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Characterization of Useful Topologies in Mathematical Utility Theory by Countable Chain Conditions

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Abstract: Under the additional assumption of complete regularity, we furnish a simple characterization of all the topologies such that every continuous total preorder is representable by a continuous utility function. In particular, we prove that a completely regular topology satisfies such property if, and only if, it is separable and every linearly ordered collection of clopen sets is countable. Since it is not restrictive to refer to completely regular topologies when dealing with this kind of problem, this is, as far as we are concerned, the simplest characterization of this sort available in the literature. All the famous utility representation theorems are corollaries of our result.

Keywords: continuous utility; useful topology; chain; complete regularity

MSC: 91B16 (Primary); 03E72; 06A06 (Secondary)



Citation: Bosi, G.; Zuanon, M.

Characterization of Useful Topologies in Mathematical Utility Theory by Countable Chain Conditions. *Axioms* **2022**, *11*, 195. <https://doi.org/10.3390/axioms11050195>

Academic Editors: Sidney A. Morris and Alexander Šostak

Received: 31 January 2022

Accepted: 21 April 2022

Published: 22 April 2022

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1. Introduction

The study of all the topologies in a set such that every *continuous total preorder* admits a *continuous utility representation* is of great interest in Mathematical Utility Theory. We recall that continuity of a total preorder on a topological space means that the *order topology* associated with the total preorder is contained in the topology under consideration. A topology of this sort is called a *useful topology*. It is noteworthy that one of the most important problems in Mathematical Economics consists of identifying conditions, implying usefulness of a topology. Clearly, in the case of a *compact topology*, usefulness implies that every continuous total preorder achieves minimum and maximum, which can be found by “optimizing” any continuous utility representation.

Herden [1] was the first to introduce the notion of a useful topology. He also inaugurated the most general and efficient approach to Mathematical Utility Theory (see, e.g., Herden [2,3]).

In particular, Herden [4,5] was able to provide the most general results concerning the existence of continuous utility representations for total preorders on a topological space, admitting as corollaries, for example, the classical and famous theorems by Eilenberg [6] (ET) and Debreu [7,8] (DT), which guarantee the continuous representability of every continuous total preorder on a *connected and separable*, and, respectively, on a *second countable* topological space. Therefore, ET and DT illustrate particular situations in order that a topology is useful.

Other authors, who studied this important concept, referred to a *continuously representable topology*, instead of a useful topology (see, e.g., Campión et al. [9] and Candeal et al. [10]).

Estévez and Hervés [11] proved that separability is a necessary condition for the usefulness of a metric topology. Since every separable metric space is second countable, this property can be combined with DT, in order to guarantee that second countability (or, equivalently, *separability*) is equivalent to usefulness when dealing with a metrizable

topology. This latter result will be referred to as Estévez–Hervés’ theorem (EHT).

The aforementioned very important theorems can be therefore restated as follows:

ET: *A topology is useful provided that it is connected and separable.*

DT: *A topology is useful provided that it is second countable.*

EHT: *For a metrizable topology, usefulness and separability (or equivalently second countability) are equivalent concepts.*

Bosi and Herden [12] introduced the notion of a *complete separable system*. They provided relatively simple characterizations of useful topologies, by proving, for example, the equivalence of usefulness of a topology on one hand, and second countability of every subtopology, whose (sub)basis is a complete separable systems, on the other hand (see Bosi and Herden (Theorem 3.1 of [12])).

More recently, Bosi and Zuanon [13] highlighted the following important facts, which intervene when analyzing useful topologies:

1. The *weak topology of continuous functions* is the coarsest topology with the property that all continuous total preorders are still continuous;
2. Since the weak topology of continuous functions is completely regular, actually it is not restrictive to limit ourselves to the consideration of completely regular topologies, when dealing with useful topologies;
3. A useful completely regular topology is necessarily separable.

In this paper, we are, therefore, primarily concerned with completely regular topologies. We introduce the concept of *strong open and closed countable chain condition* (SOCCC) of a topology (which strengthens both the concept of *open and closed countable chain condition* (OCCC), introduced by Herden and Pallack [3]), and the concept of *weak open and closed countable chain condition* (WOCCC), which was recently introduced by Bosi and Zuanon [14].

We shall say that a topology t on a given nonempty set X satisfies SOCCC if every *chain* (i.e., every nested family) \mathcal{O} of clopen subsets of X is countable. We prove that, for *completely regular* topologies, usefulness on one hand and separability plus SOCCC on the other hand are equivalent concepts. Condition SOCCC is considerably simpler than the other countable chain conditions used in the previous characterizations of useful topologies. This is the main characteristic feature of the present paper.

Since the *weak topology* of continuous functions is completely regular, and it is the coarsest topology such that all continuous total preorders are still continuous, we have that a topology is useful if, and only if, its weak topology is separable, and it satisfies SOCCC. This is, to the best of our knowledge, the simplest characterization of useful topologies. In addition, condition SOCCC is very simple and easy to be understood.

We further show that if a topology is separable, and it satisfies SOCCC, then it is useful, independently from the fact that it is completely regular or not. The famous aforementioned theorems by Eilenberg and Debreu are corollaries of this result.

2. Notation and Preliminaries

In the present paper, we shall be exclusively concerned with ZFC (Zermelo–Fraenkel + Axiom of Choice) set theory.

We recall some basic classical definitions concerning (pre)orders on topological spaces.

Definition 1. *Let \preceq be a binary relation on a nonempty set X (i.e., $\preceq \subset X \times X$). Then \preceq is defined to be*

1. *reflexive, if $x \preceq x$, for every $x \in X$;*
2. *transitive, if*

$$(x \preceq y) \text{ and } (y \preceq z) \Rightarrow (x \preceq z) \text{ for all } x, y, z \in X;$$
3. *antisymmetric, if*

$$(x \preceq y) \text{ and } (y \preceq x) \Rightarrow (x = y) \text{ for all } x, y \in X;$$

4. total, if $(x \preceq y)$ or $(y \preceq x)$, for all $x, y \in X$;
5. linear (or complete), if either $(x \preceq y)$ or $(y \preceq x)$, for all $x \neq y (x, y \in X)$;
6. a preorder, if \preceq is reflexive and transitive;
7. an order, if \preceq is an antisymmetric preorder;
8. a chain, if \preceq is a linear order.

If \preceq is a preorder on X , then its strict part (or asymmetric part) \prec , and its symmetric part \sim are defined as follows, for all $x, y \in X$:

$$x \prec y \Leftrightarrow (x \preceq y) \text{ and } \text{not}(y \preceq x),$$

$$x \sim y \Leftrightarrow (x \preceq y) \text{ and } (y \preceq x).$$

It is nearly immediate to verify that \sim is an equivalence on X . The corresponding quotient set is denoted by $X |_{\sim}$, and $[x]_{\sim} = \{z \in X : z \sim x\}$ stands for the equivalence class corresponding to the element $x \in X$. The symbol $\preceq|_{\sim}$ stands for the quotient order on the quotient set $X |_{\sim}$, which is defined as follows, for all $x, y \in X$:

$$[x]_{\sim} \preceq|_{\sim} [y]_{\sim} \Leftrightarrow x \preceq y.$$

If \preceq is a total preorder on X , then $\preceq|_{\sim}$ is a linear order on $X |_{\sim}$.

A subset D of a preordered set (X, \preceq) is defined to be decreasing if, for all $z \in X$,

$$(x \in D) \text{ and } (z \preceq x) \Rightarrow z \in D.$$

Let t be a topology on a set X . A family $\mathcal{B}' \subset t$ is defined to be a subbasis of t if the family \mathcal{B} , whose elements are all the intersections of finitely many elements of \mathcal{B}' , is a basis of t (i.e., every set $O \in t$ can be expressed as the union of a family of sets in \mathcal{B}).

Let us summarize, in the following definition, the main classical topological concepts which will be used in this paper.

Definition 2. A topology t on X is defined to be

- (i) second countable, if t has a countable basis $\mathcal{B} = \{B_n : n \in \mathbb{N}^+\}$;
- (ii) separable, if there is a countable subset D of X such that $D \cap O \neq \emptyset$ for every nonempty $O \in t$;
- (iii) completely regular, if for every $x \in X$, and every closed set $F \subseteq X$ such that $x \notin F$, there exists a continuous function $f : (X, t) \rightarrow ([0, 1], t_{nat})$ such that $f(x) = 0$ and $f(y) = 1$ for every $y \in F$.

In the sequel, we shall denote by t_{nat} the natural (interval) topology on the real line \mathbb{R} .

Given a set X , a topology t' on set X is called coarser (respectively, finer) than another topology t on X if $t' \subset t$ (respectively, $t \subset t'$). In case that t is finer than t' , then t' is defined to be a subtopology of t .

Consider any preordered set (X, \preceq) . Then we introduce the following notation, concerning the sections associated to every point $x \in X$:

$$d_{\preceq}(x) := \{z \in X : z \preceq x\}, \quad i_{\preceq}(x) := \{z \in X : x \preceq z\},$$

$$l_{\preceq}(x) := \{z \in X : z \prec x\}, \quad r_{\preceq}(x) := \{z \in X : x \prec z\}.$$

Given a preordered set (X, \preceq) , for two elements $(x, y) \in X \times X$ such that $x \prec y$ (equivalently, $(x, y) \in \prec$), we denote by $]x, y[_{\preceq}$ the (possibly empty) open interval

$$]x, y[_{\preceq} := r_{\preceq}(x) \cap l_{\preceq}(y) = \{z \in X : x \prec z \prec y\}.$$

If $x, y \lceil_{\prec} = \emptyset$, then the pair $(x, y) \in \prec$ is defined to be a *jump* in (X, \prec) .

The notation and terminology used here are the same as those adopted by Herden [4].

Let us now recall the basic definition of *continuity* relative to a total preorder on a topological space.

Definition 3. Let (X, t) be a topological space. A total preorder \prec on (X, t) is defined to be continuous if, for every element $x \in X$, the sets $l_{\prec}(x) = \{z \in X : z \prec x\}$ and $r_{\prec}(x) = \{z \in X : x \prec z\}$ are open subsets of X .

It is clear that an equivalent definition of continuity of a total preorder \prec on (X, t) could require that the sets $d_{\prec}(x) = \{z \in X : z \prec x\}$ and $i_{\prec}(x) = \{z \in X : x \prec z\}$ are closed for every element $x \in X$.

If \prec is a preorder on a set X , then denote by t^{\prec} the *order topology* on X corresponding to \prec , in the sense that t^{\prec} is the topology generated by the family $\{l_{\prec}(x) : x \in X\} \cup \{r_{\prec}(x) : x \in X\}$. This means that t^{\prec} is the topology on X whose subbasis is $\{l_{\prec}(x) : x \in X\} \cup \{r_{\prec}(x) : x \in X\}$. It is immediate to check that the continuity of a preorder \prec on a topological space (X, t) is equivalent to the requirement according to which the order topology t^{\prec} is coarser than t . This fact can be expressed by saying that the coarsest topology on X such that the sets $l_{\prec}(x)$ and $r_{\prec}(x)$ are open for every $x \in X$ coincides with t^{\prec} .

Let t be a topology on X , and consider any (nonempty) subset X' of X . The *relativized topology* $t|_{X'}$ on X' is the topology

$$t|_{X'} := \{O \cap X' : O \in t\}.$$

Definition 4. Consider a total preorder \prec on a set X . Then a mapping $u : (X, \prec) \mapsto (\mathbb{R}, \leq)$ is defined to be a *utility representation* (shortly, a *utility*) for \prec , if for all pairs $(x, y) \in X \times X$,

$$x \prec y \Leftrightarrow u(x) \leq u(y).$$

We now need to recall the definition of a *complete separable system* presented by Bosi and Herden [12].

Definition 5. Let a topology t on X be given. A family \mathcal{E} of open subsets of the topological space (X, t) , such that $\bigcup_{E \in \mathcal{E}} E = X$, is defined to be a *complete separable system* on (X, t) if the following statements are verified:

- S1:** There exist sets $E_1 \in \mathcal{E}$ and $E_2 \in \mathcal{E}$ such that $\overline{E_1} \subset E_2$.
- S2:** For all sets $E_1 \in \mathcal{E}$ and $E_2 \in \mathcal{E}$ such that $\overline{E_1} \subset E_2$, there exists some set $E_3 \in \mathcal{E}$ such that $\overline{E_1} \subset E_3 \subset \overline{E_3} \subset E_2$.
- S3:** For all sets $E \in \mathcal{E}$ and $E' \in \mathcal{E}$, at least one of the following conditions $E = E'$, or $\overline{E} \subset E'$, or $\overline{E'} \subset E$ holds.

If \prec is a preorder on X , a *complete separable system* \mathcal{E} on (X, t) is defined to be a *decreasing separable system* on the preordered topological space (X, \prec, t) when every set $E \in \mathcal{E}$ is decreasing.

Bosi and Herden (Proposition 2.1 of [12]) proved the following result, illustrating relevant properties of a complete separable system.

Proposition 1. Let \mathcal{E} to be a complete separable system on a topological space (X, t) . Then \mathcal{E} satisfies the following conditions.

- (i) $\mathcal{E}^c := \mathcal{E} \cup \{\overline{E} : E \in \mathcal{E}\}$ is linearly ordered by set inclusion;
- (ii) $E = \bigcup_{\overline{E'} \subset E, E' \in \mathcal{E}} E' = \bigcup_{\overline{E'} \subset E, E' \in \mathcal{E}} \overline{E'}$ for every $E \in \mathcal{E}$;

$$(iii) \quad \bar{E} = \bigcap_{E' \in \mathcal{E}} E' = \bigcap_{\bar{E}' \in \mathcal{E}} \bar{E}' \text{ for every } E \in \mathcal{E}.$$

The following characterization of a continuous utility holds (see Herden Theorem 3.1 (i) \Leftrightarrow (iii) of [5]).

Theorem 1. *The following statements are equivalent on a (total) preorder \succsim on a topological space (X, t) :*

- (i) *There is a continuous utility representation u for \succsim ;*
- (ii) *There is a countable decreasing complete separable system $\mathcal{E} = \{E_n\}_{n \in \mathbb{N}}$ on (X, t) with the property that, for all $x, y \in X$ with $x \prec y$, there exists $n \in \mathbb{N}$ with $x \in E_n, y \notin E_n$.*

We recall the classical definition of a *scale* in a topological space (see, e.g., Gillman and Jerison [15] and Burgess and Fitzpatrick [16]).

Definition 6. *If (X, t) is a topological space and \mathbb{S} is a dense subset of $[0, 1]$ such that $1 \in \mathbb{S}$, then a family $\mathcal{E} = \{O_r\}_{r \in \mathbb{S}}$ of open subsets of X is defined to be a *scale* in (X, t) if the following conditions hold:*

- (i) $O_1 = X$;
- (ii) $\overline{O_{r_1}} \subseteq O_{r_2}$ for every $r_1, r_2 \in \mathbb{S}$ such that $r_1 < r_2$.

It is clear that a scale is in particular a complete separable system.

3. New Characterization of Useful Topologies

Let us now present the new condition on a topology, which will be used in the novel characterization of useful and completely regular topologies we are going to prove.

Definition 7. *A topology t on a set S is defined to verify the *strong open and closed countable chain condition* (SOCCC) if every chain \mathcal{O} of clopen subsets of (X, t) is countable.*

We now recall the definition of the continuous total preorder, which is induced by a complete separable system \mathcal{E} on a topological space (X, t) .

Definition 8. *For every complete separable system \mathcal{E} on (X, t) , the *continuous total preorder* $\succsim_{\mathcal{E}}$ on (X, t) , that is induced by \mathcal{E} is defined to be*

$$\succsim_{\mathcal{E}} := \{(x, y) \in X \times X \mid \forall E \in \mathcal{E} (y \in E \Rightarrow x \in E)\}.$$

Clearly, we shall indicate by $\sim_{\mathcal{E}}$ the indifference relation induced by the total preorder $\succsim_{\mathcal{E}}$. The proof of the following lemma is found in Bosi and Zuanon Lemma 3.8 of [14].

Lemma 1. *Let \mathcal{E} be a complete separable system on a topological space (X, t) . Then there is a one-to-one correspondence between the family of all the clopen subsets $O \in \mathcal{E}$ such that*

$$O \subsetneq \bigcap_{O'' \in \mathcal{O}, O'' \in \mathcal{O}} O''$$

and the family of all the jumps $([x], [y])$ in $(X \mid \sim_{\mathcal{E}}, \succsim_{\mathcal{E} \mid \sim_{\mathcal{E}}})$.

Let us now present a new proposition, which is crucial in order to provide the desired characterization of useful topologies.

Proposition 2. *Let t be a useful and separable topology on a set X . Then t satisfies SOCCC.*

Proof. Consider a useful and separable topology t on a set X . We want to prove that t satisfies SOCCC. To this aim, let \mathcal{O} be any chain of clopen subsets of (X, t) . Then, \mathcal{O} is a complete separable system (indeed, it is not restrictive to presuppose that $X \in \mathcal{O}$); therefore, the topology $t_{\mathcal{O}}$ generated by \mathcal{O} is second countable by Bosi and Herden Theorem 3.1 of [12]. Notice that there exist only countably many sets $O \in \mathcal{O}$ such that

$$O \subsetneq \bigcap_{O' \subsetneq O, O' \in \mathcal{O}} O'$$

Indeed, by the above Lemma 1, there is a one-to-one correspondence between the set of all such sets O and the set of all the jumps $([x], [y])$ in $(X |_{\sim_{\mathcal{O}}}, \lesssim_{\mathcal{O}})$, where $\lesssim_{\mathcal{O}}$ is the continuous total preorder on (X, t) induced by \mathcal{O} (see the above Definition 8). Therefore, in order to show that \mathcal{O} is (at most) countable, it is not restrictive to presuppose that, for every $O \in \mathcal{O}$,

$$O = \bigcap_{O' \subsetneq O, O' \in \mathcal{O}} O'$$

Let $\mathcal{B} = \{O_n\}_{n \in \mathbb{N}} \subset \mathcal{O}$ be a countable basis of $t_{\mathcal{O}}$. Then (\mathcal{B}, \subset) is a countable chain of clopen sets. In addition, (\mathcal{B}, \subset) is *dense in itself* (i.e., for all $O', O'' \in \mathcal{B}$ such that $O' \subsetneq O''$, there exists $O''' \in \mathcal{B}$ such that $O' \subsetneq O''' \subsetneq O''$), since every set $O \in \mathcal{B}$ is equal to the intersection of all sets O'' in \mathcal{B} properly containing O . From Theorem 1 on page 31 in Birkhoff [17], we have that \mathcal{B} can be indexed according to the rational numbers of the real interval $]0, 1[$ in an ordered fashion, i.e., $\mathcal{B} = \{O_q\}_{q \in \mathbb{Q} \cap]0, 1[}$. Define, for every $O \in \mathcal{O}$,

$$O = O_s \Leftrightarrow O = \bigcap_{q \in \mathbb{Q} \cap]0, 1[, O \subsetneq O_q} O_q \quad (s = \inf\{q \in \mathbb{Q} \cap]0, 1[: O \subsetneq O_q\}).$$

In this way, we make up a scale $\mathcal{O} = \{O_s\}_{s \in \mathbb{S}}$ of clopen subsets of (X, t) . Assume, that \mathcal{O} has uncountably many elements. Then, there exist uncountably many pairs $(s_{\alpha}, s_{\alpha'})$ with $s_{\alpha} < s_{\alpha'}$ ($s_{\alpha}, s_{\alpha'} \in \mathbb{S}$) such that the sets of the form $O_{s_{\alpha'}} \setminus O_{s_{\alpha}}$ are pairwise disjoint and open. This clearly contradicts the separability of $t_{\mathcal{O}}$. \square

Let us now present the new characterization of completely regular and useful topologies.

Theorem 2. *Let t be a completely regular topology on a set X . Then the following statements are equivalent:*

- (i) t is useful;
- (ii) The following statements hold true:
 - (a) t is separable;
 - (b) t satisfies SOCCC.

Proof. (i) \Rightarrow (ii). If t is useful and completely regular, then it is separable (see Bosi and Zuanon Theorem 3.1 of [13], and the proof of this implication is an immediate consequence of the above Proposition 2.

(ii) \Rightarrow (i). If t satisfies SOCCC, then every chain \mathcal{O} of clopen subsets of (X, t) only contains countably many sets O such that

$$O \subsetneq \bigcap_{O' \subsetneq O, O' \in \mathcal{O}} O'$$

Therefore, the thesis follows by Bosi and Zuanon Theorem 3.1 of [13]. \square

Example 1. Consider the topology t of the Sorgenfrey line, also called the right half-open interval topology (see Steen and Seebach [18]). Then t is a zero-dimensional separable space which does not satisfy SOCCC, so the topology of the Sorgenfrey line is not useful by the above Theorem 2.

An analysis of the proof of the above Theorem 2 shows that the following proposition holds, presenting a sufficient condition in order that a topology (not necessarily completely regular) is useful.

Proposition 3. *If a topology t on a set X is separable and satisfies SOCCC, then it is useful.*

If (X, t) is any topological space, then denote by $\sigma(X, C(X, t, \mathbb{R}))$ the weak topology of continuous functions on (X, t) . It is very well known that $(X, \sigma(X, C(X, t, \mathbb{R})))$ is a completely regular space (cf., e.g., Cigler and Reichel, Satz 10, page 101 [19], and Aliprantis and Border, Theorem 2.55 and Corollary 2.56 of [20]).

Let us present a general characterization of useful topologies, which is based on the above Theorem 2, and on the consideration that any continuous total preorder on (X, t) is continuous if and only if it is continuous on $(X, \sigma(X, C(X, t, \mathbb{R})))$ (see, e.g., Bosi and Zuanon, Lemma 3.1 of [13]). We furnish a full proof of this result for the sake of clarity and completeness.

Theorem 3. *A topology t on a nonempty set X is useful if and only if $\sigma(X, C(X, t, \mathbb{R}))$ is separable and satisfies SOCCC.*

Proof. Assume that t is a useful topology on X . Then $\sigma(X, C(X, t, \mathbb{R}))$ is also useful. Indeed, every total preorder \preceq that is continuous with respect to $\sigma(X, C(X, t, \mathbb{R}))$ is also continuous with respect to t according to Bosi and Zuanon, Theorem 2.23, (9) \Rightarrow (1) of [21]. Then there exists a utility representation $u : X \rightarrow \mathbb{R}$ that is continuous with respect to t , hence u is continuous with respect to $\sigma(X, C(X, t, \mathbb{R}))$. Since the topology $\sigma(X, C(X, t, \mathbb{R}))$ is completely regular and useful, applying Theorem 2, (i) \Rightarrow (ii) for this topology, one obtains that $\sigma(X, C(X, t, \mathbb{R}))$ is separable and satisfies SOCCC.

Conversely, if $\sigma(X, C(X, t, \mathbb{R}))$ is separable and satisfies SOCCC, then from Theorem 2, (ii) \Rightarrow (i) we have that $\sigma(X, C(X, t, \mathbb{R}))$ is useful, which in turn implies that t is also useful. This consideration completes the proof. \square

The folk theorems by Eilenberg [6] and Debreu [7,8] can be now viewed as immediate consequences of Proposition 3.

Theorem 4 (Eilenberg theorem). *If a topology t on a set X is connected and separable, then it is useful.*

Proof. Observe that t is separable, and condition SOCCC is obviously verified since t is connected. \square

Theorem 5 (Debreu theorem). *If a topology t on a set X is second countable, then it is useful.*

Proof. Let t be any second countable topology. Consider any chain \mathcal{O} of clopen subsets of (X, t) . Then, the topology $t_{\mathcal{O}}$ generated by \mathcal{O} is a (completely regular) linearly ordered subtopology of t , which is therefore itself second countable by Bosi and Zuanon Lemma 3.2 of [13]. As a consequence of the property of second countability of t , we have that there are only countably many sets $O \in \mathcal{O}$ such that

$$O \subsetneq \bigcap_{O=O'', O'' \in \mathcal{O}} O''.$$

Since t is second countable, we have that t is obviously separable, and we can actually follow the proof of Proposition 2 in order to guarantee that (X, t) satisfies SOCCC. Hence, we apply the sufficient condition provided by Proposition 3. \square

Estévez and Hervés [11] theorem can be considered as a particular case of Theorem 2.

Theorem 6 (Estévez-Hervés' theorem). *Separability and usefulness are equivalent concepts on a metrizable topology t on a set X .*

Proof. If a metrizable topology is separable, then it is second countable, and therefore we can refer to the above Theorem 5. On the other hand, a metrizable topology is completely regular, and therefore, if it is useful, it is also separable by Theorem 2. \square

4. Conclusions

We have presented a new characterization of *useful topologies*, i.e., topologies on a set such that, for every continuous total preorder, there exists a continuous utility function representing it. Our main result shows that a topology is useful if, and only if, the weak topology of continuous functions is separable, and it satisfies the SOCCC condition, which is newly introduced in this paper, stating that every chain \mathcal{O} of clopen subsets of (X, t) is countable. This is, in our opinion, the simplest characterization of useful topologies which may be achieved. The famous utility representation theorems easily follow from our results.

Author Contributions: The authors contributed equally. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: Helpful suggestions for improvement by three anonymous referees are gratefully acknowledged.

Conflicts of Interest: The authors declare no conflict of interest.

References

- Herden, G. Topological spaces for which every continuous total preorder can be represented by a continuous utility function. *Math. Soc. Sci.* **1991**, *22*, 123–136. [[CrossRef](#)]
- Herden, G.; Pallack, A. On the continuous analogue of the Szpilrajn Theorem I. *Math. Soc. Sci.* **2000**, *43*, 115–134. [[CrossRef](#)]
- Herden, G.; Pallack, A. Useful topologies and separable systems. *Appl. Gen. Topol.* **2000**, *1*, 61–81. [[CrossRef](#)]
- Herden, G. On the existence of utility functions. *Math. Soc. Sci.* **1989**, *17*, 297–313. [[CrossRef](#)]
- Herden, G. On the existence of utility functions II. *Math. Soc. Sci.* **1989**, *18*, 107–111. [[CrossRef](#)]
- Eilenberg, S. Ordered topological spaces. *Am. J. Math.* **1941**, *63*, 39–45. [[CrossRef](#)]
- Debreu, G. Representation of a preference ordering by a numerical function. In *Decision Processes*; Thrall, R., Coombs, C., Davies, R., Eds.; Wiley: New York, NY, USA, 1954.
- Debreu, G. Continuity properties of Paretian utility. *Int. Econom. Rev.* **1954**, *5*, 285–293. [[CrossRef](#)]
- Cami3n, M.J.; Candeal, J.C.; Indur3in, E. Preorderable topologies and order-representability of topological spaces. *Topol. Appl.* **2009**, *156*, 2971–2978. [[CrossRef](#)]
- Candeal, J.C.; Herv3s, C.; Indur3in, E. Some results on representation and extension of preferences. *J. Math. Econom.* **1998**, *29*, 75–81. [[CrossRef](#)]
- Est3vez, M.; Herv3s, C. On the existence of continuous preference orderings without utility representation. *J. Math. Econom.* **1995**, *24*, 305–309.
- Bosi, G.; Herden, G. The structure of useful topologies. *J. Math. Econom.* **2019**, *82*, 69–73. [[CrossRef](#)]
- Bosi, G.; Zuanon, M. Topologies for the Continuous Representability of All Continuous Total Preorders. *J. Optim. Theory Appl.* **2021**, *188*, 420–431. [[CrossRef](#)]
- Bosi, G.; Zuanon, M. A simple characterization of useful topologies in mathematical utility theory. *Bull. Iranian Math. Soc.* **2022**, to appear. [[CrossRef](#)]
- Gillman, L.; Jerison, M. *Rings of Continuous Functions*; D. Van Nostrand Company: Princeton, NJ, USA, 1960.
- Burgess, D.C.J.; Fitzpatrick, M. On separation axioms for certain types of ordered topological space. *Math. Proc. Cambridge Philos. Soc.* **1977**, *82*, 59–65. [[CrossRef](#)]
- Birkhoff, G. *Lattice Theory*; American Mathematical Society Colloquium Publications: Providence, RI, USA, 1940.
- Steen, L.A.; Seebach, J.A. *Counterexamples in Topology*, 2nd ed.; Springer: Heidelberg Germany, 1978.
- Cigler, J.; Reichel, H.C. *Topologie*; Bibliographisches Institut, Mannheim-Wien-Z3rich: Gotha, Germany, 1978.
- Aliprantis, C.D.; Border, K.C. *Infinite Dimensional Analysis—A Hitchhiker's Guide*; Springer: Berlin/Heidelberg, Germany, 2006.
- Bosi, G.; Zuanon, M. Continuity and continuous multi-utility representations of nontotal preorders: Some considerations concerning restrictiveness, In *Mathematical Topics on Representations of Ordered Structures and Utility Theory*; Book in Honour of G.B. Mehta; Bosi, G., Cami3n, M.J., Candeal, J.C., Indur3in, E., Eds.; Springer: Berlin/Heidelberg, Germany, 2020; pp. 213–236.