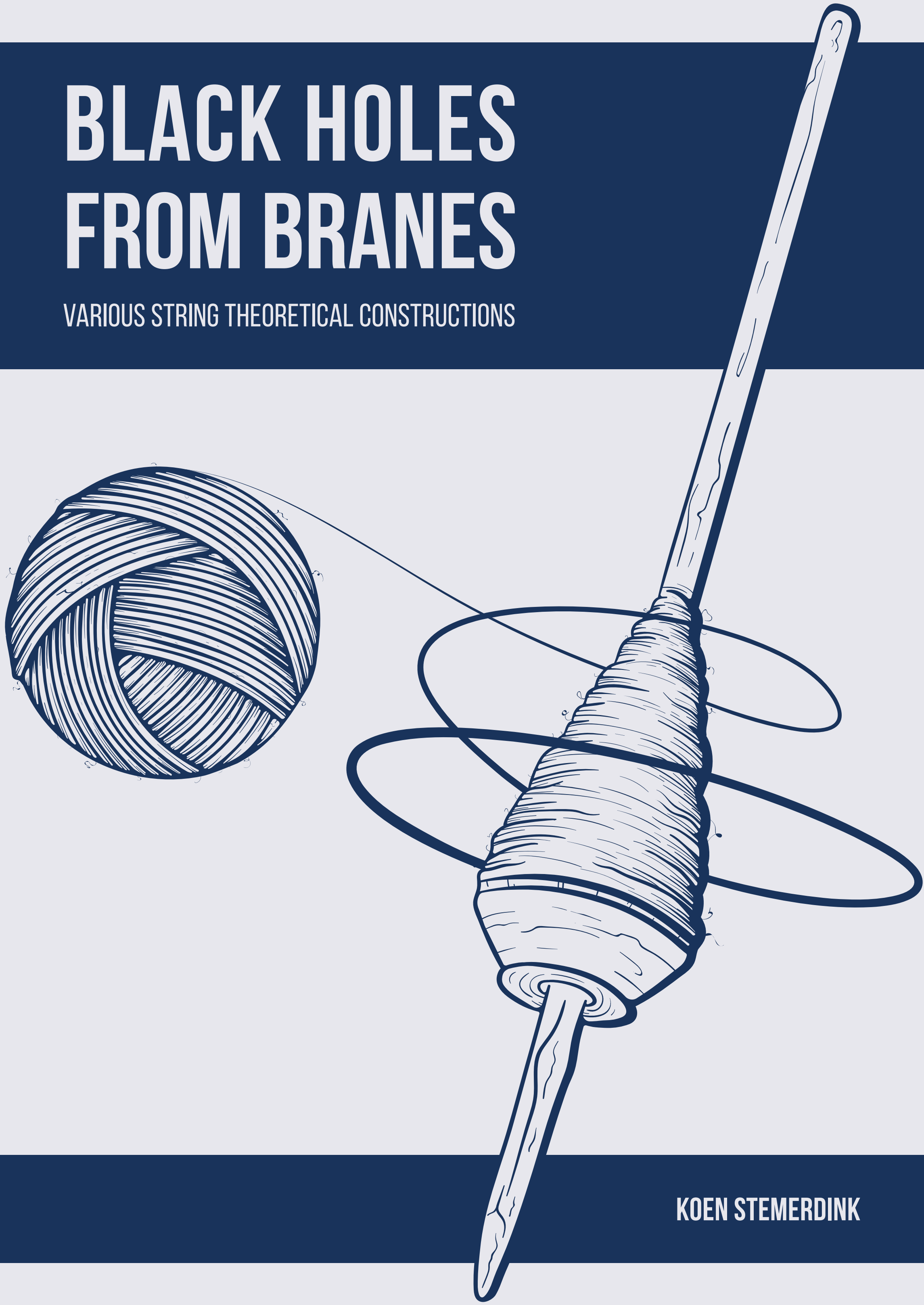
BLACK HOLES FROM BRANES

VARIOUS STRING THEORETICAL CONSTRUCTIONS



KOEN STEMERDINK

Black Holes from Branes

Various string theoretical constructions

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About the cover: The illustration depicts a stringy version of a star being 'eaten' by a black hole. The black hole is represented by a spindle (a tool used to spin fibers into yarn, Dutch: spintol), conform parts II and III of this thesis where spindle-shaped black holes are studied. Optically, the spindle is meant to resemble a rotating black hole with jets.

The star is represented by a ball of yarn, or a ball of string, in line with string theory which states that *all* matter consists of strings. A piece of string is being ripped away from the star by the black hole's gravitational pull. After orbiting the black hole a few times, the string inevitably falls into it.

Black Holes from Branes

Various string theoretical constructions

Zwarte Gaten van Branen

Verscheidene snaartheoretische constructies

(met een samenvatting in het Nederlands)

Proefschrift

ter verkrijging van de graad van doctor aan de Universiteit Utrecht op gezag van de rector magnificus, prof. dr. H.R.B.M. Kummeling, ingevolge het besluit van het college voor promoties in het openbaar te verdedigen op vrijdag 9 september 2022 des middags te 2.15 uur

door

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geboren op 2 december 1992
te Leiden

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To my parents,

for December 2001

List of publications

Part I of this thesis is based on:

[1] **Black holes in string theory with duality twists**
C. Hull, E. Marcus, K. Stemerdink, and S. Vandoren
JHEP 07 (2020) 086, arXiv:2003.11034

[2] **Black holes and freely-acting orbifolds**
G. Gkountoumis, C. Hull, E. Marcus, K. Stemerdink, and S. Vandoren
In preparation

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[3] **M2-branes on discs and multi-charged spindles**
C. Couzens, K. Stemerdink, and D. van de Heisteeg
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[4] **Universal spindles: D2's on $\boldsymbol{\Sigma}$ and M5's on $\boldsymbol{\Sigma} \times \mathbb{H}^3$**
C. Couzens and K. Stemerdink
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Part III of this thesis is based on:

[5] **The near-horizon geometry of supersymmetric rotating AdS_4 black holes in M-theory**
C. Couzens, E. Marcus, K. Stemerdink, and D. van de Heisteeg
JHEP 05 (2021) 194, arXiv:2011.07071

[6] **Embedding known AdS_4 black holes in supersymmetric classification**
C. Couzens, K. Stemerdink, and D. van de Heisteeg
In preparation

The papers [1] and [5] have also appeared in the thesis of E. Marcus. We make the following comments on the overlap between the theses:

[1] Most of the results in this paper were obtained jointly. The author focused more on the computations in section 4, while E. Marcus focused more on the computations in section 5.

[5] The author performed most of the computations in section 3, and E. Marcus performed most of the computations in section 4.

Contents

1 Context and motivation **1**

1.1 Unification in physics . . . 1

1.1.1 Modern physics . . . 2

1.1.2 A final unification . . . 4

1.1.3 Black holes . . . 6

1.1.4 Quantum gravity . . . 9

1.2 String theory . . . 10

1.2.1 Basics . . . 11

1.2.2 Superstrings . . . 13

1.2.3 Dualities . . . 14

1.2.4 Branes . . . 17

1.2.5 Compactification . . . 19

1.2.6 Supergravity . . . 20

1.3 This thesis . . . 21

I Black holes and supersymmetry breaking **25**

2 Introduction **27**

2.1 Scherk–Schwarz reduction . . . 28

2.2 Duality twists . . . 29

2.3 Toroidal orbifolds . . . 30

2.4 Outline . . . 31

3 D1/D5-branes and Scherk–Schwarz **33**

3.1 Duality covariant formulation of IIB on T^4 . . . 33

3.2 Scherk–Schwarz reduction to 5d . . . 40

3.3 5d black hole solutions . . . 61

3.4 Quantum corrections . . . 71

3.A Conventions and notation . . . 80

3.B Group theory . . . 81

3.C 5d scalar and tensor masses . . . 86

4 D1/D5-branes and freely-acting orbifolds **91**
4.1 T-duality twists . . . 91
4.2 Orbifold constructions . . . 93
4.3 Field content . . . 95
4.4 D-branes . . . 102

II Spindle and disc near-horizons **111**

5 Introduction **113**
5.1 Spindles and topological discs . . . 114
5.2 Twists and anti-twists . . . 115
5.3 Outline . . . 116

6 M2-branes on discs and multi-charge spindles **119**
6.1 4d black hole near-horizons . . . 119
6.2 Uplift to 11d supergravity . . . 130
6.A Smeared M2-branes . . . 144
6.B Killing spinors of the multi-charge solution . . . 145

7 Rotating D2-branes on spindles **149**
7.1 4d black hole solution . . . 149
7.2 Uplift to massive type IIA . . . 151
7.A Review of M5-branes on spindles . . . 158

III Supersymmetric classification **163**

8 Introduction **165**
8.1 *G*-structures . . . 167
8.2 GK geometry . . . 168
8.3 Outline . . . 169

9 Classification of rotating M2-brane solutions **171**
9.1 Conditions of the classification . . . 171
9.2 Action for the theory . . . 182
9.3 Black strings in type IIB . . . 187
9.A Black hole near-horizons and observables . . . 192

10 Embedding of known solutions in classification **201**
10.1 The Kerr–Newman solution 201
10.2 The spinning spindle solution 208
10.3 The Klemm solution 210

11 Conclusion **215**
11.1 Summary 215
11.2 Outlook 217
11.3 Samenvatting in het Nederlands 221

Acknowledgements **225**

About the author **229**

Bibliography **231**

Chapter 1

Context and motivation

"Physics is really nothing more than a search for ultimate simplicity, but so far all we have is a kind of elegant messiness."

– Bill Bryson, A Short History of Nearly Everything

1.1 Unification in physics

A strong case could be made for summarizing the history of physics as a gradual process of understanding the plethora of natural phenomena that humanity has come across, and unifying them in as few physical theories as possible. The ultimate goal of this process would be a single theory that incorporates all of physics, often dubbed the 'Theory of Everything'. Although we are not there yet, physicists have made tremendous progress towards this goal. The research presented in this thesis is intended to bring us an infinitesimal step closer.

We begin this introductory chapter with some history and some contemplation of unification in physics. In this we will not strive for completion, as one could dedicate an entire thesis to this topic alone. Rather, we will discuss particular examples and follow certain lines of thought. This serves to place the content of this thesis into context, and to motivate why the questions that we ask are important. A more specialized introduction of the research presented in this thesis is given later on.

Perhaps one of the most impressive and well-known unifying theories in physics is the theory of electromagnetism, discovered by James Clerk Maxwell in the 19th century. It brought together electricity, magnetism and light into a single framework. Through this theory it was understood that these three phenomena, that were thought of as separate until then, were actually different manifestations of the same phenomenon. All three could henceforth be described by a single set of formulas:

$$\mathrm{d}F = 0\,, \qquad \mathrm{d}\star F = J\,. \tag{1.1}$$

These are known as Maxwell's equations.

This remarkable development illustrates some of the main reasons why physicists like unification so much. First of all, and of crucial importance, note that while new insights were obtained, none were lost. The birth of electromagnetism did not mean the death of the achievements in the individual subfields. Earlier results such as Gauss's flux theorem and Faraday's law of induction were not invalidated by Maxwell's electromagnetism; rather they were incorporated into a bigger framework.

In addition, having a single theory to describe (what used to be) multiple phenomena is often considered more elegant, or even more beautiful, than having multiple different theories to work with.

Perhaps the most important and arguably the most scientific reason for unification in physics is to deepen the understanding of the phenomena that are being unified. Again, this is illustrated well by the example of electromagnetism. Not only did this unification allow us to describe electricity, magnetism and light in a single framework, it taught us that these three are merely separate manifestations of the same thing. This is a profound notion that has had tremendous impact on physicists since Maxwell.

1.1.1 Modern physics

From this point forward, things sped up. Multiple breakthroughs in the early 20th century brought about what we now refer to as 'modern physics'. Two cornerstones of modern physics are the theories of relativity and the theory of quantum mechanics. We discuss aspects of both of these in this section.

In 1905, Albert Einstein published his special theory of relativity. While this theory is most famous for unifying the concept of energy with the concept of mass through the acclaimed formula $E = mc^2$, it also unites the notions of space and time into a single continuum called 'spacetime'. Historically, one can think of special relativity as the unification of electrodynamics and Newtonian mechanics. It is no coincidence that the title of Einstein's 1905 paper is *On the Electrodynamics of Moving Bodies*, translated from the original German *Zur Elektrodynamik bewegter Körper*.

Special relativity has a predominant flaw, namely that it is only valid in the absence of gravity. Einstein was able to reconcile relativity and gravity in 1915 by introducing his general theory of relativity. In this theory, gravity is no longer described as a force in the conventional sense but as a consequence of the curvature of spacetime. This curvature is caused by the presence of energy, often in the form of mass, via the formula

$$G_{\mu\nu} + \Lambda\, g_{\mu\nu} = \frac{8\pi G_N}{c^4}\, T_{\mu\nu}\,, \tag{1.2}$$

which is called the Einstein field equation. This equation describes how matter tells space to curve; the curvature of spacetime in turn tells matter how to move.

In the century that has passed since Einstein first proposed general relativity, it has withstood many high-precision tests. Famous examples of phenomena explained or predicted by general relativity are the perihelion precession of the orbit of Mercury, gravitational lensing by heavy astronomical objects and gravitational redshifting of light moving in a strong gravitational field. One more impressive check of general relativity is the observation of gravitational waves by the LIGO detectors in 2015 [7]. This is yet another confirmation of general relativity as our most reliable and extensive theory of gravitation.

General relativity is also the framework in which models of the largest-scale structure and evolution of the universe are built. Such cosmological models are used to study questions about the origin and the ultimate fate of the universe that we live in. The leading cosmological model is called the ΛCDM (Lambda cold dark matter) model, according to which the universe consists of a cosmological constant Λ, cold dark matter and ordinary matter. The positive cosmological constant explains the accelerating expansion of the universe, while the conjectured existence of dark matter explains gravitational effects observed for example in galaxy formation and evolution. Because of the success of the ΛCDM model in explaining various properties of the universe, it is often referred to as the ‘standard model of Big Bang cosmology’.

Around the same time as relativity, another revolutionary theory emerged: quantum mechanics. This theory collects some of the most counterintuitive notions in all of physics, while also giving some of the best tested predictions of any theory.

One of the early inducements for the theory of quantum mechanics was the so-called ‘ultraviolet catastrophe’. This is the name given to the mismatch between the theoretical prediction and the experimental result of the amount of short-wavelength radiation emitted by a black body in thermal equilibrium. Theory predicted that this amount would go to infinity (for the wavelength going to zero), but experiments showed that it went to zero. While theorists have familiarized themselves with some discrepancy between theory and experiment – experimental results are only as good as the available equipment – an infinitely big mismatch is obviously a problem. This conundrum was solved by Max Planck in 1900 by assuming that electromagnetic radiation consists of quantized packets. These discrete packets of electromagnetic radiation are what we now call photons.

From this point onward, it took over 30 years and many radical insights before the quantum theory was mathematically formulated. The result was a theory in which objects can be both a particle and a wave, but cannot have both a well-defined position and velocity. In which some types of identical particles cannot occupy the same position (read: quantum state), while other types can occupy the same position with an arbitrary number of identical particles. In which objects can 'tunnel' through barriers without sufficient energy to surmount them, and in which cats can be both dead and alive as long as the owner is not paying attention.

Quantum mechanics can be unified with special relativity in a framework called quantum field theory. This theory was developed in a period spanning more than half of the 20th century, starting with the work of Paul Dirac in the 1920s and climaxing with the creation of the Standard Model of particle physics in the 1970s. The Standard Model is a quantum field theory that describes all known elementary particles (omitting the graviton as a particle) and their interactions. Some of these particles had not yet been observed at the time the Standard Model was developed, but as of 2012, with the detection of the Higgs boson in the Large Hadron Collider (LHC) [8, 9], physicists have discovered them all experimentally. Although the Standard Model has its problems – notable examples include the hierarchy problem, the strong CP problem and the inability to explain neutrino masses, dark matter and dark energy – it is a powerful model offering many predictions that agree with experimental measurements to astounding precision. This can be seen as a proof of concept for the entire framework of quantum field theory.

The many accomplishments of both general relativity (GR) and quantum field theory (QFT) have made them established and well-tested pillars of modern physics. Due to their success, all physics that is non-relativistic and non-quantum is now referred to as 'classical physics'. Loosely speaking, classical physics can be seen as the part of physics that is relevant for things that people encounter in everyday life. It is precisely for this reason that many aspects of physics beyond classical physics are so counterintuitive: they are detectable only by performing intricate experiments or by diligently studying the universe.

1.1.2 A final unification

From the perspective of unification, it seems natural to question whether general relativity and quantum field theory can be unified. After all, this would neatly collect all physics that we have discussed so far into a single theory: a theory of 'quantum gravity'.

We will go one step further, and claim that this final unification is not only a nice-to-have, it is a must-have. In order to present this line of reasoning, it is important to first understand in which regimes the theories that we have discussed so far are valid. These can be visualized nicely in a so-called Bronstein cube, see figure 1.1.

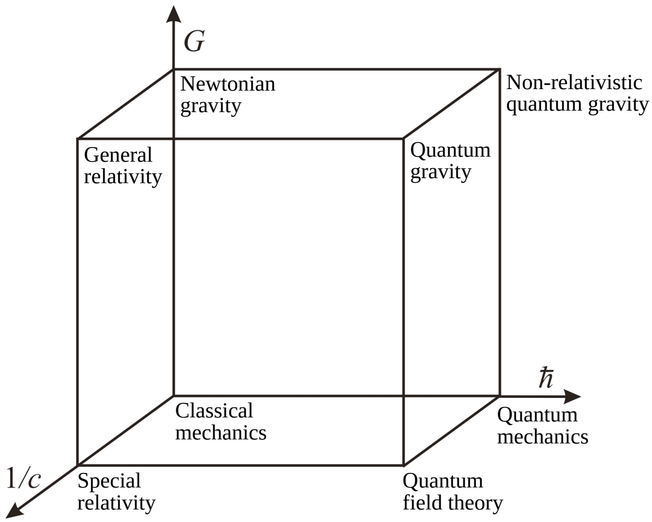


Figure 1.1: *The Bronstein cube, depicting regions of validity of certain theories. On the three axes we see the fundamental constants associated with quantum mechanics, relativity and gravity.*

Here we see three constants whose magnitudes govern the relevance of physical theories. For example, relativity is governed by the speed of light c. For observers moving at velocities close to the speed of light, taking into account relativity is quite important, while for slower observers relativistic effects can be neglected. Equivalently said, relativity becomes significant when the speed of light is small compared to the scales of an experiment. This is why special relativity plays a big role in space exploration, but not so much for cyclists getting to work. The $1/c$ axis in the Bronstein cube illustrates this transition.

Similar narratives hold for the other axes of the cube, parametrizing the relevance of quantum mechanics with the (reduced) Planck constant $\hbar$ and the relevance of gravity with the Newton constant G (often denoted as G_N).

We can clearly see some of the unifications that we have discussed already. General relativity, the unification of special relativity and gravitation, can be found in the corner of the Bronstein cube where both $1/c$ and G, associated with relativity and gravity respectively, are turned on. Similarly, we find quantum field theory in the corner where both special relativity and quantum mechanics are relevant.

A potential unification of general relativity and quantum field theory into a theory of quantum gravity would yield a framework that can handle gravity, relativity and quantum mechanics all at the same time. Again, this is shown as a corner in the Bronstein cube[1].

Now we can ask whether this unification would actually be useful. Does it give us the ability to describe phenomena that we cannot describe already with the theories at hand? After all, general relativity governs the 'extremely large', where gravitational fields are strong, while quantum field theory governs the 'extremely small', where quantum effects play a role. Where do these two meet?

The answer is simple: inside a black hole[2]. We will dedicate the next section to the introduction of these extreme objects.

1.1.3 Black holes

The conceptual notion of a black hole is relatively old. In 1784 John Michell proposed the existence of so-called 'dark stars', astrophysical objects that are so heavy that their escape velocity is bigger than the speed of light. This was before it was known that light was affected by gravity, so not much more came from this than interesting speculation.

All this changed with the development of general relativity in 1915. A few months after its publication, Karl Schwarzschild found the following solution to Einstein's equations:

$$\mathrm{d}s^2 = -\left(1 - \frac{2G_N M}{c^2\, r}\right) c^2\, \mathrm{d}t^2 + \left(1 - \frac{2G_N M}{c^2\, r}\right)^{-1} \mathrm{d}r^2 + r^2\, \mathrm{d}\Omega_2^2\,, \tag{1.3}$$

now appropriately called the Schwarzschild solution. This solution describes the deformation of spacetime around a spherically symmetric point mass.

[1] Another valid path to this corner is via non-relativistic quantum gravity, which could then be made relativistic. As this is not the approach that is followed in this thesis, we will not spend too much time on it here and leave it as a comment.

[2] Another valid answer would be: in the early stages of the Big Bang.

Initially, few people believed that this solution would be the appropriate description of actual objects appearing in nature, due to its singular behavior at $r = 2G_N M$ and at $r = 0$. The former singularity was shown to be a coordinate singularity, i.e. a flaw of the chosen coordinate system rather than something physical, but the latter singularity persisted. Nevertheless, a series of discoveries in stellar evolution and in observational astrophysics slowly forced people to take the Schwarzschild solution more seriously as something that could form from the gravitational collapse of a star.

The surface at $r = 2G_N M$ was identified as an event horizon, through which things could pass only in one direction, by David Finkelstein in 1958. This meant that objects moving around in the spacetime described by the Schwarzschild solution could only move inwards, not outwards, once they had come sufficiently close to the center. This inability of things - including light - to escape the gravitational attraction within a certain radius around the point mass is what earned this region the name 'black hole'.

For a while, it was thought that the singularity at the center of a black hole was merely a mathematical artifact, arising from the assumption of an unreasonable amount of symmetry that would never arise in nature. However, in 1965 a theorem by Roger Penrose showed that a singularity would inevitably appear in black hole formation under quite general assumptions [10]. Hence, if black holes exist in nature, they would necessarily have a singularity at their center according to the laws of general relativity.

In the 1970s physicists started to think about black holes as thermodynamical systems. Based on the second law of thermodynamics, Jacob Bekenstein conjectured that black holes have entropy and that this entropy should scale with the area of the horizon [11]. Shortly after this, it was shown by Stephen Hawking that black holes emit thermal radiation[3], similar to radiation coming from a black body [13]. This radiation is called Hawking radiation, and the corresponding black body temperature is called the Hawking temperature. For a Schwarzschild black hole, this temperature can be expressed in terms of the black hole mass as

$$T_{\mathrm{H}} = \frac{\hbar c^3}{8\pi G_N k_B M} \,. \tag{1.4}$$

In addition, Hawking fixed the coefficient in Bekensteins relation between black hole entropy and horizon area. Accordingly, this formula now carries both their

[3]This finding led to the black hole information paradox. We will not get into this interesting topic here, but refer the interested reader to the popular science book *The Black Hole War* by Leonard Susskind [12].

names as the Bekenstein–Hawking area law. It reads

$$S_{\mathrm{BH}} = \frac{c^3 k_B}{4 G_N \hbar} A \,. \tag{1.5}$$

In these formulas we keep the fundamental constants $\hbar$, c and k_B visible. The fact that all three appear, as well as the gravitational constant G_N, gives these equations a very eclectic look that we would like to preserve.

Technically, the (theoretical) discovery of Hawking radiation means that black holes are not really black. Moreover, it means that they are losing mass to their environment. This process is called black hole evaporation. However, in reality these processes are so slow that they are practically irrelevant for astrophysics. The temperature of black holes appearing in the universe is many orders of magnitude lower than the temperature of the cosmic microwave background radiation, which means that measuring it is virtually impossible.

The first astronomical object to be identified as a black hole was an X-ray source called Cygnus X-1. Of course a black hole cannot emit X-rays; by definition it cannot emit anything (neglecting Hawking radiation). The X-rays are generated by an accretion disc, a cloud of extremely hot gas surrounding the black hole, which originates from the fact that Cygnus X-1 forms a binary system together with a blue supergiant star. The black hole draws in, or ‘accretes’, matter from the star which replenishes matter that has fallen beyond the event horizon. In this way a hot disc of gas is sustained outside the horizon, emitting the X-rays.

Indirectly, many black holes have been observed. This can be done for example by studying the motion of stars orbiting a point in space where no heavy object is visible. A considerable leap forward in black hole observation was made in 2019, when the Event Horizon Telescope collaboration released the first-ever direct image of a black hole [14], shown in figure 1.2. Here we see a bright ring of light, and a black shadow in the middle where the black hole itself is.

By now, it is widely accepted that black holes can and do form in nature. Penrose’s theorem, that we mentioned earlier, tells us that they have a singularity at their center: a point where the curvature of spacetime becomes infinite. Generally, such singularities are considered unphysical, a signal that the theory that is used (in this case general relativity) is taken into a regime where it loses validity. And indeed, this is precisely what happens. An infinitely small point where the curvature of spacetime becomes infinitely large demands the inclusion of quantum effects. In other words, we need a unification of general relativity with quantum theory in order to figure out what happens here.

Figure 1.2: *The Event Horizon Telescope image of the black hole M87*.*

This takes us to the quantum gravity corner of the Bronstein cube, figure 1.1. Only this theory can tell us whether there really are singularities inside black holes, and if so, how to properly describe them.

1.1.4 Quantum gravity

As we have argued in the previous sections, having a theory of quantum gravity would not only be neat from the perspective of unification, it is also essential for describing certain extreme places in the universe, such as the interiors of black holes.

Unfortunately, formulating such a theory has proved to be rather difficult. One of the reasons for this is that there is virtually no experimental data available to guide theorists. It is expected that quantum gravity effects show up at length scales near the Planck scale, around 10^{-35} meters. For comparison, our most advanced experiments in particle physics can probe length scales around 10^{-19} meters, so we would need to go roughly 10 quadrillion times smaller. One often hears the statement that, to probe the Planck length in a conventional circular particle collider, it would need to be approximately the size of the Milky Way. Although the precise length needed depends on some assumptions, all estimates agree that it is far beyond reasonable to expect such an experiment to be carried out anywhere in the near future.

A straightforward way to do quantum gravity experiments without building a monstrous collider would be to jump into a black hole, wait until you hit the singularity, and see what happens there. Skipping over obvious practical issues, like traveling to a black hole thousands of light-years away and finding a brave enough physicist for the job, this approach will always strand on a fundamental problem. If this plan was carried out, and if the brave (or unwilling) physicist survived the trip and the experiment, he or she would never be able to report the results back to earth from inside the black hole. Hence, this approach also does not seem to be very fruitful.

For now, we will have to do without experimental data, meaning that the only thing we can do is construct theoretical frameworks. These cannot be checked experimentally, so they are judged by features like internal consistency and consistency with known and tested theories. Sometimes features like (mathematical) elegance and beauty are also taken into account, but this line of thinking is also subject to criticism[4].

One of the more prominent candidate theories of quantum gravity is called string theory[5]. The entirety of research presented in this thesis is set in this theory or limits thereof. Therefore, we dedicate the following section to introducing this field.

1.2 String theory

While the technical details of string theory are often thought of as very complicated, the basic idea is remarkably simple. String theory postulates that the fundamental degrees of freedom of our universe are described by one-dimensional objects, rather than by pointlike particles.

These so-called strings are assumed to be very small. So small in fact, that their size is many orders of magnitude away from length scales that we can currently probe in experiments. For this reason it is unlikely that string theory will be experimentally proved or disproved in the foreseeable future. As we have explained in the previous section, this drawback is not unique to string theory but ubiquitous in the field of quantum gravity.

We would like to point out that string theory rejects none of the established frameworks on the axes of the Bronstein cube. When we say that string theory replaces point-particles by strings, what we really mean is that string theory replaces

[4] For an accessible book advocating against taking into account mathematical beauty in judging theories, we refer to *Lost in Math* by Sabine Hossenfelder [15].

[5] Notable alternatives to string theory are loop quantum gravity and causal dynamical triangulation.

relativistic quantum mechanical point-particles by relativistic quantum mechanical strings. Only the dimension of the object changes. As it turns out, gravity comes in automatically, making string theory a theory of quantum gravity.

In this section, we present a brief introduction to string theory. Note that we aim neither for completeness nor for excessive depth. There are many excellent books and lecture notes on string theory that are suitable for readers looking for a more comprehensive introduction. Instead, we cherry-pick a few topics, and highlight aspects that we deem relevant for this thesis.

1.2.1 Basics

A one-dimensional string propagating in time can be described by a two-dimensional worldsheet, parametrized by coordinates $\sigma^\alpha = (\tau, \sigma)$. Here τ is the timelike coordinate and σ is the spacelike coordinate. The latter is periodic in the case of a closed string and defined on an interval in the case of an open string.

The string worldsheet is embedded in the spacetime where the string lives, usually called the target space. For simplicity, we assume this to be flat Minkowski space here. The embedding of the worldsheet is given by D coordinates $X^\mu(\tau, \sigma)$, $\mu = 0, \ldots, D-1$, describing the position of each point of the worldsheet in the D-dimensional target space.

These coordinates are scalar fields in the Polyakov action [16–18]

$$S_{\mathrm{P}} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g}\, g^{\alpha\beta} \eta_{\mu\nu}\, \partial_\alpha X^\mu \partial_\beta X^\nu\,, \tag{1.6}$$

which describes the dynamics of the string. Here we denote the worldsheet metric by $g_{\alpha\beta}$ and the target space metric by $\eta_{\mu\nu}$. We can fix the worldsheet metric to be flat using gauge freedom of the Polyakov action, simplifying the following analysis.

The equations of motion for this theory are quite simple. We have the free wave equation

$$\partial_\alpha \partial^\alpha X^\mu = 0\,, \tag{1.7}$$

from the variation with respect to the coordinates X^μ. In addition, from varying with respect to the metric $g_{\alpha\beta}$ we find the constraint that the stress-energy tensor must vanish:

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \tfrac{1}{2}\eta_{\alpha\beta}\, \partial_\gamma X^\mu \partial^\gamma X_\mu = 0\,.$$

In lightcone coordinates $\sigma^\pm = \tau \pm \sigma$ the equation of motion (1.7) reads

$$\partial_+ \partial_- X^\mu = 0\,. \tag{1.8}$$

From this it follows that the embedding coordinates can be decomposed into two parts, one depending only on σ^+ and the other depending only on σ^-. These parts are called left-moving and right-moving respectively. We write

$$X^\mu(\sigma^+,\sigma^-) = X^\mu_{\mathrm{L}}(\sigma^+) + X^\mu_{\mathrm{R}}(\sigma^-)\,. \tag{1.9}$$

These left and right-moving coordinates can be expanded in Fourier modes. The precise form of this expansion depends on the boundary conditions of X^μ in the σ direction. In the case of a closed string, the coordinates are simply periodic

$$X^\mu(\tau,\sigma+2\pi) = X^\mu(\tau,\sigma)\,, \tag{1.10}$$

and in the open string case there are two options for the boundary conditions at the endpoint of the string, namely

$$\begin{aligned} &\text{Neumann:} \qquad \partial_\sigma X^\mu(\tau,\sigma)\big|_{\sigma=0,\pi} = 0\,, \\ &\text{Dirichlet:} \qquad \partial_\tau X^\mu(\tau,\sigma)\big|_{\sigma=0,\pi} = 0\,. \end{aligned} \tag{1.11}$$

If the target space is more complicated than Minkowski space, different boundary conditions are possible. We will encounter examples of this in chapter 4, where we discuss string theory on orbifolded target spaces.

Once the appropriate boundary conditions have been identified, one can Fourier expand the X^μ's. This expansion contains an infinite tower of left-moving modes $\tilde{\alpha}^\mu_n$ and right-moving modes α^μ_n. The boundary conditions of an open string relate the left and right-moving modes, so in that case one only has a single set of independent modes. The $n=0$ modes can be identified as the momentum of the string, while the $n \neq 0$ modes act as creation and annihilation operators of vibrations on the string. By acting on the vacuum state with these operators, an infinite set of states can be constructed, corresponding to a string carrying various levels of oscillations. This collection of states is understood as the spectrum of particles that emerges in the target space.

From this spectrum, we quickly highlight a state of big importance:

$$\tilde{\alpha}^\mu_{-1}\alpha^\nu_{-1}\left|0;p^\mu\right\rangle\,. \tag{1.12}$$

This is a massless state appearing in the spectrum of a closed string. The symmetric part in the indices μ,ν gives a familiar particle: the graviton. It turns out that this particle emerges in all string theories, making gravitation a universal feature of string theory.

1.2.2 Superstrings

As written in the previous section, string theory has a lot of problems. Primarily, it contains only bosons and no fermions, both on the worldsheet and in the target space. Unsurprisingly, this version of string theory is known as bosonic string theory. The absence of fermions is a major issue for making contact with the real world, in which fermions surely exist.

In addition, the vacuum of the bosonic string is tachyonic, meaning that it has an imaginary mass. This indicates that the theory is described around a maximum in the potential and is therefore unstable.

Both of these problems can be resolved by considering a supersymmetric version of string theory. In the Ramond–Neveu–Schwarz formalism [19, 20] of the superstring, fermions are introduced on the worldsheet and the bosonic Polyakov action is replaced by a supersymmetric version. These fermionic coordinates $\psi^\mu(\tau, \sigma)$ can be Fourier expanded in a way similar to the bosonic ones. This yields a set of left and right-moving fermionic oscillators $\tilde{b}^\mu_r$ and b^μ_r that create and annihilate fermionic excitations on the string. Again, in the open string case there is only one independent set of oscillators.

On closed string worldsheets the fermions can be either periodic or anti-periodic around the σ-direction, defining two distinct sectors

$$\begin{aligned} &\text{Ramond:} && \psi^\mu(\tau, \sigma + 2\pi) = \psi^\mu(\tau, \sigma)\,, \\ &\text{Neveu–Schwarz:} && \psi^\mu(\tau, \sigma + 2\pi) = -\psi^\mu(\tau, \sigma)\,. \end{aligned} \tag{1.13}$$

As it turns out, states in the R-sector give rise to target space fermions, while states in the NS-sector yield target space bosons. In fact, the left and right-moving parts of ψ^μ can be in either sector independently. This results in four sectors: (NS,NS) and (R,R) giving spacetime bosons, and (NS,R) and (R,NS) giving spacetime fermions.

From here on it looks like we can carry on just like in the case of the bosonic string. We now have both bosonic and fermionic oscillators, and we can construct both bosonic and fermionic states in the target space. However, there are still two problems left: the tachyon in the spectrum remains, and taking into account all states does not lead to a modular invariant partition function. Both can be solved by taking a truncation of the spectrum called the GSO projection [21]. This truncation projects out the tachyon, and ensures that the partition function is modular invariant.

Within the framework of superstring theory various consistent models can be constructed:

- Type I string theory consists of open and closed strings, and produces $\mathcal{N} = (1,0)$ supersymmetry in the target space.
- Type II string theory consists of closed strings. Depending on the GSO-projection it gives rise to $\mathcal{N} = (1,1)$ or $\mathcal{N} = (2,0)$ supersymmetry in the target space. In these cases it is known as type IIA and type IIB string theory respectively. A significant part of the research presented in this thesis takes place in these models.
- Type 0 string theory is a model of closed strings that is often omitted in classifications of consistent superstring theories, as it is supersymmetric only on the level of the worldsheet. The choice of the GSO-projection renders the target space theory non-supersymmetric. Again, there are multiple choices for this projection leading to the type 0A and type 0B theories.
- Heterotic string theory is a remarkable model of closed strings in which the left-movers are purely bosonic and the right-movers are a supersymmetric combination of both bosons and fermions. Consistency requires the target space gauge group to be either $\mathrm{Spin}(32)/\mathbb{Z}_2$ or $\mathrm{E}_8 \times \mathrm{E}_8$. Both of these options give rise to $\mathcal{N} = (1,0)$ supersymmetry.

Type I string theory was the first model that was shown to be free of anomalies, by Michael Green and John Schwarz [22]. This discovery convinced people that string theory had the potential to be a fundamental description of elementary particle physics, and led to the 'first superstring revolution'.

All of the theories listed above have one thing in common: their target space is required to be ten-dimensional[6]. This requirement comes from anomaly cancellation. Obviously this is a problem for phenomenology, since nature only appears to have four spacetime dimensions. The most popular solution for this is to assume that the six extra dimensions are compact and extremely small; we will discuss this topic of 'compactification' in more detail in section 1.2.5.

1.2.3 Dualities

At first sight, the list of consistent string theories in the previous section may look a bit disorganized. They appear to be unrelated, with each one of them being a

[6]The case of the heterotic string is slightly more involved, as the bosonic sector is required to be 26-dimensional while the superstring sector is required to be 10-dimensional. This is resolved by compactifying the bosonic sector on a 16-dimensional manifold, leaving 10 dimensions for the rest of the heterotic string target space.

well-defined theory in its own right. This, however, turns out not to be the case. In fact, the different superstring theories are intimately connected through dualities. A duality is a relation between two seemingly different theories that indicates that these theories actually describe the same physics, just in a different guise.

T-duality

First we discuss T-duality, short for target space duality. As the name implies, this is a duality relating theories with different target spaces. It was discovered in 1987 by Bala Sathiapalan [23]; for a comprehensive review we refer to [24].

The easiest example of T-duality relates string theory propagating in target spaces containing a circle of different radius. If on one side of the duality the circle has radius R, it has radius α'/R on the other side. The theories defined on these different target spaces turn out to be equivalent, meaning they yield the exact same observable quantities. In particular, T-duality relates string theory defined on a target space that contains a very large circle to string theory on that same target space but with a very small circle.

If one applies this notion of target space duality to backgrounds containing a torus, the set of transformations that yield a duality enlarges. In the case of a d-dimensional torus, these transformations form the group $\mathrm{O}(d, d, \mathbb{Z})$. This group is known as the T-duality group and it contains the inversions of the radii of the d circles in the torus T^d, as well as other more complicated transformations on the torus.

The type of string theory on both sides of the duality is not necessarily the same. For example, T-duality relates type IIA and type IIB, as well as the two kinds of heterotic string theory.

S-duality

Another duality that is important in string theory is S-duality, short for strong-weak duality. While examples of strong-weak duality were already known outside of string theory, e.g. Montonen–Olive duality in super-Yang–Mills theory, it wasn't until the early 1990s that an application in string theory was found [25, 26].

S-duality relates string theories at coupling g_s to string theories at coupling $1/g_s$. This immediately explains the name strong-weak duality: if one of the theories is at strong coupling, the other is at weak coupling. The ability to relate strongly and weakly coupled theories is extremely useful, as one needs a weakly coupled theory to make meaningful predictions using perturbation theory. In a broader sense the term S-duality is used for the duality group $\mathrm{SL}(2,\mathbb{Z})$. The transformation that inverts the string coupling constant g_s is then simply an element of this group.

Again, this duality relates different string theories to one another. Type I is S-dual to heterotic $\mathrm{Spin}(32)/\mathbb{Z}_2$, while type IIB is self-dual under S-duality.

U-duality

The term U-duality, or unified duality, is used for a collection of dualities that includes both T and S-dualities. Its existence was proposed by Chris Hull and Paul Townsend in [27]. Although it is not yet entirely clear how U-duality works on string theory in general, there are clear predictions about its group theoretical structure. Namely, type II string theory on a d-torus is subject to the U-duality group $\mathrm{E}_{d+1}(\mathbb{Z})$, which is the maximal discrete subgroup of the maximally non-compact exceptional group $\mathrm{E}_{d+1,d+1}$.

Some evidence for the validity of this conjecture is seen in the fact that the T-duality group $\mathrm{O}(d,d,\mathbb{Z})$ and the S-duality group $\mathrm{SL}(2,\mathbb{Z})$ of type II on T^d are both subgroups of this proposed U-duality group $\mathrm{E}_{d+1}(\mathbb{Z})$. Furthermore, it is known that the low-energy limit of these string theory setups is maximal supergravity in $10-d$ dimensions, which has a continuous $\mathrm{E}_{d+1,d+1}$ duality symmetry. To string theorists, it is a familiar phenomenon that continuous symmetries in supergravity lift to discrete symmetries in the corresponding string theory setup.

M-theory

The rise of dualities in string theory inspired Edward Witten in 1995 to make the bold claim that the known superstring theories were not different theories at all, but that they were limits of a single overarching 11-dimensional theory that he gave the name ‘M-theory’ [28]. The dualities relating the various string theories could then be interpreted as moving through some parameter space of M-theory from one of these limits to another. A complete fundamental description of M-theory is yet to be worked out, but its fundamental degrees of freedom should be two and five-dimensional membranes and in the low energy limit it should reduce to 11d supergravity.

Witten’s proposal of M-theory provoked a surge of research, which is now known as the ‘second superstring revolution’.

Holography and AdS/CFT

The last duality that we discuss here is not a duality between string theories, but rather a duality inspired by string theory. It states that string theory (or M-theory) defined on a background that has a $(d+1)$-dimensional anti-de Sitter factor, AdS_{d+1}, is dual to a non-gravitational conformal field theory in d dimensions. Sometimes, but not always, the CFT is thought of as living on the boundary of the AdS space.

This duality is known as the AdS/CFT correspondence, and was proposed by Juan Maldacena in 1997 [29]. It is a realization of the holographic principle, invented by Gerard 't Hooft in 1993 [30], which states that all information about a region in space in a theory of quantum gravity can be described by degrees of freedom living on the boundary of that region. Hence, according to the holographic principle, the feature that information in $d+1$ dimensions can be captured in a d-dimensional theory is not unique to AdS/CFT, but universal in quantum gravity.

It can easily be understood why the AdS/CFT correspondence was a groundbreaking discovery in the quest for a theory of quantum gravity. We have introduced quantum gravity as the unification of gravity with quantum field theory (see section 1.1), but now AdS/CFT tells us that these theories may actually be different faces of one and the same theory. To stress this important notion AdS/CFT is sometimes called gauge/gravity duality.

The research presented in this thesis resides almost exclusively on the string theory or gravity side of the AdS/CFT correspondence. Although holography isn't used explicitly, much of what is done here is inspired by it. Most of the gravity solutions that are discussed contain AdS factors, or related geometries like BTZ and fibered AdS factors.

1.2.4 Branes

Now that we have introduced the various dualities in string theory, we are ready to discuss branes: higher-dimensional objects that generalize the notions of particles and strings. It turns out that these arise naturally in string theory.

In a first string theory course, D-branes are often introduced as objects on which open strings end. Recall the open string boundary conditions (1.11), and note that endpoints are fixed in directions in which they have Dirichlet boundary conditions. Consequently, if a string endpoint has Dirichlet boundary conditions in n directions, it is only allowed to move on a $(10-n)$-dimensional hypersurface lying in the remaining directions. This hypersurface is called a D-brane, where the D stands for Dirichlet. The two endpoints of an open string do not necessarily have the same boundary conditions, and therefore do not necessarily end on the same brane. A picture of open strings attached to D-branes of different dimensions is shown in figure 1.3.

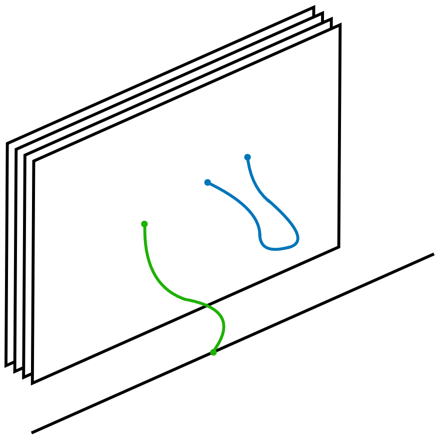


Figure 1.3: *A setup showing open strings and D-branes. One string is attached to the same stack of branes, while the other stretches between different branes. Picture taken from [31].*

These hypersurfaces arising from boundary conditions weren't taken very seriously until 1995, when Joseph Polchinski interpreted them as electric and magnetic sources of Ramond–Ramond gauge fields [32]. His argument relied on string dualities, and can be seen as one of the major breakthroughs in the second superstring revolution.

In order to see how D-branes emerge from dualities, it is useful to know that S-duality in type IIB rotates the Kalb–Ramond field B_2 and the Ramond–Ramond two-form C_2 into one another. The fundamental string is charged under B_2, so consistency demands the existence of an object charged under C_2 for the fundamental string to rotate into. This object is called a D1-brane. The number in this notation only counts the number of spatial directions; including time the object is two-dimensional.

One can now change the dimension of this brane by performing T-dualities along various circles in the target space manifold. If the D-brane is wrapping the circle on one side of the duality, it is not wrapping the circle on the other side of the duality. In this way, one finds D-branes of arbitrary dimension by performing an appropriate set of T-dualities. Each of these dualities also switches between type IIA and type IIB, so these theories only contain odd and even dimensional D-branes respectively.

In addition to these D-branes, string theory also contains so-called NS5-branes. These are magnetically charged under the Kalb–Ramond field, so they are the magnetic dual of the fundamental string. As we already mentioned in the previous section, M-theory is believed to have two and five-dimensional membranes as its fundamental degrees of freedom. These are known as M2-branes and M5-branes.

1.2.5 Compactification

As we stated earlier, the requirement of anomaly cancellation in superstring theory demands that the target space is ten-dimensional. In order to reconcile this with the observation that our universe appears to have only four spacetime dimensions, the extra dimensions are assumed to be compact and very small. If the extra dimensions are sufficiently small, they are effectively invisible for macroscopic observers, which explains why they have not been discovered in experiments. To visualize this idea one can think about a very thin cylinder, see figure 1.4.

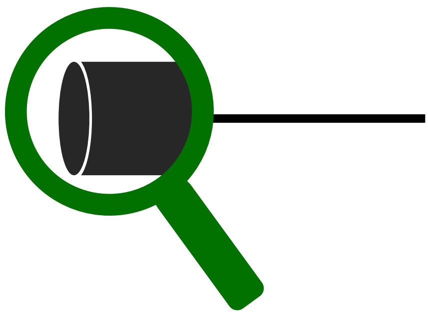


Figure 1.4: *A visualization of how small compact dimensions can be undetectable by macroscopic observers. Picture taken from [33].*

As seen from far away the cylinder looks like a line, i.e. it appears to be one-dimensional. In order to see that the object is two-dimensional (we assume that the cylinder is hollow), one has to zoom in far enough. The precise meaning of 'far enough' depends on the radius of the cylinder. As current experiments can probe distances of around 10^{-19} meters, extra dimensions with characteristic length scales much smaller than that can exist without having been observed.

Let us substantiate this qualitative argument with some formulas. Consider a massless scalar field $\phi(x^\mu, z)$ living in a d-dimensional flat space times a circle parametrized by the coordinate z. The z-dependence of this $(d+1)$-dimensional field can be expanded as

$$\phi(x^\mu, z) = \sum_{n\in\mathbb{Z}} \phi_n(x^\mu)\, \mathrm{e}^{inz/R}\,, \tag{1.14}$$

where R is the radius of the circle. This yields an infinite tower of d-dimensional fields ϕ_n. From applying the massless Klein–Gordon equation to (1.14), one finds the massive Klein–Gordon equations

$$\partial_\mu \partial^\mu \phi_n - \frac{n^2}{R^2}\, \phi_n = 0\,, \tag{1.15}$$

for the fields ϕ_n. It follows that these fields are massive with masses $m = |n/R|$. Now, if we take the radius of the circle to be small, the masses of the d-dimensional fields become large. In experiments that probe energies below $1/R$ only the $n = 0$ mode will be visible, so effectively one measures a theory that contains only a single massless d-dimensional scalar field $\phi_0(x^\mu)$. This process of obtaining a d-dimensional theory from a $(d+1)$-dimensional one is called Kaluza–Klein reduction.

This scheme can be extended to compactifications on more complicated manifolds than a circle. Examples that we will see a lot in this thesis are tori and toroidal orbifolds, as well as Sasaki–Einstein manifolds.

1.2.6 Supergravity

String theory contains infinitely many particles that can be infinitely massive. For many purposes, taking into account this excess of states is unnecessary, impossible or both. In these situations, one often relies on the low-energy limit of string theory: supergravity. For this, one truncates the infinite spectrum to the lightest collection of states, which yields a theory that can be seen as the low-energy effective description of string theory. Here, many questions that don't involve high-energy processes can be answered, without bothering with the full theory. We will often follow this approach in this thesis.

It is remarkable how much things can simplify in the supergravity limit. For example, while the full quantum description of M-theory is still unknown, it is well-established that in the low-energy limit it yields 11d supergravity. The bosonic part of the action is given by

$$S_{11} = \frac{1}{2\kappa_{11}^2} \int_{\mathcal{M}_{11}} R * 1 - \frac{1}{2} G_4 \wedge * G_4 - \frac{1}{6} C_3 \wedge G_4 \wedge G_4\,. \tag{1.16}$$

Here $G_4 = \mathrm{d}C_3$ is the field strength of a three-form potential. The fundamental degrees of freedom of M-theory, two and five-branes, can now be written down as solutions of this supergravity theory. E.g. the solution for a stack of M2-branes with worldvolume $\mathbb{R}^{1,2} \subset \mathbb{R}^{1,10}$ is

$$\begin{cases} \mathrm{d}s^2 = H(r)^{-2/3}\big(-\,\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2\big) + H(r)^{1/3}\big(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_7^2\big)\,, \\ G_4 = \partial_r H(r)^{-1}\,\mathrm{d}t \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}r\,, \end{cases} \tag{1.17}$$

where we use radial coordinates on the transverse space and

$$H(r) = 1 + \frac{\alpha}{r^6}\,, \tag{1.18}$$

is a harmonic function on $\mathbb{R}^8$. If we demand that this supergravity solution can be lifted to M-theory, we need to impose flux quantization. This restricts the values of the coefficient in $H(r)$ to $\alpha = 2^5\pi^2\ell_{11}^6 N$, where ℓ_{11} denotes the 11d Planck length and $N \in \mathbb{Z}$ counts the number of M2-branes in the stack.

These kind of solutions share a lot of properties with black hole solutions. In fact, they can be seen as higher dimensional generalizations: black branes. The solution (1.17) describes a black 2-brane; it has an horizon at $r = 0$.

Near-horizon geometry

An important feature of black brane solutions, such as the one discussed just now, is that they have anti-de Sitter near-horizon geometries. This makes them suitable as backgrounds for AdS/CFT. For the solution in (1.17) we will explicitly show how the AdS factor shows up in the near-horizon limit. Close to the horizon at $r = 0$, the leading term in the harmonic function is $H(r) = \alpha/r^6$. This simplifies the metric to

$$\mathrm{d}s^2 = \frac{r^4}{\alpha^{2/3}}\left(-\,\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2\right) + \frac{\alpha^{1/3}}{r^2}\,\mathrm{d}r^2 + \alpha^{1/3}\,\mathrm{d}\Omega_7^2\,. \tag{1.19}$$

Now, by performing the coordinate transformation $\rho = 2\alpha^{-1/2}\,r^2$, we find

$$\mathrm{d}s^2 = \frac{\alpha^{1/3}}{4}\left[\rho^2\left(-\,\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2\right) + \frac{\mathrm{d}\rho^2}{\rho^2}\right] + \alpha^{1/3}\,\mathrm{d}\Omega_7^2\,, \tag{1.20}$$

which we recognize as the metric of $\mathrm{AdS}_4 \times \mathrm{S}^7$ with radii $R_{\mathrm{AdS}} = \frac{1}{2}R_{\mathrm{S}^7} = \frac{1}{2}\alpha^{1/6}$.

1.3 This thesis

The research presented in this thesis is aimed to improve our understanding of black holes in string theory. We study various string theoretical setups that give rise to four and five-dimensional black holes. A common feature of all of these, is that they are constructed out of branes: either D-branes in string theory, or M-branes in M-theory. Hence, we title this thesis *Black Holes from Branes*.

This thesis is divided into three parts. In each of these we investigate black holes from a different perspective:

- Part I – Black holes and supersymmetry breaking

  We study black holes in vacua of string theory in which supersymmetry is spontaneously broken. The mechanisms that we employ can be used to break supersymmetry partially or completely. One of the central questions that we ask is whether the black holes that we consider survive this supersymmetry breaking, and if so, how they are affected by it.

- Part II – Spindle and disc near-horizons

  In this part we study near-horizon geometries of black holes, in which the horizon is not a regular sphere but either a spindle or a topological disc. Such black hole near-horizon geometries are not smooth – they contain conical singularities – but their uplifts to string or M-theory typically are. We consider several such near-horizons, and investigate the regularity of the uplifted geometries.

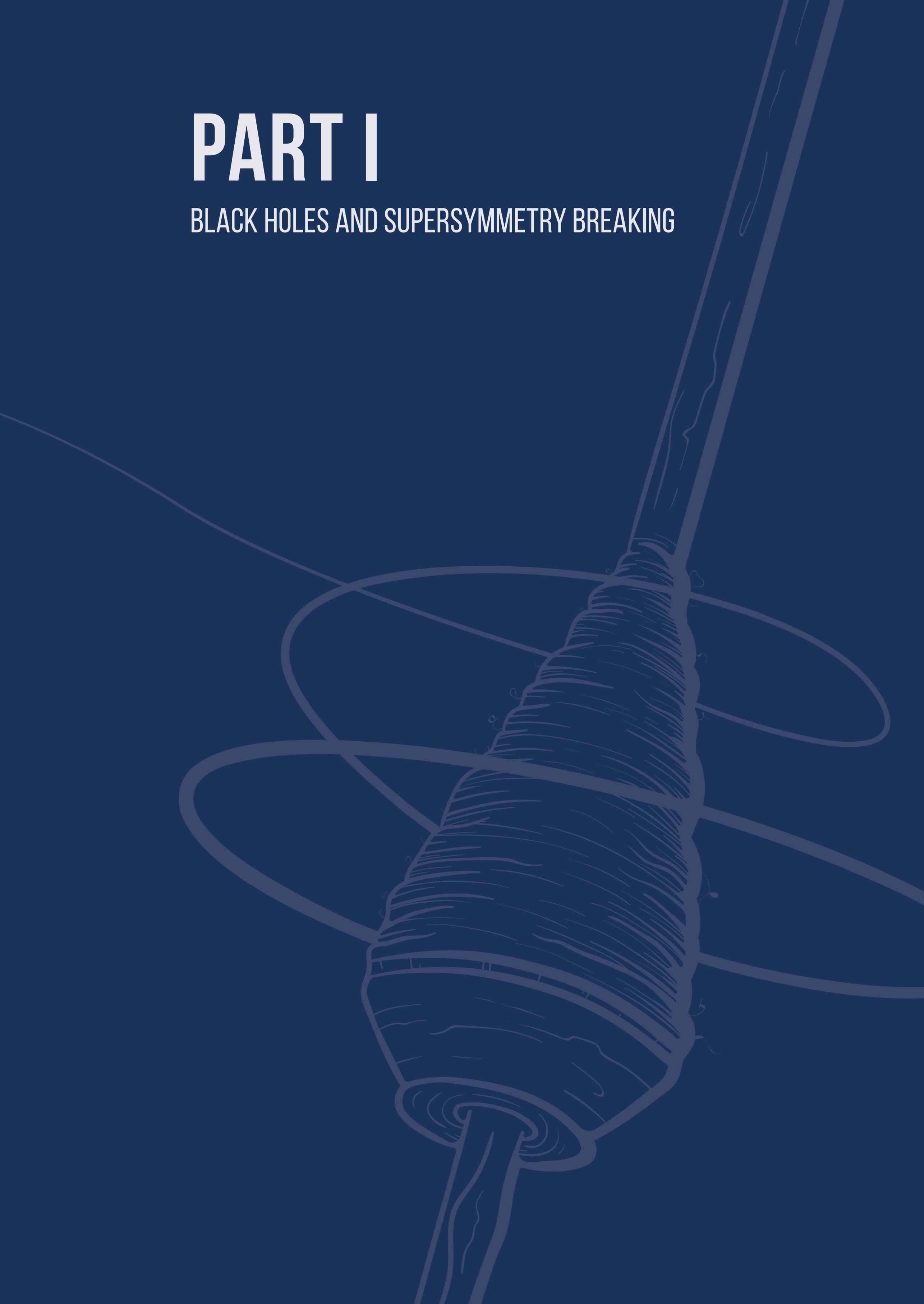
- Part III – Supersymmetric classification

  Here we develop a classification of black holes in M-theory. That is, we formulate the necessary and sufficient conditions to obtain the near-horizon geometry of an extremal supersymmetric rotating black hole constructed out of rotating M2-branes. This classification allows for a very methodical analysis of rotating black holes in M-theory and their properties.

We conclude with a summary of the main results of this thesis, and an outlook on follow-up directions to be pursued in future research.

PART I

BLACK HOLES AND SUPERSYMMETRY BREAKING



Chapter 2

Introduction

"Supersymmetry was (and is) a beautiful mathematical idea. The problem with applying supersymmetry is that it is too good for this world."

– Frank Wilczek, The Lightness of Being

In this part we study a well-known black hole construction in the context of partial supersymmetry breaking. The construction is known as the *D1-D5-P system*. It describes a brane setup in type IIB string theory on $T^4 \times S^1$ (or $K3 \times S^1$) that can be used for the study of BPS black holes in five spacetime dimensions, both microscopically and macroscopically. By compactifying the $T^4 \times S^1$, one finds an asymptotically flat three-charge $\frac{1}{8}$-BPS black hole solution of 5d $\mathcal{N} = 8$ supergravity. The entropy can be computed microscopically from a 2d $\mathcal{N} = (4,4)$ CFT dual to the near-horizon geometry of the black hole [34].

Note that while the black hole itself partially breaks supersymmetry, the theory in which it lives is maximally supersymmetric. We would like to consider extensions of this to black holes in theories with less supersymmetry. Black holes in compactifications preserving eight supersymmetries in five dimensions can be constructed in M-theory on CY_3 [35] or in F-theory on $CY_3 \times S^1$ [36, 37]. In these cases, the microscopic field theory dual to the black hole horizon geometry is a 2d $\mathcal{N} = (0,4)$ CFT. These CFTs are considerably more complicated than the $\mathcal{N} = (4,4)$ CFT of [34] as they have less supersymmetry.

Here, we consider a different way to reduce supersymmetry. The mechanism that we use carries various names, depending on the context in which it is applied. On the level of supergravity it is known as *Scherk–Schwarz reduction* [38, 39]. These reductions can be lifted to string theory, as long as certain conditions are met, and in this case the mechanism is called reduction with a *duality twist*. In the vacua of these stringy constructions, the theory can be described as a generalized orbifold. In all cases that we study in this thesis, these generalized orbifolds will be freely-acting *asymmetric orbifolds* [40, 41]. We will treat the supergravity story in chapter 3 and the string theory story in chapter 4.

These mechanisms allow for partial supersymmetry breaking and include string vacua preserving no supersymmetry at all (though these won't be the focus here). This gives rise to 5d Minkowski vacua preserving $\mathcal{N} = 6, 4, 2, 0$ supersymmetry [42]. We investigate 5d supersymmetric black holes in these theories that lift to 10d systems of branes: the D1-D5-P system as well as some U-dual systems.

This work follows up on ideas proposed earlier in [43] in an M-theory setting in which supersymmetry is completely broken. Completely broken supersymmetry is not a well controlled situation, and for that reason we will focus on twists preserving some supersymmetry. We will focus on the macroscopic description, both in supergravity and in string theory, and leave the microscopic description of the dual CFTs for future study.

2.1 Scherk–Schwarz reduction

In supergravity Scherk–Schwarz reductions are mechanisms to spontaneously break supersymmetry and to obtain massive fields from higher dimensional massless ones [38, 39]. These reductions have been extensively studied in the literature; see e.g. [42, 44–49] and references therein.

Scherk–Schwarz reduction can be seen as a generalization of Kaluza–Klein reduction, as we discussed briefly in section 1.2.5. It uses ansätze of the type

$$\hat{\psi}(x^\mu, z) = g(z)\,\psi(x^\mu)\,.$$

Here $\hat{\psi}$ and ψ are $d+1$ and d-dimensional fields respectively, and $g(z)$ is a local element of some global symmetry group G depending on the circle coordinate z. Because G is a symmetry, the d-dimensional theory will be z-independent even though the ansatz for the field $\hat{\psi}$ is not. On going round the circle $z \to z + 2\pi R$, the field picks up a monodromy $\mathcal{M} = g(2\pi R) \in G$. Such a reduction gives a consistent truncation to a gauged supergravity theory in d dimensions, in which there typically is a Scherk–Schwarz potential for the scalar fields and mass terms for all fields charged under the monodromy.

In our supergravity setup, see chapter 3, we consider type IIB supergravity compactified on T^4 which gives maximal $\mathcal{N} = (2, 2)$ supergravity in 6d [50]. This theory has a Spin(5, 5) duality symmetry, which we use to Scherk–Schwarz reduce to 5d on a circle. Such reductions have been considered before in [42]. If the twist $g(z)$ is compact, i.e. it is an element of the R-symmetry group Spin(5) × Spin(5), then the potential is non-negative and has stable 5d Minkowski vacua [47]. Such a twist can be specified by four parameters m_1, m_2, m_3, m_4 which become mass parameters in the reduced theory.

The amount of supersymmetry that is preserved in the vacuum depends on the number of parameters that are equal to zero: if r of the parameters m_i are zero, then $\mathcal{N} = 2r$ supersymmetry is preserved. This yields 5d supergravities with $\mathcal{N} = 8, 6, 4, 2, 0$ Minkowski vacua, where the case $r = 4$ is the untwisted reduction to 5d $\mathcal{N} = 8$ supergravity, and the case $r = 0$ is the twisted reduction that breaks all supersymmetry. These reductions are straightforward generalizations of the Scherk–Schwarz reduction of 5d $\mathcal{N} = 8$ supergravity to 4d with four mass parameters and $\mathcal{N} = 8, 6, 4, 2, 0$ vacua [44].

2.2 Duality twists

The lift of these supergravity reductions to full compactifications of string theory involves a number of subtle features [47]. These have been worked out in detail for compactifications of IIA string theory on K3 or the heterotic string on T^4 followed by a reduction on a circle with a duality twist in [51, 52]. Here we draw on these for our construction, which is IIB string theory compactified on $T^4 \times S^1$ with a U-duality twist around the circle.

On the level of string theory, the continuous duality symmetry $\mathrm{Spin}(5,5)$ of type IIB supergravity on T^4 is broken to the discrete U-duality subgroup $\mathrm{Spin}(5,5;\mathbb{Z})$ by quantum effects [27]. A key requirement for there to exist a lift to string theory is that the Scherk–Schwarz monodromy lies in the U-duality group $\mathrm{Spin}(5,5;\mathbb{Z})$, imposing a 'quantization' condition on the twist parameters m_i. There is still an action of the continuous group $\mathrm{Spin}(5,5)$ on the theory, but only the subgroup $\mathrm{Spin}(5,5;\mathbb{Z})$ is a symmetry.

Note that while the monodromy is required to be compact as well as an element of the discrete U-duality group $\mathrm{Spin}(5,5;\mathbb{Z})$, it is not necessarily an element of $\mathrm{Spin}(5;\mathbb{Z}) \times \mathrm{Spin}(5;\mathbb{Z})$. This is because twists in the same $\mathrm{Spin}(5,5)$ conjugacy class result in equivalent theories. Therefore, from demanding that the monodromy is compact, we only have the requirement that it is conjugate to an R-symmetry. To be explicit, the monodromy $\mathcal{M}$ should be an element of $\mathrm{Spin}(5,5;\mathbb{Z})$ that can be written as

$$\mathcal{M} = g\tilde{\mathcal{M}}g^{-1}\,, \tag{2.1}$$

where

$$g \in \mathrm{Spin}(5,5)\,, \qquad \tilde{\mathcal{M}} \in \mathrm{Spin}(5) \times \mathrm{Spin}(5) \subset \mathrm{Spin}(5,5)\,. \tag{2.2}$$

This ensures that there is a stable Minkowski vacuum. In addition, these conditions imply that $\mathcal{M}$ satisfies $\mathcal{M}^p = \mathbb{1}$ for some integer p, so that the monodromy generates a cyclic group $\mathbb{Z}_p \subset \mathrm{Spin}(5,5;\mathbb{Z})$.

The point in the scalar coset where the potential has a minimum is a fixed point under the action of the monodromy $\mathcal{M} \in \mathrm{Spin}(5,5;\mathbb{Z})$ [47]. At this critical point the construction becomes a $\mathbb{Z}_p$ generalized orbifold of IIB string theory on $T^4 \times S^1$. The corresponding orbifold action combines the action of the monodromy with a shift on the S^1 given by $z \to z + 2\pi R/p$. This shift makes the orbifold freely-acting.

The T-duality subgroup of the U-duality group is a particular embedding of $\mathrm{Spin}(4,4;\mathbb{Z}) \subset \mathrm{Spin}(5,5;\mathbb{Z})$, and when the monodromy is a T-duality, the generalized orbifold construction becomes a conventional asymmetric orbifold [40, 41]. However, this asymmetric orbifold is not modular invariant in general. The remedy is straightforward: modular invariance can be achieved if the shift in the circle coordinate z is accompanied by a shift in the coordinate of the T-dual circle. The T-dual circle has radius α'/R, and its coordinate $\tilde{z}$ undergoes a shift $\tilde{z} \to \tilde{z} + 2\pi n\alpha'/pR$ for a particular integer n which can be determined as in [41, 52].

2.3 Toroidal orbifolds

As the minimum of the potential arising from T-duality twisted reductions is described by (possibly asymmetric) toroidal orbifolds, we discuss these in a little more detail. The name toroidal orbifold, in a way, is badly chosen. It is used for geometries that are obtained from tori by taking the quotient by the action of a discrete symmetry group. It should be noted, however, that tori themselves can also be understood as orbifolds, namely as

$$T^d = \mathbb{R}^d/\Lambda\,, \tag{2.3}$$

where Λ is a d-dimensional lattice.

Roughly speaking there are two types of discrete actions that can be quotiented out of a torus, in order to obtain an orbifold thereof. First of all, one can perform rotations over certain angles that are symmetries of the torus lattice Λ. Such actions have fixed points, which leads to conical singularities in the orbifold. Secondly, one can perform shifts, which are freely acting meaning that they don't leave fixed points. Shifts in the absence of rotations are rather inconsequential, as the original torus was nothing more than a Euclidean plane with certain shifts quotiented out. Therefore, quotienting out shifts on a torus returns a torus (typically with different radii). The simplest examples of both types of orbifolds are shown in figure 2.1.

We see that on a circle (a 'one-torus') a reflection orbifold (or 'rotation' over an angle π) yields a line interval, with fixed points on both ends. A shift orbifold,

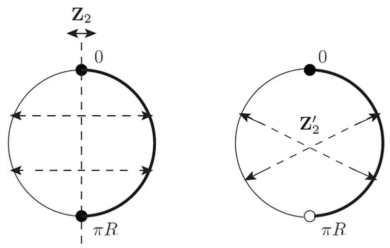


Figure 2.1: *A non-freely-acting (reflection) and a freely-acting (shift) $\mathbb{Z}_2$-orbifold of S^1. Picture taken from [53].*

however, simply yields a circle with half the original radius.

String theory on orbifolded target spaces was first considered in [54, 55]. But one can go further than that. In string theory, it is possible to put the left and right-movers on different orbifolds, in which case they are called asymmetric orbifolds [40, 41].

In the case that our duality twists are contained in the T-duality group, the theory in the vacuum can be described as an asymmetric $\mathbb{Z}_p$-orbifold of the compact space $T^4 \times S^1$. The monodromy then works as an asymmetric rotation on the torus, i.e. it gives a separate rotation on the left-moving and the right-moving torus coordinates. In principle, such rotations would leave fixed points on the torus. However, the orbifold action includes a shift on the circle $z \to z + 2\pi R/p$, which makes the combined orbifold action freely-acting.

If one considers non-perturbative duality twists, i.e. twists outside the T-duality group, one has to go beyond orbifolds that can be seen as a (possibly asymmetric) quotient of a torus. In such cases, the string theory target space is a non-geometric background called a U-fold [56]. These are generalizations of ordinary manifolds, on which the transition functions are allowed to be U-duality transformations instead of just diffeomorphisms.

2.4 Outline

As we already mentioned, we will consider our setup from two points of view. Chapter 3 contains the supergravity story, while chapter 4 contains the string theory story.

In order to perform the Scherk–Schwarz reduction on the level of supergravity, we need a duality covariant formulation of type IIB supergravity compactified on a four-torus. We present this construction in section 3.1. With this powerful framework at hand, we are ready to Scherk–Schwarz reduce to 5d, which is the topic of section 3.2. We construct mass matrices and decompose the 5D field content into massless and massive multiplets. By simply truncating to the massless sector, we can embed known BPS black holes in these five-dimensional theories. In section 3.3 we work out which choices of Scherk–Schwarz twists preserve the D1-D5-P black hole and dual setups. Next, in section 3.4, we study one-loop effects by integrating out the massive supergravity field content. We compute the corrections induced by this to the Chern-Simons terms and to the entropy of the 5d BPS black holes.

In chapter 4, we move on to string theory. Section 4.1 discusses the constraints on the Scherk–Schwarz parameters m_i that are necessary for the lift to string theory. Next, in section 4.2, we construct the orbifold that describes the duality twisted theory in its vacuum. This construction allows us to compute the 5d spectrum. In section 4.3 we do this for specific parts of the spectrum, in order to check that we have indeed constructed the orbifold corresponding to the Scherk–Schwarz reductions from chapter 3, and to analyze some states that we expect to appear in the dual CFT. Finally, in section 4.4 we discuss which D-branes survive in which orbifolds. This tells us which 5d BPS black holes we can embed in these models.

Chapter 3

D1/D5-branes and Scherk–Schwarz

3.1 Duality covariant formulation of IIB on T^4

Reducing type IIB supergravity on a four-torus gives six-dimensional maximal supergravity. This theory has $\mathcal{N} = (2,2)$ supersymmetry and a Spin$(5,5)$ duality symmetry group. The goal of this section is to write this supergravity theory in a form in which both the type IIB origin of the six-dimensional fields and the Spin$(5,5)$ symmetry are manifest. We do this explicitly for the scalar and tensor fields.

3.1.1 Ansätze for reduction to 6d

We start from type IIB supergravity. Written in Einstein frame, the bosonic terms in the Lagrangian read

$$\begin{aligned}\mathscr{L}_{\mathrm{IIB}} = \bigg(&R^{(10)} - \frac{1}{2}|\mathrm{d}\Phi|^2 - \frac{1}{2}\,e^{-\Phi}\left|H_3^{(10)}\right|^2 - \frac{1}{2}\,e^{2\Phi}\,|\mathrm{d}a|^2 - \frac{1}{2}\,e^{\Phi}\left|F_3^{(10)}\right|^2 \\ &-\frac{1}{4}\left|F_5^{(10)}\right|^2\bigg) * 1 - \frac{1}{2}\,C_4^{(10)} \wedge H_3^{(10)} \wedge F_3^{(10)}\,,\end{aligned} \tag{3.1}$$

where the field strengths are given by

$$\begin{aligned}H_3^{(10)} &= \mathrm{d}B_2^{(10)}\,,\\ F_3^{(10)} &= \mathrm{d}C_2^{(10)} - a\,\mathrm{d}B_2^{(10)}\,,\\ F_5^{(10)} &= \mathrm{d}C_4^{(10)} - \frac{1}{2}C_2^{(10)} \wedge \mathrm{d}B_2^{(10)} + \frac{1}{2}B_2^{(10)} \wedge \mathrm{d}C_2^{(10)}\,.\end{aligned} \tag{3.2}$$

The superscripts (10) indicate that the fields live in 10 dimensions. The field equations are supplemented by the self-duality constraint

$$F_5^{(10)} = *F_5^{(10)}\,. \tag{3.3}$$

In our compactification to six dimensions, the coordinates split up as $X^M = (\hat{x}^{\hat{\mu}}, y^m)$ with $M = 0, \ldots, 9$, $\hat{\mu} = 0, \ldots, 5$ and $m = 1, \ldots, 4$. We now present the ansätze that we use in our reduction. In order to arrive in Einstein frame in 6d, we decompose the ten-dimensional metric as

$$g_{MN} = \begin{pmatrix} g_4^{-1/4}\, g_{\hat{\mu}\hat{\nu}} + g_{mn}\, \mathcal{A}^m_{\hat{\mu}} \mathcal{A}^n_{\hat{\nu}} & g_{mn}\, \mathcal{A}^m_{\hat{\mu}} \\ g_{mn}\, \mathcal{A}^n_{\hat{\nu}} & g_{mn} \end{pmatrix}, \tag{3.4}$$

where $g_4 = \det(g_{mn})$. The compact part of the metric, g_{mn}, we parametrize in terms of scalar fields ϕ_i $(i = 1, \ldots, 4)$ and A_{mn} $(m < n)$ by

$$g_{mn} = \begin{cases} e^{\vec{b}_m \cdot \vec{\phi}} + \sum\limits_{k<m} e^{\vec{b}_k \cdot \vec{\phi}}\, (A_{km})^2 & \text{for } m = n \\ e^{\vec{b}_m \cdot \vec{\phi}}\, A_{mn} + \sum\limits_{k<m} e^{\vec{b}_k \cdot \vec{\phi}}\, A_{km} A_{kn} & \text{for } m < n \\ g_{nm} & \text{for } m > n\,. \end{cases} \tag{3.5}$$

Here $\vec{\phi} = (\phi_1, \phi_2, \phi_3, \phi_4)$ and the vectors $\vec{b}_m$ are given by

$$\begin{aligned} \vec{b}_1 &= (-\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}, \tfrac{1}{2})\,, \\ \vec{b}_2 &= (-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, \tfrac{1}{2})\,, \\ \vec{b}_3 &= (\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}, \tfrac{1}{2})\,, \\ \vec{b}_4 &= (\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, \tfrac{1}{2})\,. \end{aligned} \tag{3.6}$$

From this, it can be computed that $g_4 = e^{2\phi_4}$, so the scalar ϕ_4 parametrizes the volume of the T^4.

We reduce the 10d form-valued fields by simply splitting into components with different numbers of indices on the torus. For example, the Kalb-Ramond field $B_2^{(10)}$ decomposes as

$$\begin{aligned} B_2^{(10)} &= \frac{1}{2}\, B_{MN}\, \mathrm{d}X^M \wedge \mathrm{d}X^N \\ &= \frac{1}{2}\, B_{\hat{\mu}\hat{\nu}}\, \mathrm{d}x^{\hat{\mu}} \wedge \mathrm{d}x^{\hat{\nu}} + B_{\hat{\mu}m}\, \mathrm{d}x^{\hat{\mu}} \wedge \mathrm{d}y^m + \frac{1}{2}\, B_{mn}\, \mathrm{d}y^m \wedge \mathrm{d}y^n \\ &= B_2^{(6)} + B_{1,m}^{(6)} \wedge \mathrm{d}y^m + \frac{1}{2}\, B_{mn}\, \mathrm{d}y^m \wedge \mathrm{d}y^n\,, \end{aligned} \tag{3.7}$$

where $B_2^{(6)}$, $B_{1,m}^{(6)}$ and B_{mn} are 2, 1 and 0-forms defined on the six-dimensional non-compact space. The ten-dimensional scalars are simply equal to their six-dimensional descendants, e.g. $\Phi^{(10)} = \Phi^{(6)} = \Phi$. For this reason, we usually drop the superscript (D) for scalar fields.

Reduction of the self-dual five-form field strength

To find the fields that descend from the RR four-form $C_4^{(10)}$ we need to be a bit careful, since it has a self-dual field strength: $*F_5^{(10)} = F_5^{(10)}$. Because of this self-duality, the action (3.1) does not properly describe the dynamics of the RR four-form. So instead of reducing the action, we should reduce the corresponding field equations along with the self-duality constraint. The action (3.1) with field strengths (3.2) yields the following equation of motion and Bianchi identity

$$\mathrm{d}\big(* F_5^{(10)}\big) = \mathrm{d}B_2^{(10)} \wedge \mathrm{d}C_2^{(10)} \,, \tag{3.8}$$

$$\mathrm{d}F_5^{(10)} = \mathrm{d}B_2^{(10)} \wedge \mathrm{d}C_2^{(10)} \,. \tag{3.9}$$

We see that, because of the self-duality of $F_5^{(10)}$, these two equations are identical, so we only have to reduce one of them. In what follows, we choose to reduce the Bianchi identity (3.9). Subsequently, we reduce the self-duality equation and use it to rewrite the six-dimensional Bianchi identities to a system of Bianchi identities and equations of motion. By integrating this system of equations to an action, we find the proper result of the reduction of $C_4^{(10)}$. Below, we work out this reduction in detail for the scalars and the two-forms.

First, we consider the scalars. In 6d, massless four-forms can be dualized to scalars, so we need to consider the components of $F_5^{(10)}$ that have either zero or four legs on the torus. The Bianchi identities for these components following from (3.9) read

$$\begin{aligned} \mathrm{d}P_1^{(6)} &= \frac{1}{2!\,2!}\,\varepsilon^{mnpq}\,\mathrm{d}B_{mn} \wedge \mathrm{d}C_{pq} \,, \\ \mathrm{d}P_5^{(6)} &= \mathrm{d}B_2^{(6)} \wedge \mathrm{d}C_2^{(6)} \,. \end{aligned} \tag{3.10}$$

Here we have introduced the notation $P_1^{(6)} = \frac{1}{4!}\,\varepsilon^{mnpq}\,F^{(6)}_{1,mnpq}$ and $P_5^{(6)} = F_5^{(6)}$. Next, we write down the relevant components that follow from the reduction of the self-duality constraint. By using the metric ansatz (3.4), and ignoring interactions with the graviphotons $\mathcal{A}_{\hat{\mu}}^{m}$, we find

$$P_5^{(6)} = \frac{1}{g_4} * P_1^{(6)} \,. \tag{3.11}$$

We now use this constraint to eliminate $P_5^{(6)}$ from (3.10). In this way, we find the following Bianchi identity and equation of motion for the one-form field strength

$P_1^{(6)}$

$$\begin{aligned} \mathrm{d}P_1^{(6)} &= \frac{1}{2!\,2!}\,\varepsilon^{mnpq}\,\mathrm{d}B_{mn}\wedge \mathrm{d}C_{pq}\,, \\ \mathrm{d}\big(e^{-2\phi_4} * P_1^{(6)}\big) &= \mathrm{d}B_2^{(6)}\wedge \mathrm{d}C_2^{(6)}\,. \end{aligned} \tag{3.12}$$

From the first equation, we can find an expression for $P_1^{(6)}$ in terms of the corresponding scalar field that we denote by b. The second equation can be integrated to an action that contains both the kinetic term for b and interaction terms between b and other scalar and two-forms fields. These expressions can be found in (3.16) and (3.17).

Next, we look at the two-forms coming from $C_4^{(10)}$. We are interested in the action for the six-dimensional two-form fields and their interactions with scalar fields. We will ignore interactions with six-dimensional one-forms. The relevant components that follow from the reduction of (3.9) read

$$\begin{aligned} \mathrm{d}F_{3,mn}^{(6)} &= \mathrm{d}B_{mn}\wedge \mathrm{d}C_2^{(6)} + \mathrm{d}B_2^{(6)}\wedge \mathrm{d}C_{mn} \\ &= \mathrm{d}\big(B_{mn}\,\mathrm{d}C_2^{(6)} - C_{mn}\,\mathrm{d}B_2^{(6)}\big)\,. \end{aligned} \tag{3.13}$$

These are Bianchi identities for six tensors in six dimensions. We want to eliminate half of these fields in exchange for equations of motion for the residual ones. We choose to retain the components $F_{3,mn}^{(6)}$ for $mn = 12, 13, 14$ and to eliminate the ones with indices $mn = 23, 24, 34$. For this, we again use the reduced self-duality constraint. The relevant components are

$$F_{3,mn}^{(6)} = \frac{1}{2}\sqrt{g_4}\,\varepsilon_{mnpq}\,g^{pr}g^{qs} * F_{3,rs}^{(6)}\,. \tag{3.14}$$

Due to the summations over the r and s indices, each component of this equation contains a linear combination of all the dual field strengths $*F_{3,rs}^{(6)}$ (recall that the metric on T^4 is given by (3.5)). Consequently, solving (3.14) for three of the six field strengths results in unwieldy expressions. We choose not to write down these expressions here, but instead to give a step-by-step outline of the way we use them to find an action for the 6d tensors.

First, we introduce a new notation for the field strengths that we plan on retaining: $P_{3;1}^{(6)} = F_{3,12}^{(6)}$, $P_{3;2}^{(6)} = F_{3,14}^{(6)}$ and $P_{3;3}^{(6)} = F_{3,13}^{(6)}$. Here the first subscript indicates that these are three-forms, and the second subscript labels the three distinct field strengths (we will sometimes drop this label when we are talking about all three of them). The expressions for these field strengths in terms of the corresponding two-form fields can be deduced from (3.13). For example,

$$P_{3;1}^{(6)} = \mathrm{d}R_{2;1}^{(6)} + B_{12}\,\mathrm{d}C_2^{(6)} - C_{12}\,\mathrm{d}B_2^{(6)}\,, \tag{3.15}$$

where $R^{(6)}_{2;\,1}$ is then one of the two-forms that arise from compactifying the ten-dimensional 4-form. Similar expressions can be found for $P^{(6)}_{3;\,2}$ and $P^{(6)}_{3;\,3}$ in terms of fields that we call $R^{(6)}_{2;\,2}$ and $R^{(6)}_{2;\,3}$ respectively.

Next, we solve the six equations in (3.14) for $F^{(6)}_{3,mn}$ and $*F^{(6)}_{3,mn}$ (for $mn = 23, 24, 34$) in terms of the field strengths $P^{(6)}_3$ and their duals $*P^{(6)}_3$. By substituting these expressions in the components of (3.13) for $mn = 23, 24, 34$, we find the equations of motion for the tensor fields $R^{(6)}_2$ purely in terms of the (dual) field strengths $P^{(6)}_3$ and $*P^{(6)}_3$, and fields that don't descend from the RR four-form $C^{(10)}_4$. These field equations are quite unwieldy, but with some careful bookkeeping they can be integrated to an action. We will not write down this awkward version of the action here. Instead, we write down a more elegant version of the action for the six-dimensional tensor fields and their interactions with scalar fields in section 3.1.3.

3.1.2 6d scalars

The field content of maximal six-dimensional supergravity contains 25 scalars. In terms of their origin in type IIB, these are Φ, ϕ_i, A_{mn}, B_{mn}, C_{mn}, a and b. We find the action for these scalar fields by using the methods and ansätze described in the previous section. This yields

$$\begin{aligned} e^{-1}_{(6)}\,\mathscr{L}_{\rm s} \;=\; &-\frac{1}{2}\,|{\rm d}\Phi|^2 - \frac{1}{4}\,|{\rm d}\phi_4|^2 - \frac{1}{2}\,|{\rm d}g_{mn}|^2 - \frac{1}{2}\,e^{-\Phi}\,\big|H^{(6)}_{1,mn}\big|^2 \\ &-\frac{1}{2}\,e^{2\Phi}\,|{\rm d}a|^2 - \frac{1}{2}\,e^{\Phi}\,\big|F^{(6)}_{1,mn}\big|^2 - \frac{1}{2}\,e^{-2\phi_4}\,\big|P^{(6)}_1\big|^2 \,. \end{aligned} \tag{3.16}$$

Note that the absolute values apply both to the 6d Lorentz indices and to the indices on the torus. For example, $|H^{(6)}_{1,mn}|^2 = \frac{1}{2!}\,H_{\hat\mu mn}H^{\hat\mu mn} = \frac{1}{2!}\,H_{\hat\mu mn}\,g^{mp}H^{\hat\mu}{}_{pq}\,g^{pn}$. The field strengths in (3.16) are given by

$$\begin{aligned} H^{(6)}_{1,mn} &= {\rm d}B_{mn}\,, \\ F^{(6)}_{1,mn} &= {\rm d}C_{mn} - a\,{\rm d}B_{mn}\,, \\ P^{(6)}_1 &= {\rm d}b + \frac{1}{8}\,\varepsilon^{mnpq}\,(B_{mn}\,{\rm d}C_{pq} - C_{mn}\,{\rm d}B_{pq})\,. \end{aligned} \tag{3.17}$$

These 25 scalar fields together parametrize the coset Spin(5, 5)/(Spin(5) × Spin(5)) [50]. The action above has a global Spin(5, 5) and a local Spin(5) × Spin(5) symmetry. In its current form, these symmetries are not visible, so we will now write this action in a form that makes both symmetries manifest.

In order to do this, we construct a generalized vielbein (or coset representative) $\mathcal{V}$ from the scalar fields. This vielbein is an element of $\mathrm{Spin}(5,5)$ and it transforms as $\mathcal{V} \to U\,\mathcal{V}\,W(\hat{x})$, with $U \in \mathrm{Spin}(5,5)$ and $W(\hat{x}) \in \mathrm{Spin}(5) \times \mathrm{Spin}(5)$. We now define the $\mathrm{Spin}(5) \times \mathrm{Spin}(5)$ invariant field $\mathcal{H} = \mathcal{V}\,\mathcal{V}^T$, that transforms as $\mathcal{H} \to U\,\mathcal{H}\,U^T$ under global $\mathrm{Spin}(5,5)$ transformations[1]. We can now write the scalar Lagrangian in terms of $\mathcal{H}$ as

$$e_{(6)}^{-1}\,\mathscr{L}_{\mathrm{s}} \;=\; \frac{1}{8}\,\mathrm{Tr}\big[\partial_{\hat{\mu}}\mathcal{H}^{-1}\partial^{\hat{\mu}}\mathcal{H}\big]\,. \tag{3.18}$$

In this formulation, the Lagrangian is manifestly invariant under the U-duality group $\mathrm{Spin}(5,5)$.

We now specify the way we build $\mathcal{V}$ from the 25 scalar fields so that the two Lagrangians (3.16) and (3.18) are equal to one another. We choose to build $\mathcal{V} \in \mathrm{Spin}(5,5)$ in τ-frame, i.e. it satisfies $\mathcal{V}^T\tau\,\mathcal{V} = \tau$ (for the definition of τ, see appendix 3.B.1). The exact construction is as follows:

$$\begin{aligned}\mathcal{V} \;=\; &\exp\big[b\,T^b\big] \times \exp\Bigg[\sum_{1\le m<n\le 4}\big(B_{mn}\,T^B_{mn} + C_{mn}\,T^C_{mn}\big)\Bigg] \times \exp\big[a\,T^a\big] \\ &\times \Bigg(\prod_{1\le m<n\le 4}\exp\left[A_{mn}\,T^A_{mn}\right]\Bigg) \times \exp\Bigg[\Phi\,H_0 + \sum_{i=1}^{4}\phi_i\,H_i\Bigg]\,.\end{aligned} \tag{3.19}$$

Here the T's and the H's are generators of $\mathfrak{so}(5,5)$ that span the subspace of $\mathfrak{so}(5,5)$ that generates the coset $\mathrm{Spin}(5,5)/(\mathrm{Spin}(5)\times\mathrm{Spin}(5))$. The precise expressions for these generators are given in appendix 3.B.1. All the scalar fields appear under the same name as in (3.16).

Because we construct our vielbein (3.19) in τ-frame[2], the transformation matrices U and W are also written in τ-frame. Henceforth, we use this frame whenever $\mathrm{Spin}(5,5)$ and $\mathrm{Spin}(5)\times\mathrm{Spin}(5)$ groups appear (unless mentioned otherwise).

3.1.3 6d tensors

The field content of maximal supergravity in six dimensions contains five 2-form tensor gauge fields. Collectively, we denote these fields by $A^{(6)}_{2,a}$ ($a = 1,\ldots,5$), and

[1] In this section we suppress $\mathrm{Spin}(5,5)$ indices, but we will need them later on. With indices, $\mathcal{H}$ is written as $\mathcal{H}_{AB}$ and it transforms as $\mathcal{H}_{AB} \to U_A{}^C\,\mathcal{H}_{CD}\,\big(U^T\big)^D{}_B$. The inverse of $\mathcal{H}$ is written with upper indices: $\mathcal{H}^{-1} = \mathcal{H}^{AB}$.

[2] For the convenience of the reader, it might be useful to mention how this convention is related to those of other authors. The following relations hold: $\mathcal{V} = X\,U_{[\mathrm{Tanii}]}X$ where $U_{[\mathrm{Tanii}]}$ is the vielbein that is used in [50], and $\mathcal{V} = \mathcal{V}_{[\mathrm{BSS}]}X$ where $\mathcal{V}_{[\mathrm{BSS}]}$ is the vielbein that is used in [57]. The matrix X is a conjugation matrix that is defined in appendix 3.B.1.

their field strengths by $G^{(6)}_{3,a} = \mathrm{d}A^{(6)}_{2,a}$. The Lagrangian for these fields reads [50, 57]

$$\mathscr{L}_\mathrm{t} = -\frac{1}{2}\, K^{ab}\, G^{(6)}_{3,a} \wedge * G^{(6)}_{3,b} - \frac{1}{2}\, L^{ab}\, G^{(6)}_{3,a} \wedge G^{(6)}_{3,b}\,. \tag{3.20}$$

Here K^{ab} and L^{ab} are functions of the scalar fields. We define a set of dual field strengths $\tilde{G}^{(6)a}_3 = K^{ab} * G^{(6)}_{3,b} + L^{ab}\, G^{(6)}_{3,b}$ so that we can write the Lagrangian in the more compact form

$$\mathscr{L}_\mathrm{t} = -\frac{1}{2}\, G^{(6)}_{3,a} \wedge \tilde{G}^{(6)a}_3\,. \tag{3.21}$$

In this notation, we write the Bianchi identities and the equations of motion as $\mathrm{d}G^{(6)}_{3,a} = 0$ and $\mathrm{d}\tilde{G}^{(6)a}_3 = 0$. We can combine these in the more compact notation $\mathrm{d}G^{(6)}_{3,A} = 0$, where $G^{(6)}_{3,A}$ is defined as

$$G^{(6)}_{3,A} = \begin{pmatrix} G^{(6)}_{3,a} \\ \tilde{G}^{(6)a}_3 \end{pmatrix}\,. \tag{3.22}$$

The Spin(5, 5) duality symmetry acts on this ten-component vector as

$$G^{(6)}_{3,A} \;\;\to\;\; U_A{}^B\, G^{(6)}_{3,B}\,, \qquad\qquad U_A{}^B \in \mathrm{Spin}(5,5)\,. \tag{3.23}$$

Only the subgroup GL(5) $\subset$ Spin(5, 5) is a symmetry of the action. The full symmetry group is only manifest on the level of the field equations.

When we decompose our coset representative in 5×5 blocks as $\mathcal{V} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we can write the matrices K^{ab} and L^{ab} as

$$\begin{aligned} K &= \tfrac{1}{2}\big((c+d)(a+b)^{-1} - (c-d)(a-b)^{-1}\big)\,, \\ L &= \tfrac{1}{2}\big((c+d)(a+b)^{-1} + (c-d)(a-b)^{-1}\big)\,. \end{aligned} \tag{3.24}$$

Now, by making the identification

$$A^{(6)}_{2,a} = \big(R^{(6)}_{2;\,1}\,,\, R^{(6)}_{2;\,2}\,,\, R^{(6)}_{2;\,3}\,,\, C^{(6)}_2,\, -B^{(6)}_2\big)\,, \tag{3.25}$$

the Lagrangian (3.21) is exactly equal to the one that we find by explicit reduction from type IIB supergravity using the ansätze given in section 3.1.1. The advantage of (3.21) is that we have made the duality symmetry manifest.

Doubled formalism

It is a common feature of supergravity actions in even dimensions that only a subgroup of the duality group is a symmetry of the action. In such cases, one can use the so-called doubled formalism [58] to construct an action that realizes the full symmetry group. In order to do this, one needs to introduce twice the original amount of form-valued fields as well as a constraint that makes sure that the doubled theory does not contain more degrees of freedom than the original theory.

We apply this formalism to our 6d tensor fields. We promote the $\tilde{G}_3^{(6)a}$ to field strengths that correspond to the doubled fields, i.e. we write them as $\tilde{G}_3^{(6)a} = \mathrm{d}\tilde{A}_2^{(6)a}$. These doubled fields $\tilde{A}_2^{(6)a}$ are now treated as independent fields. We write down the doubled Lagrangian as

$$\mathscr{L}_{\mathrm{t}}^{(\mathrm{doubled})} = -\frac{1}{4}\,\mathcal{H}^{AB}\,G_{3,A}^{(6)} \wedge * G_{3,B}^{(6)}\,. \tag{3.26}$$

In this formulation we have ten field strengths $G_{3,A}^{(6)}$ that satisfy the Bianchi identities $\mathrm{d}G_{3,A}^{(6)} = 0$ and the equations of motion $\mathrm{d}\big(\mathcal{H}^{AB} * G_{3,B}^{(6)}\big) = 0$. Furthermore, these fields are subject to the self-duality constraint

$$G_{3,A}^{(6)} = \tau_{AB}\,\mathcal{H}^{BC} * G_{3,C}^{(6)}\,. \tag{3.27}$$

By imposing this constraint on the field equations, we see that they reduce to the ones that correspond to the undoubled action. Thus we have found a proper doubled version of (3.21). Both the action (3.26) and the constraint (3.27) are invariant under the full Spin(5, 5) duality group. This can be seen directly from the way that these transformations work on the fields:

$$\mathcal{H}^{AB} \;\to\; \left(U^{-T}\right)^{A}{}_{C}\,\mathcal{H}^{CD}\left(U^{-1}\right)_{D}{}^{B}\,, \qquad G_{3,A}^{(6)} \;\to\; U_{A}{}^{B}\,G_{3,B}^{(6)}\,, \tag{3.28}$$

where $U_A{}^B \in \mathrm{Spin}(5,5)$ and we use the notation $U^{-T} = (U^{-1})^T$.

3.2 Scherk–Schwarz reduction to 5d

In a Scherk–Schwarz reduction, one considers a $(D+1)$-dimensional supergravity theory with a global symmetry given by a Lie group G that is compactified to D dimensions. The difference between ‘ordinary’ Kaluza–Klein and Scherk–Schwarz reduction lies in the compactification ansatz. Consider a field $\hat{\psi}$ in the $(D+1)$-dimensional theory that transforms as $\hat{\psi} \to g\hat{\psi}$ with $g \in G$ (for scalars, this is typically a non-linear realization, while some fields such as the metric in Einstein frame will be invariant). The Scherk–Schwarz ansatz then gives $\hat{\psi}$ a dependence on the coordinate z on the circle, which has periodicity $z \simeq z + 2\pi R$, given by

$$\hat{\psi}(x^\mu, z) = \exp\left(\frac{Mz}{2\pi R}\right)\,\psi(x^\mu)\,, \tag{3.29}$$

where M lies in the Lie algebra of G. This ansatz is not periodic around the circle, but picks up a monodromy $\mathcal{M} = e^M \in G$. The Lie algebra element M is sometimes called the mass matrix because it appears in mass terms in the D-dimensional theory. For more details, see [38, 39, 42, 44, 45, 47, 49, 51, 52, 59–62] and references therein. A conjugate mass matrix

$$M' = gMg^{-1}\,, \tag{3.30}$$

with $g \in G$, gives a conjugate monodromy

$$\mathcal{M}' = g\mathcal{M}g^{-1}\,. \tag{3.31}$$

This conjugated monodromy gives a massive theory that is related to the one for the monodromy $\mathcal{M}$ by a field redefinition, so that it defines an equivalent theory. Thus the possible Scherk–Schwarz reductions are classified by the conjugacy classes of the duality group [47].

In our case, we reduce from 6d to 5d on a circle with a Scherk–Schwarz twist. We denote the coordinates on the five-dimensional Minkowski space by x^μ and the coordinate on the circle by z. The compact coordinate is periodic with periodicity $z \simeq z + 2\pi R$. The metric (in Einstein frame) is inert under the duality group, so we choose the conventional Kaluza–Klein metric ansatz:

$$g_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} e^{-\sqrt{1/6}\,\phi_5}\, g_{\mu\nu} + e^{\sqrt{3/2}\,\phi_5}\,\mathcal{A}^5_\mu\mathcal{A}^5_\nu & e^{\sqrt{3/2}\,\phi_5}\,\mathcal{A}^5_\mu \\ e^{\sqrt{3/2}\,\phi_5}\,\mathcal{A}^5_\nu & e^{\sqrt{3/2}\,\phi_5} \end{pmatrix}\,. \tag{3.32}$$

The factors in the exponents are chosen so that we arrive in Einstein frame in five dimensions and the scalar field ϕ_5 is canonically normalized [63].

The result of our reduction is a gauged $\mathcal{N} = 8$ supergravity theory in five dimensions in which a non-semi-simple subgroup of Spin(5, 5) is gauged. The gauge group contains an important U(1) subgroup for which $\mathcal{A}^5_\mu$ is the corresponding gauge field. For each twist, the theory has a vacuum (partially) breaking the supersymmetry where it can be described by an $\mathcal{N} < 8$ effective field theory. This reduction from 6d to 5d has been considered previously in [42, 49]. An important feature is that reducing self-dual 2-form gauge fields in 6d can result in massive self-dual 2-form fields in 5d [49]. See [60, 62] for further details.

3.2.1 Monodromies and masses

In six dimensions the global symmetry is $G = \mathrm{Spin}(5,5)$, so in principle we can choose the mass matrix to be any element of the Lie algebra of G. However, our goal is to obtain a Minkowski vacuum with partially broken supersymmetry, so, as discussed in the introduction, we restrict our twist to be conjugate to an element of the R-symmetry group

$$\mathrm{USp}(4)_{\mathrm{L}} \times \mathrm{USp}(4)_{\mathrm{R}} \;=\; \mathrm{Spin}(5)_{\mathrm{L}} \times \mathrm{Spin}(5)_{\mathrm{R}}\,, \tag{3.33}$$

that preserves the identity in $\mathrm{Spin}(5,5)$. We take then a monodromy

$$\mathcal{M} = g\tilde{\mathcal{M}}g^{-1}\,, \qquad g \in \mathrm{Spin}(5,5)\,, \qquad \tilde{\mathcal{M}} \in \mathrm{USp}(4)_{\mathrm{L}} \times \mathrm{USp}(4)_{\mathrm{R}}\,. \tag{3.34}$$

By a further conjugation, we can bring $\tilde{\mathcal{M}}$ to an element $\bar{\mathcal{M}}$ of a maximal torus $\mathbb{T} = \mathrm{U}(1)^4$ of the R-symmetry group $\mathrm{USp}(4)_{\mathrm{L}} \times \mathrm{USp}(4)_{\mathrm{R}}$

$$\tilde{\mathcal{M}} = h\bar{\mathcal{M}}h^{-1}\,, \qquad h \in \mathrm{USp}(4)_{\mathrm{L}} \times \mathrm{USp}(4)_{\mathrm{R}}\,, \qquad \bar{\mathcal{M}} \in \mathbb{T}\,. \tag{3.35}$$

The element $\bar{\mathcal{M}}$ of a maximal torus $\mathbb{T} = \mathrm{U}(1)^4$ is then specified by four angles, which we denote m_1, m_2, m_3, m_4; we take $0 \leq m_i < 2\pi$. Writing

$$\bar{\mathcal{M}} = (\mathcal{M}_{\mathrm{L}}^{\mathfrak{usp}(4)}, \mathcal{M}_{\mathrm{R}}^{\mathfrak{usp}(4)})\,, \qquad \mathcal{M}_{\mathrm{L/R}}^{\mathfrak{usp}(4)} \in \mathrm{USp}(4)_{\mathrm{L/R}}\,, \tag{3.36}$$

we can take the monodromies to be in the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ subgroup of $\mathrm{USp}(4)$ for both the left and right factors (note that $\mathrm{SU}(2) \cong \mathrm{USp}(2)$):

$$\mathrm{SU}(2)_{\mathrm{L}_1} \times \mathrm{SU}(2)_{\mathrm{L}_2} \times \mathrm{SU}(2)_{\mathrm{R}_1} \times \mathrm{SU}(2)_{\mathrm{R}_2} \;\subset\; \mathrm{USp}(4)_{\mathrm{L}} \times \mathrm{USp}(4)_{\mathrm{R}}\,. \tag{3.37}$$

We can then take, for example,

$$\mathcal{M}_{\mathrm{L}}^{\mathfrak{usp}(4)} = e^{m_1\sigma_3} \otimes e^{m_2\sigma_3}\,, \qquad \mathcal{M}_{\mathrm{R}}^{\mathfrak{usp}(4)} = e^{m_3\sigma_3} \otimes e^{m_4\sigma_3}\,, \tag{3.38}$$

where σ_3 is the usual Pauli matrix. Other choices of the monodromy are related to this by $\mathrm{USp}(4)_{\mathrm{L}} \times \mathrm{USp}(4)_{\mathrm{R}}$ conjugation.

The six-dimensional supergravity fields fit into the following representations under the R-symmetry group (see e.g. [42, 50]):

$$\begin{aligned} \text{scalars}: &\quad (\mathbf{5},\mathbf{5})\,, \\ \text{vectors}: &\quad (\mathbf{4},\mathbf{4})\,, \\ \text{tensors}: &\quad (\mathbf{5},\mathbf{1}) + (\mathbf{1},\mathbf{5})\,, \\ \text{gravitini}: &\quad (\mathbf{4},\mathbf{1}) + (\mathbf{1},\mathbf{4})\,, \\ \text{dilatini}: &\quad (\mathbf{5},\mathbf{4}) + (\mathbf{4},\mathbf{5})\,. \end{aligned} \tag{3.39}$$

We have an equal number of self-dual and anti-self-dual 2-form tensor fields, and an equal number of fermions of positive and negative chirality. In terms of the R-symmetry representations above, the self-dual tensors B_2^+ transform in the $(\mathbf{5},\mathbf{1})$ and the anti-self-dual tensors B_2^- transform in the $(\mathbf{1},\mathbf{5})$. The positive chiral gravitini ψ_μ^+ and dilatini χ^+ transform in the $(\mathbf{4},\mathbf{1})$ and $(\mathbf{5},\mathbf{4})$ respectively, and the negative chiral gravitini ψ_μ^- and dilatini χ^- transform in the $(\mathbf{1},\mathbf{4})$ and $(\mathbf{4},\mathbf{5})$.

These representations determine the charges (e_1, e_2, e_3, e_4) of each field under $\mathrm{U}(1)^4 \subset \mathrm{USp}(4)_{\mathrm{L}} \times \mathrm{USp}(4)_{\mathrm{R}}$. A field with charges (e_1, e_2, e_3, e_4) will then be an eigenvector of the mass matrix with eigenvalue $i\mu$ and will have z-dependence $e^{i\mu z/2\pi R}$ where

$$\mu = \sum_{i=1}^{4} e_i m_i \ . \tag{3.40}$$

The resulting mass for the field will turn out to be $|\mu|/2\pi R$.

3.2.2 Supersymmetry breaking and massless field content

The R-symmetry representations (3.39) decompose into the following representations under the $\mathrm{SU}(2)^4$ subgroup (3.37):

$$\begin{aligned}
\text{scalars}: &\quad (\mathbf{5},\mathbf{5}) &&\to (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2}) + (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,\\
\text{vectors}: &\quad (\mathbf{4},\mathbf{4}) &&\to (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}) + (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2}) + (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})\,,\\
\text{tensors}: &\quad (\mathbf{5},\mathbf{1}) + (\mathbf{1},\mathbf{5}) &&\to (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2}) + 2\,(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})\,,\\
\text{gravitini}: &\quad (\mathbf{4},\mathbf{1}) + (\mathbf{1},\mathbf{4}) &&\to (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})\,,\\
\text{dilatini}: &\quad (\mathbf{5},\mathbf{4}) + (\mathbf{4},\mathbf{5}) &&\to (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1}) + (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2}) + (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{2})\\
&&&\quad + (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})\,.
\end{aligned} \tag{3.41}$$

This then determines the four charges e_i under the $\mathrm{U}(1)^4$ subgroup: each doublet gives charges ± 1 and each singlet gives charge 0. For example, the sixteen vector fields in the

$$(\mathbf{4},\mathbf{4}) \ \to \ (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}) + (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2}) + (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2}) \tag{3.42}$$

have charges

$$(e_1, e_2, e_3, e_4) = (\pm 1, 0, \pm 1, 0) + (\pm 1, 0, 0, \pm 1) + (0, \pm 1, \pm 1, 0) + (0, \pm 1, 0, \pm 1)\,. \tag{3.43}$$

These charges then determine the masses through (3.40). The eight gravitini (symplectic Weyl spinors) in the $(\mathbf{4},\mathbf{1})+(\mathbf{1},\mathbf{4})$ representation of the R-symmetry group $\mathrm{USp}(4)_{\mathrm{L}} \times \mathrm{USp}(4)_{\mathrm{R}}$ decompose into four pairs, each of which has a different mass $|m_i|/2\pi R$, with $i = 1,2,3,4$. The number $\mathcal{N}$ of unbroken supersymmetries is then given by the number of massless gravitini, which is $\mathcal{N} = 2r$ where r is the number of parameters m_i that are zero. The different values of r give rise to 5d supergravities with $\mathcal{N} = 8,6,4,2,0$ Minkowski vacua, corresponding to twisting in $4-r$ of the SU(2) factors in (3.37).

In general, all fields that are charged, with at least one of the $e_i \neq 0$ corresponding to an $m_i \neq 0$, become massive in 5d. Below we give the massless field content of reductions with twists that preserve $\mathcal{N} = 8,6,4,2,0$ supersymmetry in the Minkowski vacuum and check that they fit into the relevant supermultiplets of 5d supergravities [64].

- $\mathcal{N} = 8$

  We start with the untwisted case, $m_i = 0$, where all fields remain massless. Apart from the 5d graviton, the spectrum contains 8 gravitini, 27 vectors, 48 dilatini and 42 scalars (all massless). As expected, these fields make up a single gravity multiplet of maximal 5d supergravity.

- $\mathcal{N} = 6$

  In order to end up with $\mathcal{N} = 6$ supergravity, we take only one of the four mass parameters to be non-zero so that we twist in only one of the four SU(2) subgroups. The massless spectrum from such a reduction contains a graviton, 6 gravitini, 15 vectors, 20 dilatini and 14 scalars. These fields form the gravity multiplet of the $\mathcal{N} = 6$ theory.

- $\mathcal{N} = 4$

  We obtain $\mathcal{N} = 4$ supergravity by twisting in two SU(2) groups, with two mass parameters zero. This can be done in two qualitatively different ways: either with a chiral twist, say in $\mathrm{SU}(2)_{\mathrm{R}_1}$ and $\mathrm{SU}(2)_{\mathrm{R}_2}$ with $m_1 = m_2 = 0$, or with a non-chiral twist, for example in $\mathrm{SU}(2)_{\mathrm{L}_2}$ and $\mathrm{SU}(2)_{\mathrm{R}_2}$ with $m_1 = m_3 = 0$. Both types of twists result in the same massless spectrum: the graviton, 4 gravitini, 7 vectors, 8 dilatini and 6 scalars, although as we shall see, they result in different massive spectra.

  In the $\mathcal{N} = 4$ theory, the gravity multiplet contains the graviton, 4 gravitini, 6 vectors, 4 dilatini and a single scalar field, and the vector multiplet contains

1 vector, 4 dilatini and 5 scalars [65]. We see that our massless spectrum consists of the gravity multiplet coupled to one vector multiplet.

- $\mathcal{N} = 2$

  We end up with minimal 5d supergravity by twisting in three of the four SU(2) subgroups, with just one of the mass parameters zero. The massless field content after such a twist contains the graviton, 2 gravitini, 3 vectors, 4 dilatini and 2 scalars.

  For $\mathcal{N} = 2$ supersymmetry, the gravity multiplet contains the graviton, 2 gravitini, and 1 vector field, and the vector multiplet contains 1 vector, 2 dilatini and 1 scalar field. Thus, the field content that we find from this reduction forms a gravity multiplet coupled to two vector multiplets.

- $\mathcal{N} = 0$

  By twisting in all four SU(2) groups, with all four mass parameters non-zero, we break all supersymmetry. The only fields that are not charged under such a twist are the graviton and the singlets which are completely uncharged, with all $e_i = 0$. As a result, the massless spectrum in 5d consists of the graviton, 3 vectors and 2 scalars. Note that all fermions become massive.

3.2.3 Massive field content

The charges (e_1, e_2, e_3, e_4) following from (3.41) determine the massive spectrum for the reduced theory in five dimensions. This spectrum is summarized in table 3.1. The spectrum of table 3.1 has been previously derived from Scherk–Schwarz reduction in [42] and corresponds to a gauging of $\mathcal{N} = 8$ five-dimensional supergravity.

We now give the supermultiplet structure of the massive spectra that follow from the various twists preserving different amounts of supersymmetry. All fields that acquire mass also become charged under the graviphoton $\mathcal{A}_1^5$ with covariant derivatives of the form

$$D_\mu \;=\; \partial_\mu - iq\, g\, \mathcal{A}_\mu^5\,. \tag{3.44}$$

Here the gauge coupling is $g = 1/R$, and the charge q of each 5d field is equal to $1/g = R$ times its mass. Because the massive fields are charged, the real fields that follow from the reduction have to combine into complex fields. In the spectra that we give below, we list the number of complex fields (unless stated otherwise). Furthermore, when we give the mass of a field or collection of fields we only write down $|\mu|$. In order to find the actual mass, this needs to be divided by $2\pi R$.

Fields	Representation	$\lvert\mu\rvert$ = Mass (multiplied by $2\pi R$)
Scalars	$(\mathbf{5{,}5})$	$\big\lvert \pm m_1 \pm m_2 \pm m_3 \pm m_4 \big\rvert$
		$\big\lvert \pm m_1 \pm m_2 \big\rvert$
		$\big\lvert \pm m_3 \pm m_4 \big\rvert$
		0
Vectors	$(\mathbf{4}, \mathbf{4})$	$\big\lvert \pm m_{1,2} \pm m_{3,4} \big\rvert$
Tensors	$(\mathbf{5}, \mathbf{1})$	$\big\lvert \pm m_1 \pm m_2 \big\rvert$, 0
	$(\mathbf{1}, \mathbf{5})$	$\big\lvert \pm m_3 \pm m_4 \big\rvert$, 0
Gravitini	$(\mathbf{4}, \mathbf{1})$	$\big\lvert \pm m_{1,2} \big\rvert$
	$(\mathbf{1}, \mathbf{4})$	$\big\lvert \pm m_{3,4} \big\rvert$
Dilatini	$(\mathbf{5}, \mathbf{4})$	$\big\lvert \pm m_1 \pm m_2 \pm m_{3,4} \big\rvert$
		$\big\lvert \pm m_{3,4} \big\rvert$
	$(\mathbf{4}, \mathbf{5})$	$\big\lvert \pm m_{1,2} \pm m_3 \pm m_4 \big\rvert$
		$\big\lvert \pm m_{1,2} \big\rvert$

Table 3.1: *This table gives the value of $|\mu(m_i)|$ for the 5d fields coming from the different types of 6d fields. The mass of the field is then $|\mu(m_i)|/2\pi R$. The notation $m_{i,j}$ indicates that both m_i and m_j occur. There is no correlation between the $\pm$ signs and the $_{ij}$ indices, so that e.g. $(\pm m_1 \pm m_2)$ denotes 4 different combinations of mass parameters, and $(\pm m_{1,2} \pm m_{3,4})$ denotes 16 different combinations.*

The massive multiplets we find are all BPS multiplets in five dimensions; these multiplets were analyzed and classified in [66] and are labeled by two integers (p, q). For $\mathcal{N}$ supersymmetries in five dimensions (with $\mathcal{N}$ even), the R-symmetry is $\mathrm{USp}(\mathcal{N})$. For a (p, q) massive multiplet, the choice of central charge breaks the R-symmetry to a subgroup $\mathrm{USp}(2p) \times \mathrm{USp}(2q)$, i.e. the subgroup of $\mathrm{USp}(\mathcal{N})$ preserving the central charge, where $2p + 2q = \mathcal{N}$. The nomenclature was chosen such that a massless supermultiplet of (p, q) supersymmetry in six-dimensions has, after reducing on a circle, Kaluza–Klein modes that fit into (p, q) massive supermultiplets in five dimensions. The physical states of a (p, q) massive multiplet in five dimensions then fit into representations of

$$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{USp}(2p) \times \mathrm{USp}(2q) \ , \tag{3.45}$$

where $\mathrm{SU}(2) \times \mathrm{SU}(2) \sim \mathrm{SO}(4)$ is the little group for massive representations in five dimensions. The representations of the little group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ that arise include $(3,2)$ and $(2,3)$ for massive gravitini and $(2,2)$ for massive vector fields. The representation $(3,1)$ corresponds to a massive self-dual two-form field satisfying the five-dimensional duality condition

$$\mathrm{d}B_2 = -im * B_2 \ , \tag{3.46}$$

while the $(1,3)$ representation corresponds to the anti-self dual case with $\mathrm{d}B_2 = im * B_2$. In the following, we consider the cases in which the Scherk–Schwarz reduction breaks the supersymmetry to $\mathcal{N} = 6, 4, 2$. The massless states are in the $\mathcal{N}$ supersymmetry representations given in the previous subsection, and we now give the $\mathcal{N}$ supersymmetry representations of the massive fields. It was already pointed out in section 3.2.2 that there are two qualitatively different twists that result in a theory with $\mathcal{N} = 4$ supersymmetry: a chiral one and a non-chiral one. Both theories have the same massless spectrum (see section 3.2.2), but their massive spectra are different. The non-chiral twist gives massive fields fitting into $(1,1)$ multiplets and we will refer to this as the $(1,1)$ theory. The chiral twist leads to $(0,2)$ supermultiplets and we will refer to this as the $(0,2)$ (or $(2,0)$) theory.

- $\mathcal{N} = 6$

  In order to break to $\mathcal{N} = 6$, we twist with just one of the four mass parameters non-zero. Without loss of generality, we take $m_1 \neq 0$ and the other three parameters equal to zero. The physical states will then fall into representations of

  $$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{USp}(2) \times \mathrm{USp}(4) \,. \tag{3.47}$$

  The massive field content from such a twist contains 1 gravitino, 2 self-dual tensors, 4 vectors, 13 dilatini and 10 scalars. All these fields are complex, and their mass is equal to $|m_1|$. This is a $(1,2)$ BPS supermultiplet with the representations

  $$(3,2;1,1)+(3,1;2,1)+(2,2;1,4)+(1,2;1,5)+(2,1;2,4)+(1,1;2,5) \,. \tag{3.48}$$

- $\mathcal{N} = 4$ $(0,2)$

We obtain the $(0,2)$ theory by taking chiral twist with $m_1, m_2 \neq 0$ and $m_3, m_4 = 0$. The physical states will then fall in representations of

$$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{USp}(4) \ . \tag{3.49}$$

From the reduction we find two massive $(0,2)$ spin-$\frac{3}{2}$ multiplets, one with mass $|m_1|$, and the other with mass $|m_2|$. Each consists of 1 gravitino, 4 vectors and 5 dilatini, which are in the representations

$$(3,2;1) + (2,2;4) + (1,2;5) \ . \tag{3.50}$$

Furthermore, we find two massive $(0,2)$ tensor multiplets with masses $|m_1 + m_2|$ and $|m_1 - m_2|$. Each of these contains one self-dual 2-form satisfying (3.46), 4 dilatini and 5 scalars [66], fitting in the representations

$$(3,1;1) + (2,1;4) + (1,1;5) \ . \tag{3.51}$$

We note at this point that a part of the massive spectrum above can be made massless by tuning the mass parameters. That is, if we choose $m_1 = \pm\, m_2$, one of the two (complex) tensor multiplets becomes massless. This gives two additional real vector multiplets in the massless sector of the $\mathcal{N} = 4$ theory (see section 3.2.2).

- $\mathcal{N} = 4$ $(1,1)$

For the non-chiral twist, we choose $m_1, m_3 \neq 0$ and $m_2, m_4 = 0$ in order obtain the $(1,1)$ theory. There are two massive $(1,1)$ vector multiplets, one with mass $|m_1 + m_3|$ and one with mass $|m_1 - m_3|$. Each consists of 1 vector, 4 dilatini and 4 scalars [66] corresponding to a representation of

$$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{USp}(2) \times \mathrm{USp}(2)\,, \tag{3.52}$$

given by

$$(2,2;1,1) + (2,1;2,1) + (1,2;1,2) + (1,1;2,2)\,. \tag{3.53}$$

In addition, there are two massive $(1,1)$ spin-$\frac{3}{2}$ multiplets, one with mass $|m_1|$, and one with mass $|m_3|$. Each consists of 1 gravitino, 2 (anti-)self-dual tensors, 2 vectors, 5 dilatini and 2 scalars. The one with mass $|m_1|$ is in the representation

$$(3,2;1,1)+(3,1;2,1)+(2,2;1,2)+(1,2;1,1)+(2,1;2,2)+(1,1;2,1)\,, \tag{3.54}$$

and the one with mass $|m_3|$ is in the representation

$$(2,3;1,1)+(1,3;1,2)+(2,2;2,1)+(2,1;1,1)+(1,2;2,2)+(1,1;1,2)\,. \quad (3.55)$$

As in the $(0,2)$ theory, we can tune the mass parameters in such a way that a part of this spectrum becomes massless. For $m_1 = \pm\, m_3$, one of the massive vector multiplets becomes massless, and so we get two more real vector multiplets in the massless sector of the theory (again see section 3.2.2). Note that, even though the massive tensor multiplet of the $(0,2)$ theory and the massive vector multiplet of the $(1,1)$ theory contain different fields, they give the same field content in the massless limit.

- $\mathcal{N} = 2$

  We choose $m_1, m_2, m_3 \neq 0$ and $m_4 = 0$ to obtain the $\mathcal{N} = 2$ case with massive $(0,1)$ multiplets in representations of

$$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{USp}(2)\ . \quad (3.56)$$

There are four massive hypermultiplets with masses $|m_1 \pm m_2 \pm m_3|$ consisting of 1 complex dilatino and 2 complex scalars in the

$$(2,1;1) + (1,1;2) \quad (3.57)$$

representation. The four vector multiplets with masses $|m_{1,2} \pm m_3|$ consist of 1 vector and 2 dilatini in the

$$(2,2;1) + (1,2;2) \quad (3.58)$$

representation. Furthermore, we find two tensor multiplets (1 self-dual tensor, 2 dilatini, 1 scalar) with masses $|m_1 \pm m_2|$ in the following representation of (3.56):

$$(3,1;1) + (2,1;2) + (1,1;1)\ . \quad (3.59)$$

There are also two spin-$\frac{3}{2}$ multiplets, one with mass $|m_1|$ and one with mass $|m_2|$, containing 1 gravitino, 2 vectors and 1 dilatino in the

$$(3,2;1) + (2,2;2) + (1,2;1) \quad (3.60)$$

representation. We also find another multiplet containing a spin-$\frac{3}{2}$ field: 1 gravitino, 2 anti-self-dual tensors, 1 dilatino and 2 scalars with mass $|m_3|$. This is reducible, giving one massive hypermultiplet consisting of 1 dilatino and 2 scalars with the representation (3.57) and one multiplet consisting of 1 gravitino and 2 anti-self-dual tensors in the representation:

$$(2, 3; 1) + (1, 3; 2)\ . \tag{3.61}$$

As for the $\mathcal{N} = 4$ theories, we can tune the mass parameters in order to obtain extra massless fields. Choosing $m_1 = \pm m_2$ or $m_{1,2} = \pm m_3$ would make either a tensor multiplet or a vector multiplet massless. Both of these would give two real massless vector multiplets. Another choice would be to set $m_1 = \pm m_2 \pm m_3$ so that one of the massive hypermultiplets becomes massless.

3.2.4 Mass matrices

The monodromies $\mathcal{M}_{\mathrm{L}}^{\mathfrak{usp}(4)} \in \mathrm{USp}(4)_{\mathrm{L}}$ and $\mathcal{M}_{\mathrm{R}}^{\mathfrak{usp}(4)} \in \mathrm{USp}(4)_{\mathrm{R}}$ in (3.36) are the exponentials of mass matrices in the Lie algebra of USp(4):

$$\mathcal{M}_{\mathrm{L}}^{\mathfrak{usp}(4)} = \exp(M_{\mathrm{L}}^{\mathfrak{usp}(4)})\,, \qquad \mathcal{M}_{\mathrm{R}}^{\mathfrak{usp}(4)} = \exp(M_{\mathrm{R}}^{\mathfrak{usp}(4)})\,. \tag{3.62}$$

For the monodromies (3.38), the mass matrices are given by

$$M_{\mathrm{L}}^{\mathfrak{usp}(4)} = m_1\sigma_3 \oplus m_2\sigma_3\,, \qquad M_{\mathrm{R}}^{\mathfrak{usp}(4)} = m_3\sigma_3 \oplus m_4\sigma_3\,. \tag{3.63}$$

By conjugating, as in (3.35), by an element h of the $\mathrm{SU}(2)^4$ subgroup (3.37), we can bring this to the form

$$M_{\mathrm{L}}^{\mathfrak{usp}(4)} = m_1(n_1 \cdot \sigma) \oplus m_2(n_2 \cdot \sigma)\,, \qquad M_{\mathrm{R}}^{\mathfrak{usp}(4)} = m_3(n_3 \cdot \sigma) \oplus m_4(n_4 \cdot \sigma)\,, \tag{3.64}$$

for any four unit 3-vectors n_i. Here σ is the 3-vector of Pauli matrices.

The Lie algebra of USp(4) consists of anti-hermitian 4×4 matrices $M_A{}^B$ ($M^\dagger = -M$) such that $M^{AB} = \Omega^{AC} M_C{}^B$ is symmetric ($M^{AB} = M^{BA}$), where $\Omega^{AB} = -\Omega^{BA}$ is the symplectic invariant; see appendix 3.B.2 for more details. In a basis in which $\Omega = \sigma_2 \oplus \sigma_2$ and the subgroup (3.37) is block diagonal, we have the 4×4 matrix representation

$$M_{\mathrm{L}}^{\mathfrak{usp}(4)} = \begin{pmatrix} m_1(n_1 \cdot \sigma) & 0 \\ 0 & m_2(n_2 \cdot \sigma) \end{pmatrix}\,, \qquad \Omega^{AB} = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}\,. \tag{3.65}$$

However, for our purposes, it will be useful to have mass matrices in a basis in which

$$\Omega^{AB} = \begin{pmatrix} 0 & \mathbb{1}_2 \\ -\mathbb{1}_2 & 0 \end{pmatrix}\,. \tag{3.66}$$

In this basis, we can take for example

$$M_{\mathrm{L}}^{\mathfrak{usp}(4)} = \begin{pmatrix} 0 & 0 & -m_1 & 0 \\ 0 & 0 & 0 & -m_2 \\ m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \end{pmatrix}, \tag{3.67}$$

and a similar expression for $M_{\mathrm{R}}^{\mathfrak{usp}(4)}$ that can be found by replacing $m_1 \to m_3$ and $m_2 \to m_4$. The monodromy for the above mass matrix is given by

$$\mathcal{M}_{\mathrm{L}}^{\mathfrak{usp}(4)} = \begin{pmatrix} \cos(m_1) & 0 & -\sin(m_1) & 0 \\ 0 & \cos(m_2) & 0 & -\sin(m_2) \\ \sin(m_1) & 0 & \cos(m_1) & 0 \\ 0 & \sin(m_2) & 0 & \cos(m_2) \end{pmatrix}, \tag{3.68}$$

and there is a similar expression for $\mathcal{M}_{\mathrm{R}}^{\mathfrak{usp}(4)}$. We can use the isomorphism $\mathfrak{usp}(4) \cong \mathfrak{so}(5)$ to map (3.67) to the corresponding generator in the Lie algebra of SO(5). This yields

$$M_{\mathrm{L}} = \begin{pmatrix} 0 & -(m_1+m_2) & 0 & 0 & 0 \\ m_1+m_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(m_1-m_2) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_1-m_2 & 0 & 0 \end{pmatrix}, \tag{3.69}$$

and a similar expression for M_{R} where we replace $m_1 \to m_3$ and $m_2 \to m_4$ (see appendix 3.B.2 for more information on the isomorphism $\mathfrak{usp}(4) \cong \mathfrak{so}(5)$). The corresponding SO(5) monodromy is given by

$$\mathcal{M}_{\mathrm{L}} = \begin{pmatrix} \cos(m_1+m_2) & -\sin(m_1+m_2) & 0 & 0 & 0 \\ \sin(m_1+m_2) & \cos(m_1+m_2) & 0 & 0 & 0 \\ 0 & 0 & \cos(m_1-m_2) & 0 & -\sin(m_1-m_2) \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \sin(m_1-m_2) & 0 & \cos(m_1-m_2) \end{pmatrix}. \tag{3.70}$$

The USp(4) monodromy is of course a double cover of the SO(5) monodromy: taking e.g. $m_1 = m_2 = \pi$ gives $\mathcal{M}_{\mathrm{L}} = \mathbb{1}$ but $\mathcal{M}_{\mathrm{L}}^{\mathfrak{usp}(4)} = -\mathbb{1}$.

We can use the mass matrices M_{L} and M_{R} in the algebras of $\mathrm{SO}(5)_{\mathrm{L}}$ and $\mathrm{SO}(5)_{\mathrm{R}}$ to create an $\mathfrak{so}(5,5)$ mass matrix. In the basis in which the $\mathrm{SO}(5,5)$ metric takes the form

$$\tau_{AB} = \begin{pmatrix} 0 & \mathbb{1}_5 \\ \mathbb{1}_5 & 0 \end{pmatrix}, \tag{3.71}$$

(see appendix 3.B.1) this $\mathfrak{so}(5,5)$ mass matrix is given by

$$\mathbb{M}_A{}^B = \frac{1}{2}\begin{pmatrix}(M_\mathrm{L}+M_\mathrm{R})_a{}^b & (M_\mathrm{L}-M_\mathrm{R})_{ab}\\ (M_\mathrm{L}-M_\mathrm{R})^{ab} & (M_\mathrm{L}+M_\mathrm{R})^a{}_b\end{pmatrix} \in \mathfrak{so}(5,5)\,. \tag{3.72}$$

It is this matrix that appears explicitly in the bosonic action, as we shall see in the following subsections.

In section 3.3, we consider various brane configurations that result in five-dimensional black holes. For each of these systems, we choose M_L and M_R in such a way that the fields that charge the black hole remain massless in 5d. All of these are conjugate to the ones given here. In particular, they all have the same eigenvalues and so give the same mass spectrum.

3.2.5 5d scalars

In this section, we go through the reduction of the 6d scalar fields in detail. The goal is to compute the mass that each of the 25 scalar fields obtains in 5d. For notational convenience, we set $R = \frac{1}{2\pi}$ here and in the next subsection where we reduce the 6d tensors. Consequently, the masses that we compute here carry an 'invisible' factor $\frac{1}{2\pi R}$ that can be reinstated by checking the mass dimensions.

The scalar Lagrangian in six dimensions reads (see section 3.1.2)

$$e_{(6)}^{-1}\,\mathscr{L}_\mathrm{s} = \frac{1}{8}\,\mathrm{Tr}\big[\partial_{\hat\mu}\mathcal{H}^{-1}\partial^{\hat\mu}\mathcal{H}\big]\,. \tag{3.73}$$

The global Spin(5, 5) transformations act as $\mathcal{H} \;\to\; U\,\mathcal{H}\,U^T$ with $U \in$ Spin(5, 5). This leads us to the following Scherk–Schwarz ansatz:

$$\mathcal{H}(\hat x^{\hat\mu}) = e^{\mathbb{M}z}\,\mathcal{H}(x^\mu)\,e^{\mathbb{M}^T z}\,, \tag{3.74}$$

where $\mathbb{M}$ is the mass matrix defined in (3.72). By substituting this ansatz in (3.73), we find the five-dimensional Lagrangian

$$e_{(5)}^{-1}\,\mathscr{L}_\mathrm{s} = \frac{1}{8}\,\mathrm{Tr}\big[D_\mu\mathcal{H}^{-1}D^\mu\mathcal{H}\big] - V(\mathcal{H})\,. \tag{3.75}$$

Matter that is charged under the monodromy becomes charged under the U(1) symmetry corresponding to the graviphoton $\mathcal{A}_1^5$ in 5d. The covariant derivative on $\mathcal{H}$ is given by

$$D_\mu\mathcal{H} = \partial_\mu\mathcal{H} - \mathcal{A}_\mu^5\left(\mathbb{M}\,\mathcal{H} + \mathcal{H}\,\mathbb{M}^T\right). \tag{3.76}$$

The potential in (3.75) is given by

$$V(\mathcal{H}) = \frac{1}{4}\,e^{-\sqrt{8/3}\,\phi_5}\,\mathrm{Tr}\big[\mathbb{M}^2 + \mathbb{M}^T\mathcal{H}^{-1}\mathbb{M}\,\mathcal{H}\big]\,. \tag{3.77}$$

For an R-symmetry twist, such potentials must be non-negative [47]; consequently, a global minimum can be found by solving $V = 0$. We find such a minimum by putting all 25 scalar fields to zero, so that $\mathcal{H} = \mathbb{1}$. By realizing that our mass matrix is anti-symmetric, $\mathbb{M}^T = -\mathbb{M}$, we immediately see that this gives $V = 0$.

We now compute the masses of the scalar fields in this minimum. We denote the collection of all 25 scalar fields by σ^i, with $i = 1, \ldots, 25$, and compute the mass matrix as[3]

$$m_{ij} = \left.\frac{\partial^2 V}{\partial\sigma^i \partial\sigma^j}\right|_{\sigma^k=0} . \tag{3.78}$$

We diagonalize this mass matrix as $m_{ij} = Q_i{}^k\, m^{\rm diag}_{kl}\, Q^l{}_j$, where $m^{\rm diag}$ is a diagonal matrix and Q is a conjugation matrix built from an orthonormal basis of eigenvectors. In this way, we find the mass that corresponds to each of the redefined fields $\tilde{\sigma}^i = Q^i{}_j \sigma^j$.

We have computed these masses explicitly for the mass matrices that preserve the various 6d black string configurations that we consider in section 3.3. Tables are provided in appendix 3.C.

3.2.6 5d tensors

In this section we work out the reduction of the six-dimensional tensor fields in detail, following [49, 62]. Just like in the previous subsection, we set $R = \frac{1}{2\pi}$ and neglect the Kaluza–Klein towers for notational convenience.

The Lagrangian for the six-dimensional tensor fields reads

$$\mathscr{L}_{\rm t}^{(\rm doubled)} = -\frac{1}{4}\,\mathcal{H}^{AB}\, G^{(6)}_{3,A} \wedge * G^{(6)}_{3,B} . \tag{3.79}$$

The ten three-form field strengths $G^{(6)}_{3,A}$ transform as in (3.28), so we choose our Scherk–Schwarz ansatz to be

$$G^{(6)}_{3,A}(\hat{x}^{\hat{\mu}}) = \left(e^{\mathbb{M}z}\right)_A{}^B \Big(G^{(5)}_{3,B}(x^\mu) \,+\, G^{(5)}_{2,B}(x^\mu) \wedge \left(\mathrm{d}z + \mathcal{A}^5_1\right)\Big) , \tag{3.80}$$

where $G^{(5)}_{3,A}$ and $G^{(5)}_{2,A}$ are five-dimensional field strengths that are independent of the circle coordinate z. As usual for self-dual tensor fields, we don't compactify the Lagrangian of the theory but rather its field equations. We start by reducing the

[3]The kinetic term of the sigma model is diagonal at the minimum of the potential, i.e. it takes the form $-\frac{1}{2} g_{ij}(\sigma^k)\, \partial_\mu \sigma^i \partial^\mu \sigma^j$ with $g_{ij}(0) = \delta_{ij}$.

six-dimensional Bianchi identities $\mathrm{d}G^{(6)}_{3,A} = 0$. We find

$$
\begin{aligned}
\mathrm{d}G^{(5)}_{3,A} + \mathrm{d}\big(G^{(5)}_{2,A} \wedge \mathcal{A}^5_1\big) &= 0\,, \\
\mathrm{d}G^{(5)}_{2,A} - \mathbb{M}_A{}^B\big(G^{(5)}_{3,B} + G^{(5)}_{2,B} \wedge \mathcal{A}^5_1\big) &= 0\,.
\end{aligned}
\tag{3.81}
$$

From these we deduce expressions for the five-dimensional field strengths in terms of the corresponding two-form and one-form potentials:

$$
\begin{aligned}
G^{(5)}_{3,A} &= \mathrm{d}A^{(5)}_{2,A} - G^{(5)}_{2,A} \wedge \mathcal{A}^5_1\,, \\
G^{(5)}_{2,A} &= \mathrm{d}A^{(5)}_{1,A} + \mathbb{M}_A{}^B\, A^{(5)}_{2,B}\,.
\end{aligned}
\tag{3.82}
$$

Normally at this point, we would like to shift $A^{(5)}_{2,A} \to A^{(5)}_{2,A} - (\mathbb{M}^{-1})_A{}^B\, \mathrm{d}A^{(5)}_{1,B}$ so that the field strengths in (3.82) would lose their dependence on $A^{(5)}_{1,A}$. This is not possible, however, because our mass matrix $\mathbb{M}_A{}^B$ is not invertible. We therefore need to diagonalize $\mathbb{M}_A{}^B$ and split the indices that correspond to zero and non-zero eigenvalues. In the most general case where the combinations $m_1 \pm m_2$ and $m_3 \pm m_4$ are non-zero, this splitting goes like $A \to (\alpha, \dot\alpha)$ with $\dot\alpha \in \{i, i+5\}$, where i is the index that corresponds to the row and column that we set to zero in M_{L} and M_{R}. For example, for the reduction of the D1-D5 system (see (3.113)) we have $\dot\alpha \in \{4, 9\}$. The index α takes the other eight values of the original index A. The second equation in (3.82) now separates into

$$
\begin{aligned}
G^{(5)}_{2,\alpha} &= \mathrm{d}A^{(5)}_{1,\alpha} + \mathbb{M}_\alpha{}^\beta\, A^{(5)}_{2,\beta}\,, \\
G^{(5)}_{2,\dot\alpha} &= \mathrm{d}A^{(5)}_{1,\dot\alpha}\,.
\end{aligned}
\tag{3.83}
$$

The matrix $\mathbb{M}_\alpha{}^\beta$ is invertible, so now we can shift $A^{(5)}_{2,\alpha} \to A^{(5)}_{2,\alpha} - (\mathbb{M}^{-1})_\alpha{}^\beta\, \mathrm{d}A^{(5)}_{1,\beta}$. After this shift, the five-dimensional field strengths read

$$
\begin{aligned}
G^{(5)}_{3,\alpha} &= \mathrm{d}A^{(5)}_{2,\alpha} - G^{(5)}_{2,\alpha} \wedge \mathcal{A}^5_1\,, &\qquad G^{(5)}_{3,\dot\alpha} &= \mathrm{d}A^{(5)}_{2,\dot\alpha} - G^{(5)}_{2,\dot\alpha} \wedge \mathcal{A}^5_1\,, \\
G^{(5)}_{2,\alpha} &= \mathbb{M}_\alpha{}^\beta\, A^{(5)}_{2,\beta}\,, & G^{(5)}_{2,\dot\alpha} &= \mathrm{d}A^{(5)}_{1,\dot\alpha}\,.
\end{aligned}
\tag{3.84}
$$

The six-dimensional field strengths are subject to the self-duality constraint

$$
G^{(6)}_{3,A} = \tau_{AB}\, \mathcal{H}^{BC} * G^{(6)}_{3,C}\,. \tag{3.85}
$$

We now compactify this constraint. First, we need to reduce the six-dimensional

Hodge star to five dimensions. By using the metric decomposition (3.32), we find

$$\begin{aligned} *^{(6)}G^{(6)}_{3,A} &= \left(e^{\mathbb{M}z}\right)_A{}^B \, *^{(6)}\left(G^{(5)}_{3,B} + G^{(5)}_{2,B}\wedge\left(\mathrm{d}z+\mathcal{A}^5_1\right)\right) \\ &= \left(e^{\mathbb{M}z}\right)_A{}^B \left(e^{\sqrt{2/3}\,\phi_5}\ *^{(5)}G^{(5)}_{3,B}\wedge\left(\mathrm{d}z+\mathcal{A}^5_1\right) - e^{-\sqrt{2/3}\,\phi_5}\ *^{(5)}G^{(5)}_{2,B}\right). \end{aligned} \tag{3.86}$$

This result allows us to write down the 5d self-duality constraint that follows from (3.85) as

$$G^{(5)}_{3,A} = -\,e^{-\sqrt{2/3}\,\phi_5}\,\tau_{AB}\,\mathcal{H}^{BC} * G^{(5)}_{2,C}\,. \tag{3.87}$$

Recall for the derivation of this result that $\mathcal{H}^{AB}$ with raised indices is the inverse of the matrix $\mathcal{H}$ as defined in section 3.1.2. Consequently, we use the inverse of (3.74) as Scherk–Schwarz ansatz.

Mass spectrum

In order to find the mass spectrum of the fields that descend from $G^{(6)}_{3,A}$, we put all other fields in (3.87) to zero. In particular, this means that $\mathcal{H}^{AB}=\delta^{AB}$. We find

$$\mathrm{d}A^{(5)}_{2,\alpha} = -\,\tau_\alpha{}^\beta\,\mathbb{M}_\beta{}^\gamma * A^{(5)}_{2,\gamma}\,, \qquad \mathrm{d}A^{(5)}_{2,\dot\alpha} = -\,\tau_{\dot\alpha}{}^{\dot\beta} * \mathrm{d}A^{(5)}_{1,\dot\beta}\,, \tag{3.88}$$

where we use the notation $\tau_\alpha{}^\beta = \tau_{\alpha\gamma}\,\delta^{\gamma\beta}$ and an analogous expression for the dotted indices. These are massive and massless five-dimensional self-duality conditions. From these, we can deduce the equations of motion for the corresponding fields (following [67]). They read

$$\mathrm{d}\big(*\,\mathrm{d}A^{(5)}_{2,\alpha}\big) = -\left(\tau\mathbb{M}\tau\mathbb{M}\right)_\alpha{}^\beta * A^{(5)}_{2,\beta}\,, \qquad \mathrm{d}\big(*\,\mathrm{d}A^{(5)}_{1,\dot\alpha}\big) = 0\,. \tag{3.89}$$

So in 5d, we end up with eight massive tensors and two massless vectors (again, this is for the case where $m_1 \pm m_2$ and $m_3 \pm m_4$ are non-zero). The self-duality constraint (3.87) eliminates the massless tensors $A^{(5)}_{2,\dot\alpha}$ and makes sure that the massive tensors $A^{(5)}_{2,\alpha}$ carry only half their usual degrees of freedom. The masses of the fields $A^{(5)}_{2,\alpha}$ are determined by the mass matrix $-\left(\tau\mathbb{M}\tau\mathbb{M}\right)_\alpha{}^\beta$. By diagonalizing this matrix, we find the mass corresponding to each field.

Just as for the scalar fields, we have computed these masses explicitly for the mass matrices that we use for the reduction of the D1-D5 system and the dual brane configurations in section 3.3. These masses can be found in appendix 3.C.

Graviphoton interactions

We now pay some extra attention to the interactions between the graviphoton and the vector and tensor fields that we find in this subsection. They will prove to be very important in section 3.4. As it turns out, there is a difference in the result that we find for the reduction of a self-dual 6d tensor and an anti-self-dual 6d tensor. We illustrate this difference with two simple examples.

Consider a six-dimensional (anti-)self-dual tensor field $\hat{B}_2$ with field strength $\hat{H}_3 = \hat{\mathrm{d}}\hat{B}_2$ (here hats denote 6d quantities). The field equations and self-duality constraint for this field read

$$\hat{\mathrm{d}}\hat{H}_3 = 0\,, \qquad \hat{*}\,\hat{H}_3 = \pm\,\hat{H}_3\,. \tag{3.90}$$

By decomposing this field (strength) as $\hat{H}_3 = H_3 + H_2 \wedge (\mathrm{d}z + \mathcal{A}_1)$, and by using straightforward reduction techniques and our conventions, we find the following 5d Lagrangian:

$$\mathscr{L} = -\,\tfrac{1}{2}\,H_2 \wedge * H_2 \,\pm\, \tfrac{1}{2}\,\mathcal{A}_1 \wedge H_2 \wedge H_2\,, \tag{3.91}$$

with $H_2 = \mathrm{d}B_1$. We see that a self-dual and an anti-self-dual tensor give a Chern-Simons interaction term with the graviphoton with an opposite sign.

Now take a real doublet of (anti-)self-dual tensor fields, that we Scherk–Schwarz reduce from 6d to 5d with the ansatz

$$\hat{H}_3 = \exp\left[\begin{pmatrix} 0 & -m \\ m & 0 \end{pmatrix} z\right]\big(H_3 + H_2 \wedge (\mathrm{d}z + \mathcal{A}_1)\big), \tag{3.92}$$

(apart from the ansatz and the fact that we are considering a doublet this set-up is similar to the previous one). Going through this reduction gives a complex massive tensor B_2 in five dimensions subject to the self-duality equation

$$\mathrm{d}B_2 \,-\, im\,\mathcal{A}_1 \wedge B_2 \,\pm\, im * B_2 = 0\,. \tag{3.93}$$

Again, the $\pm$ sign indicates the difference between the result for the reduction of a self-dual and an anti-self-dual tensor from six dimensions. Now, this sign is not in front of the interaction with the graviphoton, but we can still flip it by redefining $\mathcal{A}_1 \to -\mathcal{A}_1$. To see this, recall that the field B_2 is complex so that we also have the complex conjugate of (3.93). By flipping the sign of the graviphoton, we effectively switch the particle and the anti-particle $B_2 \leftrightarrow \bar{B}_2$ in order to protect the sign in the covariant derivative. The $\pm$ sign in the mass term of the equation for $\bar{B}_2$ is flipped with respect to (3.93), and so we see that redefining $\mathcal{A}_1 \to -\mathcal{A}_1$ effectively interchanges the result for a self-dual and an anti-self-dual tensor.

3.2.7 Conjugate monodromies

We have so far considered monodromies in the R-symmetry group $\mathrm{Spin}(5)_{\mathrm{L}} \times \mathrm{Spin}(5)_{\mathrm{R}}$ preserving the identity in the coset $\mathrm{Spin}(5,5)/\mathrm{Spin}(5)_{\mathrm{L}} \times \mathrm{Spin}(5)_{\mathrm{R}}$, which is the point in the moduli space at which all scalar fields vanish. Then this point in moduli space is a fixed point under the action of the Spin(5, 5) transformation $\psi \to \mathcal{M}\psi$ given by the monodromy, and as we have seen this point is a minimum of the Scherk–Schwarz potential giving a Minkowski vacuum. Conjugating by an element of the R-symmetry group

$$\mathcal{M} \to h\bar{\mathcal{M}}h^{-1}\,, \qquad h \in \mathrm{USp}(4)_{\mathrm{L}} \times \mathrm{USp}(4)_{\mathrm{R}} \tag{3.94}$$

will then preserve the fixed point in the moduli space and the minimum will remain at the origin.

However, for the embedding in string theory, we will need to consider monodromies that are related to an R-symmetry transformation by conjugation by an element of Spin(5, 5)

$$\mathcal{M} = g\tilde{\mathcal{M}}g^{-1}\,, \qquad g \in \mathrm{Spin}(5,5)\,, \qquad \tilde{\mathcal{M}} \in \mathrm{USp}(4)_{\mathrm{L}} \times \mathrm{USp}(4)_{\mathrm{R}}\,. \tag{3.95}$$

This change of monodromy can be thought of as the result of acting on the theory twisted with monodromy $\tilde{\mathcal{M}}$ by a transformation $\psi \to g\psi$. For the supergravity theory, this is just a field redefinition giving an equivalent theory, but as we shall see later this has consequences for the embedding in string theory. The fixed point is now at the coset containing g, $[g] = \{gh \mid h \in \mathrm{Spin}(5)_{\mathrm{L}} \times \mathrm{Spin}(5)_{\mathrm{R}}\}$, and this is now the location of the minimum of the potential [47]. At this point, the kinetic terms of the various fields are not conventionally normalized. On bringing these to standard form, the masses become precisely the ones given earlier for the theory with monodromy $\tilde{\mathcal{M}}$. This was of course to be expected: a field redefinition cannot change physical parameters such as masses.

3.2.8 Gauged supergravity and gauge group

The result of the Scherk–Schwarz reduction is a gauged $\mathcal{N} = 8$ supergravity theory in which a subgroup of the E_6 duality symmetry of the ungauged 5d theory is promoted to a gauge symmetry. In this subsection we discuss this gauged supergravity and its gauge group.

We start with the case in which the twist is a T-duality transformation in the T-duality subgroup Spin(4, 4) of Spin(5, 5). Consider first the bosonic NS-NS sector of the ten-dimensional supergravity theory, consisting of the metric, B-field and dilaton. Compactifying on T^4 gives a 6d theory with SO(4, 4) symmetry. There is a 6d metric, B-field and dilaton, together with 8 vector fields $A^A_{\hat{\mu}}$ in the **8** of SO(4, 4) (with $A = 1, \ldots, 8$ labelling the vector representation of SO(4, 4)) and scalars in the coset space $SO(4,4)/SO(4) \times SO(4)$. The Scherk–Schwarz compactification of this on a circle with an $SO(4,4)$ twist with mass matrix $N_A{}^B$ was given in detail in [60]. In 5d, there are then 10 gauge fields: eight A_μ^A arising from the 6d vector fields, the graviphoton vector field $\mathcal{A}_\mu^5$ from the metric and a vector field $\mathcal{B}_\mu^5$ from the reduction of the 6d B-field. Then $(A_\mu^A, \mathcal{A}_\mu^5, \mathcal{B}_\mu^5)$ are the gauge fields for a gauge group with 10 generators $T_A, T_z, T_{\tilde{z}}$ respectively. After the field redefinitions given in [60] to obtain tensorial fields transforming covariantly under duality transformations, the gauge algebra is [60]

$$[T_z, T_A] = N_A{}^B\, T_B\,, \qquad [T_A, T_B] = N_{AB}\, T_{\tilde{z}}\,, \tag{3.96}$$

with all other commutators vanishing. Here $N_{AB} = N_A{}^C \eta_{CB}$ where η_{AB} is the $SO(4,4)$-invariant metric, so that $N_{AB} = -N_{BA}$ as the mass matrix is in the Lie algebra of $SO(4,4)$. This then represents a gauging of a 10-dimensional subgroup of $SO(4,4)$, which has a $U(1)^2$ subgroup generated by $T_z, T_{\tilde{z}}$. A further $U(1)$ factor can be obtained by dualising the 2-form $b_{\mu\nu}$ to give an extra gauge field and the generator t of this $U(1)$ factor commutes with all other generators.

Next, consider reintroducing the R-R sector. In six dimensions, there are a further 8 one-form gauge fields $C_{\hat{\mu}}^\alpha$ transforming as a Weyl spinor of $\mathrm{Spin}(4,4)$ ($\alpha = 1, \ldots, 8$), which combine with the 8 NS-NS one-form gauge fields to form the **16** of $\mathrm{Spin}(5,5)$. There are also a further 4 two-form gauge fields, which split into four self-dual ones and four anti-self dual ones that transform as an **8** of $SO(4,4)$. These combine with the degrees of freedom of the NS-NS 2-form to form the **10** of $\mathrm{Spin}(5,5)$. The mass matrix acts on the spinor representation through $N_\alpha{}^\beta$ which is given as usual by $N_{\alpha\beta} = \frac{1}{4} N_{AB} (\gamma^{AB})_{\alpha\beta}$ where $N_{\alpha\beta} = N_\alpha{}^\gamma \eta_{\gamma\beta}$ and $\eta_{\alpha\beta}$ is the symmetric charge conjugation matrix. The structure in the spinor representation is related to that in the vector representation by $SO(4,4)$ triality. The gauge algebra then gains the terms

$$[T_z, T_\alpha] = N_\alpha{}^\beta\, T_\beta\,, \qquad [T_\alpha, T_\beta] = N_{\alpha\beta}\, T_{\tilde{z}}\,, \tag{3.97}$$

to give an 18-dimensional gauge group. This corresponds to gauging an 18-dimensional subgroup of E_6. For generic values of the parameters m_i, the two-form gauge fields in the **8** of $SO(4,4)$ become massive, while the 5d NS-NS two-form remains massless and can again be dualized to give a further $U(1)$ factor with generator t. For special values of the parameters, some of the two-forms in the **8** of $SO(4,4)$ can become invariant under the twist and so become massless as well. These can be dualized to give further $U(1)$ factors.

The gauge algebra can now be written

$$[T_z, T_a] = M_a{}^b \, T_b \,, \qquad [T_a, T_b] = M_{ab} \, T_{\tilde{z}} \,, \tag{3.98}$$

with all other commutators vanishing, where $T_a = (T_A, T_\alpha)$ and

$$M_a{}^b = \begin{pmatrix} N_A{}^B & 0 \\ 0 & N_\alpha{}^\beta \end{pmatrix} . \tag{3.99}$$

There is a $\mathrm{U}(1)^3$ subgroup generated by t (if the NS-NS two-form is dualized) with possible further U(1) factors coming in if some of the R-R two-forms remain massless.

In the generic case in which $M_a{}^b$ has no zero eigenvalues, then the vector fields A^a corresponding to the generators T_a all become massive, while the gauge fields corresponding to the generators $T_z, T_{\tilde{z}}, t$ remain massless. Then the gauge group is spontaneously broken to the $U(1)^3$ subgroup generated by $T_z, T_{\tilde{z}}, t$. For special values of the parameters m_i such that $M_a{}^b$ has some zero eigenvalues, there will be more massless gauge fields and the unbroken gauge group will be larger.

In section 3.3.2, we will consider a twist of this kind in the compact $\mathrm{Spin}(4) \times \mathrm{Spin}(4)$ subgroup of the $\mathrm{Spin}(4,4)$ T-duality group. The other twists we will consider are all related to this one by conjugation (see sections 3.2.7 and 3.3.2) and will give isomorphic gauge groups.

One can argue what part of the matter content is charged under each of these generators of the gauge group by Scherk–Schwarz reducing the 6d gauge transformations and seeing how the 5d fields transform under these reduced transformations. The 6d gauge transformations can be found in [50, 57]. The generators T_a come from the gauge transformations corresponding to the 16 vector fields in 6d. These transform the 6d vectors and the 6d tensors, so the 5d descendants of these fields can become charged under generators T_a. The generators $T_{\tilde{z}}$ and t come from the gauge transformation that correspond to the 6d tensor field that is a singlet under the twist. This transformation acts only on this tensor field, so after reduction no matter becomes charged under the resulting 5d transformations $T_{\tilde{z}}$ and t. The generator T_z (corresponding to the graviphoton $\mathcal{A}_1^5$) comes from 6d diffeomorphisms in the circle direction. By explicit reduction of these diffeomorphisms, we find that all fields that become massive in 5d carry U(1) charge under T_z.

3.2.9 Kaluza–Klein towers

In the previous subsections, we have constructed 5d theories with both massless and massive fields from Scherk–Schwarz reduction. However, this is not the whole story: if we consider a compactification on S^1, then each field picks up an infinite Kaluza–Klein tower[4]. We choose Scherk–Schwarz ansätze including Kaluza–Klein towers on the S^1 of the form

$$\psi(x^\mu, z) = \exp\left(\frac{Mz}{2\pi R}\right) \sum_{n\in\mathbb{Z}} e^{inz/R}\, \psi_n(x^\mu)\,, \tag{3.100}$$

where we use ψ as a schematic notation for any field in the theory that transforms in some representation of the R-symmetry group. Then if ψ has charges e_i, it is an eigenvector of M with eigenvalue $i\mu$ given by (3.40), $M\psi = i\mu\,\psi$, so that

$$\psi(x^\mu, z) = \sum_{n\in\mathbb{Z}} \exp\left(i\left(\frac{\mu}{2\pi} + n\right)\frac{z}{R}\right)\psi_n(x^\mu)\,. \tag{3.101}$$

Clearly, shifting $\frac{\mu}{2\pi}$ by an integer r can be absorbed into a shift $n \to n - r$ and so corresponds to changing the n'th Kaluza–Klein mode to the $(n-r)$'th one while leaving the sum unchanged. From table 3.1, we see that shifting the m_i by $2\pi r_i$ for any integers r_i shifts all the $\frac{\mu}{2\pi}$ by an integer and so leaves the above sum (3.101) unchanged. For this reason, there is no loss of generality in taking $m_i \in [0, 2\pi)$.

Without loss of generality, we can restrict the m_i's further by realizing that all eigenvalues $i\mu$ appear with a $\pm$ sign in front of them. By taking into account two towers of the form (3.101), one of them with a minus sign in front of μ, we see that the *combination* of these towers is unchanged under $\frac{\mu}{2\pi} \to 1 - \frac{\mu}{2\pi}$. Consequently, we can take $m_i \in [0, \pi]$ without loss of generality.

The Scherk–Schwarz ansatz is a truncation of (3.101) to the $n = 0$ mode. This gives a consistent truncation to a gauged five-dimensional supergravity theory, which is sufficient for e.g. determining which twists preserve which brane configuration in section 3.3. The full string theory requires keeping all these modes, together with stringy modes and degrees of freedom from branes wrapping the internal space.

The mass of the n'th KK-mode is given by

$$\left|\frac{\mu}{2\pi R} + \frac{n}{R}\right|\,, \qquad n \in \mathbb{Z}\,, \tag{3.102}$$

and the value of $\mu(m_i)$ for each field can be read off from table 3.1. As an example, we check this for the reduction of the 6d tensors. If the whole tower is taken into account, the Scherk–Schwarz ansatz (3.80) is extended to

$$G_3^{(6)}(\hat{x}^{\hat{\mu}}) = \exp\left(\frac{\mathbb{M}z}{2\pi R}\right) \sum_{n\in\mathbb{Z}} e^{inz/R}\left(G^{(5)}_{3,n}(x^\mu) + G^{(5)}_{2,n}(x^\mu) \wedge \left(\mathrm{d}z + \mathcal{A}^5_1\right)\right). \tag{3.103}$$

[4]Of course, even this is not the whole story. There are also Kaluza–Klein modes that come from the reduction from 10d to 6d on the four-torus, plus stringy degrees of freedom.

Note that we have restored the circle radius R in this ansatz; from now on, we will keep it manifest in all our equations. Furthermore, in (3.103) the Spin(5, 5) indices are suppressed for clarity. It can be seen directly that this extended ansatz essentially changes the mass matrix $\mathbb{M}$ as we used it in section 3.2.6 to

$$\left(\frac{\mathbb{M}}{2\pi R} + \frac{in\mathbb{1}}{R}\right), \qquad n \in \mathbb{Z}. \tag{3.104}$$

We can now use that the eigenvalue of $\mathbb{M}$ is $i\mu$ with μ given by (3.40) to see that the masses of the Kaluza–Klein modes are given by (3.102).

Note that the modes with $n = 0$ that are kept in the Scherk–Schwarz reduction are not necessarily the lightest modes in the tower. In particular, if the parameters m_i are chosen so that $\frac{\mu(m_i)}{2\pi}$ is an integer, $\frac{\mu(m_i)}{2\pi} = N$, then the mode with $n = -N$ will be massless. As an example of this, we can choose

$$m_1 = m_2 = \tfrac{\pi}{2}\,,\ m_3 = \pi\,,\ m_4 = 0\,. \tag{3.105}$$

By using table 3.1, we can see which additional massless fields arise. In this case, there are four scalars and two spin-$\frac{1}{2}$ fermions that become massless, which form a hypermultiplet of $\mathcal{N} = 2$ supergravity. For further discussion of such accidental massless modes, see [47, 51].

3.3 5d black hole solutions

In this section, we consider several 10d brane configurations that we compactify to give black holes in 5d. We do this in two steps. First, we reduce the brane configuration to a black string solution of (2, 2) supergravity in six dimensions. This solution will not be invariant under the whole Spin(5, 5) duality group, but will be preserved by a stabilizing subgroup. If we then do a standard (untwisted) compactification of this on a circle with the black string wrapped along the circle, we obtain a BPS black hole solution of $\mathcal{N} = 8$ supergravity in five dimensions. This reduction can be modified by including a duality twist on the circle. If the duality twist is in the stabilizing subgroup, the same black hole solution will remain a solution of the gauged supergravity resulting from the Scherk–Schwarz reduction, and of its truncation to an effective $\mathcal{N} < 8$ supergravity describing the massless sector. This is because the only fields that become massive as a result of the Scherk–Schwarz twist are the ones that are trivial (zero) in the black hole solution. As a consequence, the black hole will also be BPS and preserving (at least) four supercharges. Indeed, it descends from a BPS black string solution in six dimensions, and the duality twist preserves the supercharges and Killing spinors of the truncated theory that has the black hole as a solution.

Primarily, we focus on the D1-D5 system, but later in this section we also consider the dual F1-NS5 and D3-D3 systems.

3.3.1 The D1-D5-P system

The D1-D5 system, sometimes more accurately called the D1-D5-P system, consists of D1-branes, D5-branes and waves carrying momentum. The ten-dimensional configuration is as follows:

	$\mathbb{R}^{1,4}$					S^1	T^4			
	t	r	θ	φ_1	φ_2	z	y_1	y_2	y_3	y_4
D1	$-$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$-$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
D5	$-$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$-$	$-$	$-$	$-$	$-$
P	$-$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$-$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Here a line ($-$) denotes an extended direction, a dot ($\cdot$) denotes a pointlike direction, and multiple dots ($\cdots$) denote a direction in which the brane or wave is smeared.

We start from the ten-dimensional solution and reduce it to 5d with the ansätze that are given in previous sections. The D1-D5 solution of type IIB supergravity in Einstein frame reads

$$\begin{cases} \mathrm{d}s^2_{(10)} = H_1^{-\frac{3}{4}} H_5^{-\frac{1}{4}} \left[-\,\mathrm{d}t^2 + \mathrm{d}z^2 + K(\mathrm{d}t - \mathrm{d}z)^2\right] + H_1^{\frac{1}{4}} H_5^{\frac{3}{4}} \left[\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_3^2\right] \\ \qquad\qquad + H_1^{\frac{1}{4}} H_5^{-\frac{1}{4}} \left[\mathrm{d}y_1^2 + \mathrm{d}y_2^2 + \mathrm{d}y_3^2 + \mathrm{d}y_4^2\right] \\ e^{\Phi} = H_1^{\frac{1}{2}} H_5^{-\frac{1}{2}} \\ C_2^{(10)} = (H_1^{-1} - 1)\,\mathrm{d}t \wedge \mathrm{d}z + Q_5 \cos^2\theta\, \mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2\,, \end{cases} \tag{3.106}$$

where $\mathrm{d}\Omega_3^2 = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi_1^2 + \cos^2\theta\, \mathrm{d}\varphi_2^2$ is the metric on the three-sphere written in Hopf coordinates. The harmonic functions corresponding to the D1-branes, the D5-branes and the momentum modes can be written in terms of their total charges as

$$H_1 = 1 + \frac{Q_1}{r^2}\,, \qquad H_5 = 1 + \frac{Q_5}{r^2}\,, \qquad H_K = 1 + K = 1 + \frac{Q_K}{r^2}\,. \tag{3.107}$$

Reduction to six dimensions

We compactify the metric in (3.106) to 6d using the ansatz (3.4). The metric on the torus, g_{mn}, is diagonal in (3.106) so we find that

$$e^{\vec{b}_m \cdot \vec{\phi}} = H_1^{\frac{1}{4}} H_5^{-\frac{1}{4}}\,, \qquad m = 1, \ldots, 4\,. \tag{3.108}$$

By using the expressions for the vectors $\vec{b}_m$ given in (3.6), we can solve for the individual scalar fields ϕ_i. We find that only one of them is non-zero in the 6d solution:

$$e^{\phi_4} = H_1^{\frac{1}{2}} H_5^{-\frac{1}{2}}\,, \qquad \phi_1 = \phi_2 = \phi_3 = 0\,. \tag{3.109}$$

The rest of the reduction is straightforward. The six-dimensional Einstein frame metric is related to the ten-dimensional one by a Weyl rescaling with $g_4^{1/4} = H_1^{\frac{1}{4}} H_5^{-\frac{1}{4}}$, which is incorporated in the ansatz (3.4). The dilaton Φ and the R-R two-form $C_2^{(10)}$ have no non-zero components on the torus, so they reduce trivially. The result reads

$$\begin{cases} \mathrm{d}s^2_{(6)} = H_1^{-\frac{1}{2}} H_5^{-\frac{1}{2}} \left[-\,\mathrm{d}t^2 + \mathrm{d}z^2 + K(\mathrm{d}t - \mathrm{d}z)^2 \right] + H_1^{\frac{1}{2}} H_5^{\frac{1}{2}} \left[\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_3^2 \right] \\ e^{\phi_+} = H_1^{\sqrt{\frac{1}{2}}} H_5^{-\sqrt{\frac{1}{2}}} \\ C_2^{(6)} = (H_1^{-1} - 1)\,\mathrm{d}t \wedge \mathrm{d}z + Q_5 \cos^2\theta\, \mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2\,. \end{cases} \tag{3.110}$$

Here, we have defined the scalar field $\phi_+ = \frac{1}{\sqrt{2}}(\phi_4 + \Phi)$. This solution describes a black string in six dimensions.

When we use the doubled formalism (see section 3.1.3) we can rewrite the solution above in terms of the doubled tensor fields. To derive the contributions of these doubled fields to the black string solution, recall that the dual field strengths are defined as $\tilde{G}_3^{(6)a} = K^{ab} * G_{3,b}^{(6)} + L^{ab}\, G_{3,b}^{(6)}$. By putting all scalar fields except ϕ_+ to zero, this reduces to $\tilde{G}_3^{(6)a} = K^{ab} * G_{3,b}^{(6)}$ with $K^{ab} = \mathrm{diag}\,(1, 1, 1, e^{\sqrt{2}\,\phi_+}, 1)$. Hence, we find that the doubled tensors to which the black string solution couples are given by

$$\begin{aligned} C_2^{(6)} &= (H_1^{-1} - 1)\,\mathrm{d}t \wedge \mathrm{d}z + Q_5 \cos^2\theta\, \mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2\,, \\ \tilde{C}_2^{(6)} &= (H_5^{-1} - 1)\,\mathrm{d}t \wedge \mathrm{d}z + Q_1 \cos^2\theta\, \mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2\,. \end{aligned} \tag{3.111}$$

In the doubled formalism, the degrees of freedom of both these fields are halved by the self-duality constraint (3.27) so the total number of degrees of freedom of the fields that the black string couples to remain unchanged.

Scherk–Schwarz reduction to five dimensions

The last step is to Scherk–Schwarz reduce the six-dimensional black string solution, which results in a black hole in five dimensions. We choose the twist matrices to be in the stabilizing subgroup of the R-symmetry group, i.e. the subgroup of the R-symmetry that preserves the solution. As a result, all the fields that are non-constant in the black hole remain massless.

For the D1-D5 system, we choose the following $\mathfrak{usp}(4)$ mass matrices:

$$M_{\mathrm{L}}^{\mathfrak{usp}(4)} = \begin{pmatrix} 0 & 0 & -m_1 & 0 \\ 0 & 0 & 0 & -m_2 \\ m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \end{pmatrix}, \quad M_{\mathrm{R}}^{\mathfrak{usp}(4)} = \begin{pmatrix} 0 & 0 & -m_3 & 0 \\ 0 & 0 & 0 & -m_4 \\ m_3 & 0 & 0 & 0 \\ 0 & m_4 & 0 & 0 \end{pmatrix}. \tag{3.112}$$

Here m_1, m_2, m_3 and m_4 are real mass parameters, each corresponding to one SU(2) in the R-symmetry subgroup (3.37). The isomorphism $\mathfrak{usp}(4) \cong \mathfrak{so}(5)$ of appendix 3.B.2 maps these to $\mathfrak{so}(5)$ mass matrices. We find

$$\begin{aligned} M_{\mathrm{L}} &= \begin{pmatrix} 0 & -(m_1+m_2) & 0 & 0 & 0 \\ m_1+m_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(m_1-m_2) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_1-m_2 & 0 & 0 \end{pmatrix}, \\ M_{\mathrm{R}} &= \begin{pmatrix} 0 & -(m_3+m_4) & 0 & 0 & 0 \\ m_3+m_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(m_3-m_4) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_3-m_4 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.113}$$

Note that the size of M_{L} is disproportionately large, for which the reason should be obvious. The embedding of these $\mathfrak{so}(5)$ matrices in the $\mathfrak{so}(5,5)$ mass matrix $\mathbb{M}_A{}^B$ is given in (3.72).

We can now follow the techniques of sections 3.2.5 and 3.2.6 to determine the masses that each of the scalar and tensor fields acquires due to this twist. The results of these calculations for the mass matrices (3.113) are presented in appendix 3.C.1. In particular, we find that the fields that appear in the six-dimensional black string solution (ϕ_+, $C_2^{(6)}$ and $\tilde{C}_2^{(6)}$) do not become massive in this Scherk–Schwarz reduction, as required. This means that the reduction of the solution (3.110) to a 5d black hole is the same as in the untwisted case.

The two self-dual tensors that charge the black string solution, $C_2^{(6)}$ and $\tilde{C}_2^{(6)}$, yield two tensors and two vector fields in 5d. We denote these by $C_2^{(5)}$, $C_1^{(5)}$, $\tilde{C}_2^{(5)}$, $\tilde{C}_1^{(5)}$. We now consider the self-duality conditions for these fields from (3.87), where we only take along fields that are non-zero in the 5d black hole solution. We find that they are pairwise dual by the relations

$$\mathrm{d}C_1^{(5)} = e^{\sqrt{2/3}\,\phi_5}\, e^{-\sqrt{2}\,\phi_+} * \mathrm{d}\tilde{C}_2^{(5)}\,, \qquad \mathrm{d}\tilde{C}_1^{(5)} = e^{\sqrt{2/3}\,\phi_5}\, e^{\sqrt{2}\,\phi_+} * \mathrm{d}C_2^{(5)}\,. \tag{3.114}$$

We use these to write the contributions of $C_2^{(5)}$ and $\tilde{C}_2^{(5)}$ to the black hole solution in terms of the dual one-forms. In doing so, we move to an undoubled formalism. The full five-dimensional black hole solution is then given by

$$\begin{cases} \mathrm{d}s^2_{(5)} = -\,(H_1 H_5 H_K)^{-\frac{2}{3}}\,\mathrm{d}t^2 + (H_1 H_5 H_K)^{\frac{1}{3}}\left[\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_3^2\right] \\ e^{\phi_+} = H_1^{\sqrt{\frac{1}{2}}} H_5^{-\sqrt{\frac{1}{2}}} \\ e^{\phi_5} = H_1^{-\sqrt{\frac{1}{6}}} H_5^{-\sqrt{\frac{1}{6}}} H_K^{\sqrt{\frac{2}{3}}} \\ C_1^{(5)} = (H_1^{-1} - 1)\,\mathrm{d}t \\ \tilde{C}_1^{(5)} = (H_5^{-1} - 1)\,\mathrm{d}t \\ \mathcal{A}_1^5 = (H_K^{-1} - 1)\,\mathrm{d}t\,. \end{cases} \tag{3.115}$$

Here $C_1^{(5)}$ and $\tilde{C}_1^{(5)}$ are full vector fields, meaning that they are not subject to a self-duality constraint and carry the usual number of degrees of freedom. Note that this compactification can be generalized by adding angular momentum in directions transverse to the 10d branes to give a rotating black hole in five dimensions.

This three-charge black hole has been well studied in the literature. Its charges are quantized as $Q_i = c_i N_i$, where N_i are integers and the the basic charges are given by [68]

$$c_1 = \frac{4G_N^{(5)} R}{\pi \alpha' g_s}\,, \qquad c_5 = \alpha' g_s\,, \qquad c_K = \frac{4G_N^{(5)}}{\pi R}\,. \tag{3.116}$$

The entropy of this black hole can be computed with the Bekenstein-Hawking formula, which yields

$$S_{\mathrm{BH}} = \frac{A}{4G_N^{(5)}} = \frac{\pi^2}{2G_N^{(5)}}\sqrt{Q_1 Q_5 Q_K} = 2\pi\sqrt{N_1 N_5 N_K}\,. \tag{3.117}$$

3.3.2 Dual brane configurations

The F1-NS5-P system

We now study the F1-NS5-P system, which consists of F1 and NS5-branes arranged as follows:

	t	r	θ	φ_1	φ_2	z	y_1	y_2	y_3	y_4
F1	–	·	·	·	·	–	···	···	···	···
NS5	–	·	·	·	·	–	–	–	–	–

Again there are waves with momentum in the z-direction. This system is related to the D1-D5 system via S-duality. As in the previous case, we start by considering the supergravity solution in ten dimensions. It can be written in Einstein frame as

$$\begin{cases} \mathrm{d}s^2_{(10)} = H_F^{-\frac{3}{4}} H_N^{-\frac{1}{4}} \left[- \mathrm{d}t^2 + \mathrm{d}z^2 + K(\mathrm{d}t - \mathrm{d}z)^2\right] + H_F^{\frac{1}{4}} H_N^{\frac{3}{4}} \left[\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_3^2\right] \\ \qquad\qquad + H_F^{\frac{1}{4}} H_N^{-\frac{1}{4}} \left[\mathrm{d}y_1^2 + \mathrm{d}y_2^2 + \mathrm{d}y_3^2 + \mathrm{d}y_4^2\right] \\ e^{\Phi} = H_F^{-\frac{1}{2}} H_N^{\frac{1}{2}} \\ B_2^{(10)} = (H_F^{-1} - 1)\, \mathrm{d}t \wedge \mathrm{d}z + Q_N \cos^2\theta \, \mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2 \,, \end{cases} \tag{3.118}$$

where we have the harmonic functions

$$H_F = 1 + \frac{Q_F}{r^2} \,, \qquad H_N = 1 + \frac{Q_N}{r^2} \,, \tag{3.119}$$

and H_K is as before. Note that this solution can be obtained from the D1-D5 solution (3.106) by an S-duality transformation, which sends $\Phi \to -\Phi$ and $C_2^{(10)} \to B_2^{(10)}$. After reduction on T^4, we obtain a very similar six-dimensional solution, given by (3.110) with the replacements $\phi_+ \to \phi_- = \frac{1}{\sqrt{2}}(\phi_4 - \Phi)$ and $C_2^{(6)} \to B_2^{(6)}$. In the doubled formalism, the black string couples to the two-forms

$$\begin{aligned} B_2^{(6)} &= (H_F^{-1} - 1)\, \mathrm{d}t \wedge \mathrm{d}z + Q_N \cos^2\theta \, \mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2 \,, \\ \tilde{B}_2^{(6)} &= (H_N^{-1} - 1)\, \mathrm{d}t \wedge \mathrm{d}z + Q_F \cos^2\theta \, \mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2 \,. \end{aligned} \tag{3.120}$$

Again these fields carry only half their usual degrees of freedom due to the self-duality constraint (3.27).

To reduce to five dimensions, we need to specify the mass matrices. We choose the Scherk–Schwarz twist to be in the stabilizing subgroup of the R-symmetry group. Since the F1-NS5 system couples to B_2 instead of C_2, the twist is chosen to preserve B_2. We choose the $\mathfrak{so}(5)$ matrices

$$M_{\rm L} = \begin{pmatrix} 0 & -(m_1+m_2) & 0 & 0 & 0 \\ m_1+m_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(m_1-m_2) & 0 \\ 0 & 0 & m_1-m_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} , \qquad (3.121)$$

and $M_{\rm R}$ similar with $m_1 \to m_3$ and $m_2 \to m_4$. By using the isomorphism in appendix 3.B.2, these map to $\mathfrak{usp}(4)$ generators of the form

$$M_{\rm L}^{\mathfrak{usp}(4)} = \begin{pmatrix} 0 & 0 & -\frac{m_1+m_2}{2} & \frac{m_1-m_2}{2} \\ 0 & 0 & \frac{m_1-m_2}{2} & -\frac{m_1+m_2}{2} \\ \frac{m_1+m_2}{2} & \frac{-m_1+m_2}{2} & 0 & 0 \\ \frac{-m_1+m_2}{2} & \frac{m_1+m_2}{2} & 0 & 0 \end{pmatrix} . \qquad (3.122)$$

The masses of the scalar and tensor fields that follow from the reduction with these mass matrices are given in appendix 3.C.2.

The resulting five-dimensional black hole is given by (3.115) with the field redefinitions $\phi_+ \to \phi_-$, $C_2^{(5)} \to B_2^{(5)}$ and $\tilde{C}_2^{(5)} \to \tilde{B}_2^{(5)}$. It is not surprising that these black holes are related by field redefinitions. After all, the D1-D5 and F1-NS5 systems are related by U-duality, and the corresponding mass matrices are related by conjugation

$$\mathbb{M}_{\text{F1-NS5}} = C\,\mathbb{M}_{\text{D1-D5}}\,C^{-1} , \qquad C \in \mathrm{Spin}(5,5) . \qquad (3.123)$$

This conjugation matrix C is given by

$$C = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} , \qquad c = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} . \qquad (3.124)$$

Essentially this conjugation matrix interchanges the fourth and fifth row and column and the ninth and tenth row and column in the mass matrix (and monodromy).

The D3-D3-P systems

Finally, we consider the reduction of the D3-D3-P system of branes. We specify the brane configuration:

	t	r	θ	φ_1	φ_2	z	y_1	y_2	y_3	y_4
D3	$-$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$-$	$-$	$-$	$\cdots$	$\cdots$
D3$'$	$-$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$-$	$\cdots$	$\cdots$	$-$	$-$

As before, we also have momentum in the z-direction. We start with the supergravity solution in ten dimensions, in Einstein frame it can be written as

$$\begin{cases} \mathrm{d}s^2_{(10)} = H_3^{-\frac{1}{2}} H_{3'}^{-\frac{1}{2}} \big[-\mathrm{d}t^2 + \mathrm{d}z^2 + K(\mathrm{d}t - \mathrm{d}z)^2\big] + H_3^{\frac{1}{2}} H_{3'}^{\frac{1}{2}} \big[\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_3^2\big] \\ \qquad\qquad + H_3^{-\frac{1}{2}} H_{3'}^{\frac{1}{2}} \big[\mathrm{d}y_1^2 + \mathrm{d}y_2^2\big] + H_3^{\frac{1}{2}} H_{3'}^{-\frac{1}{2}} \big[\mathrm{d}y_3^2 + \mathrm{d}y_4^2\big] \\ C_4^{(10)} = (H_3^{-1} - 1)\, \mathrm{d}t \wedge \mathrm{d}z \wedge \mathrm{d}y_1 \wedge \mathrm{d}y_2 + (H_{3'}^{-1} - 1)\, \mathrm{d}t \wedge \mathrm{d}z \wedge \mathrm{d}y_3 \wedge \mathrm{d}y_4 \,, \end{cases} \tag{3.125}$$

where the harmonic functions are given by

$$H_3 = 1 + \frac{Q_3}{r^2}\,, \qquad H_{3'} = 1 + \frac{Q_{3'}}{r^2}\,. \tag{3.126}$$

On compactifying to six dimensions on T^4 by taking the coordinates $y_1, \ldots, y_4$ periodic, this brane configuration is related to the D1-D5 system by T-duality. This means that the black string solution for the D3-D3 system can be obtained from that for the D1-D5 system (3.110) by a field redefinition. We find this field redefinition as $C_2^{(6)} \to R_{2;\,1}^{(6)}$ and $\phi_+ \to \phi_1$. In the doubled formalism, the six-dimensional black string arising from the D3-D3 system couples to the two-forms

$$\begin{aligned} R_{2;\,1}^{(6)} &= (H_3^{-1} - 1)\, \mathrm{d}t \wedge \mathrm{d}z + Q_{3'} \cos^2\theta \, \mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2 \,, \\ \tilde{R}_{2;\,1}^{(6)} &= (H_{3'}^{-1} - 1)\, \mathrm{d}t \wedge \mathrm{d}z + Q_3 \cos^2\theta \, \mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2 \,. \end{aligned} \tag{3.127}$$

Different D3-D3 systems can be constructed by arranging the D3-branes differently on the torus. These would be charged under the two-forms coming from the reduction of $C_4^{(10)}$ in such systems. All of these systems are related by T-duality.

In the last step of the reduction we need to ensure the fields that are non-trivial in the black hole solution remain massless in 5d. For this twisted reduction we choose $\mathfrak{so}(5)$ mass matrices of the form

$$M_{\mathrm{L}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(m_1 + m_2) & 0 \\ 0 & 0 & 0 & 0 & -(m_1 - m_2) \\ 0 & m_1 + m_2 & 0 & 0 & 0 \\ 0 & 0 & m_1 - m_2 & 0 & 0 \end{pmatrix}, \tag{3.128}$$

which results in $R^{(6)}_{2;\,1}$ remaining massless. In $\mathfrak{usp}(4)$ this mass matrix reads

$$M_{\rm L}^{\mathfrak{usp}(4)} = \begin{pmatrix} 0 & 0 & -\frac{m_1-m_2}{2} & \frac{i(m_1+m_2)}{2} \\ 0 & 0 & \frac{i(m_1+m_2)}{2} & \frac{m_1-m_2}{2} \\ \frac{m_1-m_2}{2} & \frac{i(m_1+m_2)}{2} & 0 & 0 \\ \frac{i(m_1+m_2)}{2} & -\frac{m_1-m_2}{2} & 0 & 0 \end{pmatrix} . \tag{3.129}$$

The scalar and tensor masses that follow from the reduction with these mass matrices are given in appendix 3.C.3. The resulting five-dimensional black hole is given by making the field redefinitions $\phi_+ \to \phi_1$, $C^{(5)}_2 \to R^{(5)}_{2;\,1}$ and $\tilde{C}^{(5)}_2 \to \tilde{R}^{(5)}_{2;\,1}$ in the solution (3.115). The mass matrices are again conjugate to those of the dual D1-D5 and F1-NS5 solutions. The relation is similar to the F1-NS5 result in (3.123), except now the matrix C switches the first and fourth rows and columns instead of the fourth and fifth ones.

3.3.3 Preserving further black holes by tuning mass parameters

In the previous subsection, we chose twist matrices with four arbitrary real parameters m_i. For each black hole solution (D1-D5, F1-NS5, D3-D3), we chose this matrix in such a way that the fields that source the black hole are left unchanged by the Scherk–Schwarz twist. Consequently, the black hole remains a valid solution of the 5d theory for all values of the mass parameters.

Here, we treat the special cases in which the mass parameters can be tuned in such a way that, in addition to the original black hole, other black hole solutions are also preserved by the same twist. For example, we consider twists that preserve both the D1-D5 and F1-NS5 black holes. As it turns out, this can only be done in the $\mathcal{N}=4$ $(0,2)$ theory and in the $\mathcal{N}=0$ theory. Since we are interested mostly in partial supersymmetry breaking, we treat an example of the $\mathcal{N}=4$ $(0,2)$ case in detail below.

Preserving D1-D5 with T-duality twist in $\mathcal{N}=4$ $(0,2)$

For this example, we consider mass matrices of the form given in (3.121) that preserve the F1-NS5 black hole solution. In order to twist to the $\mathcal{N}=4$ $(0,2)$ theory, we choose $m_1, m_2 \neq 0$ and $m_3, m_4 = 0$.

Suppose that, in addition to the F1-NS5 solution, we also want to preserve the D1-D5 solution with this twist. Then the fields

$$\left\{\phi_+ = \tfrac{1}{\sqrt{2}}(\phi_4 + \Phi)\,,\ C^{(5)}_2,\ \tilde{C}^{(5)}_2\right\} \tag{3.130}$$

have to remain massless as well. The masses of these fields for this twist matrix can be found in appendix 3.C.2. By setting $m_3, m_4 = 0$, we see that each field either becomes massive with mass $|m_1 - m_2|$ or remains massless. It is therefore straightforward to tune the mass parameters in such a way that all of these fields remain massless by taking $m_1 = m_2$.

We thus see that the D1-D5 solution can be preserved in a reduction to $\mathcal{N} = 4$ $(0, 2)$ with the twist matrix that was originally proposed to preserve the F1-NS5 solution, simply by taking the two mass parameters to be equal. This particular example offers some interesting possibilities. On the one hand, we note that the twist that preserves the F1-NS5 solution lies in the perturbative $\mathrm{SO}(4, 4)$ subgroup of the duality group. From the perspective of the full string theory this is a T-duality twist. Since T-duality is a perturbative symmetry, we can in principle work out the corresponding orbifold compactification of the perturbative string theory explicitly. On the other hand, the microscopic description of the D1-D5 black hole, the D1-D5 CFT, is understood reasonably well. Therefore, it should be possible to study this particular reduction thoroughly both from the perspective of the full string theory, and from the perspective of the black hole microscopics. We will return to this elsewhere.

Other possibilities in $\mathcal{N} = 4$ $(0, 2)$

By taking $m_1 = m_2$ in the example above, we managed to keep the fields that couple to the D1-D5 black hole massless, and so we could preserve this particular solution. It turns out, however, that this choice kept more fields massless than just the ones that charge the D1-D5 solution. In particular, the fields

$$\big\{\phi_3\,,\, R^{(5)}_{2;3}\,,\, \tilde{R}^{(5)}_{2;3}\big\} \tag{3.131}$$

also remain massless (as can be checked from the tables in appendix 3.C.2). These are exactly the fields that are non-trivial in one of the three possible D3-D3 black holes. So not only the D1-D5 and F1-NS5 black holes, but also one of the D3-D3 black holes is preserved in this reduction.

Suppose now that, instead of $m_1 = m_2$, we choose $m_1 = -m_2$ in this reduction to $\mathcal{N} = 4$ $(0, 2)$. This choice does not preserve the D1-D5 solution and the D3-D3 solution charged under $R^{(5)}_{2;3}$, but instead other solutions are preserved. Now the fields coupling to the two other D3-D3 black holes remain massless:

$$\big\{\phi_1\,,\, R^{(5)}_{2;1}\,,\, \tilde{R}^{(5)}_{2;1}\big\} \quad \text{and} \quad \big\{\phi_2\,,\, R^{(5)}_{2;2}\,,\, \tilde{R}^{(5)}_{2;2}\big\}\,. \tag{3.132}$$

These are all the possibilities for preserving multiple black hole solutions with a T-duality twist to the $\mathcal{N}=4$ $(0,2)$ theory. The same game can be played, however, with the other twist matrices given in sections 3.3.1 and 3.3.2. For each case, we find that taking either $m_1 = m_2$ or $m_1 = -m_2$ in the reduction to $\mathcal{N}=4$ $(0,2)$ results in the preservation of two additional black hole solutions.

Preserving further black holes in $\mathcal{N}=0$

The only other theory in which we can preserve several black hole solutions by tuning mass parameters is the one in which we break all supersymmetry: the $\mathcal{N}=0$ case. Now all four mass parameters are non-zero. As an example, let's consider the geometric F1-NS5 twist (3.121) again. If we take $m_1 = m_2$ and $m_3 = m_4$ in this reduction, all fields (3.130) that are non-trivial in the D1-D5 black hole solution remain massless. Consequently, the D1-D5 solution is preserved. Other examples can be worked out for similar reductions to $\mathcal{N}=0$.

3.4 Quantum corrections

So far, we have considered five-dimensional supergravity theories with both massless and massive fields. For the purpose of finding black hole solutions in these theories, we truncated (consistently) to the $n=0$ modes of the Kaluza–Klein towers and identified black hole solutions in the massless sector after this truncation.

Under certain conditions, which we discuss in the next chapter, the black hole solutions we have been considering lift to solutions of the full string theory. In the string theory, the effective supergravity theory receives quantum corrections. In particular, there are quantum corrections to the coefficients of the 5d Chern-Simons terms which in turn lead to modifications of the black hole solutions and hence to quantum corrections to their entropy.

In this section, we consider corrections to the coefficients of the 5d Chern-Simons terms that result from integrating out the massive spectrum. It is a little unusual that it is massive fields that contribute to these parity-violating terms. This is because in five dimensions massive fields can be in chiral representations of the little group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and so can contribute to the parity-violating Chern-Simons terms. First, we consider these quantum corrections in a general setting and then discuss their origins and consequences for the entropy of the black holes solutions of section 3.3. Subsequently, we compute the quantum corrections to the Chern-Simons terms from integrating out massive supergravity fields. This is of course not the full story: there are in principle further corrections from stringy modes.

3.4.1 Corrections to Chern-Simons terms

In five dimensions massive fields can be *chiral* as they are in representations (s, s') of the little group $\mathrm{SU}(2) \times \mathrm{SU}(2)$, and we will refer to them as chiral if $s \neq s'$. In the supergravity theory we have been discussing, the chiral massive field content consists of the gravitino in the $(3,2)$ representation, the self-dual two-form field in the $(3,1)$ representation and the spin-half dilatino in the $(2,1)$ representation (together with their anti-chiral counterparts $(2,3)$, $(1,3)$ and $(1,2)$). As we have seen in section 3.2.3, these massive fields fit into (p,q) BPS supermultiplets. By integrating out this chiral matter, we can obtain corrections to the 5d Chern-Simons terms [69]. In principle, one would need to integrate out the entire chiral massive spectrum; the fields that we found in our supergravity calculation, as well as massive stringy modes. We focus on the supergravity fields here.

From the fields that we obtain in our duality-twisted compactification of 6d supergravity, only the self-dual tensors, gravitini (spin-$\frac{3}{2}$ fermions) and dilatini (spin-$\frac{1}{2}$ fermions) contribute to the Chern-Simons terms. Integrating out other types of massive fields does not yield Chern-Simons couplings [69]. The origin of this lies in parity: since the Chern-Simons terms violate parity, they can only be generated by integrating out parity-violating fields.

The non-abelian gauge symmetry of the 5-dimensional gauged supergravity is spontaneously broken to an abelian subgroup with massless abelian gauge field one-forms A^I with field strengths $F^I = \mathrm{d}A^I$, with the index I running over the number of massless vector fields in the theory. The pure gauge and the mixed gauge-gravitational Chern-Simons terms involving these fields are of the form

$$S_{AFF} = \frac{-g^3}{48\pi^2} \int k_{IJK} A^I \wedge F^J \wedge F^K \,, \qquad S_{ARR} = \frac{-g}{48\pi^2} \int k_I A^I \wedge \mathrm{Tr}\,(R \wedge R) \tag{3.133}$$

for some coefficients k_{IJK}, k_I. Here g denotes the gauge coupling and R denotes the curvature two-form. Integrating out the chiral massive fields yields quantum corrections to the coefficients k_{IJK}, k_I.

Consider first the Chern-Simons terms in the classical 5d supergravity obtained by Scherk–Schwarz reduction from maximal 6d supergravity. By explicit reduction, we find that there are no $A \wedge R \wedge R$ terms. There are $A \wedge F \wedge F$ terms present however. For example, in the reduction with the Scherk–Schwarz twist that preserves the

D1-D5 black hole, we find the term

$$\frac{1}{2\kappa_{(5)}^2}\int \mathcal{A}_1^5 \wedge \mathrm{d}C_1^{(5)} \wedge \mathrm{d}\tilde{C}_1^{(5)}\,, \tag{3.134}$$

so that we have $k_{IJK} = -\frac{4\pi^2}{\kappa_{(5)}^2 g^3}$ for the indices I, J, K corresponding to the three gauge fields in (3.134). This Chern-Simons term (and other similar terms) can be found from the reduction of the 6d tensor fields (following section 3.2.6). There are also Chern-Simons terms coming from the reduction of the 6d vectors.

Quantum corrections to the Chern-Simons terms are only allowed for certain amounts of unbroken supersymmetry. The coefficients of the $A \wedge F \wedge F$ term are fixed for $\mathcal{N} > 2$ supersymmetry, so corrections to this terms are only allowed in the $\mathcal{N} = 2$ (and 0) theories. For $\mathcal{N} = 2$, the supersymmetric completion of the $A \wedge R \wedge R$ term exists and is known [70], but this is not the case for theories with more supersymmetry. However, in the chiral $\mathcal{N} = 4$ $(0, 2)$ theory a $A \wedge R \wedge R$ term is generated by quantum corrections, leading to the conjecture that a supersymmetric completion of this term should exist [71]. There is no such quantum $A \wedge R \wedge R$ term for the non-chiral $\mathcal{N} = 4$ $(1, 1)$ theory, nor for the $\mathcal{N} = 8, 6$ theories. We will see in section 3.4.3 that the corrections that we find from integrating out the massive fields that come from our duality-twisted compactification of 6d supergravity (including the Kaluza–Klein towers from the circle compactification) are in agreement with the above: we find corrections to the $A \wedge F \wedge F$ term only for $\mathcal{N} = 2$ supersymmetry and a quantum $A \wedge R \wedge R$ term is induced only for $\mathcal{N} = 2$ and the chiral $\mathcal{N} = 4$ $(0, 2)$ theory.

For our purposes, we will focus on the Chern-Simons terms $\mathcal{A}^5 \wedge \mathrm{d}\mathcal{A}^5 \wedge \mathrm{d}\mathcal{A}^5$ and $\mathcal{A}^5 \wedge R \wedge R$ that involve the graviphoton $\mathcal{A}^5$. This is because the black holes that we consider couple only to the graviphoton and to vectors descending from the 6d tensors (see section 3.3). The chiral massive field content that we find from duality-twisted compactification is not charged under the gauge symmetries corresponding to the vectors that descend from 6d tensors, so for the purposes of studying corrections to the black hole solutions we only need to consider couplings of this chiral matter to the graviphoton; these then lead to corrections to the coefficients of the $\mathcal{A}^5 \wedge \mathrm{d}\mathcal{A}^5 \wedge \mathrm{d}\mathcal{A}^5$ and $\mathcal{A}^5 \wedge R \wedge R$ terms.

We introduce the notation k_{AFF} for the coefficient of the $\mathcal{A}^5 \wedge \mathrm{d}\mathcal{A}^5 \wedge \mathrm{d}\mathcal{A}^5$ term and k_{ARR} for the coefficient of the $\mathcal{A}^5 \wedge R \wedge R$ term. Neither of these terms are present in the classical theory – there is no $\mathcal{A}^5 \wedge \mathrm{d}\mathcal{A}^5 \wedge \mathrm{d}\mathcal{A}^5$ term for the graviphoton. As a result, both k_{AFF} and k_{ARR} have no classical contributions and arise only from quantum corrections.

3.4.2 Corrections to black hole entropy

We now study the effect that the corrections to the Chern-Simons terms have on the black holes that we studied in section 3.3. As it turns out, both the coefficients k_{AFF} and k_{ARR} affect the black hole solutions. In particular, the entropy of these black holes is modified by the corrections to these coefficients.

In [72,73] general BPS black hole solutions were found for $\mathcal{N} = 2$ supergravity with both pure gauge and gauge-gravitational Chern-Simons terms (3.133). These general results then give BPS black hole solutions for our $\mathcal{N} = 2$ supergravity models, with the specific values of the Chern-Simons coefficients obtained in the next subsection. In particular, these BPS black holes are preserved by four supersymmetries, and these are the black holes for which we compute the entropy.

We can also apply this to the black holes in the $\mathcal{N} = 4$ $(0, 2)$ theory. As discussed in the previous subsection, the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ $(0, 2)$ theories are the only ones for which corrections to the Chern-Simons coefficients are allowed, and so these are the only theories in which we find corrected black hole solutions.

Consider the $\mathcal{N} = 4$ $(0, 2)$ theory. By integrating out the massive field content we obtain a non-zero coefficient k_{ARR} for the gauge-gravitational Chern-Simons term. In order to compute corrected BPS black hole solutions in this theory, we use the framework of [72, 73] for $\mathcal{N} = 2$ supergravity. We can consistently truncate this $\mathcal{N} = 4$ theory to an $\mathcal{N} = 2$ theory by decomposing fields into representations of an USp(2) × USp(2) subgroup of the R-symmetry group and removing all fields that transform non-trivially under one of these USp(2)'s. For each of the black hole solutions we have considered, we make a corresponding choice of the embedding of the USp(2) × USp(2) subgroup so that all the fields that are non-trivial in the black hole solution survive the truncation. As a result, the black hole solutions of the effective theory with an $A \wedge R \wedge R$ term given in [72, 73] will also be solutions of the quantum-corrected $\mathcal{N} = 4$ $(0, 2)$ theory that we have been considering here.

We now briefly review the procedure to compute the entropy of BPS black holes in these quantum corrected theories. It is given by the formula [72, 73]

$$S = \frac{\pi}{6}\, k_{IJK}\, X^I X^J X^K\,, \tag{3.135}$$

where X^I are the (rescaled) moduli corresponding to the three gauge fields A^I that couple to the black hole charges and k_{IJK} are the Chern-Simons coefficients from (3.133). The values of these moduli in the solution are found by solving the attractor equation, which in the near-horizon limit is

$$-\frac{1}{2}\, k_{IJK}\, X^J X^K \;=\; \frac{\pi}{2g\, G_N^{(5)}}\, Q_I + 2\, k_I\,, \tag{3.136}$$

More comprehensive studies of these solutions can be found in [72, 73].

We now apply this to our setup. When we solve (3.136) and compute (3.135) for general coefficients k_{AFF} and k_{ARR} (to the Chern-Simons terms that contain the graviphoton $\mathcal{A}^5$), we find the entropy of the corrected D1-D5-P black hole solution in terms of its three charges to be

$$S_{\rm BH} = \frac{2\pi^2}{4G_N^{(5)}}\sqrt{Q_1 Q_5 \hat{Q}_K\,\frac{2\left(1+\sqrt{1+k_{AFF}\,\frac{4G_N^{(5)}}{\pi R^3}\,\frac{Q_1Q_5}{\hat{Q}_K^2}}+k_{AFF}\,\frac{4G_N^{(5)}}{3\pi R^3}\,\frac{Q_1Q_5}{\hat{Q}_K^2}\right)^2}{\left(1+\sqrt{1+k_{AFF}\,\frac{4G_N^{(5)}}{\pi R^3}\,\frac{Q_1Q_5}{\hat{Q}_K^2}}\right)^3}}\,. \tag{3.137}$$

Here the charge arising from momentum in the z direction is shifted

$$\hat{Q}_K = Q_K + \frac{4G_N^{(5)}}{\pi R}\,k_{ARR}\,. \tag{3.138}$$

It can easily be checked that for $k_{AFF} = k_{ARR} = 0$ this expression for the black hole entropy reduces to the uncorrected result

$$S_{\rm BH} = \frac{\pi^2}{2G_N^{(5)}}\,\sqrt{Q_1Q_5Q_K}\,. \tag{3.139}$$

Just as was done for the uncorrected expression for the entropy, we can express the three charges in terms of integers N_i times the basic charges as $Q_i = c_i N_i$ with the basic charges c_i as given in (3.116). This yields

$$S_{\rm BH} = 2\pi\sqrt{N_1N_5\hat{N}_K\,\frac{2\left(1+\sqrt{1+k_{AFF}\,\frac{N_1N_5}{\hat{N}_K^2}}+\frac{1}{3}\,k_{AFF}\,\frac{N_1N_5}{\hat{N}_K^2}\right)^2}{\left(1+\sqrt{1+k_{AFF}\,\frac{N_1N_5}{\hat{N}_K^2}}\right)^3}}\,, \tag{3.140}$$

where the shifted momentum charge number is given by

$$\hat{N}_K = N_K + k_{ARR}\,. \tag{3.141}$$

The expression (3.140) can be expanded for small k_{AFF} as

$$S_{\rm BH} = 2\pi\sqrt{N_1N_5\hat{N}_K} + \frac{\pi}{12}\,k_{AFF}\left(\frac{N_1N_5}{\hat{N}_K}\right)^{\frac{3}{2}} + \mathcal{O}\big(k_{AFF}^2\big)\,. \tag{3.142}$$

The first term is equal to the uncorrected black hole entropy (3.117) and the second term is the correction to first order in k_{AFF}.

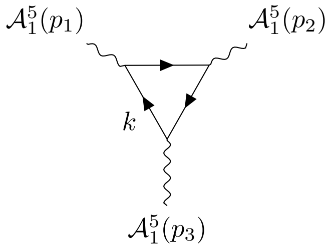


Figure 3.1: *This diagram generates corrections to the $A \wedge F \wedge F$ Chern-Simons coupling. The external lines represent the graviphoton whilst the solid internal lines represent a massive self-dual tensor, gravitino or dilatino running in the loop.*

3.4.3 One-loop calculation of the Chern-Simons coefficients

In this section we compute the contributions to k_{AFF} and k_{ARR} that come from integrating out chiral massive fields arising from the duality-twisted compactification of 6d supergravity. Similar calculations have been done in different setups, see [74,75]. While this is a well-defined calculation, some caution is needed since there will also be contributions from the chiral spectrum of stringy modes to the Chern-Simons coefficients. The coefficients that we compute here come purely from the supergravity modes.

Contributions are only obtained from integrating out massive self-dual tensors, gravitini (spin-$\frac{3}{2}$ fermions) and dilatini (spin-$\frac{1}{2}$ fermions). The relevant diagrams for corrections to the couplings (3.133) have been computed in [69]. As an example, we show the diagram that contributes to the $A \wedge F \wedge F$ term in figure 3.1. The diagrams that contribute to the $A \wedge R \wedge R$ term can be found in [69]. The results of these computations are shown in the table below.

	self-dual tensor B_2	gravitino ψ_μ	dilatino χ
k_{AFF}	$-4\, c_B\, q^3$	$5\, c_\psi\, q^3$	$c_\chi\, q^3$
k_{ARR}	$c_B\, q$	$-\frac{19}{8}\, c_\psi\, q$	$\frac{1}{8}\, c_\chi\, q$

We see that the contribution of a massive field to each of the Chern-Simons couplings consists of three parts: a prefactor that depends on the field type, a constant c_{field} (equal to ± 1) that depends on the field's representation under the massive little group, and the field's U(1) charge q under the graviphoton $\mathcal{A}^5_1$.

In order to find the corrections to the Chern-Simons terms that are induced by the massive spectra of our 5d theories, we need to know two things about each of the massive fields: the sign of c_{field} and the charge q. We always take $q \geq 0$ and absorb any minus signs into the corresponding c_{field}.

The conventions in this work are such that 5d tensors that descend from 6d self-dual tensors and 5d fermions that descend from 6d positive chiral fermions have $c_{\text{field}} = -1$, while tensors descending from 6d anti-self-dual tensors and fermions descending from 6d negative chiral fermions have $c_{\text{field}} = +1$. In terms of the six-dimensional R-symmetry representations, the signs of c_{field} of the corresponding five-dimensional massive fields are

$$\begin{aligned} (\mathbf{5},\mathbf{1}): &\quad c_B = -1\,, & (\mathbf{1},\mathbf{5}): &\quad c_B = +1\,, \\ (\mathbf{4},\mathbf{1}): &\quad c_\psi = -1\,, & (\mathbf{1},\mathbf{4}): &\quad c_\psi = +1\,, \\ (\mathbf{5},\mathbf{4}): &\quad c_\chi = -1\,, & (\mathbf{4},\mathbf{5}): &\quad c_\chi = +1\,. \end{aligned} \tag{3.143}$$

We know from section 3.2.9 that each 6d field produces a Kaluza–Klein tower of 5d fields for which the sum of the charges is given by

$$\sum_{n=-\infty}^{\infty} \left| \frac{\mu(m_i)}{2\pi} + n \right| . \tag{3.144}$$

We need to regularize such sums (and similar sums in which we take the sum of the cube of the charges). Following [76], the regularized expressions are

$$s_1[m] = \sum_{n=-\infty}^{\infty} \left| \frac{m}{2\pi} + n \right| = \left| \frac{m}{2\pi} \right| (2k+1) - k(k+1) - \frac{1}{6}\,, \tag{3.145}$$

$$\begin{aligned} s_3[m] = \sum_{n=-\infty}^{\infty} \left| \frac{m}{2\pi} + n \right|^3 = & \left| \frac{m}{2\pi} \right|^3 (2k+1) - 3\left(\frac{m}{2\pi}\right)^2 \left(k(k+1) + \frac{1}{6} \right) \\ & + 3\left| \frac{m}{2\pi} \right| \left(\frac{k(k+1)(2k+1)}{3} \right) - \frac{k^2(k+1)^2}{2} + \frac{1}{60}\,. \end{aligned} \tag{3.146}$$

Here we use the notation

$$k \equiv \left\lfloor \left| \tfrac{m}{2\pi} \right| \right\rfloor ,$$

where $\lfloor x \rfloor$ is the integer part of x.

We now have all the information that we need to compute the corrections to the Chern-Simons terms (3.133) that are generated by integrating out our massive five-dimensional spectra. Now, for a general twist (i.e. all twist parameters are turned on) we find the correction to the pure gauge term

$$\begin{aligned} k_{AFF} = {}& 4\big(- s_3[m_1] - s_3[m_2] + s_3[m_3] + s_3[m_4] \\ &\quad + s_3[m_1+m_2] + s_3[m_1-m_2] - s_3[m_3+m_4] - s_3[m_3-m_4]\big) \\ & - s_3[m_1+m_2+m_3] - s_3[m_1+m_2-m_3] - s_3[m_1-m_2+m_3] \\ & - s_3[m_1-m_2-m_3] - s_3[m_1+m_2+m_4] - s_3[m_1+m_2-m_4] \\ & - s_3[m_1-m_2+m_4] - s_3[m_1-m_2-m_4] + s_3[m_1+m_3+m_4] \\ & + s_3[m_1+m_3-m_4] + s_3[m_1-m_3+m_4] + s_3[m_1-m_3-m_4] \\ & + s_3[m_2+m_3+m_4] + s_3[m_2+m_3-m_4] + s_3[m_2-m_3+m_4] \\ & + s_3[m_2-m_3-m_4]\,, \end{aligned} \tag{3.147}$$

and the correction to the mixed gauge-gravitational term

$$\begin{aligned} k_{ARR} = {}& \tfrac{5}{2}\big(s_1[m_1] + s_1[m_2] - s_1[m_3] - s_1[m_4]\big) \\ & - s_1[m_1+m_2] - s_1[m_1-m_2] + s_1[m_3+m_4] + s_1[m_3-m_4] \\ & + \tfrac{1}{8}\big(- s_1[m_1+m_2+m_3] - s_1[m_1+m_2-m_3] - s_1[m_1-m_2+m_3] \\ & \quad - s_1[m_1-m_2-m_3] - s_1[m_1+m_2+m_4] - s_1[m_1+m_2-m_4] \\ & \quad - s_1[m_1-m_2+m_4] - s_1[m_1-m_2-m_4] + s_1[m_1+m_3+m_4] \\ & \quad + s_1[m_1+m_3-m_4] + s_1[m_1-m_3+m_4] + s_1[m_1-m_3-m_4] \\ & \quad + s_1[m_2+m_3+m_4] + s_1[m_2+m_3-m_4] + s_1[m_2-m_3+m_4] \\ & \quad + s_1[m_2-m_3-m_4]\big)\,. \end{aligned} \tag{3.148}$$

The above formulae give the contributions from summing over all Kaluza Klein modes arising from the reduction from 6d to 5d. The Scherk–Schwarz reduction to 5d supergravity keeps only the $n = 0$ modes and not the whole KK-towers, and on restricting to the $n = 0$ modes the functions s_1 and s_3 reduce to

$$s_1[m] = \left|\tfrac{m}{2\pi}\right|\,, \qquad s_3[m] = \left|\tfrac{m}{2\pi}\right|^3\,. \tag{3.149}$$

Then the quantum corrections to the Chern-Simons coefficients k_{AFF} and k_{ARR} from integrating out only the massive modes of the 5d supergravity that arises from Scherk–Schwarz reduction are given by (3.147) and (3.148) with the simpler expressions (3.149) for s_1, s_3.

The expressions (3.147) and (3.148) are the quantum corrections for general values of the mass parameters. The results for twists that preserve supersymmetry can be found by taking certain parameters in (3.147) and (3.148) equal to zero. We work out some interesting cases below.

- $\mathcal{N}=8$, $\mathcal{N}=6$ and $\mathcal{N}=4$ $(1,1)$

  By twisting to any of these cases we find that $k_{AFF}=0$ and $k_{ARR}=0$, as can be checked straightforwardly by setting the appropriate mass parameters equal to zero in (3.147) and (3.148). This is consistent with expectations based on supersymmetry and chirality, as was explained earlier in this section.

- $\mathcal{N}=4$ $(0,2)$

  For the case where we choose a chiral twist to the $\mathcal{N}=4$ theory, say with $m_1, m_2 \neq 0$ and $m_3=m_4=0$, we find that k_{AFF} vanishes but k_{ARR} does not. For such a twist, we find the correction from the $n=0$ modes to be

$$k_{ARR} \;=\; \tfrac{1}{2\pi}\left(3\,|m_1| + 3\,|m_2| - \tfrac{3}{2}\,|m_1+m_2| - \tfrac{3}{2}\,|m_1-m_2|\right)\,, \tag{3.150}$$

  and by taking into account the Kaluza–Klein towers as well we find

$$k_{ARR} \;=\; \tfrac{1}{2} + 3\,s_1[m_1] + 3\,s_1[m_2] - \tfrac{3}{2}\,s_1[m_1+m_2] - \tfrac{3}{2}\,s_1[m_1-m_2]\;. \tag{3.151}$$

- $\mathcal{N}=2$ $(0,1)$

  In the minimal $\mathcal{N}=2$ theory corrections to both the Chern-Simons coefficients are allowed, and the supersymmetric extension of the $A\wedge R\wedge R$ term is known [70]. The coefficients k_{AFF} and k_{ARR} can be computed from the general formulas (3.147) and (3.148) by taking $m_4=0$ and the other parameters non-zero. The general expressions are quite unwieldy, but if we take $m_1=m_2=m_3=m$ they simplify substantially. For this choice of mass parameters the corrections due to the $n=0$ modes are

$$k_{AFF} \;=\; 36\left|\frac{m}{2\pi}\right|^3\,, \qquad\qquad k_{ARR} \;=\; \frac{9}{4}\left|\frac{m}{2\pi}\right|\,, \tag{3.152}$$

  and the corrections due to both the $n=0$ modes and the Kaluza–Klein towers read

$$k_{AFF} \;=\; \tfrac{1}{6} - 15\,s_3[m] + 6\,s_3[2m] - s_3[3m]\,, \tag{3.153}$$

$$k_{ARR} \;=\; \tfrac{13}{24} + \tfrac{33}{8}\,s_1[m] - \tfrac{3}{4}\,s_1[2m] - \tfrac{1}{8}\,s_1[3m]\,. \tag{3.154}$$

The expressions for the coefficients k_{AFF} and k_{ARR} that we found in this subsection are computed from the supergravity fields that come from the duality-twisted compactification. A more thorough calculation would be needed to include all the stringy modes as well. The embedding into string theory is discussed in the next chapter. The full string theory calculation of the coefficients k_{AFF} and k_{ARR}, however, is beyond the scope of this work and left for future study.

Appendices

3.A Conventions and notation

Throughout this work, we set $c = \hbar = k_B = 1$, and we work in the 'mostly plus' convention for the metric, i.e. $\eta_{\mu\nu} = \mathrm{diag}(-,+,\dots,+)$. The notations that we use for the coordinates and indices in various dimensions are summarized in the table below.

Space	Coordinate	Indices
$D=10$	$X^M = \left(\hat{x}^{\hat{\mu}}, y^m\right)$	$M, N, \ldots = 0, 1, \ldots, 9$
$D=6$	$\hat{x}^{\hat{\mu}} = (x^\mu, z)$	$\hat{\mu}, \hat{\nu}, \ldots = 0, 1, \ldots, 5$
$D=5$	x^μ	$\mu, \nu, \ldots = 0, 1, \ldots, 4$
T^4	y^m	$m, n, \ldots = 1, \ldots, 4$

In general, we denote form-values fields as $A_p^{(d)}$, where p is the rank of the form and d is the dimension in which it lives. We define the Hodge star operator on forms as

$$*A_p^{(d)} = \frac{1}{p!(d-p)!} \sqrt{g_{(d)}}\, \varepsilon_{\mu_1\ldots\mu_p\nu_1\ldots\nu_{d-p}}\, A^{\mu_1\cdots\mu_p}\, \mathrm{d}x^{\nu_1} \wedge \ldots \wedge \mathrm{d}x^{\nu_{d-p}}\,. \tag{3.155}$$

We use the subscript or superscript (d) more often to indicate the number of spacetime dimensions where necessary, e.g. $R^{(d)}$, $e_{(d)}$, etc. In all dimensions, we normalize Lagrangians such that the corresponding actions are given by

$$S^{(d)} = \frac{1}{2\kappa_{(d)}^2} \int \mathscr{L}^{(d)}\,, \tag{3.156}$$

where $\kappa_{(d)}^2 = 8\pi G_N^{(d)}$ is the d-dimensional Newton's constant.

We use $A, B, \ldots = 1, \ldots, 10$ to denote Spin(5,5) indices that transform in τ-frame (as explained in appendix 3.B.1), and we use $a, b, \ldots = 1, \ldots, 5$ for indices transforming under the subgroup $\mathrm{GL}(5) \subset \mathrm{Spin}(5,5)$. For example, in 6d we have ten tensor fields (subject to a self-duality constraint), whose field strengths we write as

$$G_{3,A}^{(6)} = \begin{pmatrix} G_{3,a}^{(6)} \\ \tilde{G}_3^{(6)a} \end{pmatrix}\,. \tag{3.157}$$

The GL(5) subgroup works on the index a of the (dual) field strengths $G_{3,a}^{(6)}$ and $\tilde{G}_3^{(6)a}$. For more information on how this subgroup works, see appendix 3.B.1.

3.B Group theory

3.B.1 The group SO(5, 5) and its algebra

In this appendix we discuss some details and our conventions concerning the group $\mathrm{SO}(5,5)$ and its algebra $\mathfrak{so}(5,5)$. In particular, we construct two bases in which $\mathrm{SO}(5,5)$ can be written down; we call these the η-frame and the τ-frame. Furthermore, we build an explicit basis for the algebra that we use to construct a vielbein $\mathcal{V} \in \mathrm{SO}(5,5)$ in the main text.

Canonically, an element $g \in \mathrm{SO}(5,5)$ is represented by a 10×10 matrix, satisfying the conditions

$$g^T \eta\, g \;=\; \eta\,, \qquad\qquad \eta = \begin{pmatrix} \mathbb{1}_5 & 0 \\ 0 & -\mathbb{1}_5 \end{pmatrix}\,, \tag{3.158}$$

and $\det(g) = 1$. Henceforth, we refer to group elements satisfying these conditions as being written in the η-frame of $\mathrm{SO}(5,5)$. In the η-frame, a generator of the Lie algebra $M \in \mathfrak{so}(5,5)$ can be written in 5×5 blocks as

$$M = \begin{pmatrix} a & b \\ b^T & c \end{pmatrix}\,, \tag{3.159}$$

where a and c are antisymmetric and b is unconstrained.

There is another (isomorphic) way of writing down the group $\mathrm{SO}(5,5)$. We construct this other basis by conjugating the group elements as $\tilde{g} = X^{-1} g X$, where X is the matrix

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1}_5 & \mathbb{1}_5 \\ \mathbb{1}_5 & -\mathbb{1}_5 \end{pmatrix}\,. \tag{3.160}$$

Note that $X = X^{-1} = X^T$. We can now rewrite (3.158) in terms of $\tilde{g}$, which yields the following conditions on the conjugated group elements:

$$\tilde{g}^T \tau\, \tilde{g} \;=\; \tau\,, \qquad\qquad \tau = \begin{pmatrix} 0 & \mathbb{1}_5 \\ \mathbb{1}_5 & 0 \end{pmatrix}\,. \tag{3.161}$$

We see that the conjugated matrices $\tilde{g}$ preserve the matrix τ (instead of η), and therefore we refer to these matrices as being written in the τ-frame of $\mathrm{SO}(5,5)$. It is clear from the conjugation relation $\tilde{g} = X^{-1} g X$ that the two frames are isomorphic. The general block structure for generators of the Lie algebra $\mathfrak{so}(5,5)$ in the τ-frame is of the form

$$\tilde{M} = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}\,. \tag{3.162}$$

Here A is unconstrained and B and C are antisymmetric.

There is a subgroup $\mathrm{GL}(5) \subset \mathrm{SO}(5,5)$ that is embedded diagonally in the τ-frame matrices $\tilde{g}$. Generators of GL(5) can be represented by unconstrained 5×5 matrices, and these can be embedded diagonally in the block structure (3.162) by taking $B = C = 0$ and A equal to the $\mathfrak{gl}(5)$ generator. By exponentiating, we find the corresponding group element to be of the form

$$\begin{pmatrix} P & 0 \\ 0 & (P^T)^{-1} \end{pmatrix} \in \mathrm{GL}(5) \subset \mathrm{SO}(5,5)\,, \tag{3.163}$$

where P is an invertible five by five matrix. The embedding in the η-frame can be found by conjugating (3.163) with the matrix X given in (3.160).

A basis for the algebra $\mathfrak{so}(5,5)$

Using the general form of M, we build a basis of generators. Since $\mathfrak{so}(5,5)$ has rank five, we have five Cartan generators, denoted by H_n $(n = 0, \ldots, 4)$. We choose the Cartan subalgebra to be block-diagonal in the τ-frame, so that when written in the form (3.162), they all have $B = C = 0$. Furthermore, for convenience we choose the following form for the A matrices for the H_n:

$$\begin{aligned}
A_{H_0} &= \frac{1}{2}\mathrm{diag}\,(0,0,0,-1,1) \\
A_{H_1} &= \frac{1}{\sqrt{2}}\mathrm{diag}\,(-1,0,0,0,0)\;, \\
A_{H_2} &= \frac{1}{\sqrt{2}}\mathrm{diag}\,(0,-1,0,0,0)\;, \\
A_{H_3} &= \frac{1}{\sqrt{2}}\mathrm{diag}\,(0,0,-1,0,0)\;, \\
A_{H_4} &= \frac{1}{2}\mathrm{diag}\,(0,0,0,-1,-1)\;.
\end{aligned}$$

Apart from these Cartan generators, there are 20 root generators with $B = C = 0$. We denote them by $E^{A,+}_{nm}$ and $E^{A,-}_{nm}$ $(n, m = 1, \ldots, 5$ and $n < m)$. The $E^{A,+}_{nm}$ together fill the upper triangular part of A and the $E^{A,-}_{nm}$ fill the lower-triangular part. They do so in such a way that $\left(E^{A,+}_{nm}\right)^T = E^{A,-}_{nm}$. For example, we have

$$A_{E^{A,+}_{12}} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_{E^{A,-}_{12}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\;. \tag{3.164}$$

Finally, there are ten root generators E^B_{nm} with $A = C = 0$, and ten root generators E^C_{nm} with $A = B = 0$. The generators E^B_{nm} have

$$B_{E^B_{12}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} , \quad B_{E^B_{13}} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} , \quad \text{etc.} \tag{3.165}$$

The generators E^C_{nm} are constructed in the same way as E^B_{nm}, but now we have $B = 0$ and $C \neq 0$. The matrix C that corresponds to E^C_{nm} is equal to the matrix B that defined E^B_{nm} in the construction above. Note that this implies that $\left(E^B_{nm}\right)^T = -\,E^C_{nm}$.

The set of matrices defined above $\left\{H_n,\, E^{A,+}_{nm},\, E^{A,-}_{nm},\, E^B_{nm},\, E^C_{nm}\right\}$ gives a complete basis of generators of $\mathfrak{so}(5{,}5)$. When we mention E^A_{mn} below we always mean $E^{A,+}_{mn}$.

Let us now discuss the notation T^F_{ij} used in the text. These matrices T are certain generators of the $\mathfrak{so}(5{,}5)$ algebra described above. In particular if we let $\vec{T}^F_{ij} := (T^F_{12}, T^F_{13}, \ldots, T^F_{34})$, then we have the following definitions for T:

$$\begin{aligned}
\vec{T}^A_{ij} &= \left(E^C_{23},\, E^C_{12},\, -E^C_{13},\, -(E^A_{13})^T,\, -(E^A_{12})^T,\, -E^A_{23} \right) \\
\vec{T}^B_{ij} &= \left(E^A_{14},\, E^A_{34},\, E^A_{24},\, -E^C_{24},\, E^C_{34},\, -E^C_{14} \right) \\
\vec{T}^C_{ij} &= \left(E^A_{15},\, E^A_{35},\, E^A_{25},\, -E^C_{25},\, E^C_{35},\, -E^C_{15} \right) \\
T^a &= E^A_{45} \\
T^b &= E^C_{45}
\end{aligned}$$

3.B.2 The isomorphism $\mathfrak{usp}(4) \cong \mathfrak{so}(5)$

The group USp(4) is the group of 4×4 matrices g satisfying

$$g^\dagger = g^{-1} , \qquad \Omega\, g\, \Omega^{-1} = \left(g^{-1}\right)^T \tag{3.166}$$

where Ω is the symplectic metric, given by the block matrix

$$\Omega^{AB} = \begin{pmatrix} 0_{2\times2} & \mathbb{1}_{2\times2} \\ -\mathbb{1}_{2\times2} & 0_{2\times2} \end{pmatrix} . \tag{3.167}$$

The Lie algebra $\mathfrak{usp}(4)$ is represented by 4×4 matrices ${M_A}^B$ satisfying

$$M^\dagger = -M, \qquad \Omega\, M\, \Omega^{-1} = -M^T , \tag{3.168}$$

The isomorphism USp(4) ≅ Spin(5) can be made explicit by introducing five 4×4 gamma matrices, that satisfy the Euclidean Clifford algebra

$$\{\Gamma_a, \Gamma_b\}_A{}^B = 2\,\delta_{ab}\,\delta_A^B. \tag{3.169}$$

Here $a, b = 1, \ldots, 5$ are the indices corresponding to Spin(5), and $A, B = 1, \ldots, 4$ are the indices corresponding to USp(4). An explicit basis of (Hermitian and traceless) gamma matrices, that satisfies (3.169), is given by

$$\Gamma_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \Gamma_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.170}$$

It can easily be checked that the gamma matrices with upper indices, defined as $(\tilde{\Gamma}_a)^{AB} = \Omega^{AC}(\Gamma_a)_C{}^B$, are antisymmetric[5], i.e. $(\tilde{\Gamma}_a)^T = -\tilde{\Gamma}_a$. Using this, we deduce that

$$(\Gamma_a)^T = (\Omega^{-1}\tilde{\Gamma}_a)^T = -\tilde{\Gamma}_a\,(\Omega^{-1})^T = \Omega\,\Gamma_a\,\Omega^{-1}. \tag{3.171}$$

Hence, the symplectic metric Ω acts on the gamma matrices as a charge conjugation matrix. We now define $\Gamma_{ab} = \frac{1}{2}[\Gamma_a, \Gamma_b]$. From (3.171) and the Hermitian property of the Dirac matrices, it follows directly that Γ_{ab} satisfies the conditions (3.168). Furthermore, using the Clifford algebra, it is straightforward to check that the commutator of Γ_{ab} reads

$$\big[\Gamma_{ab}, \Gamma_{cd}\big] = -2\,\delta_{ac}\Gamma_{bd} + 2\,\delta_{ad}\Gamma_{bc} + 2\,\delta_{bc}\Gamma_{ad} - 2\,\delta_{bd}\Gamma_{ac}. \tag{3.172}$$

This is exactly the commutator of the basis elements of the $\mathfrak{so}(5)$ algebra. We conclude that the ten matrices Γ_{ab} form a set of generators of USp(4) ≅ Spin(5). Using these gamma matrices the explicit form of the isomorphism between the algebras can be derived [77]

$$M_{ab} = -\frac{1}{2}\,\mathrm{Tr}\big[M_A{}^B(\Gamma_{ab})_B{}^C\,\big]\,. \tag{3.173}$$

[5]This property is used in what follows, but it is not generally true for other choices of Ω and Γ_a.

The special orthogonal Lie algebra $\mathfrak{so}(5)$ consists of real antisymmetric matrices. We can check these properties for the found generators (3.173). The antisymmetry follows immediately from the antisymmetry in the gamma matrices $\Gamma_{ab} = -\Gamma_{ba}$. To prove the reality condition we use that both ${M_A}^B$ and ${(\Gamma_{ab})_A}^B$ satisfy the conditions (3.168). Using these constraints we find

$$\begin{aligned}(M_{ab})^* &= -\frac{1}{2}\,\mathrm{Tr}\big[M^*(\Gamma_{ab})^*\big]\\ &= -\frac{1}{2}\,\mathrm{Tr}\big[\Omega\, M\, \Omega^{-1}\,\Omega\,\Gamma_{ab}\,\Omega^{-1}\big]\\ &= -\frac{1}{2}\,\mathrm{Tr}\big[M\,\Gamma_{ab}\big] \;=\; M_{ab}.\end{aligned} \tag{3.174}$$

Thus we find that M_{ab}, as given in (3.173), is a real antisymmetric matrix, and therefore a suitable generator of SO(5). For completeness we also mention the inverse of the isomorphism (3.173) which maps $\mathfrak{so}(5)$ to $\mathfrak{usp}(4)$:

$${M^A}_B = \frac{1}{4}\, M_{ab}\, {(\Gamma^{ab})^A}_B \;. \tag{3.175}$$

3.C 5d scalar and tensor masses

3.C.1 The D1-D5 system

Here we show the masses of the fields for the D1-D5 set-up, corresponding to the mass matrices shown in (3.112) and (3.113). The scalar and tensor masses are as follows:

Field $\tilde{\sigma}^i$	Mass
$\frac{1}{\sqrt{2}}(\phi_4 + \Phi)$	0
$\frac{1}{2}(\phi_4 - \Phi + \sqrt{2}\,\phi_3)$	$\lvert m_1 - m_2 - m_3 + m_4\rvert$
$\frac{1}{2}(\phi_4 - \Phi - \sqrt{2}\,\phi_3)$	$\lvert m_1 - m_2 + m_3 - m_4\rvert$
$\frac{1}{\sqrt{2}}(\phi_1 + \phi_2)$	$\lvert m_1 + m_2 - m_3 - m_4\rvert$
$\frac{1}{\sqrt{2}}(\phi_1 - \phi_2)$	$\lvert m_1 + m_2 + m_3 + m_4\rvert$
$\frac{1}{2}(A_{12} + A_{34} + C_{12} + C_{34})$	$\lvert m_1 + m_2 - m_3 + m_4\rvert$
$\frac{1}{2}(A_{12} + A_{34} - C_{12} - C_{34})$	$\lvert m_1 + m_2 + m_3 - m_4\rvert$
$\frac{1}{2}(A_{12} - A_{34} + C_{12} - C_{34})$	$\lvert m_1 - m_2 + m_3 + m_4\rvert$
$\frac{1}{2}(A_{12} - A_{34} - C_{12} + C_{34})$	$\lvert m_1 - m_2 - m_3 - m_4\rvert$
$\frac{1}{2}(A_{14} + A_{23} + C_{14} - C_{23})$	$\lvert m_1 - m_2 + m_3 + m_4\rvert$
$\frac{1}{2}(A_{14} + A_{23} - C_{14} + C_{23})$	$\lvert m_1 - m_2 - m_3 - m_4\rvert$
$\frac{1}{2}(A_{14} - A_{23} + C_{14} + C_{23})$	$\lvert m_1 + m_2 - m_3 + m_4\rvert$
$\frac{1}{2}(-A_{14} + A_{23} + C_{14} + C_{23})$	$\lvert m_1 + m_2 + m_3 - m_4\rvert$
A_{13}	$\lvert m_1 + m_2 - m_3 - m_4\rvert$
A_{24}	$\lvert m_1 + m_2 + m_3 + m_4\rvert$
C_{13}	$\lvert m_1 - m_2 + m_3 - m_4\rvert$
C_{24}	$\lvert m_1 - m_2 - m_3 + m_4\rvert$
$\frac{1}{\sqrt{2}}(B_{12} + B_{34})$	$\lvert m_1 + m_2\rvert$
$\frac{1}{\sqrt{2}}(B_{12} - B_{34})$	$\lvert m_3 + m_4\rvert$
$\frac{1}{\sqrt{2}}(B_{13} + B_{24})$	$\lvert m_3 - m_4\rvert$
$\frac{1}{\sqrt{2}}(B_{13} - B_{24})$	$\lvert m_1 - m_2\rvert$
$\frac{1}{\sqrt{2}}(B_{14} + B_{23})$	$\lvert m_1 + m_2\rvert$
$\frac{1}{\sqrt{2}}(B_{14} - B_{23})$	$\lvert m_3 + m_4\rvert$
$\frac{1}{\sqrt{2}}(a + b)$	$\lvert m_1 - m_2\rvert$
$\frac{1}{\sqrt{2}}(a - b)$	$\lvert m_3 - m_4\rvert$

Field $A^{(5)}_{2,A}$	Mass
$C^{(5)}_2$	0
$\tilde{C}^{(5)}_2$	0
$\frac{1}{\sqrt{2}}\left(B^{(5)}_2 + \tilde{B}^{(5)}_2\right)$	$\lvert m_1 - m_2\rvert$
$\frac{1}{\sqrt{2}}\left(B^{(5)}_2 - \tilde{B}^{(5)}_2\right)$	$\lvert m_3 - m_4\rvert$
$\frac{1}{\sqrt{2}}\left(R^{(5)}_{2;1} + \tilde{R}^{(5)}_{2;1}\right)$	$\lvert m_1 + m_2\rvert$
$\frac{1}{\sqrt{2}}\left(R^{(5)}_{2;1} - \tilde{R}^{(5)}_{2;1}\right)$	$\lvert m_3 + m_4\rvert$
$\frac{1}{\sqrt{2}}\left(R^{(5)}_{2;2} + \tilde{R}^{(5)}_{2;2}\right)$	$\lvert m_1 + m_2\rvert$
$\frac{1}{\sqrt{2}}\left(R^{(5)}_{2;2} - \tilde{R}^{(5)}_{2;2}\right)$	$\lvert m_3 + m_4\rvert$
$\frac{1}{\sqrt{2}}\left(R^{(5)}_{2;3} + \tilde{R}^{(5)}_{2;3}\right)$	$\lvert m_1 - m_2\rvert$
$\frac{1}{\sqrt{2}}\left(R^{(5)}_{2;3} - \tilde{R}^{(5)}_{2;3}\right)$	$\lvert m_3 - m_4\rvert$

3.C.2 The F1-NS5 system

For the reduction of the F1-NS5 system, we chose mass matrices as in (3.121) and (3.122). The scalar and tensor masses are:

Field $\tilde{\sigma}^i$	Mass
$\frac{1}{\sqrt{2}}(\phi_4 - \Phi)$	0
$\frac{1}{2}(\phi_4 + \Phi + \sqrt{2}\,\phi_3)$	$\lvert m_1 - m_2 - m_3 + m_4\rvert$
$\frac{1}{2}(\phi_4 + \Phi - \sqrt{2}\,\phi_3)$	$\lvert m_1 - m_2 + m_3 - m_4\rvert$
$\frac{1}{\sqrt{2}}(\phi_1 + \phi_2)$	$\lvert m_1 + m_2 - m_3 - m_4\rvert$
$\frac{1}{\sqrt{2}}(\phi_1 - \phi_2)$	$\lvert m_1 + m_2 + m_3 + m_4\rvert$
$\frac{1}{2}(A_{12} + A_{34} + B_{12} + B_{34})$	$\lvert m_1 + m_2 - m_3 + m_4\rvert$
$\frac{1}{2}(A_{12} + A_{34} - B_{12} - B_{34})$	$\lvert m_1 + m_2 + m_3 - m_4\rvert$
$\frac{1}{2}(A_{12} - A_{34} + B_{12} - B_{34})$	$\lvert m_1 - m_2 + m_3 + m_4\rvert$
$\frac{1}{2}(A_{12} - A_{34} - B_{12} + B_{34})$	$\lvert m_1 - m_2 - m_3 - m_4\rvert$
$\frac{1}{2}(A_{14} + A_{23} + B_{14} - B_{23})$	$\lvert m_1 - m_2 + m_3 + m_4\rvert$
$\frac{1}{2}(A_{14} + A_{23} - B_{14} + B_{23})$	$\lvert m_1 - m_2 - m_3 - m_4\rvert$
$\frac{1}{2}(A_{14} - A_{23} + B_{14} + B_{23})$	$\lvert m_1 + m_2 - m_3 + m_4\rvert$
$\frac{1}{2}(-A_{14} + A_{23} + B_{14} + B_{23})$	$\lvert m_1 + m_2 + m_3 - m_4\rvert$
A_{13}	$\lvert m_1 + m_2 - m_3 - m_4\rvert$
A_{24}	$\lvert m_1 + m_2 + m_3 + m_4\rvert$
B_{13}	$\lvert m_1 - m_2 + m_3 - m_4\rvert$
B_{24}	$\lvert m_1 - m_2 - m_3 + m_4\rvert$
$\frac{1}{\sqrt{2}}(C_{12} + C_{34})$	$\lvert m_1 + m_2\rvert$
$\frac{1}{\sqrt{2}}(C_{12} - C_{34})$	$\lvert m_3 + m_4\rvert$
$\frac{1}{\sqrt{2}}(C_{13} + C_{24})$	$\lvert m_3 - m_4\rvert$
$\frac{1}{\sqrt{2}}(C_{13} - C_{24})$	$\lvert m_1 - m_2\rvert$
$\frac{1}{\sqrt{2}}(C_{14} + C_{23})$	$\lvert m_1 + m_2\rvert$
$\frac{1}{\sqrt{2}}(C_{14} - C_{23})$	$\lvert m_3 + m_4\rvert$
$\frac{1}{\sqrt{2}}(a + b)$	$\lvert m_3 - m_4\rvert$
$\frac{1}{\sqrt{2}}(a - b)$	$\lvert m_1 - m_2\rvert$

Field $A^{(5)}_{2,A}$	Mass
$B^{(5)}_2$	0
$\tilde{B}^{(5)}_2$	0
$\frac{1}{\sqrt{2}}\big(C^{(5)}_2 + \tilde{C}^{(5)}_2\big)$	$\lvert m_1 - m_2\rvert$
$\frac{1}{\sqrt{2}}\big(C^{(5)}_2 - \tilde{C}^{(5)}_2\big)$	$\lvert m_3 - m_4\rvert$
$\frac{1}{\sqrt{2}}\big(R^{(5)}_{2;1} + \tilde{R}^{(5)}_{2;1}\big)$	$\lvert m_1 + m_2\rvert$
$\frac{1}{\sqrt{2}}\big(R^{(5)}_{2;1} - \tilde{R}^{(5)}_{2;1}\big)$	$\lvert m_3 + m_4\rvert$
$\frac{1}{\sqrt{2}}\big(R^{(5)}_{2;2} + \tilde{R}^{(5)}_{2;2}\big)$	$\lvert m_1 + m_2\rvert$
$\frac{1}{\sqrt{2}}\big(R^{(5)}_{2;2} - \tilde{R}^{(5)}_{2;2}\big)$	$\lvert m_3 + m_4\rvert$
$\frac{1}{\sqrt{2}}\big(R^{(5)}_{2;3} + \tilde{R}^{(5)}_{2;3}\big)$	$\lvert m_1 - m_2\rvert$
$\frac{1}{\sqrt{2}}\big(R^{(5)}_{2;3} - \tilde{R}^{(5)}_{2;3}\big)$	$\lvert m_3 - m_4\rvert$

3.C.3 The D3-D3 system

For the D3-D3 brane set-up that we consider in section 3.3.2, we use the mass matrices given in (3.128) and (3.129). The scalars and tensors masses are:

Field $\tilde{\sigma}^i$	Mass
ϕ_1	0
$\frac{1}{2}(\Phi + \sqrt{2}\,\phi_2 + \phi_4)$	$\lvert m_1 + m_2 - m_3 - m_4\rvert$
$\frac{1}{2}(\Phi + \sqrt{2}\,\phi_3 - \phi_4)$	$\lvert m_1 - m_2 + m_3 - m_4\rvert$
$\frac{1}{2}(\Phi - \sqrt{2}\,\phi_3 - \phi_4)$	$\lvert m_1 - m_2 - m_3 + m_4\rvert$
$\frac{1}{2}(\Phi - \sqrt{2}\,\phi_2 + \phi_4)$	$\lvert m_1 + m_2 + m_3 + m_4\rvert$
$\frac{1}{2}(b + a + A_{12} - A_{34})$	$\lvert m_1 - m_2 - m_3 - m_4\rvert$
$\frac{1}{2}(b - a - A_{12} - A_{34})$	$\lvert m_1 + m_2 + m_3 - m_4\rvert$
$\frac{1}{2}(b - a + A_{12} + A_{34})$	$\lvert m_1 + m_2 - m_3 + m_4\rvert$
$\frac{1}{2}(b + a - A_{12} + A_{34})$	$\lvert m_1 - m_2 + m_3 + m_4\rvert$
$\frac{1}{2}(C_{23} - C_{14} - B_{24} + B_{13})$	$\lvert m_1 - m_2 - m_3 - m_4\rvert$
$\frac{1}{2}(C_{23} + C_{14} + B_{24} + B_{13})$	$\lvert m_1 + m_2 + m_3 - m_4\rvert$
$\frac{1}{2}(C_{23} + C_{14} - B_{24} - B_{13})$	$\lvert m_1 + m_2 - m_3 + m_4\rvert$
$\frac{1}{2}(C_{23} - C_{14} + B_{24} - B_{13})$	$\lvert m_1 - m_2 + m_3 + m_4\rvert$
B_{14}	$\lvert m_1 + m_2 + m_3 + m_4\rvert$
B_{23}	$\lvert m_1 + m_2 - m_3 - m_4\rvert$
C_{13}	$\lvert m_1 - m_2 + m_3 - m_4\rvert$
C_{24}	$\lvert m_1 - m_2 - m_3 + m_4\rvert$
$\frac{1}{\sqrt{2}}(C_{12} + C_{34})$	$\lvert m_3 - m_4\rvert$
$\frac{1}{\sqrt{2}}(C_{12} - C_{34})$	$\lvert m_1 - m_2\rvert$
$\frac{1}{\sqrt{2}}(A_{13} + A_{24})$	$\lvert m_3 + m_4\rvert$
$\frac{1}{\sqrt{2}}(A_{13} - A_{24})$	$\lvert m_1 + m_2\rvert$
$\frac{1}{\sqrt{2}}(A_{14} + A_{23})$	$\lvert m_1 - m_2\rvert$
$\frac{1}{\sqrt{2}}(A_{14} - A_{23})$	$\lvert m_3 - m_4\rvert$
$\frac{1}{\sqrt{2}}(B_{12} + B_{34})$	$\lvert m_3 + m_4\rvert$
$\frac{1}{\sqrt{2}}(B_{12} - B_{34})$	$\lvert m_1 + m_2\rvert$

Field $A^{(5)}_{2,A}$	Mass
$R^{(5)}_{2;\,1}$	0
$\tilde{R}^{(5)}_{2;\,1}$	0
$\frac{1}{\sqrt{2}}\big(C^{(5)}_2 + \tilde{C}^{(5)}_2\big)$	$\|m_1 + m_2\|$
$\frac{1}{\sqrt{2}}\big(C^{(5)}_2 - \tilde{C}^{(5)}_2\big)$	$\|m_3 + m_4\|$
$\frac{1}{\sqrt{2}}\big(B^{(5)}_2 + \tilde{B}^{(5)}_2\big)$	$\|m_1 - m_2\|$
$\frac{1}{\sqrt{2}}\big(B^{(5)}_2 - \tilde{B}^{(5)}_2\big)$	$\|m_3 - m_4\|$
$\frac{1}{\sqrt{2}}\big(R^{(5)}_{2;\,2} + \tilde{R}^{(5)}_{2;\,2}\big)$	$\|m_1 + m_2\|$
$\frac{1}{\sqrt{2}}\big(R^{(5)}_{2;\,2} - \tilde{R}^{(5)}_{2;\,2}\big)$	$\|m_3 + m_4\|$
$\frac{1}{\sqrt{2}}\big(R^{(5)}_{2;\,3} + \tilde{R}^{(5)}_{2;\,3}\big)$	$\|m_1 - m_2\|$
$\frac{1}{\sqrt{2}}\big(R^{(5)}_{2;\,3} - \tilde{R}^{(5)}_{2;\,3}\big)$	$\|m_3 - m_4\|$

Chapter 4

D1/D5-branes and freely-acting orbifolds

Before we can investigate the black holes and branes, we first will consider the orbifolds themselves. This section will start by discussing the orbifold action on both the bosonic and fermionic fields. These orbifold models show a string-theoretic interpretation of the Scherk-Schwarz reductions discussed in chapter 3. Not all of those reductions have an orbifold description, however. When we consider twists lying in the perturbative symmetry T-duality, we can compute stringy quantum corrections. In the more general U-duality reductions, the counterparts are generalized orbifolds that quotient by a non-perturbative symmetry. For the remainder of this work, we will solely consider the T-duality twists.

The corresponding orbifolds always have the target spaces of the form $\mathbb{R}^{1,4} \times (S^1 \times T^4)/\mathbb{Z}_p$. As we explain later in more detail, the orbifold action works, in general, as an asymmetric rotation on the four-torus and as a shift on the circle. Although we will later consider mostly symmetric orbifolds, we take a more widespread view in the remainder of this section.

4.1 T-duality twists

We consider an orbifolding of the background $\mathbb{R}^{1,4} \times S^1 \times T^4$, in the T-duality group $\mathrm{SO}(4,4,\mathbb{Z})$ of the four-torus. The compact subgroup $\mathrm{SO}(4) \times \mathrm{SO}(4) \cong \mathrm{SU}(2)^4$ works as asymmetric rotations on the worldsheet coordinates in the torus directions, and we will use these rotations to construct asymmetric orbifolds. The maximal torus of this group is $\mathrm{U}(1)^4$, so we can rotate in four independent directions. We embed these rotations in the T-duality group as

$$\mathrm{SO}(2)^4 \subset \mathrm{SU}(2)^4 \cong \mathrm{SO}(4) \times \mathrm{SO}(4) \subset \mathrm{SO}(4,4)\,. \tag{4.1}$$

By this we mean that we choose a single rotation parameter in an SO(2) subgroup of each of the SU(2) factors of the compact T-duality subgroup. We denote these parameters by m_i, $i = 1, \ldots, 4$.

In string theory, the T-duality group is quantized, which constrains the possible values that the parameters m_i can take. In order to be an element of the quantized T-duality group, an SO(4, 4) matrix is required to be integer valued in the appropriate frame. This frame is the one where the group preserves the metric

$$\tau = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \tag{4.2}$$

because this is how the group works on the integer valued lattice of winding and momentum numbers. As said before, we are interested in asymmetric rotations on the torus coordinates. To this end, we restrict ourselves to T-duality elements that are or are conjugate to rotations. That is, elements $\mathcal{M} \in \mathrm{SO}(4, 4, \mathbb{Z})$ that can be written as

$$\mathcal{M} = g\tilde{\mathcal{M}}g^{-1}\,; \qquad \tilde{\mathcal{M}} \in \mathrm{SO}(4) \times \mathrm{SO}(4)\,, \quad g \in \mathrm{SO}(4, 4)\,. \tag{4.3}$$

Note that the conjugation matrix g is an element of the continuous group. This conjugation will manifest as a field redefinition, and will not have any physical implications. The question now is which T-duality elements satisfy these conditions, and what the angles of the conjugate rotation matrices are. This essentially gives us the allowed values for the rotation parameters m_i.

Solving in general for all possible T-duality elements that are conjugate to a rotation is a very difficult problem, that we will not attempt to solve here. Instead, we consider a subgroup for which results are known in the literature, namely

$$\mathrm{SL}(2)^4 \cong \mathrm{SO}(2, 2) \times \mathrm{SO}(2, 2) \subset \mathrm{SO}(4, 4)\,. \tag{4.4}$$

In [47, 78] all conjugations of the type

$$\mathcal{M} = g\tilde{\mathcal{M}}g^{-1}\,; \qquad \mathcal{M} \in \mathrm{SL}(2, \mathbb{Z})\,, \quad \tilde{\mathcal{M}} \in \mathrm{SO}(2)\,, \quad g \in \mathrm{SL}(2) \tag{4.5}$$

have been worked out. It was found that all the possibilities for $\tilde{\mathcal{M}}$ satisfying such a conjugation are rotation matrices over angles $\alpha \in \left\{0, \pm\frac{\pi}{3}, \pm\frac{\pi}{2}, \pm\frac{2\pi}{3}, \pi\right\}$. Therefore rotations in the SO(2) subgroups of the SL(2)'s in (4.4) over one of these angles can, possibly via a conjugation, be embedded in the quantized T-duality group $\mathrm{SO}(4, 4, \mathbb{Z})$. Such rotations would then generate a $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$ or $\mathbb{Z}_6$ subgroup of SL(2). Again, we would like to stress that these are not necessarily all possible rotations that can be conjugated to integer valued elements of the T-duality group, but these are the ones that we use for the purposes of this work.

It is tempting to assume that each of the four $\mathrm{SO}(2) \subset \mathrm{SU}(2)$ subgroups in (4.1) and each of the $\mathrm{SO}(2) \subset \mathrm{SL}(2)$ subgroups in (4.4) correspond with each other one to one. This would imply that the parameters m_i are quantized precisely to the values given in the paragraph above (recall that the m_i's are the rotation parameters in the SU(2) subgroups of the T-duality group). Unfortunately, the group theory doesn't work out like that. Instead there is mixing between the SO(2)'s. A careful analysis of how the rotations in both of the subgroups (4.1) and (4.4) are embedded in SO(4, 4) yields that the linear combinations

$$\begin{aligned} &\tfrac{1}{2}(m_1 + m_2 + m_3 + m_4)\,, \qquad && \tfrac{1}{2}(m_1 + m_2 - m_3 - m_4)\,, \\ &\tfrac{1}{2}(m_1 - m_2 + m_3 - m_4)\,, && \tfrac{1}{2}(m_1 - m_2 - m_3 + m_4)\,, \end{aligned} \tag{4.6}$$

are the ones that rotate in the SO(2) subgroups of the SL(2)'s in (4.4). Therefore these linear combinations are required to take one of the values $\{0, \pm\frac{\pi}{3}, \pm\frac{\pi}{2}, \pm\frac{2\pi}{3}, \pi\}$ in order for the rotation to be a part of the quantized T-duality group.

Now, by taking appropriate sums of the linear combinations in (4.6), we find that the four mass parameters can be written as

$$m_i = \tfrac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\,, \tag{4.7}$$

where the α's take one of the values $\{0, \pm\frac{\pi}{3}, \pm\frac{\pi}{2}, \pm\frac{2\pi}{3}, \pi\}$. It will often be useful to rewrite these parameters as

$$m_i = \frac{2\pi N_i}{p}\,. \tag{4.8}$$

Here the N_i are integers, and p is the smallest positive integer such that all four m_i can be written like this. This integer p is the rank of the T-duality element, and therefore the rank of the $\mathbb{Z}_p$ orbifold that we construct with it. Note from (4.7) and (4.8) that the quantization of the α's allows for the values $p \in \{2, 3, 4, 6, 8, 12, 24\}$.

4.2 Orbifold constructions

We are now ready to discuss the action of the orbifold on the worldsheet coordinates. We construct orbifold backgrounds of the type

$$\mathbb{R}^{1,4} \times \big(S^1 \times T^4\big)/\mathbb{Z}_p\,, \tag{4.9}$$

where the $\mathbb{Z}_p$ group works as an asymmetric rotation (a T-duality) on the torus, and as a shift on the circle. Notationally, we split up the bosonic coordinates as $X^M \to (\hat{X}^{\hat{\mu}}, Y^m) \to (X^\mu, Z, Y^m)$, where Y^m $(m = 1, \ldots, 4)$ are the T^4 coordinates, Z is the circle coordinate, X^μ $(\mu = 0, \ldots, 4)$ are the $\mathbb{R}^{1,4}$ coordinates, and $\hat{X}^{\hat{\mu}}$ $(\hat{\mu} = 0, \ldots, 5)$ are the coordinates on $\mathbb{R}^{1,4} \times S^1$. We often work in complex coordinates on the torus, which we denote by $W^i = \frac{1}{\sqrt{2}}(Y^{2i-1} + iY^{2i})$ with $i = 1, 2$. Furthermore, we split up the left and right-moving parts of these coordinates as

$$W^i(\tau, \sigma) = W^i_{\mathscr{L}}(\tau + \sigma) + W^i_{\mathscr{R}}(\tau - \sigma)\,. \tag{4.10}$$

We denote the oscillators of all of these bosonic coordinates by $\tilde{\alpha}^M_n$ and α^M_n where the tilde indicates a left-mover, and we use different indices ($\hat{\mu}$, μ, z, m or i) to distinguish the submanifolds in (4.9). The fermionic modes are denoted by $\tilde{b}^M_n$ and b^M_n with a similar index structure. In the case of complex modes we use a bar to denote the complex conjugate.

Now that we have set up our notation, we are ready to present the orbifold action. It works on the bosonic torus coordinates with asymmetric rotations

$$\begin{aligned} W^1_{\mathscr{L}} &\to e^{i(m_1+m_3)}\, W^1_{\mathscr{L}}\,, \\ W^2_{\mathscr{L}} &\to e^{i(m_1-m_3)}\, W^2_{\mathscr{L}}\,, \\ W^1_{\mathscr{R}} &\to e^{i(m_2+m_4)}\, W^1_{\mathscr{R}}\,, \\ W^2_{\mathscr{R}} &\to e^{i(m_2-m_4)}\, W^2_{\mathscr{R}}\,, \end{aligned} \tag{4.11}$$

and with the same action on the fermionic torus coordinates. Note that the m_i here are chosen such that each parametrizes a rotation in an $\mathrm{SO}(2) \subset \mathrm{SU}(2)$ subgroup of the T-duality group, as was discussed in the previous section. Furthermore, all mass parameters can be written as $m_i = 2\pi N_i/p$ with $N_i \in \mathbb{Z}$ and p the rank of the orbifold.

The rotations on the torus are accompanied by a shift along the circle coordinate

$$Z \;\to\; Z + 2\pi r/p\,, \tag{4.12}$$

which makes the orbifold freely-acting. Here r is the circle radius ($Z \sim Z + 2\pi r$). Due to this shift states that carry momentum in the Z-direction obtain a phase $e^{2\pi i n/p}$ under the orbifold action, where n is the momentum number of the state.

We would like to point out that these asymmetric orbifold constructions, as they are presented in this section, are not modular invariant in general. To restore modular invariance, one needs to include a shift on the T-dual circle in the orbifold action. Quotienting out this shift would introduce a phase dependent on the winding number on the S^1, similar to the phase depending on the momentum number that we discussed above. More information can be found in [41,52]. As we do not include winding modes in this chapter, we do not work out the shift on the T-dual circle and its consequences in detail.

4.3 Field content

We construct explicitly part of the field content arising from these orbifolds. We are particularly interested in the lowest-excited states in the closed string spectrum, as these should match with the spectrum found in the supergravity calculation. By matching these spectra, we perform a non-trivial consistency check, showing that we have the correct string theory lift of the reductions from chapter 3. In addition, we are interested in certain open string excitations, as they are associated with the field content of the microscopic CFT dual to the near-horizon geometry of our setup.

4.3.1 Closed string spectrum

First we work out a part of the closed string spectrum that arises from our orbifold constructions. In order for strings to close in our geometry, they need to satisfy the boundary conditions

$$\begin{aligned} X^\mu(\tau,\sigma+2\pi) &= X^\mu(\tau,\sigma)\,, \\ Z(\tau,\sigma+2\pi) &= Z(\tau,\sigma) + 2\pi r\,(w+k/p)\,, \\ W^1_{\mathscr{L}}(\tau,\sigma+2\pi) &= \left(e^{i(m_1+m_3)}\right)^k W^1_{\mathscr{L}}(\tau,\sigma)\,, \\ W^2_{\mathscr{L}}(\tau,\sigma+2\pi) &= \left(e^{i(m_1-m_3)}\right)^k W^2_{\mathscr{L}}(\tau,\sigma)\,, \\ W^1_{\mathscr{R}}(\tau,\sigma+2\pi) &= \left(e^{i(m_2+m_4)}\right)^k W^1_{\mathscr{R}}(\tau,\sigma)\,, \\ W^2_{\mathscr{R}}(\tau,\sigma+2\pi) &= \left(e^{i(m_2-m_4)}\right)^k W^2_{\mathscr{R}}(\tau,\sigma)\,. \end{aligned} \tag{4.13}$$

Here $k = 0,\ldots,p-1$ is an integer that distinguishes between the various sectors and $w \in \mathbb{Z}$ is the winding number along the S^1 (we omit winding modes on the torus). We have the untwisted sector for $k = 0$, and $p-1$ twisted sectors for the other values of k in which case the string closes only under application of the orbifold action.

We focus on the untwisted sector, i.e. the sector with $k = 0$ boundary conditions. Furthermore, we only focus on the lowest excited states. That is, the states that are massless without the addition of momentum and/or winding modes. The reason for this is that our goal here is not to present the full orbifold spectrum, but rather the part that appears as well in supergravity. By reproducing the spectrum found from Scherk-Schwarz reduction in chapter 3, we reaffirm our assertion that the orbifold constructions in this work are the correct string theory uplift.

In this sector the Neveu-Schwarz vacuum is a scalar and the Ramond vacuum is a spinor in all target space directions. We denote the NS-vacua by $|0\rangle_{\mathscr{L}/\mathscr{R}}$ and the R-vacua by $|s_1, s_2, s_3, s_4\rangle_{\mathscr{L}/\mathscr{R}}$ with $s_k = \pm\frac{1}{2}$. The subscript $\mathscr{L}/\mathscr{R}$ is used to distinguish the left and the right-moving vacua. We choose the GSO projection in such a way that both R-vacua have to satisfy

$$\sum_{k=1}^{4} s_k \in 2\mathbb{Z} \,. \tag{4.14}$$

The NS-vacua are invariant under the orbifold action, and the action on the R-vacua depends on the values of s_3, s_4 (these give the spin in the two complex direction of the T^4). Because the R-vacua are 10d spinors, we know how they transform under rotations, so in particular under the orbifold action. In general we have

$$|s_1, s_2, s_3, s_4\rangle \to \exp\left(2\pi i \sum_{k=1}^{4} u_k\, S_k\right) |s_1, s_2, s_3, s_4\rangle = e^{2\pi i\, \vec{u}\cdot\vec{s}}\, |s_1, s_2, s_3, s_4\rangle \,, \tag{4.15}$$

where the $S_k = J_{2k-1,2k}$ are the Cartan generators of the little group SO(8) with eigenvalues s_k. The u_k denotes a rotation in the $2k-1$ and $2k$ directions over an angle $2\pi u_k$. Whenever the rotation works asymmetrically on left and right-movers, the formula above applies to spinors in each sector individually. In this case we use v_k and w_k for the left and right-moving rotation parameters respectively. Using this notation, we read off from (4.11) that our orbifold action is a rotation with

$$\begin{aligned} v_3 &= \frac{m_1 + m_3}{2\pi}\,, \qquad & w_3 &= \frac{m_2 + m_4}{2\pi}\,, \\ v_4 &= \frac{m_1 - m_3}{2\pi}\,, \qquad & w_4 &= \frac{m_2 - m_4}{2\pi}\,, \end{aligned} \tag{4.16}$$

and the other rotation parameters equal to zero. We find that the orbifold action on an R-vacuum depends on the values of s_3 and s_4, so we invent the following notation for the possible values of these spins:

$$\begin{aligned} |a_1\rangle_{\mathscr{L}/\mathscr{R}} &= \left|s_1, s_1, \tfrac{1}{2}, \tfrac{1}{2}\right\rangle_{\mathscr{L}/\mathscr{R}}\,, \\ |a_2\rangle_{\mathscr{L}/\mathscr{R}} &= \left|s_1, s_1, -\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle_{\mathscr{L}/\mathscr{R}}\,, \\ |a_3\rangle_{\mathscr{L}/\mathscr{R}} &= \left|s_1, -s_1, \tfrac{1}{2}, -\tfrac{1}{2}\right\rangle_{\mathscr{L}/\mathscr{R}}\,, \\ |a_4\rangle_{\mathscr{L}/\mathscr{R}} &= \left|s_1, -s_1, -\tfrac{1}{2}, \tfrac{1}{2}\right\rangle_{\mathscr{L}/\mathscr{R}}\,. \end{aligned} \tag{4.17}$$

Here the relative sign between s_1 and s_2 is fixed by the GSO projection. The orbifold action on each of these is

$$\begin{aligned}
|a_1\rangle_{\mathscr{L}} &\to e^{im_1}\,|a_1\rangle_{\mathscr{L}}\,, & |a_1\rangle_{\mathscr{R}} &\to e^{im_2}\,|a_1\rangle_{\mathscr{R}}\,,\\
|a_2\rangle_{\mathscr{L}} &\to e^{-im_1}\,|a_2\rangle_{\mathscr{L}}\,, & |a_2\rangle_{\mathscr{R}} &\to e^{-im_2}\,|a_2\rangle_{\mathscr{R}}\,,\\
|a_3\rangle_{\mathscr{L}} &\to e^{im_3}\,|a_3\rangle_{\mathscr{L}}\,, & |a_3\rangle_{\mathscr{R}} &\to e^{im_4}\,|a_3\rangle_{\mathscr{R}}\,,\\
|a_4\rangle_{\mathscr{L}} &\to e^{-im_3}\,|a_4\rangle_{\mathscr{L}}\,, & |a_4\rangle_{\mathscr{R}} &\to e^{-im_4}\,|a_4\rangle_{\mathscr{R}}\,.
\end{aligned} \tag{4.18}$$

Having found all the actions on the vacua, we can discuss the resulting spectrum.

First, we present in Table 4.1 the NS and R-sector states of the lowest level that survive the GSO projection (all of these are massless in the absence of momentum and/or winding modes). We write down general states that appear both in a left-moving and in a right-moving version, and we write down the orbifold charges that both of these versions carry. Furthermore, we table the representations of these states under both the massless little group SO(3) ≅ SU(2) and the massive little group SO(4) ≅ SU(2) × SU(2) in five dimensions. The latter is important when adding momenta or windings such that the state becomes massive.

Sector	State	$\mathscr{L}$ charge	$\mathscr{R}$ charge	SO(3) rep	SO(4) rep
NS	$b^{\hat\mu}_{-1/2}\,\|0\rangle$	1	1	$\mathbf{3}+\mathbf{1}$	$(\mathbf{2},\mathbf{2})$
	$b^{i}_{-1/2}\,\|0\rangle$	$e^{i(m_1\pm m_3)}$	$e^{i(m_2\pm m_4)}$	$2\times\mathbf{1}$	$2\times(\mathbf{1},\mathbf{1})$
	$\bar b^{i}_{-1/2}\,\|0\rangle$	$e^{-i(m_1\pm m_3)}$	$e^{-i(m_2\pm m_4)}$	$2\times\mathbf{1}$	$2\times(\mathbf{1},\mathbf{1})$
R	$\|a_{1,2}\rangle$	$e^{\pm im_1}$	$e^{\pm im_2}$	$2\times\mathbf{2}$	$2\times(\mathbf{2},\mathbf{1})$
	$\|a_{3,4}\rangle$	$e^{\pm im_3}$	$e^{\pm im_4}$	$2\times\mathbf{2}$	$2\times(\mathbf{1},\mathbf{2})$

Table 4.1: *Here we write down all states that are massless in the absence of momentum and/or winding modes, including their charges under the orbifold action and their representations under the massless and massive little groups in 5d. We write down general states that appear both left-moving and right-moving. The tildes on the oscillators in the left-moving sector, and the subscripts $\mathscr{L}$ and $\mathscr{R}$ on the vacua are omitted.*

We construct string states by tensoring the left and right-moving states from Table 4.1. In general such states will carry a non-trivial orbifold charge, which means that they are projected out of the orbifold spectrum. This can be fixed by adding momentum along the circle to the state, as we will see in a minute. In Table 4.2 we give the spectrum of lowest excited string states, including their orbifold charge and little group representations. For the construction of this table, we used the rules

$$\mathbf{3} \times \mathbf{3} = \mathbf{5} + \mathbf{3} + \mathbf{1}\,, \qquad \mathbf{2} \times \mathbf{2} = \mathbf{3} + \mathbf{1}\,, \qquad \mathbf{3} \times \mathbf{2} = \mathbf{4} + \mathbf{2}\,, \tag{4.19}$$

for tensoring SU(2) representations.

We can now find the field content of our orbifold construction from this table. As an example, take the string state $|a_1\rangle_{\mathscr{L}} \times b^{\hat{\mu}}_{-1/2}\,|0\rangle_{\mathscr{R}}$, which has orbifold charge e^{im_1}. First recall that we can always write $m_1 = 2\pi N_1/p$ where N_1 is an integer and p is the rank of the orbifold. To make the state invariant under the orbifold action, we add momentum along the S^1. States with momentum on the circle obtain a phase $e^{2\pi i n/p}$ with n the number of modes. If we now choose $n = -N_1$, this phase becomes $e^{-2\pi i N_1/p}$ which cancels exactly against the phase that the string state had before the addition of momentum. In other words, the state $|a_1; -N_1, 0\rangle_{\mathscr{L}} \times b^{\hat{\mu}}_{-1/2}\,|0\rangle_{\mathscr{R}}$ is invariant under the orbifold action and therefore survives in the spectrum. Here we use the notational convention to denote the momentum and winding numbers on the S^1 as $|\quad; n, w\rangle$ on the left-moving vacuum.

At this point, we would like to point out that for the states in Table 4.2 it is always possible to find an integer-valued momentum number that cancels the phase due to the orbifold action. All mass parameters can be written as $m_i = 2\pi N_i/p$ with $N_i \in \mathbb{Z}$; any sum or difference of mass parameters can thus also be written as $2\pi/p$ times an integer. If we take this integer with the sign flipped as the momentum number, the total phase cancels.

Next, we can use the little group representations in Table 4.2 to determine what kind of fields the spectrum consists of. We return to the example state $|a_1; -N_1, 0\rangle_{\mathscr{L}} \times b^{\hat{\mu}}_{-1/2}\,|0\rangle_{\mathscr{R}}$. Due to the addition of momentum, the state has become massive with mass $|N_1/r|$. From the table we then read off the representation as $(\mathbf{3}, \mathbf{2}) + (\mathbf{1}, \mathbf{2})$, i.e. it corresponds to a massive gravitino and a massive dilatino. For convenience, we table the (massless and massive) representations that correspond to various supergravity fields in five dimensions in Table 4.3.

We can rewrite the mass $|N_1/r|$ slightly in order to make contact with the Scherk-Schwarz supergravity spectrum. We know that $N_1 = p\,m_1/2\pi$, and we know that the radius of the orbifold circle r and the radius of the Scherk-Schwarz circle R are related by $r = pR$. The mass of the state is therefore equal to $|m_1/2\pi R|$. Masses of this form are precisely what was found in chapter 3, we refer in particular to section 3.2.3.

This systematic approach can be used to construct the entire field content coming from the lowest excited string states. Each of the states in Table 4.2 gives fields whose mass can be read off from their orbifold charge (it is always the absolute value

Sector	State	Orbifold charge	SO(3) rep	SO(4) rep
NS-NS	$\tilde{b}^{\hat{\mu}}_{-1/2}\lvert 0\rangle_{\mathscr{L}} \times b^{\hat{\nu}}_{-1/2}\lvert 0\rangle_{\mathscr{R}}$	1	$\mathbf{5}+3\times\mathbf{3}+2\times\mathbf{1}$	$(\mathbf{3}+\mathbf{1},\mathbf{3}+\mathbf{1})$
	$\tilde{b}^{\hat{\mu}}_{-1/2}\lvert 0\rangle_{\mathscr{L}} \times b^{i}_{-1/2}\lvert 0\rangle_{\mathscr{R}}$	$e^{i(m_2\pm m_4)}$	$2\times\mathbf{3}+2\times\mathbf{1}$	$2\times(\mathbf{2},\mathbf{2})$
	$\tilde{b}^{\hat{\mu}}_{-1/2}\lvert 0\rangle_{\mathscr{L}} \times \bar{b}^{i}_{-1/2}\lvert 0\rangle_{\mathscr{R}}$	$e^{-i(m_2\pm m_4)}$	$2\times\mathbf{3}+2\times\mathbf{1}$	$2\times(\mathbf{2},\mathbf{2})$
	$\tilde{b}^{i}_{-1/2}\lvert 0\rangle_{\mathscr{L}} \times b^{\hat{\mu}}_{-1/2}\lvert 0\rangle_{\mathscr{R}}$	$e^{i(m_1\pm m_3)}$	$2\times\mathbf{3}+2\times\mathbf{1}$	$2\times(\mathbf{2},\mathbf{2})$
	$\bar{\tilde{b}}^{i}_{-1/2}\lvert 0\rangle_{\mathscr{L}} \times b^{\hat{\mu}}_{-1/2}\lvert 0\rangle_{\mathscr{R}}$	$e^{-i(m_1\pm m_3)}$	$2\times\mathbf{3}+2\times\mathbf{1}$	$2\times(\mathbf{2},\mathbf{2})$
	$\tilde{b}^{i}_{-1/2}\lvert 0\rangle_{\mathscr{L}} \times b^{j}_{-1/2}\lvert 0\rangle_{\mathscr{R}}$	$e^{i(m_1\pm m_3)+i(m_2\pm m_4)}$	$4\times\mathbf{1}$	$4\times(\mathbf{1},\mathbf{1})$
	$\tilde{b}^{i}_{-1/2}\lvert 0\rangle_{\mathscr{L}} \times \bar{b}^{j}_{-1/2}\lvert 0\rangle_{\mathscr{R}}$	$e^{i(m_1\pm m_3)-i(m_2\pm m_4)}$	$4\times\mathbf{1}$	$4\times(\mathbf{1},\mathbf{1})$
	$\bar{\tilde{b}}^{i}_{-1/2}\lvert 0\rangle_{\mathscr{L}} \times b^{j}_{-1/2}\lvert 0\rangle_{\mathscr{R}}$	$e^{-i(m_1\pm m_3)+i(m_2\pm m_4)}$	$4\times\mathbf{1}$	$4\times(\mathbf{1},\mathbf{1})$
	$\bar{\tilde{b}}^{i}_{-1/2}\lvert 0\rangle_{\mathscr{L}} \times \bar{b}^{j}_{-1/2}\lvert 0\rangle_{\mathscr{R}}$	$e^{-i(m_1\pm m_3)-i(m_2\pm m_4)}$	$4\times\mathbf{1}$	$4\times(\mathbf{1},\mathbf{1})$
R-R	$\lvert a_{1,2}\rangle_{\mathscr{L}} \times \lvert a_{1,2}\rangle_{\mathscr{R}}$	$e^{\pm im_1\pm im_2}$	$4\times\mathbf{3}+4\times\mathbf{1}$	$4\times(\mathbf{3}+\mathbf{1},\mathbf{1})$
	$\lvert a_{1,2}\rangle_{\mathscr{L}} \times \lvert a_{3,4}\rangle_{\mathscr{R}}$	$e^{\pm im_1\pm im_4}$	$4\times\mathbf{3}+4\times\mathbf{1}$	$4\times(\mathbf{2},\mathbf{2})$
	$\lvert a_{3,4}\rangle_{\mathscr{L}} \times \lvert a_{1,2}\rangle_{\mathscr{R}}$	$e^{\pm im_3\pm im_2}$	$4\times\mathbf{3}+4\times\mathbf{1}$	$4\times(\mathbf{2},\mathbf{2})$
	$\lvert a_{3,4}\rangle_{\mathscr{L}} \times \lvert a_{3,4}\rangle_{\mathscr{R}}$	$e^{\pm im_3\pm im_4}$	$4\times\mathbf{3}+4\times\mathbf{1}$	$4\times(\mathbf{1},\mathbf{3}+\mathbf{1})$
NS-R	$\tilde{b}^{\hat{\mu}}_{-1/2}\lvert 0\rangle_{\mathscr{L}} \times \lvert a_{1,2}\rangle_{\mathscr{R}}$	$e^{\pm im_2}$	$2\times\mathbf{4}+4\times\mathbf{2}$	$2\times(\mathbf{3}+\mathbf{1},\mathbf{2})$
	$\tilde{b}^{\hat{\mu}}_{-1/2}\lvert 0\rangle_{\mathscr{L}} \times \lvert a_{3,4}\rangle_{\mathscr{R}}$	$e^{\pm im_4}$	$2\times\mathbf{4}+4\times\mathbf{2}$	$2\times(\mathbf{2},\mathbf{3}+\mathbf{1})$
	$\tilde{b}^{i}_{-1/2}\lvert 0\rangle_{\mathscr{L}} \times \lvert a_{1,2}\rangle_{\mathscr{R}}$	$e^{i(m_1\pm m_3)\pm im_2}$	$4\times\mathbf{2}$	$4\times(\mathbf{2},\mathbf{1})$
	$\tilde{b}^{i}_{-1/2}\lvert 0\rangle_{\mathscr{L}} \times \lvert a_{3,4}\rangle_{\mathscr{R}}$	$e^{i(m_1\pm m_3)\pm im_4}$	$4\times\mathbf{2}$	$4\times(\mathbf{1},\mathbf{2})$
	$\bar{\tilde{b}}^{i}_{-1/2}\lvert 0\rangle_{\mathscr{L}} \times \lvert a_{1,2}\rangle_{\mathscr{R}}$	$e^{-i(m_1\pm m_3)\pm im_2}$	$4\times\mathbf{2}$	$4\times(\mathbf{2},\mathbf{1})$
	$\bar{\tilde{b}}^{i}_{-1/2}\lvert 0\rangle_{\mathscr{L}} \times \lvert a_{3,4}\rangle_{\mathscr{R}}$	$e^{-i(m_1\pm m_3)\pm im_4}$	$4\times\mathbf{2}$	$4\times(\mathbf{1},\mathbf{2})$
R-NS	$\lvert a_{1,2}\rangle_{\mathscr{L}} \times b^{\hat{\mu}}_{-1/2}\lvert 0\rangle_{\mathscr{R}}$	$e^{\pm im_1}$	$2\times\mathbf{4}+4\times\mathbf{2}$	$2\times(\mathbf{3}+\mathbf{1},\mathbf{2})$
	$\lvert a_{3,4}\rangle_{\mathscr{L}} \times b^{\hat{\mu}}_{-1/2}\lvert 0\rangle_{\mathscr{R}}$	$e^{\pm im_3}$	$2\times\mathbf{4}+4\times\mathbf{2}$	$2\times(\mathbf{2},\mathbf{3}+\mathbf{1})$
	$\lvert a_{1,2}\rangle_{\mathscr{L}} \times b^{i}_{-1/2}\lvert 0\rangle_{\mathscr{R}}$	$e^{\pm im_1+i(m_2\pm m_4)}$	$4\times\mathbf{2}$	$4\times(\mathbf{2},\mathbf{1})$
	$\lvert a_{3,4}\rangle_{\mathscr{L}} \times b^{i}_{-1/2}\lvert 0\rangle_{\mathscr{R}}$	$e^{\pm im_3+i(m_2\pm m_4)}$	$4\times\mathbf{2}$	$4\times(\mathbf{1},\mathbf{2})$
	$\lvert a_{1,2}\rangle_{\mathscr{L}} \times \bar{b}^{i}_{-1/2}\lvert 0\rangle_{\mathscr{R}}$	$e^{\pm im_1-i(m_2\pm m_4)}$	$4\times\mathbf{2}$	$4\times(\mathbf{2},\mathbf{1})$
	$\lvert a_{3,4}\rangle_{\mathscr{L}} \times \bar{b}^{i}_{-1/2}\lvert 0\rangle_{\mathscr{R}}$	$e^{\pm im_3-i(m_2\pm m_4)}$	$4\times\mathbf{2}$	$4\times(\mathbf{1},\mathbf{2})$

Table 4.2: *The spectrum of lowest excited string states including their orbifold charge and representations under the massless and massive little group in 5d.*

of this linear combination of m_i's times $p/2\pi r$). The type of fields that this state gives can then be found from its massless or massive little group representation, depending on whether the aforementioned mass is zero or not. In this way, the

Massless field	SO(3) rep
$g_{\mu\nu}$	$\mathbf{5}$
ψ_μ	$\mathbf{4}$
A_μ	$\mathbf{3}$
χ	$\mathbf{2}$
ϕ	$\mathbf{1}$

Massive field	SO(4) rep
$B^+_{\mu\nu}$ / $B^-_{\mu\nu}$	$(\mathbf{3},\mathbf{1})$ / $(\mathbf{1},\mathbf{3})$
ψ^+_μ / ψ^-_μ	$(\mathbf{2},\mathbf{3})$ / $(\mathbf{3},\mathbf{2})$
A_μ	$(\mathbf{2},\mathbf{2})$
χ^+ / χ^-	$(\mathbf{1},\mathbf{2})$ / $(\mathbf{2},\mathbf{1})$
ϕ	$(\mathbf{1},\mathbf{1})$

Table 4.3: *Here we show the various massless and massive 5d fields and their representations under the appropriate little group.*

entire supergravity spectrum from chapter 3 can be reproduced.

As an aside, we mention that it is not difficult to extend this procedure to include the Kaluza-Klein towers on the circle. Simply add $p\mathbb{Z}$ momentum modes to the ones that were added following the procedure above. This addition doesn't change the orbifold charge, so all of these states survive as well. The masses shift by $p\mathbb{Z}/r = \mathbb{Z}/R$, e.g. the masses of the KK-tower on our example state become $|m_1/2\pi R + \mathbb{Z}/R|$. Again, this agrees with the supergravity calculation.

4.3.2 Open string spectrum

In this section, we consider open string states in our orbifold construction. In particular, we consider those that end on the D1 and D5-branes that we use to build black holes. We expect that both types of branes only survive in symmetric orbifolds (see section 4.4.1 for clarification on this). Therefore, we restrict ourselves to these orbifolds for the remainder of this section, i.e. we put $m_1 = m_2$ and $m_3 = m_4$.

The moding of open string oscillators depends on its boundary conditions: Neumann ($\partial_\sigma X = 0$) or Dirichlet ($\delta X = 0$). If both ends of the string have the same type of boundary condition (NN or DD) the bosonic modes in that direction are integer-valued, while mixed boundary conditions (ND or DN) give half-integer valued modes. The fermionic oscillators in the Ramond sector follow the moding of the bosonic modes, and the fermionic modes in the NS sector carry the 'opposite' moding. We summarize the consequences of each type of boundary condition in the table below.

We are interested in open strings that end on the D1 and D5-branes in our setup. These lie in the time and S^1 directions, and in the time, S^1 and T^4 directions

Boundary conditions	$\mathscr{L}$–$\mathscr{R}$ relation	α_n	b_n (NS)	b_n (R)
NN	$\tilde{\alpha}_n = \alpha_n$	$n \in \mathbb{Z}$	$n \in \mathbb{Z} + \frac{1}{2}$	$n \in \mathbb{Z}$
DD	$\tilde{\alpha}_n = -\alpha_n$	$n \in \mathbb{Z}$	$n \in \mathbb{Z} + \frac{1}{2}$	$n \in \mathbb{Z}$
ND	$\tilde{\alpha}_n = \alpha_n$	$n \in \mathbb{Z} + \frac{1}{2}$	$n \in \mathbb{Z}$	$n \in \mathbb{Z} + \frac{1}{2}$
DN	$\tilde{\alpha}_n = -\alpha_n$	$n \in \mathbb{Z} + \frac{1}{2}$	$n \in \mathbb{Z}$	$n \in \mathbb{Z} + \frac{1}{2}$

respectively. Therefore all strings ending on these branes have NN boundary conditions in the time and S^1 directions, and DD boundary conditions in the $\mathbb{R}^4$ directions. We distinguish different types of strings by their boundary conditions in the torus directions:

$$1-1: \quad \text{DD on } T^4\,, \tag{4.20}$$

$$5-5: \quad \text{NN on } T^4\,, \tag{4.21}$$

$$1-5: \quad \text{DN on } T^4\,, \tag{4.22}$$

$$5-1: \quad \text{ND on } T^4\,. \tag{4.23}$$

It follows that the 1–1 and 5–5 strings ('pure' boundary conditions) have the same moding in all directions, namely integer-valued bosonic modes. The 'mixed' boundary conditions, those of the 1–5 and 5–1 strings, give rise to half-integer-valued bosonic modes in the torus directions and integer-valued ones in the other six directions.

In the limit where the torus is much smaller than the circle, the open string excitations on the brane stack describe an effectively two-dimensional field theory. This field theory is well studied in the literature (see e.g. [34, 79–83] and references therein) so we will mention here only some of the essential features that we will need for the purposes of this work. From the 1–1 and 5–5 strings we find vector and hypermultiplets[1] in the adjoint representation of the gauge group $\mathrm{U}(N_1) \times \mathrm{U}(N_5)$. From the mixed boundary condition strings (1–5 and 5–1) we find hypermultiplets in the bi-fundamental representation of the gauge group. We consider the setup where the branes sit at the same point in the transverse directions, i.e. we are on the Higgs branch. Here many of the adjoint multiplets become massive through the Higgs mechanism. Furthermore, by taking the IR limit we find a fixed point where the theory becomes an SCFT, which is the dual of the near-horizon limit of

[1]We follow the common convention in the literature to organize the field content in terms of 4D $\mathcal{N} = 2$ multiplets, in order to make contact with the familiar Coulomb / Higgs branch terminology.

our geometry. The field content of this SCFT originates from the bi-fundamental hypermultiplets that come from the 1–5 and 5–1 strings, and so we will study these in some more detail here.

Just like we did for the closed strings, we begin by investigating the vacuum structure. The spinorial degeneracy of the vacua comes from acting on a highest weight state with fermionic zero-modes b_0^M. For 1–5 and 5–1 strings we find such modes in both sectors. The NS-sector has fermionic zero-modes in the torus directions and therefore the vacuum is a spinor in these directions, while the R-vacuum is a spinor in the remaining $\mathbb{R}^{1,4}$ and S^1 directions. We write these vacua as

$$\begin{aligned} \text{NS}:\ & |s_3, s_4\rangle_{\text{NS}} \ , \\ \text{R}:\ & |s_1, s_2\rangle_{\text{R}} \ , \end{aligned} \tag{4.24}$$

where we use the same notation for the spin labels as in the closed string section. Both vacua are massless, so we don't need any more oscillators to construct massless states. The NS-vacuum will give the bosonic degrees of freedom in the 2d CFT, and the R-vacuum will give the fermionic ones.

These fields will obtain boundary conditions along the spatial circle of the CFT due to the orbifold action. This is because this circle is the one along which the orbifold works with the shift $z \to z + 2\pi r/p$. The boundary conditions follow directly from the phases that the vacua (4.24) obtain under the orbifold action, and these are computed in the same way as for the closed string states. It can easily be seen that the NS-vacua will obtain a phase $e^{\pm\, i m_{1,3}}$ around the circle (depending on how the GSO-projection is imposed), and that the R-vacua are periodic.

From this finding, we would expect that the SCFT dual to the near-horizon geometry of our setup contains twisted bosons and untwisted fermions. Confirming this statement by explicitly writing down the CFT is left for future research.

4.4 D-branes

We now turn our attention to D-branes and determine the properties of the branes that survive in the orbifolds. Firstly, we engage with the question of which branes survive the orbifold, and secondly, we discuss the supersymmetry that these branes preserve. Afterward, we investigate if our predictions of surviving branes coincide with our expectations from supergravity.

4.4.1 Preservation in the orbifold

Here we will argue for the existence of certain types of branes in our orbifold constructions based on the boundary conditions of open strings ending on these branes. The argument we use is that if the boundary conditions of an open string are invariant under the orbifold action, such a string is allowed in the orbifolded background, and therefore the brane that it ends on exists in the spectrum.

An open string ending on a D-brane has Neumann boundary conditions in the extended directions and Dirichlet boundary conditions in the pointlike directions of the brane. These boundary conditions give a relation between the left and right-moving oscillators, these are shown in the table in section 4.3.2.

Our main brane setup of interest for the construction of black holes is the D1-D5 system, so we focus on these branes. Let us start with a D5-brane that wraps the $S^1 \times T^4$. A string that ends with both ends on this brane has NN boundary conditions in the four torus directions. The left and right-moving oscillators are related to each other via

$$\tilde{\alpha}^i_n = \alpha^i_n \,, \qquad i = 1, 2 \,. \tag{4.25}$$

From the orbifold action (4.11), we know that these oscillators transform as

$$\tilde{\alpha}^i_n \;\to\; e^{i(m_1 \pm m_3)}\, \tilde{\alpha}^i_n \,, \qquad\qquad \alpha^i_n \;\to\; e^{i(m_2 \pm m_4)}\, \alpha^i_n \,, \tag{4.26}$$

where the $+$ signs corresponds to $i = 1$ and the $-$ signs to $i = 2$. It is easy to see that the boundary conditions (4.25) are only invariant under the orbifold action if $m_1 = m_2$ and $m_3 = m_4$. In other words, the D5-brane that wraps the $S^1 \times T^4$ only survives the orbifolding in the case of a symmetric orbifold.

Now on to a D1-brane that wraps just the S^1. A string ending on this brane has DD boundary conditions in the torus directions, which implies that

$$\tilde{\alpha}^i_n = -\alpha^i_n \,, \qquad i = 1, 2 \,. \tag{4.27}$$

We see that the same argument holds as for the D5-brane: the boundary conditions are invariant under the orbifold action only for $m_1 = m_2$ and $m_3 = m_4$, so the D1-brane also survives only in symmetric orbifolds. Consequently, the entire D1-D5 system survives in the symmetric orbifolds that we study. Because we want to study the field theory living on this brane stack, we will focus on symmetric orbifolds in the remainder of this work.

We can now study the dual D3-D3 systems without much extra effort. In order to study D3-branes wrapping $S^1 \times T^4$, we decompose the T^4 into $T^2 \times T^2$ where the first T^2 is parametrized by W^1 and the second T^2 by W^2. We now distinguish two types of D3-branes: those wrapping the S^1 and one of the two T^2's, and those wrapping the S^1 and a one-cycle in both of the T^2's. We start by studying a D3-brane of the first type, namely

							W^1		W^2	
	X^0	X^1	X^2	X^3	X^4	Z	Y^1	Y^2	Y^3	Y^4
D3	$-$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$-$	$-$	$-$	$\cdots$	$\cdots$

The boundary conditions for a string ending on this brane imply that the left and right-movers are related by

$$\tilde{\alpha}_n^{i=1} = \alpha_n^{i=1}\,, \qquad \tilde{\alpha}_n^{i=2} = -\alpha_n^{i=2}\,. \tag{4.28}$$

These boundary conditions are again invariant under the orbifold action only for $m_1 = m_2$ and $m_3 = m_4$, i.e. under the action corresponding to a symmetric orbifold.

Now we consider the second type of D3-brane. As an example we take the following setup

							W^1		W^2	
	X^0	X^1	X^2	X^3	X^4	Z	Y^1	Y^2	Y^3	Y^4
D3	$-$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$-$	$-$	$\cdots$	$-$	$\cdots$

A string ending on this D3-brane has boundary conditions that relate the left and right-moving oscillators on the torus via

$$\tilde{\alpha}_n^{m=1} = \alpha_n^{m=1}\,, \qquad \tilde{\alpha}_n^{m=2} = -\alpha_n^{m=2}\,, \qquad \tilde{\alpha}_n^{m=3} = \alpha_n^{m=3}\,, \qquad \tilde{\alpha}_n^{m=4} = -\alpha_n^{m=4}\,. \tag{4.29}$$

Note that here the indices $m = 1, \ldots, 4$ are the ones corresponding to the real torus coordinates Y^m. We can find the relations between the oscillators of the complex coordinates $W^i = \frac{1}{\sqrt{2}}(Y^{2i-1} + iY^{2i})$ by taking linear combinations of the relations above. This gives

$$\tilde{\alpha}_n^{i=1} = \bar{\alpha}_n^{i=1}\,, \qquad \tilde{\alpha}_n^{i=2} = \bar{\alpha}_n^{i=2}\,, \tag{4.30}$$

where the bar denotes complex conjugation. These boundary conditions are invariant under different orbifold actions than the ones we saw before. Recall that the orbifold action works as in (4.26). It follows that the boundary conditions at hand are invariant under the orbifold action for $m_1 = -m_2$ and $m_3 = -m_4$. One might refer to this as an anti-symmetric orbifold, as it works with an opposite phase on the left and right-moving coordinates, but there is no fundamental difference between this and a symmetric orbifold (they are related by a field redefinition of the coordinates).

4.4.2 Supersymmetry

Our orbifold constructions break supersymmetry (partially). Here we investigate whether the supersymmetry that is preserved by the orbifold and the supersymmetry that is preserved by the brane stack are compatible. In other words, we study whether our brane setup is BPS in the orbifolded background. For this, we follow the approach laid out in [84]. We focus on symmetric orbifolds, as the D1 and D5-branes in our setup don't survive in the asymmetric ones (as we explained in section 4.4.1).

Type IIB string theory contains two 10d Killing spinors, which we label by $\varepsilon_{\mathscr{L}}$ and $\varepsilon_{\mathscr{R}}$. The subscript denotes the worldsheet chirality of these spinors. Since type IIB is a chiral $(2,0)$ theory, both have the same chirality in 10d. We choose conventions such that this can be written as

$$\Gamma_{11}\, \varepsilon_{\mathscr{L}} = \varepsilon_{\mathscr{L}}\ , \qquad \Gamma_{11}\, \varepsilon_{\mathscr{R}} = \varepsilon_{\mathscr{R}}\ , \tag{4.31}$$

where $\Gamma_{11} = \Gamma_0\Gamma_1 \ldots \Gamma_9$ is the chirality operator. Another dimension in which chirality will be important is in six dimensions: the number of dimensions left after compactification on the T^4. Both of the 10d spinors (Majorana–Weyl) decompose into two 6d spinors (symplectic Weyl), one of either chirality. These are the supersymmetry spinors of the 6d $\mathcal{N} = (2{,}2)$ theory that is obtained from reducing type IIB on an untwisted torus. We denote these four spinors by $\varepsilon^+_{\mathscr{L}/\mathscr{R}}$ and $\varepsilon^-_{\mathscr{L}/\mathscr{R}}$, where the $+/-$ sign denotes the 6d chirality, meaning

$$\Gamma_*\, \varepsilon^+_{\mathscr{L}/\mathscr{R}} = \varepsilon^+_{\mathscr{L}/\mathscr{R}}\ , \qquad \Gamma_*\, \varepsilon^-_{\mathscr{L}/\mathscr{R}} = -\,\varepsilon^-_{\mathscr{L}/\mathscr{R}}\ , \tag{4.32}$$

with $\Gamma_* = \Gamma_0\Gamma_1 \ldots \Gamma_5$. Note that here we have chosen the 10d Dirac matrices such that the $\Gamma_{\hat{\mu}}$, with $\hat{\mu} = 0, 1, \ldots, 5$, form a 6d Clifford algebra. By combining the conditions (4.31) and (4.32), we find that

$$\Gamma_{6789}\, \varepsilon^+_{\mathscr{L}/\mathscr{R}} = \varepsilon^+_{\mathscr{L}/\mathscr{R}}\ , \qquad \Gamma_{6789}\, \varepsilon^-_{\mathscr{L}/\mathscr{R}} = -\,\varepsilon^-_{\mathscr{L}/\mathscr{R}}\ , \tag{4.33}$$

where we use a Γ with multiple indices to denote an anti-symmetrized product of Γ-matrices.

In order to write the amount of supersymmetry that is preserved by the orbifold in terms of projections on the Killing spinors, we need to identify the string states of section 4.3.1 that correspond to 6d gravitini with the 6d Killing spinors that we use here. We choose this identification as

$$\begin{aligned}
\varepsilon_{\mathscr{L}}^{+} &\leftrightarrow |a_{3,4}\rangle_{\mathscr{L}} \times b^{\hat{\mu}}_{-1/2}\,|0\rangle_{\mathscr{R}}\,, \\
\varepsilon_{\mathscr{L}}^{-} &\leftrightarrow |a_{1,2}\rangle_{\mathscr{L}} \times b^{\hat{\mu}}_{-1/2}\,|0\rangle_{\mathscr{R}}\,, \\
\varepsilon_{\mathscr{R}}^{+} &\leftrightarrow \tilde{b}^{\hat{\mu}}_{-1/2}\,|0\rangle_{\mathscr{L}} \times |a_{3,4}\rangle_{\mathscr{R}}\,, \\
\varepsilon_{\mathscr{R}}^{-} &\leftrightarrow \tilde{b}^{\hat{\mu}}_{-1/2}\,|0\rangle_{\mathscr{L}} \times |a_{1,2}\rangle_{\mathscr{R}}\,.
\end{aligned} \tag{4.34}$$

Note that the Ramond vacua $|a_{1,2}\rangle$ and $|a_{3,4}\rangle$ give rise to different chiralities in 6d, and that we have chosen these to correspond to the $-$ and $+$ chiralities respectively.

Now we can express the preserved supersymmetries of our various orbifold reductions in terms of projections on the Killing spinors. Take for example a symmetric orbifold with $m_1 = m_2 \neq 0$ and $m_3 = m_4 = 0$, which gives an $\mathcal{N} = 4$ $(2,0)$ theory in 5d. We know from section 4.3.1 that the gravitini corresponding to $\varepsilon_{\mathscr{L}}^{-}$ and $\varepsilon_{\mathscr{R}}^{-}$ in (4.34) obtain mass, and therefore the corresponding supersymmetries are broken. We can project to the remaining supersymmetries by imposing

$$\mathcal{N} = 4\ (2,0): \quad \Gamma_{6789}\,\varepsilon_{\mathscr{L}} = \varepsilon_{\mathscr{L}} \quad \text{and} \quad \Gamma_{6789}\,\varepsilon_{\mathscr{R}} = \varepsilon_{\mathscr{R}}\,. \tag{4.35}$$

As an asymmetric example, let's take the orbifold with $m_1 \neq 0$ and the other parameters zero. This gives an $\mathcal{N} = 6$ theory in 5d; only the gravitino corresponding to $\varepsilon_{\mathscr{L}}^{-}$ becomes massive. In terms of projections on the Killing spinors we can write the preserves supersymmetry as

$$\mathcal{N} = 6: \quad \Gamma_{6789}\,\varepsilon_{\mathscr{L}} = \varepsilon_{\mathscr{L}} \quad \text{and} \quad \varepsilon_{\mathscr{R}}\ \text{unconstrained}\,. \tag{4.36}$$

Next we look at the supersymmetry that is preserved by the brane stack. As always our prime example is the D1-D5 system, which gives the following conditions on the supercharges

$$\mathrm{D5}: \quad \Gamma_{05}\,\Gamma_{6789}\,\varepsilon_{\mathscr{L}} = \varepsilon_{\mathscr{R}}\,, \qquad\qquad \mathrm{D1}: \quad \Gamma_{05}\,\varepsilon_{\mathscr{L}} = \varepsilon_{\mathscr{R}}\,. \tag{4.37}$$

In the untwisted torus reduction, this configuration preserves eight supercharges, i.e., it is $\frac{1}{4}$-BPS. By substituting these two conditions in one another we find that they imply both of the conditions on the unbroken supersymmetries of the symmetric $\mathcal{N} = 4$ orbifold (4.35). In other words, the D1-D5 system also preserves eight supercharges in this orbifold model, and it is a $\frac{1}{2}$-BPS configuration in the resulting $\mathcal{N} = 4$ $(2,0)$ theory.

The only other qualitatively different symmetric orbifold that we can construct is the one where all mass parameters are non-zero and $m_1 = m_2$ and $m_3 = m_4$. This background breaks supersymmetry completely, so the D1-D5 system cannot be a BPS configuration here.

Similar arguments can be used to argue whether other brane configurations of interest, such as D3-D3 systems, preserve supersymmetry in the various orbifolds that we consider.

Note that even though D1-branes and D5-branes don't survive in asymmetric orbifolds, the supercharges of the brane stack (4.37) and those of for example the asymmetric $\mathcal{N}=6$ orbifold (4.36) are compatible. There is an interesting story to be told here, which involves supersymmetric bound states of branes, see e.g. [84, 85].

4.4.3 Consistency with supergravity

So far, we have discussed the branes purely from a string theoretic point of view. However, the orbifolds that we study in this chapter are the string theory uplifts of the supergravity Scherk-Schwarz reductions studied in chapter 3. There, the 5d spectrum was worked out and masses were found for gauge fields coupling to various branes wrapped on the internal manifold. In this section, we will check that the results about which D-branes survive in our orbifold constructions agree with the supergravity calculation. The premise of this check is that a D-brane survives in the orbifold when the supergravity gauge field charging the corresponding p-brane remains massless in the Scherk-Schwarz reduction. We will see that this supergravity point of view aligns nicely with the results from section 4.4.1.

Let us first consider D1 and D5-branes as usual. In chapter 3 it was found that after diagonalizing the mass matrix, the gauge fields charging D1 and D5 don't obtain masses individually. Instead linear combinations of these fields obtain the masses

$$\begin{aligned} C_2^{\mathrm{D1}} + C_2^{\mathrm{D5}} \ &: \quad |m_1 - m_2| \,, \\ C_2^{\mathrm{D1}} - C_2^{\mathrm{D5}} \ &: \quad |m_3 - m_4| \,. \end{aligned} \tag{4.38}$$

Note that these two-form gauge fields can be dualized to vectors if they are massless, so that they can properly charge pointlike objects (the branes only wrap the internal manifold and are therefore pointlike in 5d). We can immediately see that if we want both D1 and D5-branes to survive, i.e., if we want both gauge fields in (4.38) to remain massless, we need $m_1 = m_2$ and $m_3 = m_4$. This is in agreement with the results from section 4.4.1. Both lines of reasoning lead to the conclusion that D1 and D5-branes only survive in symmetric orbifolds.

Similar arguments can be made for the various D3-brane setups. For example, let D3 denote a brane in Y^1, Y^2 directions, and D3' a brane in the Y^3, Y^4 directions (of course, both branes also wrap the circle). From the supergravity computation we found the same masses

$$\begin{aligned} C_2^{\text{D3}} + C_2^{\text{D3'}} &: \quad |m_1 - m_2|\,, \\ C_2^{\text{D3}} - C_2^{\text{D3'}} &: \quad |m_3 - m_4|\,. \end{aligned} \tag{4.39}$$

Again, the gauge fields are massless for $m_1 = m_2$ and $m_3 = m_4$, corresponding to a symmetric orbifold and therefore in line with the results from section 4.4.1. In another scenario, where the D3 and D3' lie in the Y^1, Y^3 and Y^2, Y^4 directions on the torus respectively, the masses obtained in the Scherk-Schwarz reduction are

$$\begin{aligned} C_2^{\text{D3}} + C_2^{\text{D3'}} &: \quad |m_1 + m_2|\,, \\ C_2^{\text{D3}} - C_2^{\text{D3'}} &: \quad |m_3 + m_4|\,. \end{aligned} \tag{4.40}$$

Now the gauge fields remain massless when we have $m_1 = -m_2$ and $m_3 = -m_4$. Again we find agreement between the string theory and supergravity results (recall that the D3-branes in this last example are of the 'second type' in the terminology of section 4.4.1).

One can ask what happens to e.g. the D1-D5 system when $m_1 \neq m_2$ and $m_3 = m_4$, or the other way around. In this case, one of the supergravity gauge fields in (4.38) remains massless, and is therefore expected to charge some object. In some cases this object can be understood as a bound state of D1 and D5-branes [84].

PART II

SPINDLE AND DISC NEAR-HORIZONS

Chapter 5

Introduction

"A black hole really is an object with very rich structure, just like Earth has a rich structure of mountains, valleys, oceans, and so forth. Its warped space whirls around the central singularity like air in a tornado."

– Kip Thorne

One of the best tools for constructing new field theories in lower dimensions is to compactify a higher dimensional theory on a compact manifold. If the parent theory has a brane construction one can view the compactified theory as arising from wrapping the branes on certain cycles. Following the seminal work in [86], one may probe these constructions holographically by constructing interpolating flows across dimensions: that is AdS_d geometries flowing to $\mathrm{AdS}_{d-p} \times \Sigma_p$ geometries.

The canonical method of preserving supersymmetry in these constructions was to perform a so-called *topological twist* of the theory [87]. The smoking gun of a topological twist is a Killing spinor which is constant on the compactification manifold. The twist is engineered so that background R-symmetry gauge fields cancel off the spin connection such that the spinor is constant.

Recently, a new class of supersymmetric solutions has been studied, where the compactification manifold Σ_p is either a *spindle* or a *topological disc*. These manifolds admit orbifold singularities, which implies that they do not allow globally defined constant Killing spinors. Consequently, supersymmetry must be preserved through a mechanism other than the canonical topological twist.

Since the first constructions featuring a spindle [88] and a disc [89, 90], many brane setups wrapped on these compact spaces have been considered. These include D3-branes [88, 91, 92], M2-branes [93–95], M5-branes [92, 96] and D4-D8 systems [97] on spindles, as well as D3-branes [98, 99], M5-branes [89, 90, 100] and D4-D8 systems [101] on discs. In this thesis, we focus on M2-branes on both spindles and discs (chapter 6) and on D2-branes in massive type IIA wrapped on spindles (chapter 7).

In chapter 6 we consider multi-charge $\mathrm{AdS}_2 \times \Sigma$ solutions in 4d $\mathcal{N} = 2$ $\mathrm{U}(1)^4$ STU supergravity. These solutions can be uplifted to 11d supergravity on S^7 and their local forms were originally found in this 11d guise in [102]. We perform a full global analysis of these solutions in both 4d and 11d, in a similar manner to the regularity analysis performed in [93] and [90]. Different global completions of the solution give rise to two classes of solutions, one where Σ is a spindle and another where Σ is a topological disc. In the spindle class of the solution the internal manifold Y_9 is smooth. For the disc solutions we show that the solution takes the form of a $S^5 \times S^1_z \times S^1_{\phi_4}$ fibration over a rectangle. On the edges of the rectangle the metric degenerates smoothly while at two of the four corners there are singularities, which are associated to the presence of smeared M2-branes and a monopole.

Furthermore, in chapter 7, we consider rotating $\mathrm{AdS}_2 \times \Sigma$ solutions in 4d $\mathcal{N} = 2$ Einstein–Maxwell supergravity, where Σ is a spindle. Such 4d solutions can be lifted to massive type IIA where they describe D2-branes wrapped on the spindle. We find that the 8d internal space in the uplift is a $\mathrm{SE}_5 \times S^1_z$ fibration over a rectangle. This geometry is smooth except for singularities associated to monopoles at the corners of the rectangle.

There are some interesting comments to be made about this D2-brane setup in relation to other setups that have been studied in the literature. Firstly, the singularity structure that we find is similar to that of the setup with M5-branes on spindles [92, 96]. The manner of preserving supersymmetry, however, is different. In the D2-brane case supersymmetry is preserved with an anti-twist, while the M5-brane case uses a topological topological twist (the difference between these is discussed later in this introduction).

In addition, it should be noted that the Einstein–Maxwell truncation of the 4d solutions of chapter 6 yields the same as the non-rotating limit of the 4d solutions of chapter 7. In other words, these are part of the same larger family of 4d near-horizon solutions. Within the scope of this thesis the non-rotating single-charge solution can be lifted both to 11d supergravity and to massive type IIA, yielding either M2 or D2-branes wrapped on a spindle. Interestingly enough, in the former uplift the full geometry is smooth, while the latter contains monopole singularities. Hence we find that desingularization is uplift dependent.

5.1 Spindles and topological discs

A spindle is topologically a sphere with orbifold singularities at both poles, characterized by relatively prime integers $n_\pm$ labeling the deficit angles $2\pi(1 - n_\pm^{-1})$. Mathematically, it can be described as the weighted projective space $\mathbb{WCP}^1_{[n_-,n_+]}$. Note that for $n_+ = n_- = 1$ this becomes a smooth two-sphere.

We can write the metric on a spindle in the form

$$\mathrm{d}s^2 = \mathrm{d}w^2 + F(w)\,\mathrm{d}z^2\,, \tag{5.1}$$

with $w \in [w_-, w_+]$ and z a U(1) coordinate with period 2π. In order to describe an object that is topologically a sphere, we require $F(w_-) = F(w_+) = 0$ and $F(w) > 0$ on the open interval (w_-, w_+). The conical deficit angles become manifest by expanding the metric around the poles. For a spindle $\mathbb{WCP}^1_{[n_-,n_+]}$, these expansions would yield

$$F(w) = \frac{(w - w_\pm)^2}{n_\pm^2} + \mathcal{O}(|w - w_\pm|^3)\,, \tag{5.2}$$

leading to the metric close to the poles being that of $\mathbb{R}^2/\mathbb{Z}_{n_\pm}$ in polar coordinates. If we would work out these expansions for a round sphere, i.e. for $w \in [0, \pi]$ and $F(w) = \sin^2 w$, we would indeed find $n_+ = n_- = 1$.

One finds a topological disc from the same local metric (5.1) as the spindle. The assumption that changes the global topology is that the function $F(w)$ is no longer taken to be zero at one of the poles. Consequently, the geometry is no longer topologically a sphere, but rather it is a sphere that has been 'opened up' at one of the poles. Topologically this is a disc. Because there can still be a conical singularity at the other pole, this object is sometimes called a 'half-spindle'. One can also think about such a topological disc as a sphere with an irregular puncture at one pole, and a regular puncture at the other. The irregular puncture then opens up the sphere giving rise to the disc topology, while the regular one remains as a conical singularity.

5.2 Twists and anti-twists

The presence of orbifold singularities in spindle and disc geometries rules out the existence of global constant spinors. Instead, Killing spinors are sections of non-trivial bundles of the spindle. As these are non-constant, it is impossible to preserve supersymmetry with a conventional topological twist.

Schematically, the Killing spinor equation reads

$$\left(\partial_\mu + \omega_\mu{}^{ab}\gamma_{ab} - A_\mu\right)\zeta = 0\,, \tag{5.3}$$

where we have omitted prefactors for simplicity. In the case of a topological twist, the Killing spinor ζ is constant and the spin connection and gauge field contributions cancel, satisfying the equation. From this it follows that the flux of the gauge field through the spindle or disc is proportional to the Euler character of the surface:

$$\frac{1}{2\pi}\int_{\Sigma} \mathrm{d}A = \chi\,. \tag{5.4}$$

There are two ways to preserve supersymmetry that involve non-constant Killing spinors. These are the mechanisms at work in supersymmetric spindle and disc geometries [103, 104].

First, there is a generalization of the topological twist case. Now the gauge field and the spin connection contribution in (5.3) don't cancel, but they do cancel if one would integrate over the Riemann surface. This topological cancellation has led to the unlikely nomenclature 'topologically a topological twist'. The bit that does not cancel between A and ω^{ab} in (5.3) is removed by making the Killing spinor ζ non-constant. For such twists the relation (5.4) still holds, and one can see the topological twist as a specific example of the topological topological twist.

The alternative to this is called the *anti-twist*, and when both are discussed the topological topological twist is usually simply called *twist*. One can take as the defining property of the anti-twist that (5.4) is not satisfied. This makes it recognizably different from the (topological) twist case. Perhaps a more intuitive characterization of the anti-twist (in the case of a spindle) is that the corresponding Killing spinor has opposite chirality at the two poles, see appendix 6.B.

5.3 Outline

This part is organized as follows. In section 6.1 we study the $\mathrm{AdS}_2 \times \Sigma$ solutions in 4d $\mathcal{N} = 2$ $\mathrm{U}(1)^4$ gauged STU supergravity. We study the regimes in which the solutions are well-defined, and in which they describe spindle or disc near-horizons. Here we also show explicitly that supersymmetry is realized with an anti-twist. In section 6.2 we consider the uplifted 11d solutions, presenting the solutions in both the canonical $\mathrm{AdS}_2 \times Y_9$ form of [105] and in the canonical uplift form. In the spindle class we show that the solutions are regular, while for the disc class we show that there are singularities that can be explained by the presence of smeared M2-branes and monopoles. Finally we quantize the flux and compute the on-shell Newton constant, giving a prediction for the entropy of a putative asymptotically AdS_4 black hole with horizon Σ. We relegate some technical material on smeared M2-branes and on Killing spinors in our setup to the appendices.

In chapter 7 we perform a similar analysis as in chapter 6, but for a setup involving D2-branes in massive type IIA. First, in section 7.1, we study a family of

4d black hole solutions with spindle horizons. We present both the full black hole and the near-horizon solution. In section 7.2 we perform the uplift, and we study the regularity of the 10d solution. We find that there remain singularities, which can be interpreted as coming from monopoles. We present the quantization of the fluxes and compute the 2d Newton constant. In the appendix of this chapter we review the setup involving M5-branes on a spindle, which shows similarity to our D2-brane setup.

Chapter 6

M2-branes on discs and multi-charge spindles

6.1 4d black hole near-horizons

In this section we will study a family of supersymmetric AdS_2 solutions in 4d $\mathrm{U}(1)^4$ gauged supergravity which may be uplifted to solutions of 11d supergravity on an S^7. The local form of the 11d solutions we study was originally found in [102] using dualities between multi-charge superstar solutions. We will study in detail the 4d $\mathrm{AdS}_2 \times \Sigma$ solutions in this section. One finds that there are two distinct ways of extending the local solutions globally, distinguished by the different properties of Σ. The first class gives rise to a spindle, $\mathbb{WCP}^1_{[n_-,n_+]}$. Whilst in the second class of solution, Σ is a topological disc and is (naively) singular. As we will see in section 6.2, this singularity arises due to the presence of smeared M2-branes in the full 11d solution and is therefore physical.

6.1.1 Multi-charge solutions of 4d $\mathrm{U}(1)^4$ gauged supergravity

The solutions we will study arise in 4d $\mathcal{N}=2$ $\mathrm{U}(1)^4$ gauged supergravity which is a consistent truncation of $\mathcal{N}=8$ $\mathrm{SO}(8)$ gauged supergravity. The action, following the conventions in [106], with which we may uplift solutions to 11d on S^7, is[1]

$$S=\frac{1}{16\pi G_{(4)}}\int\left(R-\frac{1}{2}\sum_{I=1}^{4}\left(X^I\right)^{-2}\left(\mathrm{d}X^I\right)^2+\sum_{I<J}X^IX^J-\frac{1}{2}\sum_{I}\left(X^I\right)^{-2}\left|F^I\right|^2\right)*1\,, \tag{6.1}$$

subject to $X^{(1)}X^{(2)}X^{(3)}X^{(4)}=1$. One may obtain the above action from the general 4d $\mathcal{N}=2$ supergravity action by using the pre-potential[2]

$$F=-\mathrm{i}\sqrt{X^{(1)}X^{(2)}X^{(3)}X^{(4)}}\,, \tag{6.2}$$

[1]For ease of notation we have set the coupling constants for each of the gauge fields to 1 without loss of generality.
[2]See for example [107].

see appendix 6.B for further details. There are three further consistent truncations of the theory of interest to us named T^3, X^0X^1 and Einstein–Maxwell and are specified by reducing the independent scalars and gauge fields as given in table 6.1.

Theory	Scalars	Gauge Fields
T^3	$X^{(1)} = X^{(2)} = X^{(3)}$	$A_1 = A_2 = A_3$
X^0X^1	$X^{(1)} = X^{(3)}\,,\ X^{(2)} = X^{(4)}$	$A_1 = A_3\,,\ A_2 = A_4$
Einstein–Maxwell	$X^{(1)} = X^{(2)} = X^{(3)} = X^{(4)} = 1$	$A_1 = A_2 = A_3 = A_4$

Table 6.1: *The three consistent truncations of 4d $\mathcal{N} = 2$ $U(1)^4$ gauged supergravity. We will largely ignore the Einstein–Maxwell truncation since it has appeared previously in the literature, and can be viewed as a special case of the other two truncations.*

The local AdS_2 solutions that we will consider here are obtained from truncating the 11d solutions in [102] to 4d. Truncating the aforementioned solution of the seven-sphere, one obtains an AdS_2 solution to 4d $\mathrm{U}(1)^4$ gauged supergravity. The bosonic sector is given by

$$\mathrm{d}s_4^2 = \sqrt{P(w)}\Big(\mathrm{d}s^2(\mathrm{AdS}_2) + \mathrm{d}s^2(\Sigma)\Big)\,, \tag{6.3}$$

$$\mathrm{d}s^2(\Sigma) = \frac{f(w)}{P(w)}\mathrm{d}z^2 + f(w)^{-1}\mathrm{d}w^2\,, \tag{6.4}$$

$$A_I = -\frac{w}{2(w-q_I)}\mathrm{d}z\,, \tag{6.5}$$

$$X^{(I)} = \frac{P(w)^{1/4}}{w-q_I}\,, \tag{6.6}$$

where the polynomials $f(w)$ and $P(w)$ are

$$P(w) = \prod_{I=1}^{4}(w-q_I)\,, \qquad f(w) = P(w) - w^2\,. \tag{6.7}$$

The solution is specified by four constants, q_I, which are in principle independent. In the following section we will extend this local solution to a well-defined global solution which requires constraints on the parameters to be imposed. We will show that the topology of Σ depends on the choice of parameters q_I giving rise to two distinct classes of solutions.

6.1.2 Global analysis

In order to extend the local solution (6.3)-(6.6) to a globally well defined one, one must impose a number of additional constraints. Firstly, we must require that the metric is both real and has the correct signature. This implies that both functions $f(w)$ and $P(w)$ must be positive definite. Moreover, we must fix the domain of the coordinate w so that Σ is a compact space. To do this we identify two zeroes of the function $f(w)$ between which both $f(w)$ and $P(w)$ are strictly positive. Since $f(w)$ is a quartic polynomial it admits four roots. Clearly we need at least two real roots, in fact we can immediately rule out $f(w)$ admitting only two real roots. This follows since $f(w)$ tends to infinity as $w \to \pm\infty$, and therefore the only domain where it is positive in this case is between the larger root and ∞ or $-\infty$ and the smaller root, consequently the domain is non-compact and therefore also Σ. We conclude that we must require four real roots.

Let us denote the roots by w_I, $I = 1, \ldots, 4$, with the labels chosen such that $w_1 \leq w_2 < w_3 \leq w_4$. The domain of w is then $[w_2, w_3]$, inside which, by construction, both $P(w)$ and $f(w)$ are non-negative. Note that we do not allow $w_2 = w_3$ as this would not give a finite size domain for w, and therefore the geometry is not well-defined. Depending on the choice of q_I we will find two classes of solutions characterised by the behaviour of the metric at the end-point w_2. When $w_2 \neq 0$ we find that Σ is a spindle, whilst for $w_2 = 0$ we encounter a topological disc. The main distinctions between these two classes have been summarised in table 6.2.

	Spindle	Topological disc
$f(w_2)/P(w_2)$	0	1
$f(w_3)/P(w_3)$	0	0
parameters	$q_I < w_2$ $q_I \neq 0$	$q_i < 0$ $q_4 = 0$

Table 6.2: *Summary of the main properties that distinguish the spindle and topological disc solutions. The ratio $f(w)/P(w)$ indicates whether the circle S^1_z degenerates or remains of finite size at the endpoints of the domain $[w_2, w_3]$. Schematic graphs of the functions $f(w)$ and $P(w)$ for the two respective classes of solutions have also been included in figures 6.1 and 6.2.*

Before we study the metric around such end-points, let us look at the constraints for well-defined scalars. Since they are related to dilatons they must be non-negative.[3] This requires $w - q_I \geq 0$ on the domain of w. We may reduce this condition to $w_2 \geq \max\{q_I\}$ which, if we label the q_I in ascending order, simply becomes $w_2 \geq q_4$. Below we will study the constraints on the parameters q_I such that the function $f(w)$ and the scalars have the properties discussed above. Schematically, all the constraints discussed above are equivalent to finding constraints on the q_I such that the functions $f(w)$ and $P(w)$ take the schematic form given in figure 6.1.

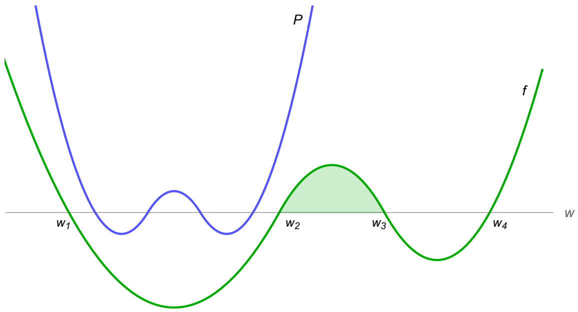


Figure 6.1: *A schematic plot showing the root structure of f and P. Here the shaded green region indicates the domain of w where the metric is defined.*

Before moving on to studying the constraints on the parameters q_I let us now look at the charges of the solution. The solution admits four magnetic charges. Using (6.5), and parametrising our temporary ignorance of the period of the z coordinate by denoting it by Δz, we find the magnetic charges

$$Q_I = \frac{1}{2\pi}\int_\Sigma \mathrm{d}A_I = \frac{q_I(w_3 - w_2)}{2(w_3 - q_I)(w_2 - q_I)}\frac{\Delta z}{2\pi}\,. \tag{6.8}$$

Since the charge depends explicitly on the period Δz we will postpone quantizing the fluxes for the moment. However we may still compute the sum of the charges

$$\sum_{I=1}^{4} Q_I = \frac{\Delta z}{4\pi}\left(\mathrm{sgn}(w_3)\frac{|f'(w_3)|}{|w_3|} + \mathrm{sgn}(w_2)\frac{|f'(w_2)|}{|w_2|}\right). \tag{6.9}$$

[3]As we will see later, it is in fact consistent to allow the dilatons to vanish at an end-point of the interval. The apparent singularity due to vanishing dilatons will be interpreted as the presence of a smeared M2 brane and is thus of physical nature.

Finally we can also compute the Euler character of the space[4]

$$\begin{aligned}\chi(\Sigma) &= \frac{1}{4\pi}\int_\Sigma R\mathrm{dvol}_\Sigma = -\frac{w^2 f'(w) - 2wf(w)}{2P(w)^{3/2}}\frac{\Delta z}{2\pi}\bigg|_{w=w_2}^{w=w_3} \\ &= \frac{\Delta z}{4\pi}\left(\frac{|f'(w_3)|}{|w_3|} + \frac{|f'(w_2)|}{|w_2|}\right),\end{aligned} \tag{6.10}$$

where, in the last line, we have used that $f'(w_2) > 0 > f'(w_3)$. Note in the two results above we have implicitly assumed that there is not a root at 0. As we will see later this is a special point in the parameter space of the solution and requires a separate treatment. Note that the sum of the charges and Euler character can only be the same if the roots are both of the same sign, in this case we would have a solution of type 'topological topological twist'.[5] For roots with different sign, which we will study here, these two quantities are not equal and therefore the solution is not dual to a CFT which has been (topologically) topologically twisted.

6.1.3 Spindle

We now want to study how the metric degenerates around an end-point of the domain of w. First let us assume that the roots w_2 and w_3 are single roots[6] and non-zero[7]. Around such an end-point the metric on Σ becomes

$$\begin{aligned}\mathrm{d}s^2(\Sigma) &= \frac{1}{f'(w_*)(w-w_*)}\mathrm{d}w^2 + \frac{f'(w_*)(w-w_*)}{w_*^2}\mathrm{d}z^2 \\ &= \frac{4}{|f'(w_*)|}\left(\mathrm{d}R^2 + \frac{|f'(w_*)|^2}{4w_*^2}R^2\mathrm{d}z^2\right),\end{aligned} \tag{6.11}$$

where we changed coordinates to $R^2 = \pm(w - w_*)$ in the last line, taking the plus sign for the expansion around w_2 and the minus sign for w_3. Note that $f'(w_2) > 0 > f'(w_3)$ and the sign introduced in the definition of the new radial coordinate has been absorbed by introducing the norm. From (6.11) we can see that the metric around the end-point looks locally like that of $\mathbb{R}^2$ if z has period

$$\frac{\Delta z}{2\pi} = \frac{2|w_*|}{|f'(w_*)|}\,. \tag{6.12}$$

[4]We assume that there is no boundary to Σ here. As we will see later we must amend this, however it turns out that the boundary contribution is trivial.
[5]Solutions of this form will be addressed in [104].
[6]For a double root the metric around the root looks locally like hyperbolic space, see for example [108].
[7]We will study the case where one of the roots is zero later in section 6.1.4, since this corresponds to the class of topological disc solutions rather than spindle solutions.

Since we have the same type of degeneration at both poles the space takes the form of a topological sphere. For this to be a round sphere we must avoid conical singularities at both poles, i.e. that the period given in (6.12) is the same at both end-points

$$\left|\frac{f'(w_2)}{w_2}\right| = \left|\frac{f'(w_3)}{w_3}\right| . \tag{6.13}$$

Using the explicit form of $f(w)$ this is equivalent to the roots satisfying

$$|w_3|(w_2 - w_1)(w_4 - w_2) = |w_2|(w_3 - w_1)(w_4 - w_3) . \tag{6.14}$$

For the two roots having the same sign the condition reduces to[8]

$$w_1 w_4 = w_2 w_3 , \tag{6.15}$$

whilst for opposite sign we have

$$w_4 = \frac{w_2 w_3(2w_1 - w_2 - w_3)}{w_1 w_2 + w_1 w_3 - 2w_2 w_3} . \tag{6.16}$$

Since the roots of the quartic are particularly unwieldy, in order to study the solutions analytically we will consider the solution in the truncated theories.

X^0X^1 truncation

First consider the solution in the X^0X^1 truncation. To truncate (6.3)-(6.6) to a solution of the X^0X^1 theory we must set $X^{(1)} = X^{(2)}$, $X^{(3)} = X^{(4)}$ and $A_1 = A_2$, $A_3 = A_4$ as in table 6.1. This is equivalent to setting the constants q_I to satisfy $q_1 = q_2$ and $q_3 = q_4$. In this truncation the function $f(w)$ takes the simplified form

$$f(w) = (w - q_1)^2(w - q_3)^2 - w^2 . \tag{6.17}$$

Both q_1 and q_3 are non-zero, since a constant equal to zero would automatically imply a common root between $f(w)$ and $P(w)$, which we will study in the section 6.1.4. To proceed, it is useful to define

$$s = q_1 + q_3 , \qquad p = 4q_1 q_3 , \tag{6.18}$$

which allows us to write the four roots of $f(w)$ in the compact form

$$\frac{1}{2}\left(s + 1 \pm \sqrt{(s+1)^2 - p}\right) , \qquad \frac{1}{2}\left(s - 1 \pm \sqrt{(s-1)^2 - p}\right) . \tag{6.19}$$

[8]We drop any solutions which sets roots equal or to 0.

Note that we have not assigned these roots the names w_I with $I = 1, \dots, 4$ yet, because we must still determine their order. There are two regimes where all the roots are real, distinguished by the sign of s. We find these regimes to be

$$s \geq 0\,, \quad p \leq (1-s)^2 \qquad \text{or} \qquad s \leq 0\,, \quad p \leq (1+s)^2\,. \tag{6.20}$$

It turns out that there is only one ordering of the roots which is consistent with the positive scalar condition, $w_2 - q_3 > 0$:

$$\begin{aligned} w_1 &= \frac{1}{2}\Big(s-1-\sqrt{(s-1)^2-p}\Big)\,, \qquad & w_2 &= \frac{1}{2}\Big(s-1+\sqrt{(s-1)^2-p}\Big)\,, \\ w_3 &= \frac{1}{2}\Big(s+1-\sqrt{(s+1)^2-p}\Big)\,, \qquad & w_4 &= \frac{1}{2}\Big(s+1+\sqrt{(s+1)^2-p}\Big)\,, \end{aligned} \tag{6.21}$$

along with the additional constraints, either

$$-1 < s \leq -\tfrac{1}{2}\,, \quad 0 < p < (1+s)^2\,, \quad \text{or} \quad -\tfrac{1}{2} < s < 0\,, \quad 0 < p \leq s^2\,, \tag{6.22}$$

which are a further restriction of the conditions in (6.20). In terms of q_1 and q_3 ($q_1 \leq q_3$) these conditions reduce to

$$-\tfrac{1}{4} < q_3 < 0\,, \qquad -\big(1-\sqrt{-q_3}\big)^2 < q_1 \leq q_3\,. \tag{6.23}$$

One may naturally wonder if these regimes are compatible with a spherical horizon, i.e. one where the conical singularities can be removed. It is simple to show that this is not possible and one always obtains a spindle. Plugging the roots (6.21) into either of (6.15) or (6.16), we find that the only possibilities to satisfy either of them are to set one of q_1 or q_3 to zero or $q_1 = -q_3$. Both of these options are incompatible with the region in (6.23). The case where either q_1 or q_3 vanish needs a more careful treatment since we introduce a different degeneration of the solution as we will see in the next section.

We conclude that it is not possible to find a spherical horizon in the X^0X^1 truncation. The solutions we have found here are spindles and admit conical singularities. Following [93], instead of making a single choice for the period Δz we impose

$$\Delta z = \frac{4\pi|w_2|}{n_-|f'(w_2)|} = \frac{4\pi|w_3|}{n_+|f'(w_3)|}\,, \tag{6.24}$$

which exhibits the space as the spindle $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$, with conical deficit angles $2\pi(1-n_\pm^{-1})$ at the two poles.

We can now return to the quantization of the magnetic charges. Following [93] the correct quantization condition to impose is

$$Q_I = \frac{1}{2\pi}\int_\Sigma F_I = \frac{p_I}{n_-n_+} \quad \text{with} \;\; p_I \in \mathbb{Z}\,. \tag{6.25}$$

As explained in [93] with this quantization, and for charges p_I coprime to both $n_\pm$, this gives rise to a well-defined and smooth orbifold circle fibration of the seven-sphere over the spindle

$$S^7 \hookrightarrow Y_9 \to \mathbb{WCP}^1_{[n_-,n_+]} \,. \tag{6.26}$$

The twist parameters of the fibration are p_I and is such that it leads to a compact space Y_9. From (6.8) and (6.25) we find that the twist parameters are given by

$$p_I = n_- n_+ \frac{q_I(w_3 - w_2)}{2(w_3 - q_I)(w_2 - q_I)} \frac{\Delta z}{2\pi} \,. \tag{6.27}$$

Inserting the expression for the period, (6.24) into the sum of the roots (6.9) and the Euler character (6.10) we find

$$\sum_{I=1}^{4} Q_I = \frac{1}{n_+} - \frac{1}{n_-} \,, \tag{6.28}$$

$$\chi(\Sigma) = \frac{1}{n_+} + \frac{1}{n_-} \,, \tag{6.29}$$

confirming that this solution does not involve the usual (topological) topological twist.

6.1.4 Topological disc

Let us now consider the second way of obtaining a degeneration of the surface: a common root between $P(w)$ and $f(w)$. We can see immediately from the form of $f(w)$, namely $f(w) = P(w) - w^2$, that if $f(w)$ and $P(w)$ have a common root, then this root must be located at $w = 0$. Since the roots of $P(w)$ are given by the parameters q_I, this means that (at least) one of the charges must be zero. In order to find a topological disc as the black hole horizon, we want the zero root of $f(w)$ to be a boundary of the domain of w, i.e. it should be either w_2 or w_3 in the notation above. Recall that for the scalars to be positive we need to impose $w_2 \geq \max\{q_I\}$, which for at least one $q_I = 0$ implies $w_2 \geq 0$. It is clear that the only consistent specification of the roots is to set $w_2 = 0$, and $w_3 > 0$ with $q_4 = 0$ (all other q's are then ≤ 0). It follows that in order for the solution to have a root at 0 we must require the second root of $f(w)$ and the fourth root of $P(w)$ to be equal to zero. A schematic plot of this setup is given in figure 6.2.

Now consider the possibility of a double root at zero. In order for this to work we have to set either w_1 or w_3 equal to zero. We can see immediately that the option

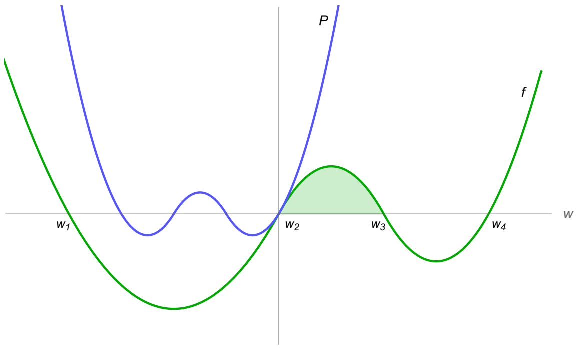


Figure 6.2: *A schematic plot showing the root structure of $f(w)$ and $P(w)$ such that we find a topological disc. As before the shaded green region indicates the domain of w where the metric is defined.*

$w_3 = 0$ is not allowed since a finite domain requires $w_3 > w_2 = 0$. This leaves $w_1 = 0$. We can see from figure 6.2 that this option would set all four q_I to zero by requiring $f(w) \leq P(w)$ everywhere. Consequently in this case we would simply have $P(w) = w^4$ and $f(w) = P(w) - w^2 = w^2(w-1)(w+1)$. We see that this expression for $f(w)$ has one negative and one positive root, which is in contradiction with our choice that $w_1 = w_2 = 0$. We conclude that the only setup with a single root at zero, i.e. setups with a root structure as in figure 6.2, are the relevant ones to consider, all others do not give rise to well-defined solutions.

T^3 truncation

We saw above that we must fix only one root of $f(w)$ to be zero. Consequently, this completion of the space is not possible in either the Einstein–Maxwell theory nor the X^0X^1 truncation. In order to analyze this setup somewhat analytically we study this completion in the T^3 truncation, fixing the three non-zero charges equal, i.e. $q_1 = q_2 = q_3 \equiv q < 0$. This essentially pushes the three negative roots of $P(w)$ in figure 6.2 together into a triple root. In this case the functions simplify to

$$\begin{aligned} P(w) &= w\,(w-q)^3\,, \\ f(w) &= w\left((w-q)^3 - w\right). \end{aligned} \tag{6.30}$$

We now require the cubic polynomial $f(w)/w = (w-q)^3 - w$ to have one negative and two positive roots, so that the root equal to zero is indeed w_2.

It is straightforward to compute the discriminant of the polynomial $f(w)/w$, and is given by $4-27q^2$. A cubic polynomial has three real roots if its discriminant is non-negative, which gives us the constraint $-\frac{2}{3\sqrt{3}} \leq q \leq \frac{2}{3\sqrt{3}}$ for the allowed values of q. We have already imposed that q must be negative for the positivity of the scalars, and so we restrict to the range $-\frac{2}{3\sqrt{3}} \leq q < 0$.

The equation $f(w)/w = 0$ can be solved analytically, but since the results are surprisingly bulky for such a simple polynomial equation, we will not present the explicit values here. Instead, we plot the three roots in figure 6.3 to show that indeed one is negative and two are positive as required. We see that for all values of q in the range $\left[-\frac{2}{3\sqrt{3}}, 0\right)$ we indeed obtain one negative and two positive roots, therefore for all of these values we have a solution with a horizon admitting a completion with a root at 0.

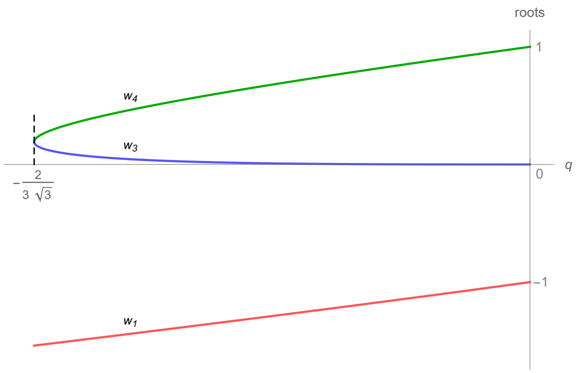


Figure 6.3: *The values of the three roots of $f(w)/w$ as a function of q in the T^3 truncation. The root w_3 approaches zero as $q \to 0$ but is positive for all values in the range $q \in \left[-\frac{2}{3\sqrt{3}}, 0\right)$.*

We now want to determine the global form of the metric. At w_3 we may use our results in the previous section to determine that the metric on Σ degenerates as the orbifold $\mathbb{R}^2/\mathbb{Z}_k$ if the period of z is fixed to be

$$\Delta z = \frac{4\pi w_3}{k|f'(w_3)|} \,. \tag{6.31}$$

Consider now the degeneration at $w = 0$. Note that we must consider the full 4d-solution now since the overall warp factor $P(w)$ vanishes here. Around $w = 0$ the metric takes the form

$$\mathrm{d}s_4^2 = \sqrt{|q|^3 w}\left(\mathrm{d}s^2(\mathrm{AdS}_2) + \mathrm{d}z^2 + \frac{1}{w|q|^3}\mathrm{d}w^2\right) . \tag{6.32}$$

This is conformal to the direct product of AdS_2 with a cylinder, however note that the conformal factor vanishes in this limit and the metric has a singularity at $w = 0$. To add to the apparent misery, in this limit three of the four $X^{(I)}$'s vanish despite being dilatons. Fortunately all is not lost: this singularity is physical! Since this is best seen from the 11d uplifted solution we will postpone this discussion until section 6.2.4 and proceed unabated. We now want to ascertain what space Σ is. We see that at $w = w_3$ the space looks like the orbifold $\mathbb{R}^2/\mathbb{Z}_k$ whilst at $w = 0$ we have a circle which does not contract. This is describing a topological disc with an orbifold singularity at the centre. Consider the Euler characteristic of Σ. We find[9]

$$\chi(\Sigma) = \frac{1}{k} \,. \tag{6.33}$$

Recall that for an orbifold O, the Euler characteristic is given by $\chi(O) = \frac{1}{d}\chi(M)$ where M is a compact oriented manifold providing a finite covering of O of degree d. We therefore see that this is precisely the expected result for a $\mathbb{Z}_k$ orbifold of a disc. Next let us examine the sum of the magnetic charges of the gauge fields. Since we set $q_4 = 0$ we see that A_4 is now pure gauge, whilst the remaining three are all equal. We find that the sum of the charges is[10]

$$\sum_{I=1}^{4} Q_I = \frac{1}{k} - \frac{\Delta z}{4\pi} \,. \tag{6.34}$$

Note in particular that the sum of the charges does not equal the Euler characteristic of the disc, and in fact can never be made to. This implies that the mechanism for preserving supersymmetry, like in the spindle case studied above, is not the usual topological twist. One can also see this from computing the explicit Killing spinors on the four-dimensional solution; they depend on the disc coordinates which is not the case for a standard topological twist, see appendix 6.B.

Beyond the $\mathbf{T^3}$ truncation

We have argued above that one cannot obtain a disc from our local solution in either the X^0X^1 truncation nor the Einstein–Maxwell truncation. One may wonder

[9]Note that since Σ has a boundary one must be slightly more careful with this computation than in the spindle case. The Gauss–Bonnet theorem contains a contribution from the boundary

$$\chi(\Sigma) = \frac{1}{4\pi}\int_{\Sigma} R\text{dvol}(\Sigma) + \frac{1}{2\pi}\int_{\partial\Sigma} \kappa\text{dvol}(\partial\Sigma) \,,$$

which we neglected earlier. However in the present case the geodesic curvature κ vanishes and therefore the boundary does not contribute to the Euler characteristic.

[10]One sees that this is the same as the analogous result for M5 branes on a disc as in [90].

if it is possible to have a solution keeping the remaining non-zero q_I's distinct. Studying this more general solution near to the singular point one finds that the uplift interpretation is not simply a smeared M2-brane as in the T^3 truncation. It would be interesting to see if it is possible in this case to give the singularity a physical interpretation, we have not been able to rule out a more complicated completion involving a more exotic brane configuration.

6.2 Uplift to 11d supergravity

In this section we discuss the 11d uplift of the 4d AdS_2 solutions given in section 6.1. We find that the class of spindle solutions is lifted to a smooth 11d geometry. For the other class we find that the boundary of the topological disc gives rise to a stack of smeared M2-branes, making this singularity of physical nature.

6.2.1 STU truncation uplift

The local solution we considered in section 6.1 was originally found in 11d in [102]. The full metric after a little rewriting is given by

$$\begin{aligned} \mathrm{d}s^2 = \mathrm{e}^{2A}\bigg(&\mathrm{d}s^2(\mathrm{AdS}_2) + \frac{f(w)}{P(w)}\mathrm{d}z^2 + \frac{1}{f(w)}\mathrm{d}w^2 \\ &+ \frac{4}{P(w)Y}\sum_{I=1}^{4}(w-q_I)\Big[\mathrm{d}\mu_I^2 + \mu_I^2\Big(\mathrm{d}\phi_I - \frac{w}{2(w-q_I)}\mathrm{d}z\Big)^2\Big]\bigg)\,, \end{aligned} \tag{6.35}$$

where the μ_I's are the embedding coordinates of an S^7 satisfying

$$\sum_{I=1}^{4}\mu_I^2 = 1\,. \tag{6.36}$$

The functions $f(w)$ and $P(w)$ are as in (6.7), whilst Y and e^A are given by

$$\begin{aligned} Y &= \sum_{I=1}^{4}\mu_I^2(w-q_I)^{-1}\,, \\ \mathrm{e}^{3A} &= P(w)Y\,. \end{aligned} \tag{6.37}$$

We see that the constraint on the scalars being positive definite is equivalent to the metric having correct signature in 11d.

It is also convenient to rewrite the metric in the form of the classification of AdS_2 solutions in [105]. Using the results in [102] it takes the form

$$\mathrm{d}s^2 = \mathrm{e}^{2A}\Big(\mathrm{d}s^2(\mathrm{AdS}_2) + (\mathrm{d}z+\sigma)^2 + \mathrm{e}^{-3A}\mathrm{d}s_8^2\Big)\,, \tag{6.38}$$

with

$$\sigma = -\frac{2w}{P(w)Y}\sum_I \mu_I^2\mathrm{d}\phi_I\,, \tag{6.39}$$

$$\mathrm{d}s_8^2 = \frac{P(w)Y}{f(w)}\mathrm{d}w^2 + 4\sum_I (w-q_I)(\mathrm{d}\mu_I^2 + \mu_I^2\mathrm{d}\phi_I^2) - \frac{4w^2}{P(w)Y}\Big(\sum_I \mu_I^2\mathrm{d}\phi_I\Big)^2\,. \tag{6.40}$$

In this rewriting the R-symmetry vector is manifest. It is important to note that the R-symmetry vector is

$$R_{1\mathrm{d}} = \partial_z\,. \tag{6.41}$$

A change in the gauge of the gauge fields, which can be reabsorbed by a coordinate shift of the ϕ_I's will alter as we will see later. Note that the Killing spinors have charge $\frac{1}{2}$ under z. The 8d space is Kähler with Kähler form

$$J_8 = 2\mathrm{d}w\wedge\sum_I \mu_I^2\,\mathrm{d}\phi_I + 4\sum_I (w-q_I)\,\mu_I\mathrm{d}\mu_I\wedge d\phi_I\,. \tag{6.42}$$

6.2.2 General analysis

We start by computing some general quantities of interest for the family of solutions that we study. We compute the free-energy/entropy of the dual CFT, which for AdS_2 solutions is given by

$$S = \mathcal{F} = \frac{1}{4G_2}\,, \tag{6.43}$$

and we check that the M2-brane fluxes are properly quantized. Furthermore, we check the regularity of the 11d solutions in both the spindle and the disc cases, and examine some interesting regions of the solutions.

Flux quantization

As usual, we need to check that the flux of the solution is quantized so that the solution can be properly lifted to M-theory. The four-form flux of the solution is given by

$$G_4 = L^3\,\mathrm{dvol}_{\mathrm{AdS}_2}\wedge\Big[\mathrm{d}\big(e^{3A}(\mathrm{d}z+\sigma)\big) - J_8\Big]\,. \tag{6.44}$$

There are no non-trivial four-cycles in our geometry, but there are non-trivial seven-cycles, so we focus on the Hodge dual flux which can be written as [109]

$$*G_4 = L^6\,(\mathrm{d}z+\sigma)\wedge \mathrm{d}\sigma \wedge \frac{J_8^2}{2} + \text{closed piece}\,. \tag{6.45}$$

Since the closed piece does not contribute to any of the integrals we refrain from presenting it here.

Both the spindle and the topological disc have a seven-cycle given by the S^7. In order to compute the flux through this cycle we go to one of the endpoints of the w interval, because there the seven-sphere decouples from the rest of the geometry which allows us to integrate over this cycle. We compute this integral as

$$\frac{1}{(2\pi\ell_p)^6}\int_{\mathrm{S}^7} *\,G_4 = \frac{2L^6}{\pi^2\ell_p^6} \equiv N\,, \tag{6.46}$$

and hence find the quantization condition $N\in\mathbb{Z}$.

For the spindle this is the only non-trivial seven-cycle. However, for the topological disc we find an additional cycle in the simultaneous $\mu_4, w\to 0$ limit, associated with the presence of smeared M2-branes. We will discuss the quantization condition arising from this cycle in section 6.2.4.

Free energy

In order to compute the free energy, it is convenient to use the metric in the form of (6.38). We find that the two-dimensional Newton's constant is given by the simple expression

$$\frac{1}{G_2} = \frac{L^9}{G_{11}}\int_{Y_9} e^{9A}\,\mathrm{dvol}_{Y_9} = \frac{8L^9}{3\pi^3\ell_p^9}\,(w_3-w_2)\,\Delta z\,, \tag{6.47}$$

where we above used that $G_{11}=\frac{(2\pi)^8\ell_p^9}{16\pi}$, and ℓ_p denotes the 11d Planck length. Using the quantization of the flux and the definition of N above we have

$$S = \frac{1}{4G_2} = \frac{1}{3\sqrt{2}}N^{3/2}\,(w_3-w_2)\,\Delta z\,. \tag{6.48}$$

6.2.3 Spindle

Here we study the regularity of the spindle solutions in the uplift to 11d. We will find that the singularities from 4d have disappeared in the uplift, and that the solution is regular. This is a well-known feature of M2- and D3-branes on spindles (though not for M5's), see for example [88, 92, 93], and is sometimes referred to as desingularization.

In 4d we found conical singularities at $w = w_2$ and $w = w_3$, so here we again focus on these points. Our 11d solution has five U(1) Killing vectors, namely ∂_z and ∂_{ϕ_I}. In order to check for conical singularities, we are looking for linear combinations of these Killing vectors that have vanishing norm at w_2 or w_3. It turns out that the only such Killing vectors are given by[11]

$$\partial_{\psi_i} = c_i \left(\partial_z + \sum_I \frac{w_i}{2(w_i - q_I)} \, \partial_{\phi_I} \right) , \tag{6.49}$$

where $i = 2, 3$. For these, we find that $||\partial_{\psi_2}||^2(w_2) = ||\partial_{\psi_3}||^2(w_3) = 0$. We can now fix the normalization constants c_i by imposing that these Killing vectors have the appropriate periodicity of 2π around the point where they degenerate. We expand the norm of the ∂_{ψ_i} in $\Delta w = \pm(w - w_i)$ which gives

$$||\partial_{\psi_i}||^2 = \mathrm{e}^{2A} \, \frac{c_i^2 \, |f'(w_i)|}{P(w_i)} \, \Delta w + \mathcal{O}\big(\Delta w^2\big) \, , \tag{6.50}$$

and therefore the metric spanned by w and ψ_i close to w_i can be written as

$$\frac{1}{|f'(w_i)| \, \Delta w} \, \mathrm{d}w^2 + c_i^2 \, \frac{|f'(w_i)|}{P(w_i)} \, \Delta w \, \mathrm{d}\psi_i^2 = \frac{4}{|f'(w_i)|} \left[\mathrm{d}R^2 + \frac{c_i^2 \, f'(w_i)^2}{4P(w_i)} \, R^2 \mathrm{d}\psi_i^2 \right] . \tag{6.51}$$

Here we have omitted the overall warping e^{2A}, and we have made the change of coordinates $\Delta w = R^2$. We see that in order to get the correct periodicity, we must fix the normalization constants as

$$c_i = \left(\frac{f'(w_i)}{2\sqrt{P(w_i)}} \right)^{-1} = \left(\sum_I \frac{w_i}{2(w_i - q_I)} - 1 \right)^{-1} . \tag{6.52}$$

These coefficients are identical to the ones found in [102], where the regularity of this family of solutions was previously studied.

We now have six Killing vectors that degenerate somewhere in the manifold, but there are only five isometries. We therefore know that they must be related by the constraint[12]

$$a_2 \, \partial_{\psi_2} + a_3 \, \partial_{\psi_3} + p_1 \, \partial_{\phi_1} + \ldots + p_4 \, \partial_{\phi_4} = 0 \, , \tag{6.53}$$

[11] Note that the z circle, that gave rise to the conical singularities in the 4d solution, does not pinch anywhere in the uplifted manifold. The norm of the corresponding Killing vector is simply given by $||\partial_z||^2 = \mathrm{e}^{2A}$.

[12] Since it turns out that the constant coefficients of the ϕ_I Killing vectors become the magnetic charges p_I defined in (6.25), we have chosen to call them p_I from the outset. Note that there was no assumption that these are magnetic charges in reaching this conclusion.

for some coprime integers $a_2, a_3, p_1, \ldots, p_4$. Notice that the constants c_i may be rewritten as

$$c_2 = \frac{n_- \Delta z}{2\pi}\,, \qquad c_3 = -\frac{n_+ \Delta z}{2\pi} \tag{6.54}$$

by using (6.24). Expanding the terms in the sum (6.53) we find the constraints

$$0 = a_2 n_- - a_3 n_+\,, \tag{6.55}$$

$$0 = \sum_{i=2}^{3} \frac{a_i c_i w_i}{2(w_i - q_I)} + p_I\,. \tag{6.56}$$

where the latter must be satisfied for all I. We may solve the first by

$$a_2 = n_+\,, \qquad a_3 = n_-\,, \tag{6.57}$$

where we have used that a_2 and a_3 should be coprime. Plugging this into the second constraint we find

$$p_I = n_+ n_- Q_I\,, \tag{6.58}$$

which is precisely the quantization condition we imposed on the fluxes in (6.25). We conclude that provided the fluxes p_I (defined in (6.25)) are relatively prime to both n_+ and n_-, which are relatively prime themselves, then the 11d solution is smooth and free of conical singularities.

R-symmetry vector

Let us now consider the R-symmetry of the solution. Using the general results in [105, 109] and the form presented in (6.38) we can identify the 1d R-symmetry vector as

$$R_{1\mathrm{d}} = \partial_z\,. \tag{6.59}$$

However, this leads to a Killing spinor which has charge $\frac{1}{2}$ under the isometry of the spindle. To see why this is true, note that the 11d Killing spinor is the tensor product of the Killing spinor of the 4d solution with the Killing spinor on S^7. The uplift of our solution to 11d, as given in [102] which uses the gauge fields in (6.5), is in the correct form of the classification in [105] (we follow the conventions in [109] though). This implies that the 11d Killing spinor has charge $\frac{1}{2}$ under z and therefore the 4d Killing spinor has the same charge. Now, since the Killing spinor on S^7 has charge $\frac{1}{2}$ under each of the U(1)'s we may absorb the z dependence by a coordinate shift:

$$\phi_I \to \tilde{\phi}_I = \phi_I - \frac{1}{4} z\,. \tag{6.60}$$

The R-symmetry vector is then

$$R_{1\mathrm{d}} = \partial_z + \frac{1}{4}\sum_{I=1}^{4} \partial_{\tilde{\phi}_I} \,, \tag{6.61}$$

and we identify the summand with the 3d superconformal R-symmetry of ABJM before compactification. Note the similarity with the D3 case in [92].

Entropy

Finally Newton's constant is given in equation (6.48),

$$\frac{1}{G_2} = \frac{2\sqrt{2}}{3} N^{3/2} \,(w_3 - w_2)\, \Delta z \,. \tag{6.62}$$

We now want to write this in terms of the charges Q_I given in (6.25). Since the result for the Newton's constant takes the same form in the general STU solution as in the truncated solution we will present the general result for the four charges. To proceed it is useful to note that the term $(w_3 - w_2)\Delta z$ appears in the expression for the charges. A simple but tedious computation using the properties of the two polynomials $P(w)$ and $f(w)$ and the four charges allows us to express the roots implicitly in terms of the charges[13]

$$\begin{aligned}
\hat{Q}^{(4)} &\equiv \prod_{I=1}^{4} Q_I = \left(\frac{\Delta z(w_3 - w_2)}{4\pi}\right)^4 \frac{w_1 w_4}{w_2 w_3}\,,\\
\hat{Q}^{(3)} &\equiv \sum_{I=1}^{4} \prod_{J\neq I} Q_J = \left(\frac{\Delta z(w_3 - w_2)}{4\pi}\right)^3 \frac{w_1 w_4}{w_2 w_3}\left[(w_1 + w_4)\frac{w_2 w_3}{w_1 w_4} + (w1 + w2 - 2w_2 - 2w_3)\right],\\
\hat{Q}^{(2)} &\equiv \sum_{I<J} Q_I Q_J = \left(\frac{\Delta z(w_3 - w_2)}{4\pi}\right)^2 \frac{w_1 w_4}{w_2 w_3}\Bigg[1 + 3(w_2 + w_3)^2 - 2(w_2 + w_3)(w_1 + w_4) + w_1 w_4\\
&\qquad - 3w_2 w_3 + \frac{1}{w_1 w_4}\Big(w_2 w_3 + w_2 w_3 \sum_{1\leq I<J\leq 4} w_I w_J + (w_1 + w_4 - 2w_2 - 2w_3)\sum_{I=1}^{4}\prod_{J\neq I} w_I\Big)\Bigg],\\
\hat{Q}^{(1)} &\equiv \sum_{I=1}^{4} Q_I = \left(\frac{\Delta z(w_3 - w_2)}{4\pi}\right)\Big(2(w_1 + w_4) - (w_2 + w_3) - \frac{w_1 w_4}{w_2 w_3}(w_2 + w_3)\Big)\,.
\end{aligned} \tag{6.63}$$

Note that $Q^{(1)}$ can be expressed in terms of $n_\pm$ as in (6.28) once the constraint on the periods, (6.24) is taken into account. To proceed it is useful to define

$$x = \frac{\Delta z(w_3 - w_2)}{4\pi}\,, \qquad \alpha = w_1 + w_4\,, \qquad \beta = w_1 w_4\,, \tag{6.64}$$

[13]See [104] for further details on these expressions in terms of the quartic invariant.

where, as in all good algebra problems, 'x' is what we want to compute.

We first solve the constraint on the periods, namely equation (6.24), for the variable β which has solution

$$\beta = -\frac{w_2 w_3\big(n_-(w_2-\alpha)+n_+(w_3-\alpha)\big)}{n_+ w_2 + n_- w_3}\,. \tag{6.65}$$

With this solution we see that both $\hat{Q}^{(1)}$ and χ take the expected form. We next eliminate α in favour of the new variable x which gives

$$\alpha = \frac{n_+ w_2(1-n_- w_2 x)+n_- w_3(1+n_+ w_3 x)}{n_- n_+ x(w_3-w_2)}\,. \tag{6.66}$$

Substituting the new variables and constraints into the functions of the charges $\hat{Q}^{(A)}$, for $A=2,3,4$ gives three equations for the three unknowns x, w_2, w_3. It is again convenient to change variables, introducing

$$\gamma = w_3 + w_2\,, \qquad \delta = w_3 - w_2\,. \tag{6.67}$$

We may now solve for x, γ, δ in terms of $n_\pm$, and $\hat{Q}^{(A)}$, $A=2,3,4$ giving[14]

$$x = \sqrt{\frac{1+n_-n_+\hat{Q}^{(2)}-\sqrt{\big(1+n_-n_+\hat{Q}_2\big)^2-4n_-^2n_+^2\hat{Q}^{(4)}}}{2n_-n_+}}\,, \tag{6.68}$$

$$\delta = \frac{(n_-+n_+)x}{\sqrt{\big(1+n_-n_+\hat{Q}_2\big)^2-4n_-^2n_+^2\hat{Q}^{(4)}}}\,, \tag{6.69}$$

$$\gamma = \frac{\Big((n_+-n_-)\big(1+n_-n_+\hat{Q}^{(2)}\big)+2n_-^2n_+^2\hat{Q}^{(3)}\Big)x}{\big(1+n_-n_+\hat{Q}_2\big)^2-4n_-^2n_+^2\hat{Q}^{(4)}}\,. \tag{6.70}$$

In principle we can now invert to solve for the roots in terms of the charges however this leads to ugly expressions and so we will spare the reader this pain. Instead, we can now express the entropy in terms of the charges and Euler characteristic as

$$S = \frac{1}{4G_2} = \frac{\pi}{3}N^{\frac{3}{2}}\sqrt{4\hat{Q}^{(2)}+\chi^2-\big(\hat{Q}^{(1)}\big)^2-\sqrt{\Big(4\hat{Q}^{(2)}+\chi^2-\big(\hat{Q}^{(1)}\big)^2\Big)^2-64\hat{Q}^{(4)}}}\,, \tag{6.71}$$

[14]One actually obtains four solutions when inverting these results however the constraints $\delta > 0$ and $x > 0$ which follow from regularity leave a single physically sensible solution.

with $\hat{Q}^{(A)}$ the symmetric combination of the charges defined above. This result is completely general and valid for the full STU solution, not just the X^0X^1 truncated solution we studied in detail in this section.[15]

6.2.4 Topological disc

In this section we will study the different regimes of the topological disc solution. The internal metric takes the form of an $S^5 \times S^1_{\phi_4} \times S^1_z$ fibration over the rectangle given by (μ_4^2, w) with $\mu_4^2 \in [0,1]$ and $w \in [0, w_3]$. Within the rectangle the manifold does not degenerate, but on each of the boundaries some part of the metric degenerates, see figure 6.4 for a pictorial representation of the solution.

Before we begin it is convenient to reparametrise the S^7 embedding coordinates as

$$\mu_i = \sqrt{1-\mu_4^2}\, m_i\,, \qquad \text{with} \quad \sum_{i=1}^{3} m_i^2 = 1\,, \tag{6.72}$$

and to define

$$Y = \frac{1}{w}\Big(\mu_4^2 + w(1-\mu_4^2)\,\hat{Y}\Big)\,, \quad \hat{Y} = \sum_i \frac{m_i^2}{w-q_i}\,, \quad P(w) = w\hat{P}(w)\,, \quad f(w) = w\hat{f}(w). \tag{6.73}$$

Note that wY is non-zero except at $w = \mu_4 = 0$ and $\hat{Y}$ is strictly positive definite. With these definitions the metric takes the form

$$\begin{aligned} \mathrm{d}s^2 =& \Big[\hat{P}(w)\Big(\mu_4^2 + w(1-\mu_4^2)\,\hat{Y}\Big)\Big]^{2/3}\Bigg(\mathrm{d}s^2(\mathrm{AdS}_2) + \frac{\hat{f}(w)}{\hat{P}(w)}\mathrm{d}z^2 + \frac{1}{w\hat{f}(w)}\mathrm{d}w^2 \\ &+ \frac{4}{\hat{P}(w)\Big(\mu_4^2 + w(1-\mu_4^2)\,\hat{Y}\Big)}\Bigg[(1-\mu_4^2)\sum_{i=1}^{3}(w-q_i)\Big[\mathrm{d}m_i^2 + m_i^2 D\phi_i^2\Big] \\ &+ \Big[\mathrm{d}\mu_4^2 + \mu_4^2 D\phi_4^2 + \sum_{i=1}^{3}(w-q_i)\Big(\frac{m_i^2\mu_4^2}{1-\mu_4^2}\mathrm{d}\mu_4^2 - 2m_i\mu_4\mathrm{d}\mu_4\mathrm{d}m_i\Big)\Big]\Bigg]\Bigg)\,. \end{aligned} \tag{6.74}$$

Note that the cross terms $\mathrm{d}\mu_4\mathrm{d}m_i$ vanish if all q_i are set equal.

[15] In [95] the authors study solutions in the X^0X^1 truncation and present an expression for the entropy in that truncation. The form given there does not obviously agree with the one presented here when restricted the X^0X^1 case, however one can show that the two expressions give numerically equivalent results. Despite this one can show that this alternative expression for the entropy relies on a relation between the roots w_I which is not true for the full STU solution and therefore cannot be extended to the full multi-charge solution.

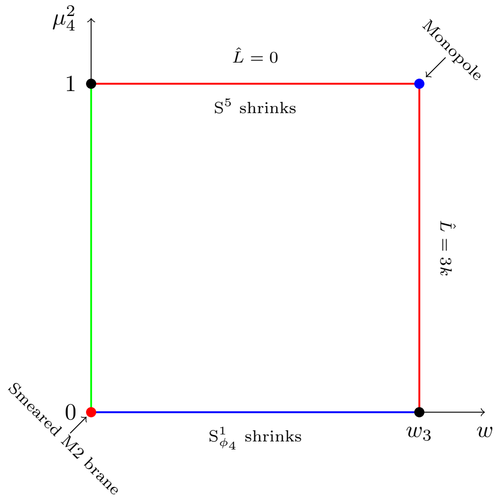


Figure 6.4: *A schematic plot of the rectangle over which the $S^5 \times S^1_{\phi_4} \times S^1_z$ is fibered. The blue line indicates that the ϕ_4 circle pinches smoothly to $\mathbb{R}^2$. Along the green edge the metric is smooth, but has a singularity consistent with a smeared M2 brane at the intersection with the blue line. At the opposite corner sits a monopole leading to a $\mathbb{R}^8/\mathbb{Z}_k$ orbifold.*

Smeared M2-branes

Let us begin with the line $\mu_4^2 = 1$ away from the end-points of w. We see that this point corresponds to the degeneration of the (squashed) S^5, with both the S^1_z and $S^1_{\phi_4}$ circles remaining of finite size. However, it is only for $q_1 = q_2 = q_3 \equiv q$ that the degeneration is smooth and gives $\mathbb{R}^6$. We will therefore restrict to the T^3 truncation from now on.[16] Conversely, consider the degeneration at $\mu_4 = 0$ away from the end-points of w. We see that the only part of the metric which degenerates is the circle $S^1_{\phi_4}$, degenerating smoothly if ϕ_4 has period 2π. This is of course the required period for the round S^7 in the uplift.

Next let us consider the degeneration at $w = 0$ away from both end-points of μ_4. Since all the hatted objects, $\hat{P}(w)$, $\hat{f}(w)$ and $\hat{Y}$ are non-zero and positive at $w = 0$

[16]Note that this simplifies the metric considerably. In particular we have $\hat{Y} = (w - q)^{-1}$ and the cross terms $\mathrm{d}\mu_4 \mathrm{d}m_i$ drop out.

the only degeneration of the metric is the line-interval and the metric is therefore smooth. In particular, note that the singularity that we encountered at $w = 0$ in 4d has been removed in the uplift. Instead, we have a singularity at $w = \mu_4 = 0$. To investigate this properly it is convenient to make the change of coordinates

$$\hat{P}(0)w = -qr\cos^2\left(\frac{\theta}{2}\right), \qquad \hat{P}(0)\mu_4^2 = r\sin^2\left(\frac{\theta}{2}\right), \tag{6.75}$$

and then expand around $r = 0$, which gives the metric in the form

$$\mathrm{d}s^2 = r^{2/3}\Big(\mathrm{d}s^2_{\mathrm{AdS}_2} + \mathrm{d}z^2\Big) + \frac{|q|}{r^{1/3}}\Big(\mathrm{d}r^2 + r^2\Big(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi_4^2\Big) + \sum_i\Big[\mathrm{d}m_i^2 + m_i^2 D\phi_i^2\Big]\Big). \tag{6.76}$$

In comparing with appendix 6.A, where we study the metric of various smeared M2-branes, we see that this takes the form of an M2-brane with world-volume $\mathrm{AdS}_2 \times S^1_z$, localised at the centre of $\mathbb{R}^3$ and smeared on a round S^5. The singularity is meaningful, it arises due to a flavour M2-brane in the geometry. Similar results were found in the M5 [90,99], D3 [98,99] and D4-D8 system [101] cases. Note that we could have taken the limits separately, and in either order, and obtained the same result, the change of coordinates was introduced for convenience. It is also simple to see that the limit $w = 0, \mu_4 = 1$ is smooth given the discussion above. It is interesting to note that the location of the smeared brane for both D3- and M2-branes is the bottom most corner of the diagram in figure 6.4, for the M5-branes it is located along the $w = 0$ side [90]. This seems to be a general feature of odd dimensional spheres, the smeared brane is located at a single point in the uplift, whilst for even dimensional spheres it is located along a line. One also sees a different behaviour between the uplift of spindle solutions on even and odd-dimensional spheres. Uplifting on odd-dimensional spheres allows for the orbifold singularities to be resolved whilst on even-dimensional spheres orbifold singularities remain in the uplifted solution.

Monopoles

We have left the most subtle limits for last. We first want to consider the $w = w_3$ limit before taking the simultaneous $w = w_3$ $\mu_4^2 = 1$ limit. For this it is best to rewrite the metric as a fibration of the S^1_z over the S^7. In doing this rewriting it is convenient to allow an arbitrary gauge choice for the four gauge fields of the form $\delta A_I = n_I \mathrm{d}z$. There are two convenient choices that one can make. The first is such that the gauge field vanishes at the orbifold point, that is we take

$$\delta A_I = \frac{w_3}{2(w_3 - q_I)} \mathrm{d}z \,. \tag{6.77}$$

This choice of gauge has the benefit of leading to a well defined gauge field in 4d when going to the orbifold point, recall that z shrinks there, however it leads to a Killing spinor which is charged under the isometry of the disc. Following our earlier discussion above we may perform a gauge transformation of the gauge fields so that the spinor is uncharged under the rotation generator of the disc. It turns out that this is the best choice of gauge to make.

First note that A_4 is pure gauge at the moment. We should therefore first perform a gauge transformation to set $A_4 = 0$, that is we have $A_4 \to A_4 + \frac{1}{2}\mathrm{d}z = 0$. This is equivalent to performing a change of coordinates $\phi_4 \to \tilde{\phi}_4 = \phi_4 - \frac{1}{2}z$, since this preserves the form of $\mathrm{d}\phi_4 + A_4$. Now consider the phase of the Killing spinor after this transformation: it is proportional to

$$\frac{z}{2} + \sum_{i=1}^{3} \phi_i + \tilde{\phi}_4 \,. \tag{6.78}$$

We may now perform the coordinate shift $\phi_i \to \tilde{\phi}_i = \phi_i - \frac{1}{6}z$ which removes the z dependence of the phase of the Killing spinor. We can ensure that the one-forms $\mathrm{d}\phi_i + A_i$ are invariant under this coordinate transformation provided we perform a gauge transformation

$$A_i \to A_i + \frac{1}{6}\mathrm{d}z \,, \qquad A_4 \to A_4 + \frac{1}{2}\mathrm{d}z \,. \tag{6.79}$$

After this slightly round-about argument we conclude that the gauge transformation above is equivalent to removing the z dependence in the 4d Killing spinor and the expressions above we should set

$$n_i = \tfrac{1}{6} \qquad \text{and} \qquad n_4 = \tfrac{1}{2} \,. \tag{6.80}$$

Notice that the gauge choice we make here will not change the analysis that we have performed earlier in this section, however it has an important effect here.

Rewriting the metric in the form of an S^1_z fibration over the seven sphere we have

$$\begin{aligned}
\mathrm{d}s^2 =& \mathrm{e}^{2A}\bigg[\mathrm{d}s^2(\mathrm{AdS}_2) + \frac{1}{w\hat{f}(w)}\mathrm{d}w^2 + \frac{4(w-q)(1-\mu_4^2)}{\hat{P}(w)(w-q\mu_4^2)}\sum_{i=1}^{3}\mathrm{d}m_i^2 \\
&+ R_z\Big(\mathrm{d}z - L\sum_{i=1}^{3} m_i^2\mathrm{d}\phi_i\Big)^2 + R_1 m_1^2\Big(\mathrm{d}\phi_1 - S_1\sum_{i=2}^{3} m_i^2\mathrm{d}\phi_i\Big)^2 \\
&+ R_2 m_2^2\Big(\mathrm{d}\phi_2 - S_2 m_3^2\mathrm{d}\phi_3\Big)^2 + R_3 m_3^2\mathrm{d}\phi_3^2 + \frac{4w(w-q)\mu_4^2}{\hat{P}(w)(w-q\mu_4^2)}\mathrm{d}\phi_4^2\bigg]\,,
\end{aligned} \tag{6.81}$$

where

$$\begin{aligned}
R_z =& \frac{(q+2w)^2(1-\mu_4^2) + 9(w-q\mu_4^2)\hat{f}(w)}{9\hat{P}(w)}\,, \\
R_1 =& \frac{4(w-q)^2(1-\mu_4^2)\Big((1-m_1^2)(2w+q)^2(1-\mu_4^2) + 9(w-q\mu_4^2)\hat{f}(w)\Big)}{(w-q\mu_4^2)\hat{P}(w)\Big((2w+q)^2(1-\mu_4^2) + 9(w-q\mu_4^2)\hat{f}(w)\Big)}\,, \\
R_2 =& \frac{4(w-q)^2(1-\mu_4^2)\Big(m_3^2(2w+q)^2(1-\mu_4^2) + 9(w-q\mu_4^2)\hat{f}(w)\Big)}{(w-q\mu_4^2)\hat{P}(w)\Big((1-m_1^2)(2w+q)^2(1-\mu_4^2) + 9(w-q\mu_4^2)\hat{f}(w)\Big)}\,, \\
R_3 =& \frac{36(w-q)^2(1-\mu_4^2)\hat{f}(w)}{\hat{P}(w)\Big(m_3^2(2w+q)^2(1-\mu_4^2) + 9(w-q\mu_4^2)\hat{f}(w)\Big)}\,,
\end{aligned} \tag{6.82}$$

and

$$\begin{aligned}
L =& \frac{6(w-q)(2w+q)(1-\mu_4^2)}{(2w+q)^2(1-\mu_4^2) + 9(w-q\mu_4^2)\hat{f}(w)}\,, \\
S_1 =& \frac{(2w+q)^2(1-\mu_4^2)}{(1-m_1^2)(2w+q)^2(1-\mu_4^2) + 9(w-q\mu_4^2)\hat{f}(w)}\,, \\
S_2 =& \frac{(2w+q)^2(1-\mu_4^2)}{m_3^2(2w+q)^2(1-\mu_4^2) + 9(w-q\mu_4^2)\hat{f}(w)}\,.
\end{aligned} \tag{6.83}$$

From the form of the functions it is clear that all the R_i vanish as $\mu_4^2 \to 1$, whilst R_z remains finite. In addition the fibration functions $S_{1,2}$ vanish in this limit and we have

$$L(\mu_4^2 = 1, w) = 0\,. \tag{6.84}$$

This leads to the smooth shrinking of the S^5 that we saw previously. Conversely, at $w = w_3$ the R_i are all finite, $S_{1,2}$ become constant (in particular the μ_4 dependence drops out) and the S^5 has a non-zero radius but is twisted. Note that the function

L is a non-zero constant at $w = w_3$,

$$L(\mu_4^2, w = w_3) = \frac{6(w_3 - q)}{2w_3 + q} = -\frac{3k\Delta z}{2\pi} \equiv \frac{\hat{L}\Delta z}{2\pi}\,. \tag{6.85}$$

Clearly there is a jump at the corner ($\mu_4^2 = 1, w = w_3$) of the rectangle depending on the direction we approach the corner from. This signifies the existence of a monopole source located there. The charge of the monopole is computed by evaluating the Chern number of the line-bundle,

$$Q_m = \frac{1}{\Delta z}\int \mathrm{d}Dz\,. \tag{6.86}$$

For the case at hand we find that the monopole charge is

$$Q_m^i = \hat{L} = 3k\,, \tag{6.87}$$

and this accounts for the singularity at the origin of the disc in 4d.

Since this is somewhat subtle let us study this in a different way. Lets take the simultaneous $\mu_4^2 = 1, w = w_3$ limit by changing to the coordinates

$$\mu_4^2 = 1 - r^2\sin^2\xi\,, \qquad w = w_3 + (w_3 - q)\hat{f}'(w_3)r^2\cos^2\xi\,. \tag{6.88}$$

In this limit the metric becomes

$$\begin{aligned}\mathrm{d}s^2 = w_3^{2/3}\bigg[&\mathrm{d}s^2(\mathrm{AdS}_2) + 4\mathrm{d}\phi_4^2 + \frac{4}{w_3^{2/3}}\bigg\{\mathrm{d}r^2 \\ &+ r^2\bigg(\mathrm{d}\xi^2 + \frac{\hat{f}'(w_3)^2}{4}\cos^2\xi\mathrm{d}z^2 + \sin^2\xi\sum_{i=1}^{3}\Big[\mathrm{d}m_i^2 + m_i^2\Big(\mathrm{d}\phi_i + \frac{\hat{f}'(w_3)}{6}\mathrm{d}z\Big)^2\Big]\bigg)\bigg\}\bigg]\,.\end{aligned} \tag{6.89}$$

This is $\mathrm{AdS}_2 \times S^1_{\phi_4} \times \mathbb{R}^8/\mathbb{Z}_k$ where we have used (6.31), and we see that the orbifold singularity at the centre of the disc in 4d arises in 11d from a quotient space $\mathbb{R}^8/\mathbb{Z}_k$, i.e. it is a monopole. A natural interpretation is that this corresponds to a regular puncture whilst the smeared M2 brane arises due to an irregular puncture. It would be interesting to understand this from a field theory computation.

R-symmetry vector

We conclude the regularity analysis by identifying the R-symmetry of the solution. From the general form of the metric without the gauge shifts the R-symmetry vector is simply

$$R_{1\mathrm{d}} = \partial_z\,. \tag{6.90}$$

However, recall that we performed a gauge transformation above in order for the Killing spinor to be independent of the spindle U(1) coordinate. Taking into account the gauge transformation and denoting the new coordinates $\tilde{\phi}_I$ we have

$$R_{1\text{d}} = \partial_z = \partial_z + \frac{1}{6}\sum_{i=1}^{3}\partial_{\tilde{\phi}_i} - \frac{1}{2}\partial_{\tilde{\phi}_4}\,. \tag{6.91}$$

Note that this is different to the result of the spindle. The $\tilde{\phi}$ terms do not give the canonical R-symmetry for the parent ABJM theory as in the spindle example.

Flux quantization

As mentioned earlier, there are two more cycles that we should consider when quantizing the flux. The first is the $S^7/\mathbb{Z}_k$ lying at the top right rectangle in figure 6.4 where the monopole is located. We find that the quantization of the flux imposes that N/k is integer. We shall therefore define

$$N = k\hat{N}\,, \qquad \hat{N} \in \mathbb{Z}. \tag{6.92}$$

The second extra cycle we will consider is the one located at the left-most corner of the rectangle, that is the limiting point where the smeared M2 brane is located. This cycle is given by $S^5 \times S^1_z \times I_\theta$, using the parametrisation (6.75). Integration of $*\,G_4$ over this cycle yields the quantization condition

$$\frac{1}{(2\pi\ell_p)^6}\int_{S^5\times S^1_z\times I_\theta} *\,G_4 = \frac{L^6}{\pi^2\ell_p^6}\frac{\Delta z}{2\pi} = \frac{N}{2}\frac{\Delta z}{2\pi} \in \mathbb{Z}\,. \tag{6.93}$$

We may rewrite this in terms of the charges which gives

$$N\left(\frac{1}{k} + 3Q\right) \in \mathbb{Z}\,. \tag{6.94}$$

We see that this is generically satisfied once N is defined as in (6.92).

Entropy

We may now compute the 2d Newton's constant and thereby the entropy. We find

$$S = \frac{1}{4G_2} = \frac{2\sqrt{2}\pi N^{3/2}}{3k}\sqrt{\frac{p^3}{1+3p}}\,, \tag{6.95}$$

with $Q = pk$. Contrast this expression with the entropy for the spindle in (6.71).

Appendices

6.A Smeared M2-branes

In this appendix we will study the form of the metric for a single stack of M2-branes smeared over internal manifolds of various dimensions, see [110] for D-branes of this form. Our interest is in the singularity structure of the solutions, so that we may compare with the singularities we find in the topological disc solutions of section 6.2.4.

To begin, recall that the metric of a single stack of M2 branes in flat space, takes the form

$$\mathrm{d}s^2 = H^{-2/3}\mathrm{d}s^2(\mathcal{M}^{1,2}) + H^{1/3}\mathrm{d}s^2_{8d}\,. \tag{6.96}$$

The M2-branes lie along $\mathcal{M}^{1,2}$ whilst the function H is harmonic on the transverse 8d space. It is typical to impose that the transverse space takes the form of a cone with base a compact manifold and that the function H depends only on the radial coordinate. Indeed writing the transverse space as a cone

$$\mathrm{d}s^2_{8d} = \mathrm{d}r^2 + r^2\mathrm{d}s^2(X_7)\,, \tag{6.97}$$

and taking H to be a function of the radial coordinate only, the harmonic function is given by

$$H(r) = \alpha + \frac{\beta}{r^6}\,, \tag{6.98}$$

with α and β two integration constants. This solution describes a stack of M2-branes localised at the tip of the cone, however this is not the most general solution one can construct.

Rather than considering a cone, one may decompose the space into the direct product of two pieces of dimensions s and $8-s$ respectively. The first part is a space over which we will smear the M2 brane, whilst the second will be taken to be a cone. Concretely we decompose the transverse 8d space as

$$\mathrm{d}s^2_{8d} = \mathrm{d}r^2 + r^2\mathrm{d}s^2(X_{7-s}) + \mathrm{d}s^2(Y_s)\,, \tag{6.99}$$

and impose that the function H is depends only on the radial coordinate once again. The function H now takes the form

$$H = \alpha + \frac{\beta}{r^{6-s}}\,. \tag{6.100}$$

For $s=0$ this clearly reduces to the usual M2 brane solution.

We may now insert the expressions for the harmonic function into the full brane solution and study the solution close to the brane in the $r \to 0$ limit. Close to $r = 0$ the fully localized M2 brane metric looks like

$$\mathrm{d}s^2 \sim r^4 \mathrm{d}s^2(\mathcal{M}^{1,2}) + r^{-2}\Big(\mathrm{d}r^2 + \mathrm{d}s^2(X_7)\Big)\,, \tag{6.101}$$

whilst the smeared metric looks like

$$\mathrm{d}s^2 \sim r^{4-2s/3} \mathrm{d}s^2(\mathcal{M}^{1,2}) + r^{s/3-2}\Big(\mathrm{d}r^2 + r^2\mathrm{d}s^2(X_{7-s}) + \mathrm{d}s^2(Y_s)\Big)\,. \tag{6.102}$$

We will use this result to show that the naively singular solution obtained in section 6.2.4 is in fact indicating the presence of a smeared M2 brane.

6.B Killing spinors of the multi-charge solution

In this appendix we give the Killing spinors of the 4d $\mathrm{AdS}_2 \times \Sigma$ solution. In order to use a consistent set of conventions for the supersymmetry transformations and the equations of motion that our solution solve we will first review how the action (6.1) is embedded into the general classification of 4d $\mathcal{N} = 2$ gauged supergravity in the presence of vector multiplets. We will mostly follow the conventions in [107] for 4d $\mathcal{N} = 2$ gauged supergravity. The bosonic part of the Lagrangian is[17]

$$16\pi G_N \mathcal{L} = \big(R - 2g_{\alpha\bar{\beta}}\partial_\mu z^\alpha \partial^\mu \bar{z}^{\bar{\beta}} - 2\mathcal{V}\big) \star 1 + \mathrm{Im}[\mathcal{N}_{IJ}] F^I \wedge \star F^J - \mathrm{Re}[\mathcal{N}_{IJ}] F^I \wedge F^J\,, \tag{6.103}$$

where

$$\begin{aligned}
\mathrm{e}^{-\mathcal{K}} &= \mathrm{i}\big(X^{(I)}\bar{\mathcal{F}}_I - \bar{X}^{(I)}\mathcal{F}_I\big)\,, && (6.104)\\
g_{\alpha\bar{\beta}} &= \partial_\alpha \partial_{\bar{\beta}} \mathcal{K}\,, && (6.105)\\
\mathcal{N}_{IJ} &= \bar{\mathcal{F}}_{IJ} + \mathrm{i}\frac{N_{IK}N_{JL}X^{(K)}X^{(L)}}{N_{PQ}X^{(P)}X^{(Q)}}\,, && (6.106)\\
N_{IJ} &= 2\,\mathrm{Im}\,\mathcal{F}_{IJ}\,, \quad \mathcal{F}_I = \partial_I F\,, \quad \mathcal{F}_{IJ} = \partial_I\partial_J F\,, && (6.107)\\
\mathcal{V} &= -3X^{(I)}\bar{X}^{(J)}\vec{P}_I \cdot \vec{P}_J + g^{\alpha\bar{\beta}}\nabla_\alpha\nabla_{\bar{\beta}}\bar{X}^{(J)}\vec{P}_I \cdot \vec{P}_J && (6.108)
\end{aligned}$$

and F is the prepotential. For the prepotential

$$F = -\mathrm{i}\sqrt{X^{(1)}X^{(2)}X^{(3)}X^{(4)}}\,, \tag{6.109}$$

[17] Note that we redefine the fields in [107] in order for the Newton's constant to be an overall prefactor.

this agrees with the Lagrangian in (6.1). It is convenient to introduce the three independent scalars (we set axions to vanish in the following) via

$$\frac{X^{(1)}}{X^{(4)}} = \tau_2\tau_3\,, \quad \frac{X^{(2)}}{X^{(4)}} = \tau_1\tau_3\,, \quad \frac{X^{(3)}}{X^{(4)}} = \tau_1\tau_2\,, \quad \tau_i = \mathrm{e}^{-\phi_i}\,. \tag{6.110}$$

The Lagrangian presented in the main text was understood to be subject to the gauge condition

$$X^{(1)}X^{(2)}X^{(3)}X^{(4)} = 1\,. \tag{6.111}$$

The gravitino supersymmetry variation is

$$\begin{aligned}\delta\psi^i_\mu = &\Big(\nabla_\mu - \frac{\mathrm{i}}{2}\mathcal{A}_\mu\Big)\epsilon^i - \frac{\mathrm{i}}{2}\zeta_I A^I_\mu \sigma^3{}_j{}^i\epsilon^j \\ &+ \mathrm{e}^{\mathcal{K}/2}X^{(I)}\,\mathrm{Im}[\mathcal{N}_{IJ}]\not{F}^J\epsilon^{ij}\gamma_\mu\epsilon_j + \frac{\mathrm{i}}{2}\mathrm{e}^{\mathcal{K}/2}\zeta_I X^{(I)}\sigma^{3\,ij}\gamma_\mu\epsilon_j\,,\end{aligned} \tag{6.112}$$

whilst for the gaugino it is

$$\delta\chi^\alpha_i = \not{\partial} z^\alpha\epsilon_i + \frac{1}{4}g^{\alpha\bar{\beta}}\nabla_{\bar{\beta}}\bar{X}^{(I)}N_{IJ}\big(F^J - \mathrm{i}\star F^J\big)_{ab}\gamma^{ab}\epsilon_{ij}\epsilon^j - \mathrm{i}g^{\alpha\bar{\beta}}\nabla_{\bar{\beta}}\bar{X}^{(I)}\sigma^3{}_{ij}\epsilon^j\,. \tag{6.113}$$

Spinors are taken to be chiral with raised indices and anti-chiral for lowered indices, that is

$$\gamma_5\epsilon^i = \epsilon^i\,, \qquad \gamma_5\epsilon_i = -\epsilon_i\,. \tag{6.114}$$

The supersymmetry parameters are taken to be Majorana, with charge conjugation defined as

$$\epsilon^i = \epsilon_i^C\,. \tag{6.115}$$

Following [111] we combine the Majorana spinors into a Dirac spinor, $\psi \equiv \psi^1 + \psi_2$ and $\epsilon = \epsilon^1 + \epsilon_2$ which leads to the supersymmetry variations

$$\begin{aligned}\delta\psi = &\Big[\nabla_\mu - \frac{\mathrm{i}}{2}\mathcal{A}_\mu\gamma_5 + \frac{\mathrm{i}}{2}\zeta_I A^I_\mu + \mathrm{e}^{\mathcal{K}/2}\,\mathrm{Im}\,\mathcal{N}_{IJ}\not{F}^J\gamma_\mu\Big(\mathrm{Re}\,X^{(I)} - \mathrm{i}\,\mathrm{Im}\,X^{(I)}\gamma_5\Big) \\ &- \frac{\mathrm{i}}{2}\gamma_\mu\zeta_I\Big(\mathrm{Re}\,X^{(I)} - \mathrm{i}\,\mathrm{Im}\,X^{(I)}\gamma_5\Big)\Big]\epsilon\,.\end{aligned} \tag{6.116}$$

In the following we will not need the gravitini Killing spinor equation in this form so we suppress the details.

In order to solve the Killing spinor equations we will take the following basis of 4d gamma matrices,

$$\gamma_0 = \mathrm{i}\sigma_2\otimes\sigma_3\,, \quad \gamma_1 = \sigma_3\otimes\sigma_3\,, \quad \gamma_2 = 1_{2\times2}\otimes\sigma_1\,, \quad \gamma_3 = 1_{2\times2}\otimes\sigma_2\,, \tag{6.117}$$

and take the unit radius metric on AdS_2 to be

$$\mathrm{d}s^2(\mathrm{AdS}_2) = -r^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2}\,. \tag{6.118}$$

The Killing spinors on AdS_2, η satisfy the Killing spinor equation

$$\Big[\hat{\nabla}_a - \frac{\sigma}{2}\rho_a\Big]\eta = 0\,, \tag{6.119}$$

with σ a sign that we will determine later, and ρ_a the 2d gamma matrices which can be obtained from the above 4d gamma matrices. Note that if η solves the Killing spinor equation for σ then $\rho_3\eta$ solves the Killing spinor equation for $-\sigma$. It is simple to show that the solutions to the Killing spinor equations are

$$\eta_+ = \Big(a_2 r^{-\frac{1}{2}}, \sqrt{r}(a_1 + a_2 t)\Big)\,, \qquad \eta_- = \Big(\sqrt{r}(a_1 + a_2 t), a_2 r^{-\frac{1}{2}}\Big)\,. \tag{6.120}$$

We can decompose the 4d spinors, depending on the sign σ, in terms of the tensor product of the spinor on AdS_2 and the spinor on Σ, whether it be the spindle or disc. We have

$$\epsilon_\pm = \eta_\pm \otimes \theta_\pm\,, \tag{6.121}$$

and the two-component spinor $\theta_\pm$ on Σ satisfies the projection condition

$$\sigma_3\theta_\pm = \pm\theta_\pm\,. \tag{6.122}$$

We may now insert this spinor ansatz into the supersymmetry condition (6.116). Reducing on AdS_2 we find

$$\delta\Psi_a = \Bigg[\hat{\nabla}_a - \frac{\mathrm{i}\sigma}{2}\gamma_{23}\gamma_a + \frac{P'(w)}{4\sqrt{P(w)}}\Big(\frac{\sigma}{2} - \frac{\sqrt{f(w)}}{2\sqrt{P(w)}}\gamma_2 + \frac{\mathrm{i}\sigma w}{2\sqrt{P(w)}}\gamma_{23}\Big)\gamma_a\Bigg]\epsilon\,, \tag{6.123}$$

where we have introduced the sign α which satisfies $w - q_I = \sigma|w - q_I|$. The large bracketed term is a projection condition that we must impose on the spinors $\theta_\pm$. The resultant Killing spinor equation becomes

$$\Big[\hat{\nabla}_a - \frac{\mathrm{i}\sigma}{2}\gamma_a\gamma_{23}\Big]\eta_\pm \otimes \theta_\pm = 0\,, \tag{6.124}$$

which is immediately satisfied by our decomposition if we take η_+ for $\sigma = 1$ and η_- for $\sigma = -1$. It remains to solve the two remaining components of the gravitino Killing spinor equation. We introduce an arbitrary gauge shift for the gauge fields of the form

$$A^I \to A^I + n^I\mathrm{d}z\,, \tag{6.125}$$

which is the same gauge we introduced to study the R-symmetry vector in the main text. Inserting the ansatz into the Killing spinor equation we find the solution

$$\theta_+ = P(w)^{-1/8} \mathrm{e}^{\frac{\mathrm{i}z}{2}\left(1-\sum_I n^I\right)} \left(\sqrt{\sqrt{P(w)}+w}\,,\ -\sqrt{\sqrt{P(w)}-w} \right), \tag{6.126}$$

and

$$\theta_- = \sigma_3 \cdot \theta_+ = P(w)^{-1/8} \mathrm{e}^{\frac{\mathrm{i}z}{2}\left(1-\sum_I n^I\right)} \left(\sqrt{\sqrt{P(w)}+w}\,,\ \sqrt{\sqrt{P(w)}-w} \right). \tag{6.127}$$

One can then check that this solves the gravitini Killing spinor equations. Note that for the spindle we may remove the phase by taking $n^I = \frac{1}{4}$ as claimed in the main text. For the disc, note that the gauge transformation for A^4 is further shifted as discussed in section 6.2.4. Taking this into account we indeed find the R-symmetry vector given in (6.91). We see that the Killing spinors are manifestly non-constant and therefore we see immediately that supersymmetry on either the spindle or the disc is not preserved by the usual topological twist.

First let us focus on the spindle case. Note that we have the identity

$$\sqrt{f(w)} = \sqrt{\sqrt{P(w)}+w} \cdot \sqrt{\sqrt{P(w)}-w}\,, \tag{6.128}$$

moreover at a root of $f(w)$ we have

$$P(w_*) = w_*^2\,. \tag{6.129}$$

We therefore see that the Killing spinor is never vanishing. Only at the poles of the spindle does a component of the spinor vanish and for our solutions which preserve supersymmetry via the anti-twist, where one root is positive and the second is negative, different components of the spinor vanish. At the poles of the Killing spinor the preserved Killing spinor become chiral and for the anti-twist are of different chiralities on the two halves of the spindle.

The disc is slightly more subtle. At the non-zero root we have the same discussion as for the spindle at the positive root. We obtain a chiral spinor. At $w = 0$ the spinor actually vanishes as $w^{1/8}$. Dividing through by this vanishing conformal factor the resultant spinor is the sum of a chiral and anti-chiral spinor.

Chapter 7

Rotating D2-branes on spindles

7.1 4d black hole solution

In this section we will consider the spindle solution of 4d $\mathcal{N}=2$ Einstein-Maxwell supergravity. This was first discussed in the context of a spindle geometry in [93], having been originally found in [112], and we will briefly review the analysis of the former paper. First, we discuss the full black hole solution, and afterward we discuss the near-horizon limit.

7.1.1 Full solution

The action for 4d $\mathcal{N}=2$ Einstein–Maxwell supergravity we will use is

$$S = \frac{1}{16\pi G_{(4)}} \int \Big((R+6) \star 1 - F \wedge \star F \Big) \,, \tag{7.1}$$

where we have normalised the cosmological constant for simplicity. A solution to this action is[1]

$$\begin{aligned} \mathrm{d}s^2 = \frac{1}{H(r,\theta)^2} \Big[&- \frac{Q(r)}{S(r,\theta)} \Big(\mathrm{d}t - a \sin^2\theta \, \mathrm{d}\phi\Big)^2 + \frac{S(r,\theta)}{Q(r)} \, \mathrm{d}r^2 \\ &+ \frac{P(\theta)}{S(r,\theta)} \sin^2\theta \Big(a \, \mathrm{d}t - (r^2+a^2) \, \mathrm{d}\phi \Big)^2 + \frac{S(r,\theta)}{P(\theta)} \, \mathrm{d}\theta^2 \Big] \,, \\ A = - \frac{er}{S(r,\theta)} \big(&\mathrm{d}t - a \sin^2\theta \mathrm{d}\phi\big) + \frac{g\cos\theta}{S(r,\theta)} \big(a\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi \big) \,, \end{aligned} \tag{7.2}$$

where

$$\begin{aligned} H(r,\theta) &= 1 - \alpha \, r \cos\theta \,, \\ S(r,\theta) &= r^2 + a^2 \cos^2\theta \,, \\ P(\theta) &= 1 - 2\alpha m \cos\theta + \Big(\alpha^2 (a^2 + e^2 + g^2) - a^2 \Big) \cos^2\theta \,, \\ Q(r) &= \Big(r^2 - 2mr + a^2 + e^2 + g^2 \Big) (1 - \alpha^2 r^2) + (a^2 + r^2) r^2 \,. \end{aligned} \tag{7.3}$$

[1] We follow the presentation of the solution in [93] for easy comparison with existing literature.

We demand that the metric has the correct signature. This requires that the functions $S(r,\theta)$, $Q(r)$ and $P(\theta)$ are all positive. The range of θ is taken to be $[0,\pi]$ and it is convenient to define $\theta_- = 0$, $\theta_+ = \pi$. The requirement of the correct metric signature then implies

$$\begin{aligned} P(\theta_-) &= 1 - 2\alpha m + \Big(\alpha^2(a^2+e^2+g^2) - a^2\Big) > 0\,, \\ P(\theta_+) &= 1 + 2\alpha m + \Big(\alpha^2(a^2+e^2+g^2) - a^2\Big) > 0\,. \end{aligned} \tag{7.4}$$

Following [93] we study the metric on constant (t,r) slices. Expanding around $\theta_\pm$ the θ,ϕ terms of the metric become

$$\mathrm{d}s^2 \sim \frac{1}{H(r,\theta_\pm)^2}\frac{r^2+a^2}{P(\theta_\pm)}\Big[\mathrm{d}\theta^2 + (\theta-\theta_\pm)^2 P(\theta_\pm)^2\,\mathrm{d}\phi^2\Big]\,. \tag{7.5}$$

We are interested in knowing whether this geometry can be made regular. Since

$$P(\theta_+) - P(\theta_-) = 4\alpha m\,, \tag{7.6}$$

it is not possible to assign a period to ϕ such that (7.5) is smooth at both poles when $\alpha \neq 0$, i.e. it cannot be a smooth S^2. Instead one must contend with conical singularities at the poles. We may define the period of ϕ to be

$$\Delta\phi = \frac{2\pi}{P(\theta_+)\,n_+} = \frac{2\pi}{P(\theta_-)\,n_-}\,. \tag{7.7}$$

If $n_\pm$ are positive relatively prime integers, this gives the weighted projective space $\mathbb{WCP}^1_{[n_-,n_+]}$, often called a spindle.

7.1.2 Near-horizon limit

If we impose extremality and take the near-horizon limit of the black hole solution presented earlier in this section, we obtain a fibered AdS_2 space, see [93]. We will be rather short in presenting the details of the near-horizon solution here, as it is the rotating version of the 4d solution presented in chapter 6, provided that we set all four charges there equal. Because of this, large parts of the analysis are similar or equivalent.

The near-horizon geometry is given by

$$\begin{aligned} \mathrm{d}s^2_{\mathrm{NH}} &= \sqrt{P(w)}\left[-r^2\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + \frac{1}{f(w)}\,\mathrm{d}w^2 + \frac{f(w)}{P(w)}\,(\mathrm{d}z + jr\,\mathrm{d}t)^2\right], \\ A &= h(w)\big(\mathrm{d}z + jr\,\mathrm{d}t\big)\,, \end{aligned} \tag{7.8}$$

with

$$
\begin{aligned}
P(w) &= \left[(w-q)^2 + \frac{j^2}{4}\right]^2 , \\
f(w) &= P(w) - \big(1-j^2\big)w^2 , \\
h(w) &= -\sqrt{1-j^2}\left[\frac{4(w-q)(w+q)-j^2}{4(w-q)^2+j^2}\right] .
\end{aligned}
\tag{7.9}
$$

Again, we find that the geometry cannot be made smooth at both poles of the spindle. The period of z is fixed by

$$
\frac{\Delta z}{2\pi} = \frac{2\sqrt{P(w_\pm)}}{|f'(w_\pm)|\, n_\pm} . \tag{7.10}
$$

It is sometimes convenient to define the 2π-periodic coordinate $\hat{z}$

$$
\hat{z} = \frac{2\pi}{\Delta z}\, z . \tag{7.11}
$$

We omit the explicit computation of the integrated fluxes and the Euler characteristic here, as it is quite similar to the one presented in the previous chapter. In particular, the finding of section 6.1.3 that supersymmetry is preserved via an anti-twist still holds.

7.2 Uplift to massive type IIA

In this section we will consider the holographic duals of 3d Chern–Simons SCFTs arising from D2-branes wrapped on a spindle. Massive type IIA supergravity on a topological six-sphere gives rise to 4d $\mathcal{N}=8$ ISO(7) supergravity. As shown in [113] there exists a consistent truncation of this theory to 4d $\mathcal{N}=2$ Einstein–Maxwell supergravity. We may therefore uplift the known solutions containing spindles within Einstein–Maxwell to massive type IIA and study their properties. Using the formulas presented in [114], the uplift of a solution of 4d $\mathcal{N}=2$ Einstein–Maxwell in Einstein frame is

$$
\begin{aligned}
\mathrm{d}s^2_{10} &= L^2\,(2+\cos^2\alpha)^{1/8}\sqrt{1+\cos^2\alpha}\left[\frac{1}{3}\mathrm{d}s^2_4 + \mathrm{d}s^2_{\mathrm{Y}_6}\right] , \\
\mathrm{d}s^2_{\mathrm{Y}_6} &= \frac{1}{2}\,\mathrm{d}\alpha^2 + \frac{3\sin^2\alpha}{4+2\cos^2\alpha}\,\hat{\eta}^2 + \frac{\sin^2\alpha}{1+\cos^2\alpha}\,\mathrm{d}s^2_{\mathrm{KE}_4} ,
\end{aligned}
\tag{7.12}
$$

where

$$\hat{\eta} = \eta + \frac{1}{3}A = \mathrm{d}\psi + \sigma + \frac{1}{3}A\,, \tag{7.13}$$

$$\mathrm{d}\eta = 2\,J_{\mathrm{KE}_4}\,, \tag{7.14}$$

and A is the 4d gauge field. The coordinate α is defined on an interval $[0, \pi]$, and the metric on Y_6 is that of a sine cone, which is squashed due to the α-dependent prefactors. The Kähler–Einstein metric is normalized to satisfy $R_{mn} = 6g_{mn}$ and J_{KE_4} is the Kähler form. Together η and $\mathrm{d}s^2_{\mathrm{KE}_4}$ form a squashed 5d Sasaki–Einstein manifold.

The other fields in the uplift to massive type IIA are given by

$$\mathrm{e}^{\Phi} = \frac{1}{m^{4/5}}\frac{(2+\cos^2\alpha)^{3/4}}{1+\cos^2\alpha}\,, \tag{7.15}$$

$$H_3 = \frac{L^2}{m^{2/5}}\left[\frac{2\sin^3\alpha}{(1+\cos^2\alpha)^2}\,J_{\mathrm{KE}_4}\wedge\mathrm{d}\alpha + \frac{1}{2\sqrt{3}}\sin\alpha\,\mathrm{d}\alpha\wedge\star_4 F\right], \tag{7.16}$$

$$F_0 = \frac{m}{L}\,, \tag{7.17}$$

$$\begin{aligned} F_2 = L\,m^{3/5}\bigg[&-\frac{\sin^2\alpha\,\cos\alpha}{(2+\cos^2\alpha)(1+\cos^2\alpha)}\,J_{\mathrm{KE}_4} - \frac{3\sin\alpha\,(2-\cos^2\alpha)}{2(2+\cos^2\alpha)^2}\,\mathrm{d}\alpha\wedge\hat{\eta} \\ &+\frac{\cos\alpha}{2(2+\cos^2\alpha)}\,F - \frac{1}{2\sqrt{3}}\cos\alpha\,\star_4 F\bigg]\,, \end{aligned} \tag{7.18}$$

$$\begin{aligned} F_4 = L^3\,m^{1/5}\bigg[&\frac{\sin^4\alpha\,(2+3\cos^2\alpha)}{2(1+\cos^2\alpha)^2}J^2_{\mathrm{KE}_4} + \frac{3\sin^3\alpha\,\cos\alpha\,(4+\cos^2\alpha)}{2(1+\cos^2\alpha)(2+\cos^2\alpha)}J_{\mathrm{KE}_4}\wedge\mathrm{d}\alpha\wedge\hat{\eta} \\ &+\frac{1}{\sqrt{3}}\,\mathrm{dvol}_4 - \frac{1}{4}\sin\alpha\,\cos\alpha\left(\frac{2\sin\alpha\,\cos\alpha}{1+\cos^2\alpha}\,J_{\mathrm{KE}_4} + \mathrm{d}\alpha\wedge\hat{\eta}\right)\wedge F \\ &-\frac{1}{4\sqrt{3}}\left(\frac{2\sin^2\alpha}{1+\cos^2\alpha}\,J_{\mathrm{KE}_4} + \frac{3\sin\alpha\,\cos\alpha}{2+\cos^2\alpha}\,\mathrm{d}\alpha\wedge\hat{\eta}\right)\wedge\star_4 F\bigg]\,. \end{aligned} \tag{7.19}$$

We have redefined L and m with respect to some of the existing literature. Where necessary we will provide the dictionary.

7.2.1 Metric regularity

We now want to study the regularity of the uplifted 10d solution. The discussion proceeds in a similar manner to [93]. That is, we write the 10d geometry (7.12) as

$$\mathrm{d}s^2_{10} = L^2\,(2+\cos^2\alpha)^{1/8}\sqrt{1+\cos^2\alpha}\,\left[\frac{\sqrt{P(w)}}{3}\,\mathrm{d}s^2_{\mathrm{AdS}_2} + \mathrm{d}s^2_8\right], \tag{7.20}$$

and study the regularity of the 8d metric at fixed (t, r)

$$\mathrm{d}s_8^2 = \frac{\sqrt{P(w)}}{3f(w)}\mathrm{d}w^2 + \frac{f(w)}{3\sqrt{P(w)}}Dz^2 + \frac{1}{2}\mathrm{d}\alpha^2 + \frac{3\sin^2\alpha}{4+2\cos^2\alpha}\hat{\eta}^2 + \frac{\sin^2\alpha}{1+\cos^2\alpha}\mathrm{d}s^2_{\mathrm{KE}_4}\,, \tag{7.21}$$

where we have defined

$$Dz = \mathrm{d}z + jr\mathrm{d}t\,. \tag{7.22}$$

We find this 8d space to be an $S^1_z \times \mathrm{SE}_5$ fibration over a rectangle parametrized by the coordinates w and α. A plot of this rectangle is provided in figure 7.1 where we indicate the regions where we will study regularity in detail. In the interior of the rectangle regularity is ensured, since none of the coefficients in (7.21) vanish there.

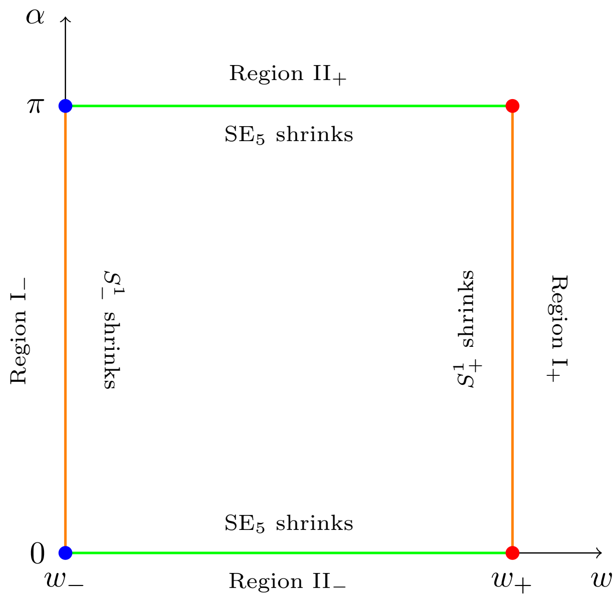


Figure 7.1: *A schematic plot of the rectangle over which there is a* $\mathrm{SE}_5 \times S^1_z$ *fibration. A monopole is located at each corner of the rectangle and they are called region III in the main text. The red monopoles have weight* n_+ *whilst the blue monopoles have weight* n_-.

Region I: the poles of the spindle

We will first focus on the metric away from $\alpha = 0, \pi$ and study the degeneration at the two poles of the spindle. To that end we construct the following U(1) Killing vectors, each of which has vanishing norm at one of the poles:

$$\partial_{\phi_\pm} = c_\pm \left(\partial_z - \frac{1}{3} h(w_\pm)\, \partial_\psi \right) , \tag{7.23}$$

with $c_\pm$ constants that we will fix momentarily. By expanding the norm of these Killing vectors around $w_\pm$ we find that[2]

$$||\partial_{\phi_\pm}||^2 = \frac{c_\pm^2\, |f'(w_\pm)|}{3\sqrt{1-j^2}\, |w_\pm|}\, \Delta w + \mathcal{O}(\Delta w^2) , \tag{7.24}$$

where $\Delta w = \mp(w - w_\pm)$ is the positive distance away from the pole. Consequently, the metric spanned by w and $\phi_\pm$ close to $w_\pm$ can be written as

$$\begin{aligned} &\frac{\sqrt{1-j^2}\, |w_\pm|}{3|f'(w_\pm)|\Delta w}\, \mathrm{d}w^2 + \frac{c_\pm^2\, |f'(w_\pm)|\Delta w}{3\sqrt{1-j^2}\, |w_\pm|}\, \mathrm{d}\phi_\pm^2 \\ &= \frac{4\sqrt{1-j^2}\, |w_\pm|}{3|f'(w_\pm)|} \left[\mathrm{d}R^2 + \frac{c_\pm^2\, f'(w_\pm)^2}{4(1-j^2)w_\pm^2}\, R^2 \mathrm{d}\phi_\pm^2 \right] , \end{aligned} \tag{7.25}$$

where we have introduced the radial coordinate as $R^2 = \Delta w$. In order for the Killing vector $\partial_{\phi_\pm}$ to degenerate smoothly at the pole, we see that we need to normalize it as

$$c_\pm = \frac{2\sqrt{1-j^2}\, w_\pm}{f'(w_\pm)} . \tag{7.26}$$

The coordinates $\phi_\pm$ now have periodicity 2π around $w = w_\pm$ respectively and the conical singularities are resolved. This works in an analogous way to the discussion in [93] for the uplift of this solution to 11d supergravity on a Sasaki–Einstein 7-manifold.

Let us study the metric close to the poles in a bit more detail. We switch to the coordinates $\phi_\pm$ using the transformations

$$\begin{aligned} z &= 2\sqrt{1-j^2} \left(\frac{w_+}{f'(w_+)}\, \phi_+ - \frac{w_-}{f'(w_-)}\, \phi_- \right) , \\ \psi &= -\tfrac{2}{3}\sqrt{1-j^2} \left(\frac{w_+\, h(w_+)}{f'(w_+)}\, \phi_+ - \frac{w_-\, h(w_-)}{f'(w_-)}\, \phi_- \right) . \end{aligned} \tag{7.27}$$

[2]Here we use the leading order expansions $f(w) = |f'(w_\pm)|\, \Delta w$ and $P(w) = (1-j^2)w_\pm^2$ around $\Delta w = 0$.

We may rewrite the metric in these new coordinates as

$$\mathrm{d}s_8^2 = \frac{\sqrt{P(w)}}{3f(w)}\mathrm{d}w^2 + \tfrac{1}{2}\mathrm{d}\alpha^2 + F_+ D\phi_+^2 + F_- D\phi_-^2 + G D\phi_+ D\phi_- + \frac{\sin^2\alpha}{1+\cos^2\alpha}\mathrm{d}s^2_{\mathrm{KE}_4}, \tag{7.28}$$

where

$$D\phi_\pm = \mathrm{d}\phi_\pm + \frac{3f'(w_\pm)}{2\sqrt{1-j^2}\, w_\pm (h(w_\mp) - h(w_\pm))}\left(\sigma + \tfrac{1}{3} jr\, h(w_\mp)\,\mathrm{d}t\right), \tag{7.29}$$

and

$$\begin{aligned} F_\pm &= \frac{2(1-j^2)w_\pm^2\Big(2(2+\cos^2\alpha)f(w) + \sin^2\alpha\big(h(w)-h(w_\pm)\big)^2\sqrt{P(w)}\Big)}{3(2+\cos^2\alpha)\sqrt{P(w)}f'(w_\pm)^2}, \\ G &= \frac{4(1-j^2)w_+w_-\Big(2(2+\cos^2\alpha)f(w) + \sin^2\alpha\big(h(w)-h(w_+)\big)\big(h(w)-h(w_-)\big)\sqrt{P(w)})\Big)}{3(2+\cos^2\alpha)\sqrt{P(w)}f'(w_+)f'(w_-)}. \end{aligned} \tag{7.30}$$

Note that $F_+(w_+) = G(w_+) = 0$ and $F_-(w_-) = G(w_-) = 0$.

Region II

In order to study the metric close to $\alpha = 0, \pi$, we simply Taylor expand the trigonometric functions in the metric around these points. This yields

$$\mathrm{d}s_8^2 = \frac{\sqrt{P(w)}}{3f(w)}\,\mathrm{d}w^2 + \frac{f(w)}{3\sqrt{P(w)}}\,Dz^2 + \frac{1}{2}\Big(\mathrm{d}R^2 + R^2\left(\hat{\eta}^2 + \mathrm{d}s^2_{\mathrm{KE}_4}\right)\Big), \tag{7.31}$$

where we use $R = \alpha, \pi - \alpha$ for the distance away from $\alpha = 0, \pi$ respectively. We see that this is the metric of a 6d cone fibered over the spindle. When $\mathrm{KE}_4 = \mathbb{CP}^2$ this is $\mathbb{R}^6$ and free of singularities, while for other Kähler–Einstein manifolds it has a singularity at the tip of the cone. This singularity appears also in the AdS_4 vacuum solution.

Region III: monopoles

In addition to the limits to the endpoints of the w and α intervals, we should also take the limit where both w and α go to an endpoint simultaneously, i.e. to a corner of the rectangle in figure 7.1.

We denote the endpoints of the intervals by $\alpha \in [\alpha_-, \alpha_+] = [0, \pi]$ and $w \in [w_-, w_+]$. In order to take the combined limit, we need a coordinate transformation to 'polar coordinates' around the corner of the rectangle. A proper set of such coordinates is given by (R, ζ) defined by

$$\alpha = \alpha_\pm \mp R\cos\zeta\,, \qquad w = w_\pm \mp \left|\frac{3f'(w_\pm)}{8\sqrt{1-j^2}\,w_\pm}\right| R^2\sin^2\zeta\,, \tag{7.32}$$

where the signs are chosen such that R is a positive radial coordinate for each of the four corners. The scalings and powers in (7.32) are chosen such that we find the appropriate metric $(\mathrm{d}R^2 + R^2\mathrm{d}\zeta^2)$ in polar coordinates for large R. The combined limit $\alpha \to \alpha_\pm, w \to w_\pm$ is now simply $R \to 0$.

Changing to these new coordinates and expanding around $R = 0$ yields the 8d metric

$$\mathrm{d}s_8^2 = \frac{1}{2}\left[\mathrm{d}R^2 + R^2\left(\mathrm{d}\zeta^2 + \cos^2\zeta\left(\left(\mathrm{d}\psi + \sigma + \tfrac{1}{3}A\right)^2 + \mathrm{d}s^2_{\mathrm{KE}_4}\right) + \sin^2\zeta\,\frac{f'(w_\pm)^2}{4(1-j^2)w_\pm^2}\,Dz^2\right)\right]. \tag{7.33}$$

Here the gauge field is given by $A = h(w_\pm)\,(\mathrm{d}z + jr\,\mathrm{d}t)$. The $\mathrm{d}z$ leg is constant so it can be absorbed by a shift in ψ. Furthermore, the period of z at the poles of the spindle is given by

$$\Delta z = \frac{4\pi\sqrt{1-j^2}\,|w_\pm|}{n_\pm|f'(w_\pm)|}\,. \tag{7.34}$$

If we rescale z such that it is 2π periodic, i.e. use the coordinate $\hat{z}$ introduced in (7.11), we find the metric

$$\mathrm{d}s_8^2 = \frac{1}{2}\left[\mathrm{d}R^2 + R^2\left(\mathrm{d}\zeta^2 + \cos^2\zeta\left(\left(\mathrm{d}\hat{\psi} + \sigma + \tfrac{1}{3}jr\,h(w_\pm)\,\mathrm{d}t\right)^2 + \mathrm{d}s^2_{\mathrm{KE}_4}\right) + \frac{1}{n_\pm^2}\sin^2\zeta\,D\hat{z}^2\right)\right], \tag{7.35}$$

where $\hat{\psi}$ and $\hat{z}$ are the shifted and rescaled coordinates with correspondingly rescaled one-form $D\hat{z}$. Note that when the Kähler–Einstein space is $\mathbb{CP}^2$ this is $\mathbb{R}^8/\mathbb{Z}_{n_\pm}$. We recognize this geometry as a monopole of weight $n_\pm$. The presence of these monopoles at the corners of the (w, α) rectangle explains the singularities there. These singularities cannot be resolved. This is most reminiscent of the case of M5-branes where the uplifted theory still admits a conical singularity [89, 90, 96]. In appendix 7.A we have analyzed the M5-brane solution studied there in this light and found the same type of singularity structure.

Note that these monopoles and their corresponding singularities have not been found in the uplift of this 4d setup to 11d supergravity, see [3, 93, 95, 104] and chapter 6 of this thesis. Instead in the M2-brane setup the singularities of the 4d theory are removed in the uplifted theory (under suitable conditions on the fluxes, which are the same ones we take here). Here we find the novel feature that the singularities cannot be removed in a different uplift of the same black hole solution.

7.2.2 Flux quantization

We now want to consider flux quantization. This is most simply studied in string frame as it removes the need to take into account powers of the dilaton. Consider first the Romans mass, it must satisfy

$$F_0 = \frac{n}{2\pi \ell_s}\,, \qquad n \in \mathbb{Z}\,, \tag{7.36}$$

in order for the theory to be well-defined. Here n can be interpreted as the number of D8-branes present in the setup.

In order to work out the quantization of the remaining fluxes, we first compute the corresponding Page fluxes. These are closed, and are therefore the appropriate ones to quantize. The Page fluxes, using polyform notation, are defined as

$$\hat{f} = F \wedge \mathrm{e}^{-B}\,, \tag{7.37}$$

where F is understood to be the magnetic part of the flux[3]

$$F = F_0 + F_2 + F_4 + F_6 + F_8 + F_{10}\Big|_{\text{magnetic}}\,, \tag{7.38}$$

in string frame. The quantization conditions on the Page fluxes $\hat{f}_p$ read

$$\frac{1}{(2\pi\ell_s)^{p-1}} \int_{\Sigma_p} \hat{f}_p \,\in\, \mathbb{Z} \tag{7.39}$$

where Σ_p is an integral cycle in the compact 8d space. Let us consider all possible cycles that we can construct in the geometry. There is a single two-cycle which is simply the spindle at a fixed point on the squashed sine cone Y_6 which we can take to be at either of the endpoints of the α interval. There are no topological four-cycles in the geometry that we can integrate the four-form flux over. There is a single six-cycle which is given by Y_6 at a fixed point on the spindle and a single eight-cycle given by the full internal space.

To proceed we need to choose a gauge for the B-field. We take

$$B_2 = F_2/F_0\,, \tag{7.40}$$

which can be recognized as a proper choice for B_2 directly from the Bianchi identities of massive type IIA, and can also be checked from (7.16)-(7.18). Using this gauge choice we can compute the Page fluxes, e.g. as

$$\hat{f}_6 = -\mathrm{e}^{\Phi/2} * F_4 - F_4 \wedge B_2 + F_2 \wedge \frac{B_2^2}{2} - F_0\,\frac{B_2^3}{6}\,. \tag{7.41}$$

[3]Note that in this notation some of these fluxes F_p obtain a minus sign with respect to the Hodge star of the corresponding $(10-p)$ form. This is the case if $p/2$ is odd.

Note that we have added the appropriate power of e^{Φ} to $*F_4$ in order to go to string frame. The quantization of this six-form flux yields the condition

$$\frac{1}{(2\pi\ell_s)^5}\int_{Y_6}\hat{f}_6 = \frac{1}{(2\pi\ell_s)^5}\,\frac{16L^5}{3m^{1/5}}\,\mathrm{Vol}_{\mathrm{SE}_5} = N\,\in\,\mathbb{Z}\,. \tag{7.42}$$

Here this N can be interpreted as the number of D2-branes. Together with the condition (7.36) on F_0, this leads to the following quantization conditions on the parameters L and m:

$$L = \frac{2^{1/6}3^{5/24}\,\pi\ell_s\,n^{1/24}N^{5/24}}{\mathrm{Vol}_{\mathrm{SE}_5}^{5/24}}\,,\qquad\qquad m = \frac{3^{5/24}\,n^{25/24}N^{5/24}}{2^{5/6}\,\mathrm{Vol}_{\mathrm{SE}_5}^{5/24}}\,. \tag{7.43}$$

Note that these expressions agree with [115], taking into account the redefinitions $L_{\mathrm{here}} = 2^{5/16}3^{1/2}\,L_{\mathrm{there}}$ and $m_{\mathrm{here}} = 2^{5/16}\,(\mathrm{e}^{-5\Phi_0/4})_{\mathrm{there}}$. It turns out that the two-form Page flux vanishes. The eight-form flux $\hat{f}_8$ does not vanish, but its integral over the internal space does. The quantization conditions for these fluxes are therefore trivially satisfied.

We now compute the holographic free energy, which is simply given by one over the 2d Newton constant. We find this by integrating over the volume of the internal space

$$\frac{1}{G_2} = \frac{1}{G_{10}}\int_{\mathcal{M}_8}\mathrm{dvol}_8 = \frac{L^8\,\mathrm{Vol}_{\mathrm{SE}_5}}{3^{3/2}\,5\,\pi^6\ell_s^8}\,\Delta z\,(w_3-w_2)\,, \tag{7.44}$$

where we used that $G_{10} = 2^3\pi^6\ell_s^8$. We now put in the quantization of L (7.43), which allows us to express the 2d Newton constant as

$$\frac{1}{G_2} = \frac{2^{4/3}3^{1/6}\pi^2}{5\,\mathrm{Vol}_{\mathrm{SE}_5}^{2/3}}\,\Delta z\,(w_3-w_2)\,n^{1/3}\,N^{5/3}\,, \tag{7.45}$$

which is consistent with other results in the literature.

Appendices

7.A Review of M5-branes on spindles

We will review the AdS_5 solutions of 11d supergravity corresponding to M5-branes on a spindle, as first studied in [96]. Our interest in this solution is to study the singular behaviour of the uplifted theory and to compare with the results we find for D2-branes on a spindle. The 7d metric of [96] is

$$\mathrm{d}s^2 = \Big(wP(w)\Big)^{1/5}\left[4\mathrm{d}s^2(\mathrm{AdS}_5) + \frac{w}{f(w)}\mathrm{d}w^2 + \frac{f(w)}{P(w)}\mathrm{d}z^2\right]. \tag{7.46}$$

The metric on AdS_5 is taken to be the one with unit radius which implies that it is Einstein satisfying $R_{\mu\nu} = -4g_{\mu\nu}$. The functions appearing in the metric are the simple polynomials

$$h_i(w) = w^2 - l_i\,, \qquad P(w) = h_1(w)h_2(w)\,, \qquad f(w) = P(w) - w^3\,. \tag{7.47}$$

The solution is supported by two real scalars and two abelian gauge fields

$$A_i = \frac{l_i}{h_i(w)}\mathrm{d}z\,, \qquad X_i(w) = \frac{\Big(wP(w)\Big)^{2/5}}{h_i(w)}\,, \tag{7.48}$$

As explained in [96] the w coordinate is bounded between two positive roots, $w_\pm$ of the function $f(w)$. The period of the circle direction z is fixed to be

$$\frac{\Delta z}{2\pi} = \frac{2w_\pm^2}{n_\pm|f'(w_\pm)|}\,, \tag{7.49}$$

with $n_\pm$ orbifold weights for the spindle at the respective poles. The two magnetic charges are

$$Q_i = \frac{1}{2\pi}\int_\Sigma \mathrm{d}A_i = \frac{\Delta z}{2\pi}\Big[\frac{l_i}{h_i(w_+)} - \frac{l_i}{h_i(w_-)}\Big]\,. \tag{7.50}$$

The solution can be uplifted to 11d supergravity on an S^4, with resultant 11d metric

$$\begin{aligned}\mathrm{d}s^2_{11} =&\,\Omega^{1/3}\Big(wP(w)\Big)^{1/5}\Bigg[4\mathrm{d}s^2(\mathrm{AdS}_5) + \frac{w}{f(w)}\mathrm{d}w^2 + \frac{f(w)}{P(w)}\mathrm{d}z^2\\ &+ \frac{1}{\Omega\big(wP(w)\big)^{1/5}}\Big(X_0^{-1}\mathrm{d}\mu_0^2 + \sum_{i=1}^{2} X_i^{-1}(\mathrm{d}\mu_i^2 + \mu_i^2(\mathrm{d}\phi_i + A_i)^2)\Big)\Bigg]\,.\end{aligned} \tag{7.51}$$

Here, μ_I are embedding coordinates for the S^4 and satisfy $\mu_0^2 + \mu_1^2 + \mu_2^2 = 1$. The new functions appearing in the metric are

$$X_0 = X_1^{-2}X_2^{-2}\,, \qquad \Omega = \sum_{I=0}^{2} X_I\mu_I^2\,. \tag{7.52}$$

It is useful to reparametrise the μ_I as

$$\mu_1 = \sqrt{1-\mu_0^2}\sin\theta\,, \quad \mu_2 = \sqrt{1-\mu_0^2}\cos\theta\,. \tag{7.53}$$

With these coordinates the metric takes the form

$$\begin{aligned} ds_{11}^2 =& \Omega^{1/3}(wP(w))^{1/5}\Bigg[4ds^2(\mathrm{AdS}_5) + \frac{w}{f(w)}dw^2 + \frac{f(w)}{P(w)}dz^2 \\ &+ \frac{1}{\Omega(wP(w))^{1/5}}\Bigg\{(1-\mu_0)^2\Bigg[\frac{\sin^2\theta}{X_1}D\phi_1^2 + \frac{\cos^2\theta}{X_2}D\phi_2^2 + \Big(\frac{\cos^2\theta}{X_1} + \frac{\sin^2\theta}{X_2}\Big)d\theta^2\Bigg] \\ &+ \frac{2\mu_0\sin\theta\cos\theta(X_1-X_2)}{X_1X_2}d\theta d\mu_0 + \Bigg[X_1^2X_2^2 + \frac{\mu_0^2}{1-\mu_0^2}\Big(\frac{\sin^2\theta}{X_1} + \frac{\cos^2\theta}{X_2}\Big)\Bigg]d\mu_0^2\Bigg\}\Bigg]\,, \end{aligned} \tag{7.54}$$

with

$$D\phi_i = d\phi_i + A_i\,, \tag{7.55}$$

and Ω takes the form

$$\Omega = \frac{\mu_0^2}{X_1^2X_2^2} + (1-\mu_0^2)\Big(X_1\sin^2\theta + X_2\cos^2\theta\Big)\,. \tag{7.56}$$

To simplify the analysis we can take $l_1 = l_2$ which sets the two gauge fields and scalars equal. This is then similar to the Einstein–Maxwell solution we studied in the main text. Note that this simplification removes the θ, μ_0 cross term and the (fibered) S^3 is now round as opposed to squashed. The four-sphere remains squashed still. Since keeping the analysis general does not present much additional difficulty, especially with access to mathematica, we will just keep the analysis as general as possible.

Near to $\theta = 0, \pi$ an S^2 in the metric shrinks smoothly when ϕ_i have period 2π, this is of course the expected degeneration of a three-sphere. Next consider a degeneration at the root of $f(w)$ at $w_\pm$. The degenerating Killing vector with 2π period at one of the poles of the spindle (away from $\mu_0^2 = 1$) is

$$k_\pm = \pm\frac{2w_\pm^2}{|f'(w_\pm)|}\left(\partial_z - \sum_{i=1}^{2}\frac{l_i}{h_i(w_\pm)}\partial_{\phi_i}\right)\,. \tag{7.57}$$

Note that the prefactor satisfies

$$\frac{2w_\pm^2}{|f'(w_\pm)|} = \frac{n_\pm\Delta z}{2\pi}\,. \tag{7.58}$$

We have four Killing vectors which degenerate at $\theta = 0, \pi$ and $w = w_\pm$ and therefore they must satisfy a linear relation

$$a_+ k_+ + a_- k_- + p_1 \partial_{\phi_1} + p_2 \partial_{\phi_2} = 0\,. \tag{7.59}$$

This implies three constraints on the parameters:

$$n_+ a_+ - n_- a_- = 0\,, \quad n_+ a_+ Q_i = p_i\,, \tag{7.60}$$

where we have used the first condition to simplify the second and third and the explicit form of the magnetic charges in (7.50). Clearly we should solve the first condition by taking

$$a_+ = n_-\,, \tag{7.61}$$

then we find the quantization condition

$$n_+ n_- Q_i = p_i \in \mathbb{Z}\,. \tag{7.62}$$

We conclude that the manifold is smooth and singularity free away from the four points $(\mu_0, w) = (\pm 1, w_\pm)$. Let us consider these points and for simplicity set $X_1 = X_2$.[4] We should change coordinates to

$$w = w_\pm \mp \frac{|f'(w_\pm)| X_1(w_\pm)}{2 w_\pm^{9/5}} r^2 \cos^2\zeta\,, \qquad \mu_0 = \pm 1 \mp r^2 \sin^2\zeta\,. \tag{7.63}$$

Then

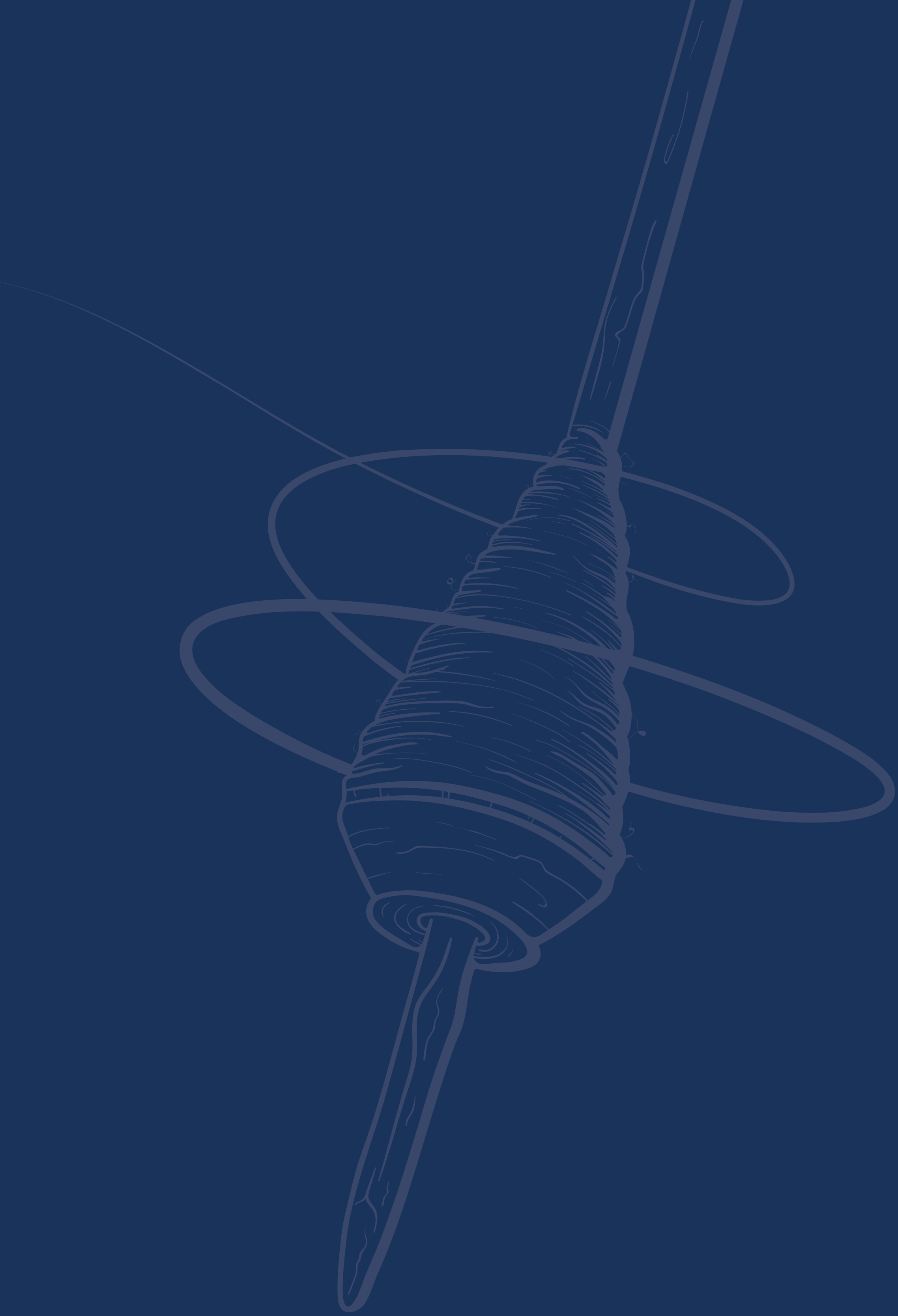
$$\mathrm{d}s_6^2 = \frac{2X_1(w_\pm)}{w_\pm^{4/5}} \Big[\mathrm{d}r^2 + r^2 \Big\{ \mathrm{d}\zeta^2 + \frac{1}{n_\pm^2} \cos^2\zeta \mathrm{d}\hat{z}^2 + \sin^2\zeta \Big(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi_1^2 + \cos^2\theta \mathrm{d}\phi_2^2 \Big) \Big\} \Big]\,, \tag{7.64}$$

where we defined the 2π periodic coordinate $\hat{z}$ as

$$z = \frac{\Delta z}{2\pi}\, \hat{z}\,, \tag{7.65}$$

and we have performed a linear shift of the ϕ_i coordinates in the final result. This is the metric on $\mathbb{R}^6/\mathbb{Z}_{n_\pm}$ and implies the existence of monopoles located at the four points $(\mu_0, w) = (\pm 1, w_\pm)$, with the orbifold weight correlated to the $\pm$ of $w_\pm$. Compare this to the D2-brane case studied in the main text, in particular equation (7.35). Note that both have the same phenomenon of resolve the potential line of singularities in the uplift at the cost of singularities at isolated points instead. This should be contrasted with the M2-brane and D3-brane solutions where the uplifted metric can be made smooth everywhere.

[4]For $X_1 \neq X_2$ the coordinate transformation is more complicated due to the mixed term between μ_0 and θ and the coordinate transformation requires a coordinate transformation for the three coordinates rather than 2.



PART III

SUPERSYMMETRIC CLASSIFICATION

Chapter 8

Introduction

"A picture may be worth a thousand words,
a formula is worth a thousand pictures."

– Edsger W. Dijkstra[1]

The idea of extremization principles playing a fundamental role in physics has a long history since the advent of the Lagrangian and the principle of least action. More recently extremal problems have also been shown to play a role in both quantum field theory and supergravity. On the field theory side a-maximization [116], F-maximization [117], c-extremization [118, 119] and $\mathcal{I}$-extremization [120] have been successfully used to compute observables in SCFTs in 4, 3, 2 and 1 dimension(s) respectively.

Via AdS/CFT it is natural to conjecture that there are dual extremization principles on the gravity side. Indeed such *geometric extremization* principles have been found for all of the field theory principles mentioned above. In [121, 122] a geometric dual to a-maximization and F-maximization was given whilst in [109] an analogous proposal for c-extremization and $\mathcal{I}$-extremization was given for certain classes of theories. The classes of solutions tackled in [109] and in the later works [123–129] are AdS_3 solutions in type IIB and AdS_2 solutions in 11d supergravity. Subclasses of these arise as the near-horizon of static black strings and black holes embedded in the respective theories[2].

For example, the near-horizon limit of a static asymptotically AdS_4 extremal black hole in 4d gauged supergravity contains an AdS_2 factor, see e.g. [131–134] and references therein. The staticity of the black hole requires that the transverse directions of the geometry are not fibered over AdS_2 but merely form a warped product. If one further restricts to magnetically charged black holes and uplifts the near-horizon solution to 11d supergravity, one obtains a supersymmetric solution with an AdS_2 factor and electric four-form charge. Solutions of this form were classified in [105, 135] and later extended in [136, 137] to include additional magnetic

[1]The author wonders whether he was referring to formulas like (9.75).
[2]See also [130] for an extremization principle for $\mathcal{N} = (0, 4)$ AdS_2 solutions in type IIB.

flux. The geometries are a warped product of AdS_2 with a nine-dimensional internal manifold which is locally a U(1) bundle over a conformally Kähler space. Such 9d manifolds are part of a class of geometries, sometimes called Gauntlett–Kim (GK) geometries, that exist in all odd dimensions larger than or equal to seven [105, 138, 139].

From the 11d perspective one constructs these geometries by placing M2-branes in an asymptotic geometry of $\mathbb{R} \times CY_5$ and wraps them on a curve inside the Calabi–Yau five-fold. The near-horizon of this setup then gives rise to the AdS_2 geometry which in turn is seen to be the near-horizon of a black hole.

In order to obtain an 11d AdS_2 solution in this example it was important that the 4d black hole was both static and only magnetically charged. Adding rotation to the four-dimensional black hole leads to the internal space being fibered over the AdS_2 in the near-horizon, which will clearly persist in the uplift. Though not as obvious, if the 4d black hole has electric charges which are identified as arising from gauged flavour symmetries, this will also lead to a fibered AdS_2 in the 11d uplift. A gauge field in the truncation can have two sources, either it comes from gauging an isometry of the compactification manifold, or from the expansion of a p-form potential on $(p-1)$-cycles of the compactification manifold. The former gauge fields are dual to flavour symmetries whilst the latter are dual to baryonic symmetries. For the flavour symmetries the uplift will lead to the isometries being fibered over AdS_2 in the 11d solution.

In summary, in order to incorporate more general black holes which rotate and have electric charges, one must relax the warped product structure of the 11d solution and allow for the internal manifold to be fibered over AdS_2. Such solutions are not included in the classifications of [105, 137], and are not suitable for the geometric extremization procedure of [109].

In this part of the thesis we lay the groundwork for extending the geometric dual of $\mathcal{I}$-extremization and c-extremization to theories arising from the near-horizon of rotating black holes and black strings respectively. Concretely we will classify a large class of supersymmetric solutions of 11d supergravity containing an internal manifold arbitrarily fibered over AdS_2. With such a general ansatz we cover the black holes considered in [114, 140–143]. We find that the 9d internal manifold is a U(1) fibration over an 8d space admitting a balanced metric. These can be seen as rotating generalizations of GK geometries. The balanced metric satisfies a *master equation* which is the analogue of the one found in the non-rotating case [105, 138], see also [136, 144–147]. Through dualities we also classify a class of rotating black string near-horizons in type IIB.

8.1 G-structures

In differential geometry, a G*-structure* is a property of a manifold that says something about the transition functions that map between the patches that make up the manifold. The transition functions make up a group, which is called the structure group, and a d-dimensional manifold is said to have a G-structure if the tangent frame bundle can be reduced such that its transition functions form the group $G \subset \mathrm{GL}(d, \mathbb{R})$.

An intuitive way to think about G-structures is in terms of globally defined tensors or spinors. The existence of a globally defined object implies that this object is left invariant by all transition functions, i.e. it is invariant under the structure group; otherwise the object would by definition not be globally defined. Therefore the possibility of constructing globally defined tensors or spinors on a manifold constrains the structure group of that manifold. See figure 8.1 for an example: a globally defined vector reduces the structure group $\mathrm{O}(d)$ to $\mathrm{O}(d-1)$.

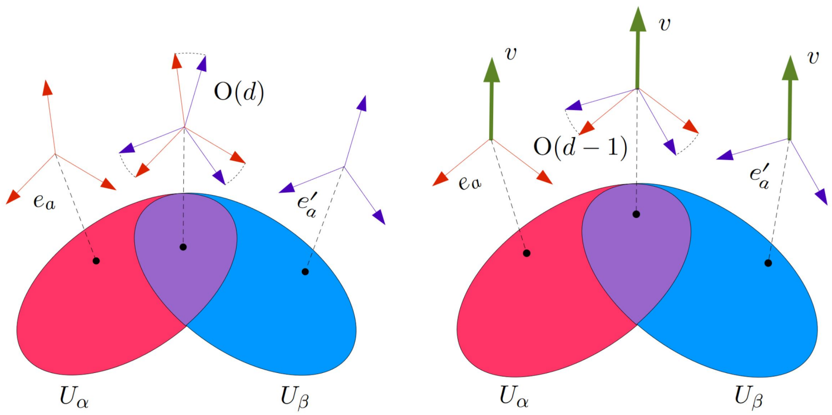


Figure 8.1: *An intuitive picture showing how a globally defined object can reduce the structure group. On the left we see a manifold that is assumed to have the structure group* $\mathrm{O}(d)$*. On the right a globally defined vector* v *is added, whose existence implies that the structure group can be reduced to the subgroup leaving the vector invariant:* $\mathrm{O}(d-1)$ *rotations orthogonal to* v*. Here* $U_\alpha,, U_\beta$ *are two patches with local frames* e_a, e'_a*. Picture taken from [148].*

A manifold that can be covered by a single patch has a trivial structure group consisting only of the identity. Such manifolds are called parallelizable. In the context of parallelizable manifolds, such as flat Euclidean or Minkowski space, one typically doesn't think about G-structures, but the fact that these manifolds have a trivial structure is crucial for being able to define all sorts of tensor and spinor fields on it. In other words, this property is crucial for most of what is done in (flat space) field theory.

In this part of the thesis, we will mostly encounter $\mathrm{SU}(d/2)$ structure manifolds. These are almost complex manifolds, and their structure can be characterized by a globally defined $(1,1)$-form j and $(d/2,0)$-form ω. Many properties of such manifolds can be deduced from the exterior derivatives of these forms. In general we can write these as

$$\mathrm{d}j = \frac{1}{8} w_1 \lrcorner \mathrm{Im}[\omega] + w_3 + \frac{1}{4} w_4 \wedge j\,, \tag{8.1}$$

$$\mathrm{d}\,\mathrm{Re}[\omega] = \frac{1}{3} w_1 \wedge \frac{j^2}{2!} + w_2 \wedge j - \frac{1}{8} w_5 \wedge \mathrm{Re}[\omega]\,, \tag{8.2}$$

where we have chosen normalizations convenient for an $\mathrm{SU}(5)$ structure as that is the one we will encounter in chapter 9. The w_i are called torsion modules: w_1 is a real $(2,0)+(0,2)$-form, w_2 a real primitive $(3,1)+(1,3)$-form, w_3 a real primitive $(2,1)+(1,2)$-form and w_4 and w_5 are real one-forms. Depending on which of these torsion modules are zero and which are non-zero, such $\mathrm{SU}(d/2)$ structure manifolds can be divided into various subclasses. We have listed some of these in table 8.1.

Torsion modules	Type of geometry
$w_1 = w_2 = 0$	Complex
$w_1 = w_3 = w_4 = 0$	Symplectic
$w_1 = w_2 = w_4 = 0$	Balanced
$w_1 = w_2 = w_3 = w_4 = 0$	Kähler
$w_1 = w_2 = w_3 = w_4 = w_5 = 0$	Calabi-Yau

Table 8.1: *Various subclasses of* $\mathrm{SU}(d/2)$ *structure geometries.*

8.2 GK geometry

We give a brief review of GK geometries, as our internal manifolds can be seen as rotating generalizations thereof. These were first constructed as 7d and 9d manifolds appearing in supersymmetric AdS_3 and AdS_2 solutions in type IIB and 11d supergravity respectively [105, 138]. Later it was realized that the structure of these geometries can be generalized to arbitrary odd dimension (≥ 5) [139, 149].

GK geometry in $2n+1$ dimensions is defined on a manifold Y_{2n+1} that has a U(1) Killing vector, called the R-symmetry vector. The one-form dual to this vector is called η and the metric on Y_{2n+1} can be written as

$$\mathrm{d}s_Y^2 = \eta^2 + \mathrm{e}^B \, \mathrm{d}s_{\mathcal{B}}^2 \,. \tag{8.3}$$

Here B is a scalar field, and $\mathcal{B}$ is a $2n$-dimensional base space that is Kähler. In addition to the scalar B, this geometry is accompanied by a two-form F. Both can be expressed in terms of geometric quantities, as[3]

$$\mathrm{e}^B = \tfrac{1}{2} R \,, \qquad\qquad F = -J + \mathrm{d}(\mathrm{e}^{-B}\eta) \,, \tag{8.4}$$

with R and J the Ricci scalar and Kähler form on the base space $\mathcal{B}$. For these geometries to be interpreted as solutions of string or M-theory, the Kähler metric on the base space has to satisfy the equation

$$\Box R = \tfrac{1}{2} R^2 - R_{ij} R^{ij} \,, \tag{8.5}$$

which is known as the *master equation*. Here R_{ij} is the Ricci tensor.

GK geometry can be seen as a generalization of Sasaki–Einstein (SE) geometry. While SE manifolds appear as the bases of Ricci-flat Kähler cones, GK manifolds appear as the bases of Ricci-flat complex cones that carry an $\mathrm{SU}(n+1)$ structure but are not Kähler. The R-symmetry vector in GK geometry is the analogue of the Reeb vector in SE geometry. The $2n$-dimensional base space $\mathcal{B}$ that we discussed earlier is Kähler; the corresponding base in a SE manifold would be both Kähler and Einstein, meaning that its Ricci tensor is proportional to its metric.

For convenience, we summarize some of the properties of SE and GK geometry, and of the geometry that we present in this part of the thesis, in table 8.2.

8.3 Outline

The outline of this part is as follows. We will present our classification for rotating near-horizon geometries in chapter 9. Essentially this is a set of necessary and sufficient conditions that solutions must satisfy in order to fall in the category of near-horizon geometries that we consider. In chapter 10 we take known black hole solutions from the literature, and embed them in our classification.

[3] The prefactors in these expressions depend on the dimension of the GK geometry. Here we present the ones for Y_9, in order to make the comparison between the static case here and the rotating case in chapter 9 easier. Note, however, that in chapter 9 a different normalization for B is used.

Geometry	$(2n+2)$-dim cone	$2n$-dim base of U(1) fibration
SE_{2n+1}	Ricci-flat Kähler	Kähler–Einstein
GK_{2n+1}	Ricci-flat complex	Kähler
Rotating GK_{2n+1}	Ricci-flat complex	Balanced

Table 8.2: *We list some properties of SE, GK and rotating GK geometry. In particular, we list the cones that they are a base of, and the bases that they are a* U(1) *fibration over.*

More specifically, in section 9.1 we study the necessary and sufficient conditions for a supersymmetric solution with time fibered over the transverse directions and consistent with preserving an $\mathrm{SO}(2,1)$ symmetry. In section 9.2 we give an action from which the equations of motion found in section 9.1 may be derived. In particular we show that when supersymmetry is imposed on the action it reduces to a simple form which computes the entropy of the black hole/string. Section 9.3 discusses the conditions on the geometry of rotating black strings in type IIB by using dualities with the 11d geometry. A discussion on general black hole near-horizons and the computation of observables of the solutions is presented in appendix 9.A.

The black holes that we embed in our classification in chapter 10 are the AdS_4 Kerr–Newman (section 10.1), the spinning spindle (section 10.2) and the Klemm (secton 10.3) solutions. We find that for some of these the 8d base space is balanced while for others it is Kähler.

Chapter 9

Classification of rotating M2-brane solutions

9.1 Conditions of the classification

In this section we will explain the general procedure for obtaining the conditions for preserving supersymmetry of near-horizon solutions of rotating black holes. In general the conditions we find are necessary and sufficient conditions that must be satisfied by the near-horizon of any rotating black hole in 11d supergravity arising from rotating M2-branes. We will determine these conditions by using the results in [150] which classified all 11d supergravity backgrounds preserving supersymmetry and admitting a timelike Killing vector. Using [150] we can reduce the 11d supersymmetry conditions into differential conditions on a 10d base space. This base space must be non-compact and upon imposing the natural condition that the 10d space is a cone we can reduce the conditions further to a compact 9d base, Y_9. This 9d base is a U(1) fibration over an 8d base, $\mathcal{B}$. In general the 8d base is not conformally Kähler, which is true for the non-rotating AdS_2 case studied in [105], but instead is a conformally balanced space.

One of the guiding principles that we will use is to impose that the near-horizon solution possesses an $\mathrm{SO}(2,1)$ symmetry dual to the conformal group in the 1d superconformal quantum mechanical theory. Generally the ansatz that we will use when reducing the supersymmetry conditions does not possess this full symmetry but only a subset of it. However, from the point of view of imposing supersymmetry it is more convenient to work with this more general setup and then further constrain the geometry to preserve the full conformal group later. We will find that the additional constraints that we need to impose for the existence of an $\mathrm{SO}(2,1)$ symmetry are specified by giving a constant vector with entries corresponding to each of the Killing vectors of the metric. These constants are related to the near-horizon angular velocities of the black hole along the Killing directions.

We begin this section by reviewing the conditions for a supersymmetric geometry in 11d supergravity to admit a timelike Killing vector following [150]. We discuss in detail the ansatz we will use in performing the reduction and subsequently reduce the conditions to an 8d base space. Up until this point we have not imposed the existence of an $\mathrm{SO}(2,1)$ symmetry and in the final part of this section we discuss the additional constraints one must impose for such a symmetry using the results in appendix 9.A.

9.1.1 Timelike structures in 11d supergravity

In [150] the conditions for a solution of 11d supergravity to admit a timelike Killing spinor were derived. Here we summarize the most important results for our purposes. The metric takes the general form

$$\mathrm{d}s_{11}^2 = -\Delta^2(\mathrm{d}t + a)^2 + \Delta^{-1}\mathrm{e}^{2\phi}\mathrm{d}s_{10}^2 \tag{9.1}$$

where Δ and $\mathrm{e}^{2\phi}$ are functions defined on the 10d base. Note that we use a rescaling $\mathrm{e}^{2\phi}$ of the 10d metric compared to [150]. The 10d base admits a canonical $\mathrm{SU}(5)$ structure which we denote by (j,ω)[1]. We normalize this structure such that

$$\omega \wedge \bar{\omega} = (-2i)^5\,\frac{j^5}{5!}\,. \tag{9.2}$$

The exterior derivatives of the structure forms satisfy

$$\mathrm{d}j = \frac{1}{8}w_1 \lrcorner\, \mathrm{Im}[\omega] + w_3 + \frac{1}{4}w_4 \wedge j\,, \tag{9.3}$$

$$\mathrm{d}\,\mathrm{Re}[\omega] = \frac{1}{3}w_1 \wedge \frac{j^2}{2!} + w_2 \wedge j - \frac{1}{8}w_5 \wedge \mathrm{Re}[\omega]\,. \tag{9.4}$$

Here the w_i are the torsion modules of the $\mathrm{SU}(5)$ structure: w_1 is a real $(2,0)+(0,2)$-form, w_2 a real primitive $(3,1)+(1,3)$-form, w_3 a real primitive $(2,1)+(1,2)$-form and w_4 and w_5 are real one-forms. The 11d four-form flux is decomposed into 10d fluxes as

$$\mathcal{G}_4 = (\mathrm{d}t + a) \wedge f_3 + h_4\,. \tag{9.5}$$

Following the results of [150], imposing supersymmetry yields the following conditions relating the fluxes to the structure forms

$$\mathrm{d}(\mathrm{e}^{2\phi}j) = f_3\,, \tag{9.6}$$

$$\mathrm{d}\big(\Delta^{-3/2}\mathrm{e}^{5\phi}\,\mathrm{Re}[\omega]\big) = \mathrm{e}^{2\phi}\star_{10} h_4 - \mathrm{e}^{2\phi}\,h_4 \wedge j - \mathrm{e}^{4\phi}\,\mathrm{d}a \wedge \frac{j^2}{2}\,. \tag{9.7}$$

[1] In comparison to [150] one should identify $(a, \mathrm{e}^{2\phi}j, \mathrm{e}^{5\phi}\omega, \mathrm{e}^{5\phi}\,\mathrm{Re}[\omega])_{\text{here}} \leftrightarrow (\omega, \Omega, \theta, \chi)_{\text{there}}$. In particular this transforms the torsion modules as $(\mathrm{e}^{\phi}w_1, \mathrm{e}^{3\phi}w_2, \mathrm{e}^{2\phi}w_3, w_4 + 8\,\mathrm{d}\phi, w_5 - 40\,\mathrm{d}\phi)_{\text{here}} \leftrightarrow (w_1, w_2, w_3, w_4, w_5)_{\text{there}}$.

Moreover it follows that the 11d flux takes the form

$$\begin{aligned}\mathcal{G}_4 = (\mathrm{d}t + a) \wedge \mathrm{d}(\mathrm{e}^{2\phi} j) - \Big[\frac{3}{4}\,\mathrm{d}a^{(0)} j + \mathrm{d}a^{(2,0)} + \mathrm{d}a^{(0,2)} + \frac{1}{3}\,\mathrm{d}a_0^{(1,1)}\Big] \wedge \mathrm{e}^{2\phi} j \\ + \frac{1}{2}\mathrm{e}^{-2\phi} \star_{10} \mathrm{d}\big(\Delta^{-3/2}\mathrm{e}^{5\phi}\,\mathrm{Re}[\omega]\big) - \frac{1}{2}\mathrm{e}^{-2\phi} \star_{10} \Big[j \wedge \mathrm{d}\big(\Delta^{-3/2}\mathrm{e}^{5\phi}\,\mathrm{Re}[\omega]\big)\Big] \wedge j \\ - \frac{1}{16}\Delta^{-3/2}\mathrm{e}^{3\phi} \star_{10} \left([w_5 + 4w_4 - 8\mathrm{d}\phi] \wedge \mathrm{Re}[\omega]\right) + h_0^{(2,2)}\,,\end{aligned} \tag{9.8}$$

where $\mathrm{d}a$ decomposes as $\mathrm{d}a = \mathrm{d}a^{(0)} j + \mathrm{d}a_0^{(1,1)} + \mathrm{d}a^{(2,0)} + \mathrm{d}a^{(0,2)}$, and $h_0^{(2,2)}$ is the primitive $(2,2)$ part of h_4 and is unconstrained by supersymmetry. Additionally the torsion module w_5 is fixed by supersymmetry to be

$$w_5 = -12\,\mathrm{d}\log\Delta + 40\,\mathrm{d}\phi\,. \tag{9.9}$$

For a supersymmetric solution to exist these conditions must be supplemented by the Bianchi identity and Maxwell equation

$$\mathrm{d}h_4 = -\mathrm{d}a \wedge \mathrm{d}(\mathrm{e}^{2\phi} j)\,, \tag{9.10}$$

$$\mathrm{d}\big(\Delta^{-3}\mathrm{e}^{4\phi} \star_{10} \mathrm{d}(\mathrm{e}^{2\phi} j)\big) = \mathrm{e}^{2\phi}\,\mathrm{d}a \wedge \star_{10}\, h_4 + \frac{1}{2}h_4 \wedge h_4\,. \tag{9.11}$$

The set of equations as given above are both necessary and sufficient for a solution to admit a timelike Killing spinor.

Our main motivation is to obtain the near-horizon geometries of rotating M2-branes wrapped on Riemann surfaces, which may give rise to the near-horizon of rotating black holes. It is also possible to engineer black holes using M5-branes, see for example [114, 151–154] however we will not consider this possibility in this work and restrict exclusively to solutions without M5 branes. This implies that we must make some assumptions about the form of the solution. It would be interesting in the future to relax these assumptions. To engineer such solutions one should place the rotating M2-branes in an asymptotic geometry of the form $\mathbb{R}_t \ltimes CY_5$ and then wrap the M2-brane on a Riemann surface inside the Calabi–Yau five-fold. Note that the rotation of the M2-brane leads to the non-trivial fibration of the 11d spacetime, with the time direction fibered over the five-fold. Since the asymptotic geometry is Calabi–Yau it is natural to expect that our 10d base space is complex, which requires that $w_1 = w_2 = 0$. This is indeed how the rotating M2-brane solution is embedded in the classification of [150] however we have not been able to prove that restricting to just M2-branes implies the complex condition. We will be satisfied with using the complex condition as a well-motivated ansatz in the following though it would certainly be interesting to lift this restriction. In addition to requiring the complex condition we also want to eliminate the possibility of having flux sourcing M5-branes. For this reason we will remove any terms appearing in the flux which are of Hodge type $(4,0)+(0,4)$, since these would not come from M2-branes wrapped on a Riemann surface.[2] From (9.8) and (9.9) we see that this assumption implies $w_4 = 3\,\mathrm{d}\log\Delta - 8\,\mathrm{d}\phi$.

Under these assumptions the 10d torsion conditions are

$$\begin{aligned} \mathrm{d}(\mathrm{e}^{2\phi}\, j) &= f_3\,, \\ \mathrm{d}(\Delta^{-3}\mathrm{e}^{8\phi}\, j^4) &= 0\,, \\ \mathrm{d}(\Delta^{-3/2}\mathrm{e}^{5\phi}\, \omega) &= 0\,. \end{aligned} \tag{9.12}$$

The last unspecified torsion module is given by the primitive part of the three-form flux: $w_3 = \mathrm{e}^{-2\phi}\, f_{3,0}$. The 11d flux can now be succinctly written as

$$\begin{aligned} \mathcal{G}_4 =&(\mathrm{d}t + a)\wedge \mathrm{d}(\mathrm{e}^{2\phi} j) - \Big[\frac{3}{4}\mathrm{d}a^{(0)} j + \mathrm{d}a^{(2,0)} + \mathrm{d}a^{(0,2)} + \frac{1}{3}\mathrm{d}a_0^{(1,1)}\Big]\wedge \mathrm{e}^{2\phi} j + h_0^{(2,2)} \\ =& -\mathrm{d}\big[(\mathrm{d}t + a)\wedge \mathrm{e}^{2\phi} j\big] + \tilde{h}^{(2,2)}\,, \end{aligned} \tag{9.13}$$

where we define the shifted four-form flux

$$\begin{aligned} \tilde{h}^{(2,2)} &= h_4 + \mathrm{d}a\wedge \mathrm{e}^{2\phi} j\,, \\ &= h_0^{(2,2)} + \frac{1}{2}\,\mathrm{e}^{2\phi}\,\mathrm{d}a^{(0)}\,\frac{j^2}{2!} + \frac{2}{3}\,\mathrm{e}^{2\phi}\,\mathrm{d}a_0^{(1,1)}\wedge j\,. \end{aligned} \tag{9.14}$$

The Bianchi identity (9.10) and Maxwell equation (9.11) can now be rewritten in terms of $\tilde{h}^{(2,2)}$ as

$$\mathrm{d}\tilde{h}^{(2,2)} = 0\,, \tag{9.15}$$

$$\mathrm{d}\big(\Delta^{-3}\mathrm{e}^{4\phi}\star_{10}\mathrm{d}(\mathrm{e}^{2\phi} j)\big) = \frac{1}{2}\tilde{h}^{(2,2)}\wedge\tilde{h}^{(2,2)}\,. \tag{9.16}$$

For future reference, we give a few useful identities containing $\tilde{h}^{(2,2)}$:

$$\star_{10}\,\tilde{h}^{(2,2)} = \tilde{h}^{(2,2)}\wedge j - \mathrm{e}^{2\phi}\,\mathrm{d}a^{(0)}\,\frac{j^3}{3!} - 2\,\mathrm{e}^{2\phi}\,\mathrm{d}a_0^{(1,1)}\wedge\frac{j^2}{2!}\,, \tag{9.17}$$

$$j\lrcorner\,\tilde{h}^{(2,2)} = 2e^{2\phi}\,\mathrm{d}a^{(1,1)}\,. \tag{9.18}$$

Here $\mathrm{d}a^{(1,1)} = \mathrm{d}a^{(0)} j + \mathrm{d}a_0^{(1,1)}$, i.e. we omit the $(0,2)$ and $(2,0)$ contributions.

[2]Lifting this assumption would open up the possibility of studying the near-horizon of rotating asymptotically AdS_7 black holes [151] arising from wrapping M5-branes on SLAG five-cycles in the Calabi–Yau five-fold. This would give the 11d geometric setting for the computations performed in [114, 152–154].

9.1.2 Ansatz

To proceed we must now insert an ansatz for the 10d base space. It was shown in [150] that the base is necessarily non-compact (the argument uses some smoothness conditions but these should hold in the present setting), and so we impose that the base is conformally a cone. The metric we take is

$$\mathrm{d}s^2_{11} = -\Delta^2(\mathrm{d}t + a)^2 + \Delta^{-1}\mathrm{e}^{2\phi}\big(\mathrm{d}r^2 + r^2\mathrm{d}s^2_9\big)\,. \tag{9.19}$$

Next we need to specify how the scalar fields Δ, ϕ, connection one-form a and fluxes scale with respect to the radial coordinate. Ultimately we want to be able to recover a warped AdS_2 factor and an r-independent 9d space. This fixes the scaling of Δ and ϕ to be

$$\Delta = \frac{\mathrm{e}^{B+C}}{r}\,, \qquad \mathrm{e}^{2\phi} = \frac{\mathrm{e}^{3B+C}}{r^3}\,, \tag{9.20}$$

where we have introduced two new scalars B and C which are independent of the radial coordinate. For general scalar C this will not lead to a geometry admitting an $\mathrm{SO}(2,1)$ isometry generating the conformal group in 1d. As discussed earlier one must impose additional constraints. Rather than imposing them now it is more convenient to impose them later and leave the scalar C unconstrained for the moment.

The conical geometry naturally gives rise to an R-symmetry vector ξ defined by

$$\xi = j\cdot(r\partial_r)\,. \tag{9.21}$$

As can be easily checked by explicit computation the norm squared of the vector is r^2. On the link of the cone at $r=1$ this translates to the existence of a unit-norm vector generating a holomorphic foliation over an 8d base admitting an $\mathrm{SU}(4)$ structure inherited from the parent $\mathrm{SU}(5)$ structure. We denote this 8d base by $\mathcal{B}$. Introducing coordinates for this vector

$$\xi = \partial_z\,, \tag{9.22}$$

we can write the dual one-form as

$$\eta = \mathrm{d}z + P\,, \tag{9.23}$$

where P is a one-form on $\mathcal{B}$. We may now decompose the $\mathrm{SU}(5)$ structure (j,ω) in terms of the $\mathrm{SU}(4)$ structure, which we denote by (J,Ω), as

$$\begin{aligned} j &= r\eta\wedge\mathrm{d}r + r^2\,\mathrm{e}^{-3B-C/3}\,J\,,\\ \omega &= r^4\,\mathrm{e}^{-6B-2C/3}\,\mathrm{e}^{\mathrm{i}z}(\mathrm{d}r - \mathrm{i}r\eta)\wedge\Omega\,. \end{aligned} \tag{9.24}$$

Here we include a scaling $\mathrm{e}^{-3B-C/3}$ of the 8d base, and a phase along the z-direction. The choice of scaling has been chosen so that the two form is balanced rather than conformally balanced as will become clear in the following section. While the phase is required by supersymmetry and implies that the holomorphic volume form has unit charge under the vector ξ.

The scaling of the connection one-form appearing in the time-fibration is fixed to be

$$a = r(\alpha\,\eta + A)\,, \tag{9.25}$$

where α and A denote an 8d scalar and one-form respectively. Note that we did not include a term with a leg on $\mathrm{d}r$ in this decomposition because such a term could be absorbed by redefinitions and coordinate changes for a near-horizon geometry. It will turn out that imposing the $\mathrm{SO}(2,1)$ symmetry will further constrain the one-form a and scalar C however we postpone this discussion to later. The field strength $\mathrm{d}a$ is

$$\mathrm{d}a = \alpha\,\mathrm{d}r\wedge\eta + \mathrm{d}r\wedge A - r\,\eta\wedge\mathrm{d}\alpha + r\,(\alpha\,\mathrm{d}\eta + \mathrm{d}A)\,. \tag{9.26}$$

With these ansätze the 11d metric becomes

$$\mathrm{d}s^2_{11} = \mathrm{e}^{2B}\Big[-\mathrm{e}^{2C}\Big(\frac{\mathrm{d}t}{r} + \alpha\,\eta + A\Big)^2 + \frac{\mathrm{d}r^2}{r^2} + \eta^2 + \mathrm{e}^{-3B-C/3}\,\mathrm{d}s^2_8\Big]\,. \tag{9.27}$$

We recover the non-rotating case by setting $\alpha = 0$, $A = 0$ and $\mathrm{e}^{2C} = 1$.[3]

Finally we must fix the r-scaling of the flux. The scaling is fixed by regularity as $r\to 0$ and preserving the $\mathrm{SO}(2,1)$ symmetry which requires the radial dependence to only appear in the one-forms

$$\frac{\mathrm{d}t}{r} \quad \text{and} \quad \frac{\mathrm{d}r}{r}\,. \tag{9.28}$$

It follows that the 10d fluxes f_3 and $\tilde{h}^{(2,2)}$ decompose in terms of 8d fluxes as

$$f_3 = r^{-1}\Big[\frac{\mathrm{d}r}{r}\wedge\eta\wedge F_1 + \frac{\mathrm{d}r}{r}\wedge F_2 + F_3\Big]\,, \tag{9.29}$$

$$\tilde{h}^{(2,2)} = H^{(2,2)} + \frac{\mathrm{d}r}{r}\wedge(H^{(2,1)} + H^{(1,2)}) + \eta\wedge\mathrm{i}(H^{(2,1)} - H^{(1,2)}) + \frac{\mathrm{d}r}{r}\wedge\eta\wedge H^{(1,1)}\,. \tag{9.30}$$

In principle one could include a piece of f_3 with one leg on η and two legs on $\mathcal{B}$, but we omit it here because it will be put to zero by supersymmetry. Note that we

[3] In comparison to [105] we identify $B_{\text{here}} \leftrightarrow A_{\text{there}}$, and comparing with [139] we identify $B_{\text{here}} \leftrightarrow -B_{\text{there}}/3$.

keep track of the Hodge type of the components of $\tilde{h}^{(2,2)}$, where the holomorphic and anti-holomorphic one-form associated with $\mathrm{d}r$ and η are given by $e^1 = \mathrm{d}r - \mathrm{i}r\eta$ and its conjugate respectively.

9.1.3 8d supersymmetry conditions

We can now derive the 8d conditions by reducing their 10d counterparts using the ansätze presented in the previous section. Let us begin by reducing the SU(5) structure torsion conditions to SU(4) structure conditions. From decomposing (9.12) we find

$$
\begin{aligned}
F_1 &= -\,\mathrm{d}\mathrm{e}^{3B+C}\,, && (9.31)\\
F_2 &= \mathrm{e}^{3B+C}\,\mathrm{d}\eta - \mathrm{e}^{2C/3}\,J\,, && (9.32)\\
F_3 &= \mathrm{d}(\mathrm{e}^{2C/3}\,J)\,, && (9.33)\\
\mathrm{d}J^3 &= 0\,, && (9.34)\\
\mathrm{d}\eta\wedge\frac{J^3}{3!} &= \mathrm{e}^{-3B-C/3}\,\frac{J^4}{4!}\,, && (9.35)\\
\mathrm{d}\mathrm{e}^{-3B-C/3}\wedge J^4 &= 0\,, && (9.36)\\
\mathrm{d}\Omega &= \mathrm{i}\big(P+\tfrac{1}{3}\mathrm{d}^cC\big)\wedge\Omega\,. && (9.37)
\end{aligned}
$$

Recall that Ω has unit charge under the vector ∂_z which is evident from (9.24). From these equations we can deduce the SU(4) torsion modules W_i. From (9.37) we immediately see that the 8d base is complex: $W_1 = W_2 = 0$. Furthermore, from (9.34) we see that $W_4 = 0$, i.e. the base is balanced. Fixing the two-form to be balanced as opposed to conformally balanced fixed the choice of scaling of the 8d base in (9.24). In particular the base is not Kähler: the third torsion module is related to the primitive part of F_3 as $W_3 = \mathrm{e}^{-2C/3}\,F_{3,0}$. However, for some black holes known in the literature this part of the flux vanishes, and the 8d base is Kähler. From (9.37) we find $W_5 = -4J\cdot P - \frac{4}{3}\mathrm{d}C$ and this fixes the Ricci-form of the base in terms of the connection P and the scalar C as we show below. Before proceeding it is useful to rewrite the three-form flux f_3 as

$$
f_3 = r^{-1}\left[\frac{\mathrm{d}r}{r}\wedge\hat{F} - \mathrm{d}\hat{F}\right], \quad \text{where} \quad \hat{F} = -\mathrm{e}^{2C/3}J + \mathrm{d}(\mathrm{e}^{3B+C}\eta) \qquad (9.38)
$$

which puts it into a form more reminiscent of the non-rotating case [105].

Let us turn our attention to the other identities following from (9.31)-(9.37). Firstly, from (9.36) we find

$$
\mathcal{L}_\xi\mathrm{e}^{-3B-C/3} = 0\,. \qquad (9.39)
$$

In fact, we will take ξ to be a symmetry of each of the scalars B, C individually, though supersymmetry does not require this. This assumption is natural since we want ξ to play the role of the R-symmetry vector of the solution. Note that these conditions imply that it is a Killing vector of the 10d space and by imposing $\mathcal{L}_\xi \alpha = 0$, it is in fact a Killing vector for the full 11d metric. Taking the exterior derivative of (9.37) implies

$$\mathrm{d}\eta \wedge \Omega = 0\,, \tag{9.40}$$

hence $\mathrm{d}\eta$ is a $(1,1)$-form on the base. Moreover from (9.35) we find that

$$J \lrcorner\, \mathrm{d}\eta = \mathrm{e}^{-3B-C/3}\,. \tag{9.41}$$

Finally from (9.37) we can read off the Ricci form ρ on the 8d space to be

$$\rho = \mathrm{d}\eta + \tfrac{1}{3}\mathrm{d}\mathrm{d}^c C\,. \tag{9.42}$$

Note that the second term is exact since we require the scalar C to be globally well-defined.

This in turn allows us to compute the Chern–Ricci scalar[4]

$$R_C \equiv 2J \lrcorner\, \rho = 2e^{-3B-C/3} - \tfrac{2}{3}\Box C\,. \tag{9.43}$$

The Chern–Ricci scalar is related to the more common 8d Ricci scalar via[5]

$$R_8 = R_C - \tfrac{1}{2}|\mathrm{d}J|^2\,. \tag{9.44}$$

It is clear from the above relation that the two scalars coincide when the manifold is Kähler.

So far we have only imposed supersymmetry and not the equations of motion. Integrability of the Killing spinor equations implies that the Einstein equations are satisfied so long as the Bianchi identity (9.15) and Maxwell equation (9.16) are imposed. Imposing these gives us additional constraints on the geometry and fluxes. From reducing the Bianchi identity we find

$$\begin{aligned}
\mathrm{d}H^{(2,2)} &= -\mathrm{i}\mathrm{d}\eta \wedge (H^{(2,1)} - H^{(1,2)})\,,\\
\partial H^{(2,1)} &= \bar{\partial} H^{(1,2)} = 0\,,\\
\bar{\partial} H^{(2,1)} &= \partial H^{(1,2)} = -\tfrac{1}{2}\,\mathrm{d}\eta \wedge H^{(1,1)}\,,\\
\mathrm{d}H^{(1,1)} &= 0\,.
\end{aligned} \tag{9.45}$$

[4]Here we find the d'Alembertian operator through the short computation:

$$J \lrcorner\, \mathrm{d}\mathrm{d}^c C = *(\mathrm{d}\mathrm{d}^c C \wedge *J) = *\,\mathrm{d}\,(\mathrm{d}^c C \wedge *J) = *\,\mathrm{d}*(\mathrm{d}^c C \lrcorner\, J) = -*\mathrm{d}*\mathrm{d}C = -\Box C\,,$$

where we use that $\frac{1}{3!}\mathrm{d}J^3 = \mathrm{d}*J = 0$.

[5]Note that this is equivalent to the identity $R_8 = R_C - \frac{1}{2}|\mathrm{d}^c J|^2$ that is also used in the literature.

From this decomposition it is simple to show that the R-symmetry vector ξ is not just a symmetry of the metric, but also for the 10d flux $\tilde{h}^{(2,2)}$, i.e.

$$\mathcal{L}_\xi \tilde{h}^{(2,2)} = 0\,. \tag{9.46}$$

In fact, we find that ξ is a symmetry for the full 11d flux G_4 as well, since by using (9.26) and that the scalar α has vanishing Lie-derivative along ξ one can show that

$$\mathcal{L}_\xi G_4 = 0\,. \tag{9.47}$$

This is then consistent with our interpretation of ξ as being the Killing vector dual to the R-symmetry of a putative dual field theory.

From the 10d Maxwell equation we find the set of equations

$$\begin{aligned}
-\mathrm{d} *_8 \mathrm{d}\mathrm{e}^{-3B-C} + \mathrm{e}^{-2C/3}\,\mathrm{d}\eta\wedge\mathrm{d}\eta\wedge\frac{J^2}{2!} &= \frac{1}{2}H^{(2,2)}\wedge H^{(2,2)}\,, && (9.48)\\
\mathrm{e}^{-4C/3}\,\mathrm{d}\eta\wedge *_8\,\mathrm{d}(\mathrm{e}^{2C/3}J) &= H^{(2,2)}\wedge\big(H^{(2,1)}+H^{(1,2)}\big)\,, && (9.49)\\
-\mathrm{d}\eta\wedge\mathrm{d}\Big(\mathrm{e}^{-2C/3}\,\frac{J^2}{2!}\Big) &= H^{(2,2)}\wedge \mathrm{i}\big(H^{(2,1)}-H^{(1,2)}\big)\,, && (9.50)\\
-\mathrm{d}\mathrm{d}^c\Big(\mathrm{e}^{-2C/3}\frac{J^2}{2!}\Big) &= H^{(2,2)}\wedge H^{(1,1)} + 2\mathrm{i}H^{(2,1)}\wedge H^{(1,2)}\,. && (9.51)
\end{aligned}$$

It can be shown that the second and third equation are equivalent by acting with the operator $J\cdot$ which acts by contracting the complex structure into each index of the form. For a (p,q)-form this acts by multiplying the form by i^{p-q}. By applying the 8d Hodge star to (9.48), and by inserting (9.42) and (9.43), we can rewrite it as

$$\begin{aligned}
-\mathrm{e}^{2C/3}\Box\big(\mathrm{e}^{-2C/3}(R_C+\tfrac{2}{3}\Box C)\big) + \tfrac{1}{2}\big(R_C+\tfrac{2}{3}\Box C\big)^2 - 2\big|\rho-\tfrac{1}{3}\mathrm{d}\mathrm{d}^c C\big|^2&\\
= \mathrm{e}^{2C/3} *_8 \big(H^{(2,2)}\wedge H^{(2,2)}\big)\,.&
\end{aligned}\tag{9.52}$$

This is the rotating version of the master equation [105, 138]. It reduces to the familiar non-rotating master equation of [105] by setting $e^{2C}=1$, $H^{(2,2)}=0$ and $\mathrm{d}J=0$ (so that $R_C=R_8$).

One can be slightly more explicit with the form of the flux terms and determine them up to primitive pieces. From (9.26) and by decomposing $\mathrm{d}a$ in term of its

Hodge type we find

$$
\begin{aligned}
\mathrm{d}a^{(0)} &= \tfrac{4}{5}\, r^{-1} \mathrm{e}^{3B+C/3}\, \mathrm{d}A^{(0)}\,, \\
\mathrm{d}a^{(2,0)} &= \tfrac{1}{2} e^1 \wedge (A^{(1,0)} - \mathrm{i}\partial\alpha) + r\, \mathrm{d}A^{(2,0)}\,, \\
\mathrm{d}a^{(0,2)} &= \tfrac{1}{2} \bar{e}^1 \wedge (A^{(0,1)} + \mathrm{i}\bar{\partial}\alpha) + r\, \mathrm{d}A^{(0,2)}\,, \\
\mathrm{d}a_0^{(1,1)} &= \mathrm{d}r \wedge \eta \left(\alpha + \tfrac{4}{5}\, \mathrm{e}^{3B+C/3}\, \mathrm{d}A^{(0)}\right) + r\big(\alpha\, \mathrm{d}\eta + \tfrac{1}{5}\, \mathrm{d}A^{(0)} J + \mathrm{d}A_0^{(1,1)}\big) \\
&\quad + \tfrac{1}{2}\mathrm{d}r \wedge (A - \mathrm{d}^c\alpha) + \tfrac{1}{2} r\eta \wedge (J \cdot A - \mathrm{d}\alpha)\,.
\end{aligned}
\tag{9.53}
$$

We can use these decompositions to reduce (9.18) which implies:

$$
\begin{aligned}
e^{-2C/3}\, J \lrcorner H^{(1,1)} &= 2\alpha\,, \\
e^{-2C/3}\, J \lrcorner H^{(2,1)} &= \mathrm{i}\partial\alpha + A^{(1,0)}\,, \\
e^{-2C/3}\, J \lrcorner H^{(2,2)} - e^{-3B-C}\, H^{(1,1)} &= 2\mathrm{d}A^{(1,1)} + 2\alpha \mathrm{d}\eta\,.
\end{aligned}
\tag{9.54}
$$

Therefore we may rewrite the fluxes as

$$
\begin{aligned}
H^{(1,1)} &= \tfrac{1}{2} e^{2C/3} \alpha J + H_0^{(1,1)}\,, \\
H^{(2,1)} &= \tfrac{1}{3} e^{2C/3} J \wedge (i\partial\alpha + A^{(1,0)}) + H_0^{(2,1)}\,, \\
H^{(2,2)} &= \tfrac{1}{2} J \wedge \big(e^{-3B-C/3} H^{(1,1)} + 2e^{2C/3}(\mathrm{d}A^{(1,1)} + \alpha \mathrm{d}\eta)\big) \\
&\quad - \tfrac{1}{3}(2e^{2C/3}\mathrm{d}A^{(0)} + e^{-3B+C/3}\alpha) J^2 + H_0^{(2,2)}\,,
\end{aligned}
\tag{9.55}
$$

where $H_0^{(p,q)}$ denotes the primitive piece. In principle one could now substitute these expressions into the Bianchi identities and Maxwell equations however this is not particularly enlightening and so we refrain from presenting them here. Note that the primitive pieces are essential for satisfying the Bianchi identities.

9.1.4 Imposing the SO(2,1) isometry

So far our analysis has been for general scalars C, α and one-form A. However, in order to construct the near-horizon of a black hole we need to impose that there is an $\mathrm{SO}(2,1)$ isometry, which leads to constraints on these fields, see also [132] which proves that the near-horizon of a black hole in M-theory necessarily has this symmetry. In appendix 9.A we have given the general metric for the near-horizon of a rotating black hole with a manifest AdS_2 factor over which the internal manifold is fibered and seen the constraints that this imposes on the geometry. In particular the fibration is governed by a vector of constants k^i associated to each Killing vector of the internal manifold fibered over AdS_2. As we reviewed in the appendix the necessity for these parameters to be constant arises in order that there is an

SO(2, 1) isometry. From the analysis of appendix 9.A we find that the scalar and one-form take the form[6]

$$\alpha\eta + A = -k^i g(\partial_{\phi_i}, \cdot)\,, \qquad \mathrm{e}^{-2C} = 1 + |\alpha\eta + A|_9^2\,, \tag{9.56}$$

where ∂_{ϕ_i} are the Killing vectors of the internal manifold and the metric g_{ij} is the metric on $\mathrm{d}s_9^2$, as defined in (9.19), restricted to the angular coordinates. Denoting by

$$\eta_i \equiv g(\partial_{\phi_i}, \cdot)\,, \tag{9.57}$$

the dual one-form of the Killing vector ∂_{ϕ_i} using the metric on $\mathrm{d}s_9^2$. Then the one-form a is simply

$$a \equiv r(\alpha\eta + A) = -rk^i\eta_i\,. \tag{9.58}$$

In the remainder of this section let us assume that the 8d base is Kähler since this will allow for more explicit expressions. In addition we will assume that the base is toric, with the 9d space Y_9 admitting a $\mathrm{U}(1)^5$ action with Killing vectors ∂_{ϕ_i}.[7] We may write the one form η as[8]

$$\eta = 2\sum_i w_i \mathrm{d}\phi_i\,, \tag{9.59}$$

where the w_i are the moment map coordinates of the cone restricted to Y_9. Moreover the Kähler two-form on the base may be expanded as

$$J = \sum_i \mathrm{d}x_i \wedge \mathrm{d}\phi_i\,, \tag{9.60}$$

where x_i are global functions on Y_9 since $b_1(Y_9) = 0$ for a toric contact structure. Note that

$$\partial_{\phi_i} \lrcorner\, J = -\mathrm{d}x_i\,. \tag{9.61}$$

With this short (and very incomplete) review of toric geometry we may proceed with writing the scalars and one-form in terms of the global functions of the toric geometry defined above. It follows that

$$\alpha = -k^i\partial_{\phi_i} \lrcorner\, \eta = -2k^i w_i\,. \tag{9.62}$$

[6]We use the math literature notation such that $g(\partial_{\phi_i}, \cdot)$ is a one-form.

[7]We need not require the full space to be toric for our arguments to hold, we merely do so for simplicity of exposition. An interesting case to consider, which requires a minor generalization, is to consider a Riemann surface embedded into Y_9 as $Y_9 \equiv O(\vec{n})_{\Sigma_g} \times_{U(1)^4} Y_7$ with $\vec{n}$ a four-vector of constant twist parameters which are the Chern numbers of the U(1) bundle over the Riemann surface [123].

[8]We follow the toric geometry notational conventions of [123].

Next consider A, we find the simple result

$$A = -\mathrm{e}^{-3B-C/3} k^i \mathrm{d}^c x_i \,. \tag{9.63}$$

Finally we may evaluate (9.56) which implies

$$\mathrm{e}^{-2C} = 1 + (2k^i w_i)^2 + \mathrm{e}^{-3B-C/3} |k^i \mathrm{d}^c x_i|_8^2 \,, \tag{9.64}$$

where $|\cdot|_8^2$ is the norm with respect to the Kähler metric. In principle one could try to solve this for the scalar C, however this is a sextic equation to solve. One could use (9.64) as defining the combination $\mathrm{e}^{-3B-C/3}$ which appears ubiquitously in the geometry.

Note that this last comment only applies when the gauge field A is non-zero. When it vanishes and the fibration is only along the R-symmetry direction, it turns out that C is constant. To see this it is more insightful to use the parametrization employed in appendix 9.A where the z-coordinate is assigned its own constant k^z, i.e. we do not use the basis ∂_{ϕ_i} used previously in this section. In this basis the Killing vectors are the four U(1) isometries of the base and the R-symmetry vector ∂_z. It is then clear that for A to vanish each of the four constants associated to the U(1)'s of the base must be zero. It follows from (9.129) that α is precisely the constant $-k^z$. Moreover e^{-2C} takes the constant value,

$$\mathrm{e}^{-2C} = 1 + (k^z)^2 \,. \tag{9.65}$$

The natural interpretation of this subcase is that of the near-horizon of a non-rotating black hole equipped with an electric component for the graviphoton and possibly including magnetic charges for each of the gauge fields in the 4d theory.

9.2 Action for the theory

One of the essential ingredients for performing the extremization in [109] was the existence of an action which gave rise to the equations of motion of the theory. This action was derived in [139] for the near-horizon geometry of static black holes and strings in M-theory and type IIB respectively. As a first step towards performing the extremization in the rotating case we will construct the analogous rotating action. Thereafter we impose the supersymmetry constraints on this action and show that it reduces to a simple and familiar form. The action computes the entropy of these black holes.

9.2.1 Non-supersymmetric action

The simplest method for constructing an action for the 9d geometry is to reduce the 11d action using our ansätze. By construction the equations of motion of the resulting 9d action will match the ones obtained in the section 9.1.3. We start from the action of eleven-dimensional supergravity

$$S_{11} = \frac{1}{2\kappa_{11}^2} \int R_{11} *_{11} 1 - \frac{1}{2} \mathcal{G}_4 \wedge *_{11} \mathcal{G}_4 - \frac{1}{6} \mathcal{C}_3 \wedge \mathcal{G}_4 \wedge \mathcal{G}_4 \,. \tag{9.66}$$

Here $\mathcal{C}_3$ is the three-form potential and $\mathcal{G}_4 = \mathrm{d}\mathcal{C}_3$ is its field strength. Using the ansätze

$$\begin{aligned} \mathrm{d}s_{11}^2 &= -\Delta^2(\mathrm{d}t + a)^2 + \Delta^{-1} e^{2\phi}\, \mathrm{d}s_{10}^2 \,, \\ \mathcal{G}_4 &= \Delta^{-1} e^0 \wedge f_3 + h_4 \,, \end{aligned} \tag{9.67}$$

we reduce this action to 10d. The Bianchi identity $\mathrm{d}\mathcal{G}_4 = 0$ implies that

$$\mathrm{d}f_3 = 0 \,, \qquad \mathrm{d}h_4 + \mathrm{d}a \wedge f_3 = 0 \,. \tag{9.68}$$

We write these field strengths in terms of their potentials as

$$f_3 = \mathrm{d}c_2 \,, \qquad h_4 = \mathrm{d}c_3 - \mathrm{d}a \wedge c_2 \,. \tag{9.69}$$

Now we can write $\mathcal{G}_4$ into the convenient form

$$\mathcal{G}_4 = -\,\mathrm{d}\big[(\mathrm{d}t + a) \wedge c_2\big] + \tilde{h}^{(2,2)} \,, \tag{9.70}$$

where we introduce the shifted four-form field strength

$$\tilde{h}^{(2,2)} = h_4 + \mathrm{d}a \wedge c_2 = \mathrm{d}c_3 \,. \tag{9.71}$$

Note that although we add a superscript to indicate that upon imposing supersymmetry this field strength is a $(2,2)$-form, at the moment we have not imposed supersymmetry yet so we have to treat $\tilde{h}^{(2,2)}$ as a general four-form. The 11d potential $\mathcal{C}_3$ can now be expressed in terms of the 10d potentials as

$$\mathcal{C}_3 = -(\mathrm{d}t + a) \wedge c_2 + c_3 \,. \tag{9.72}$$

By using these ansätze and definitions, we find the 10d Lagrangian

$$\begin{aligned} \mathscr{L}_{10} =& \Delta^{-3} e^{8\phi} \big(R_{10} - 72(\partial_\mu\phi)^2 - 18\,\nabla^2\phi - 12(\partial_\mu \log\Delta)^2 + 7\,\nabla^2 \log\Delta + 56\,\partial_\mu\phi\,\partial^\mu \log\Delta\big) *_{10} 1 \\ &+ \tfrac{1}{2}\, e^{6\phi}\, \mathrm{d}a \wedge *_{10}\, \mathrm{d}a + \tfrac{1}{2}\, \Delta^{-3} e^{4\phi}\, f_3 \wedge *_{10} f_3 - \tfrac{1}{2}\, e^{2\phi}\, \tilde{h}^{(2,2)} \wedge *_{10}\, \tilde{h}^{(2,2)} \\ &+ e^{2\phi}\, c_2 \wedge \mathrm{d}a \wedge *_{10}\, \tilde{h}^{(2,2)} - \tfrac{1}{2}\, e^{2\phi}\, c_2 \wedge \mathrm{d}a \wedge *_{10}\, (c_2 \wedge \mathrm{d}a) \\ &+ \tfrac{1}{2}\, c_2 \wedge \tilde{h}^{(2,2)} \wedge \tilde{h}^{(2,2)} - \tfrac{1}{2}\, (c_2)^2 \wedge \mathrm{d}a \wedge \tilde{h}^{(2,2)} + \tfrac{1}{6} (c_2)^3 \wedge (\mathrm{d}a)^2 \,. \end{aligned} \tag{9.73}$$

Next we want to consider the reduction of this Lagrangian to 9d, by using the cone ansatz presented in (9.1.2). In addition, we want to split off the η-direction from the 8d space $\mathcal{B}$ so that we end up with a 9d Lagrangian density of the form $\mathscr{L}_9 = \eta \wedge (\ldots)$ where the dots represent an expression in terms of fields defined on $\mathcal{B}$. The relevant ansätze for this reduction are[9]

$$\begin{aligned}
\mathrm{d}s_{10}^2 &= \mathrm{d}r^2 + r^2\eta^2 + r^2\,\mathrm{e}^{-3B-C/3}\,\mathrm{d}s_8^2\,,\\
\mathrm{e}^{2\phi} &= r^{-3}\,\mathrm{e}^{3B+C}\,,\\
\Delta &= r^{-1}\,\mathrm{e}^{B+C}\,,\\
\mathrm{d}a &= r\,(\alpha\,\mathrm{d}\eta + \mathrm{d}A) + \mathrm{d}r\wedge A - r\,\eta\wedge\mathrm{d}\alpha + \alpha\,\mathrm{d}r\wedge\eta\,,\\
c_2 &= r^{-1}\,C_2 + r^{-1}\,\eta\wedge C_1 + r^{-2}\,C_0\,\mathrm{d}r\wedge\eta\,,\\
f_3 &= r^{-1}\,F_3 + r^{-2}\,\mathrm{d}r\wedge F_2 + r^{-1}\,\eta\wedge\hat{F}_2 + r^{-2}\,\mathrm{d}r\wedge\eta\wedge F_1\,,\\
\tilde{h}^{(2,2)} &= H^{(2,2)} + r^{-1}\mathrm{d}r\wedge(H^{(2,1)}+H^{(1,2)}) + i\eta\wedge(H^{(2,1)}-H^{(1,2)}) + r^{-1}\mathrm{d}r\wedge\eta\wedge H^{(1,1)}.
\end{aligned} \tag{9.74}$$

Performing this reduction is a lengthy but in principle straightforward calculation. We find the 9d Lagrangian[10]

$$\begin{aligned}
\mathscr{L}_9 = \eta\wedge\Big[&\big(R_8 - \tfrac{9}{2}(\partial_\mu B)^2 - \tfrac{7}{6}(\partial_\mu C)^2 - 3\,\partial_\mu B\,\partial^\mu C - 2\,e^{-3B-C/3}\big)\ast_8 1 - \tfrac{1}{2}\,e^{3B+C/3}\,\mathrm{d}\eta\wedge\ast_8\mathrm{d}\eta\\
&+ \tfrac{1}{2}\,e^{-3B+5C/3}\alpha^2\ast_8 1 + \tfrac{1}{2}\,e^{2C}A\wedge\ast_8 A + \tfrac{1}{2}\,e^{2C}\mathrm{d}\alpha\wedge\ast_8\mathrm{d}\alpha + \tfrac{1}{2}\,e^{3B+7C/3}(\alpha\,\mathrm{d}\eta+\mathrm{d}A)\wedge\ast_8(\alpha\,\mathrm{d}\eta+\mathrm{d}A)\\
&+ \tfrac{1}{2}\,e^{-6B-2C}F_1\wedge\ast_8 F_1 + \tfrac{1}{2}\,e^{-3B-5C/3}F_2\wedge\ast_8 F_2 + \tfrac{1}{2}\,e^{-3B-5C/3}\hat{F}_2\wedge\ast_8\hat{F}_2 + \tfrac{1}{2}\,e^{-4C/3}F_3\wedge\ast_8 F_3\\
&- \tfrac{1}{2}\,e^{3B+C}\,H^{(2,2)}\wedge\ast_8 H^{(2,2)} - 2e^{2C/3}H^{(2,1)}\wedge\ast_8 H^{(1,2)} - \tfrac{1}{2}\,e^{-3B+C/3}H^{(1,1)}\wedge\ast_8 H^{(1,1)}\\
&+ e^{3B+C}\,C_2\wedge(\alpha\,\mathrm{d}\eta+\mathrm{d}A)\wedge\ast_8 H^{(2,2)} + ie^{2C/3}\big(C_1\wedge(\alpha\,\mathrm{d}\eta+\mathrm{d}A) - C_2\wedge\mathrm{d}\alpha\big)\wedge\ast_8(H^{(2,1)}-H^{(1,2)})\\
&+ e^{2C/3}C_2\wedge A\wedge\ast_8(H^{(2,1)}+H^{(1,2)}) + e^{-3B+C/3}\big(\alpha\,C_2 + C_1\wedge A + C_0\,(\alpha\,\mathrm{d}\eta+\mathrm{d}A)\big)\wedge\ast_8 H^{(1,1)}\\
&- \tfrac{1}{2}\,e^{3B+C}\,C_2\wedge(\alpha\,\mathrm{d}\eta+\mathrm{d}A)\wedge\ast_8(C_2\wedge(\alpha\,\mathrm{d}\eta+\mathrm{d}A)) - \tfrac{1}{2}\,e^{2C/3}C_2\wedge A\wedge\ast_8(C_2\wedge A)\\
&- \tfrac{1}{2}\,e^{2C/3}\big(C_1\wedge(\alpha\,\mathrm{d}\eta+\mathrm{d}A) - C_2\wedge\mathrm{d}\alpha\big)\wedge\ast_8\big(C_1\wedge(\alpha\,\mathrm{d}\eta+\mathrm{d}A) - C_2\wedge\mathrm{d}\alpha\big)\\
&- \tfrac{1}{2}\,e^{-3B+C/3}\big(\alpha\,C_2 + C_1\wedge A + C_0\,(\alpha\,\mathrm{d}\eta+\mathrm{d}A)\big)\wedge\ast_8\big(\alpha\,C_2 + C_1\wedge A + C_0\,(\alpha\,\mathrm{d}\eta+\mathrm{d}A)\big)\\
&- C_2\wedge H^{(2,2)}\wedge H^{(1,1)} - 2iC_2\wedge H^{(2,1)}\wedge H^{(1,2)} - \tfrac{1}{2}C_0\,H^{(2,2)}\wedge H^{(2,2)} - C_1\wedge H^{(2,2)}\wedge(H^{(2,1)}+H^{(1,2)})\\
&+ \tfrac{1}{2}\,(C_2)^2\wedge\alpha\,H^{(2,2)} - \tfrac{1}{2}\,(C_2)^2\wedge\mathrm{d}\alpha\wedge(H^{(2,1)}+H^{(1,2)}) - \tfrac{1}{2}\,i(C_2)^2\wedge A\wedge(H^{(2,1)}-H^{(1,2)})\\
&+ \tfrac{1}{2}\,(C_2)^2\wedge(\alpha\,\mathrm{d}\eta+\mathrm{d}A)\wedge H^{(1,1)} + C_0\,C_2\wedge(\alpha\,\mathrm{d}\eta+\mathrm{d}A)\wedge H^{(2,2)} + C_2\wedge C_1\wedge A\wedge H^{(2,2)}\\
&+ C_2\wedge C_1\wedge(\alpha\,\mathrm{d}\eta+\mathrm{d}A)\wedge(H^{(2,1)}+H^{(1,2)}) - (C_2)^2\wedge C_1\wedge A\wedge(\alpha\,\mathrm{d}\eta+\mathrm{d}A)\\
&- \tfrac{1}{3}(C_2)^3\wedge\alpha\,(\alpha\,\mathrm{d}\eta+\mathrm{d}A) - \tfrac{1}{3}(C_2)^3\wedge A\wedge\mathrm{d}\alpha - \tfrac{1}{2}\,C_0\,(C_2)^2\wedge(\alpha\,\mathrm{d}\eta+\mathrm{d}A)^2\Big]\,.
\end{aligned} \tag{9.75}$$

From this action one can derive the equation of motions that define the solutions discussed in the previous section 9.1.3. Note that we have not imposed any supersymmetry in deriving this action.

[9]Note that we omitted the part of c_2 that has one leg on $\mathrm{d}r$ and one leg on $\mathcal{B}$ only. The reason for this is that such a term can be absorbed in C_2 by a gauge transformation.

[10]We split off the r-coordinate as $\mathscr{L}_{10} = \mathscr{L}_9\wedge r^{-2}\mathrm{d}r$. Splitting off $\mathrm{d}r$ on the left side would give an overall minus sign.

9.2.2 Supersymmetric action

Here we consider the restriction of the Lagrangian obtained above to off-shell supersymmetric geometries. We say these 9d geometries are off-shell because we do not impose the equations of motion such as (9.52), and supersymmetric since we do impose the supersymmetry constraints discussed in section 9.1. We will see that the Lagrangian (9.75) becomes quite simple once supersymmetry has been imposed. The simplest method is to impose supersymmetry in 10d and subsequently reduce to 9d, instead of starting from the 9d Lagrangian (9.75). We begin with the 10d non-supersymmetric Lagrangian

$$\begin{aligned}\mathscr{L}_{10} =& \Delta^{-3} e^{8\phi}\big(R_{10} - 72(\partial_\mu\phi)^2 - 18\,\nabla^2\phi - 12(\partial_\mu \log\Delta)^2 + 7\,\nabla^2\log\Delta + 56\,\partial_\mu\phi\,\partial^\mu\log\Delta\big) *_{10} 1 \\ &+ \tfrac{1}{2}\, e^{6\phi}\, \mathrm{d}a \wedge *_{10}\, \mathrm{d}a + \tfrac{1}{2}\,\Delta^{-3} e^{4\phi}\, f_3 \wedge *_{10}\, f_3 - \tfrac{1}{2}\, e^{2\phi}\, \tilde{h}^{(2,2)} \wedge *_{10}\, \tilde{h}^{(2,2)} \\ &+ e^{2\phi}\, c_2 \wedge \mathrm{d}a \wedge *_{10}\, \tilde{h}^{(2,2)} - \tfrac{1}{2}\, e^{2\phi}\, c_2 \wedge \mathrm{d}a \wedge *_{10}\, (c_2 \wedge \mathrm{d}a) \\ &+ \tfrac{1}{2}\, c_2 \wedge \tilde{h}^{(2,2)} \wedge \tilde{h}^{(2,2)} - \tfrac{1}{2}\,(c_2)^2 \wedge \mathrm{d}a \wedge \tilde{h}^{(2,2)} + \tfrac{1}{6}(c_2)^3 \wedge (\mathrm{d}a)^2\,.\end{aligned} \tag{9.76}$$

Here we can readily plug in the susy conditions

$$c_2 = \mathrm{e}^{2\phi}\, j\,, \qquad f_3 = \mathrm{d}(\mathrm{e}^{2\phi}\, j)\,. \tag{9.77}$$

Furthermore, we use the decompositions

$$\mathrm{d}a = \mathrm{d}a^{(0)}\, j + \mathrm{d}a_0^{(1,1)} + \mathrm{d}a^{(2,0)} + \mathrm{d}a^{(0,2)}\,, \tag{9.78}$$

$$\tilde{h}^{(2,2)} = \tilde{h}_0^{(2,2)} + \tfrac{1}{2}\, \mathrm{e}^{2\phi}\, \mathrm{d}a^{(0)}\, \frac{j^2}{2!} + \tfrac{2}{3}\, \mathrm{e}^{2\phi}\, \mathrm{d}a_0^{(1,1)} \wedge j\,, \tag{9.79}$$

to write out the Hodge stars

$$*_{10}\, \mathrm{d}a = \mathrm{d}a^{(0)}\, \frac{j^4}{4!} - \mathrm{d}a_0^{(1,1)} \wedge \frac{j^3}{3!} + \big(\mathrm{d}a^{(2,0)} + \mathrm{d}a^{(0,2)}\big) \wedge \frac{j^3}{3!}\,, \tag{9.80}$$

$$*_{10}\, (j \wedge \mathrm{d}a) = 2\, \mathrm{d}a^{(0)}\, \frac{j^3}{3!} - \mathrm{d}a_0^{(1,1)} \wedge \frac{j^2}{2!} + \big(\mathrm{d}a^{(2,0)} + \mathrm{d}a^{(0,2)}\big) \wedge \frac{j^2}{2!}\,, \tag{9.81}$$

$$*_{10}\, \tilde{h}^{(2,2)} = \tilde{h}^{(2,2)} \wedge j - \mathrm{e}^{2\phi}\, \mathrm{d}a^{(0)}\, \frac{j^3}{3!} - 2\, \mathrm{e}^{2\phi}\, \mathrm{d}a_0^{(1,1)} \wedge \frac{j^2}{2!}\,. \tag{9.82}$$

By combining all these results, we find the 10d supersymmetric Lagrangian

$$\begin{aligned}\mathscr{L}_{10}^{\mathrm{SUSY}} =& \Delta^{-3}\mathrm{e}^{8\phi}\big(R_{10} - 80(\partial_\mu\phi)^2 - 12(\partial_\mu\log\Delta)^2 + 62\,\partial_\mu\phi\,\partial^\mu\log\Delta \\ &- \nabla^2(18\,\phi - 7\,\log\Delta)\big) *_{10} 1 + \tfrac{1}{2}\,\Delta^{-3}\mathrm{e}^{8\phi}\, \mathrm{d}j \wedge *_{10}\, \mathrm{d}j\,.\end{aligned} \tag{9.83}$$

Here we also used that $w_4 = j \lrcorner \mathrm{d}j = 3\,\mathrm{d}\log\Delta - 8\,\mathrm{d}\phi$.

We reduce this Lagrangian to 9d using the ansätze

$$\begin{aligned} \mathrm{d}s^2_{10} &= \mathrm{d}r^2 + r^2\eta^2 + r^2\,\mathrm{e}^{-3B-C/3}\,\mathrm{d}s^2_8\,, \\ \mathrm{e}^{2\phi} &= r^{-3}\,\mathrm{e}^{3B+C}\,, \\ \Delta &= r^{-1}\,\mathrm{e}^{B+C}\,, \\ j &= r\,\eta\wedge\mathrm{d}r + r^2\,e^{-3B-C/3}\,J\,, \end{aligned} \tag{9.84}$$

and find (again using $\mathscr{L}_{10} = \mathscr{L}_9 \wedge r^{-2}\mathrm{d}r$)

$$\begin{aligned} \mathscr{L}_9^{\mathrm{SUSY}} = \eta\wedge\Big[&\big(R_8 - \tfrac{3}{2}(\partial_\mu(3B+\tfrac{1}{3}C))^2 - 11\,\mathrm{e}^{-3B-C/3}\big)\,*_8 1 + 2\,J\wedge *_8\,\mathrm{d}\eta \\ &+ 2\mathrm{e}^{-3B-C/3}J\wedge *_8\,J + \tfrac{1}{2}\mathrm{e}^{6B+2C/3}\mathrm{d}(\mathrm{e}^{-3B-C/3}\,J)\wedge *_8\,\mathrm{d}(\mathrm{e}^{-3B-C/3}\,J)\Big]. \end{aligned} \tag{9.85}$$

We simplify this expression using the supersymmetry conditions

$$\begin{aligned} J\lrcorner\,\mathrm{d}\eta &= \mathrm{e}^{-3B-C/3}\,, \\ W_4 = J\lrcorner\,\mathrm{d}J &= 0\,, \\ R_C &= 2\,\mathrm{e}^{-3B-C/3} - \tfrac{2}{3}\,\Box C\,, \end{aligned} \tag{9.86}$$

as well as the relation between the Ricci and the Chern–Ricci scalar

$$R_C = R_8 + \tfrac{1}{2}\,|\mathrm{d}J|^2\,. \tag{9.87}$$

This yields the remarkably simple result

$$\begin{aligned} \mathscr{L}_9^{\mathrm{SUSY}} &= \eta\wedge\mathrm{e}^{-3B-C/3}\,*_8\,1 \\ &= \eta\wedge\mathrm{d}\eta\wedge\frac{J^3}{3!}\,. \end{aligned} \tag{9.88}$$

Note that this is the same expression for the 9d supersymmetric action as was obtained in the non-rotating case in [109]. A subtle difference is that $\mathrm{d}\eta \neq \rho$ here, but rather $\rho = \mathrm{d}\eta + \frac{1}{3}\mathrm{d}\mathrm{d}^c C$. However since the forms ρ and $\mathrm{d}\eta$ are in the same cohomology class this distinction does not matter. Observe that

$$\int_{Y_9}\eta\wedge\mathrm{d}\mathrm{d}^c C\wedge\frac{J^3}{3!} = \int_{Y_9}\eta\wedge\left(\mathrm{d}\Big(\mathrm{d}^c C\wedge\frac{J^3}{3!}\Big) + \mathrm{d}^c C\wedge\frac{\mathrm{d}J^3}{3!}\right) = 0\,, \tag{9.89}$$

where the first term equality uses the fact that $J, \mathrm{d}J$ and $\mathrm{d}^c C$ are basic[11] with respect to to the R-symmetry vector ξ, and the second equality follows since the

[11] A form β is basic with respect to ξ if it satisfies both $\xi\lrcorner\beta = 0$ and $\mathcal{L}_\xi\beta = 0$.

first term is a total derivative and the second vanishes because J is balanced. We conclude that we may replace $\mathrm{d}\eta$ by ρ in expression (9.88) and therefore the integrals for computing the supersymmetric action, and therefore the entropy

$$S_{\mathrm{BH}} = \frac{1}{4G_{11}} \int_{Y_9} \mathscr{L}_9^{\mathrm{SUSY}} \,, \tag{9.90}$$

in both the rotating and non-rotating cases are exactly the same.

Later in section 9.3 we will discuss how one can obtain near-horizon geometries of rotating black strings in type IIB from the 11d setup considered so far. In anticipation of this, let us reduce the 9d action for geometries on the M-theory side to a 7d action for geometries on the type IIB side. These 7d geometries can be obtained from the 9d geometries by requiring that the 9d geometry admits a two-torus. By using the ansatz (9.94) we find the supersymmetric Lagrangian for the 7d geometry to be

$$\mathscr{L}_7^{\mathrm{SUSY}} = \eta \wedge \mathrm{d}\eta \wedge \frac{J_{(6)}^2}{2!} \,. \tag{9.91}$$

Let us point out that one can replace $\mathrm{d}\eta$ by $\rho_{(6)}$ in the 7d Lagrangian only when τ is constant. Namely, for a non-trivial axio-dilaton profile the term $\mathrm{d}Q$ appearing in (9.98) is only locally exact, and therefore cannot be interpreted as a total derivative term as was the case for the $\mathrm{d}\mathrm{d}^c C$ term. As studied in [129] it is more convenient to view these near-horizon geometries from an 11d perspective rather than a 10d one. The central charge of the dual 2d SCFT is given by

$$c = \frac{3}{(2\pi)^6 \mathrm{g}_s^2 \ell_s^8} \int_{Y_7} \eta \wedge \mathrm{d}\eta \wedge \frac{J_{(6)}^2}{2!} \,. \tag{9.92}$$

9.3 Black strings in type IIB

Having studied our 11d setup we now turn our attention to rotating black string solutions in type IIB supergravity. We take our 11d setup and require that the internal space admits a two-torus, T^2. The 8d balanced manifold then breaks up as a semidirect product of this torus and a 6d manifold. Wherever 8d quantities split up in components on the torus and the 6d manifold, we simply denote this with the subscripts (2) and (6). Under this assumption of a torus in the internal space we can apply dualities to arrive in type IIB, where we find a classification of rotating black string solutions that can be interpreted as rotating D3-branes wrapped on a Riemann surface.

If we add a warp factor acting homogeneously on the torus, the balanced condition of the 8d manifold implies that the 6d manifold is conformally balanced. For simplicity we do not take into account such a warping which gives a balanced 6d manifold. As such we take the metric ansatz

$$
\begin{aligned}
\mathrm{d}s_{11}^2 = \mathrm{e}^{2B}\Big[&- \mathrm{e}^{2C}\Big(\frac{\mathrm{d}t}{r} + \alpha\eta + A_{(6)} + A_{(2)}\Big)^2 + \frac{\mathrm{d}r^2}{r^2} + \eta^2 \\
&+ \mathrm{e}^{-3B-C/3}\Big(\mathrm{d}s_6^2 + \frac{1}{\tau_2}(\mathrm{d}x + \tau_1 \mathrm{d}y)^2 + \tau_2 \mathrm{d}y^2\Big)\Big]\,,
\end{aligned}
\tag{9.93}
$$

where τ_1 and τ_2 are scalars valued on the 6d base and the complex combination $\tau = \tau_1 + \mathrm{i}\tau_2$ is a holomorphic function ($\bar{\partial}\tau = 0$). In principle we can take the two U(1)'s of the two-torus to be fibered over AdS_2, i.e. in the language of appendix 9.A we can introduce constants k^x, k^y which are related to the angular momenta in these directions. However, introducing these parameters leads to the system becoming unreasonably complicated[12] once we arrive in type IIB, and therefore we shall just proceed with these parameters set to zero, which in (9.93) implies that $A_{(2)} = 0$. In addition, we also assume that η has no dependence on the T^2. The final piece of the solution we need to specify is the dependence of the flux on the torus: we take $\tilde{h}^{(2,2)}$ to have no legs along the torus directions.[13] Note that this is consistent with setting the rotation of the solution along the torus directions to zero, through the condition (9.18). In addition to this, we assume that the scalars B, C are independent of the torus coordinates, and are hence defined on the 6d base.

We now reduce the 8d conditions from section 9.1.3 with this assumption of a torus in the internal space onto a set of conditions on the inherited 6d base space that has an SU(3) structure. We decompose the two-form as

$$
J = J_{(6)} + J_{(2)}\,, \qquad\qquad J_{(2)} = \mathrm{d}x \wedge \mathrm{d}y\,, \tag{9.94}
$$

which (using (9.34)) implies that $J_{(6)}$ is a balanced two-form: $\mathrm{d}J_{(6)}^2 = 0$. Furthermore from (9.35) we find that

$$
\mathrm{d}\eta \wedge \frac{J_{(6)}^2}{2!} = \mathrm{e}^{-3B-C/3}\,\frac{J_{(6)}^3}{3!}\,, \tag{9.95}
$$

[12] In type IIB these extra parameters will lead to a further warping of the metric. In particular, the dilaton will not be simply the dilaton one would get from the F-theory picture, i.e. τ_2^{-1}. In addition, since we must satisfy (9.18) it is clear that turning these on will lead to turning on additional fluxes other than the self-dual five-form in type IIB. It would be interesting to fully work out the details of this more general case.

[13] The primitive piece of this part of flux (with legs on the torus) will give rise to a transgression term like in [136]. Again for our purposes such a term is an unnecessary complication, and so we set it to zero here, although it is certainly interesting to consider.

which implies $J_{(6)} \lrcorner \mathrm{d}\eta = \mathrm{e}^{-3B-C/3}$. We write the holomorphic four-form as

$$\Omega = \Omega_{(6)} \wedge \Omega_{(2)}\,, \qquad \Omega_{(2)} = \frac{1}{\sqrt{\tau_2}}\big(\mathrm{d}x + \tau \mathrm{d}y\big)\,. \tag{9.96}$$

From (9.37) it now follows that

$$\mathrm{d}\Omega_{(6)} = \mathrm{i}\big(P + \tfrac{1}{3}\mathrm{d}^c C - Q\big) \wedge \Omega_{(6)}\,, \tag{9.97}$$

where $Q = -\frac{1}{2\tau_2}\mathrm{d}\tau_1$ and we have used the holomorphicity of τ. This gives us the Ricci form on the 6d space as

$$\rho_{(6)} = \mathrm{d}\eta + \tfrac{1}{3}\mathrm{d}\mathrm{d}^c C - \mathrm{d}Q\,, \tag{9.98}$$

which is the generalization of equation (2.57) of [145] to the rotating case. The additional term changes the expression for the Chern–Ricci scalar to

$$R_{C(6)} = 2\mathrm{e}^{-3B-C/3} - \tfrac{2}{3}\Box_6 C + \frac{1}{2\tau_2^2}\,|\mathrm{d}\tau|^2\,. \tag{9.99}$$

With our ansatz the 8d Bianchi identities (9.45) for the fluxes $H^{(p,q)}$ remain the same but should be understood as 6d conditions. The expansions of these fluxes as in (9.55) require slight modifications in the numerical coefficients but are otherwise the same after the replacement $J \to J_{(6)}$. From reducing the Maxwell equations (9.48 –9.51) we find

$$\begin{aligned} -\mathrm{d} \star_6 \mathrm{d}\mathrm{e}^{-3B-C} + \mathrm{e}^{-2C/3}\mathrm{d}\eta \wedge \mathrm{d}\eta \wedge J_{(6)} &= 0\,, \\ \mathrm{d}\eta \wedge \mathrm{d}(\mathrm{e}^{-2C/3}J_{(6)}) &= 0\,, \\ \mathrm{d}\mathrm{d}^c(\mathrm{e}^{-2C/3}J_{(6)}) &= 0\,, \\ \mathrm{d} \star_6 \mathrm{d}\mathrm{e}^{-2C/3} &= H^{(2,2)} \wedge H^{(1,1)} + 2\mathrm{i}H^{(2,1)} \wedge H^{(1,2)}\,. \end{aligned} \tag{9.100}$$

We can now proceed by reducing along the A-cycle ($\mathrm{d}x+\tau_1\mathrm{d}y$) of the torus to type IIA supergravity. Note that the Ricci form is independent of the T^2-coordinates and therefore so is the one-form η. This leads to a standard reduction of 11d supergravity to massless type IIA. One finds that the metric in string frame is given by

$$\mathrm{d}s^2_{\mathrm{IIA}} = \mathrm{e}^{2B+2\phi_{\mathrm{IIA}}/3}\Big[-\mathrm{e}^{2C}\Big(\frac{\mathrm{d}t}{r} + \alpha\eta + A_{(6)}\Big)^2 + \frac{\mathrm{d}r^2}{r^2} + \eta^2 + \mathrm{e}^{-3B-C/3}\big(\mathrm{d}s_6^2 + \tau_2\mathrm{d}y^2\big)\Big]\,, \tag{9.101}$$

and is supplemented by

$$\mathrm{e}^{4\phi_{\mathrm{IIA}}/3} = \frac{1}{\tau_2}\mathrm{e}^{-B-C/3}\,, \tag{9.102}$$

$$C_1^{\mathrm{IIA}} = \tau_1 \mathrm{d}y\,, \tag{9.103}$$

$$C_3^{\mathrm{IIA}} = c_3 - \mathrm{e}^{3B+C}\Big(\frac{\mathrm{d}t}{r} + \alpha\eta + A_{(6)}\Big)\wedge\Big(\eta\wedge\frac{\mathrm{d}r}{r} + \mathrm{e}^{-3B-C/3}J_{(6)}\Big)\,, \tag{9.104}$$

$$B_2^{\mathrm{IIA}} = \mathrm{e}^{2C/3}\Big(\frac{\mathrm{d}t}{r} + \alpha\eta + A_{(6)}\Big)\wedge\mathrm{d}y\,. \tag{9.105}$$

Recall that we can decompose the 11d gauge potential as (9.72), where c_3 is the potential corresponding to $\tilde{h}^{(2,2)}$, and $c_2 = \mathrm{e}^{2\phi}j$ is fixed by supersymmetry.

By performing a T-duality along the y-direction we land in type IIB. The metric in Einstein frame reads

$$\begin{aligned}\mathrm{d}s^2_{\mathrm{IIB}} = \mathrm{e}^{3B/2-C/6}\Big[&-\mathrm{e}^{2C}\Big(\frac{\mathrm{d}t}{r}+\alpha\eta+A_{(6)}\Big)^2 + \mathrm{e}^{2C/3}\Big(\mathrm{d}y + \mathrm{e}^{2C/3}\Big(\frac{\mathrm{d}t}{r}+\alpha\eta+A_{(6)}\Big)\Big)^2\\ &+\frac{1}{r^2}\Big(\mathrm{d}r^2 + r^2\big(\eta^2 + \mathrm{e}^{-3B-C/3}\,\mathrm{d}s^2_6\big)\Big)\Big]\,.\end{aligned} \tag{9.106}$$

Here we have made explicit a cone in the geometry. It is useful to redefine the scalar B in the form $B = -\widetilde{B}/3 + C/9$ which puts the metric in the form

$$\begin{aligned}\mathrm{d}s^2_{\mathrm{IIB}} = \mathrm{e}^{-\widetilde{B}/2}\Big[&-\mathrm{e}^{2C}\Big(\frac{\mathrm{d}t}{r}+\alpha\eta+A_{(6)}\Big)^2 + \mathrm{e}^{2C/3}\Big(\mathrm{d}y + \mathrm{e}^{2C/3}\Big(\frac{\mathrm{d}t}{r}+\alpha\eta+A_{(6)}\Big)\Big)^2\\ &+\frac{\mathrm{d}r^2}{r^2} + \eta^2 + \mathrm{e}^{\widetilde{B}-2C/3}\,\mathrm{d}s^2_6\Big]\,.\end{aligned} \tag{9.107}$$

If we take $C = \alpha = A_{(6)} = 0$, the first line gives precisely the metric for AdS_3 written as a U(1) fibration over AdS_2. The effect of a non-trivial scalar C and connection pieces $\alpha, A_{(6)}$ is to make the black string rotate. Note that this is precisely the form of the near-horizon of the black string found in [143] uplifted to a 10d solution of type IIB. The fluxes consist of an axio-dilaton and five-form flux given by

$$C_0^{\mathrm{IIB}} + \mathrm{i}\mathrm{e}^{-\phi_{\mathrm{IIB}}} = \tau_1 + \mathrm{i}\tau_2\,, \tag{9.108}$$

$$\begin{aligned}F_5^{\mathrm{IIB}} = (1+\star_{10})\,\mathrm{d}\Big[&c_3\wedge\Big(\mathrm{d}y + \mathrm{e}^{2C/3}\Big(\frac{\mathrm{d}t}{r}+\alpha\eta+A_{(6)}\Big)\Big)\\ &-\mathrm{e}^{-\widetilde{B}+4C/3}\Big(\frac{\mathrm{d}t}{r}+\alpha\eta+A_{(6)}\Big)\wedge\Big(\eta\wedge\frac{\mathrm{d}r}{r} + \mathrm{e}^{\widetilde{B}-2C/3}J_{(6)}\Big)\wedge\mathrm{d}y\Big]\,.\end{aligned} \tag{9.109}$$

Having given the metric and fluxes we now specify the supersymmetry conditions that the geometry must satisfy. These can be derived from the 11d supergravity ones by reducing them on the torus. Note that the cone appearing in the metric in (9.106) has an SU(4) structure which is inherited from the SU(5) structure of our 11d solutions. We denote the corresponding two-form by $j_{(8)}$, and we can decompose it as

$$j_{(8)} = r\eta \wedge \mathrm{d}r + r^2 \mathrm{e}^{\widetilde{B}-2C/3}\, J_{(6)}\,, \tag{9.110}$$

where $J_{(6)}$ is the two-form that we found in the decomposition (9.94). This two-form corresponds to the balanced SU(3) structure of the 6d space. On this SU(3) structure, we previously found the conditions:

$$\mathrm{d}J_{(6)}^2 = 0\,, \tag{9.111}$$

$$\mathrm{d}\Omega_{(6)} = \mathrm{i}\big(P + \tfrac{1}{3}\mathrm{d}^c C - Q\big) \wedge \Omega_{(6)}\,. \tag{9.112}$$

The geometry must in addition satisfy the Bianchi identities and Maxwell equations that we discussed earlier in this section subject to the potential c_3 satisfying

$$j_{(8)} \lrcorner\, \mathrm{d}c_3 = 2r^{-3}\mathrm{e}^{-\widetilde{B}+4C/3}\mathrm{d}a^{(1,1)}\,. \tag{9.113}$$

The first of the Maxwell equations (9.100) is the master equation, which can be rewritten as

$$\begin{aligned}\mathrm{e}^{2C/3}\,\Box_6\big(\mathrm{e}^{-2C/3}\big(R_{C(6)} + \tfrac{2}{3}\Box_6 C - \tfrac{1}{2\tau_2^2}\,|\mathrm{d}\tau|^2\big)\big) - \tfrac{1}{2}\big(R_{C(6)} + \tfrac{2}{3}\Box_6 C - \tfrac{1}{2\tau_2^2}\,|\mathrm{d}\tau|^2\big)^2 \\ + 2\big|\rho_{(6)} - \tfrac{1}{3}\mathrm{d}\mathrm{d}^c C + \mathrm{d}Q\big|^2 \;=\; 0\,.\end{aligned} \tag{9.114}$$

Note that the master equation is independent of the fluxes here. Further, notice that the conditions reduce to those of [138] if one sets $C = \alpha = A_{(6)} = c_3 = 0$.

The solutions in this classification may be interpreted as the near-horizon geometries of rotating black strings. When one inserts a Riemann surface into the balanced 6d base it is natural to interpret these as arising from the compactification of rotating D3-branes on the Riemann surface. Moreover, this is not the most general setup that can be considered and it would be interesting to further investigate extensions. A possible method for doing this is to reduce the 11d setup studied here on a torus which is also fibered over the AdS_2, as we alluded to at the beginning of this section. This will necessarily lead to two free constants in the type IIB solution and also to more general fluxes. However, such solutions are far more involved than the ones presented in this section.

Appendices

9.A Black hole near-horizons and observables

In this appendix we will study the general form of the near-horizon of a black hole. This analysis serves two purposes. Firstly it will motivate the ansatz we take in section 9.1.2 for the 11d supergravity solution, in particular the warping of the metric and the temporal fibration. Despite this, in the main text we will use a more general ansatz to the one motivated here purely for convenience of the notation. It is understood that one must impose an additional constraint on the geometry in order for it to be the near-horizon of a black hole as we will show later in this section.

The second purpose for this analysis is to determine how to evaluate the physical observables for our solution. The parametrization of the metric which is most useful for obtaining the conditions arising from supersymmetry is not the one that is most useful for defining the observables such as the entropy and angular momentum of the black hole where an explicit AdS_2 factor is used. The analysis of this section will allow us to translate between the two view-points and compute observables easily from the form of the metric obtained from supersymmetry.

9.A.1 General near-horizon metric

Following [155, 156] (see also [131] and references therein) consider a spacetime containing a smooth degenerate Killing horizon, with future directed Killing field $\tilde{K}$. Let the cross section of the Killing field be H and let the unique past-directed vector field be $\hat{U}$. The vector field $\hat{U}$ is tangent to the null geodesics orthogonal to the horizon cross section and can be normalised so that $\hat{K}\cdot\hat{U}=1$. We will consider rotating black holes which imply that the solution must admit at least one rotational U(1) symmetry, i.e. it is axisymmetric. If, in addition, the spacetime has a $\mathrm{U}(1)^m$ isometry group which acts transitively on the horizon the black hole near-horizon takes the form[14]

$$\mathrm{d}s^2=\Gamma(y)\Big[-r^2\mathrm{d}t^2+\frac{\mathrm{d}r^2}{r^2}+G_{MN}(y)\mathrm{d}y^M\mathrm{d}y^N+\gamma_{\mu\nu}(y)(\mathrm{d}\phi^\mu+k^\mu r\mathrm{d}t)(\mathrm{d}\phi^\nu+k^\nu r\mathrm{d}t)\Big]\,. \tag{9.115}$$

[14]We have made some trivial redefinitions to the form of the metric appearing in [156], in particular we have changed coordinates on AdS_2 from Gaussian Null coordinates to Poincaré coordinates and extracted an overall factor from each of the sub metrics.

Here ϕ are periodic coordinates and k^μ are *constants* related to the near-horizon value of the chemical potentials of the angular momentum of the black hole. The functions of the metric all depend on the y coordinates and are independent of the ϕ's. We do not need to specify the ranges of the indices M and μ for the argument but let the range of $\mu \in \{1, \ldots, m\}$[15]. Note that the first two entries of the metric are precisely the metric on AdS_2 with unit radius. Moreover it is clear from this form that there is an $\mathrm{SO}(2,1) \times \mathrm{U}(1)^m$ isometry.[16]

The metric in this form is useful for computing the observables of the black hole however it is not as useful when trying to impose supersymmetry. Due to the gauging over AdS_2 it is finicky to try to implement SUSY preservation in this form. It is known that supersymmetry in 11d supergravity imposes that a metric admits either a timelike or null Killing vector [150, 157]. Since the form of the metric we are considering above has a timelike Killing vector we will focus on this case[17]. It is then useful to rewrite the metric so that the timelike Killing vector is manifest. This will lead to the time-direction being fibered over the remaining directions. A small rearrangement puts the metric into the form

$$\begin{aligned}\mathrm{d}s^2 =&\Gamma(y)\bigg[-(1-\gamma_{\tau\kappa}k^\tau k^\kappa)\Big(r\mathrm{d}t - \frac{k^\mu\gamma_{\mu\nu}\mathrm{d}\phi^\nu}{1-\gamma_{\sigma\rho}k^\sigma k^\rho}\Big)^2 + \frac{\mathrm{d}r^2}{r^2} + G_{mn}(y)\mathrm{d}y^m\mathrm{d}y^n \\ &+\Big(\gamma_{\mu\nu} + \frac{k^\sigma\gamma_{\sigma\mu}k^\rho\gamma_{\rho\nu}}{1-\gamma_{\kappa\tau}k^\tau k^\kappa}\Big)\mathrm{d}\phi^\mu\mathrm{d}\phi^\nu\bigg]\,. \end{aligned} \tag{9.116}$$

The metric now exhibits the timelike Killing vector in a simple form. It is then

[15] Note that n cannot be zero otherwise the black hole is not rotating and we fall into the class of solutions given in [105].

[16] The $\mathrm{SO}(2,1)$ algebra of the metric in these coordinates is realised by the three Killing vectors

$$H = \partial_t\,, \quad D = t\partial_t + r\partial_r\,, \quad K = (t^2 + r^{-2})\partial_t - 2tr\partial_r - 2r^{-1}k^i\partial_{\psi^i}\,,$$

where the ψ_i denote the $\mathrm{U}(1)$ symmetries of the internal manifold, see also [132]. Note that the generators are twisted with respect to the $\mathrm{U}(1)$ symmetries of the internal manifold which are gauged over the AdS_2. It is important that the twisting parameters, the k^i's are constant otherwise the $\mathrm{SO}(2,1)$ algebra is broken. The Killing vectors satisfy the algebra

$$[H, D] = H\,, \quad [K, D] = -K\,, \quad [H, K] = 2D$$

which is precisely the algebra of the conformal group in 1d and commutes with the isometries of the internal manifold.

[17] One could also have attacked the problem using the null Killing vector of AdS_2. The benefit of using the timelike Killing vector is that it is transferable to the case of black strings in type IIB and so we pursue this choice here.

natural to take as ansatz[18]

$$\mathrm{d}s^2 = \mathrm{e}^{2B}\Big[-\mathrm{e}^{2C}\Big(\frac{\mathrm{d}t}{r}+\hat{A}\Big)^2+\frac{\mathrm{d}r^2}{r^2}+\mathrm{d}s_9^2\Big] \tag{9.117}$$

for the near-horizon, with $\hat{A}$ an r-independent one-form on the 9d base. In this rotated form the AdS_2 factor is obscured, however as we mentioned previously this form is far more amenable to imposing supersymmetry. However this ansatz does come with some downsides. Firstly computing observables, such as the horizon area are not nearly as clear as in the form given in the ansatz (9.115). Moreover it is not clear which solutions can be identified with the near-horizon of a rotating black hole from the form in (9.117), in particular the scalar C is arbitrary in our ansatz whilst its analogue in (9.116) is constrained. We shall study this constraint shortly however in the main text we shall refrain from imposing it for as long as possible. We will see that we can proceed unabated in the classification without needing to impose such a condition.

9.A.2 Constraints from the near-horizon

In this section we shall look at the additional constraints imposed on the metric ansatz used in the main text which follow from it being the near-horizon of a black hole. We shall compare our ansatz with the general form of the near-horizon given in the previous section, rewriting the expressions in terms of quantities adapted to the metric in the form of the classification. The classification implies that the metric takes the form

$$\mathrm{d}s^2 = \mathrm{e}^{2B}\Big[-\mathrm{e}^{2C}\Big(\frac{\mathrm{d}t}{r}+\alpha\eta+A\Big)^2+\frac{\mathrm{d}r^2}{r^2}+G_{mn}(y)\mathrm{d}y^m\mathrm{d}y^n+g_{\mu\nu}\mathrm{d}\phi^\mu\mathrm{d}\phi^\nu\Big] \tag{9.118}$$

where we have written the metric with the same splitting as earlier. It is trivial to identify

$$\begin{aligned} \mathrm{e}^{2B} &= \Gamma(y)\,, & \mathrm{e}^{2C} &= 1-|k|_\gamma^2\,, \\ -\mathrm{e}^{-2C}k^\mu\gamma_{\mu\nu}\mathrm{d}\phi^\nu &= \alpha\eta + A\,, & g_{\mu\nu} &= \gamma_{\mu\nu}+\frac{k^\sigma\gamma_{\sigma\mu}k^\rho\gamma_{\rho\nu}}{1-\gamma_{\kappa\tau}k^\tau k^\kappa}\,. \end{aligned} \tag{9.119}$$

Note that we have defined $|\cdot|_\gamma$ to be the norm with respect to the metric γ, similarly we let $|\cdot|_g$ denote the norm with respect to g. Simple manipulations of these definitions gives

$$g_{\mu\nu}k^\nu = \mathrm{e}^{-2C}\gamma_{\mu\nu}k^\nu\,,\qquad |k|_g^2 = \mathrm{e}^{-2C}|k|_\gamma^2\,,\qquad \mathrm{e}^{2C} = (1+|k|_g^2)^{-1}\,. \tag{9.120}$$

[18]We change the radial coordinate as $r\to r^{-1}$ in order to write the transverse directions to the timelike foliation as a cone in the main text.

Note that this implies we can constrain the scalar C in terms of data of the fibration, in particular

$$\mathrm{e}^{-2C} = 1 + |\alpha\eta + A|_9^2\,. \tag{9.121}$$

Finally rewriting this in terms of the full metric of the classification we find the condition

$$\mathrm{e}^{-2C} = 1 + \alpha^2 + \mathrm{e}^{3B+C/3}|A|^2 \tag{9.122}$$

where the final norm is with respect to the metric on the balanced manifold. Let us further analyze the condition on the fibration in the time-direction. We have

$$\alpha\eta + A = -k^\mu g_{\mu\nu}\mathrm{d}\phi^\nu\,. \tag{9.123}$$

Therefore in order to specify α and A we should specify η, the metric $g_{\mu\nu}$ and a set of constants k^μ. These constants k^μ are related to the near-horizon values of the chemical potentials of the angular momentum of the black hole (when viewed from 11d). As a final step let us rewrite the metric used in the arguments above so that the R-symmetry vector is manifest. We want to identify

$$G_{mn}\mathrm{d}y^m\mathrm{d}y^n + g_{\mu\nu}\mathrm{d}\phi^\mu\mathrm{d}\phi^\nu \equiv (\mathrm{d}z + P)^2 + \mathrm{e}^D\mathrm{d}s_8^2\,. \tag{9.124}$$

Clearly the G_{mn} part fits in trivially after extracting out the required warp factor. The angular part can be written as

$$\begin{aligned} g_{\mu\nu}\mathrm{d}\phi^\mu\mathrm{d}\phi^\nu &= g_{zz}\mathrm{d}z^2 + 2g_{z\hat{\mu}}\mathrm{d}z\mathrm{d}\phi^{\hat{\mu}} + g_{\hat{\mu}\hat{\nu}}\mathrm{d}z^{\hat{\mu}}\mathrm{d}z^{\hat{\nu}} && (9.125)\\ &= (\mathrm{d}z + g_{z\hat{\mu}}\mathrm{d}\phi^{\hat{\mu}})^2 + (g_{\hat{\mu}\hat{\nu}} - g_{z\hat{\mu}}g_{z\hat{\nu}})\mathrm{d}\phi^{\hat{\mu}}\mathrm{d}\phi^{\hat{\nu}} && (9.126)\end{aligned}$$

where we have used that $g_{zz} = 1$ and we should identify $g_{z\hat{\mu}}\mathrm{d}\phi^{\hat{\mu}} = P$. Therefore we have

$$\mathrm{d}s^2 = (\mathrm{d}z+P)^2+\mathrm{e}^D\mathrm{d}s_8^2 = (\mathrm{d}z+P)^2+G_{mn}\mathrm{d}y^m\mathrm{d}y^n+(g_{\hat{\mu}\hat{\nu}}-P_{\hat{\mu}}P_{\hat{\nu}})\mathrm{d}\phi^{\hat{\mu}}\mathrm{d}\phi^{\hat{\nu}}\,. \tag{9.127}$$

Inserting the decomposition into the connection piece of the timelike fibration we have

$$k^\mu g_{\mu\nu}\mathrm{d}\phi^\nu = k^z\mathrm{d}z + k^{\hat{\mu}}g_{\hat{\mu}z}\mathrm{d}z + k^{\hat{\mu}}g_{\hat{\mu}\hat{\nu}}\mathrm{d}\phi^{\hat{\nu}} + k^z g_{z\hat{\mu}}\mathrm{d}\phi^\mu \tag{9.128}$$

from which we find

$$-\alpha(\mathrm{d}z+P)-A = k^\mu g_{\mu\nu}\mathrm{d}\phi^\nu = (k^z+k^{\hat{\mu}}P_{\hat{\mu}})(\mathrm{d}z+P)+(g_{\hat{\mu}\hat{\nu}}-P_{\hat{\mu}}P_{\hat{\nu}})k^{\hat{\mu}}\mathrm{d}\phi^{\hat{\nu}}\,. \tag{9.129}$$

Therefore given a vector of constants parametrising the rotation and the internal metric one can construct $\alpha\eta+A$. In fact if one imposes that the internal manifold is toric one may write the gauge field in a simple way as we have explained in section 9.1.4.

9.A.3 Observables

Let us now use the near-horizon solution to study what observables we can compute. The three main observables are the entropy of the black hole, the angular momentum and its electric/magnetic charges, all of which can be computed in the near-horizon. One may also ask if it is possible to compute the electrostatic potential and angular velocity, however these observables require some knowledge of the UV data since they are defined as

$$\mathcal{O}_{\rm BH} = \mathcal{O}_{\rm NH} - \mathcal{O}_{\infty}\,. \tag{9.130}$$

In this section we will focus on rephrasing the computation of the entropy, electric charges and angular momentum in terms of integrals over various cycles of the internal manifold.

Entropy

First consider the entropy of the black hole. The entropy is given up to normalization by the area of the horizon of the black hole. In order to compute the horizon area one should write the metric so that a bona-fide AdS_2 factor appears in the metric and the internal manifold is fibered over this. Clearly in order to compute the entropy in this way the metric of use to us is the one given in (9.115) and not the one that naturally comes out from supersymmetry. With this rewriting the horizon is manifest and the entropy is given simply the surface area of the horizon which implies

$$S_{\rm BH} = \frac{1}{4G_2}\,, \tag{9.131}$$

where the Newton's constant is that of a 2d theory admitting the AdS_2 near-horizon as a vacuum solution. In order to compute the Newton's constant (at leading order, we will not make any comments about subleading corrections though these are certainly very interesting) we should look at reducing the 11d Einstein-Hilbert term of 11d supergravity on the AdS_2 background in (9.115). We have[19]

$$\begin{aligned}\frac{1}{G_{11}}\int_{M_{11}} R\,\mathrm{dvol}_{M_{11}} &= \frac{1}{G_{11}}\int_{\mathcal{M}_2} R_2\mathrm{dvol}_2 \int_{Y_9}\Gamma(y)^{\frac{9}{2}}\sqrt{\det(G)}\sqrt{\det(\gamma)}\mathrm{d}y\wedge\mathrm{d}\phi\\ &\equiv \frac{1}{G_2}\int_{\mathcal{M}_2} R_2\mathrm{dvol}_2\,,\end{aligned} \tag{9.132}$$

[19]To save cluttering the notation we let $\mathrm{d}y\wedge\mathrm{d}\phi$ denote $\bigwedge \mathrm{d}y^m \wedge \bigwedge \mathrm{d}\phi^\mu$.

from which we identify

$$\begin{aligned}\frac{1}{G_2} &= \frac{1}{G_{11}}\int_{Y_9}\Gamma(y)^{\frac{9}{2}}\sqrt{\det(G)}\sqrt{\det(\gamma)}\mathrm{d}y\wedge\mathrm{d}\phi \\ &= \frac{1}{G_{11}}\int_{Y_9}\Gamma(y)^{\frac{9}{2}}\mathrm{dvol}_9\,.\end{aligned} \tag{9.133}$$

Let us now translate this result into the notation of the metric arising from supersymmetry, namely (9.116). We expect that the difference is precisely a warping of the volume form which indeed turns out to be the case. To this end let us compute the volume of the internal manifold. We distinguish between the two volume forms by writing $\mathrm{dvol}_{\mathrm{SUSY}}$ for the volume form in the form natural from supersymmetry. We have

$$\mathrm{dvol}_{\mathrm{SUSY}} = \Gamma(y)^{\frac{9}{2}}\sqrt{\det(G)}\sqrt{\det\left(\gamma_{\mu\nu}+\frac{k^\sigma\gamma_{\sigma\mu}k^\rho\gamma_{\rho\nu}}{1-\gamma_{\kappa\tau}k^\tau k^\kappa}\right)}\mathrm{d}y\wedge\mathrm{d}\phi\,. \tag{9.134}$$

We can expand the determinant second determinant. Using the fact that for an invertible matrix A and vectors v, w one has

$$\det(A+vw^T) = \det(A)(1+w^TA^{-1}v)\,, \tag{9.135}$$

we have

$$\begin{aligned}\det\left(\gamma_{\mu\nu}+\frac{k^\sigma\gamma_{\sigma\mu}k^\rho\gamma_{\rho\nu}}{1-\gamma_{\kappa\tau}k^\tau k^\kappa}\right) &= \det(\gamma_{\mu\nu})\left(1+\frac{1}{1-\gamma_{\kappa\tau}k^\tau k^\kappa}k^\sigma\gamma_{\sigma\mu}\gamma^{\mu\nu}k^\rho\gamma_{\rho\nu}\right) \\ &= \frac{\det(\gamma_{\mu\nu})}{1-\gamma_{\kappa\tau}k^\tau k^\kappa}\,.\end{aligned} \tag{9.136}$$

It follows that

$$\mathrm{dvol}_{\mathrm{SUSY}} = \frac{\Gamma(y)^{\frac{9}{2}}\sqrt{\det(G)}\sqrt{\det(\gamma)}}{\sqrt{1-\gamma^\kappa\gamma^\tau\gamma_{\kappa\tau}}}\mathrm{d}y\wedge\mathrm{d}\phi\,, \tag{9.137}$$

and therefore

$$\frac{1}{G_2} = \int_{Y_9}\Gamma(y)^{\frac{9}{2}}\sqrt{1-\gamma_{\kappa\tau}k^\kappa k^\tau}\mathrm{dvol}_{\mathrm{SUSY}}\,. \tag{9.138}$$

Our proposal for computing the entropy is therefore

$$\begin{aligned}S_{\mathrm{BH}} =& \frac{1}{4G_{11}}\int_{Y_9}\mathrm{e}^{-3B-C/3}\eta\wedge\frac{J^4}{4!} \\ =& \frac{1}{4G_{11}}\int_{Y_9}\eta\wedge\mathrm{d}\eta\wedge\frac{J^3}{3!}\,,\end{aligned} \tag{9.139}$$

where we used (9.41) in the final equality. As discussed in section 9.2.2 this is precisely the same formula as the entropy in the non-rotating case. One should view this section as a proof that the quantity computed in section 9.2.2 really is the entropy of the black hole.

Electric charges

Next let us consider the quantization of the four-form flux which will give rise to the electric charges of the theory. In the presence of a Chern–Simons term there is more than one definition of a charge. One can consider the gauge-invariant but non-conserved charge

$$Q = \frac{1}{(2\pi\ell_p)^6}\int_{\Sigma_7} *_{11}G_4\,, \tag{9.140}$$

where we integrate over all compact seven-cycles of the geometry. Alternatively the Page charge

$$Q = \frac{1}{(2\pi\ell_p)^6}\int_{\Sigma_7}\Big(*_{11}\,G_4 + \frac{1}{2}C_3\wedge G_4\Big)\,, \tag{9.141}$$

is conserved by application of the Maxwell equation but is not gauge invariant due to the bare potential appearing in the definition. In the following we will consider only the Page charge since it defines a conserved charge. In order to be able to write this charge we must be able to at least locally write the four-form flux in terms of a potential three-form. This is equivalent to the requirement that $\tilde{h}^{(2,2)}$ as defined in (9.14) can be written (at least locally) in terms of a potential. In fact, if we demand that it is exact, i.e. that the potential is a globally defined three-form, it follows that there is no M5-brane charge. Substituting our ansatz into the Page charge we find

$$\begin{aligned} Q = \frac{1}{(2\pi\ell_p)^6}\int_{\Sigma_7}\eta\wedge\Big[\mathrm{e}^{-2C/3}\mathrm{d}\eta\wedge\frac{J^2}{2} - \frac{1}{2}\Big(C^{(2)}\wedge H^{(2,2)} + C^{(3)}\wedge \mathrm{d}C^{(2)} \\ + \mathrm{e}^{2C/3}J\wedge\Big(C^{(3)}\wedge\mathrm{d}\alpha - C^{(2)}\wedge(\alpha\mathrm{d}\eta+\mathrm{d}A)\Big)\Big)\Big]\,, \end{aligned} \tag{9.142}$$

where we have introduced the potentials

$$\begin{aligned} H^{(1,1)} &= \mathrm{d}C^{(1)}\,,\\ \mathrm{i}(H^{(2,1)} - H^{(1,2)}) &= \mathrm{d}C^{(2)}\,,\\ H^{(2,2)} &= \mathrm{d}C^{(3)} = \mathrm{d}\eta\wedge C^{(2)}\,. \end{aligned} \tag{9.143}$$

Angular momentum

We now want to find a similar formulation for computing the angular momentum of the black hole. To such an end we may use the results of [133], (see also [158] for the analogous computation for 5d black rings), which gives the formula for computing the Komar integral for the Noether current of a Killing vector, ξ in 11d supergravity. By an abuse of notation we will also call the dual one-form ξ. The angular momentum is then given by

$$J_\xi = \frac{1}{\mathrm{S}_{\mathrm{SUSY}}} \int_{Y_9} \left[*_{11}\, \mathrm{d}\xi + (\xi \cdot C_3) \wedge *_{11} G_4 + \frac{1}{3} (\xi \cdot C_3) \wedge C_3 \wedge G_4 \right] \tag{9.144}$$

where the three-form potential C_3 should be chosen so that it has vanishing Lie derivative along the given isometry. Since this formula is dependent on the choice of Killing vector we will refrain from writing this more explicitly and just include it for completeness.

Chapter 10

Embedding of known solutions in classification

In this chapter, we study the uplift of various known asymptotically AdS_4 supersymmetric rotating black hole solutions to 11d supergravity, putting each of the solutions into the form of the classification of the previous chapter. In particular, we uplift the AdS_4 Kerr–Newman (KN) [159], the spinning spindle [93] and the Klemm [160] black hole solutions. We leave the black hole that was found in [161] for future study.

10.1 The Kerr–Newman solution

First, we study the supersymmetric limit of the AdS_4 Kerr–Newman black hole solution found in [159] and further studied in [162,163]. This black hole is a solution of Einstein–Maxwell supergravity, and can be uplifted to 11d supergravity on a 7d Sasaki–Einstein manifold.

Black hole solution

We begin by considering the black hole in four dimensions before studying the eleven-dimensional uplift. The four-dimensional black hole is given by

$$
\begin{aligned}
\mathrm{d}s^2 &= -\frac{\Delta_r}{W}\left(\mathrm{d}t - \frac{\gamma\sin^2\theta}{\Xi}\mathrm{d}\phi\right)^2 + W\left(\frac{\mathrm{d}r^2}{\Delta_r} + \frac{\mathrm{d}\theta^2}{\Delta_\theta}\right) + \frac{\Delta_\theta\sin^2\theta}{W}\left(\gamma\mathrm{d}t - \frac{\tilde{r}^2+\gamma^2}{\Xi}\mathrm{d}\phi\right)^2, \\
A &= \frac{2m\tilde{r}\sinh^2\delta}{W}\left(\mathrm{d}t - \frac{\gamma\sin^2\theta}{\Xi}\mathrm{d}\phi\right) + \alpha_{\mathrm{gauge}}\mathrm{d}t\,,
\end{aligned} \tag{10.1}
$$

where

$$
\begin{aligned}
\tilde{r} &= r + 2m\sinh^2\delta\,, \\
\Delta_r &= r^2 + \gamma^2 - 2mr + \tilde{r}^2(\tilde{r}^2 + \gamma^2)\,, \\
\Delta_\theta &= 1 - \gamma^2\cos^2\theta\,, \\
W &= \tilde{r}^2 + \gamma^2\cos^2\theta\,, \\
\Xi &= 1 - \gamma^2\,.
\end{aligned} \tag{10.2}
$$

The solution is characterised by three constants (γ, δ, m) whilst the parameter α_{gauge} is related to a pure gauge transformation and is therefore not a parameter of the solution. The solution describes a non-extremal black hole provided that $\gamma^2 < 1$ and m is bounded from below. The exact value of the bound is not important for our purposes, but it is derived in [164]. Without loss of generality we have $m, \delta, \gamma > 0$. The black hole is characterized by its energy E, electric charge Q and angular momentum J:

$$
E = \frac{m}{G_{(4)}\Xi^2}\cosh 2\delta\,, \qquad Q = \frac{m}{G_{(4)}\Xi}\sinh 2\delta\,, \qquad J = \frac{m\gamma}{G_{(4)}\Xi^2}\cosh 2\delta\,. \tag{10.3}
$$

The Bekenstein-Hawking entropy of the black hole can be found by computing the area of the outer horizon, resulting in

$$
S = \left.\frac{\pi(\tilde{r}^2 + \gamma^2)}{G_{(4)}\Xi}\right|_{r=r_+}\,, \tag{10.4}
$$

where r_+ denotes the largest positive root of $\Delta_r = 0$, and therefore describes the location of the outer horizon. For arbitrary values of the parameters (γ, δ, m), the black hole is neither extremal nor supersymmetric. The BPS limit is defined by first imposing supersymmetry and then extremality. Supersymmetry is attained by imposing

$$
\mathrm{e}^{4\delta} = 1 + \frac{2}{\gamma}\,. \tag{10.5}
$$

The solution is now supersymmetric but not extremal, in fact it has timelike closed curves and a naked singularity. To remedy this and obtain an extremal black hole we further identify

$$
m = \gamma(1+\gamma)\sqrt{2+\gamma}\,. \tag{10.6}
$$

There is now only a single parameter left in the solution, namely γ. With these identifications the function Δ_r acquires a double root at

$$
r^* = \gamma\sqrt{2+\gamma}\left(1 + \gamma - \sqrt{\gamma(2+\gamma)}\right)\,, \tag{10.7}
$$

with the other two roots becoming complex.

Near-horizon limit

We now want to take the near-horizon limit of the solution. It is convenient to perform a change of coordinates to shift the double root location in Δ_r to 0, and to rewrite the function as

$$\Delta_r = \rho^2 f(\rho)\,, \qquad f(\rho) = (\rho + r^* - r^-)(\rho + r^* - r^+)\,, \tag{10.8}$$

where

$$\rho = r - r^*\,. \tag{10.9}$$

Since we will need to evaluate the function f at the horizon often, we note that

$$f(0) = 1 + \gamma\,(6 + \gamma)\,. \tag{10.10}$$

In the metric, the change of the r to ρ coordinate results only in changes in the functions (10.2), since the dr term is invariant. To simplify notation we therefore redefine the functions such that an argument of 0 means we evaluate at the horizon. In particular we now take

$$\tilde{r}(\rho) = \rho + r^* + 2m\sinh^2\delta\ , \tag{10.11}$$

such that $\tilde{r}(0)$ is evaluated at the horizon. Similarly $W(0,\theta)$ evaluates W at the horizon; for notational convenience we denote the functions $W(0,\theta) = W(\theta)$ and $f(0) = f_0$. Furthermore, in the BPS limit one can derive that $\tilde{r}(0) = \sqrt{\gamma}$. To take the near-horizon limit we perform the changes of coordinates

$$\rho \to \epsilon\rho\,, \qquad t \to \frac{t}{\epsilon}\,, \qquad \phi \to \phi + \frac{\beta t}{\epsilon}\,, \tag{10.12}$$

where β is a constant that we will determine shortly. The near-horizon limit is now obtained by taking $\epsilon \to 0$. The dθ^2 term will clearly be sent to $W(\theta)/\Delta_\theta$, and we can ignore this term for the time being. We find the other terms as

$$\begin{aligned}
\frac{\Delta_r}{W(r,\theta)}\left(\mathrm{d}t - \frac{\gamma\sin^2\theta}{\Xi}\mathrm{d}\phi\right)^2 &\to \frac{\rho^2 f_0}{W(\theta)}\left(1 - \frac{\beta\gamma\sin^2\theta}{\Xi}\right)^2\mathrm{d}t^2\,,\\
\frac{W(r,\theta)}{\Delta_r}\mathrm{d}r^2 &\to \frac{W(\theta)}{f_0}\frac{\mathrm{d}\rho^2}{\rho^2}\,,\\
\frac{\Delta_\theta\sin^2\theta}{W(r,\theta)}\left(\gamma\mathrm{d}t - \frac{\tilde{r}^2+\gamma^2}{\Xi}\mathrm{d}\phi\right)^2 &\to \frac{\Delta_\theta\sin^2\theta}{W(\theta)}\left[\frac{\mathrm{d}t}{\epsilon}\left(\gamma - \frac{\tilde{r}(\epsilon\rho)^2+\gamma^2}{\Xi}\beta\right) - \frac{\gamma+\gamma^2}{\Xi}\mathrm{d}\phi\right]^2.
\end{aligned} \tag{10.13}$$

In the last line we can expand $\tilde{r}(\epsilon\rho) \sim \sqrt{\gamma} + \epsilon\rho\tilde{r}'(0) + \mathcal{O}(\epsilon^2)$, resulting in a term which diverges as ϵ^{-1}, proportional to the constant

$$1 - \frac{1+\gamma}{\Xi}\beta\,. \tag{10.14}$$

The existence of this term is the reason we introduced the shift in the ϕ coordinate. It can be set to zero by fixing the constant β as

$$\beta = \frac{\Xi}{1+\gamma}\,. \tag{10.15}$$

For this choice of β we combine the results from above and write down the final result for the near-horizon geometry as

$$\begin{aligned} \mathrm{d}s^2|_{\mathrm{NH}} = {} & \frac{W(\theta)}{f_0}\Big(-\rho^2\mathrm{d}t^2 + \frac{\mathrm{d}\rho^2}{\rho^2}\Big) \\ & + \frac{W(\theta)}{\Delta_\theta}\mathrm{d}\theta^2 + \frac{\sin^2\theta\,\Delta_\theta}{W(\theta)}\Big(\frac{\gamma+\gamma^2}{\Xi}\Big)^2\Big(\mathrm{d}\phi + \frac{2\sqrt{\gamma}\,\Xi}{(1+\gamma)f_0}\,\rho\,\mathrm{d}t\Big)^2\,, \end{aligned} \tag{10.16}$$

where, in order to make the AdS_2 factor manifest, we rescaled the time-coordinate:

$$t \to \frac{\gamma(1+\gamma)}{f_0}t\,. \tag{10.17}$$

Consider now the gauge field. By performing the same near-horizon and BPS limit, we find a divergent term in the gauge field, proportional to

$$\frac{dt}{\epsilon}(\alpha_{\mathrm{gauge}} - 2)\,. \tag{10.18}$$

This term is purely gauge and we can remove it by making a suitable choice for the gauge parameter. The resulting near-horizon vector field is

$$A|_{\mathrm{NH}} = \frac{2\sqrt{\gamma}\,(1+\gamma)}{W(\theta)}\Big(\frac{2\gamma - W(\theta)}{f_0}\rho\,\mathrm{d}t + \frac{\gamma\sqrt{\gamma}\sin^2\theta}{\Xi}\mathrm{d}\phi\Big)\,, \tag{10.19}$$

where the time coordinate has been rescaled with the same factor as in the metric.

Uplift to 11d

Now that we have derived the near-horizon metric and gauge field of the AdS_4 KN solution in minimal supergravity we can consider the uplift to 11d supergravity. The uplift of the metric and flux to eleven dimensions are given by

$$\begin{aligned} \mathrm{d}s^2_{11} &= \mathrm{d}s^2_4 + \big(\eta + \tfrac{1}{4}A\big)^2 + \mathrm{d}s^2_6\,, \\ \mathcal{G}_4 &= \frac{3}{8}\,\mathrm{dvol}(\mathrm{AdS}_4) - \frac{1}{4}\star_4 F\wedge J\,, \end{aligned} \tag{10.20}$$

where $F = dA$ is the field strength of A, $\mathrm{d}s_4^2$ is the near-horizon metric we just derived (10.16) and $\mathrm{d}s_6^2$ is the base of the Sasaki-Einstein manifold with $\eta = \mathrm{d}z + \sigma$ dual to the Reeb-vector ∂_z. Furthermore, we have $\mathrm{d}\eta = 2J_6$ where J_6 is the Kähler form on $\mathrm{d}s_6^2$.

We now want to rewrite the metric and flux in (10.20) into the form of our classification, as presented in section 9.1.3. To recover this form, we write the metric in (10.20) such that it becomes a time-fibration over a base. It is also necessary to perform some coordinate redefinitions

$$z \to \frac{\gamma\,(3+\gamma)}{2\,f_0}\,z\,, \qquad \phi \to \phi - \frac{\Xi}{f_0}\,z\,. \tag{10.21}$$

After completing the straightforward but tedious rotations of the vielbeins, the metric we find is of the form

$$\mathrm{d}s_{11}^2 = \mathrm{e}^{2B}\Big[-\frac{\mathrm{e}^{2C}}{r^2}(\mathrm{d}t + a)^2 + \frac{\mathrm{d}r^2}{r^2} + \eta^2 + Y(\theta)\,\mathrm{D}\phi^2 + \frac{f_0}{\Delta_\theta}\,\mathrm{d}\theta^2 + \mathrm{e}^{-2B}\mathrm{d}s_6^2\Big]\,. \tag{10.22}$$

We will now clarify the several notational conventions used in this metric. Firstly, we have renamed the coordinate ρ to r, in order to conform with the conventions of the classification. We have also introduced Y, $\mathrm{D}\phi$ and redefined η as follows

$$\begin{aligned} Y(\theta) &= \frac{\gamma^2\,(3+\gamma)^2\,f_0\,(1-\gamma^2\cos^2\theta)\,\sin^2\theta}{\Xi^2\cos^2\theta\,(2\gamma + W(\theta))^2}\,, \\ \mathrm{D}\phi &= \mathrm{d}\phi + \frac{2\,\Xi}{\gamma\,(3+\gamma)}\,\sigma\,, \\ \eta &= \mathrm{d}z + \frac{2\,f_0}{\gamma\,(3+\gamma)}\,\sigma + \frac{f_0\,(\gamma\sin\theta)^2}{\Xi\;(2\gamma + W(\theta))}\,\mathrm{D}\phi\,. \end{aligned} \tag{10.23}$$

Note that the coordinate redefinitions we made in (10.21) were necessary to ensure that the metric ends up with $\mathrm{d}z^2$, with its coefficient being exactly equal to one. The scalars e^C and e^B are found to be

$$\mathrm{e}^C = \frac{\gamma(1+\gamma)\,\cos\theta\;(W(\theta) + 2\gamma)}{W(\theta)\sqrt{f_0\,W(\theta)}}\,, \qquad \mathrm{e}^B = \sqrt{\frac{W(\theta)}{4\,f_0}}\,. \tag{10.24}$$

Recall that these scalars can also be used to compute Δ via (9.20). The last unspecified piece of the metric is the fibration a, which is given by

$$a = \sqrt{\gamma}\,r\left(\frac{2\,\gamma^2\cos^2\theta - \gamma^2 - 1}{(1+\gamma)\;(W(\theta) + 2\gamma)}\,\eta + \frac{\gamma\,(3+\gamma)\,\tan^2\theta\,(\gamma^2\,\cos^2\theta - 1)f_0}{(1+\gamma)\,\Xi\,(W(\theta) + 2\gamma)^2}\,\mathrm{D}\phi\right)\,. \tag{10.25}$$

The fibration is of the expected form $a = r(\alpha\,\eta + A)$, and this specification of a completes the endeavour of writing the metric in the classification form. Now we can move on to consider the flux; recall that in the classification we wrote it as

$$\begin{aligned}\mathcal{G}_4 &= -\mathrm{d}\left((\mathrm{d}t + a) \wedge \mathrm{e}^{2\phi} j\right) + \tilde{h}^{(2,2)}\,,\\ \tilde{h}^{(2,2)} &= \mathrm{d}c_3\,.\end{aligned} \tag{10.26}$$

We have already found the fibration a in (10.25) and $\mathrm{e}^{2\phi}$ is given in terms of the scalars e^B and e^C via (9.20). The ten-dimensional complex structure form j can be found from the vielbeins of the metric we found in (10.22). Our remaining tasks thus consists of finding an expression for $\tilde{h}^{(2,2)}$, which in its turn is determined by the potential c_3. After carefully rewriting the flux we obtain from (10.20), we find an appropriate expression for the potential given by

$$\begin{aligned}c_3 = \frac{\gamma^2\sqrt{\gamma}\,(3+\gamma)}{8\,\Xi\,\left(2\gamma + W(\theta)\right)}&\Bigg(\sin\theta\,\mathrm{d}\theta \wedge \eta \wedge \mathrm{D}\phi + \frac{(3+\gamma)(2\gamma - W(\theta))}{\gamma\,\Xi\,\cos\theta}\,\mathrm{D}\phi \wedge \mathrm{d}(\mathrm{D}\phi)\\ &+ \frac{\sin^2\theta\,(2\gamma - W(\theta))}{r\,f_0\,\gamma\cos\theta}\,\mathrm{d}r \wedge \mathrm{D}\phi \wedge \eta\Bigg)\,.\end{aligned} \tag{10.27}$$

We have checked that the uplifted solution satisfies all the conditions of our classification.

In addition to this time-fibration form, we write the metric in the form of (9.115) which gives

$$\begin{aligned}\mathrm{d}s^2 = \mathrm{e}^{2B}&\left(-r^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2}\right) + \gamma_{\theta\theta}\,\mathrm{d}\theta^2 + \mathrm{d}s_6^2\\ &+ \gamma_{\mu\nu}\left(\mathrm{d}\psi^\mu + M^\mu(\theta)\,\sigma + k^\mu r\mathrm{d}t\right)\left(\mathrm{d}\psi^\nu + M^\nu(\theta)\,\sigma + k^\nu r\mathrm{d}t\right)\,.\end{aligned} \tag{10.28}$$

In this form the AdS_2 is clearly visible. The metric on the base of the Sasaki–Einstein manifold $\mathrm{d}s_6$ and the one-form σ are defined as before. Apart from these already familiar notions we established several new notational conventions; first of all we have introduced

$$\gamma_{\theta\theta} = \frac{W(\theta)}{4\,\Delta_\theta}\,, \qquad M^z = \frac{2\,f_0}{\gamma\,(3+\gamma)}\,, \qquad M^\phi = \frac{2\,\Xi}{\gamma\,(3+\gamma)}\,. \tag{10.29}$$

We introduced indices $\mu, \nu \in \{z, \phi\}$ along with a metric $\gamma_{\mu\nu}$ that we will specify below and, finally, we defined $\mathrm{d}\psi$ as

$$\mathrm{d}\psi^\mu = (\mathrm{d}z,\, \mathrm{d}\phi)\,. \tag{10.30}$$

The metric (10.28) shows that only the ϕ and z coordinates are gauged over the AdS_2 space. We could have expected this, since the original AdS_4 black hole had rotation only in the ϕ direction, and in (10.20) we have gauged the Reeb-vector with the four-dimensional gauge field. The metric, $\gamma_{\mu\nu}$, that we introduced for these two coordinates, has the following components

$$\gamma_{zz} = \kappa^2 - \frac{\gamma^2\,(1+\gamma)\sin^2\theta}{f_0\,W(\theta)}\left(\kappa - \frac{\gamma\,(1+\gamma)\,N(\theta)}{4\,f_0\,W(\theta)}\right), \tag{10.31}$$

$$\gamma_{z\phi} = \frac{\gamma^2\,(1+\gamma)\,\sin^2\theta}{2\,\Xi\,W(\theta)}\left(\kappa - \frac{\gamma\,(1+\gamma)\,N(\theta)}{2\,f_0\,W(\theta)}\right), \tag{10.32}$$

$$\gamma_{\phi\phi} = \frac{\gamma^3\,(1+\gamma)^2\,\sin^2\theta\,N(\theta)}{4\,\Xi^2\,W(\theta)^2}, \tag{10.33}$$

where, to alleviate notational clutter, we have introduced

$$\kappa = \frac{\gamma\,(3+\gamma)}{2\,f_0}, \qquad N(\theta) = 1 + \gamma - \gamma^2\cos^2\theta - \gamma^3\,\cos^4\theta\,. \tag{10.34}$$

Now that we have specified the $\gamma_{\mu\nu}$ in (10.28), the description of the metric is almost complete. The last remaining unknowns are the constants k^i which specify the gauging over the AdS_2. We find

$$k^z = k^\phi = \frac{1-\gamma}{\sqrt{\gamma}\,(3+\gamma)}\,. \tag{10.35}$$

Note that the precise value of k^ϕ depends on how we scale the ϕ coordinate; the fact that both k^i are equal arises due to our conventions for the coordinates. Since the above k^i are the only non-zero ones, the black hole rotates only in the z and ϕ directions.

We can now check the identities (9.62) and (9.63). In order to do this we need to compute the scalars w_i and x_i appearing in (9.59) and (9.60) for our solution. We find

$$\begin{aligned} x_\phi &= -\frac{\gamma^{5/3}(3+\gamma)(1+\gamma)^{1/3}(3+\gamma\cos^2\theta)^{1/3}}{8\Xi f_0^{2/3}}, & x_z &= 0\,, \\ w_\phi &= \frac{\gamma f_0\sin^2\theta}{2\Xi(3+\gamma\cos^2\theta)}, & w_z &= \frac{1}{2}\,. \end{aligned} \tag{10.36}$$

It is now a simple matter to see that both (9.62) and (9.63) are satisfied.

10.2 The spinning spindle solution

Let us consider the spinning spindle solution [93] in 4d Einstein–Maxwell supergravity. We directly consider the near-horizon limit of the full black hole solution, which is

$$
\begin{aligned}
\mathrm{d}s^2_{\mathrm{NH}} &= \frac{(j^2+y^2)^2-q(y)}{4(j^2+y^2)}r^2Dt^2 + \frac{y^2+j^2}{4}\frac{\mathrm{d}r^2}{r^2} + \frac{y^2+j^2}{q(y)}\mathrm{d}y^2 \\
&\quad + \frac{(j^2+y^2)q(y)}{4((j^2+y^2)^2-j^2q(y))}\mathrm{d}\tilde{\phi}^2\,, \\
A &= 2h(y)\Big(jrDt + \frac{(j^2+y^2)^2}{(j^2+y^2)^2-j^2q(y)}\mathrm{d}\tilde{\phi}\Big)\,, \\
rDt &= r\mathrm{d}t - \frac{jq(y)}{(j^2+y^2)^2-j^2q(y)}\mathrm{d}\tilde{\phi}\,.
\end{aligned}
\tag{10.37}
$$

The two functions q and h appearing in the solution are given by

$$
\begin{aligned}
q(y) &= (y^2+j^2)^2 - (a-2y\sqrt{1-j^2})^2\,, \\
h(y) &= \frac{\sqrt{1-j^2}}{2} - \frac{1}{2(y^2+j^2)}(ay+2j^2\sqrt{1-j^2})\,.
\end{aligned}
\tag{10.38}
$$

Since this is a solution of Einstein–Maxwell supergravity, it can be uplifted to 11d supergravity on a generic Sasaki–Einstein manifold using the uplift formulas as given in [165]:

$$
\begin{aligned}
\mathrm{d}s^2 &= \frac{1}{4}\mathrm{d}s^2_4 + \Big(\mathrm{d}\psi + \sigma + \frac{1}{4}A\Big)^2 + \mathrm{d}s^2_{\mathrm{KE}_6}\,, \\
G_4 &= \frac{3}{8}\mathrm{dvol}_4 - \frac{1}{4}(\star_4\mathrm{d}A)\wedge J_{\mathrm{KE}_6}\,, \\
\mathrm{d}\sigma &= 2J_{\mathrm{KE}_6}\,.
\end{aligned}
\tag{10.39}
$$

The curvature of the Kähler–Einstein space is normalised in the usual manner to satisfy $R_{mn} = 6g_{mn}$.

Uplifting this solution to 11d supergravity and putting it into the form of the classification yields the result

$$\begin{aligned}
\mathrm{d}s_{11}^2 =& \mathrm{e}^{2B}\Bigg[-\mathrm{e}^{2C}\left(r\mathrm{d}t + k_z\frac{\sqrt{1-j^2}y(j^2+y^2)-(1-j^2)H(y)}{j^2H(y)}\eta + k_\phi\frac{(1-j^2)q(y)}{H(y)^2}D\phi\right)^2 \\
&+\frac{\mathrm{d}r^2}{r^2} + \eta^2 + \mathrm{e}^{-3B-C/3}\left(\frac{(j^2+y^2)H(y)^{1/3}}{16q(y)}\mathrm{d}y^2\right. \\
&\left.+\frac{(1-j^2)(j^2+y^2)q(y)}{64H(y)^{5/3}}D\phi^2 + \frac{H(y)^{1/3}}{4}\mathrm{d}s^2_{\mathrm{KE}_6}\right)\Bigg]\,,
\end{aligned} \tag{10.40}$$

where we have defined

$$\begin{aligned}
D\phi &= \mathrm{d}\phi - 4\sigma\,, \\
\eta &= \mathrm{d}z + \frac{y(j^2+y^2)\sqrt{1-j^2} - H(y)}{j^2H(y)}D\phi\,, \\
H(y) &= \sqrt{1-j^2}y(3j^2+y^2) - aj^2\,, \\
\mathrm{e}^{2B} &= \frac{j^2+y^2}{16}\,, \qquad \mathrm{e}^{C} = \frac{H(y)}{(j^2+y^2)^{3/2}}\,,
\end{aligned} \tag{10.41}$$

and we have made the coordinate redefinitions

$$\tilde{\phi} = \sqrt{1-j^2}z\,, \qquad \psi = \frac{1}{4}(z-\phi)\,. \tag{10.42}$$

Note that according to these coordinate transformations the vector dual to the R-symmetry is the U(1) of the spindle and not the U(1) Reeb vector of the SE_7 on which we uplift. This is a difference between the Kerr–Newman solution studied in the previous section, and it is an expected feature of spindles solutions. The constants k_i, see (9.56), are found to be

$$k_z = k_\phi = -\frac{j}{\sqrt{1-j^2}}\,. \tag{10.43}$$

Let us now examine the 8d base in this solution. The SU(4) structure two-form is found as

$$J_8 = \frac{H(y)^{1/3}}{4}J_{\mathrm{KE}_6} + \frac{\sqrt{1-j^2}(j^2+y^2)}{32H(y)^{2/3}}D\phi\wedge\mathrm{d}y\,, \tag{10.44}$$

which is in fact closed meaning that the 8d base is Kähler. We found the same result for the Kerr–Newman solution studied previously. The $(4,0)$-form can be written as

$$\Omega_8 = e^{-i\phi}\frac{H(y)^{1/2}}{2^3}\Omega_{\mathrm{KE}_6}\wedge\left(\frac{\sqrt{j^2+y^2}H(y)^{1/6}}{4\sqrt{q(y)}}dy - i\frac{\sqrt{1-j^2}\sqrt{j^2+y^2}\sqrt{q(y)}}{8H(y)^{5/6}}D\phi\right). \tag{10.45}$$

Since the non-primitive pieces of the fluxes are fixed completely by supersymmetry, we will only present the primitive pieces here. We find

$$H_0^{(1,1)} = \frac{1}{4}\frac{jyh(y)(\sqrt{1-j^2}+2(1-j^2)h(y)}{8j^2\sqrt{1-j^2}h(y)+4(1-j^2)(j^2+y^2)}\left(J_{\mathrm{KE}_6} - \frac{3\sqrt{1-j^2}(j^2+y^2)}{8H(y)}D\phi\wedge \mathrm{d}y\right), \tag{10.46}$$

while $H_0^{(2,1)} = H_0^{(2,2)} = 0$.

This completes the embedding of the uplift of the rotating spindle solution in our classification. We have found that the 8d base is Kähler, and that only the $(1,1)$-flux is has a primitive piece.

10.3 The Klemm solution

Let us now consider the uplift of the black hole found in [160] as a solution in the X^0X^1 truncation of the $\mathrm{U}(1)^4$ gauged STU model. The extremal black hole metric reads

$$\begin{aligned}4g_0g_1l^2\,\mathrm{d}s_4^2 = &-\frac{\Delta_q(q)}{\Xi^2(q^2+j^2\cosh^2\theta-\delta^2)}\left(\mathrm{d}T + j\sinh^2\theta\mathrm{d}\tilde{\phi}\right)^2\\ &+\frac{q^2+j^2\cosh^2\theta-\delta^2}{\Delta_q(q)}\mathrm{d}q^2 + \frac{q^2+j^2\cosh^2\theta-\delta^2}{\Delta_\theta(\theta)}\mathrm{d}\theta^2\\ &+\frac{\Delta_\theta(\theta)\sinh^2\theta(q^2+j^2-\delta^2)}{\Xi^2(q^2+j^2\cosh^2\theta-\delta^2)}\left(\mathrm{d}\tilde{\phi} - \frac{j}{q^2+j^2-\delta^2}\mathrm{d}T\right)^2,\end{aligned} \tag{10.47}$$

where the functions are given by

$$\Delta_q(q) = \frac{1}{l^2}(q+r_h)^2(q-r_h)^2\,,\qquad \Delta_\theta(\theta) = 1+\frac{j^2}{l^2}\cosh^2\theta\,,\qquad \Xi = 1+\frac{j^2}{l^2}\,. \tag{10.48}$$

The solution is supported by two gauge fields and a complex scalar

$$A_I = \frac{\cosh\theta}{4g_I(q^2+j^2\cosh^2\theta-\delta^2)}\left(j\mathrm{d}T - (q^2+j^2-\delta^2)\mathrm{d}\tilde{\phi}\right), \tag{10.49}$$

$$\tau = \frac{j^2\cosh^2\theta + q^2 - \delta^2 + 2\mathrm{i}j\delta\cos\theta}{j^2\cosh^2\theta + (q-\delta)^2} \equiv e^{-\phi(\theta)} - \mathrm{i}\chi(\theta)\,. \tag{10.50}$$

In order to take the near-horizon limit we make the redefinitions

$$T = t\frac{q_0}{\epsilon}\,, \qquad q = r_h + \epsilon q_0 r\,, \qquad \tilde{\phi} = \phi + t\tilde{\Omega}\frac{q_0}{\epsilon}\,, \tag{10.51}$$

where we have defined

$$\tilde{\Omega} = \frac{j}{r_h^2 + j^2 - \delta^2}\,, \qquad q_0 = \frac{l^2\Xi}{2\sqrt{2}r_h}\,. \tag{10.52}$$

One now finds the near-horizon limit by taking taking $\epsilon \to 0$ and setting the horizon location r_h to

$$r_h = \frac{\sqrt{l^2 - j^2 + 2\delta^2}}{\sqrt{2}}\,. \tag{10.53}$$

At this point it is useful to introduce some new notation:

$$r_h = \frac{1}{2}(R_+ + R_-)\,, \qquad \delta = \frac{1}{2}(R_+ - R_-)\,, \tag{10.54}$$

with

$$R_+ = \frac{l^2 - j^2}{2R_-}\,, \tag{10.55}$$

in order to satisfy (10.53). We may then define

$$S_{\pm\pm}(\theta) = R_\pm R_\pm + j^2\cosh^2\theta\,, \tag{10.56}$$

with the two $\pm$ signs being independent. Note that $S_{\pm\pm}$ is symmetric in its two indices. The resultant near-horizon solution takes the form

$$\begin{aligned}
\mathrm{d}s^2_{\mathrm{NH}} = {} & \frac{S_{+-}(\theta)}{4(R_+ + R_-)^2 g_0 g_1}\Big(- r^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + \frac{(R_+ + R_-)^2}{l^2\Delta_\theta(\theta)}\mathrm{d}\theta^2\Big) \\
& + \frac{l^2\sinh^2\theta\Delta_\theta(\theta)}{16 g_0 g_1 S_{+-}(\theta)}\Big(\mathrm{d}\phi + \frac{2j}{R_+ + R_-}r\mathrm{d}t\Big)^2\,, && (10.57) \\
A_I = {} & -\frac{(l^2 + j^2)\cosh\theta}{8 g_I S_{+-}(\theta)}\Big(\mathrm{d}\phi + \frac{2j}{R_+ + R_-}r\mathrm{d}t\Big)\,, && (10.58) \\
\tau = {} & \frac{R_+ - \mathrm{i}j\cosh\theta}{R_- - \mathrm{i}j\cosh\theta}\,. && (10.59)
\end{aligned}$$

We may now proceed to uplift this solution to 11d supergravity on an S^7 using the uplift formulas presented in [106]. The uplifted metric takes the form

$$\begin{aligned}
\mathrm{d}s^2_{11} = \Lambda^{1/3}\Big[& \mathrm{d}s_4^2 + \frac{1}{g_1 g_2}\Big(\mathrm{d}\xi^2 + \frac{\cos^2\xi}{4Z_1}\Big(\mathrm{d}\theta_1^2 + \sin^2\theta_1\mathrm{d}\phi_1^2 + D\psi_1^2\Big) \\
& + \frac{\sin^2\xi}{4Z_2}\Big(\mathrm{d}\theta_2^2 + \sin^2\theta_2\mathrm{d}\phi_2^2 + D\psi_2^2\Big)\Big)\Big]\,, && (10.60)
\end{aligned}$$

where

$$\begin{aligned}
D\psi_I &= \mathrm{d}\psi_I + \cos\theta_I \mathrm{d}\phi_I - 2g_I A_I\,, \\
\Lambda &= Z_1 Z_3\,, \\
Z_1 &= \mathrm{e}^{\phi(\theta)}\cos^2\xi + \sin^2\xi\,, \\
Z_2 &= \cos^2\xi + \sin^2\xi\big(\mathrm{e}^{-\phi} + \chi^2\mathrm{e}^{\phi}\big)\,.
\end{aligned} \tag{10.61}$$

We can now put the metric into the form of the classification via a lengthy and time-consuming but otherwise straightforward computation. We find the two scalars to be

$$\mathrm{e}^{2B} = \frac{S_{+-}(\theta)(Z_1Z_2)^{1/3}}{4g_0g_1(R_+ + R_-)^2}\,, \qquad \mathrm{e}^{2C} = \frac{(R_+\sin^2\xi + R_-\cos^2\xi)^2}{4S_{+-}(\theta)^3 Z_1 Z_2}\,. \tag{10.62}$$

To identify the R-symmetry direction we must perform the change of coordinates

$$\psi_1 \to \psi + \frac{R_-}{R_+ + R_-}z\,, \quad \psi_2 \to \psi + \frac{R_+}{R_+ + R_-}z\,, \quad \phi \to (j^2 + l^2)\chi\,, \quad t \to \frac{-1}{j^2+l^2}t\,, \tag{10.63}$$

after which the the R-symmetry direction is ∂_z. Note that this coordinate comes only from S^7 angles, and does not mix with the directions of the 4d black hole, in contrast to the Kerr–Newman and spindle solutions. The metric in the form of the classification reads

$$\begin{aligned}
\mathrm{d}s^2_{11} =& \mathrm{e}^{2B}\bigg[-\mathrm{e}^{2C}(r\mathrm{d}t - \mathcal{A})^2 + \frac{\mathrm{d}r^2}{r^2} + Dz^2 \\
&+ \mathrm{e}^{-3B-C/3}\frac{(R_+\sin^2\xi + R_-\cos^2\xi)^{1/3}(Z_1Z_2)^{1/3}}{2^{1/3}8(g_0g_1)^{3/2}(R_+ + R_-)}\bigg(4\mathrm{d}\xi^2 + \frac{S_{+-}(\theta)}{l^2\Delta_\theta(\theta)}\mathrm{d}\theta^2 \\
&+ \frac{\cos^2\xi\sin^2\xi(R_+ - R_-)^2}{(R_+\sin^2\xi + R_-\cos^2\xi)^2}D\psi^2 + l^2\sinh^2\theta\Delta_\theta(\theta)S_{+-}(\theta)\mathrm{d}\chi^2 \\
&+ \frac{\cos^2\xi S_{+-}(\theta)}{j^2\cosh^2\theta + R_-(R_+\sin^2\xi + R_-\cos^2\xi)}\mathrm{dvol}(S^2_1) \\
&+ \frac{\sin^2\xi S_{+-}(\theta)}{j^2\cosh^2\theta + R_+(R_+\sin^2\xi + R_-\cos^2\xi)}\mathrm{dvol}(S^2_2)\bigg)\bigg]\,,
\end{aligned} \tag{10.64}$$

where

$$
\begin{aligned}
D\psi =& \mathrm{d}\psi + \frac{R_+}{R_+ - R_-}\cos\theta_1 \mathrm{d}\phi_1 - \frac{R_-}{R_+ - R_-}\cos\theta_2 \mathrm{d}\phi_2 - \cosh\theta\ S_{+-}(\theta)\mathrm{d}\chi\,,\\
Dz =& \mathrm{d}z + \frac{R_+ + R_-}{R_+\sin^2\xi + R_-\cos^2\xi}\bigg[D\psi - \Big(\sin^2\xi + \frac{R_-}{R_+ - R_-}\Big)\cos\theta_1\mathrm{d}\phi_1\\
&- \Big(\cos^2\xi - \frac{R_+}{R_+ - R_-}\Big)\cos\theta_2\mathrm{d}\phi_2\bigg]\,,\\
\mathcal{A} =& 2j\bigg[\frac{\cosh\theta S_{+-}(\theta)}{R_+\sin^2\xi + R_-\cos^2\xi}Dz + \frac{(R_+ - R_-)^2(R_+ + R_-)\cos^2\xi\sin^2\xi\cosh\theta}{(R_+\sin^2\xi + R_-\cos^2\xi)^2}D\psi\\
&- 2jl^2(R_+ + R_-)\sinh^2\theta\Delta_\theta(\theta)\mathrm{d}\chi\bigg]\,.
\end{aligned}
\tag{10.65}
$$

The two-form of the SU(4) structure base is

$$
\begin{aligned}
J_8 =& \frac{\Lambda^{1/3}(R_+\sin^2\xi + R_-\cos^2\xi)^{1/3}}{2^{1/3}8(g_0g_1)^{3/2}(R_+ + R_-)}\bigg[- 2\frac{\cos\xi\sin\xi(R_+ - R_-)}{R_+\sin^2\xi + R_-\cos^2\xi}\mathrm{d}\xi\wedge D\psi\\
&+ S_{+-}(\theta)\sinh\theta\mathrm{d}\theta\wedge\mathrm{d}\chi - \frac{S_{+-}(\theta)\cos^2\xi}{j^2\cosh^2\theta + R_-(R_+\sin^2\xi + R_-\cos^2\xi)}\sin\theta_1\mathrm{d}\theta_1\wedge\mathrm{d}\phi_1\\
&- \frac{S_{+-}(\theta)\sin^2\xi}{j^2\cosh^2\theta + R_+(R_+\sin^2\xi + R_-\cos^2\xi)}\sin\theta_2\mathrm{d}\theta_2\wedge\mathrm{d}\phi_2\bigg]\,.
\end{aligned}
\tag{10.66}
$$

It can be checked that this two-form is balanced, as it should be, but in fact *not* Kähler. This is in contrast with the spindle and Kerr–Newman solutions where the two-form was closed and the base was Kähler.

We can now check the constraints from the $SO(2,1)$ isometry of the near-horizon. The scalar α is found to be

$$
\alpha = -\frac{2j\cosh\theta S_{+-}(\theta)}{R_+\sin^2\xi + R_-\cos^2\xi}\,,
\tag{10.67}
$$

and from (9.129) we have

$$
k^z = k^\psi = k^{\phi_1} = k^{\phi_2} = 0\,, \qquad k^\chi = -\frac{2j}{R_+ + R_-}\,.
\tag{10.68}
$$

We may then also identify the one-form A in the notation of chapter 9 to be

$$
\begin{aligned}
A =& -2j\bigg[\frac{(R_+ - R_-)^2(R_+ + R_-)\cos^2\xi\sin^2\xi\cosh\theta}{(R_+\sin^2\xi + R_-\cos^2\xi)^2}D\psi\\
&- 2jl^2(R_+ + R_-)\sinh^2\theta\Delta_\theta(\theta)\mathrm{d}\chi\bigg]\,.
\end{aligned}
\tag{10.69}
$$

It can then be checked that the scalar C takes the correct form (9.56).

Finally, we consider the fluxes. These may be written, in the notation of the classification, as

$$
\begin{aligned}
H^{(1,1)}=&\mathrm{d}\Bigg[\frac{j(R_+-R_-)^2\cos^2\xi\sin^2\xi\cosh\theta}{8g_0^{3/2}g_1^{3/2}(R_++R_-)^2(R_+\sin^2\xi+R_-\cos^2\xi)}D\psi\\
&-\frac{jl^2(R_+\sin^2\xi+R_-\cos^2\xi)\sinh^2\theta\Delta_\theta(\theta)}{8g_0^{3/2}g_1^{3/2}(R_++R_-)^2}\mathrm{d}\chi\Bigg]\,, \qquad (10.70)\\
H^{(3)}=&\frac{j}{8g_0^{3/2}g_1^{3/2}(R_++R_-)}\Bigg[\Big(\cos^2\xi\mathrm{dvol}(S_1^2)+\sin^2\xi\mathrm{dvol}(S_2^2)\Big)\\
&\wedge\left(\frac{(R_+-R_-)^2\cosh\theta\cos^2\xi\sin^2\xi}{(R_+\sin^2\xi+R_-\cos^2\xi)^2}D\psi-l^2\sinh^2\theta\Delta_\theta(\theta)\mathrm{d}\chi\right)\\
&-D\psi\wedge\mathrm{d}\chi\wedge\Bigg(\frac{(R_+-R_-)^2\sinh\theta\cosh\theta\cos^2\xi\sin^2\xi(3S_{+-}(\theta)-2R_+R_-)}{(R_+\sin^2\xi+R_-\cos^2\xi)^2}\mathrm{d}\theta\\
&+\frac{2l^2(R_+-R_-)\sinh^2\theta\Delta_\theta(\theta)\cos\xi\sin\xi}{R_+\sin^2\xi+R_-\cos^2\xi}\mathrm{d}\xi\Bigg)\Bigg]\,, \qquad (10.71)\\
H^{(2,2)}=&\frac{j\cosh\theta}{8g_0^{3/2}g_1^{3/2}(T_+\sin^2\xi+R_-\cos^2\xi)}\Bigg[-2\cos^2\xi\sin^2\xi\mathrm{dvol}(S_1^2)\wedge\mathrm{dvol}(S_2^2)\\
&+\frac{(R_+-R_-)\sin2\xi\sinh2\theta\big(2S_{+-}-R_+R_-+(R_+\sin^2\xi+R_-\cos^2\xi)^2\big)}{2(R_+\sin^2\xi+R_-\cos^2\xi)}\mathrm{d}\theta\wedge\mathrm{d}\chi\wedge D\psi\wedge\mathrm{d}\xi\\
&-\sinh\theta\big(R_+\sin^2\xi+R_-\cos^2\xi\big)\mathrm{d}\theta\wedge\mathrm{d}\chi\wedge\Big(R_+\cos^2\xi\mathrm{dvol}(S_1^2)+R_-\sin^2\xi\mathrm{dvol}(S_2^2)\Big)\\
&+\left(\frac{2(R_+-R_-)\sin2\xi}{R_+\sin^2\xi+R_-\cos^2\xi}D\psi\wedge\mathrm{d}\xi-\sinh\theta(R_+R_-+2j^2\cosh^2\theta\mathrm{d}\theta\wedge\mathrm{d}\chi\right)\\
&\wedge(\cos^2\xi\mathrm{dvol}(S_1^2)+\sin^2\xi\mathrm{dvol}(S_2^2)\Big)\,. \qquad (10.72)
\end{aligned}
$$

Here $H^{(3)}=H^{(2,1)}+H^{(1,2)}$ is the total three-form flux. It is now straightforward to check that the Bianchi identities (9.45) and the constraints on the non-primitive pieces (9.55) are satisfied. Finally, we check the Maxwell equations (9.48)-(9.51), finding that they are all satisfied.

Chapter 11

Conclusion

> *"All of physics is either impossible or trivial. It is impossible until you understand it, and then it becomes trivial."*
>
> – Ernest Rutherford

11.1 Summary

In this dissertation we have studied black holes from various perspectives in string theory. One common theme of all the black holes that we have studied, is that they are constructed from branes.

In part I we considered supersymmetry breaking Scherk–Schwarz duality twists and their effect on black holes in string theory. Our setup was type IIB string theory compactified on a four-torus and then further compactified on a circle with a duality twist along the circle. To work out the latter compactification, we have explicitly constructed a duality covariant formulation of type IIB supergravity on a four-torus in which the 10d origin of the 6d fields was manifest. We then reduced this six-dimensional theory on a circle with a duality twist. We chose a monodromy in the R-symmetry group depending on four independent twist parameters. These reductions yield gauged 5d $\mathcal{N} = 8$ supergravity, with Minkowski vacua preserving $\mathcal{N} = 6, 4, 2, 0$ supersymmetry, depending on the chosen twist.

In these reductions we have studied several different brane configurations, the D1-D5-P system and dual configurations, that give rise to five-dimensional black holes in the standard untwisted reduction. For these brane configurations, we have deduced the conditions on the twist that need to be satisfied in order to ensure that the corresponding black hole is a solution of the reduced theory.

Our reduction scheme yields a rich spectrum of massive modes in 5d. We have integrated out the chiral massive fields which induces quantum corrections to the coefficients of the pure gauge and mixed gauge-gravitational Chern-Simons terms. These corrections modify the BPS black hole solutions that we study. We worked out these modifications, in particular to the expression for the black hole entropy.

Scherk–Schwarz reductions can be lifted to string theory so long as the monodromy is an element of the discrete U-duality group $\mathrm{Spin}(5,5;\mathbb{Z})$. We have worked out the quantization conditions that this requirement imposes on the twist parameters. If these are satisfied, all our constructions can be embedded in string theory.

Moreover, when the duality twist is a T-duality, the theory at the minimum of the potential can be described as an asymmetric orbifold. We have explicitly constructed this orbifold, and computed a part of the field content that arises from it. Furthermore, we have argued what conditions the survival of certain D-brane configurations puts on the orbifold, finding agreement between these conditions and the ones that were imposed on the Scherk–Schwarz twist in the corresponding supergravity setup. In particular, D1 and D5-branes survive only in symmetric orbifolds.

In part II we studied M2-branes and D2-branes wrapping Riemann surfaces with non-constant curvature: spindles and topological discs. These give rise to 4d black hole solutions in $\mathcal{N}=2$ STU supergravity, whose near-horizon is a warped product of AdS_2 with the Riemann surface. In general, such solutions can have four charges and can rotate, in which case the AdS_2 is fibered. We have shown that the disc and spindle solutions can be obtained from different global completions of the same local solution, and we have analyzed their properties in detail. In particular, we have shown that such solutions preserve supersymmetry with an anti-twist, by comparing the Euler characteristic of the Riemann surface to the sum of the charges.

We have uplifted various truncations of this family of near-horizon solutions to M-theory and to massive type IIA. In our 4d analysis, we studied non-rotating two-charge spindle solutions and non-rotating single-charge disc solutions. We performed the uplift of both of these to 11d supergravity, and we found that the spindle solution is smooth, while the disc solution has singularities associated to the presence of smeared M2-branes and monopoles. In a similar manner, we studied the rotating Einstein–Maxwell spindle solution, and uplifted it to massive type IIA. The resulting 10d solution contains monopole singularities, similar to those found in the analogous M5-brane setup. Remarkably, these singularities are absent in the M-theory uplift of the same 4d solution, and the geometry is completely smooth.

In part III we have classified the necessary and sufficient conditions for near-horizon geometries of extremal supersymmetric rotating black holes in 11d supergravity, which are associated to rotating M2-branes. These near-horizon geometries contain an AdS_2 factor which is fibered by the internal geometry. We have allowed for the most general fibration and flux configuration supporting rotating M2-branes. Due to the generality of our ansatz the black holes covered by our classification can include both electric and magnetic charges as well as angular momentum in 4d.

We have written the conditions of our classification in terms of differential equations on an 8d balanced space. The full 9d internal space is a U(1) fibration over this 8d base. We have constructed a Lagrangian from which the equations of motion that these solutions must satisfy can be derived, one of which is the so-called master equation. By use of dualities, we have also presented necessary and sufficient conditions for the near-horizon geometry of a class of rotating black string solutions in type IIB.

Finally, we have embedded several known 4d black hole solutions from the literature into our classification. This serves both as a non-trivial check of the correctness of our classification and as a way of better understanding certain properties of these solutions, such as the difference between AdS_4 and $mAdS_4$ black holes.

11.2 Outlook

Of course, the lines of research discussed here do not end with the completion of this thesis. Below, we mention several follow-up directions, that will hopefully be investigated in future research.

Microscopics of D1/D5-branes in freely-acting orbifolds

One interesting topic for future research would be to study the microscopic side of the macroscopic story laid out in part I of this work. This would involve working out the effects of the duality twist on the D1-D5 CFT dual to the near-horizon geometry of our setup. We expect this to give rise to twisted boundary conditions along the spatial circle of the SCFT, and the corresponding superconformal algebra to be a ρ-twisted algebra [166]. In order to work out this microscopic story, an in-depth understanding of the map between symmetry groups of the bulk and boundary theories is required.

CFT duals of spindle and disc solutions

The spindle and disc solutions that we considered in part II have the natural interpretation of M2- and D2-branes wrapped on the Riemann surface. It would be interesting to improve the understanding of the field theories living on these branes, and on other branes wrapping spindles and discs.

M5-branes on a disc have recently been shown to be the holographic duals of a class of Argyres–Douglas theories [89, 90]. The 4d $\mathcal{N} = 2$ SCFTs arise from considering the compactification of the 6d $\mathcal{N} = (2, 0)$ worldvolume theory on a twice punctured sphere; a regular puncture is located at one pole and an irregular puncture located at the other.

It is therefore tempting to conjecture that the M2-brane disc solutions we consider in this thesis are the holographic duals of ABJM compactified on a twice punctured sphere. Similarly, D3-branes wrapped on a disc should be the holographic duals of $\mathcal{N} = 4$ SYM wrapped on a similarly twice punctured sphere. It would be interesting to confirm these suspicions in the future

Studying the field theory duals of spindle and disc geometries would be interesting more generally. The universal features of these solutions, that supersymmetry is not preserved with a conventional topological twist but with either a twist or an anti-twist, is a recent discovery. Research towards a better understanding of these mechanisms on the field theory side could reveal new and exciting features of SCFTs.

Geometric extremization for rotating black holes

Part III of this thesis can be seen as a first step towards formulating a geometric extremization principle for rotating black holes, analogous to the one presented in [109] for static black holes. It would certainly be interesting to pursue this avenue of research further. Such an extremization principle could then be used to determine the entropy of black holes whose near-horizon geometries fall within our classification, i.e. a class of rotating black holes with both electric and magnetic charges. This would be the geometric version of the $\mathcal{I}$-extremization principle, which can be used to compute this entropy microscopically in the field theories dual to the near-horizon geometries.

A similar extremization program could be set up for rotating black strings in type IIB. One could extend our work to a more general classification of black string near-horizons, and pursue a geometric extremization principle for that class of solutions.

New black hole solutions

Finding new solutions, either full black holes or near-horizon geometries, would be an interesting direction of future research in its own right. The search for undiscovered black hole solutions can be seen as an extension of both parts II and III of this work.

One could, for example, consider the rotating generalizations of the four-charge solutions of 4d $\mathcal{N} = 2$ U$(1)^4$ supergravity that we studied in chapter 6. This would allow for the construction of spinning disc solutions following the ideas presented in this thesis on different global completions of local solutions. Such rotating disc solutions have not appeared in the literature previously. The uplifts of such solutions to M-theory would then fall within the classification of chapter 9.

Furthermore, understanding how the classification of chapter 9 seems to encompass both magnetic AdS_4 and Kerr–Newman-like (non-magnetic) AdS_4 solutions [167] would be interesting, and could potentially be used to generate new rotating black hole solutions.

Thoughts on the future of string theory

Let us conclude by making some remarks about the current status and the future of string theory. For the most part this is a personal perspective, and it should be seen in that light. This passage is intended to provoke reflection and debate.

It is uncertain what the future of string theory will bring. A lot of research is happening in numerous promising directions, although some of those seem to directly contradict one another[1]. With the first and second superstring revolutions taking place in the 1980s and the 1990s respectively, a third revolution to give the field a renewed unified direction is certainly making us wait.

One event that would undoubtedly have caused a tsunami of innovation in the field, is the experimental confirmation of broken supersymmetry in nature. Unfortunately, the LHC has been deafeningly quiet on this matter. Low-energy supersymmetry is not a requirement of string theory, so the lack of experimental detection thereof does not discredit the field. This does mean, however, that there is less direction for how string theory should make contact with phenomenology[2]. Of course one can hold on to the hope that 'supersymmetry may be right around the corner', but the fact that this reasoning can be repeated time and time again with each new experiment makes it somewhat pointless.

There is a perspective that one can take to get rid of phenomenological concerns. All these disappear by considering string theory to be a branch of mathematics rather than a branch of physics. Irrespective of whether string theory really is a

[1] One example of this would be lines of research aiming to either defend or challenge the KKLT mechanism.

[2] Another place that string theorists can turn to in anticipation of experimental data to be guided by is 'beyond GR' physics that could potentially be found e.g. in gravitational wave observation experiments or in direct black hole observation experiments.

fundamental description of nature, it has had significant impact on various branches of mathematics. A prominent example is the homological mirror symmetry program. From this perspective one could classify large portions of string theory as 'physical mathematics', a subfield aiming for progress in mathematics based on intuition from physics. The quality of string theory to induce developments in other fields can be used in its own right to justify its existence and further study, as well as to steer its direction.

Despite the unquestionable rationality behind this argument, most string theorists stick with the view that string theory should be seen as a physical theory. A pessimist might argue that this is because there is no viable alternative: string theory is the only game in town[3]. Perhaps one of the reasons for people to pursue string theory as a theory of quantum gravity is simply because that is what they know, or because that is what 'everybody else' is doing[4].

An optimist, however, may emphasize the many remarkable successes of string theory: the natural way gravity emerges on a quantum string, the microscopic explanation of the Bekenstein–Hawking entropy, and the AdS/CFT correspondence, to name just a few examples. Edward Witten once expressed this point of view nicely in the following quote.

> *"I just think too many nice things have happened in string theory for it to be all wrong. Humans do not understand it very well, but I just don't believe there is a big cosmic conspiracy that created this incredible thing that has nothing to do with the real world."*

In addition, one can argue that string theory, at its very core, is nothing more than quantum field theory with higher dimensional objects. This conceptually slight extension solves some problems that ordinary QFT has, such as UV-completeness. The elegance and naturalness of this point of view make string theory a very appealing theory to study.

It is also worth noting that it is a mistake to assume that string theory has no relevance for experiments whatsoever. String theory and AdS/CFT have led to a bound on the sheer viscosity to entropy density ratio for a class of strongly coupled theories. It is experimentally verified that quark-gluon plasma in the LHC, which

[3]We ignore loop quantum gravity and related theories here, as string theorists often do.

[4]In his book *The Road to Reality* [168] Roger Penrose called this a bandwagon effect, which is a psychological phenomenon whereby people tend to adopt certain behaviors simply because others are doing so.

falls in this class, satisfies this bound.

Irrespective of which perspective one takes – string theory being physics or mathematics, and it being a useful theory to study or not – profound new insights are needed in order to make progress in our fundamental understanding of nature. The author would like to encourage researchers, even in this highly competitive job market with a strong focus on publications and citations, to stray from the beaten track and to boldly ask the big questions.

11.3 Samenvatting in het Nederlands

In dit proefschrift hebben we zwarte gaten vanuit verscheidene perspectieven in snaartheorie bestudeerd. Een gemeenschappelijke eigenschap van alle zwarte gaten die we hebben bestudeerd, is dat ze zijn opgebouwd uit branen.

In deel I hebben we gekeken naar supersymmetrie brekende Scherk–Schwarz dualiteitswendingen en hun effect op zwarte gaten in snaartheorie. Onze opstelling was type IIB snaartheorie gecompactificeerd op een vier-torus en vervolgens verder gecompactificeerd op een cirkel met een dualiteitswending langs de cirkel. Om de laatstgenoemde compactificatie te kunnen uitwerken, hebben we expliciet een dualiteitscovariante formulering van type IIB supergravitatie geconstrueerd op een vier-torus waarin de 10d oorsprong van de 6d velden manifest was. Vervolgens hebben we deze zes-dimensionale theorie gereduceerd op een cirkel met een dualiteitswending, waarbij we een monodromie in de R-symmetriegroep hebben gekozen, afhankelijk van vier onafhankelijke wendingsparameters. Deze reductie resulteert in geijkte 5d $\mathcal{N} = 8$ supergravitatie, met Minkowski vacua die $\mathcal{N} = 6, 4, 2, 0$ supersymmetrie behouden.

In deze reducties hebben we verscheidene braanconfiguraties bestudeerd, het D1-D5-P systeem en duale configuraties, die resulteren in vijf-dimensionale zwarte gaten in de standaard reductie zonder wending. Voor deze braanconfiguraties hebben we de condities op de wending afgeleid waaraan moet worden voldaan om ervoor te zorgen dat het corresponderende zwarte gat een oplossing is van gereduceerde theorie.

Onze reductie levert een rijk spectrum aan massieve toestanden op in 5d. We hebben de chirale massieve velden uitgeïntegreerd, wat kwantumcorrecties geeft aan de coëfficiënten van de zuivere ijk en de gemengde ijk-gravitatie Chern–Simons termen. Deze correcties modificeren de BPS zwart gatoplossingen die we bestuderen. We hebben deze modificaties uitgewerkt, in het bijzonder die aan de entropie van het zwarte gat.

Scherk–Schwarz reducties kunnen worden verheven naar snaartheorie zolang de monodromie een element is van de discrete U-dualiteitsgroep $\mathrm{Spin}(5,5;\mathbb{Z})$. We hebben de kwantisatiecondities uitgewerkt die deze eis oplegt aan de wendingsparameters. Als aan deze condities voldaan wordt, kunnen al onze constructies worden ingebed in snaartheorie.

Wanneer de dualiteitstwist een T-dualiteit is, kan de theorie in het minimum van de potentiaal worden beschreven als een asymmetrische orbi-variëteit. We hebben deze orbi-variëteit expliciet geconstrueerd, en een deel van de veldinhoud berekend die eruit voortkomt. Verder hebben we betoogd welke voorwaarden het voortbestaan van bepaalde D-braanconfiguraties aan de orbi-variëteit stelt, waarbij we overeenstemming hebben gevonden tussen deze en de voorwaarden die werden opgelegd aan de Scherk–Schwarz wending in de overeenkomstige supergravitatieopstelling. In het bijzonder overleven D1- en D5-branen alleen in symmetrische orbi-variëteiten.

In deel II hebben we M2-branen en D2-branen bestudeerd die gewikkeld zijn om Riemann-oppervlakken met niet-constante kromming: spintollen en topologische schijven. Deze geven 4d zwart gatoplossingen in $\mathcal{N} = 2$ STU supergravitatie, waarvan de nabij-horizon een vervormd product is van AdS_2 met het Riemann-oppervlak. In het algemeen kunnen dergelijke oplossingen vier ladingen hebben en kunnen ze roteren, in welk geval de AdS_2 gevezeld is. We hebben aangetoond dat de schijf- en spintoloplossingen kunnen worden verkregen uit verschillende mondiale voltooiingen van dezelfde lokale oplossing, en we hebben hun eigenschappen in detail geanalyseerd. In het bijzonder hebben we aangetoond dat dergelijke oplossingen supersymmetrie behouden met een anti-wending, door de Eulerkarakteristiek van het Riemann-oppervlak te vergelijken met de som van de ladingen.

We hebben verscheidene afknottingen van deze familie van nabij-horizon oplossingen verheven naar M-theorie en naar massieve type IIA. In onze 4d analyse hebben we niet-roterende twee-lading spintoloplossingen en niet-roterende enkel-lading schijfoplossingen bestudeerd. We hebben de verheffing van beide naar 11d supergravitatie uitgevoerd, en we ontdekten dat de spintoloplossing glad is, terwijl de schijfoplossing singulariteiten heeft die geassocieerd worden met de aanwezigheid van uitgesmeerde M2-branen en monopolen. Op een vergelijkbare manier hebben we roterende Einstein–Maxwell spintoloplossingen bestudeerd en deze verheven naar massieve type IIA. De resulterende 10d oplossing bevat monopoolsingulariteiten, vergelijkbaar met degenen die gevonden zijn in de analoge M5-braanopstelling. Opmerkelijk is dat deze singulariteiten afwezig zijn in de M-theorie verheffing van dezelfde 4d oplossing, en dat daar de geometrie volledig glad is.

In deel III hebben we de noodzakelijke en toereikende voorwaarden geclassificeerd voor de nabij-horizongeometrieën van extreme supersymmetrische roterende zwarte gaten in 11d supergravitatie, die geassocieerd worden met roterende M2-branen. Deze nabij-horizongeometrieën bevatten een AdS_2-factor die gevezeld wordt door de interne geometrie. We hebben rekening gehouden met de meest algemene vezeling en met fluxconfiguraties die roterende M2-branen toestaan. Vanwege de algemeenheid van ons ansatz kunnen de zwarte gaten die in onze classificatie vallen in 4d zowel elektrische als magnetische ladingen bevatten, evenals impulsmoment.

We hebben de voorwaarden van onze classificatie in termen van differentiaalvergelijkingen op een 8d gebalanceerde ruimte geschreven. De volledige 9d interne ruimte in een U(1)-vezeling over deze 8d basis. We hebben een Lagrangiaan geconstrueerd waaruit de bewegingsvergelijkingen kunnen worden afgeleid waaraan deze oplossingen moeten voldoen. Een hiervan is de zogenaamde meestervergelijking. Door gebruik te maken van dualiteiten hebben we ook noodzakelijke en toereikende voorwaarden gepresenteerd voor de nabij-horizongeometrie van een klasse van roterende zwarte snaaroplossingen in type IIB.

Ten slotte hebben we verscheidene bekende 4d zwart gatoplossingen uit de literatuur in onze classificatie ingebed. Dit dient als een niet-triviale controle van de juistheid van onze classificatie, en als een manier om bepaalde eigenschappen van deze oplossingen, zoals het verschil tussen AdS_4 en $mAdS_4$ zwarte gaten, beter te begrijpen.

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About the author

Koen Christiaan Stemerdink was born on December 2, 1992 in Leiden, the Netherlands. He obtained his secondary school diploma from Stedelijk Gymnasium Leiden in 2011.

That same year he started a bachelor's program in physics at Utrecht University, which he supplemented with a program in mathematics one year later. In 2015 he wrote a bachelor's thesis on thermal spin transport and electron-magnon interactions in easy-plane ferromagnets under supervision of prof. dr. Rembert Duine, thereby successfully completing his double bachelor's degree.

In the academic year 2015-2016 he took a break from studying in order to take part in the university's student participation. He got elected to the university council of Utrecht University, an advisory body to the executive board, in which he served as chairman of the student-part.

He resumed his studies in 2016, starting a master's program in theoretical physics, which he completed cum laude in 2018. His master's thesis was on black holes and supersymmetry breaking Scherk–Schwarz reductions in string theory, and was supervised by prof. dr. Stefan Vandoren.

He continued this line of research in his PhD, again under supervision of prof. dr. Stefan Vandoren, now in the role of promotor. In addition, he studied spindle and disc solutions as well as classifications of black hole near-horizon geometries, in collaboration with his co-promotor dr. Christopher Couzens. His research culminated in this PhD thesis, titled *Black Holes from Branes: various string theoretical constructions*, which he intends to defend on September 9, 2022.

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