (see appendix A.4.1)

$$\begin{aligned}\delta_{(\tau,Y)}\mathcal{T}_{AB} \mathrel{\hat{=}} &[\tau\partial_u + \mathcal{L}_Y + \dot{\tau}]\mathcal{T}_{AB} + 4\mathcal{P}_{\langle A}\partial_{B\rangle}\tau \\ &+ \frac{3}{2}\left(C^{CD}D_CC_{AB} - C_{AB}D_CC^{CD}\right)\partial_D\tau - 3D_{\langle A}C_{B\rangle C}C^{CD}\partial_D\tau + \frac{3}{4}\partial_{\langle A}(C_{CD}C^{CD})\partial_{B\rangle}\tau \\ &- 4\mathsf{E}_bD_{\langle A}\partial_{B\rangle}\tau\,.\end{aligned} \tag{3.32}$$

We can simplify this expression considerably using the identity

$$\frac{1}{2}\left(C_D{}^CD_CC_{AB} - C_{AB}D_CC^C{}_D\right) = D_{\langle A}C_{B\rangle C}C^C{}_D - \frac{1}{4}\partial_{\langle A}(C_{CE}C^{CE})q_{B\rangle D}\,, \tag{3.33}$$

that can be proven using complex coordinates. Finally, this means that we simply have

$$\delta_{(\tau,Y)}\mathcal{T}_{AB} \mathrel{\hat{\hat{=}}} [\tau\partial_u + \mathcal{L}_Y + \dot{\tau}]\,\mathcal{T}_{AB} + 4\mathcal{P}_{\langle A}\partial_{B\rangle}\tau. \tag{3.34}$$

Again a drastic simplification from the original transformation (2.22).

JHEP04(2022)125

3.5 Covariant tensors and Weyl scalars

In order to elucidate the relation between the covariant observables introduced above and the Weyl scalars in the Newman-Penrose formalism [34, 35] at null infinity, let us introduce a doubly-null tetrad $(\ell, t, m, \bar{m})$ adapted to the 2 + 2 foliation defined by two null vectors ℓ, t transverse to the sphere and a complex dyad $m, \bar{m}$ tangent to the sphere, with $q^{AB} = 2m^{(A}\bar{m}^{B)}$. Explicitly, in the Bondi coordinates (u, r) on $\mathcal{I}$ these vectors are given by

$$\ell = \partial_r\,, \quad t = \partial_u \quad m = m^A\partial_A\,. \tag{3.35}$$

By contracting the Weyl tensor $W_{\mu\nu\rho\sigma}$ with the tetrad field above, we obtain the 5 Weyl scalars $(\Psi_4, \Psi_3, \Psi_2, \Psi_1, \Psi_0)$. The asymptotic values of the Weyl scalars, which are determined by the peeling theorem, are respectively given by $(\frac{1}{2}\dot{N}^{AB}, \mathcal{J}^A, \mathcal{M} - i\tilde{\mathcal{M}}, \mathcal{P}_A, \mathcal{T}_{AB})$ (see appendix D of [17])

$$\Psi_4 := -W_{t\bar{m}t\bar{m}} = \frac{1}{2r}\,\dot{N}^{AB}\bar{m}_A\bar{m}_B + o(r^{-1})\,, \tag{3.36a}$$

$$\Psi_3 := -W_{t\ell t\bar{m}} = \frac{1}{r^2}\,\mathcal{J}^A\bar{m}_A + o(r^{-2})\,, \tag{3.36b}$$

$$\Psi_2 := -\frac{1}{2}\,(W_{\ell t\ell t} + W_{\ell t m\bar{m}}) = \frac{1}{r^3}\,(\mathcal{M} + i\tilde{\mathcal{M}}) + o(r^{-3}), \tag{3.36c}$$

$$\Psi_1 := -W_{\ell t\ell m} = \frac{1}{r^4}\,\mathcal{P}_A m^A + o(r^{-4})\,, \tag{3.36d}$$

$$\Psi_0 := -W_{\ell m\ell m} = \frac{1}{r^5}\,\mathcal{T}_{AB}m^A m^B + o(r^{-5})\,. \tag{3.36e}$$

This means that the covariant observables are simply, and up to normalisation, the asymptotic Weyl scalars.

4 EOM from symmetry

In the previous section we have constructed the tensor $\dot{N}^{AB}$ and the pseudo-tensors $(\mathcal{J}^A, \mathcal{M}, \tilde{\mathcal{M}}, \mathcal{P}_A, \mathcal{T}_{AB})$. By design these are covariant under the kinematical part of the bmsw algebra and they represent the metric data up to order $1/r$ in the metric expansion. We now want to explore their covariance properties under supertranslations and derive from it the asymptotic evolution equations, that appear as restrictions on the free data.

The strategy that defines our symmetry argument is as follows. For a given scale weight s and a given spin, we first identify the combinations of free data that transform with that weight and are covariant under the action of the BMSW infinitesimal transformations. The associated asymptotic EEs are then obtained by setting those combinations to zero. To be more precise the first step of the argument requires identifying quantities denoted $\mathcal{E}$ which are now tensorial under the full symmetry group including supertranslaions. One can then argue that in the absence of sources, that is for pure gravity, the only possible consistent equation is $\mathcal{E} = 0$. The reason we put the r.h.s. of the equation equal to zero instead of 1 say is that $\mathcal{E}$ can be understood as transforming in the coadjoint representation of the BMSW group. It would be inconsistent to fixed $\mathcal{E} = 1$ as the transformed value $g\mathcal{E}g^{-1}$, for g an element of the BMSW group, would now be different from 1. The only option is to have an equality of the form $\mathcal{E} = O$ where O is an object that transforms under the coadjoint representation of BMSW. In the absence of matter no such object exists and the only admissible coadjoint orbit that can source the equation is $O = 0$. Later in the section we identify, in the presence of matter, which combination of the energy-momentum tensor transforms in the same orbit as $\mathcal{E}$. This leads to a proposal for the asymptotic equations of motion in the presence of matter.

Let us emphasize that the derivation of Einstein's equation from symmetry is only valid for the asymptotic equations of motion that arises in a $1/r$ expansion of the metric. The rest of the equations that allow to reconstruct the bulk metric, through the radial evolution, are not derived in that way but they are assumed to hold. This is consistent with the holographic perspective where the boundary is assumed to have a unique bulk reconstruction. Let us also emphasize that going from the identification of a tensor $\mathcal{E}$ to the imposition $\mathcal{E} = 0$ as an equation of motion is not new. It is the same strategy that Einstein used to derive the equation $G_{\mu\nu} = 0$ from a *gauge* symmetry argument [67].

Before focusing on the derivation of the set of Einstein's equations, we need to distinguish the equations that are derived from a symmetry argument form the ones that are imposed as boundary conditions. The only equations that we impose as boundary conditions are listed in (2.3), which in particular contain the equation (2.5). It is possible to relax these boundary conditions and perform a more full fledge analysis where the equation (2.5) is also derived from symmetry, but we do not do this here.

We now look systematically at the metric components that transform homogeneously under the full BMSW group. One starts by the scalar data of weight $s = 1$. At this weight there is only one datum of weight $s = 1$ that transforms homogeneously under BMSW transformations. It is the scalar given by b_1 which labels the term of order b_1/r in the expansion of β, which transforms as $\delta_{(\tau,Y)} b_1 = [\tau\partial_u + Y^A\partial_A + \dot{\tau}]b_1$. Since this is the only datum of $s = 1$ that transforms homogeneously, the covariant relation $b_1 = 0$ falls in the set of covariant relations to impose in the absence of external sources and it is thus included in the set of asymptotic EEs derived using exclusively BMSW transformation properties.

For the next steps we analyze the two scalar invariants of weight $s = 2$ and the spin one invariant of weight $s = 3$, representing evolution equations for covariant observables. This allows us to apply our symmetry argument for an immediate derivation of the Einstein's equations (2.6), (2.7), (2.8). More precisely, from the transformations of the 2d Ricci scalar under the metric rescaling $q_{AB} \to e^{-2\dot{\tau}} q_{AB}$ and (2.21f) it is immediate to see that the combination

$$\mathcal{E}_F = F - \frac{R(q)}{4} \tag{4.1}$$

transforms homogeneously as a scalar of weight $s = 2$, namely

$$\delta_{(\tau,Y)} \mathcal{E}_F = [\mathcal{L}_Y + 2\dot{\tau}]\, \mathcal{E}_F\,. \tag{4.2}$$

Hence the EE (2.6) is recovered as the covariant expression $\mathcal{E}_F = 0$. From the transformations (2.21e), (2.21i), it is straightforward to show that the combination

$$\mathcal{E}_U^A := U^A + \frac{1}{2} D_B C^{AB} \tag{4.3}$$

transforms homogeneously as a vector of weight $s = 3$, namely

$$\delta_{(\tau,Y)} \mathcal{E}_U^A = [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}] \mathcal{E}_U^A \,. \tag{4.4}$$

This shows that the EE (2.7) is also recovered as the covariant expression $\mathcal{E}_U^A = 0$.

Similarly to the case for $\mathcal{E}_U^A$, from the transformations (2.21b), (2.21h) we see right away that the scalar combination

$$\mathcal{E}_b := b + \frac{1}{32} C_{AB} C^{AB} \tag{4.5}$$

transforms as

$$\delta_{(\tau,Y)} \mathcal{E}_b = [\tau\partial_u + \mathcal{L}_Y + 2\dot{\tau}]\, \mathcal{E}_b + \frac{1}{4} \mathcal{E}_U^A \partial_A \tau \,. \tag{4.6}$$

Therefore, on-shell of the previously just derived EE $\mathcal{E}_U^A = 0$, we recover (2.8) as well from the requirement of covariance under the BMSW group action.

Now that we have shown that our symmetry argument can be applied to derive both Einstein's equations (2.7), (2.8), in the following we will at times go on-shell of these two equations in order to simplify some of the expressions. We recall that imposition of (2.7) alone is denoted by a single hat while impositions of (2.8) as well by a double hat. In particular, we point out already that, when including matter sources in our analysis, we will refrain from imposing (2.8) as the combination $b + \frac{1}{32} C_{AB} C^{AB}$ picks up a stress-energy tensor contribution in the Einstein's equations and this needs to be taken properly into account when studying covariance properties of matter terms as well. This means that some of the transformations derived in sections 4.1, 4.2, 4.3 will need to be generalized to include terms proportional to $\mathcal{E}_b$; this is done in appendix A.

4.1 Mass evolution from symmetry

The goal of this section is to show that, quite remarkably, the symmetry transformation of $\mathcal{M}$ completely determines its equation of motion. To see this, one evaluates the transformation of the covariant mass time derivative and the current divergence

$$\begin{aligned} \delta_{(\tau,Y)} \dot{\mathcal{M}} &\mathrel{\hat{=}} [\tau\partial_u + \mathcal{L}_Y + 4\dot{\tau}]\, \dot{\mathcal{M}} + \partial_u (\mathcal{J}^A \partial_A \tau) \,, \\ \delta_{(\tau,Y)} D_A \mathcal{J}^A &= [\tau\partial_u + \mathcal{L}_Y + 4\dot{\tau}]\, D_A \mathcal{J}^A + 2\partial_u (\mathcal{J}^A D_A \tau) + \frac{1}{2} \dot{N}^{AB} D_A \partial_B \tau \,, \end{aligned} \tag{4.7}$$

where we used that $\frac{1}{2} D_A \dot{N}^{AB} = \dot{\mathcal{J}}^B$. This means that the quantity

$$\mathcal{E} := \dot{\mathcal{M}} - \frac{1}{2} D_A \mathcal{J}^A - \frac{1}{8} \dot{N}^{AB} C_{AB} \tag{4.8}$$

transforms homogeneously under the symmetry transformation. Therefore, the covariant conservation equation is $\mathcal{E}=0$ or

$$\dot{M}-\frac{1}{2}D_A\mathcal{J}^A=-\frac{1}{8}N_{AB}N^{AB}\,, \tag{4.9}$$

when written in terms of the original variables.

This is one of the Einstein's equation which is derived purely from symmetry principle. If one uses a fluid analogy where $\mathcal{M}$ plays the role of the energy density, we see that $-\mathcal{J}^A$ is the energy transport current while $\frac{1}{8}N_{AB}N^{AB}$ plays the role of the entropy production. The expression (4.8) in terms of the covariant quantities shows that when no radiation is present, which corresponds to the condition $\dot{N}_{AB}=0$, then the covariant mass is conserved

$$\dot{\mathcal{M}}-\frac{1}{2}D_A\mathcal{J}^A=0. \tag{4.10}$$

The same can be followed for the dual mass, using that

$$\begin{aligned}\delta_{(\tau,Y)}\dot{\tilde{\mathcal{M}}}&=[\tau\partial_u+\mathcal{L}_Y+4\dot{\tau}]\tilde{\mathcal{M}}+\partial_u(\tilde{\mathcal{J}}^A\partial_A\tau),\\ \delta_{(\tau,Y)}D_A\tilde{\mathcal{J}}^A&=[\tau\partial_u+\mathcal{L}_Y+4\dot{\tau}]\,D_A\tilde{\mathcal{J}}^A+2\partial_u(\tilde{\mathcal{J}}^AD_A\tau)+\frac{1}{2}\dot{\tilde{N}}^{AB}D_A\partial_B\tau\,,\end{aligned} \tag{4.11}$$

from which we deduce that the quantity

$$\tilde{\mathcal{E}}:=\dot{\tilde{\mathcal{M}}}-\frac{1}{2}D_A\tilde{\mathcal{J}}^A-\frac{1}{8}\dot{\tilde{N}}^{AB}C_{AB} \tag{4.12}$$

transforms homogeneously under the symmetry transformation. Therefore, the covariant conservation equation for the dual mass is $\tilde{\mathcal{E}}=0$.

4.2 Momentum evolution from symmetry

We can establish in a similar manner that the asymptotic evolution equation for the momentum can be written as $\mathcal{E}_A=0$, with (see also details in appendix C)

$$\mathcal{E}_A:=\dot{\mathcal{P}}_A-\partial_A\mathcal{M}-\tilde{\partial}_A\tilde{\mathcal{M}}-C_{AB}\mathcal{J}^B\,, \tag{4.13}$$

a vector that transforms homogeneously under the BMSW group. The goal is show that the anomaly of $\mathcal{E}_A$ vanishes when $\mathcal{E}$ and $\tilde{\mathcal{E}}$ vanish, namely

$$\Delta_\tau\mathcal{E}_A\,\hat{\hat{=}}\,0\,. \tag{4.14}$$

We rely on the duality between the covariant mass and dual mass to simplify the proof.

Let us first focus on the quadratic anomalies, which we denote $\Delta_\tau^{(2)}$. From the transformation properties derived in appendix B, we can write

$$\begin{aligned}\Delta_\tau^{(2)}\partial_A\mathcal{M}&\hat{=}\mathcal{J}^BD_A\partial_B\tau=\mathcal{J}^BD_{\langle A}\partial_{B\rangle}\tau+\frac{1}{2}\mathcal{J}_A\Delta\tau\,,\\ \Delta_\tau^{(2)}\tilde{\partial}_A\tilde{\mathcal{M}}&=\tilde{\mathcal{J}}^B\tilde{D}_A\partial_B\tau=-\mathcal{J}^B\tilde{D}_{\langle A}\tilde{\partial}_{B\rangle}\tau+\frac{1}{2}\tilde{\mathcal{J}}^B\epsilon_{AB}\Delta\tau\\ &=\mathcal{J}^BD_{\langle A}\partial_{B\rangle}\tau-\frac{1}{2}\mathcal{J}_A\Delta\tau\,,\\ \Delta_\tau^{(2)}C_{AB}\mathcal{J}^B&=-2\mathcal{J}^BD_{\langle A}\partial_{B\rangle}\tau\,,\end{aligned} \tag{4.15}$$

from which it is straightforward to see that

$$\Delta_\tau^{(2)}\mathcal{E}_A \hat{\hat{=}} 0\,, \tag{4.16}$$

since $\Delta_\tau^{(2)}\dot{\mathcal{P}}_A \hat{\hat{=}} 0$ as immediate from (3.30).

Let us now look at the linear anomaly $\Delta_\tau^{(1)}$. Using again the results of appendix B, we can write

$$\Delta_\tau^{(1)}\dot{\mathcal{P}}_A \hat{\hat{=}} 3\dot{\mathcal{M}}\partial_A\tau + 3\mathcal{M}\partial_A\dot{\tau} + 3\dot{\tilde{\mathcal{M}}}\tilde{\partial}_A\tau + 3\tilde{\mathcal{M}}\tilde{\partial}_A\dot{\tau}\,, \tag{4.17}$$

$$\Delta_\tau^{(1)}\partial_A\mathcal{M} \hat{=} \partial_A\tau\dot{\mathcal{M}} + 3\partial_A\dot{\tau}\mathcal{M} + D_A\mathcal{J}^B\partial_B\tau, \tag{4.18}$$

$$\Delta_\tau^{(1)}\tilde{\partial}_A\tilde{\mathcal{M}} = \tilde{\partial}_A\tau\dot{\tilde{\mathcal{M}}} + 3\tilde{\partial}_A\dot{\tau}\tilde{\mathcal{M}} + \tilde{D}_A\tilde{\mathcal{J}}^B\partial_B\tau$$

$$\Delta_\tau^{(1)}C_{AB}\mathcal{J}^B = \frac{1}{2}C_{AB}\dot{N}^{BC}\partial_C\tau = \frac{1}{4}(C_{BC}\dot{N}^{BC})\partial_A\tau + \frac{1}{4}(C_{BC}\dot{\tilde{N}}^{BC})\tilde{\partial}_A\tau\,. \tag{4.19}$$

Moreover, given the identity

$$D_A\mathcal{J}_B + \tilde{D}_A\tilde{\mathcal{J}}_B = q_{AB}(D_C\mathcal{J}^C) + \epsilon_{AB}(\tilde{D}_C\tilde{\mathcal{J}}^C)\,, \tag{4.20}$$

we have

$$\begin{aligned}\Delta_\tau^{(1)}\mathcal{E}_A &\hat{\hat{=}} \left[2\dot{\mathcal{M}} - (D_C\mathcal{J}^C) - \frac{1}{4}(C_{BC}\dot{N}^{BC})\right]\partial_A\tau + \left[2\dot{\tilde{\mathcal{M}}} - (D_C\tilde{\mathcal{J}}^C) - \frac{1}{4}(C_{BC}\dot{\tilde{N}}^{BC})\right]\tilde{\partial}_A\tau \\ &= 2\mathcal{E}\partial_A\tau + 2\tilde{\mathcal{E}}\tilde{\partial}_A\tau = 0\,,\end{aligned} \tag{4.21}$$

on-shell of $\mathcal{E} = 0 = \tilde{\mathcal{E}}$. We thus see that also the equation of motion for the momentum, $\mathcal{E}_A = 0$, can be derived from purely symmetry principles.

4.3 Stress tensor evolution from symmetry

Let us show that the same strategy applies to the spin-2 equation of motion as well. For the spin-2 sphere metric component we have seen in (3.34) that the combination $\mathcal{T}_{AB} = 3\left(E_{AB} - \frac{1}{16}C_{AB}C_{CD}C^{CD}\right)$ has no quadratic anomaly. The goal is two show that the combination

$$\mathcal{E}_{AB} := \dot{\mathcal{T}}_{AB} - D_{\langle A}\mathcal{P}_{B\rangle} - \frac{3}{2}\left(C_{AB}\mathcal{M} + \tilde{C}_{AB}\tilde{\mathcal{M}}\right) \tag{4.22}$$

is also free of anomaly on-shell of the momentum evolution equation $\mathcal{E}_A = 0$ and therefore it determines the tensorial equation of motion. Since the quadratic anomaly of $\dot{\mathcal{T}}_{AB}$ vanishes, one only needs to evaluate the following quadratic anomalies

$$\begin{aligned}\Delta_\tau^{(2)}\left(D_{\langle A}\mathcal{P}_{B\rangle} + \frac{3}{2}\left(C_{AB}\mathcal{M} + \tilde{C}_{AB}\tilde{\mathcal{M}}\right)\right) &\hat{\hat{=}} 3\Big(\tilde{\mathcal{M}}D_{\langle A}\tilde{\partial}_{B\rangle}\tau + \mathcal{M}D_{\langle A}\partial_{B\rangle}\tau \\ &\qquad - \mathcal{M}D_{\langle A}\partial_{B\rangle}\tau - \tilde{\mathcal{M}}\tilde{D}_{\langle A}\partial_{B\rangle}\tau\Big) \\ &= 0\,,\end{aligned} \tag{4.23}$$

where we used the result in appendix B for the anomaly of the quantity $D_A\mathcal{P}_B$, as well as (2.21b).

JHEP04(2022)125

We are thus left to show that the linear anomaly vanishes too. For the same quantities as in (4.23), we have

$$\Delta^{(1)}_\tau\left(D_{\langle A}\mathcal{P}_{B\rangle}\right) \hat{\hat{=}} 4\mathcal{P}_{\langle A}\partial_{B\rangle}\dot{\tau} + \dot{\mathcal{P}}_{\langle A}\partial_{B\rangle}\tau + 3D_{\langle A}\tilde{\mathcal{M}}\tilde{\partial}_{B\rangle}\tau + 3D_{\langle A}\mathcal{M}\partial_{B\rangle}\tau\,, \tag{4.24}$$

$$\Delta^{(1)}_\tau\left(\frac{3}{2}C_{AB}\mathcal{M}\right) \hat{=} \frac{3}{2}C_{AB}\mathcal{J}^C\partial_C\tau\,, \tag{4.25}$$

$$\Delta^{(1)}_\tau\left(\frac{3}{2}\tilde{C}_{AB}\tilde{\mathcal{M}}\right) = \frac{3}{2}\tilde{C}_{AB}\tilde{\mathcal{J}}^C\partial_C\tau\,. \tag{4.26}$$

Therefore, combining these together we have

$$\begin{aligned}
&\Delta^{(1)}_\tau\left(D_{\langle A}\mathcal{P}_{B\rangle} + \frac{3}{2}\left(C_{AB}\mathcal{M} + \tilde{C}_{AB}\tilde{\mathcal{M}}\right)\right)\\
&\hat{\hat{=}}\, 4\mathcal{P}_{\langle A}\partial_{B\rangle}\dot{\tau} + \dot{\mathcal{P}}_{\langle A}\partial_{B\rangle}\tau + 3D_{\langle A}\tilde{\mathcal{M}}\tilde{\partial}_{B\rangle}\tau + 3D_{\langle A}\mathcal{M}\partial_{B\rangle}\tau + \frac{3}{2}C_{AB}\mathcal{J}^C\partial_C\tau + \frac{3}{2}\tilde{C}_{AB}\tilde{\mathcal{J}}^C\partial_C\tau\\
&= 4\partial_u(\mathcal{P}_{\langle A}\partial_{B\rangle}\tau) - 3\left(\dot{\mathcal{P}}_{\langle A} - \tilde{D}_{\langle A}\tilde{\mathcal{M}} + D_{\langle A}\mathcal{M}\right)\partial_{B\rangle}\tau + \frac{3}{2}C_{AB}\mathcal{J}^C\partial_C\tau + \frac{3}{2}\tilde{C}_{AB}\tilde{\mathcal{J}}^C\partial_C\tau\\
&= 4\partial_u(\mathcal{P}_{\langle A}\partial_{B\rangle}\tau) - 3\mathcal{E}_{\langle A}\partial_{B\rangle}\tau.
\end{aligned} \tag{4.27}$$

This was simplified by using the identity

$$C_{AB}\partial_C\tau - \tilde{C}_{AB}\tilde{\partial}_C\tau = C_{C\langle A}\partial_{B\rangle}\tau - \tilde{C}_{C\langle A}\tilde{\partial}_{B\rangle}\tau = 2C_{C\langle A}\partial_{B\rangle}\tau. \tag{4.28}$$

We also used the definition of the momentum equation of motion (4.13). From this, taking the time derivative of (3.34), we conclude that $\Delta^{(1)}_\tau\mathcal{E}_{AB} \hat{\hat{=}} 0$ as well once we use $\mathcal{E}_A = 0$.

4.4 Matter sources

The previous results show that the multiplet $(\mathcal{E}, \tilde{\mathcal{E}}, \mathcal{E}_A, \mathcal{E}_{AB})$ of evolution equations transforms homogeneously under the BMSW symmetry group. More precisely, we have

$$\delta_{(\tau,Y)}\mathcal{E} \hat{=} [\tau\partial_u + \mathcal{L}_Y + 4\dot{\tau}]\mathcal{E}\,, \tag{4.29}$$

$$\delta_{(\tau,Y)}\tilde{\mathcal{E}} = [\tau\partial_u + \mathcal{L}_Y + 4\dot{\tau}]\tilde{\mathcal{E}}\,, \tag{4.30}$$

$$\delta_{(\tau,Y)}\mathcal{E}_A \hat{\hat{=}} [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}]\mathcal{E}_A + 2\mathcal{E}\partial_A\tau + 2\tilde{\mathcal{E}}\tilde{\partial}_A\tau\,, \tag{4.31}$$

$$\delta_{(\tau,Y)}\mathcal{E}_{AB} \hat{\hat{=}} [\tau\partial_u + \mathcal{L}_Y + 2\dot{\tau}]\mathcal{E}_{AB} + 3\mathcal{E}_{\langle A}\partial_{B\rangle}\tau\,. \tag{4.32}$$

We have seen as well that the asymptotic Einstein's equations constraining the metric functions F, U, b transform as

$$\begin{aligned}
\delta_{(\tau,Y)}\mathcal{E}_F &= [\mathcal{L}_Y + 2\dot{\tau}]\,\mathcal{E}_F\,,\\
\delta_{(\tau,Y)}\mathcal{E}_U^A &= [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}]\mathcal{E}_U^A,\\
\delta_{(\tau,Y)}\mathcal{E}_b &= [\tau\partial_u + \mathcal{L}_Y + 2\dot{\tau}]\mathcal{E}_b + \frac{1}{4}\mathcal{E}_U^A\partial_A\tau\,.
\end{aligned} \tag{4.33}$$

This is the first main result of this paper, namely the derivation of asymptotic Einstein's equations at null infinity from the unique demand of constructing tensorial operators starting from the time derivative of the pseudo-tensors $(\mathcal{M}, \tilde{\mathcal{M}}, \mathcal{P}_A, \mathcal{T}_{AB})$. We now want to extend our symmetry argument also to the case where matter sources are present and use it to derive the asymptotic EEs coupled to matter.

Let us start by showing that a transformation structure similar to (4.29) for the asymptotic vacuum Einstein's equations is reproduced by the conservation equations of matter. We consider matter sources with stress-energy tensor (SET) $T_{\mu\nu}$. We restrict our analysis to stress-energy tensor components that preserve the following asymptotic Einstein's equations

$$\mathsf{E}_F \mathrel{\hat{=}} 0 \mathrel{\hat{=}} \mathsf{E}_U^A. \tag{4.34}$$

These conditions mean (see [17], section 8, for more general conditions) that the stress-energy tensor components have the following expansions [24]

$$T_{AB} = \frac{1}{r}\hat{T} q_{AB} + \frac{1}{r^2}\hat{T}_{AB} + o(r^{-2}), \tag{4.35a}$$

$$T_{uu} = \frac{1}{r^2}\hat{T}_{uu} + o(r^{-2}), \qquad T_{uA} = \frac{1}{r^2}\hat{T}_{uA} + o(r^{-2}), \tag{4.35b}$$

$$T_{rA} = \frac{1}{r^3}\hat{T}_{rA} + o(r^{-3}), \qquad T_{ru} = o(r^{-3}), \tag{4.35c}$$

$$T_{rr} = \frac{1}{r^4}\hat{T}_{rr} + o(r^{-3}). \tag{4.35d}$$

As shown in appendix D.1, the asymptotic components of the stress-energy tensor transform as

$$\delta_{(\tau,Y)}\hat{T}_{uu} = [\tau\partial_u + \mathcal{L}_Y + 4\dot{\tau}]\hat{T}_{uu}, \tag{4.36a}$$

$$\delta_{(\tau,Y)}\hat{T} = [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}]\hat{T}, \tag{4.36b}$$

$$\delta_{(\tau,Y)}\hat{T}_{rr} = [\tau\partial_u + \mathcal{L}_Y + 2\dot{\tau}]\hat{T}_{rr}, \tag{4.36c}$$

$$\delta_{(\tau,Y)}\hat{T}_{uA} = [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}]\hat{T}_{uA} - \hat{T}\partial_A\dot{\tau} + \hat{T}_{uu}\partial_A\tau, \tag{4.36d}$$

$$\delta_{(\tau,Y)}\hat{T}_{rA} = [\tau\partial_u + \mathcal{L}_Y + 2\dot{\tau}]\hat{T}_{rA} - \hat{T}_{rr}\partial_A\dot{\tau} + \hat{T}\partial_A\tau, \tag{4.36e}$$

$$\begin{aligned}\delta_{(\tau,Y)}\hat{T}_{\langle AB\rangle} &= [\tau\partial_u + \mathcal{L}_Y + 2\dot{\tau}]\hat{T}_{\langle AB\rangle} \\ &\quad + 2\hat{T}_{u\langle A}\partial_{B\rangle}\tau - 2\hat{T}_{r\langle A}\partial_{B\rangle}\dot{\tau} - 2\hat{T}D_{\langle A}\partial_{B\rangle}\tau,\end{aligned} \tag{4.36f}$$

where the r^{-2} component of T_{AB} is given by $\hat{T}_{AB} := \hat{T}_2 q_{AB} + \hat{T}_{\langle AB\rangle}$. From these transformations we can identify the conservation equations as the combinations that transform as pseudo-tensors.[14] These are (see appendix D.2 for their derivation)

$$\mathcal{C} := \partial_u\hat{T}_{rr} + 2\hat{T}, \qquad \mathcal{C}_A^1 := \partial_u\hat{T}_{rA} - \partial_A\hat{T}, \qquad \mathcal{C}_A^2 := \partial_A\hat{T}_{rr} + 2\hat{T}_{rA}, \tag{4.37}$$

which transform as

$$\begin{aligned}\delta_{(\tau,Y)}\mathcal{C} &= [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}]\mathcal{C}, \\ \delta_{(\tau,Y)}\mathcal{C}_A^1 &= [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}]\mathcal{C}_A^1 - \mathcal{C}\partial_A\dot{\tau}, \\ \delta_{(\tau,Y)}\mathcal{C}_A^2 &= [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}]\mathcal{C}_A^2 + \mathcal{C}\partial_A\tau.\end{aligned} \tag{4.38}$$

[14] Strictly speaking its is the combination $\mathcal{C}_A^1 + \dot{\mathcal{C}}_A^2$ which transforms as a pseudo tensor.

JHEP04(2022)125

The proof follows from

$$\begin{aligned}\delta_{(\tau,Y)}\partial_u\hat{T}_{rA} &= [\tau\partial_u+\mathcal{L}_Y+3\dot{\tau}]\partial_u\hat{T}_{rA}-(\partial_u\hat{T}_{rr}-\hat{T})\partial_A\dot{\tau}+\partial_u\hat{T}\partial_A\tau\,,\\ \delta_{(\tau,Y)}\partial_A\hat{T} &= [\tau\partial_u+\mathcal{L}_Y+3\dot{\tau}]\partial_A\hat{T}+3\hat{T}\partial_A\dot{\tau}+\partial_u\hat{T}\partial_A\tau\,,\\ \delta_{(\tau,Y)}\partial_A\hat{T}_{rr} &= [\tau\partial_u+\mathcal{L}_Y+2\dot{\tau}]\partial_A\hat{T}_{rr}+\dot{\hat{T}}_{rr}\partial_A\tau+2\hat{T}_{rr}\partial_A\dot{\tau}\,.\end{aligned} \tag{4.39}$$

Applying our symmetry argument, one then recovers the result that the conservation equations are given by $\mathcal{C}=0,\mathcal{C}^1_A=0,\mathcal{C}^2_A=0$. Using that the leading order of the Einstein equation component $G_{rr}=T_{rr}$ (see, e.g., [17]) implies

$$T_{rr}=-8\mathsf{E}_b\,, \tag{4.40}$$

we see that the conservation equations mean

$$\hat{T}=4\dot{\mathsf{E}}_b,\qquad \hat{T}_{rA}=4\partial_A\mathsf{E}_b\,. \tag{4.41}$$

The asymptotic EEs coupled to matter read[15]

$$\tilde{\mathcal{E}}=0\,,\quad \mathcal{E}+\mathcal{S}=0\,,\quad \mathcal{E}_A+\mathcal{S}_A=0\,,\quad \mathcal{E}_{AB}+\mathcal{S}_{AB}=0\,, \tag{4.42}$$

where the sources are given by

$$\mathcal{S}:=\frac{1}{2}\hat{T}_{uu},\qquad \mathcal{S}_A:=\hat{T}_{uA}+\frac{1}{2}\partial_A\hat{T},\qquad \mathcal{S}_{AB}:=\frac{3}{2}\hat{T}_{\langle AB\rangle}-\frac{1}{4}D_{\langle A}\partial_{B\rangle}\hat{T}_{rr}-\frac{3}{2}\hat{T}C_{AB}\,. \tag{4.43}$$

The sources $\mathcal{S},\mathcal{S}_A$ above agree with the ones written in [24], once we recall that the momentum N_A used by Flanagan and Nichols relates to our through $N_A=P_A+\partial_A b$; they also agree with the expansion of the Einstein tensor components near null infinity in terms of the asymptotic EEs written in [17]. The source $\mathcal{S}_{AB}$ matches the one written in [64], again after proper translation of the momentum definition.[16]

This form of the sources is derived in appendix D.3, again applying our symmetry argument. More precisely, the expressions (4.43) are obtained by demanding that EEs coupled to matter transform homogeneously under the BMSW symmetry group as

$$\delta_{(\tau,Y)}(\mathcal{E}+\mathcal{S}) \mathrel{\hat{=}} [\tau\partial_u+\mathcal{L}_Y+4\dot{\tau}](\mathcal{E}+\mathcal{S})\,, \tag{4.44}$$

$$\delta_{(\tau,Y)}(\mathcal{E}_A+\mathcal{S}_A) \mathrel{\hat{=}} [\tau\partial_u+\mathcal{L}_Y+3\dot{\tau}](\mathcal{E}_A+\mathcal{S}_A)+2(\mathcal{E}+\mathcal{S})\partial_A\tau+2\tilde{\mathcal{E}}\tilde{\partial}_A\tau\,, \tag{4.45}$$

$$\delta_{(\tau,Y)}(\mathcal{E}_{AB}+\mathcal{S}_{AB}) \mathrel{\hat{=}} [\tau\partial_u+\mathcal{L}_Y+2\dot{\tau}](\mathcal{E}_{AB}+\mathcal{S}_{AB})+3(\mathcal{E}_{\langle A}+\mathcal{S}_{\langle A})\partial_{B\rangle}\tau\,. \tag{4.46}$$

Notice that we have removed the double hat symbol over the equal signs as the E_b asymptotic equation is sourced by the SET component T_{rr} (4.40) and it thus contributes to the definition of the covariant tensors encoding the EEs in the presence of matter.

[15] Given the topological nature of the dual mass charge, the corresponding EE does not acquire a source term.

[16] The expansion of the $\langle AB\rangle$-component of the Einstein tensor containing $\dot{\mathcal{E}}_{AB}$ was not completed in [17], thus we cannot compare the expression of the $\mathcal{S}_{AB}$ source with that reference.

5 Properties of the covariant observables

Let us now summarize the covariance properties that we have revealed so far and the nested structure that organizes them. The covariant observables are the radiative observable

$$\mathcal{N}^{AB} := \dot{N}^{AB}\,, \tag{5.1}$$

corresponding to the truly free data at $\mathcal{I}^+$ encoding the two polarizations of the outgoing gravitational radiation, and the corner observables $(\mathcal{J}^A, \mathcal{M}, \tilde{\mathcal{M}}, \mathcal{P}_A, \mathcal{T}_{AB})$ defined as

$$\mathcal{J}^A := \frac{1}{2} D_B N^{AB} + \frac{1}{4}\partial^A R(q)\,, \tag{5.2a}$$

$$\mathcal{M} := M + \frac{1}{8} N^{AB} C_{AB}\,, \tag{5.2b}$$

$$\tilde{\mathcal{M}} := \frac{1}{4} D_B (D\cdot\tilde{C})^B + \frac{1}{8}\tilde{N}^{AB} C_{AB}\,, \tag{5.2c}$$

$$\mathcal{P}_A := P_A + \frac{1}{16} D_A (C^{BC} C_{BC}) + \frac{1}{4} C_{AB} (D\cdot C)^B\,, \tag{5.2d}$$

$$\mathcal{T}_{AB} := 3\left(E_{AB} - \frac{1}{16} C_{AB} (C_{CD} C^{CD})\right)\,, \tag{5.2e}$$

and corresponding to initial data. They transform as follows

$$\delta_{(\tau,Y)}\mathcal{N}^{AB} = [\tau\partial_u + \mathcal{L}_Y + 5\dot{\tau}]\,\mathcal{N}^{AB}, \tag{5.3a}$$

$$\delta_{(\tau,Y)}\mathcal{J}^A = [\tau\partial_u + \mathcal{L}_Y + 4\dot{\tau}]\,\mathcal{J}^A + \frac{1}{2}\mathcal{N}^{AB}\partial_B\tau, \tag{5.3b}$$

$$\delta_{(\tau,Y)}\mathcal{M} \,\hat{=}\, [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}]\,\mathcal{M} + \mathcal{J}^A\partial_A\tau, \tag{5.3c}$$

$$\delta_{(\tau,Y)}\tilde{\mathcal{M}} = [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}]\tilde{\mathcal{M}} + \tilde{\mathcal{J}}^A\partial_A\tau, \tag{5.3d}$$

$$\delta_{(\tau,Y)}\mathcal{P}_A \,\hat{\hat{=}}\, [\tau\partial_u + \mathcal{L}_Y + 2\dot{\tau}]\,\mathcal{P}_A + 3\left(\mathcal{M} q_{AB} + \tilde{\mathcal{M}}\epsilon_{AB}\right)\partial^B\tau, \tag{5.3e}$$

$$\delta_{(\tau,Y)}\mathcal{T}_{AB} \,\hat{\hat{=}}\, [\tau\partial_u + \mathcal{L}_Y + \dot{\tau}]\,\mathcal{T}_{AB} + 4\mathcal{P}_{\langle A}\partial_{B\rangle}\tau\,. \tag{5.3f}$$

These transformation properties show a clear pattern where the linear anomaly of each covariant quantity of conformal dimension s is uniquely determined by the ones of conformal dimension $s-1$.

The requirement of covariance under the full BMSW group gives the equations of motion

$$\dot{\mathcal{J}}^A = \frac{1}{2} D_B \mathcal{N}^{AB}\,, \tag{5.4a}$$

$$\dot{\mathcal{M}} = \frac{1}{2} D_A \mathcal{J}^A + \frac{1}{8} C_{AB}\mathcal{N}^{AB}\,, \tag{5.4b}$$

$$\dot{\tilde{\mathcal{M}}} = \frac{1}{2} D_A \tilde{\mathcal{J}}^A + \frac{1}{8} C_{AB}\tilde{\mathcal{N}}^{AB}\,, \tag{5.4c}$$

$$\dot{\mathcal{P}}_A = D_A\mathcal{M} + \tilde{D}_A\tilde{\mathcal{M}} + C_{AB}\mathcal{J}^B\,, \tag{5.4d}$$

$$\dot{\mathcal{T}}_{AB} = D_{\langle A}\mathcal{P}_{B\rangle} + \frac{3}{2}\left(C_{AB}\mathcal{M} + \tilde{C}_{AB}\tilde{\mathcal{M}}\right). \tag{5.4e}$$

We can now combine the action on the soft observable with the equations of motion to write down the action of the symmetry group on the *corner phase space* variables. That is, the phase space obtained after imposition of the constraint equations. This gives us the on-shell actions

$$\delta_{(\tau,Y)}\mathcal{J}^A = [\mathcal{L}_Y + 4\dot{\tau}]\,\mathcal{J}^A + \frac{1}{2}\mathcal{N}^{AB}\partial_B\tau + \frac{\tau}{2}D_B\mathcal{N}^{AB}, \tag{5.5a}$$

$$\delta_{(\tau,Y)}\mathcal{M} \hat{=} [\mathcal{L}_Y + 3\dot{\tau}]\,\mathcal{M} + \mathcal{J}^A\partial_A\tau + \frac{\tau}{2}\left(D_A\mathcal{J}^A + \frac{1}{4}C_{AB}\mathcal{N}^{AB}\right), \tag{5.5b}$$

$$\delta_{(\tau,Y)}\tilde{\mathcal{M}} = [\mathcal{L}_Y + 3\dot{\tau}]\tilde{\mathcal{M}} + \tilde{\mathcal{J}}^A\partial_A\tau + \frac{\tau}{2}\left(D_A\tilde{\mathcal{J}}^A + \frac{1}{4}C_{AB}\tilde{\mathcal{N}}^{AB}\right), \tag{5.5c}$$

$$\delta_{(\tau,Y)}\mathcal{P}_A \hat{\hat{=}} [\mathcal{L}_Y + 2\dot{\tau}]\,\mathcal{P}_A + 3(\mathcal{M}\partial_A\tau + \tilde{\mathcal{M}}\tilde{\partial}_A\tau) + \tau\left(\partial_A\mathcal{M} + \tilde{\partial}_A\tilde{\mathcal{M}} + C_{AB}\mathcal{J}^B\right), \tag{5.5d}$$

$$\delta_{(\tau,Y)}\mathcal{T}_{AB} \hat{\hat{\hat{=}}} [\mathcal{L}_Y + \dot{\tau}]\,\mathcal{T}_{AB} + 4\mathcal{P}_{\langle A}\partial_{B\rangle}\tau + \tau\left(D_{\langle A}\mathcal{P}_{B\rangle} + \frac{3}{2}C_{AB}\mathcal{M} + \frac{3}{2}\tilde{C}_{AB}\tilde{\mathcal{M}}\right). \tag{5.5e}$$

This information is enough to recover the equations of motion which are given by

$$\dot{\mathcal{J}}^A = \delta_{(1,0)}\mathcal{J}^A\,, \quad \dot{\mathcal{M}} = \delta_{(1,0)}\mathcal{M}\,, \qquad \dot{\mathcal{P}}_A = \delta_{(1,0)}\mathcal{P}_A\,, \quad \dot{\mathcal{T}}_{AB} = \delta_{(1,0)}\mathcal{T}_{AB}\,, \tag{5.6}$$

where $\tau = 1$ denotes the constant function on the sphere.

5.1 Non-radiative phase space

The non-radiative phase space is obtained after imposing the no-radiation condition[17]

$$\mathcal{N}^{AB} = 0\,. \tag{5.7}$$

This means that we have

$$C_{AB} = un_{AB} + c_{AB}\,, \tag{5.8}$$

where n_{AB} and c_{AB} are time independent. Once the no-radiation condition is imposed, we can solve the evolution equations (5.4) explicitly as follows (we use the label NR to denote

[17] The fact that this is a correct quantity to set to zero in the absence of radiation follows from the fact the it transforms covariantly under the BMSW transformations (as well as under the standard BMS group). Our choice of no-radiation condition is motivated by the demand that the corner charges be conserved in time, as shown below, and it corresponds to a Weyl tensor of Petrov type I. However, one can argue that this choice is conventional. Another accepted choice is to call radiation $\text{Im}\Psi_2, \Psi_3, \Psi_4$ since they depends explicitely on the shear and its time derivative only and to include in the charges $\text{Re}\Psi_2, \Psi_1, \Psi_0$, which represent independent data. This is the choice adopted in [25], where it was demanded the vanishing of the shifted news tensor (3.7), also transforming with no anomaly, and of the dual mass. In this second perspective the conventional notion of no radiation would be to impose that $\text{Im}\Psi_2, \Psi_3, \Psi_4$ vanish, with a Weyl tensor of type II. Ultimately understanding what is the right notion of no-radiation should be related to whether the charges not included in the radiation are not only conserved but also form a coadjoint orbit. So far it only has been proven that the even stronger non-radiative condition $\text{Im}\Psi_2 = \Psi_3 = \Psi_4 = \Psi_0 = 0$, which is of Petrov type D, forms a coadjoint orbit (see e.g. [49]). Understanding whether the no-radiation condition we work with here qualifies under this criterion still needs to be investigated.

JHEP04(2022)125

the non-radiative solutions)

$$\mathcal{J}_A^{\rm NR} = j_A\,, \tag{5.9a}$$

$$\mathcal{M}^{\rm NR} = m + \frac{u}{2} D_A j^A\,, \tag{5.9b}$$

$$\tilde{\mathcal{M}}^{\rm NR} = \tilde{m} + \frac{u}{2} D_A \tilde{j}^A\,, \tag{5.9c}$$

$$\mathcal{P}_A^{\rm NR} = 2p_A + u(D_A m + \tilde{D}_A \tilde{m} + c_{AB} j^B) + \frac{u^2}{2}(D_{\langle A} D_{B\rangle} + n_{AB}) j^B\,, \tag{5.9d}$$

$$\begin{aligned}\mathcal{T}_{AB}^{\rm NR} &= 3t_{AB} + u\left(2D_{\langle A} p_{B\rangle} + \frac{3}{2}(m c_{AB} + \tilde{m}\tilde{c}_{AB})\right) \\ &\quad + \frac{3u^2}{2}\left(\left[\frac{1}{3} D_{\langle A} D_{B\rangle} + \frac{1}{2} n_{AB}\right] m + \left[\frac{1}{3} D_{\langle A} \tilde{D}_{B\rangle} + \frac{1}{2}\tilde{n}_{AB}\right]\tilde{m} + \frac{1}{3} D_{\langle A} c_{B\rangle C} j^C + \frac{5}{6} c_{C\langle A} D_{B\rangle} j^C\right) \\ &\quad + u^3\left(\frac{1}{6} D_{\langle A} D_B D_{C\rangle} j^C + \frac{1}{6} D_{\langle A} n_{B\rangle C} j^C + \frac{2}{3} n_{C\langle A} D_{B\rangle} j^C\right),\end{aligned} \tag{5.9e}$$

where $(j_A, m, \tilde{m}, p_A, t_{AB})$ are constant tensors on the sphere. In fact, given the evolution equations (5.4a), (5.4b), (5.4c), it is immediate to see that $\dot{j}^A = \dot{m} = \dot{\tilde{m}} = 0$. For p_A and t_{AB} the proof goes as follows. From the evolution equation (5.4d) and the expansions (5.9b), (5.9c), we get

$$\begin{aligned}\dot{\mathcal{P}}_A &= D_A m + \tilde{D}_A \tilde{m} + c_{AB}\mathcal{J}^B + \frac{u}{2}(D_A D_B j^B - \tilde{D}_A \tilde{D}_B j^B + 2n_{AB}\mathcal{J}^B) \\ &= (D_A m + \tilde{D}_A \tilde{m} + c_{AB}\mathcal{J}^B) + u(D_{\langle A} D_{B\rangle} + n_{AB}) j^B\,,\end{aligned} \tag{5.10}$$

where we used the identity

$$D_A D_B - \tilde{D}_A \tilde{D}_B = 2D_{\langle A} D_{B\rangle}\,. \tag{5.11}$$

Next, from the evolution equation (5.4e) and also the expansion (5.9d), we get

$$\begin{aligned}\dot{\mathcal{T}}_{AB} &= 2D_{\langle A} p_{B\rangle} + u(D_{\langle A} D_{B\rangle} m + D_{\langle A} \tilde{D}_{B\rangle}\tilde{m} + D_{\langle A}[c_{B\rangle C} j^C]) \\ &\quad + \frac{u^2}{2}(D_{\langle A} D_B D_{C\rangle} j^C + D_{\langle A}[n_{B\rangle C} j^C]) \\ &\quad + \frac{3}{2} m c_{AB} + \frac{3}{2}\tilde{m}\tilde{c}_{AB} + \frac{3u}{2}\left(\frac{1}{2} c_{AB} D_C j^C + \frac{1}{2}\tilde{c}_{AB} D_C \tilde{j}^C + m n_{AB} + \tilde{m}\tilde{n}_{AB}\right) \\ &\quad + \frac{3u^2}{4}(n_{AB} D_C j^C + \tilde{n}_{AB} D_C \tilde{j}^C)\,,\end{aligned} \tag{5.12}$$

and we use that

$$\frac{1}{4} c_{AB} D_C j^C + \frac{1}{4}\tilde{c}_{AB} D_C \tilde{j}^C = \frac{1}{2} c_{C\langle A} D_{B\rangle} j^C \tag{5.13}$$

to arrive at

$$\begin{aligned}\dot{\mathcal{T}}_{AB} &= 2D_{\langle A} p_{B\rangle} + \frac{3}{2}(m c_{AB} + \tilde{m}\tilde{c}_{AB}) \\ &\quad + u\left(\left[D_{\langle A} D_{B\rangle} + \frac{3}{2} n_{AB}\right] m + \left[D_{\langle A}\tilde{D}_{B\rangle} + \frac{3}{2}\tilde{n}_{AB}\right]\tilde{m} + D_{\langle A} c_{B\rangle C} j^C + \frac{5}{2} c_{C\langle A} D_{B\rangle} j^C\right) \\ &\quad + u^2\left(\frac{1}{2} D_{\langle A} D_B D_{C\rangle} j^C + \frac{1}{2} D_{\langle A} n_{B\rangle C} j^C + 2n_{C\langle A} D_{B\rangle} j^C\right).\end{aligned} \tag{5.14}$$

The conserved quantities $(j^A, m, \tilde{m}, p_A, t_{AB})$ represent the charges parametrizing the non-radiative corner phase space. They encode on $\mathcal{I}$ the physical content of the spacetime. We come back to this important point in a moment.

Let us first remark that, in the asymptotic analysis of symmetry, it is often customary to define a stronger version of the no radiation condition to be specified by

$$\mathcal{N}^{AB} = 0\,, \qquad \mathcal{J}^A = 0\,, \tag{5.15}$$

and the spacetime is said to be strongly non-radiative. The second condition means that $n_{AB} + \frac{1}{2}Rq_{AB}$ is the Liouville energy-momentum tensor. In this case, the conservation equations look simpler, with $\mathcal{M}^{\rm NR} = m$ and $\tilde{\mathcal{M}}^{\rm NR} = \tilde{m}$ independent of time and

$$\mathcal{P}_A^{\rm NR} = 2p_A + u(D_A m + \tilde{D}_A\tilde{m})\,, \tag{5.16}$$

$$\begin{aligned}\mathcal{T}_{AB}^{\rm NR} &= 3t_{AB} + u\left(2D_{\langle A}p_{B\rangle} + \frac{3}{2}(mc_{AB} + \tilde{m}\tilde{c}_{AB})\right)\\ &\quad + \frac{u^2}{2}\left(\left[D_{\langle A}D_{B\rangle} + \frac{3}{2}n_{AB}\right]m + \left[D_{\langle A}\tilde{D}_{B\rangle} + \frac{3}{2}\tilde{n}_{AB}\right]\tilde{m}\right).\end{aligned} \tag{5.17}$$

5.2 Symmetry transformations

Under a symmetry transformation the shear components transform as

$$\delta_{(T,W,Y)}c_{AB} = [\mathcal{L}_Y - W]\,c_{AB} - \left[2D_{\langle A}D_{B\rangle} - n_{AB}\right]T\,, \tag{5.18}$$

$$\delta_{(T,W,Y)}n_{AB} = \mathcal{L}_Y n_{AB} - 2D_{\langle A}D_{B\rangle}W\,, \tag{5.19}$$

while the symmetry transformations of the conserved charge aspects are given by

$$\delta_{(T,W,Y)}j^A = [\mathcal{L}_Y + 4W]\,j^A\,, \tag{5.20a}$$

$$\delta_{(T,W,Y)}m \mathrel{\hat{=}} [\mathcal{L}_Y + 3W]\,m + j^A\partial_A T + \frac{T}{2}D_A j^A\,, \tag{5.20b}$$

$$\delta_{(T,W,Y)}\tilde{m} = [\mathcal{L}_Y + 3W]\tilde{m} + \tilde{j}^A\partial_A T + \frac{T}{2}D_A\tilde{j}^A\,, \tag{5.20c}$$

$$\delta_{(T,W,Y)}p_A \mathrel{\hat{\hat{=}}} [\mathcal{L}_Y + 2W]\,p_A + \frac{3}{2}(m\partial_A T + \tilde{m}\tilde{\partial}_A T) + \frac{T}{2}\left(\partial_A m + \tilde{\partial}_A\tilde{m} + c_{AB}j^B\right)\,, \tag{5.20d}$$

$$\delta_{(T,W,Y)}t_{AB} \mathrel{\hat{\hat{=}}} [\mathcal{L}_Y + W]\,t_{AB} + \frac{8}{3}p_{\langle A}\partial_{B\rangle}T + T\left(\frac{2}{3}D_{\langle A}p_{B\rangle} + \frac{1}{2}c_{AB}m + \frac{1}{2}\tilde{c}_{AB}\tilde{m}\right)\,. \tag{5.20e}$$

These transformation properties represent the second main result of the paper. There are two important aspects related to them we now highlight.

First, we see that the conserved charge aspects parametrizing the corner phase space transform homogeneously for the asymptotic corner symmetry group (see section 2.3) where $T = 0$. Second, and most importantly, the transformations (5.20) are conjectured to define a moment map between the corner phase space at $\mathcal{I}$ and the dual Lie algebra of the extended corner symmetry group of null infinity. This fundamental conjecture, that will be investigated in [68], gives a precise meaning to our claim above that the conserved charges $(j^A, m, \tilde{m}, p_A, t_{AB})$ parametrize the corner phase space at $\mathcal{I}$.

6 An impulsive wave solution

Now that we have found the non-radiative solutions $\mathcal{N}_{AB} = 0$ we investigate the nature of the non-linear impulsive solutions that describe the fundamental transitions among vacua. An impulsive gravitational wave or gravitational impulse is an *exact* solution of the vacuum Einstein's equations of motion. Their study goes back to the work of Aichelburg and Sexl [52] and of Szekeres, Khan and Penrose [53, 54]. Their mathematical study through a cut and paste approach started with the work of Penrose [55]. The study of spherical impulsive waves has continued and followed many formal mathematical developments since then, see [56–60], supplemented by the study of their collisions [60, 69, 70] and the relationship with the memory effects [71, 72]. It is important to appreciate that impulsive waves that are asymptotically flat have to be spherical; this excludes the extensively studied pp-waves, which are planar.

A gravitational impulse is analogous to the gravitational shock wave studied thoroughly by Dray and 't Hooft [73–75], in the sense that they are, by definition, solutions of the Einstein's equations that produce radiation localized on a null hypersurface, the hypersurface $u =$ cste. Gravitational impulses[18] are fundamentally different in nature from shock waves though, in the sense that a shock wave needs a non-vanishing energy-momentum source while a gravitational impulse does not need any energy-momentum source. Gravitational impulses are made of pure geometry. It is interesting to realize that the gravitational impulse solution we are constructing here is a solution of full non-linear gravity. Its linearization is related to the so called gravitational soft mode introduced by Strominger et al. in [76, 77] and studied more thoroughly in [78–83]. Let us also point out that plane-fronted gravitational impulses were considered in [84] as solutions to the gluing conditions between interfaces of bounded finite regions in a discrete gravitational context.

6.1 Impulsive wave phase space

A shock wave localized at $u = 0$ describes the transition between an initial vacuum labelled by $O^- = (c^-_{AB}, n^-_{AB}, \mathcal{J}^{A-}, \mathcal{M}^-, \tilde{\mathcal{M}}^-, \mathcal{P}^-_A, \mathcal{T}^-_{AB})$ to the corresponding out vacuum labelled by O^+. By definition a non-expanding gravitational impulse solution satisfies[19]

$$\begin{aligned} C_{AB} &= (c^+_{AB} + un^+_{AB})\theta(u) + (c^-_{AB} + un^-_{AB})\theta(-u), \\ &= c_{AB} + un_{AB} + (\mathring{c}_{AB} + u\mathring{n}_{AB})\epsilon(u)\,, \end{aligned} \tag{6.1}$$

where we denote $\mathring{c}_{AB} := (c^+_{AB} - c^-_{AB})$ the jump across the impulse and $c_{AB} := \frac{1}{2}(c^+_{AB} + c^-_{AB})$ the average value (similarly for $\mathring{n}_{AB}$ and n_{AB}). In the last line above, we have introduced the step function

$$\epsilon(u) := \frac{1}{2}[\theta(u) - \theta(-u)], \qquad \dot{\epsilon}(u) = \delta(u)\,, \tag{6.2}$$

where $\dot{\delta}(u) := \partial_u \delta(u)$ is the derived delta function.

[18] Here we defer from the accepted nomenclature of Penrose [55] who calls an impulsive gravitational wave a gravitational wave whose metric is continuous but not C_1 on some (null) hypersurface, while shock waves refer, for him, to metrics which are C_1. The curvature tensor of an impulsive gravitational wave is proportional to a delta function while the curvature tensor of a shock wave is proportional to a step function.

[19] We used that $\theta(u) + \theta(-u) = 1$.

An impulsive wave corresponds to the choice where the induced metric is continuous. This means that we impose $\mathring{c}_{AB} = 0$. This condition is necessary in order to ensure that the energy flux is finite.[20] This continuity condition means that

$$N_{AB} = n_{AB} + \mathring{n}_{AB}\epsilon(u)\,, \tag{6.3}$$
$$\mathcal{N}_{AB} = \mathring{n}_{AB}\delta(u)\,, \tag{6.4}$$

and we see that the Weyl tensor component $\mathcal{N}_{AB}$ is proportional to a delta function.

We can now easily integrate out the evolution equations and express the evolution of the covariant quantities $(\mathcal{J}^A, \mathcal{M}, \tilde{\mathcal{M}}, \mathcal{P}_A, \mathcal{T}_{AB})$ in terms of the conserved quantities $(j_A, m, \tilde{m}, p_A, t_{AB})$, which have the property that they are constant in time in the non-radiative zone $u < 0$ and $u > 0$ before and after the gravitational impulse, and the impulse strength $\mathring{n}_{AB}$.

Let us start with the current $\mathcal{J}^A$. It is immediate to see that the solution to (5.4a) can be written as

$$\begin{aligned}\mathcal{J}^A(u) &= \mathcal{J}_A^{\rm NR} + \mathcal{J}_A^{\rm R1}\\ &= \frac{1}{4}D^A R(q) + \frac{1}{2}D_B n^{AB} + \frac{1}{2}\epsilon(u) D_B \mathring{n}^{AB}\,,\end{aligned} \tag{6.5}$$

where we made explicit its structure as a sum of the non-radiative solution (5.9a) and a distributional radiative component linear in $\mathring{n}_{AB}$.

To get the mass evaluation one integrates the evolution equation (5.4b)

$$\dot{\mathcal{M}} = \frac{1}{2}D_A\mathcal{J}^A + \frac{1}{8}C_{AB}\mathcal{N}^{AB}. \tag{6.6}$$

The first term can be easily integrated if one uses that

$$\partial_u[u\epsilon(u)] = \epsilon(u) + u\delta(u) = \epsilon(u). \tag{6.7}$$

The product $C_{AB}\mathcal{N}^{AB}$ contains product of distribution which are evaluated using

$$u\delta(u) = 0, \qquad \epsilon(u)\delta(u) = 0, \qquad u\epsilon(u)\dot{\delta}(u) = 0, \qquad \epsilon(u)\delta(u) = 0. \tag{6.8}$$

Explicitly, we get[21]

$$C_{AB}\mathcal{N}^{AB} = [c_{AB} + un_{AB} + u\epsilon(u)\mathring{n}_{AB}]\mathring{n}^{AB}\delta(u) = c_{AB}\mathring{n}^{AB}\delta(u) = \partial_u[c_{AB}\mathring{n}^{AB}\epsilon(u)]. \tag{6.9}$$

This means that we obtain the solution

$$\begin{aligned}\mathcal{M}(u) &= \mathcal{M}^{\rm NR} + \mathcal{M}^{\rm R1}\\ &= m + \frac{u}{2}D_A j^A + \frac{1}{8}\epsilon(u)\, c_{AB}\mathring{n}^{AB} + \frac{1}{4}u\epsilon(u)\, D_A D_B\mathring{n}^{AB}.\end{aligned} \tag{6.10}$$

[20] When $\mathring{c}_{AB} \neq 0$, we have that $N_{AB}N^{AB} = \mathring{c}_{AB}\mathring{c}^{AB}\delta(u)^2 + \cdots$ which is ill defined.

[21] We use the following regularization

$$\delta(u)\theta(u) = \frac{1}{2}\delta(u), \qquad \delta(u)\theta(-u) = \frac{1}{2}\delta(-u).$$

We see that the covariant mass is the sum of a non-distributional component $\mathcal{M}^{\mathrm{NR}}$, which agrees with the non-radiative expression (5.9b), and a distributional radiative component $\mathcal{M}^{\mathrm{R1}}$, which is linear in the impulse radiative news $\mathring{n}_{AB}$. A similar analysis gives the expression for the dual mass in the presence of an impulse

$$
\begin{aligned}
\tilde{\mathcal{M}}(u) &= \tilde{\mathcal{M}}^{\mathrm{NR}} + \tilde{\mathcal{M}}^{\mathrm{R1}} \\
&= \tilde{m} + \frac{u}{2}\, D_A \tilde{j}^A + \frac{1}{8}\epsilon(u)\, c_{AB} \mathring{\tilde{n}}^{AB} + \frac{1}{4} u\epsilon(u)\, D_A D_B \mathring{\tilde{n}}^{AB}.
\end{aligned}
\tag{6.11}
$$

The expression for the covariant momentum can be obtained by integrating out (5.4d)

$$
\dot{\mathcal{P}}_A = D_A \mathcal{M} + \tilde{D}_A \tilde{\mathcal{M}} + C_{AB} \mathcal{J}^B \,. \tag{6.12}
$$

To perform the integration one uses the expansion

$$
C_{AB}\mathcal{J}^B = [c_{AB} + u n_{AB}] j^B + \frac{1}{2}\epsilon(u) c_{AB} D_C \mathring{n}^{CB} + u\epsilon(u) \left[\frac{1}{2} n_{AB} D_C \mathring{n}^{CB} + \mathring{n}_{AB} j^B\right] + \frac{1}{2} u \mathring{n}_{AB} D_C \mathring{n}^{CB} , \tag{6.13}
$$

where we used that $\epsilon^2(u) = 1$ as a distribution. The final expression is given by

$$
\begin{aligned}
\mathcal{P}_A &= \mathcal{P}_A^{\mathrm{NR}} + \mathcal{P}_A^{\mathrm{R1}} + \mathcal{P}_A^{\mathrm{R2}} \\
&= 2p_A + u(D_A m + \tilde{D}_A \tilde{m} + c_{AB} j^B) + \frac{u^2}{2}(D_{\langle A} D_{B\rangle} + n_{AB}) j^B \\
&\quad + \frac{1}{8} u\epsilon(u) \left[D_A(c_{BC}\mathring{n}^{BC}) + D_A(c_{BC}\mathring{\tilde{n}}^{BC}) + 4 c_{AB} D_C \mathring{n}^{CB}\right] \\
&\quad + \frac{1}{8} u^2 \epsilon(u) \left[D_A D_B D_C \mathring{n}^{BC} + D_A D_B D_C \mathring{\tilde{n}}^{BC} + 2 n_{AB} D_C \mathring{n}^{CB} + 4 \mathring{n}_{AB} j^B\right] \\
&\quad + \frac{u^2}{4} \mathring{n}_{AB} D_C \mathring{n}^{CB} \,.
\end{aligned}
\tag{6.14}
$$

We see that the covariant momentum is the sum of the non-radiative expression $\mathcal{P}_A^{\mathrm{NR}}$ given in (5.9d), a distributional expression $\mathcal{P}_A^{\mathrm{R1}}$ proportional to the impulse news $\mathring{n}_{AB}$ and a secular component $\mathcal{P}_A^{\mathrm{R2}}$ quadratic in the impulse news.

Finally, the expression for the stress tensor can be obtained by integrating out (5.4e)

$$
\dot{\mathcal{T}}_{AB} = D_{\langle A} \mathcal{P}_{B\rangle} + \frac{3}{2}\left(C_{AB}\mathcal{M} + \tilde{C}_{AB}\tilde{\mathcal{M}}\right) . \tag{6.15}
$$

A similar analysis shows that the solution can be written as

$$
\mathcal{T}_{AB} = \mathcal{T}_{AB}^{\mathrm{NR}} + \mathcal{T}_{AB}^{\mathrm{R1}} + \mathcal{T}_{AB}^{\mathrm{R2}} , \tag{6.16}
$$

showing that the stress tensor can be decomposed into a non-radiative component $\mathcal{T}_{AB}^{\mathrm{NR}}$, a distributional radiative component $\mathcal{T}_{AB}^{\mathrm{R1}}$ linear in $\mathring{n}_{AB}$ and a secular radiative component $\mathcal{T}_{AB}^{\mathrm{R2}}$ quadratic in $\mathring{n}_{AB}$. The non-radiative component is already given in (5.9e). The distributional radiative component reads

$$
\begin{aligned}
\mathcal{T}^{\mathrm{R1}}_{AB}=&\frac{3}{16}u\epsilon(u)\left[c_{AB}c_{CD}\mathring{n}^{CD}+\tilde{c}_{AB}c_{CD}\mathring{\tilde{n}}^{CD}\right]\\
&+\frac{1}{16}u^2\epsilon(u)\Big[D_{\langle A}\big(D_{B\rangle}(c_{CD}\mathring{n}^{CD})+D_{B\rangle}(c_{CD}\mathring{\tilde{n}}^{CD})+4c_{B\rangle C}D_D\mathring{n}^{DC}\big)\\
&+3(c_{AB}D_CD_D\mathring{n}^{CD}+\tilde{c}_{AB}D_CD_D\mathring{\tilde{n}}^{CD})+\frac{3}{2}(n_{AB}c_{CD}\mathring{n}^{CD}+\tilde{n}_{AB}c_{CD}\mathring{\tilde{n}}^{CD})+12(m\mathring{n}_{AB}+\tilde{m}\mathring{\tilde{n}}_{AB})\Big]\\
&+\frac{1}{24}u^3\epsilon(u)\Big[D_{\langle A}\big(D_{B\rangle}D_CD_D\mathring{n}^{CD}+D_{B\rangle}D_CD_D\mathring{\tilde{n}}^{CD}+2n_{B\rangle C}D_D\mathring{n}^{CD}+4\mathring{n}_{B\rangle C}j^C\big)\\
&+3(n_{AB}D_CD_D\mathring{n}^{CD}+\tilde{n}_{AB}D_CD_D\mathring{\tilde{n}}^{CD})+6(D_Cj^C\mathring{n}_{AB}+D_C\tilde{j}^C\mathring{\tilde{n}}_{AB})\Big],
\end{aligned}
\tag{6.17}
$$

while the secular radiative components is

$$
\begin{aligned}
\mathcal{T}^{\mathrm{R2}}_{AB}=&\frac{3u^2}{32}\left[\mathring{n}_{AB}c_{CD}\mathring{n}^{CD}+\mathring{\tilde{n}}_{AB}c_{CD}\mathring{\tilde{n}}^{CD}\right]\\
&+\frac{u^3}{8}\left[\mathring{n}_{AB}D_CD_D\mathring{n}^{CD}+\mathring{\tilde{n}}_{AB}D_CD_D\mathring{\tilde{n}}^{CD}+\frac{2}{3}D_{\langle A}(\mathring{n}_{B\rangle C}D_D\mathring{n}^{DC})\right].
\end{aligned}
\tag{6.18}
$$

6.2 Recovering Penrose's solution

The solution first described by Penrose in [55], and obtained by a holomorphic gluing along a null-cone of two portions of flat space, is a particular example of the construction we have just given. Penrose's solution can be revealed by imposing that

$$
c_{AB}=0, \qquad D_B\mathring{n}^{AB}=0. \tag{6.19}
$$

Under these conditions, we see that the radiative components of the current, mass, momentum and stress tensor all vanish

$$
\mathcal{J}^{\mathrm{R}}_A=0\,, \qquad \mathcal{M}^{\mathrm{R}}=0=\tilde{\mathcal{M}}^{\mathrm{R}}\,, \qquad \mathcal{P}^{\mathrm{R}}_A=0\,, \qquad \mathcal{T}^{\mathrm{R}}_{AB}=0\,. \tag{6.20}
$$

The Penrose's solution is characterized by demanding that the non-radiative component is also flat. This means that the only non-vanishing component is the radiative one $\mathcal{N}^{AB}=\mathring{n}^{AB}\delta(u)$. This solution is integrable in the bulk exactly. It is obtained by patching up two flat space solutions

$$
\mathrm{d}s^2=-2\mathrm{d}u\mathrm{d}r+\mathrm{d}u^2+\frac{4r^2}{(1+|z|^2)^2}\mathrm{d}z\mathrm{d}\bar{z}, \tag{6.21}
$$

along the sphere at $u=0$. The key element is to recognize that the asymptotic news can be written as a Schwarzian derivative

$$
\mathring{n}_{zz}=\{h,z\}=\frac{h'''}{h'}-\frac{3}{2}\left(\frac{h''}{h'}\right)^2, \qquad \mathring{n}_{\bar{z}\bar{z}}=\{\bar{h},\bar{z}\}\,, \tag{6.22}
$$

where h is holomorphic. The full solution can then be obtained by the following matching condition at $u=0$

$$
(r,z,\bar{z})_+=\left(\frac{r}{|h'|}\frac{1+|z|^2}{1+|h|^2},h(z),\bar{h}(\bar{z})\right)_-. \tag{6.23}
$$

7 Conclusions

Exploiting the BMSW extension [17] of the residual diffeomorphism symmetry of null infinity, we have constructed in section 3 charges associated to all the Weyl scalars and that transform semi-covariantly, i.e. with only linear anomaly appearing, under the action of the BMSW group. The characterization of the full phase space of $\mathcal{I}$ led us to the introduction of a duality transform and in particular to the definition of the dual covariant mass (3.21). We have shown in section 4 how the sole demand of anomaly freedom is enough to recover the asymptotic Einstein's equations coupled to matter, written as evolution equations for the covariant charges, by identifying the quantities that transform homogeneously under the symmetry transformations.

This derivation of the gravitational dynamics from purely a symmetry principle highlights the central role of the extended corner symmetry algebra, revealed in [14, 15], in providing a local holographic description of gravity. In particular, borrowing the terminology from representation theory, we have seen how the evolution equations can be understood as *intertwiners* for the BMSW group, as they imply that a given combination is left invariant by the action of the this group, whose Lie algebra represents a subalgebra of the extended corner symmetry one [15].

More precisely, our derivation of the asymptotic evolution Einstein's equations as the functionals of the gravitational phase space variables left invariant by the asymptotic symmetry group opens a new way to think about the quantization of gravitational dynamics in terms of representation theory structures associated to the quantization of this group. Among these, the intertwiner space represents the subspace of invariant tensors in the tensor product of a given set of irreducible representations of the quantum symmetry algebra. One can then envisage a regularization procedure where a notion of intertwiner can be used to fuse tensor products of irreducible representations associated to corners at consecutive instants of time at $\mathcal{I}$, so that a quantum version of constraint equations is holographically implemented. In order for this strategy to correctly capture the gravitational dynamics at the quantum level it is crucial to identify a basis where the propagating degrees of freedom of the radiation for general spacetimes can be represented explicitly and possibly in a non-perturbative manner.

Within this program of describing the gravitational dynamics starting from the representation of the corner symmetry group, we have taken here a further step in this direction in section 5 by identifying the conserved charges that define the non-radiative corner phase space. We have shown that they transform under a representation of the extended corner symmetry group. This statement is supported by the transformation properties (5.20). We have then studied a fundamental vacua transition process by solving the evolution equations in the presence of an impulsive gravitational wave, representing an exact solution of the vacuum Einstein's equations. Interestingly, we found that all the Weyl scalars in the asymptotic corner phase space are non-vanishing. The solutions consist of a vacuum component, given by the conserved charges describing the non-radiative phase space, and a radiative component. The latter contains a distributional contribution linear in the gravitational impulse news and a secular contribution quadratic in it. This opens the

JHEP04(2022)125

way towards a description of an arbitrary signal as a succession of gravitational impulses. The next step in the program is to ensure that the representation that we have identified for the conserved charges can be understood as a coadjoint representation. Imposition of asymptotic dynamics at the quantum level can then be phrased in terms of a notion of intertwiners between the irreducible representations of the asymptotic symmetry group and the quantum numbers associated to radiation in an impulsive wave basis.

Let us conclude by pointing out an interesting implication of our strategy in recovering the asymptotic dynamics of gravity. A natural question is whether our symmetry argument can implement any constraint on modifications of gravitational dynamics beyond Einstein's theory. An answer to this question can be provided by relying on the relatively recent discovery of the equivalence between soft graviton theorems and asymptotic symmetries (see [30–32] for reviews). In particular, the leading, subleading and sub-subleading tree-level soft theorems have been shown to be equivalent to respectively the covariant mass and dual mass EOM (5.2b), (5.2c), the covariant momentum EOM (5.2d), and the spin-2 charge EOM (5.2e) [50]. Moreover, it was shown in [85, 86] that tree-level soft graviton theorems at leading and subleading orders do not receive higher derivative corrections, while the sub-subleading soft graviton theorem corrections vanish for pure gravity. One exception where we expect corrections to the sub-subleading soft theorem, which is beyond the scope of our analysis here, is when gravity is coupled to a dilaton field. We can thus conclude that our strategy to derive asymptotic evolution equations at leading order in the large-r expansion around null infinity uniquley determined by symmetry is unaffected by higher derivative corrections to vacuum general relativity at leading, subleading and sub-subleading orders.

Acknowledgments

We would like to thank Glenn Barnich, Roberto Oliveri, Simone Speziale for helpful discussions and insights. Research at Perimeter Institute is supported in part by the Government of Canada through the Department of Innovation, Science and Economic Development Canada and by the Province of Ontario through the Ministry of Colleges and Universities. This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No. 841923.

A Action of the symmetry

We want to find $\delta_{(\tau,Y)}\Phi^i$ such that

$$\mathcal{L}_{\xi_{(\tau,Y)}} g_{\mu\nu}[\Phi^i] = \partial_\epsilon g_{\mu\nu}[\Phi^i + \epsilon\delta_{(\tau,Y)}\Phi^i]\Big|_{\epsilon=0} \,. \tag{A.1}$$

In this section we concentrate on the case where $\Phi^i = \{b, F, M, U_A, P_A, q_{AB}, C_{AB}, E_{AB}\}$. It will prove convenient to express the BMSW vector fields $\xi_{(\tau,Y)}$ (2.15) in the form

$$\xi_{(\tau,Y)} = \bar{\xi}_{(\tau,Y)} + \frac{1}{r}\xi_1 + \frac{1}{r^2}\xi_2 + \frac{1}{r^3}\xi_3 \,, \tag{A.2}$$

where $\bar{\xi}_{(\tau,Y)} := \tau\partial_u + Y^A\partial_A - \dot{\tau}r\partial_r$ is the asymptotic component given in (2.13). The lower order vector fields $\xi_i = y_i^A\partial_A + \rho_i r\partial_r$ only have tangential and radial components. Their expression is derived from (2.15b), (2.15c) and the expansion of

$$\begin{aligned}
I^{AB} &= \int_r^\infty \frac{\mathrm{d}r'}{r'^2} e^{2\beta}\gamma^{AB} \\
&= \frac{1}{r}q^{AB} - \frac{1}{2r^2}C^{AB} + \frac{q^{AB}}{r^3}\left(\frac{2}{3}b + \frac{1}{12}C^{CD}C_{CD}\right) + o(r^{-4})\,.
\end{aligned} \tag{A.3}$$

The expansion of the tangential vector is

$$y_1^A = -\partial^A\tau, \qquad y_2^A = \frac{1}{2}C^{AB}\partial_B\tau, \qquad y_3^A = \left(2b - \frac{8}{3}\mathsf{E}_b\right)\partial^A\tau\,, \tag{A.4}$$

while the radial components are given by

$$\rho_1 = \frac{1}{2}\Delta\tau\,, \tag{A.5}$$

$$\rho_2 = -\frac{1}{2}\left(D_A C^{AB}\partial_B\tau + \frac{1}{2}C^{AB}D_A\partial_B\tau - \mathsf{E}_U^A\partial_A\tau\right)\,, \tag{A.6}$$

$$\rho_3 = -\left(\frac{4}{3}\partial_A b\partial^A\tau - \frac{4}{3}\partial_A\mathsf{E}_b\partial^A\tau + \left(b - \frac{4}{3}\mathsf{E}_b\right)\Delta\tau + \frac{1}{3}P^A\partial_A\tau + \frac{1}{3}C^{AC}U_C\partial_A\tau\right)\,. \tag{A.7}$$

The metric components read as

$$\begin{aligned}
g_{uu} &= -2\Phi e^{2\beta} + \frac{1}{r^2}\gamma_{AB}\Upsilon^A\Upsilon^B \\
&= -2F + \frac{2M}{r} + \frac{1}{r^2}\left(q_{AB}U^AU^B - 4Fb\right) + o(r^{-2})\,,
\end{aligned} \tag{A.8}$$

$$g_{ur} = -e^{2\beta} = -1 - \frac{2}{r^2}b + o(r^{-2})\,, \tag{A.9}$$

$$g_{rr} = 0\,, \tag{A.10}$$

$$g_{Au} = -\gamma_{AB}\Upsilon^B = -U_A + \frac{2}{3r}\left(P_A - \frac{1}{2}C_{AB}U^B + \partial_A b\right) + o(r^{-1})\,, \tag{A.11}$$

$$g_{Ar} = 0\,, \tag{A.12}$$

$$g_{AB} = r^2\gamma_{AB} = r^2 q_{AB} + rC_{AB} + \frac{1}{4}q_{AB}C_{CD}C^{CD} + \frac{1}{r}E_{AB} + o(r^{-1})\,. \tag{A.13}$$

In order to compute their symmetry transformations, we use the field expansion (A.2) and write a general field transformation as

$$\delta_{(\tau,Y)} = \delta_{\bar{\xi}} + \Delta_\tau\,, \tag{A.14}$$

where

$$\delta_{\bar{\xi}} := \tau\partial_u + \mathcal{L}_Y + s\dot{\tau}\,, \tag{A.15}$$

with s the conformal weight of the given quantity and Δ_τ its anomaly.

A.1 g_{ur}

We start with

$$\mathcal{L}_\xi g_{ur} = \xi^\nu \partial_\nu g_{ur} + g_{uu}\partial_r \xi^u + g_{ur}\partial_r \xi^r + g_{uA}\partial_r \xi^A + g_{ru}\partial_u \xi^u \,. \tag{A.16}$$

The dominant contribution is obtained by replacing $\xi \to \bar{\xi}$ and we find that g_{ur} transforms as a scalar of weight $[-r\partial_r]$, since

$$\delta_{\bar{\xi}} g_{ur} = \tau \dot{g}_{ur} + \mathcal{L}_Y g_{ur} - \dot{\tau} r \partial_r g_{ur} \,. \tag{A.17}$$

Given the notation $y^A = \sum_i y_i^A / r^i$, $\rho = \sum_i \rho_i / r^i$, the anomaly is given by

$$\begin{aligned}
\Delta_\tau g_{ur} &= y^A \partial_A g_{ur} + \rho r \partial_r g_{ur} + g_{ur}\partial_r (r\rho) + g_{uA}\partial_r y^A \\
&= -\partial_r \left(\frac{1}{r}\rho_2\right) - U_A \partial_r \left(\frac{1}{r} y_1^A\right) + o(r^{-2}) \\
&= -\frac{1}{r^2}\left[\frac{1}{2}\left(D_A C^{AB}\partial_B \tau + \frac{1}{2} C^{AB} D_A \partial_B \tau - \mathsf{E}_U^A \partial_A \tau\right) + U^A \partial_A \tau\right] + o(r^{-2}) \\
&= -\frac{2}{r^2}\left[\frac{1}{8} C^{AB} D_A \partial_B \tau + \frac{1}{4} E_U^A \partial_A \tau\right] + o(r^{-2}) \,.
\end{aligned} \tag{A.18}$$

Now, since $g_{ur} = -2(1 + b/r^2) + o(r^{-2})$, this means that we have

$$\delta_{(\tau,Y)} b = [\tau \partial_u + Y^A \partial_A + 2\dot{\tau}] b + \frac{1}{8} C^{AB} D_A \partial_B \tau + \frac{1}{4} \mathsf{E}_U^A \partial_A \tau \,, \tag{A.19}$$

where we used that

$$\mathsf{E}_U^A := U^A + \frac{1}{2} D_B C^{AB} \,\hat{=}\, 0 \,. \tag{A.20}$$

A.2 g_{uu}

Next, we look at

$$\mathcal{L}_\xi g_{uu} = \xi^u \partial_u g_{uu} + \xi^A \partial_A g_{uu} + \xi^r \partial_r g_{uu} + 2 g_{uu}\partial_u \xi^u + 2 g_{uA}\partial_u \xi^A + 2 g_{ur}\partial_u \xi^r . \tag{A.21}$$

This means that g_{uu} transforms as a scalar of weight $[2 - r\partial_r]$, since

$$\delta_{\bar{\xi}} g_{uu} = \tau \dot{g}_{uu} + \mathcal{L}_Y g_{uu} + \dot{\tau}(2 - r\partial_r) g_{uu}. \tag{A.22}$$

The anomaly is given by

$$\begin{aligned}
\Delta_\tau g_{uu} &= y^A \partial_A g_{uu} + \rho\, r \partial_r g_{uu} + 2 g_{uA}\partial_u y^A + 2 r g_{ur} \dot{\rho} \\
&= -\frac{2}{r}\left(y_1[F] + U_A \dot{y}_1^A\right) - 2\left(\dot{\rho}_1 + \frac{1}{r}\dot{\rho}_2\right) + o(r^{-1}) \\
&= -\Delta\dot{\tau} + \frac{2}{r}\left(\partial^A \tau \partial_A F + \frac{1}{2} D_A N^{AB} \partial_B \tau + \frac{1}{4}\partial_u \left(C^{AB} D_A \partial_B \tau\right) + \frac{1}{2}(\mathsf{E}_U^A \partial_A \dot{\tau} - \dot{\mathsf{E}}_U^A \partial_A \tau)\right) + o(r^{-1}) \,.
\end{aligned} \tag{A.23}$$

We can thus read off the field variations

$$\delta_{(\tau,Y)}F = [\tau\partial_u + \mathcal{L}_Y + 2\dot\tau]\, F + \frac{1}{2}\Delta\dot\tau, \tag{A.24}$$

$$\begin{aligned}\delta_{(\tau,Y)}M &= [\tau\partial_u + \mathcal{L}_Y + 3\dot\tau]\, M + \left(\frac{1}{2}D_B N^{AB} + \partial^A F\right)\partial_A\tau \\ &\quad + \frac{1}{4}N^{AB}D_A\partial_B\tau + \frac{1}{4}C^{AB}D_A\partial_B\dot\tau + \frac{1}{2}(\mathsf{E}_U^A\partial_A\dot\tau - \dot{\mathsf{E}}_U^A\partial_A\tau)\,.\end{aligned} \tag{A.25}$$

A.3 g_{uA}

Next, we rewrite

$$g_{Au} = -U_A + \frac{1}{r}V_A + o(r^{-1})\,, \tag{A.26}$$

where, following (2.4c) and (2.4d), we have

$$V_A := \frac{2}{3}\left(P_A - \frac{1}{2}C_{AB}U^B + \partial_A b\right)\,, \tag{A.27}$$

and compute

$$\mathcal{L}_\xi g_{Au} = \xi^\nu\partial_\nu g_{Au} + g_{AB}\partial_u\xi^B + g_{Au}\partial_u\xi^u + g_{uu}\partial_A\xi^u + g_{uB}\partial_A\xi^B + g_{ur}\partial_A\xi^r\,. \tag{A.28}$$

This means that g_{Au} transforms as a vector of weight $[1 - r\partial_r]$, since

$$\delta_\xi g_{Au} = \tau\dot g_{Au} + \mathcal{L}_Y g_{Au} + \dot\tau(1 - r\partial_r)g_{Au}. \tag{A.29}$$

The anomaly is given by

$$\begin{aligned}\Delta_\tau g_{Au} &= \rho r\partial_r g_{Au} + y^B D_B g_{Au} + g_{AB}\partial_u y^B + g_{uu}\partial_A\tau + g_{uB}\partial_A y^B - g_{ur}\partial_A(r\dot\tau) + r g_{ur}\partial_A\rho \\ &= -\frac{1}{r}y_1^B D_B U_A + \left(r^2 q_{AB} + rC_{AB} + \frac{1}{4}q_{AB}C_{CD}C^{CD}\right)\left(\frac{1}{r}\dot y_1^B + \frac{1}{r^2}\dot y_2^B + \frac{1}{r^3}\dot y_3^B\right) \\ &\quad + \left(-2F + \frac{2M}{r}\right)\partial_A\tau - \frac{1}{r}\left(-U_B + \frac{1}{r}V_B\right)D_A\partial^B\tau - \left(1 + \frac{2}{r^2}b\right)\partial_A\left(-r\dot\tau + \rho_1 + \frac{1}{r}\rho_2\right) + o(r^{-1}), \\ &= \dot y_{2A} + C_{AB}\dot y_1^B - 2F\partial_A\tau - \partial_A\rho_1 \\ &\quad + \frac{1}{r}\left[2M\partial_A\tau + 2b\partial_A\dot\tau - \partial_A\rho_2 - y_1^B D_B U_A + U^B D_A\partial_B\tau + \dot y_{3A} + C_{AB}\dot y_2^B + \frac{1}{4}C_{CD}C^{CD}\dot y_{1A}\right] \\ &= \frac{1}{2}N^{AB}\partial_B\tau - \frac{1}{2}C^{AB}\partial_B\dot\tau - 2F\partial_A\tau - \frac{1}{2}\partial_A\Delta\tau \\ &\quad + \frac{1}{r}\Bigg[D_B U_A\partial^B\tau + U^B D_A\partial_B\tau + 2\dot b\partial_A\tau + 2b\partial_A\dot\tau - \frac{8}{3}\dot{\mathsf{E}}_b\partial_A\tau - \frac{8}{3}\mathsf{E}_b\partial_A\dot\tau \\ &\quad + \frac{1}{2}C_{AB}N^{BC}\partial_C\tau + \underbrace{\frac{1}{2}C_{AB}C^{BC}\partial_C\dot\tau - \frac{1}{4}C_{CD}C^{CD}\partial_A\dot\tau}_{=0} \\ &\quad + \frac{1}{2}\partial_A\left(D_C C^{CB}\partial_B\tau + \frac{1}{2}C^{CB}D_C\partial_B\tau - \mathsf{E}_U^B\partial_B\tau\right) + 2M\partial_A\tau + 2b\partial_A\dot\tau\Bigg],\end{aligned} \tag{A.30}$$

from which we can read off the transformations

$$\delta_{(\tau,Y)}U_A = [\tau\partial_u + \mathcal{L}_Y + \dot\tau]\,U_A + \frac{1}{2}(4F\partial_A\tau + \partial_A\Delta\tau) + \frac{1}{2}\left(C_A{}^B\partial_B\dot\tau - N_A{}^B\partial_B\tau\right), \tag{A.31}$$

$$\begin{aligned}\delta_{(\tau,Y)}V_A = {}& [\tau\partial_u + \mathcal{L}_Y + 2\dot\tau]\,V_A + \frac{1}{2}C_{AB}N^{BC}\partial_C\tau + 2M\partial_A\tau + 2\dot b\partial_A\tau + 4b\partial_A\dot\tau \\ &+ \frac{1}{2}\left(D_AD^CC_{CB} - D_BD^CC_{CA}\right)\partial^B\tau + \frac{1}{4}\partial_A\left(C^{CB}D_C\partial_B\tau\right) \\ &+ \frac{1}{2}\mathsf{E}_U^BD_A\partial_B\tau + \left(D_B\mathsf{E}_{UA} - \frac{1}{2}D_A\mathsf{E}_{UB}\right)\partial^B\tau - \frac{8}{3}\dot{\mathsf{E}}_b\partial_A\tau - \frac{8}{3}\mathsf{E}_b\partial_A\dot\tau\,. \end{aligned} \tag{A.32}$$

We can now use

$$\delta_{(\tau,Y)}(\partial_Ab) = [\tau\partial_u + \mathcal{L}_Y + 2\dot\tau](\partial_Ab) + \frac{1}{8}\partial_A(C^{BC}D_B\partial_C\tau) + \dot b\partial_A\tau + 2b\partial_A\dot\tau\,, \tag{A.33}$$

$$\begin{aligned}\delta_{(\tau,Y)}(C_{AB}U^B) = {}& [\tau\partial_u + \mathcal{L}_Y + 2\dot\tau](C_{AB}U^B) + \frac{1}{4}C_{BC}C^{BC}\partial_A\dot\tau - \frac{1}{2}C_{AB}N^{BC}\partial_C\tau \\ &+ \frac{1}{2}C_{AB}(4F\partial^B\tau + \partial^B\Delta\tau) + D_{\langle A}D_{B\rangle}\tau D_CC^{CB} \\ &- 2D_{\langle A}D_{B\rangle}\tau\mathsf{E}_U^B\,, \end{aligned} \tag{A.34}$$

to finally compute, on-shell of $\mathsf{E}_U^A \hat{=} 0$, the momentum transformation

$$\begin{aligned}\delta_{(\tau,Y)}P_A = {}& \frac{3}{2}\delta_{(\tau,Y)}V_A + \frac{1}{2}\delta_{(\tau,Y)}(C_{AB}U^B) - \delta_{(\tau,Y)}(\partial_Ab) \\ \hat{=}{}& [\tau\partial_u + \mathcal{L}_Y + 2\dot\tau]P_A + 3M\partial_A\tau - \frac{1}{8}C_{BC}N^{BC}\partial_A\tau + \frac{1}{2}C_{AB}N^{BC}\partial_C\tau \\ &+ FC_{AB}\partial^B\tau + \frac{1}{4}C_{AB}\partial^B\Delta\tau \\ &+ \frac{3}{4}\left(D_AD^CC_{CB} - D_BD^CC_{AC}\right)\partial^B\tau + \frac{1}{4}\partial_A\left(C^{CB}D_C\partial_B\tau\right) \\ &+ \frac{1}{2}D_{\langle A}D_{B\rangle}\tau D_CC^{CB} - 2\dot{\mathsf{E}}_b\partial_A\tau\,. \end{aligned} \tag{A.35}$$

We see that the momentum transformation does not contain any anomaly term proportional to $\partial_A\dot\tau$.

A.4 Sphere metric

We compute here the anomaly of the sphere metric component E_{AB} in the expansion (2.4d). The Lie derivative of the metric component g_{AB} yields

$$\mathcal{L}_\xi g_{AB} = \xi^u\partial_ug_{AB} + \xi^C\partial_Cg_{AB} + \xi^r\partial_rg_{AB} + 2g_{(Au}\partial_{B)}\xi^u + 2g_{C(A}\partial_{B)}\xi^C\,. \tag{A.36}$$

This means that g_{AB} transforms as a tensor of weight $-[r\partial_r]$, since

$$\delta_{\bar\xi}g_{AB} = \tau\dot g_{AB} + \mathcal{L}_Yg_{AB} - \dot\tau r\partial_rg_{AB}. \tag{A.37}$$

The anomaly is given by

$$\begin{aligned}
\Delta_\tau g_{AB} =& (\mathcal{L}_y+\rho r\partial_r)(r^2\gamma_{AB})+2g_{u(A}\partial_{B)}\tau \\
=& \frac{1}{r}(\mathcal{L}_{y_1}+\rho_1 r\partial_r)(r^2 q_{AB})+2g_{u(A}\partial_{B)}\tau \\
&+\frac{1}{r}(\mathcal{L}_{y_1}+\rho_1 r\partial_r)(rC_{AB})+\frac{1}{r^2}(\mathcal{L}_{y_2}+\rho_2 r\partial_r)(r^2 q_{AB}) \\
&+\frac{1}{r^3}(\mathcal{L}_{y_3}+\rho_3 r\partial_r)(r^2 q_{AB})+\frac{1}{r^2}(\mathcal{L}_{y_2}+\rho_2 r\partial_r)(rC_{AB})+\frac{1}{r}(\mathcal{L}_{y_1}+\rho_1 r\partial_r)\left(\frac{1}{4}q_{AB}C_{CD}C^{CD}\right) \\
=& r(\mathcal{L}_{y_1}q_{AB}+2\rho_1 q_{AB})+\left[(\mathcal{L}_{y_2}q_{AB}+2\rho_2 q_{AB}+\mathcal{L}_{y_1}C_{AB}+\rho_1 C_{AB})-2U_{(A}\partial_{B)}\tau\right] \\
&+\frac{1}{r}\left(\mathcal{L}_{y_3}q_{AB}+2\rho_3 q_{AB}+\mathcal{L}_{y_2}C_{AB}+\rho_2 C_{AB}+\mathcal{L}_{y_1}\left(\frac{1}{4}q_{AB}C_{CD}C^{CD}\right)\right) \\
&+\frac{4}{3r}\left(P_{(A}-\frac{1}{2}C_{C(A}U^C+\partial_{(A}b\right)\partial_{B)}\tau+o(r^{-1}).
\end{aligned} \tag{A.38}$$

Therefore, we can write

$$\Delta_\tau q_{AB}=0\,, \tag{A.39a}$$

$$\Delta_\tau C_{AB}=\mathcal{L}_{y_1}q_{AB}+2\rho_1 q_{AB}=-2D_{\langle A}\partial_{B\rangle}\tau\,, \tag{A.39b}$$

$$\Delta_\tau\left(\frac{1}{4}q_{AB}C_{CD}C^{CD}\right)=\mathcal{L}_{y_2}q_{AB}+2\rho_2 q_{AB}+\mathcal{L}_{y_1}C_{AB}+\rho_1 C_{AB}-2U_{(A}\partial_{B)}\tau\,, \tag{A.39c}$$

$$\begin{aligned}
\Delta_\tau E_{AB}=&\ \mathcal{L}_{y_3}q_{AB}+2\rho_3 q_{AB}+\mathcal{L}_{y_2}C_{AB}+\rho_2 C_{AB}+\mathcal{L}_{y_1}\left(\frac{1}{4}q_{AB}C_{CD}C^{CD}\right) \\
&+\frac{4}{3}\left(P_{(A}-\frac{1}{2}C_{C(A}U^C+\partial_{(A}b\right)\partial_{B)}\tau\,.
\end{aligned} \tag{A.39d}$$

The first two anomaly relations in (A.39) yield (2.21a), (2.21b). The third one can be evaluated to be

$$\begin{aligned}
\Delta_\tau\left(\frac{1}{4}q_{AB}C_{CD}C^{CD}\right) &\hat{=} -\frac{1}{2}q_{AB}C^{CD}D_C\partial_D\tau-C_{C(A}D_{B)}\partial^C\tau+\frac{1}{2}C_{AB}\Delta\tau \\
&=-q_{AB}C^{CD}D_C\partial_D\tau\,,
\end{aligned} \tag{A.40}$$

where we have used[22]

$$C_{C\langle A}D_{B\rangle}\partial^C\tau=\frac{1}{2}C_{AB}D_C\partial^C\tau\,, \tag{A.41}$$

and it is thus consistent with (A.39a), (A.39b), as $\Delta_\tau\left(\frac{1}{4}q_{AB}C_{CD}C^{CD}\right)=\frac{1}{2}q_{AB}C^{CD}\Delta_\tau C_{CD}$.

[22]The relation (A.41) is an application of the general property that, for any pair of 2×2 symmetric and traceless matrices A, B, the identity

$$A_{\langle A}{}^{C}B_{B\rangle C}=0$$

holds. Then (A.41) follows from $A_{AB}=C_{AB}$ and $B_{AB}=D_{\langle A}\partial_{B\rangle}\tau$.

A.4.1 Stress tensor anomaly

We can now compute our quantity of interest, namely

$$\begin{aligned}
\Delta_\tau E_{AB} =& 4\left(b-\frac{4}{3}\mathsf{E}_b\right)D_{(A}\partial_{B)}\tau+4\partial_{(A}\left(b-\frac{4}{3}\mathsf{E}_b\right)\partial_{B)}\tau \\
&-2\left(\frac{4}{3}\partial_C b\partial^C\tau-\frac{4}{3}\partial_C\mathsf{E}_b\partial^C\tau+\left(b-\frac{4}{3}\mathsf{E}_b\right)\Delta\tau+\frac{1}{3}P^C\partial_C\tau+\frac{1}{3}C^{CD}U_C\partial_D\tau\right)q_{AB} \\
&+\frac{1}{2}C^{CD}\partial_C\tau D_D C_{AB}+C_{C(A}D_{B)}(C^{CD}\partial_D\tau)-\frac{1}{2}C_{AB}\left(D_C C^{CD}\partial_D\tau+\frac{1}{2}C^{CD}D_C\partial_D\tau\right) \\
&-\frac{1}{2}C_{CD}C^{CD}D_{(A}\partial_{B)}\tau\ \underbrace{-\frac{1}{4}q_{AB}\partial^C\tau\partial_C(C_{DE}C^{DE})}_{=-8q_{AB}\partial_C\mathsf{E}_b\partial^C\tau+8q_{AB}\partial_C b\partial^C\tau} \\
&+\frac{4}{3}\left(P_{(A}-\frac{1}{2}C_{C(A}U^C+\partial_{(A}b\right)\partial_{B)}\tau \\
=& 4bD_{(A}\partial_{B)}\tau-2q_{AB}b\Delta\tau-\frac{1}{4}C_{AB}C^{CD}D_C\partial_D\tau \\
&+\frac{4}{3}\left(P_{(A}-\frac{1}{2}C_{C(A}U^C-8\partial_{(A}b\right)\partial_{B)}\tau-\frac{2}{3}q_{AB}\left(P^D-\frac{1}{2}C^{CD}U_C-8\partial^D b\right)\partial_D\tau \\
&+\frac{1}{2}\left(C^{CD}D_C C_{AB}+C_{CA}D_B C^{CD}+C_{CB}D_A C^{CD}-C_{AB}D_C C^{CD}\right)\partial_D\tau \\
&+16\partial_{(A}b\partial_{B)}\tau-q_{AB}C^{CD}U_C\partial_D\tau \\
&-\frac{16}{3}\mathsf{E}_b D_{\langle A}\partial_{B\rangle}\tau-\frac{16}{3}\partial_{\langle A}\mathsf{E}_b\partial_{B\rangle}\tau\,. \qquad \text{(A.42)}
\end{aligned}$$

On-shell of the asymptotic Einstein's equation $\mathsf{E}_U^A \mathrel{\hat=} 0$, we thus have

$$\begin{aligned}
\Delta_\tau E_{AB} \mathrel{\hat=}& 4bD_{\langle A}\partial_{B\rangle}\tau-\frac{1}{4}C_{AB}C^{CD}D_C\partial_D\tau \\
&+\frac{4}{3}\left(P_{\langle A}-\frac{1}{2}U^C C_{C\langle A}-8\partial_{\langle A}b\right)\partial_{B\rangle}\tau \\
&+\frac{1}{2}\left(C^{CD}D_C C_{AB}-C_{AB}D_C C^{CD}\right)\partial_D\tau \\
&+C_{C\langle A}D_{B\rangle}C^{CD}\partial_D\tau+16\partial_{\langle A}b\partial_{B\rangle}\tau \\
&-\frac{16}{3}\mathsf{E}_b D_{\langle A}\partial_{B\rangle}\tau-\frac{16}{3}\partial_{\langle A}\mathsf{E}_b\partial_{B\rangle}\tau\,. \qquad \text{(A.43)}
\end{aligned}$$

The last line can be simplified since

$$\begin{aligned}
C_{C\langle A}D_{B\rangle}C^{CD}\partial_D\tau+16\partial_{\langle A}b\partial_{B\rangle}\tau &= D_{\langle A}(C_{B\rangle C}C^{CD})\partial_D\tau-D_{\langle A}C_{B\rangle C}C^{CD}\partial_D\tau+16\partial_{\langle A}b\partial_{B\rangle}\tau \\
&= 16\partial_{\langle A}\mathsf{E}_b\partial_{B\rangle}\tau-D_{\langle A}C_{B\rangle C}C^{CD}\partial_D\tau\,. \qquad \text{(A.44)}
\end{aligned}$$

This means that the spin 2 anomaly is given, on-shell of $\mathsf{E}_U^A \mathrel{\hat=} 0$, by

$$\begin{aligned}
\Delta_\tau E_{AB} \mathrel{\hat=}& \frac{4}{3}\left(P_{\langle A}-\frac{1}{2}U^C C_{C\langle A}-8\partial_{\langle A}b\right)\partial_{B\rangle}\tau \\
&+\frac{1}{2}\left(C^{CD}D_C C_{AB}-C_{AB}D_C C^{CD}\right)\partial_D\tau-D_{\langle A}C_{B\rangle C}C^{CD}\partial_D\tau
\end{aligned}$$

$$
\begin{aligned}
&+4bD_{\langle A}\partial_{B\rangle}\tau-\frac{1}{4}C_{AB}C^{CD}D_C\partial_D\tau\\
&-\frac{16}{3}\mathsf{E}_b D_{\langle A}\partial_{B\rangle}\tau+\frac{32}{3}\partial_{\langle A}\mathsf{E}_b\partial_{B\rangle}\tau\,.
\end{aligned}\tag{A.45}
$$

We could use the definition of the covariant momentum $\mathcal{P}_A=P_A-\frac{1}{2}C_{CA}U^C+\frac{1}{16}\partial_A(C_{BC}C^{BC})$ to rewrite

$$
\begin{aligned}
\Delta_\tau E_{AB}\,\hat{=}\,&\frac{4}{3}\mathcal{P}_{\langle A}\partial_{B\rangle}\tau\\
&+\frac{1}{2}\left(C^{CD}D_C C_{AB}-C_{AB}D_C C^{CD}\right)\partial_D\tau\\
&+\frac{1}{4}\partial_{\langle A}(C_{CD}C^{CD})\partial_{B\rangle}\tau-D_{\langle A}C_{B\rangle C}C^{CD}\partial_D\tau\\
&-\frac{1}{8}C_{CD}C^{CD}D_{\langle A}\partial_{B\rangle}\tau-\frac{1}{4}C_{AB}C^{CD}D_C\partial_D\tau\\
&-\frac{4}{3}\mathsf{E}_b D_{\langle A}\partial_{B\rangle}\tau\,.
\end{aligned}\tag{A.46}
$$

B Variations

In this section we compute the behavior under symmetry transformation of different quantities.

- **Connection.**
  One establishes that

$$
\begin{aligned}
\delta_{(\tau,Y)}\Gamma^C_{AB}&=\frac{1}{2}q^{CD}\left(D_A\delta_{(\tau,Y)}q_{BD}+D_B\delta_{(\tau,Y)}q_{AD}-D_D\delta_{(\tau,Y)}q_{AB}\right)\\
&=\frac{1}{2}q^{CD}\left(D_AD_BY_D+D_BD_AY_D+[D_A,D_D]Y_B+[D_B,D_D]Y_A\right)\\
&\quad -q^{CD}\left(D_A\dot{\tau}q_{BD}+D_B\dot{\tau}q_{AD}-D_D\dot{\tau}q_{AB}\right)\\
&=D_{(A}D_{B)}Y^C+\frac{1}{2}\left(R_{BDA}{}^C+R_{ADB}{}^C\right)Y^D-2D_{\langle A}\dot{\tau}\delta^C_{B\rangle}\,,
\end{aligned}\tag{B.1}
$$

  where we used that $[D_A,D_B]V_C=R_{ABC}{}^DV_D$ and $[D_A,D_B]V^C=R^C{}_{DAB}V^D$. This means that the contribution to the anomaly due to τ is due to the presence of Weyl rescaling and given by

$$
\Delta_\tau\Gamma^C_{AB}=-D_A\dot{\tau}\delta^C_B-D_B\dot{\tau}\delta^C_A+D^C\dot{\tau}q_{AB}\,,\tag{B.2}
$$

$$
\Delta_\tau\Gamma^A_{AB}=-2D_B\dot{\tau}\,.\tag{B.3}
$$

  Given a vectorial section V^A of scale s it can be checked that the anomaly only depends on τ

$$
\begin{aligned}
\delta_{(\tau,Y)}D_AV^C&=D_A\delta_{(\tau,Y)}V^C+\delta_{(\tau,Y)}\Gamma^C_{AB}V^B\\
&=[\tau\partial_u+\mathcal{L}_Y+s\dot{\tau}]\,(D_AV^C)+\Delta_\tau(D_AV^C)\,.
\end{aligned}\tag{B.4}
$$

To evaluate the anomaly one establishes that

$$\begin{aligned}
\delta_\tau D_A V^C &= D_A \delta_\tau V^C + (\delta_\tau \Gamma^C_{AB}) V^B \\
&= D_A[\tau \dot{V}^C + s\dot{\tau} V^C] - \Big(D_A\dot{\tau}\delta^C_B + D_B\dot{\tau}\delta^C_A - D^C\dot{\tau} q_{AB} \Big) V^B \\
&= [\tau D_A \dot{V}^C + s\dot{\tau} D_A V^C] \\
&\quad + D_A\tau \dot{V}^C + s(D_A\dot{\tau})V^C - \Big(D_A\dot{\tau} V^C + \delta^C_A V^B D_B\dot{\tau} - D^C\dot{\tau} V_A \Big) ,
\end{aligned} \tag{B.5}$$

which means that even if V^A is a section of weight s its spatial derivative contains an anomaly given by

$$\begin{aligned}
\Delta_\tau (D_A V^C) &= D_A\tau \dot{V}^C + s(D_A\dot{\tau})V^C + \Delta_\tau \Gamma^C_{AB} V^B \\
&= D_A\tau \dot{V}^C + s(D_A\dot{\tau})V^C - \Big(D_A\dot{\tau} V^C + \delta^C_A V^B D_B\dot{\tau} - D^C\dot{\tau} V_A \Big) .
\end{aligned} \tag{B.6}$$

Similarly, the anomaly for the derivative of a form of weight s is

$$\Delta_\tau (D_A V_B) = D_A\tau \dot{V}_B + s(D_A\dot{\tau})V_B + \Big(D_A\dot{\tau} V_B + V_A D_B\dot{\tau} - V^C D_C\dot{\tau} q_{AB} \Big) . \tag{B.7}$$

- ***D · C* vector.**
One evaluates

$$\begin{aligned}
\Delta_\tau (D_B C^{CB}) &= N^{AB}\partial_B\tau + 3C^{CB}\partial_B\dot{\tau} + D_B\Delta_\tau C^{CB} + \Delta_\tau \Gamma^C_{BA} C^{AB} + \Delta_\tau \Gamma^B_{BA} C^{AC} \\
&= N^{AB}\partial_B\tau + 3C^{CB}\partial_B\dot{\tau} - 2D_B D^{\langle B}\partial^{C\rangle}\tau - 4C^{AC}\partial_A\dot{\tau}. \\
&= N^{CB}\partial_B\tau - C^{CB}\partial_B\dot{\tau} - 2\Delta D^C\tau + D^C\Delta\tau \\
&= N^{CB}\partial_B\tau - C^{CB}\partial_B\dot{\tau} - R(q) D^C\tau - D^C\Delta\tau ,
\end{aligned} \tag{B.8}$$

where we used that

$$[\Delta, D_C]\tau = [D^B, D_C] D^B\tau = R^B{}_{DBC} D^D\tau = \frac{1}{2} R(q) D_C\tau. \tag{B.9}$$

- **$\mathcal{J}^A$ vector.**
Given the definition (3.8) of the vector $\mathcal{J}^A$, and by means of (2.21e) and

$$\begin{aligned}
\delta_{(\tau,Y)}(\partial^A F) &= q^{AC}\delta_{(\tau,Y)}(\partial_C F) + \partial_C F \delta_{(\tau,Y)} q^{AC} \\
&= q^{AC}\delta_{(\tau,Y)}(\partial_C F) - (D^A Y^C + D^C Y^A)\partial_C F + 2\dot{\tau}\partial^A F \\
&= \partial^A(\delta_{(\tau,Y)} F) - (D^A Y^C + D^C Y^A)\partial_C F + 2\dot{\tau}\partial^A F \\
&= [\tau\partial_u + \mathcal{L}_Y + 4\dot{\tau}] (\partial^A F) + 2F\partial^A\dot{\tau} + \frac{1}{2}\partial^A\Delta\dot{\tau} ,
\end{aligned} \tag{B.10}$$

we compute

$$\delta_{(\tau,Y)}\mathcal{J}^A = [\tau\partial_u + \mathcal{L}_Y + 4\dot{\tau}] \mathcal{J}^A + \frac{1}{2}\dot{N}^{AB}\partial_B\tau . \tag{B.11}$$

JHEP04(2022)125

A similar calculation shows that

$$\delta_{(\tau,Y)}\tilde{\mathcal{J}}^A = [\tau\partial_u + \mathcal{L}_Y + 4\dot{\tau}]\,\tilde{\mathcal{J}}^A + \frac{1}{2}\dot{\tilde{N}}^{AB}\partial_B\tau\,. \tag{B.12}$$

Next, we can write

$$\delta_{(\tau,Y)}(D_A\mathcal{J}^A) = D_A(\delta_{(\tau,Y)}\mathcal{J}^A) + (\delta_{(\tau,Y)}\Gamma^A_{AB})\mathcal{J}^B\,, \tag{B.13}$$

from which, by means of (B.3), we get

$$\delta_{(\tau,Y)}(D_A\mathcal{J}^A) = [\tau\partial_u + \mathcal{L}_Y + 4\dot{\tau}]\,D_A\mathcal{J}^A + 2\partial_u(\mathcal{J}^A D_A\tau) + \frac{1}{2}\dot{N}^{AB}D_A\partial_B\tau\,, \tag{B.14}$$

where we have used

$$\frac{1}{2}D_A\dot{N}^{AB} = \dot{\mathcal{J}}^B \tag{B.15}$$

and

$$D_A(\mathcal{L}_Y\mathcal{J}^A) + \mathcal{J}^B D_B D_A Y^A = \mathcal{L}_Y(D_A\mathcal{J}^A)\,. \tag{B.16}$$

We also have

$$\begin{aligned}
\delta_{(\tau,Y)}(C_{AB}\mathcal{J}^B) &= \delta_{(\tau,Y)}C_{AB}\mathcal{J}^B + C_{AB}\delta_{(\tau,Y)}\mathcal{J}^B\\
&= [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}]\,C_{AB}\mathcal{J}^B\\
&\quad - 2\mathcal{J}^B D_{\langle A}D_{B\rangle}\tau + \frac{1}{2}C_{AB}\dot{N}^{BC}\partial_C\tau\\
&= [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}]\,C_{AB}\mathcal{J}^B\\
&\quad - D_C N^{BC}D_{\langle A}D_{B\rangle}\tau - 2\partial^B F D_{\langle A}D_{B\rangle}\tau + \frac{1}{2}C_{AB}\dot{N}^{BC}\partial_C\tau\\
&= [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}]\,C_{AB}\mathcal{J}^B\\
&\quad + \frac{1}{2}C_{AB}\dot{N}^{BC}\partial_C\tau - \Big(D_C N^{BC} + 2\partial^B F\Big)\,D_{\langle A}\partial_{B\rangle}\tau\,.
\end{aligned} \tag{B.17}$$

- **Covariant mass $\mathcal{M}$.**
  Given the covariant mass transformation (3.11), we have

$$\begin{aligned}
\delta_{(\tau,Y)}(\partial_A\mathcal{M}) &= \partial_A(\delta_{(\tau,Y)}\mathcal{M})\\
&\hat{=} [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}]\,\partial_A\mathcal{M}\\
&\quad + \partial_A\tau\dot{\mathcal{M}} + 3\partial_A\dot{\tau}\mathcal{M}\\
&\quad + D_A\mathcal{J}^B\partial_B\tau + \mathcal{J}^B D_A\partial_B\tau\,.
\end{aligned} \tag{B.18}$$

- **Derivative $D_{[A}(D\cdot C)_{B]}$.**
  From (2.21e) we see that $(D\cdot C)_B$ is a generalized tensor of dimension 1 and hence its anomaly is

$$\begin{aligned}
\Delta_\tau(D_{[A}(D\cdot C)_{B]})) &= D_{[A}\tau(D^C N_{B]C}) + (D_{[A}\dot{\tau})(D^C C_{B]C}) + D_{[A}(\Delta_\tau D\cdot C)_{B]}\\
&= D_{[A}\tau(D^C N_{B]C}) + (D_{[A}\dot{\tau})(D^C C_{B]C})\\
&\quad + D_{[A}(N_{B]}{}^C\partial_C\tau - C_{B]}{}^C\partial_C\dot{\tau} - R(q)D_{B]}\tau - D_{B]}\Delta\tau)
\end{aligned}$$

$$
\begin{aligned}
&= D_{[A}\tau(2D^C N_{B]C} + D_{B]}R) - (N_{[A}{}^C D_{B]}\partial_C\tau + C_{[B}{}^C D_{A]}\partial_C\dot{\tau}) \\
&= 4D_{[A}\tau\mathcal{J}_{B]} - (N_{[A}{}^C D_{B]}\partial_C\tau + C_{[B}{}^C D_{A]}\partial_C\dot{\tau})\,. \qquad \text{(B.19)}
\end{aligned}
$$

Contracting this identity with ϵ^{AB} gives the identity

$$
\Delta_\tau(D_A(D\cdot\tilde{C})^A) = 4\tilde{\mathcal{J}}^A\partial_A\tau - (N^{AB}\tilde{D}_A\partial_B\tau + \tilde{C}^{AB}D_A\partial_B\dot{\tau})\,. \qquad \text{(B.20)}
$$

- **Dual covariant mass $\tilde{\mathcal{M}}$.**
From the definition (3.21) and the relation (B.20), we can derive

$$
\begin{aligned}
\Delta_\tau\tilde{\mathcal{M}} &= \frac{1}{4}\Delta_\tau(D_A(D\cdot\tilde{C})^A) + \frac{1}{8}\left(\Delta_\tau C_{AB}\tilde{N}^{AB} + C_{AB}\Delta_\tau\tilde{N}^{AB}\right) \\
&= \tilde{\mathcal{J}}^A\partial_A\tau - \frac{1}{4}(N^{AB}\tilde{D}_A\partial_B\tau + \tilde{C}^{AB}D_A\partial_B\dot{\tau} \\
&\quad - \frac{1}{4}D_{\langle A}\partial_{B\rangle}\tau\tilde{N}^{AB} - \frac{1}{4}C_{AB}\tilde{D}_{\langle A}\partial_{B\rangle}\dot{\tau} \\
&= \tilde{\mathcal{J}}^A\partial_A\tau\,. \qquad \text{(B.21)}
\end{aligned}
$$

From this anomaly and the definition (3.8), we can further compute

$$
\begin{aligned}
\delta_{\tau,Y}(\tilde{\partial}_A\tilde{\mathcal{M}}) = \tilde{\partial}_A(\delta_{\tau,Y}\tilde{\mathcal{M}}) \\
&= [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}]\,\tilde{\partial}_A\tilde{\mathcal{M}} \\
&\quad + \tilde{\partial}_A\tau\dot{\tilde{\mathcal{M}}} + 3\tilde{\partial}_A\dot{\tau}\mathcal{M} \\
&\quad + \tilde{D}_A\tilde{\mathcal{J}}^B\partial_B\tau + \tilde{\mathcal{J}}^B\tilde{D}_A\partial_B\tau\,. \qquad \text{(B.22)}
\end{aligned}
$$

- **Covariant momentum $\mathcal{P}_A$.**
Given the covariant momentum transformation (3.30), we want to compute the anomaly of the quantity $D_A\mathcal{P}_B$. From the relation

$$
\Delta_\tau(D_A\mathcal{P}_B) = D_A(\Delta_\tau\mathcal{P}_B) - (\Delta_\tau\Gamma^D_{AB})\mathcal{P}_D\,, \qquad \text{(B.23)}
$$

and the anomaly (B.2), we see that the last term in (B.23) yields the following contribution to the anomaly

$$
\begin{aligned}
(\delta^D_B\partial_A\dot{\tau} + \delta^D_A\partial_B\dot{\tau} - q_{BA}\partial^D\dot{\tau})\mathcal{P}_D &= 2\mathcal{P}_{(A}\partial_{B)}\dot{\tau} - q_{BA}\mathcal{P}_D\partial^D\dot{\tau} \\
&= 2\mathcal{P}_{\langle A}\partial_{B\rangle}\dot{\tau}\,. \qquad \text{(B.24)}
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\Delta_\tau(D_A\mathcal{P}_B) \mathrel{\hat{\hat{=}}} &\ \partial_A\tau\dot{\mathcal{P}}_B + 2\partial_A\dot{\tau}\mathcal{P}_B + 2\mathcal{P}_{\langle A}\partial_{B\rangle}\dot{\tau} \\
&+ 3D_A\tilde{\mathcal{M}}\tilde{\partial}_B\tau + 3D_A\mathcal{M}\partial_B\tau + 3\tilde{\mathcal{M}}D_A\tilde{\partial}_B\tau + 3\mathcal{M}D_A\partial_B\tau\,. \qquad \text{(B.25)}
\end{aligned}
$$

- **Möbius derivative.**
We now want to analyse the transformations of "Möbius derivative operator" $[D_{\langle A}D_{B\rangle} + \frac{s}{2}n_{AB}]\phi$ for a section of conformal weight s, and the transformation of $[D_{\langle A}D_{B\rangle} +$

$\frac{s}{2}n_{AB}]j^B$ for a vector of weight $s+1$. One starts with the computation of the conformal anomaly

$$\begin{aligned}\delta_W D_{\langle A}D_{B\rangle}\phi &= -\delta_W \Gamma^C_{AB} D_C\phi + D_{\langle A}\delta_W D_{B\rangle}\phi \\ &= 2D_{\langle A}W D_{B\rangle}\phi + sD_{\langle A}D_{B\rangle}(W\phi) \\ &= 2(s+1)D_{\langle A}W D_{B\rangle}\phi + sWD_{\langle A}D_{B\rangle}\phi + s\phi D_{\langle A}D_{B\rangle}W. \end{aligned} \tag{B.26}$$

Combining this with the fact that $\delta_W n_{AB} = -2D_{\langle A}D_{B\rangle}W$, we find that

$$\delta_W\left[D_{\langle A}D_{B\rangle} + \frac{s}{2}n_{AB}\right]\phi = sW\left[D_{\langle A}D_{B\rangle} + \frac{s}{2}n_{AB}\right]\phi + 2(s+1)D_{\langle A}WD_{B\rangle}\phi\,, \tag{B.27}$$

which shows that the Möbius combination $\left[D_{\langle A}D_{B\rangle} + \frac{s}{2}n_{AB}\right]$ possesses no quadratic anomaly and that it is tensorial for sections of conformal weight $s=-1$.

Similarly, one evaluates

$$\begin{aligned}\delta_W D_{\langle A}D_{B\rangle}V^B &= \delta_W \Gamma^C_{AB} D_C V^B + D_{\langle A}(\delta_W\Gamma^B_{B\rangle C}V^C) + D_{\langle A}D_{B\rangle}\delta_W V^B \\ &= -2D_{\langle A}WD_{B\rangle}V^B - 2D_{\langle A}(D_{B\rangle}WV^B)) + sD_{\langle A}D_{B\rangle}(WV^B) \\ &= 2(s-2)D_{\langle A}WD_{B\rangle}V^B + sWD_{\langle A}D_{B\rangle}\phi + (s-2)\phi D_{\langle A}D_{B\rangle}W. \end{aligned} \tag{B.28}$$

Combining this with the fact that $\delta_W n_{AB} = -2D_{\langle A}D_{B\rangle}W$, we find that

$$\delta_W\left[D_{\langle A}D_{B\rangle} + \frac{(s-2)}{2}n_{AB}\right]V^B = sW\left[D_{\langle A}D_{B\rangle} + \frac{(s-2)}{2}n_{AB}\right]V^B + 2(s-2)D_{\langle A}WD_{B\rangle}V^B\,, \tag{B.29}$$

which shows that the Möbius combination $\left[D_{\langle A}D_{B\rangle} + \frac{(s-2)}{2}n_{AB}\right]V^B$ possesses no quadratic anomaly and that it is tensorial for vectorial sections of conformal weight $s=2$.

C Derivation of the momentum evolution equation

Given the metric parametrization (2.4), the asymptotic Einstein's equation for the momentum P_A is given by [17, 22, 25]

$$\begin{aligned}\dot{P}_A = {} & D_A M + \frac{1}{8}D_A(C^{BC}N_{BC}) \\ & + C_{AB}D^B F \\ & - \frac{1}{4}N^{CB}D_A C_{CB} \\ & - \frac{1}{4}\left(D_B D^B D^C C_{AC} - D_B D_A D_C C^{BC}\right) \\ & - \frac{1}{4}D_B(N_{AC}C^{CB}) + \frac{1}{4}D_B(N^{CB}C_{AC})\,. \end{aligned} \tag{C.1}$$

Recalling the definition (3.26) of the covariant momentum, we can write

$$\begin{aligned}
\dot{\mathcal{P}}_A &= D_A M + \frac{1}{8} D_A(C^{BC}N_{BC}) + C_{AB} D^B F \\
&\quad - \frac{1}{4} N^{CB} D_A C_{CB} + \frac{1}{4} N_{AB} (D\cdot C)^B + \frac{1}{4} C_{AB} (D\cdot N)^B + \frac{1}{8} \partial_A (C^{CB} N_{CB}) \\
&\quad - \frac{1}{4} D^B \left(D_B (D\cdot C)_A - D_A (D\cdot C)_B \right) \\
&\quad - \frac{1}{4} D^B (N_A{}^C C_{CB} - N_B{}^C C_{CA}) \qquad (C.2) \\
&= D_A \left(M + \frac{1}{8} C^{BC} N_{BC} \right) + C_{AB} \left(\frac{1}{2} D_B N_A{}^B + D^B F \right) + D^B J_{[AB]} \\
&\quad - \frac{1}{4} N^{CB} D_A C_{CB} + \frac{1}{4} N_{AB} (D\cdot C)^B - \frac{1}{4} C_{AB} (D\cdot N)^B + \frac{1}{8} \partial_A (C^{CB} N_{CB}) \\
&\quad - \frac{1}{8} D^B (N_A{}^C C_{CB} - N_B{}^C C_{CA}) \qquad (C.3) \\
&= D_A \mathcal{M} + \tilde{D}_A \tilde{\mathcal{M}} + C_{AB} \mathcal{J}^B \\
&\quad - \frac{1}{4} N^{CB} D_A C_{CB} + \frac{1}{4} N_{AB} (D\cdot C)^B - \frac{1}{4} C_{AB} (D\cdot N)^B + \frac{1}{8} \partial_A (C^{CB} N_{CB}) \\
&\quad - \frac{1}{8} D^B (N_A{}^C C_{CB} - N_B{}^C C_{CA}) \,, \qquad (C.4)
\end{aligned}$$

where we have used the definitions (3.8), (3.29), (3.21). The terms in the second and third lines of (C.4) can be expanded and simplified as

$$\begin{aligned}
&-\frac{1}{4} N^{CB} D_A C_{CB} + \frac{1}{4} N_{AB} (D\cdot C)^B - \frac{1}{4} C_{AB} (D\cdot N)^B + \frac{1}{8} \partial_A (C^{CB} N_{CB}) \\
&-\frac{1}{8} (D_B N_{AC} C^{CB} - N^{BC} D_B C_{CA}) - \frac{1}{8} (N_{AB} (D\cdot C)^B - (D\cdot N)^B C_{BA}) \\
&+\frac{1}{8} (N^{CB} D_B C_{CA} - C^{CB} D_B N_{AC}) \\
&\quad = \frac{1}{8} N_{AB} (D\cdot C)^B - \frac{1}{8} C_{AB} (D\cdot N)^B + \frac{1}{4} (C^{CB} D_{[A} N_{B]C} - N^{CB} D_{[A} C_{B]C}). \qquad (C.5)
\end{aligned}$$

Now we use that

$$\epsilon_{AB} \epsilon^{CD} = \delta_A^C \delta_B^D - \delta_A^D \delta_B^C \qquad (C.6)$$

to massage terms like

$$C^{CB} D_{[A} N_{B]C} = \frac{1}{2} \epsilon_{AB} C^{BC} (D\cdot \tilde{N})_C = \frac{1}{2} \tilde{C}_{AB} (D\cdot \tilde{N})^B \,, \qquad (C.7)$$

where $\tilde{N}_{BC} = \epsilon_B{}^A N_{AC}$. This means that the contributions in the second and third lines of (C.4) can be written as

$$\frac{1}{8} (N_{AB} (D\cdot C)^B - \tilde{N}_{AB} (D\cdot \tilde{C})^B] - \frac{1}{8} [C_{AB} (D\cdot N)^B - \tilde{C}_{AB} (D\cdot \tilde{N})^B] \,. \qquad (C.8)$$

Finally, since N is symmetric and traceless, we can derive the relation

$$N_{AB}(D\cdot C)^B = \tilde{N}_{AB}(D\cdot\tilde{C})^B\,, \tag{C.9}$$

and similarly

$$C_{AB}(D\cdot N)^B = \tilde{C}_{AB}(D\cdot\tilde{N})^B\,. \tag{C.10}$$

We thus arrive at the sought after expression

$$\dot{\mathcal{P}}_A = D_A\mathcal{M} + \tilde{D}_A\tilde{\mathcal{M}} + C_{AB}\mathcal{J}^B\,. \tag{C.11}$$

Note that, since $C_{AB}\mathcal{J}^B = \tilde{C}_{AB}\tilde{\mathcal{J}}^B$, we can write the momentum evolution equation in a completely self-dual manner as

$$\dot{\mathcal{P}}_A = D_A\mathcal{M} + \tilde{D}_A\tilde{\mathcal{M}} + \frac{1}{2}\left(C_{AB}\mathcal{J}^B + \tilde{C}_{AB}\tilde{\mathcal{J}}^B\right)\,. \tag{C.12}$$

D Stress-energy tensor

D.1 SET anomaly proof

We give here the explicit derivation of the transformation properties of all the SET components (4.36). From the analysis of appendix A, we can see immediately that the component $\hat{T}_{uu}$ transforms as a scalar of weight 4, since

$$\delta_{\bar{\xi}}T_{uu} = \tau\dot{T}_{uu} + \mathcal{L}_Y T_{uu} + \dot{\tau}(2 - r\partial_r)T_{uu}. \tag{D.1}$$

The anomaly is given by

$$\Delta_\tau T_{uu} = y^A\partial_A T_{uu} + \rho\, r\partial_r T_{uu} + 2T_{uA}\partial_u y^A + 2rT_{ur}\dot{\rho} = o(r^{-2})\,, \tag{D.2}$$

which implies

$$\Delta_\tau \hat{T}_{uu} = 0\,. \tag{D.3}$$

The component $\hat{T}_{uA}$ transforms as a vector of weight 3 since from

$$\mathcal{L}_\xi T_{Au} = \xi^\nu\partial_\nu T_{Au} + T_{Au}\partial_u\xi^u + T_{Ar}\partial_u\xi^r + T_{AB}\partial_u\xi^B + T_{uu}\partial_A\xi^u + T_{ur}\partial_A\xi^r + T_{uB}\partial_A\xi^B \tag{D.4}$$

we have that

$$\delta_{\bar{\xi}}T_{Au} = \tau\dot{T}_{Au} + \mathcal{L}_Y T_{Au} + \dot{\tau}(1 - r\partial_r)T_{Au}\,. \tag{D.5}$$

The anomaly is given by

$$\begin{aligned}\Delta_\tau T_{Au} &= y^B D_B T_{Au} + \rho\, r\partial_r T_{Au} + T_{Ar}\partial_u\xi^r + T_{AB}\partial_u y^B + T_{uu}\partial_A\tau + T_{ur}\partial_A\xi^r + T_{uB}\partial_A y^B\\ &= -\frac{1}{r^2}\hat{T}\partial_A\dot{\tau} + \frac{1}{r^2}\hat{T}_{uu}\partial_A\tau + o(r^{-2})\,,\end{aligned} \tag{D.6}$$

from which

$$\Delta_\tau \hat{T}_{Au} = -\hat{T}\partial_A\dot{\tau} + \hat{T}_{uu}\partial_A\tau\,. \tag{D.7}$$

From the Lie derivative of the T_{AB} component

$$\mathcal{L}_\xi T_{AB} = \xi^u\partial_u T_{AB} + \xi^C\partial_C T_{AB} + \xi^r\partial_r T_{AB} + 2T_{u(A}\partial_{B)}\xi^u + 2T_{r(A}\partial_{B)}\xi^r + 2T_{C(A}\partial_{B)}\xi^C\,, \tag{D.8}$$

it is immediate to see that $\hat{T}$ transforms as a scalar of weight 3, since

$$\delta_{\bar{\xi}} T_{AB} = \tau\dot{T}_{AB} + \mathcal{L}_Y T_{AB} - \dot{\tau}r\partial_r T_{AB}\,, \tag{D.9}$$

and q_{AB} has conformal dimension -2, while $\hat{T}_{AB}$ transforms as a tensor of weight 2. It is also easy to see that $\hat{T}$ has no anomaly, since both the leading terms in T_{AB} and q_{AB} have no anomaly. At the same time, the anomaly of the r^{-2} component of T_{AB} given by $\hat{T}_{AB} := \hat{T}_2 q_{AB} + \hat{T}_{\langle AB\rangle}$ can be read off of

$$\Delta_\tau T_{AB} = y^C\partial_C T_{AB} + r\rho\partial_r T_{AB} + 2T_{u(A}\partial_{B)}\tau - 2T_{r(A}\partial_{B)}(\dot{\tau}r) + 2T_{C(A}\partial_{B)}y^C \tag{D.10}$$

and it is given by

$$\begin{aligned}\Delta_\tau \hat{T}_{AB} &= -q_{AB}\partial^C\tau\partial_C\hat{T} - \frac{1}{2}q_{AB}\hat{T}\Delta\tau + 2\hat{T}_{u(A}\partial_{B)}\tau - 2\hat{T}_{r(A}\partial_{B)}\dot{\tau} - 2\hat{T}D_{(A}\partial_{B)}\tau\\ &= -q_{AB}\partial^C\tau\partial_C\hat{T} - \frac{3}{2}q_{AB}\hat{T}\Delta\tau + q_{AB}\hat{T}_{uC}\partial^C\tau - q_{AB}\hat{T}_{rC}\partial_C\dot{\tau}\\ &\quad + 2\hat{T}_{u\langle A}\partial_{B\rangle}\tau - 2\hat{T}_{r\langle A}\partial_{B\rangle}\dot{\tau} - 2\hat{T}D_{\langle A}\partial_{B\rangle}\tau\,.\end{aligned} \tag{D.11}$$

This means that

$$\Delta_\tau\hat{T}_2 = \partial_C\hat{T}\partial^C\tau - \frac{3}{2}\hat{T}\Delta\tau + \hat{T}_{uC}\partial^C\tau - \hat{T}_{rC}\partial_C\dot{\tau} \tag{D.12}$$

and

$$\Delta_\tau\hat{T}_{\langle AB\rangle} = 2\hat{T}_{u\langle A}\partial_{B\rangle}\tau - 2\hat{T}_{r\langle A}\partial_{B\rangle}\dot{\tau} - 2\hat{T}D_{\langle A}\partial_{B\rangle}\tau\,. \tag{D.13}$$

The component $\hat{T}_{rr}$ transforms as a vector of weight 2 since from

$$\mathcal{L}_\xi T_{rr} = \xi^\nu\partial_\nu T_{rr} + 2T_{ru}\partial_r\xi^u + 2T_{rr}\partial_r\xi^r + 2T_{rA}\partial_r\xi^A \tag{D.14}$$

we have that

$$\mathcal{L}_{\bar{\xi}} T_{rr} = \tau\dot{T}_{rr} + \mathcal{L}_Y T_{rr} - \dot{\tau}(2 + r\partial_r)T_{rr}\,. \tag{D.15}$$

The anomaly is given by

$$\Delta_\tau T_{rr} = y^A D_A T_{rr} + \rho\, r\partial_r T_{rr} + 2T_{rr}\partial_r(r\rho) + 2T_{rA}\partial_r y^A\,, \tag{D.16}$$

from which

$$\Delta_\tau \hat{T}_{rr} = 0\,. \tag{D.17}$$

The component T_{rA} transforms as a vector of weight 2 since from

$$\mathcal{L}_\xi T_{rA} = \xi^\nu \partial_\nu T_{rA} + T_{ru}\partial_A \xi^u + T_{rr}\partial_A \xi^r + T_{rB}\partial_A \xi^B + T_{Ar}\partial_r \xi^r + T_{AB}\partial_r \xi^B \tag{D.18}$$

we have that

$$\mathcal{L}_{\bar{\xi}} T_{rA} = \tau \dot{T}_{rA} + \mathcal{L}_Y T_{rA} - \dot{\tau}(1 + r\partial_r) T_{rA}\,. \tag{D.19}$$

The anomaly is given by

$$\Delta_\tau T_{rA} = y^A D_A T_{rA} + \rho\, r\partial_r T_{rA} + T_{ru}\partial_A \tau + T_{rr}\partial_A \xi^r + T_{rB}\partial_A y^B + T_{Ar}\partial_r(r\rho) + T_{AB}\partial_r y^B\,, \tag{D.20}$$

from which

$$\Delta_\tau \hat{T}_{rA} = -\hat{T}_{rr}\partial_A \dot{\tau} + \hat{T}\partial_A \tau\,. \tag{D.21}$$

Notice that this anomaly is consistent with the conservation equations $\mathcal{C}$, $\mathcal{C}^2_A$ in (4.37) as

$$\Delta_\tau \partial_A \hat{T}_{rr} = \partial_u \hat{T}_{rr}\partial_A \tau + 2\hat{T}_{rr}\partial_A \dot{\tau} = -2\hat{T}\partial_A \tau + 2\hat{T}_{rr}\partial_A \dot{\tau}\,. \tag{D.22}$$

It also follows that the vector $\dot{\hat{T}}_{rA}$ transforms with weight 3 and anomaly

$$\begin{aligned}\Delta_\tau \dot{\hat{T}}_{rA} &= -\dot{\hat{T}}_{rr}\partial_A \dot{\tau} + \dot{\hat{T}}\partial_A \tau + \hat{T}\partial_A \dot{\tau}\\ &= \dot{\hat{T}}\partial_A \tau + 3\hat{T}\partial_A \dot{\tau}\,,\end{aligned} \tag{D.23}$$

consistently with the conservation equation $\mathcal{C}^1_A$ in (4.37) as

$$\Delta_\tau \partial_A \hat{T} = \partial_u \hat{T}\partial_A \tau + 3\hat{T}\partial_A \dot{\tau}\,. \tag{D.24}$$

D.2 Conservation equations proof

Here we derive the SET conservation equations (4.37). These follow from

$$\nabla^\mu T_{\mu\nu} = g^{\mu\rho}\left(\partial_\rho T_{\mu\nu} - \Gamma^\sigma_{\mu\rho} T_{\sigma\nu} - \Gamma^\sigma_{\nu\rho} T_{\sigma\mu}\right) = 0\,, \tag{D.25}$$

and the inverse metric components given by

$$g^{uu} = 0\,, \tag{D.26a}$$

$$g^{ur} = -e^{-2\beta}\,, \tag{D.26b}$$

$$g^{rr} = 2\Phi e^{-2\beta} = 2F - \frac{2M}{r} + o(r^{-1}) \tag{D.26c}$$

$$g^{Au} = 0\,, \tag{D.26d}$$

$$g^{Ar} = -e^{-2\beta}\frac{\Upsilon^A}{r^2} = -\frac{U^A}{r^2} + o(r^{-2})\,, \tag{D.26e}$$

$$g^{AB} = \frac{1}{r^2}\gamma^{AB} = \frac{1}{r^2}q^{AB} - \frac{1}{r^3}C^{AB} + o(r)\,. \tag{D.26f}$$

Let us consider first the component $\nu = r$. We have

$$\begin{aligned} 0 &= g^{u\rho}\left(\partial_\rho T_{ur} - \Gamma^\sigma_{u\rho}T_{\sigma r} - \Gamma^\sigma_{r\rho}T_{\sigma u}\right) \\ &\quad + g^{r\rho}\left(\partial_\rho T_{rr} - \Gamma^\sigma_{r\rho}T_{\sigma r} - \Gamma^\sigma_{r\rho}T_{\sigma r}\right) \\ &\quad + g^{A\rho}\left(\partial_\rho T_{Ar} - \Gamma^\sigma_{A\rho}T_{\sigma r} - \Gamma^\sigma_{r\rho}T_{\sigma A}\right) \\ &= \frac{1}{r^4}\left[-\partial_u \hat{T}_{rr} - q^{AB}\hat{T}q_{AB}\right] + o(r^{-4})\,, \end{aligned} \tag{D.27}$$

where only the spin connection component $\Gamma^C_{rB} = \frac{1}{r}\delta^C_B$ contributes at the leading order. We thus obtain the conservation equation

$$\mathcal{C} := \partial_u \hat{T}_{rr} + 2\hat{T} = 0\,. \tag{D.28}$$

Next, we consider the component $\nu = u$ and we obtain

$$\begin{aligned} 0 &= g^{u\rho}\left(\partial_\rho T_{uu} - \Gamma^\sigma_{u\rho}T_{\sigma u} - \Gamma^\sigma_{u\rho}T_{\sigma u}\right) \\ &\quad + g^{r\rho}\left(\partial_\rho T_{ru} - \Gamma^\sigma_{r\rho}T_{\sigma u} - \Gamma^\sigma_{u\rho}T_{\sigma r}\right) \\ &\quad + g^{A\rho}\left(\partial_\rho T_{Au} - \Gamma^\sigma_{A\rho}T_{\sigma u} - \Gamma^\sigma_{u\rho}T_{\sigma A}\right) \\ &= \frac{1}{r^3}\left[2\hat{T}_{uu} - 2\hat{T}_{uu}\right] + o(r^{-3})\,, \end{aligned} \tag{D.29}$$

where only the spin connection component $\Gamma^u_{AB} = r q_{AB}$ contributes at the leading order. We thus see that this component yields a trivial relation.

Finally, the component $\nu = A$ yields

$$\begin{aligned} 0 &= g^{u\rho}\left(\partial_\rho T_{uA} - \Gamma^\sigma_{u\rho}T_{\sigma A} - \Gamma^\sigma_{A\rho}T_{\sigma u}\right) \\ &\quad + g^{r\rho}\left(\partial_\rho T_{rA} - \Gamma^\sigma_{r\rho}T_{\sigma A} - \Gamma^\sigma_{A\rho}T_{\sigma r}\right) \\ &\quad + g^{B\rho}\left(\partial_\rho T_{BA} - \Gamma^\sigma_{B\rho}T_{\sigma A} - \Gamma^\sigma_{A\rho}T_{\sigma B}\right) \\ &= \frac{1}{r^3}\left[2\hat{T}_{uA} - \partial_u \hat{T}_{rA} + q^{BC}D_C\hat{T}_{AB} - 2\hat{T}_{uA}\right] + o(r^{-3})\,, \end{aligned} \tag{D.30}$$

where again only the spin connection component $\Gamma^u_{AB} = r q_{AB}$ contributes at the leading order. We thus obtain the conservation equation

$$\mathcal{C}^1_A := \partial_u \hat{T}_{rA} - \partial_A \hat{T} = 0\,. \tag{D.31}$$

Combining the two conservation equations (D.28), (D.31) one gets the third conservation equation

$$\mathcal{C}^2_A := \partial_A \hat{T}_{rr} + 2\hat{T}_{rA} = 0\,. \tag{D.32}$$

D.3 Sources

Let us derive the explicit expressions (4.43) for the matter sources by applying our symmetry argument. We start with the covariant mass and momentum equations. By inspection of the conformal weights under the BMSW group action, we can consider the general ansatz

$$\mathcal{S} := \frac{1}{2}\left(\hat{T}_{uu} + \beta\dot{\hat{T}}\right), \qquad \mathcal{S}_A := \hat{T}_{uA} + \alpha\partial_A\hat{T}, \tag{D.33}$$

with α, β two free numerical coefficients to be determined. It follows that

$$\delta_{(\tau,Y)}\mathcal{S}_A = [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}]\mathcal{S}_A + \hat{T}_{uu}\partial_A\tau + \alpha\dot{\hat{T}}\partial_A\tau + (3\alpha-1)\hat{T}\partial_A\dot{\tau}. \tag{D.34}$$

Next, given the definition (3.26) of the covariant momentum and the transformation (2.21j), one gets that

$$\delta_{(\tau,Y)}\mathcal{P}_A \hat{=} [\tau\partial_u + \mathcal{L}_Y + 2\dot{\tau}]\mathcal{P}_A + 3\left(\mathcal{M}\partial_A\tau + \tilde{\mathcal{M}}\tilde{\partial}_A\tau\right) - 2\dot{\mathsf{E}}_b\partial_A\tau. \tag{D.35}$$

Using that in the presence of matter we have $4\dot{\mathsf{E}}_b = T$, this means

$$\Delta_\tau\mathcal{E}_A \hat{=} -\frac{1}{2}\dot{\hat{T}}\partial_A\tau - \frac{1}{2}\hat{T}\partial_A\dot{\tau}, \tag{D.36}$$

and

$$\begin{aligned}\delta_{(\tau,Y)}\left(\mathcal{E}_A + \mathcal{S}_A\right) &= [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}]\left(\mathcal{E}_A + \mathcal{S}_A\right) + 2\tilde{\mathcal{E}}\tilde{\partial}_A\tau \\ &\quad + 2\left(\mathcal{E} + \frac{1}{2}\left(\hat{T}_{uu} + \left(\alpha - \frac{1}{2}\right)\dot{\hat{T}}\right)\right)\partial_A\tau + \hat{T}\left(3\alpha - \frac{3}{2}\right)\partial_A\dot{\tau}.\end{aligned} \tag{D.37}$$

We thus see that we need $\alpha = 1/2$ to remove the anomaly and then the sources read

$$\mathcal{S} := \frac{1}{2}\hat{T}_{uu}, \qquad \mathcal{S}_A := \hat{T}_{uA} + \frac{1}{2}\partial_A\hat{T}, \tag{D.38}$$

so that

$$\delta_{(\tau,Y)}\left(\mathcal{E}_A + \mathcal{S}_A\right) = [\tau\partial_u + \mathcal{L}_Y + 3\dot{\tau}]\left(\mathcal{E}_A + \mathcal{S}_A\right) + 2\left(\mathcal{E} + \mathcal{S}\right)\partial_A\tau + 2\tilde{\mathcal{E}}\tilde{\partial}_A\tau. \tag{D.39}$$

For the spin-2 asymptotic EE, compatibility of the conformal weights suggests that we start with the ansatz

$$\mathcal{S}_{AB} := \gamma\hat{T}_{\langle AB\rangle} + \beta D_{\langle A}\partial_{B\rangle}\hat{T}_{rr} + \zeta\hat{T}C_{AB}. \tag{D.40}$$

By means of the transformations (D.13) and

$$\begin{aligned}\delta_{(\tau,Y)}D_{\langle A}\partial_{B\rangle}\hat{T}_{rr} &= [\tau\partial_u + \mathcal{L}_Y + 2\dot{\tau}]D_{\langle A}\partial_{B\rangle}\hat{T}_{rr} + 2\partial_u D_{\langle A}T_{rr}\partial_{B\rangle}\tau + 6D_{\langle A}\hat{T}_{rr}\partial_{B\rangle}\dot{\tau} \\ &\quad + \partial_u T_{rr}D_{\langle A}\partial_{B\rangle}\tau + 2\hat{T}_{rr}D_{\langle A}\partial_{B\rangle}\dot{\tau},\end{aligned} \tag{D.41}$$

where note that the coefficient 6 above involves using (4.39) and a contribution from $-\delta_{(\tau,Y)}\Gamma^C_{\langle AB\rangle}\partial_C\hat{T}_{rr}$, we have

$$
\begin{aligned}
\delta_{(\tau,Y)}\mathcal{S}_{AB} = {} & [\tau\partial_u + \mathcal{L}_Y + 2\dot{\tau}]\mathcal{S}_{AB} \\
& + \left[2\gamma\hat{T}_{u\langle A} - 4\beta D_{\langle A}\hat{T}\right]\partial_{B\rangle}\tau \\
& + \left[-2\gamma\hat{T}_{r\langle A} + 6\beta D_{\langle A}\hat{T}_{rr}\right]\partial_{B\rangle}\dot{\tau} \\
& - 2\left[\gamma+\beta+\zeta\right]\hat{T}D_{\langle A}\partial_{B\rangle}\tau \\
& + 2\beta\hat{T}_{rr}D_{\langle A}\partial_{B\rangle}\dot{\tau}\,.
\end{aligned} \tag{D.42}
$$

Next, we can use the definition (4.22) to compute the off-shell of the E_b equation of motion anomaly

$$
\Delta_\tau\mathcal{E}_{AB} \mathrel{\hat{=}} -2\dot{\mathsf{E}}_b D_{\langle A}\partial_{B\rangle}\tau - 4\mathsf{E}_b D_{\langle A}\partial_{B\rangle}\dot{\tau} + 2\partial_{\langle A}\dot{\mathsf{E}}_b\partial_{B\rangle}\tau\,, \tag{D.43}
$$

which follows from (D.35) and the covariant spin-2 pseudo-tensor transformation (see (A.46))

$$
\delta_{(\tau,Y)}\mathcal{T}_{AB} \mathrel{\hat{=}} [\tau\partial_u + \mathcal{L}_Y + \dot{\tau}]\,\mathcal{T}_{AB} + 4\mathcal{P}_{\langle A}\partial_{B\rangle}\tau - 4\mathsf{E}_b D_{\langle A}\partial_{B\rangle}\tau\,. \tag{D.44}
$$

By demanding

$$
\delta_{(\tau,Y)}(\mathcal{E}_{AB}+\mathcal{S}_{AB}) = [\tau\partial_u + \mathcal{L}_Y + 2\dot{\tau}](\mathcal{E}_{AB}+\mathcal{S}_{AB}) + 3\left(\mathcal{E}_{\langle A} + \mathcal{S}_{\langle A}\right)\partial_{B\rangle}\tau\,, \tag{D.45}
$$

we can fix the coefficients β, γ, ζ from the conditions

$$
2\beta + \frac{1}{2} = 0\,, \tag{D.46}
$$

$$
-2(\gamma+\beta+\zeta) - \frac{1}{2} = 0\,, \tag{D.47}
$$

$$
-2\gamma\hat{T}_{rA} + 6\beta D_A\hat{T}_{rr} = 0 \quad\rightarrow\quad -2\gamma = 12\beta\,, \tag{D.48}
$$

$$
2\gamma\hat{T}_{uA} + \left(\frac{1}{2} - 4\beta\right)\partial_A\hat{T} = 3\left(\hat{T}_{uA} + \frac{1}{2}\partial_A\hat{T}\right)\,. \tag{D.49}
$$

It is immediate to see that the system of equations is solved by

$$
\beta = -\frac{1}{4}\,,\quad \gamma = \frac{3}{2}\,,\quad \zeta = -\frac{3}{2}\,, \tag{D.50}
$$

from which

$$
\mathcal{S}_{AB} := \frac{3}{2}\hat{T}_{\langle AB\rangle} - \frac{1}{4}D_{\langle A}\partial_{B\rangle}\hat{T}_{rr} - \frac{3}{2}\hat{T}C_{AB}\,. \tag{D.51}
$$

References

[1] W. Donnelly and L. Freidel, *Local subsystems in gauge theory and gravity*, *JHEP* **09** (2016) 102 [arXiv:1601.04744] [INSPIRE].

[2] L. Freidel and A. Perez, *Quantum gravity at the corner*, *Universe* **4** (2018) 107 [arXiv:1507.02573] [INSPIRE].

[3] L. Freidel, A. Perez and D. Pranzetti, *Loop gravity string*, *Phys. Rev. D* **95** (2017) 106002 [arXiv:1611.03668] [INSPIRE].

[4] L. Freidel and E.R. Livine, *Bubble networks: framed discrete geometry for quantum gravity*, *Gen. Rel. Grav.* **51** (2019) 9 [arXiv:1810.09364] [INSPIRE].

[5] L. Freidel, E.R. Livine and D. Pranzetti, *Gravitational edge modes: from Kac-Moody charges to Poincaré networks*, *Class. Quant. Grav.* **36** (2019) 195014 [arXiv:1906.07876] [INSPIRE].

[6] L. Freidel, E.R. Livine and D. Pranzetti, *Kinematical Gravitational Charge Algebra*, *Phys. Rev. D* **101** (2020) 024012 [arXiv:1910.05642] [INSPIRE].

[7] L. Freidel, M. Geiller and D. Pranzetti, *Edge modes of gravity. Part I. Corner potentials and charges*, *JHEP* **11** (2020) 026 [arXiv:2006.12527] [INSPIRE].

[8] L. Freidel, M. Geiller and D. Pranzetti, *Edge modes of gravity. Part II. Corner metric and Lorentz charges*, *JHEP* **11** (2020) 027 [arXiv:2007.03563] [INSPIRE].

[9] L. Freidel, M. Geiller and D. Pranzetti, *Edge modes of gravity. Part III. Corner simplicity constraints*, *JHEP* **01** (2021) 100 [arXiv:2007.12635] [INSPIRE].

[10] W. Donnelly, L. Freidel, S.F. Moosavian and A.J. Speranza, *Gravitational edge modes, coadjoint orbits, and hydrodynamics*, *JHEP* **09** (2021) 008 [arXiv:2012.10367] [INSPIRE].

[11] E. Noether, *Invariant Variation Problems*, *Transp. Theory Statist. Phys.* **1** (1971) 186 [*Gott. Nachr.* **1918** (1918) 235] [physics/0503066] [INSPIRE].

[12] L. Freidel, *A canonical bracket for open gravitational system*, arXiv:2111.14747 [INSPIRE].

[13] L. Ciambelli, R.G. Leigh and P.-C. Pai, *Embeddings and Integrable Charges for Extended Corner Symmetry*, arXiv:2111.13181 [INSPIRE].

[14] L. Ciambelli and R.G. Leigh, *Isolated surfaces and symmetries of gravity*, *Phys. Rev. D* **104** (2021) 046005 [arXiv:2104.07643] [INSPIRE].

[15] L. Freidel, R. Oliveri, D. Pranzetti and S. Speziale, *Extended corner symmetry, charge bracket and Einstein's equations*, *JHEP* **09** (2021) 083 [arXiv:2104.12881] [INSPIRE].

[16] V. Chandrasekaran, E.E. Flanagan and K. Prabhu, *Symmetries and charges of general relativity at null boundaries*, *JHEP* **11** (2018) 125 [arXiv:1807.11499] [INSPIRE].

[17] L. Freidel, R. Oliveri, D. Pranzetti and S. Speziale, *The Weyl BMS group and Einstein's equations*, *JHEP* **07** (2021) 170 [arXiv:2104.05793] [INSPIRE].

[18] H. Bondi, *Gravitational Waves in General Relativity*, *Nature* **186** (1960) 535 [INSPIRE].

[19] H. Bondi, M.G.J. van der Burg and A.W.K. Metzner, *Gravitational waves in general relativity. Part 7. Waves from axisymmetric isolated systems*, *Proc. Roy. Soc. Lond. A* **269** (1962) 21 [INSPIRE].

[20] R.K. Sachs, *On the Characteristic Initial Value Problem in Gravitational Theory*, *J. Math. Phys.* **3** (1962) 908 [INSPIRE].

JHEP04(2022)125

[21] G. Barnich and C. Troessaert, *Symmetries of asymptotically flat 4 dimensional spacetimes at null infinity revisited*, *Phys. Rev. Lett.* **105** (2010) 111103 [arXiv:0909.2617] [INSPIRE].

[22] G. Barnich and C. Troessaert, *BMS charge algebra*, *JHEP* **12** (2011) 105 [arXiv:1106.0213] [INSPIRE].

[23] M. Campiglia and A. Laddha, *Asymptotic symmetries and subleading soft graviton theorem*, *Phys. Rev. D* **90** (2014) 124028 [arXiv:1408.2228] [INSPIRE].

[24] E.E. Flanagan and D.A. Nichols, *Conserved charges of the extended Bondi-Metzner-Sachs algebra*, *Phys. Rev. D* **95** (2017) 044002 [arXiv:1510.03386] [INSPIRE].

[25] G. Compère, A. Fiorucci and R. Ruzziconi, *Superboost transitions, refraction memory and super-Lorentz charge algebra*, *JHEP* **11** (2018) 200 [*Erratum* *JHEP* **04** (2020) 172] [arXiv:1810.00377] [INSPIRE].

[26] S. Banerjee, S. Ghosh and P. Paul, *MHV graviton scattering amplitudes and current algebra on the celestial sphere*, *JHEP* **02** (2021) 176 [arXiv:2008.04330] [INSPIRE].

[27] S. Banerjee and S. Ghosh, *MHV gluon scattering amplitudes from celestial current algebras*, *JHEP* **10** (2021) 111 [arXiv:2011.00017] [INSPIRE].

[28] S. Banerjee, S. Ghosh and S.S. Samal, *Subsubleading soft graviton symmetry and MHV graviton scattering amplitudes*, *JHEP* **08** (2021) 067 [arXiv:2104.02546] [INSPIRE].

[29] S. Banerjee, S. Ghosh and P. Paul, *(Chiral) Virasoro invariance of the tree-level MHV graviton scattering amplitudes*, arXiv:2108.04262 [INSPIRE].

[30] A. Strominger, *Lectures on the Infrared Structure of Gravity and Gauge Theory*, arXiv:1703.05448 [INSPIRE].

[31] S. Pasterski, *Lectures on celestial amplitudes*, *Eur. Phys. J. C* **81** (2021) 1062 [arXiv:2108.04801] [INSPIRE].

[32] A.-M. Raclariu, *Lectures on Celestial Holography*, arXiv:2107.02075 [INSPIRE].

[33] L. Ciambelli and C. Marteau, *Carrollian conservation laws and Ricci-flat gravity*, *Class. Quant. Grav.* **36** (2019) 085004 [arXiv:1810.11037] [INSPIRE].

[34] E. Newman and R. Penrose, *An Approach to gravitational radiation by a method of spin coefficients*, *J. Math. Phys.* **3** (1962) 566 [INSPIRE].

[35] E.T. Newman and T.W.J. Unti, *Behavior of Asymptotically Flat Empty Spaces*, *J. Math. Phys.* **3** (1962) 891 [INSPIRE].

[36] T.M. Adamo, C.N. Kozameh and E.T. Newman, *Null Geodesic Congruences, Asymptotically Flat Space-Times and Their Physical Interpretation*, *Living Rev. Rel.* **12** (2009) 6 [*Living Rev. Rel.* **15** (2012) 1] [arXiv:0906.2155] [INSPIRE].

[37] H. Godazgar, M. Godazgar and C.N. Pope, *New dual gravitational charges*, *Phys. Rev. D* **99** (2019) 024013 [arXiv:1812.01641] [INSPIRE].

[38] H. Godazgar, M. Godazgar and C.N. Pope, *Tower of subleading dual BMS charges*, *JHEP* **03** (2019) 057 [arXiv:1812.06935] [INSPIRE].

[39] H. Godazgar, M. Godazgar and C.N. Pope, *Dual gravitational charges and soft theorems*, *JHEP* **10** (2019) 123 [arXiv:1908.01164] [INSPIRE].

[40] H. Godazgar, M. Godazgar and M.J. Perry, *Hamiltonian derivation of dual gravitational charges*, *JHEP* **09** (2020) 084 [arXiv:2007.07144] [INSPIRE].

[41] H. Godazgar, M. Godazgar and M.J. Perry, *Asymptotic gravitational charges*, *Phys. Rev. Lett.* **125** (2020) 101301 [arXiv:2007.01257] [INSPIRE].

[42] U. Kol and M. Porrati, *Properties of Dual Supertranslation Charges in Asymptotically Flat Spacetimes*, *Phys. Rev. D* **100** (2019) 046019 [arXiv:1907.00990] [INSPIRE].

[43] U. Kol, *Subleading BMS charges and the Lorentz group*, *JHEP* **04** (2022) 002 [arXiv:2011.06008] [INSPIRE].

[44] R. Oliveri and S. Speziale, *A note on dual gravitational charges*, *JHEP* **12** (2020) 079 [arXiv:2010.01111] [INSPIRE].

[45] A.M. Grant, K. Prabhu and I. Shehzad, *The Wald-Zoupas prescription for asymptotic charges at null infinity in general relativity*, *Class. Quant. Grav.* **39** (2022) 085002 [arXiv:2105.05919] [INSPIRE].

[46] G. Barnich and P.-H. Lambert, *A Note on the Newman-Unti group and the BMS charge algebra in terms of Newman-Penrose coefficients*, *Adv. Math. Phys.* **2012** (2012) 197385 [*J. Phys. Conf. Ser.* **410** (2013) 012142] [arXiv:1102.0589] [INSPIRE].

[47] G. Barnich and C. Troessaert, *Finite BMS transformations*, *JHEP* **03** (2016) 167 [arXiv:1601.04090] [INSPIRE].

[48] G. Barnich, P. Mao and R. Ruzziconi, *BMS current algebra in the context of the Newman-Penrose formalism*, *Class. Quant. Grav.* **37** (2020) 095010 [arXiv:1910.14588] [INSPIRE].

[49] G. Barnich and R. Ruzziconi, *Coadjoint representation of the BMS group on celestial Riemann surfaces*, *JHEP* **06** (2021) 079 [arXiv:2103.11253] [INSPIRE].

[50] L. Freidel, D. Pranzetti and A.-M. Raclariu, *Sub-subleading Soft Graviton Theorem from Asymptotic Einstein's Equations*, arXiv:2111.15607 [INSPIRE].

[51] L. Freidel, D. Pranzetti and A.-M. Raclariu, *Higher spin dynamics in gravity and $w_{1+\infty}$ celestial symmetries*, arXiv:2112.15573 [INSPIRE].

[52] P.C. Aichelburg and H. Balasin, *Symmetries of impulsive gravitational waves*, *Helv. Phys. Acta* **69** (1966) 337 [INSPIRE].

[53] P. Szekeres, *Colliding gravitational waves*, *Nature* **228** (1970) 1183 [INSPIRE].

[54] K.A. Khan and R. Penrose, *Scattering of two impulsive gravitational plane waves*, *Nature* **229** (1971) 185 [INSPIRE].

[55] R. Penrose, *The geometry of impulsive gravitational waves*, in *General relativity: Papers in honour of J.L. Synge*, Clarendon Press, Oxford, U.K. (1972), pp. 101–115 [INSPIRE].

[56] P.A. Hogan, *A Spherical impulse gravity wave*, *Phys. Rev. Lett.* **70** (1993) 117 [INSPIRE].

[57] A.N. Aliev and Y. Nutku, *Impulsive spherical gravitational waves*, *Class. Quant. Grav.* **18** (2001) 891 [gr-qc/0011016] [INSPIRE].

[58] J. Podolsky and R. Steinbauer, *Geodesics in space-times with expanding impulsive gravitational waves*, *Phys. Rev. D* **67** (2003) 064013 [gr-qc/0210007] [INSPIRE].

[59] J. Luk and I. Rodnianski, *Local Propagation of Impulsive Gravitational Waves*, *Commun. Pure Appl. Math.* **68** (2015) 511 [arXiv:1209.1130] [INSPIRE].

[60] J. Luk and I. Rodnianski, *Nonlinear interaction of impulsive gravitational waves for the vacuum Einstein equations*, arXiv:1301.1072 [INSPIRE].

[61] R.K. Sachs, *Gravitational waves in general relativity. Part 8. Waves in asymptotically flat space-times*, *Proc. Roy. Soc. Lond. A* **270** (1962) 103 [INSPIRE].

[62] G. Barnich and C. Troessaert, *Aspects of the BMS/CFT correspondence*, *JHEP* **05** (2010) 062 [arXiv:1001.1541] [INSPIRE].

[63] T. Mädler and J. Winicour, *Bondi-Sachs Formalism*, *Scholarpedia* **11** (2016) 33528 [arXiv:1609.01731] [INSPIRE].

[64] D.A. Nichols, *Center-of-mass angular momentum and memory effect in asymptotically flat spacetimes*, *Phys. Rev. D* **98** (2018) 064032 [arXiv:1807.08767] [INSPIRE].

[65] R. Geroch, *Asymptotic structure of space-time*, in *Asymptotic Structure of Space-Time*, F.P. Esposito and L. Witten eds., Springer, Boston, MA, U.S.A. (1977).

[66] G. Compère and J. Long, *Vacua of the gravitational field*, *JHEP* **07** (2016) 137 [arXiv:1601.04958] [INSPIRE].

[67] A. Einstein, *The Foundation of the General Theory of Relativity*, *Annalen Phys.* **354** (1916) 769 [*Annalen Phys.* **49** (1916) 769] [*Annalen Phys.* **14** (2005) 517] [INSPIRE].

[68] L. Freidel, S.F. Moosavian and D. Pranzetti, *Coadjoint Orbits of null infinity*, to appear.

[69] Y. Nutku and M. Halil, *Colliding Impulsive Gravitational Waves*, *Phys. Rev. Lett.* **39** (1977) 1379 [INSPIRE].

[70] S. Chandrasekhar and B.C. Xanthopoulos, *A New Type of Singularity Created by Colliding Gravitational Waves*, *Proc. Roy. Soc. Lond. A* **408** (1986) 175 [INSPIRE].

[71] P.M. Zhang, C. Duval and P.A. Horvathy, *Memory Effect for Impulsive Gravitational Waves*, *Class. Quant. Grav.* **35** (2018) 065011 [arXiv:1709.02299] [INSPIRE].

[72] S. Bhattacharjee, S. Kumar and A. Bhattacharyya, *Memory Effect and BMS-like Symmetries for Impulsive Gravitational Waves*, *Phys. Rev. D* **100** (2019) 084010 [arXiv:1905.12905] [INSPIRE].

[73] T. Dray and G. 't Hooft, *The Gravitational Shock Wave of a Massless Particle*, *Nucl. Phys. B* **253** (1985) 173 [INSPIRE].

[74] T. Dray and G. 't Hooft, *The Effect of Spherical Shells of Matter on the Schwarzschild Black Hole*, *Commun. Math. Phys.* **99** (1985) 613 [INSPIRE].

[75] T. Dray and G. 't Hooft, *The Gravitational Effect of Colliding Planar Shells of Matter*, *Class. Quant. Grav.* **3** (1986) 825 [INSPIRE].

[76] A. Strominger, *Asymptotic Symmetries of Yang-Mills Theory*, *JHEP* **07** (2014) 151 [arXiv:1308.0589] [INSPIRE].

[77] T. He, P. Mitra and A. Strominger, *2D Kac-Moody Symmetry of 4D Yang-Mills Theory*, *JHEP* **10** (2016) 137 [arXiv:1503.02663] [INSPIRE].

[78] S. Pasterski, S.-H. Shao and A. Strominger, *Flat Space Amplitudes and Conformal Symmetry of the Celestial Sphere*, *Phys. Rev. D* **96** (2017) 065026 [arXiv:1701.00049] [INSPIRE].

[79] M. Pate, A.-M. Raclariu, A. Strominger and E.Y. Yuan, *Celestial operator products of gluons and gravitons*, *Rev. Math. Phys.* **33** (2021) 2140003 [arXiv:1910.07424] [INSPIRE].

[80] L. Donnay, A. Puhm and A. Strominger, *Conformally Soft Photons and Gravitons*, *JHEP* **01** (2019) 184 [arXiv:1810.05219] [INSPIRE].

JHEP04(2022)125

[81] A. Guevara, E. Himwich, M. Pate and A. Strominger, *Holographic symmetry algebras for gauge theory and gravity*, *JHEP* **11** (2021) 152 [arXiv:2103.03961] [INSPIRE].

[82] S. Pasterski, A. Puhm and E. Trevisani, *Celestial diamonds: conformal multiplets in celestial CFT*, *JHEP* **11** (2021) 072 [arXiv:2105.03516] [INSPIRE].

[83] S. Pasterski, A. Puhm and E. Trevisani, *Revisiting the conformally soft sector with celestial diamonds*, *JHEP* **11** (2021) 143 [arXiv:2105.09792] [INSPIRE].

[84] W. Wieland, *Discrete gravity as a topological field theory with light-like curvature defects*, *JHEP* **05** (2017) 142 [arXiv:1611.02784] [INSPIRE].

[85] H. Elvang, C.R.T. Jones and S.G. Naculich, *Soft Photon and Graviton Theorems in Effective Field Theory*, *Phys. Rev. Lett.* **118** (2017) 231601 [arXiv:1611.07534] [INSPIRE].

[86] A. Laddha and A. Sen, *Sub-subleading Soft Graviton Theorem in Generic Theories of Quantum Gravity*, *JHEP* **10** (2017) 065 [arXiv:1706.00759] [INSPIRE].

JHEP04(2022)125