State Dependent Switching Control of Affine Linear Systems With Dwell Time: Application to Power Converters

Antonio Russo, Gian Paolo Incremona, Alberto Cavallo and Patrizio Colaneri

***Abstract*— This paper addresses a state dependent switching law for the stabilization of continuous-time, switched affine linear systems satisfying dwell time constraints. Such a law is based on the solution of Lyapunov-Metzler inequalities from which stability conditions are derived. The key point of this law is that the switching rule calculation depends on the evolution forward by the dwell time of quadratic Lyapunov functions assigned to each subsystem. As such, the proposed law is readily applicable to power converters showing that it is an interesting alternative to other switching control techniques.**

***Index Terms*— Switched systems, Lyapunov-Metzler inequalities, dwell time, power converters.**

I. Introduction

Switched systems are a family of hybrid systems characterized by continuous time dynamics alternatively selected by a switching rule point-wise in time [1]. Moreover, the theory of such switched systems offers an elegant implementation of capturing dynamics which intrinsically incorporate many subsystems defined via the use of a switching signal. However, designing a stabilizing switching rule is not trivial. In fact, in most cases the switched system does not directly inherit the stability properties of the individual subsystems (see [1, Ch. 2]).

Still, as solution of the stabilizing switching problem, two different choices of the switching rule can be found in the literature. On the one hand, *arbitrary switching* is characterized by the absence of any temporal constraint on the switching rule, that is commutations among subsystems can occur arbitrarily fast. On the other hand, a *dwell time*, that is a minimum amount of time between two consecutive switching instants, can be introduced. Although an arbitrary switching approach allows an easier theoretical analysis, in practical cases, because of actuators delays or limited bandwidth, it is necessary to dwell a minimum amount of time in the currently active subsystem before being able to switch to a different one.

Such type of approach has been successfully used in several works starting from nineties, see e.g., [2]–[5]. These results, among many others, cover a subclass of dynamical models whose subsystems share a common equilibrium point. For instance, in [5], the problem of stabilizing a linear switched system with dwell time is solved through Lyapunov-Metzler inequalities.

Nevertheless, it could be more realistic to aim at stabilizing affine switched systems whose subsystems may not share a common equilibrium point. Among many others, examples of these systems are provided in [6] and [7], where the state boundedness is proved by virtue of switching signals fulfilling an average dwell time and a dwell time bound, respectively. In [8], robustness to external disturbances of switched discrete and continuous-time dynamics with multiple equilibria was discussed by resorting to the notion of Input-to-State Stability (ISS), provided that sufficient dwell time conditions are satisfied. Recently, in [9] the case with hybrid affine systems characterized by periodic time-triggered switching is addressed. Analogously, in [10], a control law for switched converters with Pulse-Width Modulation (PWM) is proposed based on a Lyapunov function candidate and on hybrid dynamical systems theory. In [11], a switching rule is generated through the solution of differential Linear Matrix Inequalities (LMIs), by taking into account a convex combination of the switching subsystems and imposing fixed dwell time to make the state reach a pre-specified limit cycle. In [12], instead, the switching law is a priori selected without taking into account any dwell time, which is a posteriori estimated. Such a law is based on a Lyapunov function, suitably regularized in terms of time and space in order to reduce the switches number and avoid infinitely fast switching.

Summarizing the above, since switched affine systems can exhibit the undesired phenomenon of high frequency switching, suitable control actions capable of guaranteeing a dwell time must be implemented to solve this problem. Most switching laws make the subsystems switch using fixed intervals or estimating a posteriori the dwell time. However, adding a dwell time constraint in the design of the switching law may ensure better results [5] in terms of actuators effort. Motivated by this fact and inspired by [5], this note aims at putting forward an alternative method to the design of a switching rule enabling the following features:

1) a dwell time which can be suitably adjusted in the switching rule by the designer to guarantee satisfactory

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performance in whatever appropriate sense;

2) a practical stabilization of the state trajectory around the so-called switching equilibrium, i.e., the prescribed reference point;
3) a guaranteed bound on the average $\mathcal{H}_2$ cost function, depending on the dwell time.

Furthermore, it is worth to mention that our interest in affine linear systems stems from their application to the control of power converters. Indeed, control of these devices has attracted large attention in the past decades due to its applicability to different fields, such as smart grids and transportation systems (see, e.g., [13]–[15], among many others). Despite their intrinsic discontinuous nature, large part of the literature on power converters focuses on the control of their averaged model. In fact, continuous time techniques can be designed to control the duty cycle which drives the PWM associated to the switches position. Such techniques focus on capturing the low-frequency behaviour of power electronic converters while neglecting high-frequency variations due to circuit switching (see e.g., [16, Ch. 4]). Moreover, typical continuous-time control laws do not take into account requirements on the boundedness of the control variable, which should instead be a null or positive signal bounded by unitary magnitude. Therefore, addressing input boundedness for continuous time control would require an additional layer of complexity. Finally, since most of power converters (for instance, buck, boost, buck-boost, Cùk and Sepic to name a few) can be modeled as affine linear switched systems [17], this attractive feature suggests the definition of a general switched control framework applicable to all these devices sharing such a common structure.

The organization of the paper is as follows. The considered problem is formulated in Section II. Our proposal is presented and theoretically analysed in Section III, while Section IV describes the application to power converters and an illustrative example based on a boost converter. Moreover, a comparison is presented with the algorithm introduced in [11], which also addresses switched affine linear systems. Finally, concluding remarks are drawn is Section V.

Notation: The transpose of a matrix A is denoted by A'. The sets of reals is notated as $\mathbb{R}$, while the sets of non-negative real and natural numbers are $\mathbb{R}_+$ and $\mathbb{N}_+$, respectively. Signals in the time domains are denoted by lower-case letters, like $x(t)$, or just x. For Hermitian matrices, $X > 0$ (resp. $X \geq 0$) indicates that X is positive (resp. semi-positive) definite, and $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ denote its minimum and maximum eigenvalue, respectively. $\mathcal{L}_\infty$ norm of a signal $f : \mathbb{R}_+ \to \mathbb{R}$ is $\|f\|_\infty := \sup_{t\in\mathbb{R}_+}|f(t)|$.

II. Problem Statement

Consider a switched affine linear system captured by

$$\dot{x}(t) = A_{\sigma(t)}x(t) + b_{\sigma(t)},\ x(0) = x_0,\ \ \sigma(0) = i_0, \qquad (1)$$

where the state $x \in \mathbb{R}^n$ is available for feedback for all $t \geq 0$, $A_\sigma \in \{A_1, \ldots, A_M\}$, and $b_\sigma \in \{b_1, \ldots, b_M\}$ are constant matrices of appropriate dimension modelling the M subsystems, for $M \in \mathbb{N}_+$. Let $\Omega := \{1, \ldots, M\}$, and $i_0 \in \Omega$. Then, denoting as $\{t_k\}$, $k = 1, \ldots, \infty$, the monotonically increasing sequence of switching time instants, the goal is to determine the switching rule $u(x(t)) : \mathbb{R}^n \to \Omega$, such that $\sigma(t) := u(x(t), t)$ is stabilizing for system (1). Moreover, let Ω_i denote the set of integers $\{1, \ldots, i-1, i+1, \ldots, M\}$, and introduce $y \in \mathbb{R}^p$ as the system output determined by $y(t) = C_0x(t)$ with $C_0 \in \mathbb{R}^{p\times n}$.

Motivated by [5], the idea underlying the proposed state dependent switching control is to obey a dwell time constraint by generating a switching rule which takes into account the forward evolution of a T-length interval of the Lyapunov Function (LF) $V(x,t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ with $V(x,t) = x'P_{\sigma(t)}x$, P_σ being positive definite. In the following, for the sake of simplicity, the dependence on the state of $V(x,t)$ will be omitted when it is clear from the context. In this way, we want to guarantee that, given the dwell time $T > 0$, if $t_0 = 0$ is the initial time instant, then $t_{k+1} - t_k > T$ for any $k \geq 1$. The present work is therefore focused on practically stabilizing this system via a state-feedback control, according to the following definition

Definition 1 ([18]): System (1) is practically stable with respect to $(\mathcal{C}_1, \mathcal{C}_2, t_0, T)$, $\mathcal{C}_1 \subset \mathcal{C}_2$, if $x(t_0) \in \mathcal{C}_1$ implies $x(t) \in \mathcal{C}_2$ for all $t \in [t_0, t_0 + T)$.

Moreover, we prove the existence of a bound on

$$\lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} x'(\tau)C_0'C_0x(\tau)\mathrm{d}\tau,$$

and we would like to provide a characterization of the region of attraction depending on the magnitude of b_σ and T. Note that, the selection of the dwell time T depends on the specific application, constrained to the fulfilment of Lyapunov-Metzler inequalities hereafter introduced.

III. The Proposed State-Feedback Switching Rule

Differently from [5], where linear unforced systems where considered, in this work the switching rule depends also on a term m_i, which is the forced motion of the state due to the term b_i, that is $m_i := \int_t^{t+T} \mathrm{e}^{A_i(t+T-\tau)}b_i\mathrm{d}\tau$.

Theorem 1: Assume that for some $T > 0$, $\varepsilon > 0$, and scalars $\lambda_{i,j} \geq 0$, $i \in \Omega$ and $j \in \Omega_i$, there exists a collection of positive definite matrices P_i of compatible dimension, such that

$$A_i'P_i + P_iA_i + \sum_{j\in\Omega_i} \lambda_{i,j}(\mathrm{e}^{A_j'T}P_j\mathrm{e}^{A_jT} - P_i) + \varepsilon P_i + C_0'C_0 < 0 \qquad (2)$$

for all $i \in \Omega$. Assume also that system (1) switched at $t = t_k$ so that $u(x(t), t) = i$. Then, the following switching law

$$u(x(t), t) = i \ \ \forall t \in [t_k, t_k + T] \qquad (3a)$$

$$u(x(t), t) = i \ \ \forall t > t_k + T, \text{ if } x(t)'(\mathrm{e}^{A_j'T}P_j\mathrm{e}^{A_jT} - P_i)x(t) + 2m_j'P_j\mathrm{e}^{A_jT}x(t) + m_j'P_jm_j \geq 0 \ \ \forall\, j \in \Omega_i \qquad (3b)$$

$$u(x(t_{k+1}), t_{k+1}) = \underset{j\in\Omega_i}{\operatorname{argmin}}\, x(t_{k+1})'(\mathrm{e}^{A_j'T}P_j\mathrm{e}^{A_jT})x(t_{k+1}) + 2m_j'P_j\mathrm{e}^{A_jT}x(t_{k+1}) + m_j'P_jm_j, \text{ otherwise}, \qquad (3c)$$

where the next switching instant is

$$t_{k+1} \coloneqq \inf_{t>t_k+T}\{t \mid \exists j\colon x(t)'(\mathrm{e}^{A_j'T}P_j\mathrm{e}^{A_jT}-P_i)x(t) + 2m_j'P_j\mathrm{e}^{A_jT}x(t)+m_j'P_jm_j<0\},$$

makes system (1) practically stable with dwell time T. Moreover, the associated cost is bounded as

$$\lim_{\bar{k}\to\infty}\frac{1}{\bar{k}}\int_0^{\tau_{\bar{k}}} x(\tau)'C_0'C_0x(\tau)\mathrm{d}\tau<\lambda_{\max}(C_0'C_0)T\|x\|_\infty^2+\delta(T) \quad (4)$$

with $\tau_k = t_k+T$,

$$\delta(T)=\frac{2}{\varepsilon}\max_i\Big(\eta_i+\frac{2\gamma_i'P_i^{-1}\gamma_i}{\varepsilon}\Big), \eta_i(b_i,T)\coloneqq\sum_{j\in\Omega_i}\lambda_{i,j}m_j'P_jm_j$$

and

$$\gamma_i(b_i,T)\coloneqq P_ib_i+\sum_{j\in\Omega_i}\lambda_{i,j}\mathrm{e}^{A_j'T}P_jm_j.$$

Proof: In line with arguments in [5], the proof is divided into two steps. Given the state switching control $\sigma(t)=u(x(t),t)$, choose the LF $V(x(t),t)=x(t)'P_{\sigma(t)}x(t)$, such that, given T, $P_{\sigma(t)}$ satisfies (2) for the ith subsystem. If $t\geq t_k+T$ the switching is allowed. Since $\sigma(t)=i$ in the interval $t\in[t_k+T,t_{k+1})$ is constant, we have that the total time derivative of V along the trajectory of the system is upper-bounded as

$$\begin{aligned}\dot{V}(x(t),t)<{}&2x(t)'P_ib_i-\varepsilon V(x(t),t)-x(t)'C_0'C_0x(t)\\&+\sum_{j\in\Omega_i}\lambda_{i,j}\big(2m_j'P_j\mathrm{e}^{A_jT}x(t)+m_j'P_jm_j\big)\\&-\sum_{j\in\Omega_i}\lambda_{i,j}\big(x(t)'(\mathrm{e}^{A_j'T}P_j\mathrm{e}^{A_jT}-P_i)x(t)\\&\quad+2m_j'P_j\mathrm{e}^{A_jT}x(t)+m_j'P_jm_j\big),\end{aligned} \quad (5)$$

with m_j being independent of the switching instant t_k since

$$\int_t^{t+T}\mathrm{e}^{A_j(t+T-\tau)}b_j\mathrm{d}\tau=\int_0^T\mathrm{e}^{A_j(T-\tau)}b_j\mathrm{d}\tau.$$

Introducing $\gamma_i(b_i,T)$ and $\eta_i(b_i,T)$, we notice that in inequality (5) the last term is negative definite by virtue of (3), so that $\dot{V}$ is upper-bounded by

$$\begin{aligned}\dot{V}(x(t),t)<{}&2x(t)'\gamma_i(b_i,T)+\eta_i(b_i,T)\\&-\varepsilon V(x(t),t)-x(t)'C_0'C_0x(t).\end{aligned} \quad (6)$$

Letting $\beta\coloneqq\frac{\varepsilon}{2}$ and $\delta_1(T)\coloneqq\max_i\big(\eta_i+\frac{\gamma_i'P_i^{-1}\gamma_i}{\beta}\big)$, completing the square, for $t\in[t_k+T,t_{k+1})$, inequality (6) in turn implies

$$\dot{V}(x(t),t)<-\beta V(x(t),t)+\delta_1(T)-x(t)'C_0'C_0x(t). \quad (7)$$

Analogously to [19], inequality (7) straightforwardly implies practical stability of the continuous dynamics. Indeed, for any $t\in[t_k+T,t_{k+1})$ and any $\nu_1\in(0,1)$ it holds that

$$\dot{V}(x(t),t)<-\nu_1\beta\underline{\theta}\|x(t)\|^2,\quad \forall\|x(t)\|\geq\sqrt{\tfrac{\delta_1(T)}{(1-\nu_1)\beta\underline{\theta}}}, \quad (8)$$

which proves that the continuous dynamics is ultimately bounded as $\|x(t)\|\leq\frac{\bar{\theta}}{\underline{\theta}}\sqrt{\frac{\delta_1(T)}{(1-\nu_1)\beta\underline{\theta}}}$, with $\bar{\theta}\coloneqq\max_i\lambda_{\max}(P_i)$ and $\underline{\theta}\coloneqq\min_i\lambda_{\min}(P_i)$.

Denote now as $V(x(t_{k+1}),t_{k+1}^-)$ the limit of the LF at the next switching instant t_{k+1} for t approaching t_{k+1} from the left, that is at t_{k+1}^-. Indicating $\delta(T)\coloneqq\frac{\delta_1(T)}{\beta}$ and integrating (7) in the interval $[t_k+T,t_{k+1})$, $V(x(t_{k+1}),t_{k+1}^-)$ is bounded as

$$\begin{aligned}V(x(t_{k+1}),t_{k+1}^-)<{}&\mathrm{e}^{-\beta(t_{k+1}-t_k-T)}V(x(t_k+T),t_k+T)\\&+\frac{1}{\beta}\big(1-\mathrm{e}^{-\beta(t_{k+1}-t_k-T)}\big)\delta_1(T)\\&-\int_{t_k+T}^{t_{k+1}}x(\tau)'C_0'C_0x(\tau)\mathrm{d}\tau\\={}&-\rho_k(V(x(t_k+T),t_k+T)-\delta(T))\\&-\int_{t_k+T}^{t_{k+1}}x(\tau)'C_0'C_0x(\tau)\mathrm{d}\tau\\&+V(x(t_k+T),t_k+T)\end{aligned} \quad (9)$$

with $\rho_k=1-\mathrm{e}^{-\beta(t_{k+1}-t_k-T)}$, $\rho_k\in(0,1)$ for all $k\geq0$ due to the property $t_{k+1}-t_k-T>0$.

Now, consider the case $t\in[t_{k+1},t_{k+1}+T)$, so that switching is not allowed, and the generic jth subsystem is active. By definition, we can write that $V(x(t_{k+1}),t_{k+1}^-)=x(t_{k+1})'P_ix(t_{k+1})$ and $V(x(t_{k+1}+T),t_{k+1}+T)=x(t_{k+1}+T)'P_jx(t_{k+1}+T)$, so that, computing their difference and exploiting (3), one has

$$\begin{aligned}&V(x(t_{k+1}+T),t_{k+1}+T)-V(x(t_{k+1}),t_{k+1}^-)\\&=x(t_{k+1})'\big(\mathrm{e}^{A_j'T}P_j\mathrm{e}^{A_jT}-P_i\big)x(t_{k+1})\\&\quad+2m_j'P_j\mathrm{e}^{A_jT}x(t_{k+1})+m_j'P_jm_j<0,\end{aligned} \quad (10)$$

that is $V(x(t_{k+1}+T),t_{k+1}+T)<V(x(t_{k+1}),t_{k+1}^-)$. Hence, combining the latter with (9), it holds that

$$\begin{aligned}&V(x(t_{k+1}+T),t_{k+1}+T)-V(x(t_k+T),t_k+T)\\&<-\rho_k(V(x(t_k+T),t_k+T)-\delta(T))\\&\quad-\int_{t_k+T}^{t_{k+1}}x(\tau)'C_0'C_0x(\tau)\mathrm{d}\tau\\&<-\rho_k(\underline{\theta}\|x(t_k+T)\|^2-\delta(T)).\end{aligned} \quad (11)$$

Thus, for any $\nu_2\in(0,1)$ it holds that

$$\begin{aligned}&V(x(t_{k+1}+T),t_{k+1}+T)-V(x(t_k+T),t_k+T)\\&<-\underline{\theta}\nu_2\rho_k\|x(t_k+T)\|^2,\\&\forall\|x(t_k+T)\|\geq\sqrt{\tfrac{\delta(T)}{(1-\nu_2)\underline{\theta}}}\end{aligned} \quad (12)$$

which proves that in the interval $[t_k+T,t_{k+1}+T]$ the dynamics is ultimately bounded as $\|x(t)\|\leq\frac{\bar{\theta}}{\underline{\theta}}\sqrt{\frac{\delta(T)}{(1-\nu_2)\underline{\theta}}}$. Consider now $\nu_1=\nu_2=\nu\in(0,1)$. Condition (12), together with (8), straightforwardly implies that the switching system (1) is practically stable with dwell time T and stability region

$$\mathcal{C}_2\coloneqq\left\{x\in\mathbb{R}^n \mid \|x\|\leq\frac{\bar{\theta}}{\underline{\theta}}\sqrt{\tfrac{\delta(T)}{(1-\nu)\underline{\theta}}}\right\}.$$

Now, for the sake of simplicity, pose $\tau_k=t_k+T$, $k\geq1$, in (11), $\tau_0=T$ and $\sigma(0)\in\Omega$, and complete the interval of the integral term in $[\tau_k,\tau_{k+1}]$ by summing and subtracting the same quantity. Therefore, for $k \geq 0$, inequality (11) becomes

$$\begin{aligned} V(x(\tau_{k+1}),\tau_{k+1}) - V(x(\tau_k),\tau_k) \\ < -\rho_k(\underline{\theta}\|x(t_k+T)\|^2 - \delta(T)) \\ - \int_{\tau_k}^{\tau_{k+1}} x(\tau)' C_0' C_0 x(\tau) \mathrm{d}\tau \\ + \int_{\tau_{k+1}-T}^{\tau_{k+1}} x(\tau)' C_0' C_0 x(\tau) \mathrm{d}\tau. \end{aligned} \tag{13}$$

Starting from $k = 0$ and iteratively summing $\bar{k} - 1$ times condition (13), one obtains

$$\begin{aligned} V(x(\tau_{\bar{k}}),\tau_{\bar{k}}) - V(x(\tau_0),\tau_0) \\ < -\underline{\theta} \sum_{k=0}^{\bar{k}-1} \rho_k \|x(\tau_k)\|^2 \\ + \bar{k}\delta(T) - \sum_{k=0}^{\bar{k}-1} \int_{\tau_k}^{\tau_{k+1}} x(\tau)' C_0' C_0 x(\tau) \mathrm{d}\tau \\ + \sum_{k=0}^{\bar{k}-1} \int_{\tau_{k+1}-T}^{\tau_{k+1}} x(\tau)' C_0' C_0 x(\tau) \mathrm{d}\tau. \end{aligned} \tag{14}$$

By virtue of the practical stability property of the switched system (1), $\|x\|_\infty^2$ is guaranteed to be limited, and one has

$$\int_{\tau_{k+1}-T}^{\tau_{k+1}} x(\tau)' C_0' C_0 x(\tau) \mathrm{d}\tau \leq \lambda_{\max}(C_0' C_0) T \|x\|_\infty^2. \tag{15}$$

Substituting the term in (15) into (14), and rearranging the inequality, one obtains

$$\begin{aligned} \int_{\tau_0}^{\tau_{\bar{k}}} x(\tau)' C_0' C_0 x(\tau) \mathrm{d}\tau < -V(x(\tau_{\bar{k}}),\tau_{\bar{k}}) + V(x(\tau_0),\tau_0) \\ + \bar{k}\lambda_{\max}(C_0' C_0) T \|x\|_\infty^2 + \bar{k}\delta(T). \end{aligned}$$

Adding and subtracting the integral of the cost in $[0, T]$, dividing left and right sides by $\bar{k}$, and computing the limit for $\bar{k}$ to infinity, one finally gets (4), which proves the proposed theorem. ■

Remark 3.1 (cost upper-bound): The upper-bound on the average cost (4) depends on the infinity norm of the state. Being the controlled system practically stable in the region $\mathcal{C}_2$, it is clear that if $x_0 \in \mathcal{C}_1$, with $\mathcal{C}_1 \subset \mathcal{C}_2$, then $\|x\|_\infty$ is bounded in $\mathcal{C}_2$. ▽

Remark 3.2 (logic of the switching rule): The switching rule presented in Theorem 1 relies on the comparison of the LF of the active subsystem at the current time with the forecast of the LFs of the remaining subsystems at T instants forward in time. In fact, the switching rule (3) can be rewritten as

$$u(x(t),t) = i \ \ \forall t \in [t_k, t_k + T] \tag{16a}$$

$$u(x(t),t) = i \ \ \forall t > t_k + T, \quad \text{if } V_j(x(t+T), t+T) \geq V_i(x(t),t),\ \forall j \in \Omega_i \tag{16b}$$

$$u(x(t_{k+1}), t_{k+1}) = \operatorname*{argmin}_{j \in \Omega_i} V_j(x(t_{k+1}+T)), \text{ otherwise,} \tag{16c}$$

where $V_j(x(t), t) := x' P_j x$, t_{k+1} is defined as in Theorem 1. However, the only way to verify conditions (16b) and (16c) is through the explicit computation of the LFs as in conditions (3b) and (3c), respectively.

Moreover, note that we restrict the switching signal to take value only in the discrete set $\Omega = \{1, \ldots, M\}$. If the *argmin* function assumes more than one value in the discrete set, we can choose it arbitrarily (for instance, a possible choice would be selecting the smallest index). As for the initial condition $\sigma(0)$ one can select it according to (16c) as $\sigma(0) = \operatorname{argmin}_{j \in \Omega} V_j(x(T), T)$. ▽

It is now interesting to note that Theorem 1 contains as a particular case the quadratic stability condition and the state switching stabilization for the associated sampled time system case. The next proposition summarizes this result. To streamline the exposition, we consider the case with $M = 2$.

Proposition 2: Consider system (1), with $i = 1, 2$, and let $\lambda_{i,j} := \alpha \bar{\lambda}_{i,j}$, with $\bar{\lambda}_{i,j}$ fixed entries of a Metzler matrix. As α goes to infinity, the following statements hold.

(i) Let $T = 0$ and assume that there exists a set of bounded positive definite matrices P_i satisfying for any $\alpha > 0$ the Lyapunov-Metzler equalities

$$A_i' P_i + P_i A_i + \alpha \sum_{j \in \Omega_i} \bar{\lambda}_{i,j}(P_j - P_i) + \varepsilon P_i + C_0' C_0 = 0. \tag{17}$$

Assume also that $\bar{\lambda}_{i,j}$ are such that $b_{\text{ave}} = \bar{\lambda}_{2,1} b_1 + \bar{\lambda}_{1,2} b_2 = 0$ and $A_{\text{ave}} = \bar{\lambda}_{2,1} A_1 + \bar{\lambda}_{1,2} A_2$ is Hurwitz stable. Then, as α goes to infinity, the switching rule (3) makes the origin of system (1) a quadratically stabilizable switched equilibrium.

(ii) Let $T > 0$ such that $F_{i,j} := \mathrm{e}^{A_i T} \mathrm{e}^{A_j T}$ is Schur stable (all eigenvalues in the open unit circle). Assume that there exists a set of bounded positive definite matrices P_i satisfying for all $\alpha > 0$ (namely, $P_i(\alpha)$) the Lyapunov-Metzler equalities

$$A_i' P_i + P_i A_i + \alpha \sum_{j \in \Omega_i} \bar{\lambda}_{i,j}(\mathrm{e}^{A_j' T} P_j \mathrm{e}^{A_j T} - P_i) + \varepsilon P_i + C_0' C_0 = 0. \tag{18}$$

Then, as α goes to infinity, it follows that

$$\begin{cases} \lim_{\alpha \to \infty} P_i(\alpha) = 0, \\ \lim_{\alpha \to \infty} \bar{P}_i(\alpha) = \hat{P}_i, \end{cases} \quad i = 1, 2, \tag{19}$$

with $\bar{P}_i(\alpha) = \alpha P_i(\alpha)$, $\hat{P}_i$ being the unique solution of the discrete-time Lyapunov equation

$$\hat{P}_i = F_{i,j}' \hat{P}_i F_{i,j} + \mathrm{e}^{A_j' T} \frac{C_0' C_0}{\bar{\lambda}_{j,i}} \mathrm{e}^{A_j T} + \frac{C_0' C_0}{\bar{\lambda}_{i,j}},$$

for $i, j = 1, 2$, $i \neq j$. Moreover, matrices $\hat{P}_i$ satisfy the discrete Lyapunov-Metzler equations

$$\hat{P}_1 = e^{A_2' T} \hat{P}_2 e^{A_2 T} + \frac{C_0' C_0}{\bar{\lambda}_{1,2}}, \tag{20a}$$

$$\hat{P}_2 = e^{A_1' T} \hat{P}_1 e^{A_1 T} + \frac{C_0' C_0}{\bar{\lambda}_{2,1}}, \tag{20b}$$

for the discrete time switched system

$$x((k+1)T) = \mathrm{e}^{A_{\sigma(kT)} T} x(kT) + m_{\sigma(kT)}. \tag{21}$$

Proof: As for statement (i), its proof directly follows from [20], [21] by observing that $P_{\text{ave}} = \lim_{\alpha \to \infty} P_i$ satisfies the Lyapunov equality $A_{\text{ave}}' P_{\text{ave}} + P_{\text{ave}} A_{\text{ave}} + \varepsilon P_{\text{ave}} + C_0' C_0 = 0$.

Moreover, the derivative of the LF $V(x(t),t) = x'(t)P_{\text{ave}}x(t)$ is

$$\dot{V}(t) = x'(t)(A'_\sigma P_{\text{ave}} + P_{\text{ave}}A_\sigma)x(t) + 2x'(t)P_{\text{ave}}b_\sigma.$$

Taking the switching rule $\sigma(t) = \operatorname{argmin}_i x'(t)(A'_i P_{\text{ave}} + P_{\text{ave}}A_i)x(t)$ (which can be proved to be equivalent to (3b) for $T = 0$), we have that

$$\dot{V}(t) \leq -x'(t)C'_0C_0x(t) - \varepsilon V(x(t),t).$$

As for statement (ii), equality (18) can be obtained from (2) by letting $\lambda_{i,j} = \alpha\bar{\lambda}_{i,j}$. Considering the case $i, j = 1, 2$, dividing all the terms in (18) by α, as α tends to infinity one obtains

$$\mathrm{e}^{A'_2T}P_2\mathrm{e}^{A_2T} - P_1 = 0, \tag{22a}$$

$$\mathrm{e}^{A'_1T}P_1\mathrm{e}^{A_1T} - P_2 = 0\ . \tag{22b}$$

Hence, combining the previous equations and exploiting the definition of $F_{i,j}$, one has

$$P_i = F'_{i,j}P_iF_{i,j}, \quad i, j = 1, 2,\ i \neq j, \tag{23}$$

with $F_{i,j}$ being Schur stable. Therefore, by inspection, one can conclude that P_i tends to zero as α goes to infinity.

Consider now matrices $\bar{P}_i = \alpha P_i$ such that the following Lyapunov-Metzler condition holds

$$A'_i\bar{P}_i + \bar{P}_iA_i + \alpha\sum_{j\in\Omega_i}\bar{\lambda}_{i,j}(\mathrm{e}^{A'_jT}\bar{P}_j\mathrm{e}^{A_jT} - \bar{P}_i) + \varepsilon\bar{P}_i + \alpha C'_0C_0 = 0, \tag{24}$$

$i = 1, 2$. Dividing (24) by α, as α goes to infinity, one has

$$\bar{\lambda}_{1,2}(e^{A'_2T}\bar{P}_2e^{A_2T} - \bar{P}_1) + C'_0C_0 = 0,$$
$$\bar{\lambda}_{2,1}(e^{A'_1T}\bar{P}_1e^{A_1T} - \bar{P}_2) + C'_0C_0 = 0.$$

Letting $\hat{P}_i$ be the solution of the above set of equations, one has (20). Finally, combining equations (20), one obtains the expression of $\hat{P}_i$. Therefore, the switching law $\sigma(kT) = \operatorname{argmin}_{i\neq j}(\mathrm{e}^{A_iT}x(kT) + m_i)'\hat{P}_i(\mathrm{e}^{A_iT}x(kT) + m_i) - x'(kT)\hat{P}_jx(kT)$ is consistent with the one in [22, Th. 3], where only linear discrete time autonomous switched systems were considered, i.e., $m_i = 0$. In our case, this switching law is applied to the discrete time affine switched system (21), making it practically stable. ∎

Remark 3.3 (limit case with $T = 0$): Note that if $T = 0$ and $b_{\text{ave}} = 0$, as α goes to infinity, then the region $\mathcal{C}$ in Theorem 1 reduces to the singleton $x = 0$, thus resorting to the results presented in [20]. Moreover, the cost $\int_0^\infty x(\tau)'C'_0C_0x(\tau)\mathrm{d}\tau < x'_0P_{\text{ave}}x_0$. ▽

IV. Application to Power Converters Control

An interesting application of the proposed theory is the control of DC-DC switched power converters. Indeed, their intrinsic switched structure and the limited commutation frequency of the switching devices recall for the design of a discontinuous control law embedding dwell time. Moreover, many converters can be mathematically modeled as affine linear switched systems. Hereafter, the case of the boost converter (see [17, Sec. 2.3]), characterized by two possible switching modes, is described.

A. Illustrative Example

Consider the converter shown in Fig. 1. It presents two switches Q_1 and Q_2 operating in counter-phase (that is $Q_1 = 1$ when $Q_2 = 0$ and vice versa), and such couple of switches can be operated only with finite frequency, which imposes a minimum required dwell time for the switching signal.

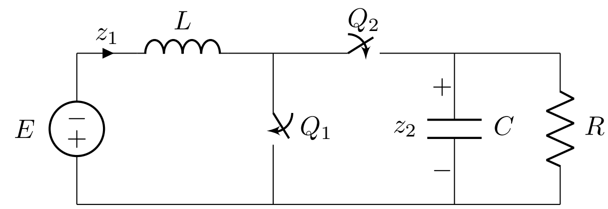

Fig. 1. Boost converter topology.

Thus, considering the two possible configurations of the switches, it is not difficult to retrieve the mathematical model of the converter as

$$\dot{z}(t) = A_{\sigma(t)}z(t) + b, \quad y(t) = C_0z(t)$$

with $\sigma \in \{1, 2\}$, while

$$A_1 = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix},\ b = \begin{bmatrix} \frac{E}{L} \\ 0 \end{bmatrix},\ C_0 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

and z_1 representing the current flowing through the inductor L, z_2 being the voltage across the capacitor C or over the load resistance R, and $\sigma(t)$ being the position of the switch Q_1, with $Q_1 = 0$ when $\sigma = 1$, and $Q_1 = 1$ when $\sigma = 2$. The boost converter presents a rather interesting property, that is one of the two configurations is unstable. In fact, when $Q_1 = 1$, it holds that $\dot{z}_1 = \frac{E}{L}$, which in turn implies that the current z_1 exponentially increases to infinity. Furthermore, the boost converter is commonly adopted to regulate the load voltage to a desired value larger than the input voltage E. Indicating with $z_2^\star$ the desired load voltage, it is possible to define the error state $x = z - z^\star$ where $z^\star = \left[\frac{z_2^{\star 2}}{RE},\ z_2^\star\right]'$, thus implying

$$\dot{x}(t) = A_{\sigma(t)}x(t) + b_{\sigma(t)},$$

where $b_{\sigma(t)} = A_{\sigma(t)}z^\star + b$. Therefore, the task of controlling the state z to a desired value is reformulated as the problem of regulating to zero the state x, consistently to (1). The MATLAB SimPowerSystem toolbox has been adopted to perform more realistic simulations.

As proved in §III, the adoption of the switching law (3) guarantees that the switched system is practically stable under dwell time switching, and, more importantly, the size of the estimated region of attraction is a monotonic function of the selected dwell time. Specifically, it is expected that the norm of x increases as the dwell time is higher. In the proposed simulation scenario, the switching law has been implemented with several values of dwell time, that is $T = 10^{-4}$ sec, $T = 5 \times 10^{-5}$ sec and $T = 10^{-5}$ sec, while $\bar{\lambda}_{1,2} = \frac{E}{z_2^\star}$ and $\bar{\lambda}_{2,1} = 1 - \bar{\lambda}_{1,2}$ (according to Proposition 2). These values are such that $F_{1,2}$ and hence $F_{2,1}$ (see

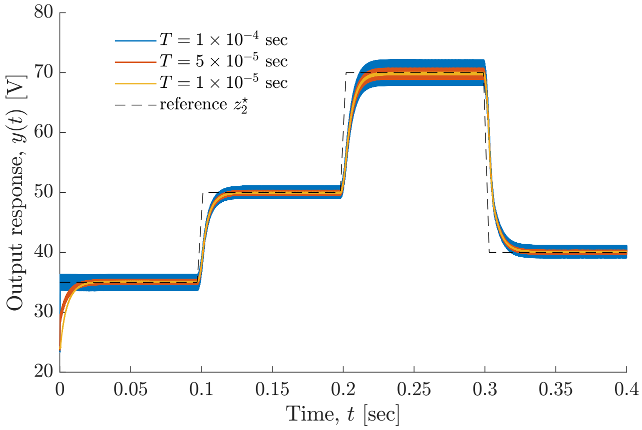

Fig. 2. Boost converter voltage for $T = 10^{-4}$ sec (blue line), $T = 5 \times 10^{-5}$ sec (red line), and $T = 10^{-5}$ sec (yellow line).

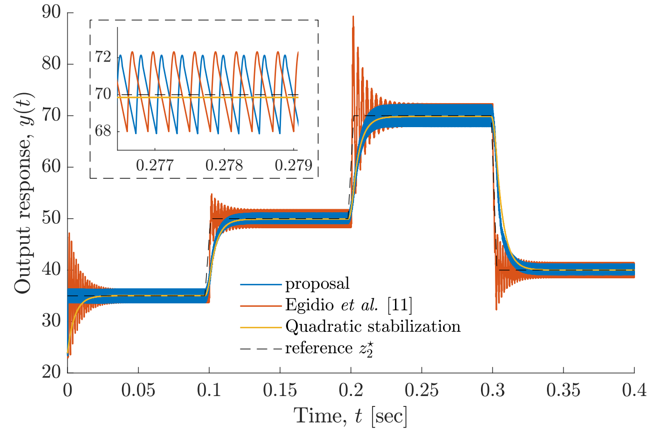

Fig. 3. Boost converter voltage when the proposed switching rule (blue line), that in [11] (red line), and an *ad hoc* quadratic stability based switching law (yellow line) are applied.

Proposition 2) are Schur stable. The converter parameters have been chosen instead as $E = 24\,\mathrm{V}$, $C = 100\,\mathrm{mF}$, $L = 300\,\mathrm{mH}$, and $R = 37.5\,\Omega$. As shown in Fig. 2, the reference value for $z_2^\star$ has been selected as a step wise constant signal taking value $35\,\mathrm{V}$ in the interval $[0, 0.1]$ sec, $50\,\mathrm{V}$ in the interval $[0.1, 0.2]$ sec, $70\,\mathrm{V}$ in the interval $[0.2, 0.3]$ sec, and finally $40\,\mathrm{V}$ in the interval $[0.3, 0.4]$ sec. As expected, Fig. 2 shows that the oscillations of the output voltage z_2 around the reference signal decreases with the dwell time value, thus confirming the theoretical results in Theorem 1.

B. Comparison

To conclude this section, we show a brief comparison of the proposed switching algorithm with some other available options. Since our main motivation for the design of our switching approach is the presence of the affine term, it is interesting to compare it with alternative strategies given by the state dependent law proposed in [11], and an *ad hoc* method based on quadratic stabilization through an arbitrary switching rule. More precisely, as for the algorithm in [11], it aims at guaranteeing global stability of a limit cycle, chosen by the designer, through a switching rule characterized by fixed switching intervals. Since our proposal generates periodic state trajectories, the limit cycle required in [11] has been selected so that its period matches with the one of the trajectory achieved using our switching rule.

Our comparison criteria are:

- the root mean square value of the output variable in steady-state, termed oRMS;
- the settling time of the output variable, termed oST.

The results are reported in Table I.

TABLE I

PERFORMANCE OF THE SWITCHING LAWS.

strategy	oRMS	oST
proposed strategy	0.899	0.021
Egidio *et al.* [11]	1.336	0.002
Quadratic stabilization	0.025	0.027

Fig. 3 shows a comparison of the load voltage for these three approaches. It is worth noting that the proposed algorithm and the one in [11] show different dynamics during the transient time, while achieving similar performance in steady state. Indeed, the algorithm in [11] provides shorter oST at the price of higher oscillations during the transient interval, thus implying higher oRMS. However, as evident from the closed-up in Fig. 3, the oscillations in steady state of the two algorithms are of comparable magnitude. Finally, as for the comparison between our proposal and the *ad hoc* alternative method based on quadratic stabilization through an arbitrary switching rule, the latter allows a reduction of oRMS with respect to the proposal due to the lack of dwell time constraint. The settling times are instead comparable. It is worth to mention that it is not possible to apply in practice an arbitrary switching method because of the ideal infinite switching frequency. Our approach instead takes into account a dwell time equal to $T = 1 \times 10^{-4}$ sec, which is consistent with possible field implementations.

V. Conclusions

The note has introduced a novel state dependent switching rule, which incorporates the input term of the considered affine linear system and the forward evolution of Lyapunov functions, in quadratic form, constrained to a pre-specified dwell time. The switching rule is enabled by the solutions of Lyapunov-Metzler inequalities, which provide sufficient conditions for the practical stability of the switched system, and for the boundedness of the cost associated to the control strategy.

Although the presented ideas look to be extendable to many application domains, in this work we study the application to power converters, keeping their natural switching nature instead of the commonly adopted averaged model. In fact, we have shown the potentiality of the proposal as a ready-to-implementable solution even in practice.

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