The Chern-Simons Topological Quantum Field Theory

and the $\mathfrak{so}_8$ Large Color R-matrix for Quantum Knot Invariants

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To D.O.G.,
without whom this work wouldn't have been possible.

Abstract

Abstract In theoretical physics, Quantum Field Theory (QFT) is an extremely successful theoretical framework combining both Special Relativity and Quantum Mechanics, enabling to design physical models of subatomic particles and quasiparticles describing the most fundamental aspects of matter with an incredibly high accuracy. Among these theories, the Chern-Simons QFT is a special one, not only describing topological phenomena in physics such as the Quantum Hall Effect, but also fitting the notion of what is known as a Topological Quantum Field Theory (TQFT). It was by using the axioms of TQFTs that Edward Witten showed back in 1989 how closely related the Chern-Simons theory is to the realm of polynomial invariants appearing in Knot Theory, such as the well-known Jones polynomial. In the past years, further research in this field has led to new and more powerful invariants of links and, by means of Dehn surgeries on them, of 3-manifolds as well. For instance, the Gukov-Manolescu series proposed recently in 2020 —denoted $F_K(x,q)$— is a conjectural invariant of knot complements that, in a sense, analytically continues the colored Jones polynomials. Shortly after, Sunghyuk Park introduced the Large Color R-matrix approach for $\mathfrak{sl}(2,\mathbb{C})$ to study F_K for some simple links, giving a definition of F_K for positive braid knots and computing F_K for various knots and links. This procedure has in turn been extended by Angus Gruen to all other Lie algebras $\mathfrak{sl}_{n+1}$ beyond $\mathfrak{sl}_2$. In this work, after a broad review on the above mentioned background, we move on to the family $\mathfrak{so}_{2n}$ of complex semisimple Lie algebras in Cartan's classification, mainly focusing on the $\mathfrak{so}_8$ case attracted by the three-fold symmetry in its Dynkin diagram D_4.

Keywords: Topological Quantum Field Theory, Knot Theory, Dehn Surgery, Chern-Simons, Lie Algebra, Representation Theory, Quantum Groups.

MSC2020: 57K16, 57R56.

Resumen En física teórica, la Teoría Cuántica de Campos (TCC) es un marco teórico extremadamente exitoso que combina la Relatividad Especial con la Mecánica Cuántica, permitiendo el diseño de modelos físicos de las partículas subatómicas y cuasipartículas que describen los aspectos más fundamentales de la materia con una precisión increíblemente alta. Entre dichas teorías, la TCC Chern-Simons es una especial, que no sólo describe fenómenos topológicos en física tales como el Efecto Hall Cuántico, sino que encaja con la noción de lo que se conoce como una Teoría Topológica de Campos Cuánticos (TTCC). Fue utilizando estos axiomas de las TTCCs que Edward Witten mostró en 1989 cómo de estrechamente relacionada está la teoría de Chern-Simons con el ámbito de invariantes polinómicos que aparecen en la Teoría de Nudos, tales como el bien conocido polinomio de Jones. En estos últimos años, investigaciones en este campo han dado lugar a nuevos y más poderosos invariantes de enlaces y, a través de cirugías de Dehn sobre ellos, así mismo de 3-variedades. Por ejemplo, la serie de Gukov-Manolescu recientemente propuesta en 2020 —denotada $F_K(x,q)$— es un invariante conjetural de complementos de nudos que, en cierto sentido, continúa analíticamente los polinomios de Jones coloreados. Poco después, Sunghyuk Park introdujo el enfoque de la Matriz R de Gran Color correspondiente a $\mathfrak{sl}(2,\mathbb{C})$ para estudiar F_K para trenzados positivos y calcular F_K para varios nudos y enlaces. Este procedimiento ha sido a su vez extendido por Angus Gruen a todas las otras álgebras de Lie $\mathfrak{sl}_{n+1}$ más allá de $\mathfrak{sl}_2$. En la presente obra, tras un extenso repaso sobre los anteriormente mencionados conceptos, abordamos la família $\mathfrak{so}_{2n}$ de álgebras de Lie semisimples sobre los complejos en la clasificación de Cartan, centrándonos principalmente en el caso $\mathfrak{so}_8$ atraídos por la simetría triple en su diagrama de Dynkin D_4.

Palabras clave: Teoría Topológica Cuántica de Campos, Teoría de Nudos, Cirugía de Dehn, Chern-Simons, Álgebra de Lie, Teoría de Representaciones, Grupos Cuánticos.

MSC2020: 57K16, 57R56.

Resum A física teòrica, la Teoria Quàntica de Camps (TQC) és un marc teòric extremadament reeixit que combina la Relativitat Especial amb la Mecànica Quàntica, permetent el diseny de models físics de les partícules subatómiques i quasipartícules que descriuen els aspectes més fundamentals de la matèria amb una precisió increïblement alta. Entre aquestes teories, la TQC Chern-Simons és una especial, que no només descriu fenòmens topològics a la física com ara l'Efecte Hall Quàntic, sinó que encaixa amb la noció del que es coneix com una Teoria Topològica de Camps Quàntics (TTCQ). Va ser fent servir aquests axiomes de les TTCQs que Edward Witten va mostrar al 1989 com d'estretament relacionada està la teoria de Chern-Simons amb l'àmbit d'invariants polinòmics que apareixen a la Teoria de Nusos, com ara el ben conegut polinomi de Jones. En aquests darrers anys, investigacions en aquest camp han donat lloc a nous i més poderosos invariants d'enllaços i, a través de cirurgies de Dehn sobre ells, de 3-varietats també. Per exemple, la sèrie de Gukov-Manolescu recentment proposta el 2020 —denotada $F_K(x,q)$— és un invariant conjectural de complements de nusos que, en cert sentit, continua analíticament els polinomis de Jones colorejats. Poc després, Sunghyuk Park va introduir l'enfoc de la Matriu R de Gran Color corresponent a $\mathfrak{sl}(2,\mathbb{C})$ per estudiar F_K per trenats positius i calcular F_K per a diversos nusos i enllaços. Aquest procediment ha estat així mateix extès per Angus Gruen a totes les altres àlgebres de Lie $\mathfrak{sl}_{n+1}$ més enllà de $\mathfrak{sl}_2$. En aquest treball, després d'un extens repàs sobre els anteriorment esmentats conceptes, abordem la família $\mathfrak{so}_{2n}$ d'àlgebres de Lie semisimples sobre els complexos a la classificació de Cartan, centrant-nos principalment en el cas $\mathfrak{so}_8$ atrets per la simetria triple al seu diagrama de Dynkin D_4.

Paraules clau: Teoria Topològica Quàntica de Camps, Teoria de Nusos, Cirurgia de Dehn, Chern-Simons, Àlgebra de Lie, Teoria de Representacions, Grups Quàntics.

MSC2020: 57K16, 57R56.

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Preface

In the broad light of day, mathematicians check their equations and their proofs, leaving no stone unturned in their search for rigour. But, at night, under the full moon, they dream, they float among the stars and wonder at the miracle of the heavens. They are inspired. Without dreams there is no art, no mathematics, no life.

Michael Atiyah

The present work is organized as follows.

We start in Chapter 1 by introducing the reader to the notion of a TQFT, both giving the categorical description as well as the underlying ideas this encapsulates in terms of cobordisms and vector spaces. The chapter is ended with a few comments on the interest of TQFTs for physicists and mathematicians.

The next chapter is at a first stage devoted to the introduction of some basic concepts in Knot theory and skein relations, jumping then into the surgery description of 3-manifolds in terms of Dehn surgery on knots and links, showing that the clasification problems of knots and of 3-manifolds are closely related. Being Dehn surgery defined through embedded tori, a review on the Mapping Class Group of the torus T^2 is carried out to then better describe surgeries on links in $\mathbb{S}^3$. Further, the topological invariant known as the *linking number* is introduced, which enables an alternative description of the *framing* concept. An alternative graphical notation in terms of plumbed graphs and the invariant moves between them —known as Kirby moves— is also presented. These notions are basic tools appearing in the construction of recently developed knot invariants.

In Chapter 3 we finally introduce the Chern-Simons theory, both the abelian and non-abelian version. We proceed by showing how the skein relation defining the Jones polynomial is recovered from the Chern-Simons theory when viewed as a TQFT as described in Chapter 1. We do so by following Witten's approach. We conclude by describing two different fields in physics in which Chern-Simons theory can play a role, namely the Quantum Hall Effect and Quantum Gravity.

Last, in Chapter 4 we present the carried-out research project, preceded by a review on the braiding construction of knot invariants through the so-called R-matrices satisfying the Yang-Baxter equation. After redoing the computations for the symmetric representation of $\mathfrak{sl}_2$, its quantum version and the resulting Large Color R-matrix, we procede to present the obtained results on the $\mathfrak{so}_{2n}$ family. The first cases are reduced to simpler ones concerning $\mathfrak{sl}_2$ or $\mathfrak{sl}_{n+1}$. Then, the approach for $\mathfrak{so}_8$ is presented, giving the corresponding symmetric representation in terms of a polynomial representation analogous to the one employed for $\mathfrak{sl}_2$, yet slightly more involved. The necessary ingredients required by the general R-matrix product-formula are almost completely obtained, leaving the not-fully-developed quantum version of the obtained representation for future steps.

An Appendix with the required notions on Smooth manifolds, Lie algebras and Representation theory, and Homology and Cohomology theory is added, along with some of the `Mathematica` codes employed throughout the development of the project.

Abbreviations

The next list describes some abbreviations used within the body of the document.

CFT **C**onformal **F**ield **T**heory

CS **C**hern-**S**imons

GR **G**eneral **R**elativity

LQG **L**oop **Q**uantum **G**ravity

QCD **Q**uantum **C**hromo**D**ynamics

QEA **Q**uantum **E**nveloping **A**lgebra

QFT **Q**uantum **F**ield **T**heory

QG **Q**uantum **G**roups

QHE **Q**uantum **H**all **E**ffect

QM **Q**uantum **M**echanics

QT **Q**uantum **T**heory

TQC **T**opological **Q**uantum **C**omputers

TQFT **T**opological **Q**uantum **F**ield **T**heory

UEA **U**niversal **E**nveloping **A**lgebra

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Topological Quantum Field Theories

1

In recent years there has been a remarkable renaissance in the relation between Geometry and Physics. This relation involves the most advanced and sophisticated ideas on each side and appears to be extremely deep. The traditional links between the two subjects, as embodied for example in Einstein's Theory of General Relativity or in Maxwell's Equations for Electro-Magnetism are concerned essentially with classical fields of force, governed by differential equations, and their geometrical interpretation. The new feature of present developments is that links are being established between quantum *physics and* topology. *It is no longer the purely* local *aspects that are involved but their* global *counterparts. In a very general sense this should not be too surprising. Both quantum theory and topology are characterized by discrete phenomena emerging from a continuous background. However, the realization that this vague philosophical view-point could be translated into reasonably precise and significant mathematical statements is mainly due to the efforts of Edward Witten who, in a variety of directions, has shown the insight that can be derived by examining the topological aspects of quantum field theories.*

Michael Atiyah [Ati89]

Being the Chern-Simons theory a Topological Quantum Field Theory (TQFT), we start by presenting this notion first introduced in 1988 by E. Witten [Wit88b] and rigorously axiomatized in 1989 by M. Atiyah [Ati89].

1.1 A First Approach to TQFTs

Let us first present the general idea of the notion of a TQFT and gain some insight in its meaning and features. We start by giving an axiomatic description and procede by extracting their consequences and interpretation. At the very end we will reformulate its definition in a categorical way. We mainly follow [Koc03; Bai].[1]

1: We refer the reader to Appendix A for a review on smooth manifolds. In particular, recall that a **closed** manifold is a compact manifold without boundary.

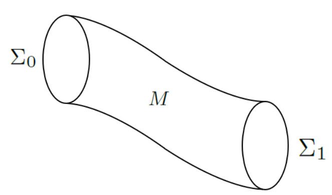

Figure 1.1: Representation of a manifold M with boundary $\partial M = \Sigma_0 \bigsqcup \Sigma_1$.

Definition 1.1.1 *Roughly, a **TQFT** in dimension $n \in \mathbb{N}$ over a field $\mathbb{K}$ is a rule $\mathcal{Z}$ that assigns:*

- *to every closed oriented $(n-1)$-dimensional manifold Σ, a $\mathbb{K}$-vector space $\mathcal{Z}(\Sigma)$,*
- *and to every oriented n-dimensional manifold M with boundary $\partial M = \Sigma$, an element[a] $\mathcal{Z}(M)$ in $\mathcal{Z}(\Sigma)$*

subject to some axioms, discussed below.

[a] Being $\mathcal{Z}(\Sigma)$ a vector space, the element $\mathcal{Z}(M)$ is sometimes referred to as a vector.

A graphical picture worth having in mind can be seen in Figure 1.1. We may think of the $(n-1)$-manifolds as representing *space* and the n-manifolds having them as boundaries (called *cobordisms*) as representing *spacetime*. There is a notion of order in cobordisms from their *left* boundary to their *right* one capturing a sense of *evolution*. For this reason, cobordisms are sometimes referred to as *arrows*.

We will see that elements in the second item are just linear maps $\mathcal{Z}(M) : \mathcal{Z}(\Sigma_0) \to \mathcal{Z}(\Sigma_1)$ from the vector space associated to the *left* boundary Σ_0 to the one associated to the *right* boundary Σ_1. In other words, we can view any cobordism M between Σ_0 and Σ_1 as inducing a linear transformation $\mathcal{Z}(M) : \mathcal{Z}(\Sigma_0) \to \mathcal{Z}(\Sigma_1)$. This makes clearer the use of the word *arrows* mentioned before (cobordisms are sent to arrows).

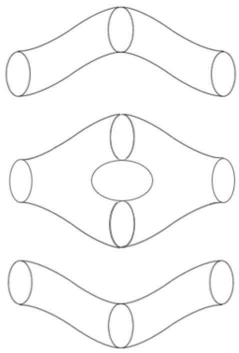

Figure 1.2: The disjoint union of cobordisms "running in parallel" corresponds to the tensor product of the associated linear maps: $f \otimes g \otimes h$.

On the other hand, we may think of these vector spaces associated to each *space* as Hilbert spaces whose vectors are quantum states of a physical system, while the linear map associated to *spacetime* as a linear operator between Hilbert spaces, representing a process from one state to another.

Thus, TQFTs present themselves as an axiomatization of an interrelation between special relativity and quantum theory, which is in fact what Quantum Field Theory (QFT) deals with. This gives some insight to why TQFTs deserve their name.

This rule $\mathcal{Z}$ is subject to a collection of axioms containing the essential information, which can be briefly given as:

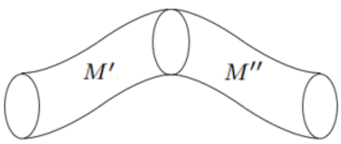

Figure 1.3: Compositions of cobordisms are given by glueing two cobordisms along a common boundary.

Definition 1.1.2 *The axioms for a TQFT are:*

(A1) $\Sigma \cong \Sigma' \Rightarrow \mathcal{Z}(\Sigma) \cong \mathcal{Z}(\Sigma')$.
(A2) $\mathcal{Z}(\Sigma \times I) = \mathrm{Id}_{\mathcal{Z}(\Sigma)}$.
(A3) $M = M' \cup M'' \Rightarrow \mathcal{Z}(M) = \mathcal{Z}(M'') \circ \mathcal{Z}(M')$.
(A4) $\mathcal{Z}(\Sigma \sqcup \Sigma') = \mathcal{Z}(\Sigma) \otimes \mathcal{Z}(\Sigma')$ *and* $\mathcal{Z}(M_1 \sqcup M_2) = \mathcal{Z}(M_1) \otimes \mathcal{Z}(M_2)$.
(A5) $\mathcal{Z}(\Sigma^*) = \mathcal{Z}(\Sigma)^*$.
(A6) $\mathcal{Z}(\emptyset_{n-1}) = \mathbb{K}$ *and* $\mathcal{Z}(\emptyset_n) = 1$.

The first axiom (A1) tells us that diffeomorphic manifolds Σ, Σ' (i.e. topologically equivalent) are sent to isomorphic vector spaces. Further, (A4) tells us that $\mathcal{Z}$ is multiplicative in the sense that disjoint unions go to tensor products: in the first case, disjoint boundaries go to the tensor product of the associated vector spaces. The second case (A4b) is discussed later on. In (A5), Σ^* denotes the same manifold with the opposite orientation[2] and $\mathcal{Z}(\Sigma)^*$ the dual vector space. Thus, the second item along with (A4)-(A5) tells us that a given oriented manifold M with oriented boundary $\partial M = \Sigma_0^* \sqcup \Sigma_1$ is assigned by $\mathcal{Z}$ to an element $\mathcal{Z}(M)$ living in the vector space $\mathcal{Z}(\partial M) = \mathcal{Z}(\Sigma_0^* \sqcup \Sigma_1) = \mathcal{Z}(\Sigma_0)^* \otimes \mathcal{Z}(\Sigma_1) \cong \mathrm{Hom}(\mathcal{Z}(\Sigma_0), \mathcal{Z}(\Sigma_1))$[3] , the vector space of linear maps from $\mathcal{Z}(\Sigma_0)$ to $\mathcal{Z}(\Sigma_1)$. Hence, the elements $\mathcal{Z}(M)$ are trully linear maps. This fourth axiom (A4b) tells us that multiplicativity must also hold for disjoint cobordisms $M_1 \sqcup M_2$; i.e. cobordisms "running in parallel" (Figure 1.2) are sent to the tensor product of linear maps. Next, (A3) comes to say that given a decomposition of a cobordism (e.g. Figure 1.3), the associated linear map is given by the composition of linear maps (preserving the order). The second axiom (A2) tells us

2: Orientations in the boundary of a cobordism are defined with respect to the n-manifold, in such a way that there are only two possible orientations. See the section on cobordisms.

3: Indeed, there is an isomorphism

$$U^* \otimes V \cong \mathrm{Hom}(U, V)$$

making this identification possible: given $u^* \otimes v \in U^* \otimes V$ define $f : U \to V$ by contraction with the first entry, i.e. by setting $f(\cdot) := u^*(\cdot)v$.

that the *identity* cobordism $\Sigma \times I$ (the cylinder in Figure 1.4) is sent to the identity map of $\mathcal{Z}(\Sigma)$. Last, axiom (A6) says that the empty $(n-1)$-manifold $\Sigma = \emptyset$ goes to the ground field $\mathbb{K}$ (whence the emtpy cylinder $\emptyset \times I$ goes to the identity map of $\mathbb{K}$) and the empty n-manifold goes to the neutral element with respect to the tensor product.

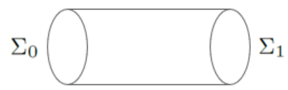

Figure 1.4: Cylinder cobordism $\Sigma \times I$, where $I = [0,1]$ and $\partial\Sigma = \Sigma_0^* \sqcup \Sigma_1$.

Axioms (A1)-(A3) yield functoriality, while (A4) and (A6) yield monoidality, as will be manifest in the coming sections introducing these concepts. Axiom (A5) states that $\mathcal{Z}$ is *involutory*.

The fact of considering general cobordisms instead of just cylinders (as done in the homotopy axiom for homology theories) is related to *relativistic invariance*. The fact of taking (A4) to be multiplicative and not additive is related to the *quantum nature of the theory*, expressing the common principle in Quantum Mechanics that the state space of two independent systems is the tensor product of the two state spaces associated to each system.

Axioms (A1)-(A2) express that the theory is topological. Hence, evolution depends only on the diffeomorphism class of spacetime and not on additional structures such as metric or curvature. It is for this reason that TQFT may serve as baby models to explore and understand the key features of the commonly complicated QFTs.

1.2 Categories and Functors

Let us now introduce the concepts of categories, functors and cobordisms, which enable a mathematically more elegant and precise description of TQFTs capturing all of the aforementioned axioms.

Category theory is in some sense a generalization of different mathematical structures, given by a collection of *objects* and a collection of *morphisms* between them. Further, the notion of morphism can be upgraded to that of *functor*: maps between categories. These concepts are then required to satisfy certain properties so as to confer the desired structure. In this setting, it turns out that a TQFT can actually be described as a functor between the category of cobordisms and the category of vector spaces.

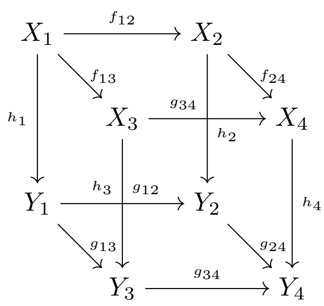

Figure 1.5: Category Theory formalizes mathematical structure and its concepts in terms of objects and morphisms, which can be viewed as the nodes and edges of a labeled directed graph (representing the category itself).

Definition 1.2.1 *A **category** $\mathcal{C}$ consists of a collection of objects $ob(\mathcal{C})$ together with sets of morphisms $\mathrm{Hom}_{\mathcal{C}}(A,B)$ from A to B for each pair of objects $A, B \in ob(\mathcal{C})$. Further, for every triple $A, B, C \in ob(\mathcal{C})$ there is a composition law*

$$\mathrm{Hom}_{\mathcal{C}}(A,B) \times \mathrm{Hom}_{\mathcal{C}}(B,C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A,C)$$
$$(f,g) \mapsto g \circ f$$

subject to the following conditions:

C1) Associativity: Given $A, B, C, D \in ob(\mathcal{C})$ and the composition of morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

we have $h \circ (g \circ f) = (h \circ g) \circ f$.

C2) *Identity: For every $B \in ob(\mathcal{C})$ there exists an identity morphism $\mathrm{Id}_B \in \mathrm{Hom}_{\mathcal{C}}(B,B)$ such that $\mathrm{Id}_B \circ f = f$ for any $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ and $g \circ \mathrm{Id}_B = g$ for any $g \in \mathrm{Hom}_{\mathcal{C}}(B,C)$.*

C3) *The sets $\mathrm{Hom}_{\mathcal{C}}(A,B)$ and $\mathrm{Hom}_{\mathcal{C}}(A',B')$ are disjoint unless $A = A'$ and $B = B'$.*

Some examples of categories are: **Sets** (where objects are sets and morphisms are functions between sets), $\mathbf{Vect}_{\mathbb{K}}$ ($\mathbb{K}$-vector spaces and linear maps), **Top** (topological spaces and continuous maps), $\mathcal{G}$ (groups and homomorphisms of groups) and **Diff** (differentiable manifolds and diffeomorphisms), among others.

Definition 1.2.2 *Let $\mathcal{B}$ and $\mathcal{C}$ be categories. A* ***covariant functor*** *$F\colon \mathcal{B} \longrightarrow \mathcal{C}$ is a rule that assigns*

i) *An object $F(A) \in ob(\mathcal{C})$ for every object $A \in ob(\mathcal{B})$.*

ii) *For every morphism $f\colon A \to B$ in $\mathcal{B}$, a morphism $F(f)\colon F(A) \to F(B)$ in $\mathcal{C}$ satisfying:*

  F1) *$F(\mathrm{Id}_A) = \mathrm{Id}_{F(A)}$ for every $A \in ob(\mathcal{B})$.*

  F2) *Given $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{B}$ we have $F(g \circ f) = F(g) \circ F(f)$.*

A ***contravariant functor*** *$F\colon \mathcal{B} \to \mathcal{C}$ satisfies i) and*

ii') *For every morphism $f\colon A \to B$ in $\mathcal{B}$, a morphism $F(f)\colon F(B) \to F(A)$ in $\mathcal{C}$ satisfying $F1$) and:*

  F2) *Given $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{B}$ we have $F(g \circ f) = F(f) \circ F(g)$.*

There are also some generalizations of the notion of isomorphism and equivalence at the level of categories, which go as follows.

Definition 1.2.3 *Given two functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$, a* ***natural transformation*** *$\alpha : F \Longrightarrow G$ consists of a morphism $\alpha_A : F(A) \longrightarrow G(A)$ in $\mathcal{D}$ for every $A \in \mathrm{ob}(\mathcal{C})$ such that for every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ the diagram in $\mathcal{D}$ given by Figure 1.6 commutes.*

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

Figure 1.6: Commutative diagram for a natural transformation α.

Definition 1.2.4 *Given two functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$, a* ***natural isomorphism*** *$\alpha : F \Longrightarrow G$ is a natural transformation with an inverse natural transformation $\beta : G \Longrightarrow F$ such that $\beta \circ \alpha = \mathrm{Id}_F$ and $\alpha \circ \beta = \mathrm{Id}_G$. That is, every component α_A of α is an isomorphism.*

Definition 1.2.5 *A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is an* ***equivalence*** *if it has a weak inverse: a functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ such that there exist two natural isomorphisms $\alpha : G \circ F \Longrightarrow \mathrm{Id}_{\mathcal{C}}$ and $\beta : F \circ G \Longrightarrow \mathrm{Id}_{\mathcal{D}}$. Two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent, denoted $\mathcal{C} \simeq \mathcal{D}$, if there is an equivalence between.*

Categories may have further structures, as for instance that of *monoidality*. We will talk about *symmetric monoidal functors*. These allow us to study topological invariants; that is, properties that remain unchanged under homeomorphisms, similar to what is done in Homology Theory in Algebraic Topology.

1.3 Monoidal Categories

Roughly, a monoidal category is one equipped with some sort of "product" satisfying certain properties and a "unit", a neutral element with respect to this product[4] .

4: These serve as background for defining different maps (monoids). For instance, a monoid in $(\mathbf{Vect}_{\mathbb{K}}, \otimes, \mathbb{K})$ is precisely a $\mathbb{K}$-algebra A, since the multiplication map is described as a $\mathbb{K}$-linear map $A \otimes A \longrightarrow A$ and the unit map as $\mathbb{K} \longrightarrow A$.

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes \mathrm{Id}_B"'] & & A \otimes (I \otimes B) \arrow[dl, "\mathrm{Id}_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

Figure 1.7: Triangle diagram.

In the following, the cartesian product of categories is defined on objects and morphisms in the natural way. The empty product category $\mathbf{1}$ is the one with a single object and no other than its identity morphism.

Definition 1.3.1 (Monoidal category) *A* ***monoidal category*** *is a category $\mathcal{C}$ equipped with*

1) *a functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ called the* ***tensor product****,*
2) *a functor $\mathrm{I} : \mathbf{1} \to \mathcal{C}$ called* ***unit***[a]*, and*
3) *three natural isomorphisms satisfying some coherence conditions (comutativity of* ***Pentagon*** *and* ***Triangle diagrams****: Figure 1.8, Figure 1.7) expressing that the tensor operation*
   - *is associative: there is a natural isomorphism α, called* ***associator****, with components*

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C),$$

   - *has I as a left and right identity: there are two natural isomorphisms λ and ρ, called* ***unit isomorphisms****, with components $\lambda_A : I \otimes A \cong A$ and $\rho_A : A \otimes I \cong A$*

[a] The image of the only element in $\mathbf{1}$ will be simply denoted by $I \in \mathrm{Ob}(\mathcal{M})$.

$$\begin{tikzcd}
& (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A,B,C \otimes D}"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A \otimes B,C,D}"] \arrow[d, "\alpha_{A,B,C} \otimes \mathrm{Id}_D"'] & & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B \otimes C,D}"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "\mathrm{Id}_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

Figure 1.8: Pentagon diagram.

A **strict monoidal category** is one for which α, λ, ρ are identities. That is, $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ and $A \otimes I = A = I \otimes A$.

A symmetric monoidal category will be one whose tensor products commute. For this purpose, define the **twist functor** $T : \mathcal{C} \times \mathcal{D} \to \mathcal{D} \times \mathcal{C}$ as:

- $T(A \times B) = B \times A$ for any objects $A \in \mathrm{Ob}(\mathcal{C}), B \in \mathrm{Ob}(\mathcal{D})$
- $T(f \times g) = g \times f$ for any morphisms f, g in $\mathcal{C}, \mathcal{D}$.

Definition 1.3.2 (Symmetric monoidal category) *A* ***symmetric monoidal category*** *is a monoidal category $(\mathcal{M}, \otimes, I)$ such that for each pair of objects $A, B \in \mathrm{Ob}(\mathcal{M})$ there is a natural isomorphism β, with components $\beta_{A,B} : A \otimes B \to B \otimes A$, such that:*

- *Inverse law or symmetric condition: β is its own inverse, i.e. $\beta_{B,A} \circ \beta_{A,B} = \mathrm{Id}_{A \otimes B}$,*

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "\beta_{A,B \otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(A \otimes B) \otimes C \arrow[u, "\alpha_{A,B,C}"] \arrow[d, "\beta_{A,B} \otimes \mathrm{Id}_C"'] & B \otimes (C \otimes A) \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[u, "\mathrm{Id}_B \otimes \beta_{A,C}"']
\end{tikzcd}$$

Figure 1.9: Hexagon diagram.

- *associativity coherence: the **Hexagon diagram** commutes (Figure 1.9),*
- *Unit coherence: $\lambda_A \circ \beta_{A,I} = \rho_A$.*

Given two monoidal categories, a monoidal functor between them will be one such that it preserves the monoidal structure.

Definition 1.3.3 (Monoidal functor) *A **monoidal functor** between two monoidal categories $(\mathcal{M}_1, \otimes_1, I_1)$ and $(\mathcal{M}_2, \otimes_2, I_2)$ is a functor $F : \mathcal{M}_1 \to \mathcal{M}_2$ equipped with:*

1) a natural isomorphism Φ, with components $\Phi_{A,B} : F(A) \otimes_2 F(B) \to F(A \otimes_1 B)$, and
2) an isomorphism $\varphi : I_2 \to F(I_1)$ in $\mathcal{M}_2$

such that the diagrams in Figure 1.10, Figure 1.11 and Figure 1.12 commute for any objects A, B, C in $\mathcal{M}_1$.

$$\begin{tikzcd}
I_2 \otimes_2 F(A) \arrow[r, "\lambda_{F(A)}"] \arrow[d, "\varphi \otimes_2 F(A)"'] & F(A) \\
F(I_1) \otimes_2 F(A) \arrow[r, "\Phi_{I_1,A}"'] & F(I_1 \otimes_1 A) \arrow[u, "F(\lambda_A)"']
\end{tikzcd}$$

Figure 1.10: Commutativity coherence condition for monoidal functor F.

$$\begin{tikzcd}
F(A) \otimes_2 I_2 \arrow[r, "\rho_{F(A)}"] \arrow[d, "F(A) \otimes_2 \varphi"'] & F(A) \\
F(A) \otimes_2 F(I_1) \arrow[r, "\Phi_{A,I_1}"'] & F(A \otimes_1 I_1) \arrow[u, "F(\rho_A)"']
\end{tikzcd}$$

Figure 1.11: Commutativity coherence condition for monoidal functor F.

$$\begin{tikzcd}
& F(A \otimes_1 B) \otimes_2 F(C) \arrow[rr, "\Phi_{A \otimes_1 B, C}"] & & F((A \otimes_1 B) \otimes_1 C) \arrow[dr, "F(\alpha_{A,B,C})"] & \\
(F(A) \otimes_2 F(B)) \otimes_2 F(C) \arrow[ur, "\Phi_{A,B} \otimes_2 \mathrm{Id}_{F(C)}"] \arrow[dr, "\alpha_{F(A),F(B),F(C)}"'] & & & & F(A \otimes_1 (B \otimes_1 C)) \\
& F(A) \otimes_2 (F(B) \otimes_2 F(C)) \arrow[rr, "\mathrm{Id}_{F(A)} \otimes_2 \Phi_{B,C}"'] & & F(A) \otimes_2 F(B \otimes_1 C) \arrow[ur, "\Phi_{A, B \otimes_1 C}"'] &
\end{tikzcd}$$

Figure 1.12: Commutativity coherence condition for monoidal functor F.

Finally, we can define the notion of a *symmetric* monoidal functor.

Definition 1.3.4 (Symmetric monoidal functor) *A **symmetric monoidal functor** is a monoidal functor F between two symmetric monoidal categories $(\mathcal{M}_1, \otimes_1, I_1)$ and $(\mathcal{M}_2, \otimes_2, I_2)$ such that the diagram in Figure 1.13 commutes for any objects $A, B \in \mathrm{ob}(\mathcal{M}_1)$.*

$$\begin{tikzcd}
& F(B) \otimes_2 F(A) \arrow[dr, "\Phi_{B,A}"] & \\
F(A) \otimes_2 F(B) \arrow[ur, "\beta_{F(A),F(B)}"] \arrow[dr, "\Phi_{A,B}"'] & & F(B \otimes_1 A) \\
& F(A \otimes_1 B) \arrow[ur, "F(\beta_{A,B})"'] &
\end{tikzcd}$$

Figure 1.13: Commutative diagram defining a symmetric monoidal functor.

This notion will be important for our definition of a TQFT.

1.4 Cobordisms

In our first approach to TQFTs we vaguely introduced the notion of *cobordisms* between two manifolds Σ_1 and Σ_2 as a manifold M having them as boundary, $\Sigma_1 \sqcup \Sigma_2 = \partial M$. We aim now to briefly present them and the category they form.

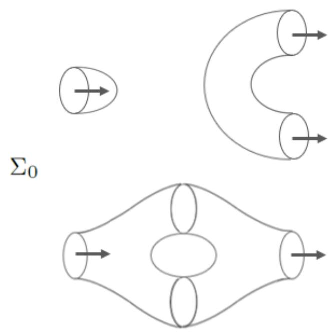

Figure 1.14: Two-dimensional cobordism from Σ_1 to Σ_2, where arrows represent the positive normal vectors, often omitted understanding them to point from left to right.

Definition 1.4.1 (Oriented cobordism) *Given two closed oriented $(m-1)$-manifolds Σ_1 and Σ_2, an **oriented cobordism** from Σ_1 to Σ_2 is an oriented m-manifold M along with two smooth maps*

$$\iota_{\text{in}} : \Sigma_1 \longrightarrow M \longleftarrow \Sigma_2 : \iota_{\text{out}}$$

such that ι_{in} (resp. ι_{out}) is an orientation-preserving diffeomorphism that maps Σ_1 (resp. Σ_2) onto the in-boundaries (resp. out-boundaries) of M.

Cobordisms are drawn placing their in-boundaries to the left and their out-boundaries to the right (Figure 1.14).

In the category of cobordisms, these are defined up to an equivalence relation as follows.

Definition 1.4.2 (Equivalent cobordisms) *Two cobordisms M and N, both from Σ_1 to Σ_2, are* ***equivalent*** *if there exists a diffeomorphism $\varphi : M \longrightarrow N$ making the diagram in Figure 1.15 commute.*

$$\begin{tikzcd} & M \arrow[d, "\cong\varphi"] & \\ \Sigma_1 \arrow[ur] \arrow[dr] & N & \Sigma_2 \arrow[ul] \arrow[dl] \end{tikzcd}$$

Figure 1.15: Equivalence of cobordisms.

This indeed gives an equivalence relation defining the equivalence classes of cobordisms between two given manifolds, which will be the morphisms in the category. Composition will be given by gluing.

Definition 1.4.3 *Let M and N be two cobordisms with a common boundary Σ , along with morphisms $\iota^M_{\text{out}} : \Sigma \longrightarrow M$ and $\iota^M_{\text{in}} : \Sigma \longrightarrow M$. Then, the cobordism resulting from* ***gluing*** *M* ***and*** *N* ***along Σ*** *is*

$$M \sqcup_\Sigma N := M \sqcup N/\sim,$$

where $\sim$ is the equivalence relation given by identifying two points $p \in M$ and $q \in N$ iff there exists a point $x \in \Sigma$ such that $\iota^M_{\text{out}}(x) = p$ and $\iota^M_{\text{in}}(x) = q$.

The identity morphisms are given by cylinders (Figure 1.16).

Figure 1.16: Cylinder cobordism.

Definition 1.4.4 *Given a closed oriented manifold Σ, the* ***cylinder*** *C_Σ is defined as $\Sigma \times [0,1]$ oriented with $\Sigma \times \{0\}$ as in-boundary and $\Sigma \times \{1\}$ as out-boundary. Thus, with the canonical maps*

$$\iota_{\text{in}} : \Sigma \xrightarrow{\sim} \Sigma \times \{0\} \hookrightarrow C_\Sigma \hookleftarrow \Sigma \times \{1\} \xleftarrow{\sim} \Sigma : \iota_{\text{out}}$$

we have that C_Σ is a cobordism from Σ to itself.

One can further show that gluing a cobordism M with a cylinder doesn't change the equivalence class, obtaining thus an equivalent cobordism.

Definition 1.4.5 *The category of cobordisms* **nCob** *consists of:*

- $\mathrm{Ob}(\mathbf{nCob})$*: objects are closed oriented $(m-1)$-manifolds Σ.*
- $\mathrm{Hom}_{\mathbf{nCob}}(\Sigma, \Sigma')$*: morphisms $M : \Sigma_1 \longrightarrow \Sigma_2$ are equivalence classes of cobordisms from Σ_1 to Σ_2.*
- *The identity morphisms $\mathrm{Id}_\Sigma : \Sigma \longrightarrow \Sigma$ are the equivalence classes of cylinders C_Σ.*
- *The composition $N \circ M : \Sigma_1 \longrightarrow \Sigma_3$ of two morphisms $M : \Sigma_1 \longrightarrow \Sigma_2$ and $N : \Sigma_2 \longrightarrow \Sigma_3$ is the equivalence class of $M \sqcup_{\Sigma_2} N$.*

The composition law is given by glueing together cobordisms along a boundary.

Notice that, opposite to what is usual in most categories, the name of this one comes from its morphisms (cobordisms) instead of its objects.

1.5 Definition of a TQFT

With all the previous machinery, one can finally summarize what a TQFT is in the following categorical definition.

Definition 1.5.1 *An n-dimensional TQFT over a field $\mathbb{K}$ is a symmetric monoidal functor $\mathcal{Z}$ from* **nCob** *to* $\mathbf{Vect}_{\mathbb{K}}$*:*

$$\mathcal{Z} : \mathbf{nCob} \to \mathbf{Vect}_{\mathbb{K}}$$

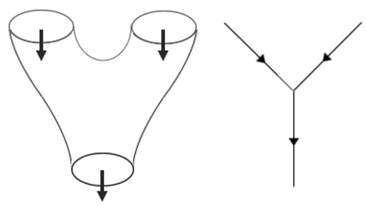

Figure 1.17: Resemblance between Feynman diagrams and cobordisms, now viewed from top to bottom. Both can be associated with the multiplication operation $A \otimes A \longrightarrow A$ in a $\mathbb{K}$-algebra A.

Here it is used the fact that $(\mathbf{Vect}_{\mathbb{K}}, \otimes, \mathbb{K}, \sigma)$ and $(\mathbf{nCob}, \sqcup, \emptyset, T)$ are symmetric monoidal categories, with (respectively) tensor product and disjoint union as tensor operations, ground field and empty set as units and twists given by $\sigma : V \otimes W \longrightarrow W \otimes V$ and $T : \Sigma \sqcup \Sigma' \xrightarrow{\sim} \Sigma' \sqcup \Sigma$.

1.6 Physical and Mathematical interests

Let us end this chapter with some further comments on the physical and mathematical interests on TQFTs, besides the ones already made in the previous pages.

As mentioned, intuitively TQFTs capture the notion of space (the closed manifolds, bordisms) and spacetime "evolving between them" (the cobordism itself) —basic ingredients in General Relativity— being assigned by the TQFT-functor to the associated vector or Hilbert spaces representing the state spaces and to an operator understood as a time-evolution operator (or more commonly known in the physics literature as transition amplitude or Feynman path integral or partition function[5]), respectively —which in turn are basic ingredients in Quantum Mechanics and Quantum Field Theory. Further, the multiplicative axiom (A4b) stating that $\mathcal{Z}(\Sigma \sqcup \Sigma') = \mathcal{Z}(\Sigma) \otimes \mathcal{Z}(\Sigma')$ expresses the common principle in quantum mechanics that the state space of two independent systems is the tensor product of the two states.

5: Recall that in quantum field theory, given the action S as a functional of field configurations φ, the partition function is given by

$$Z = \int \mathcal{D}\varphi e^{iS[\varphi]},$$

where $\mathcal{D}\varphi$ is a (not well defined) measure over all possible field configurations on all of spacetime, and then the usual time-ordered vacuum expectation value of a functional F is given by

$$\langle F \rangle := \frac{1}{Z} \int \mathcal{D}\varphi F[\varphi] e^{iS[\varphi]}$$

with Z in the denominator ensuring normalization.

Thus, although at a first glance general relativity and quantum theory use "different sorts of mathematics", one based on objects such as manifolds and the other on others such as Hilbert spaces, it turns out that both can be described categorically in a similar way, as TQFTs make manifest.

The resemblance of the TQFT-functor with the Feynman path integral formulation suggests a relation between Feynman diagrams (contributions representing terms in an expansion of the Feynman path integral or partition function) and cobordisms or even algebraic operations. For instance, the "merging of particles" diagram would correspond to the inverted pair of pants (see Figure 1.17) or the multiplication operation $A \otimes A \longrightarrow A$ in a $\mathbb{K}$-algebra. Similarly, there are analogue cobordisms for the creation, splitting and annihilation diagrams, and corresponding to the unit $\mathbb{K} \longrightarrow A$, comultiplication $A \longrightarrow A \otimes A$ and counit $A \longrightarrow \mathbb{K}$. Observe the intuitive notion of time in cobordisms accounting for the distinction between merging and splitting. Also, correspondingly, in the algebraic or categorical description, the notion of morphism involves a sense of direction: arrows from source to target.

Furthermore, TQFTs possess features one expects from a theory of quantum gravity. For instance, Feynman diagrams are replaced in string theory by so-called *worldsheets*, 2-dimensional cobordisms describing the embedding of a string in spacetime. Also, the theory of Loop Quantum Gravity (LQG) —pursuing to merge general relativity and quantum mechanics, adding matter of the Standard Model of Particle Physics to the pure quantum gravity description— can be formulated as a generalized TQFT, as shown e.g. in [Rov11].

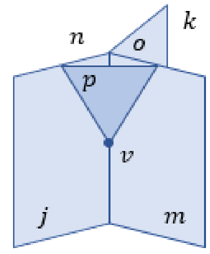

Figure 1.18: In Loop Quantum Gravity, the *spinfoam* is a topological structure representing a configuration (analogous to a Feynman diagram) taken into account in a Feynman path integral description of Quantum Gravity.

Being topological, as mentioned earlier, a TQFT serves as a baby model to do calculations and gain experience before embarking into the full theory (QFT), which is yet not fully understood and expected to be much more complicated. That the theory is topological means that transition amplitudes do *not* depend on any additional structure on spacetime like Riemannian metric or curvature, but only on the topology. In particular, there is no time-evolution along "cylindrical" spacetime.

Last, monoidality in **nCob** is given by the disjoint union and so one may understand a disjoint union of spacetimes as evolving in parallel, independent from one another. In the case of $\mathbf{Vec}_{\mathbb{K}}$, however, monoidality comes from the tensor product of vector spaces instead of the cartesian product, opening the possibility of interesting phenomenons such as quantum entanglement.

From a mathematical point of view, the classification of low dimensional manifolds has been one of the important questions attempted by both mathematicians and physicists in the last decades. A quantum field theoretic approach to these problems has shown to be an elegant technique giving consistent results and, further, topological field theories play a very important role in capturing the topological features of manifolds. In fact, TQFTs play a role in the study of knots and three-manifolds, as happens in the case of Chern-Simons theory for instance. The Hilbert space of the Chern-Simons theory is given by the space of conformal blocks of a Wess-Zumino conformal field theory, while gauge invariant topological observables are the Wilson loop operators whose expectation value give the knot invariants.

To see why TQFTs should give rise to such invariants, note that when M is a closed m-manifold so that it has no boundary, $\partial M = \emptyset$, then $\mathcal{Z}(M) \in \mathcal{Z}(\emptyset) = \mathbb{K}$ is a constant[6] (element of the ground field), being the same for any manifold in the equivalence class of M. Thus the theory produces in particular *invariants* of closed m-manifolds.

6: Alternatively, $\mathcal{Z}(M) \in \mathcal{Z}(\emptyset^* \sqcup \emptyset) = \mathbb{K}^* \otimes \mathbb{K} \cong \mathrm{Hom}(\mathbb{K}^* \otimes \mathbb{K}, \mathbb{K}^* \otimes \mathbb{K})$ giving a linear map $f : \mathbb{K} \longrightarrow \mathbb{K}$ which is equivalent to giving a constant $k = f(1) \in \mathbb{K}$.

These notions will be explored later on by studying the Chern-Simons TQFT and its relation with the well-known Jones Polynomial giving invariants of knots.

Knot Theory and Dehn Surgery | 2

So, what's the deal with 'knot theory'? Is it theory... or knot?

A knot is simply a tangled loop in ordinary three-dimensional space, such as often causes us frustration in everyday life. Knots are also the subject of a rather rich mathematical theory. In the last three decades, it has unexpectedly turned out that rather deep aspects of the theory of knots are best understood in the context of 20th and 21st century developments in quantum physics. In his talk Knots and Quantum Theory, Edward Witten —Charles Simonyi Professor in the School of Natural Sciences— attempts to explain what quantum theory has to do with knots.

Insitute for Advanced Study[1]

1: See https://www.ias.edu/video/witten-friends

Let us now introduce the reader to some relevant topological notions concerning links and 3-manifolds. As we shall see in coming chapters, this theory yields a connection between Chern-Simons theory and the Jones polynomial appearing in Knot Theory. The key result is the theorem asserting that any closed orientable 3-manifold M can be obtained from $\mathbb{S}^3$ by an integral surgery on a link $\mathcal{L} \subset \mathbb{S}^3$. We introduce the required notions to understand this fact and its consequences and applications. We show that any such 3-manifold can be described by a plumbed graph with decorated nodes, and how different graphs giving rise to the same 3-manifold are related by a sequence of moves, known as *Kirby moves*. These are useful tools for the construction of invariants of links and three-manifolds. We mainly follow [Sav12; PK16].

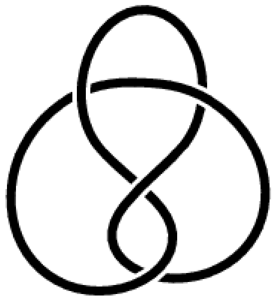

Figure 2.1: Diagram of the Figure Eight knot.

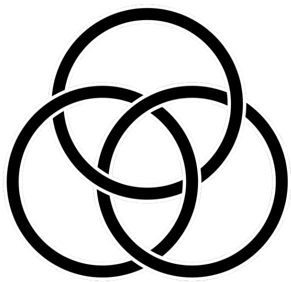

Figure 2.2: Example of a link: the borromean rings.

2.1 Knots and Links in 3-manifolds

We all are familiar with the everyday life notion of tying a knot to fix our shoelaces or the experience of finding our earphones in a tangled mess of strings. Mathematicians refer to a *knot* as an abstraction of this concept, meaning a possibly tangled loop freely floating in ordinary space. What concerns mathematicians is thus the tangle itself.

Definition 2.1.1 (Knot, link) *A finite collection of smoothly embedded closed curves in a closed orientable 3-manifold M*

$$\mathbb{S}^1 \sqcup \cdots \sqcup \mathbb{S}^1 \hookrightarrow M$$

*is called a **link**. A one-component link is called a **knot**.*

Pictorically, one has in mind knots and links as e.g. the ones in Figure 2.1 and Figure 2.2, thinking them inside $\mathbb{R}^3$. Recall that a *closed* manifold is a compact one without boundary. The requirement that each of the curves of a link be smoothly embedded[2] avoids pathological examples

2: Let M, N be smooth manifolds with dimensions $m \leq n$. A smooth map $f : M \to N$ is called an **immersion** if $d_x f : T_x M \to T_{f(x)} N$ is injective for all $x \in M$. The map f is an **embedding** if it is an immersion which further is injective and proper, i.e. the preimage of every compact set in N is compact in M.

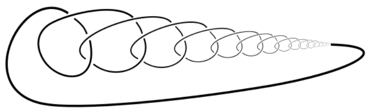

Figure 2.3: Pathological cases to be avoided when defining the mathematical notion of a knot.

like the ones in Figure 2.3. One does not distinguish between *equivalent* knots and links, the ones that can be continuously obtained from one another by just "wiggling the string" without ever snapping it.

Definition 2.1.2 (Equivalent links) *Two links $\mathcal{L}$ and $\mathcal{L}'$ in M are said to be* ***isotopically equivalent*** *if there is a smooth orientation preserving*[a] *automorphism $h : M \to M$ such that $h(\mathcal{L}) = \mathcal{L}'$.*

[a] In the case where links have two or more components, one also assigns a fixed ordering of the components and requires that h respects the ordering.

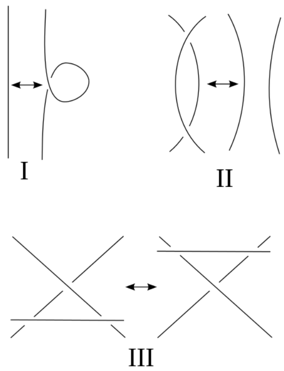

Figure 2.4: Reidemeister moves *I*, *II* and *III*.

To know whether two knots are isotopically equivalent or not, there exists a minimal set of moves called **Reidemeister moves** (the ones in Figure 2.4) which are known to preserve the isotopy class. That is, any two equivalent knots can be obtained from one another by a finite number of Reidemeister moves. These moves are applied to knot *diagrams*, i.e. a regular projection of the knot onto a plane which enables the drawing of the knot.

Definition 2.1.3 *Let P be a plane in $\mathbb{R}^3$ and $\pi : \mathbb{R}^3 \to P$ the orthogonal projection. Given a link $\mathcal{L}$ in $\mathbb{R}^3$, one says that π is a* ***regular projection*** *for $\mathcal{L}$ if every line $\pi^{-1}(x)$, $x \in P$, intersects $\mathcal{L}$ in 0, 1 or 2 points and the Jacobian $d_y\pi$ has rank 1 at every intersection point $y \in \pi^{-1}(x)$.*

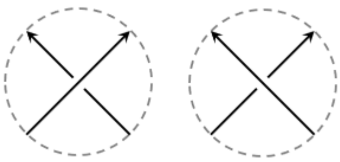

Figure 2.5: Possible intersections: overcrossing and undercrossing.

Observe that we thus only allow simple *crossings*, as the ones depicted in Figure 2.5. The nice thing about these projections is that the following holds:

Proposition 2.1.1 *Every link admits a regular projection.*

Thus, links are often described by their regular projections, and drawn as smooth curves in $\mathbb{R}^2$ with marked undercrossings and overcrossings at each double point. The simplest knot is the one whose projection is (isotopically equivalent to) a circle:

Definition 2.1.4 *Any knot equivalent to the knot* $(\cos t, \sin t, 0)$, $0 \leq t \leq 2\pi$, *is called a* ***trivial knot*** *or an* ***unknot***.

It is clear that in general distinguishing two arbitrary knots is not an easy task at all (think of knots such as the one in Figure 2.7). For this reason, one searches for powerful *invariants*. An **invariant** is a mathematical object (such as a number or a polynomial) assigned to a given knot, which remains unchanged under Reidemeister moves so that it is well defined in the isotopy class of the knot. Different knots are then known to be different if their corresponding invariants are not the same. However, in principle the assignment need not be injective, so that two knots with the same invariant may or may not be equivalent. This is a major problem in the *classification of knots*. Hence, one ideally aims for perfectly injective assignments and says that the invariant is stronger or more powerful the more unequivalent knots it can distinguish (the more closer to being injective).

One common approach is to deal with polynomial invariants associated to each knot. These are constructed ensuring they respect the Reidemeister moves so that they indeed constitute topological invariants. The earliest such polynomial known is the Alexander polynomial and a more recent and famous one is the Jones polynomial. Since Jones' times, different generalizations of these have appeared (some involving more variables) being even more powerful. These polynomials are often obtained through a recursion relation, called **skein relation**, which enables to compute a knot invariant in terms of the invariants corresponding to simpler knots, obtained in each step by replacing a specific crossing (say, a positive one) by an undercrossing or no crossing (see Figure 2.6), finally obtaining some copies of unknots. A normalization is therefore usually given by fixing a value for the unknot.[3]

3: See [GS14] for more details and examples.

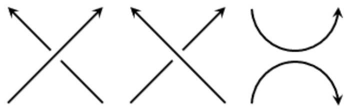

Figure 2.6: Overcrossing (+), undercrossing (−) and no crossing (0), respectively, appearing in skein relations.

For example, the Alexander polynomial in the variable q, denoted $P_+(q)$ for a link $\mathcal{L}_+$, is defined by means of the skein relation

Alexander polynomial

$$P_+(q) - P_-(q) = \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) P_0(q)$$

with normalization $P_U(q) = 1$ for the unknot U. It cannot, however, distinguish mirror knots and gives zero for disjoint unions of knots. It was around sixty years later that the stronger, Jones polynomial was found. It is given by the skein relation

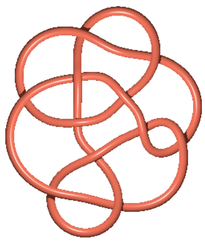

Figure 2.7: Example of a complicated knot with multiple crossings. The corresponding invariant can be computed recursively by iteratively replacing a specific crossing by the opposite one and no crossing at all, and computing the invariant corresponding to each of the two new and simpler knots obtained.

Jones polynomial

$$q^{-1}V_+(q) - qV_-(q) = \left(q^{-\frac{1}{2}} - q^{\frac{1}{2}}\right) V_0(q),$$

which is a slight modification of the Alexander one. This can, for example, distinguish a knot K from its mirror image K^* and their invariant polynomials satisfy the symmetry $V_{K^*}(q) = V_K(q^{-1})$. Further and more interestingly, the Jones polynomial for disjoint links is given by the product of invariants for each component, similar to the multiplicative expected property in QFT for the expectation value of uncorrelated observables. Since Jones' contribution, some other generalizations have been constructed and research in this field is still active. For example, a two-variable generalization known as HOMFLY-PT[4] polynomial, which is even more powerful. Its skein relation is given by

4: Name due to the different groups involved in this proposal.

HOMFLY-PT polynomial

$$w^{-\frac{1}{2}}q^{-\frac{1}{2}}P_+(w,q) - w^{\frac{1}{2}}q^{\frac{1}{2}}P_-(w,q) = \left(q^{-\frac{1}{2}} - q^{\frac{1}{2}}\right) P_0(w,q).$$

Observe that it recovers the Jones polynomial for $w = q$ and the Alexander polynomial for $w = q^{-1}$. Such polynomials are shown to be given by Chern-Simons theory, as we will present later in this thesis following E. Witten's paper [Wit89]. Unfortunately, none of these polynomials solves the *classification problem* in knot theory and further polynomial invariants are still being searched.[5]

5: In the final chapter of this thesis we will show our results on a particular example corresponding to the choice of the $\mathfrak{so}_8$ Lie algebra and the r-th symmetric representations.

Here, we will focus on topological aspects concerning the construction of manifolds through surgery on knots and links. This will lead to invariants of manifolds in the framework of the *classification of low dimensional manifolds*, closely related to the one for knots.

For this purpose, we will be interested in considering specific neighborhoods of knots, sketched in Figure 2.8 and defined below.

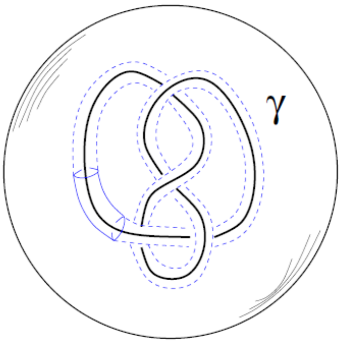

Figure 2.8: A figure eight knot γ in $\mathbb{S}^3$ with its tubular neighborhood.

Definition 2.1.5 *Every link $\mathcal{L} \subset M$ can be thickened to a* ***tubular neighborhood*** *$N(\mathcal{L})$ consisting of a collection of smoothly embedded disjoint tori, one for each link component, whose cores $\{0\} \times \mathbb{S}^1$ form the link $\mathcal{L}$.*[a]

[a] The tubular neighborhood of a knot is the embedded solid torus and thus possibly tangled. However, it is commonly identified with the solid torus $D^2 \times \mathbb{S}^1$ itself.

Links will be considered to live in the compactification of 3-dimensional euclidean space, $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$, while being thought of as links in $\mathbb{R}^3$ (see illustration in Figure 2.8). The reason for working with $\mathbb{S}^3$ is obviously its compactness (further, it is the only closed 3-manifold of Heegard genus 0).[6]

6: There is an alternative equivalent definition of the 3-sphere $\mathbb{S}^3$ in terms of quaternions. Recall their definition $\mathbb{H} := \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}$ and $i^2 = j^2 = k^2 = -1\}$ with the norm defined as $||a + bi + cj + dk||^2 = a^2 + b^2 + c^2 + d^2$. Then, the three sphere is $\mathbb{S}^3 := \{h \in \mathbb{H} \mid ||h|| = 1\} \subset \mathbb{H}$. One can show that it is a Lie group and further that $\mathbb{S}^3 \cong SU(2) \cong Sp(1)$ and $\mathbb{S}^3/\{\pm 1\} \cong \mathbb{R}P^3 \cong SO(3)$.

2.2 Surgery on Links in a 3-manifold

With these notions at hand, let us introduce the concept of surgery on links in a given 3-manifold. First we describe our working tools.

Definition 2.2.1 *Given two topological spaces X and Y and a continuous map $f : Z \subset X \to Y$, consider the disjoint union $X \cup Y$ and the equivalence relation $z \sim f(z)$ iff $z \in Z$. Then,*

$$X \cup_f Y := (X \cup Y)/\sim$$

with the quotient topology is said to be the space obtained by ***gluing X and Y along f***.[a]

[a] In most cases, f will be a homeomorphism of Z onto $f(Z) \subset Y$.

We want also to have an “inverse” operation to the gluing of spaces. This is given by *cutting* the space open along a subspace.

Definition 2.2.2 *Let X be a connected space and $Y \subset X$ a closed subspace such that the closure of $X \backslash Y$ recovers X, and let $X \backslash Y$ consist of a finite number of connected components $X_1, \ldots, X_n$. Define the space*

$$X' = \bigcup X_i \times \{i\} \subset X \times \mathbb{R},$$

“pulling the components apart”. Then, the closure of X' in the product topology on $X \times \mathbb{R}$ is the result of what is referred to as ***cutting X open along Y***.

One may visualize these pictures by considering a torus. Construct it by starting with a square and identifying opposite edges, as in Figure 2.9. Observe that the identified points along which we glue become thus the same topological point. Notice the importance of the orientation in the gluing process. Now, in this example, cutting along a circle would correspond to the converse of the last depicted step. Doing it along two circles, we would be left with two cylinders (i.e. two different connected components apart from each other).

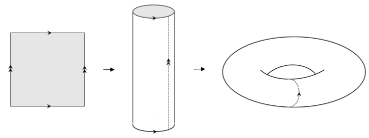

Figure 2.9: Gluing opposite edges of a square, with the shown orientation each, to give a torus.

In fact, tori will be key tools for us. In a more general setting:

Definition 2.2.3 *The n-**torus** is the product of n circles:*

$$T^n := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1.$$

It is just a generalization of the usual 2-torus $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$, already encountered as given by a "doughnut" shaped surface. We are now ready for the definition.[7]

7: Notice that we are going to *glue* and *cut along* a torus. Hence the points to be identified are the ones composing the whole surface, which is not as easy to visualize as the previous example where we glued along curves only.

Definition 2.2.4 (Dehn surgery) *Given a knot k in a closed orientable 3-manifold M and its tubular neighborhood $N(k)$, cut the manifold open along the embedded 2-torus $\partial N(k)$ obtaining two manifolds: on the one hand the **knot exterior** $K := M \backslash \mathrm{int} N(k)$, and on the other the embedded solid torus $N(k)$. In this way, K is a manifold with boundary $\partial K = T^2$ and —abusing notation— one has $M = K \cup (D^2 \times \mathbb{S}^1)$, where $D^2 \times \mathbb{S}^1$ refers to $N(k)$ identified through the embedding. Finally, using an arbitrary homeomorphism $h : \partial D^2 \times \mathbb{S}^1 \to \partial K$ to glue $D^2 \times \mathbb{S}^1$ back in K, one obtains the space*

$$Q := K \cup_h (D^2 \times \mathbb{S}^1),$$

*which is a closed orientable 3-manifold. It is then said that Q is obtained from M by **Dehn surgery** along k.*

Usually one takes $M = \mathbb{S}^3$ and then K is called the **knot complement**. Observe that a knot complement (the complement of an embedded solid torus in $\mathbb{S}^3$) is also an embedded torus in $\mathbb{S}^3$. Indeed, link the two tori through their holes and blow up one of them to all of $\mathbb{S}^3$.[8] This extends through the embedding.

8: This can be done thanks to the extra point ∞ giving the compactification of $\mathbb{R}^3$, since it is where the second torus closes back to itself.

Observe that the manifold Q depends on the chosen homeomorphism h. It turns out that the manifold Q is completely determined by the image of the *meridian* $\partial D^2 \times \{*\}$ of the solid torus $D^2 \times \mathbb{S}^1$; that is, by the curve $c = h(\partial D^2 \times \{*\})$ lying on the boundary of K.

Regarding this last statement, we pause now for a moment and devote a section to the *mapping class group* of T^2, before coming back to relevant results on surgery in the three sphere.

2.3 The Mapping Class Group of T^2

Recall first the concept of a *handlebody* and a *Heegard splitting*, for which we will need to bear in mind some other notions.[9] [10] [11]

9: Let A, B be two topological spaces and consider its cartesian product $A \times B$ with the product topology. Then, its boundary is given by:

$$\partial(A \times B) = (\partial A \times B) \cup (A \times \partial B).$$

10: The space enclosed by an n-**sphere** $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\}$ is called an $(n+1)$-**ball**, D^{n+1}. Thus:

$$\partial D^{n+1} = \mathbb{S}^n.$$

Notice that, with this at hand, we have $\mathbb{S}^3 = \partial D^4 = \partial(D^2 \times D^2) = (\mathbb{S}^1 \times D^2) \cup (D^2 \times \mathbb{S}^1)$ which shows the previously mentioned decomposition of the 3-sphere into two solid tori glued along their common boundary, the torus $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$.

11: Given an m-manifold M, a k-**handle** is defined as

$$H^k := D^k \times D^{m-k},$$

where D^k is a k-ball, with $0 \leq k \leq m$. The k-handles are then attached to the boundary of M along $(\partial D^k) \times D^{m-k}$ using an embedding $f : (\partial D^k) \times D^{m-k} \to \partial M$. The corners that arise can be smoothed out and hence $M \cup_f (D^k \times D^{m-k})$ is again a smooth manifold.

Definition 2.3.1 *The **handlebody** of **genus** g is the orientable 3-dimensional manifold given by attaching g 1-handles $H^1 = D^2 \times [-1, 1]$ to a 3-ball D^3. The gluing homeomorphisms match the $2g$ discs $D^2 \times \{\pm 1\}$ with $2g$ disjoint 2-discs in $\partial D^3 = \mathbb{S}^2$ in such a way that the resulting manifold is orientable (See Figure 2.10).*[a]

[a] Remark: the boundary of a handlebody of genus g turns out to be homeomorphic to a Riemann surface of genus g, hence the word *genus*.

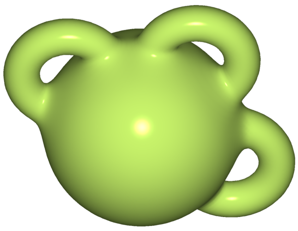

Figure 2.10: Representation of a handlebody.

Interestingly, any closed orientable 3-manifold M can be obtained by gluing together two handlebodies with the same genus.

Definition 2.3.2 *Given a closed oriented 3-manifold M, a **Heegaard splitting** of genus g is $M = H \cup H'$, where H and H' are handlebodies of genus g such that $H \cap H' = \partial H = \partial H'$.*

Theorem 2.3.1 *Any closed orientable 3-manifold admits a Heegaard splitting of some genus.*

This is interesting because it allows us to talk about closed orientable 3-manifolds and surgeries on them in terms of the handlebodies, as was actually done in the definition of Dehn surgery. One just has to study the gluing homeomorphisms. Given a Heegard splitting $H \cup_f H'$, meaning that the handlebodies H and H' are glued by a homeomorphism $f : F \to F$ along their common boundary F, the *mapping class group* will measure which of the different homeomorphisms give rise to the "same" manifold. Before giving a precise definition, recall first the necessary notion of *isotopy*.

Definition 2.3.3 *Two homeomorphisms $f_0, f_1 : F \to F$ are said to be **isotopic** whenever there exists a homotopy f_t, $0 \le t \le 1$, between them such that each f_t is a homeomorphism.*

Observe that all homeomorphisms isotopic to an orientation preserving (reversing) homeomorphism f are orientation preserving (reversing, respectively). It can be shown that gluing two handlebodies H and H' by isotopic homeomorphisms produces homeomorphic manifolds. This observation justifies the following definition.

Definition 2.3.4 (Mapping class group) *Denote by* $\mathrm{Homeo}(F)$ *the group*[a] *of orientation preserving homeomorphisms $f : F \to F$ of a closed oriented surface F, and* $\mathrm{Homeo}_0(F)$ *for the normal subgroup consisting of the homeomorphisms that are isotopic to the identity. Then, the **mapping class group** of the surface F is the quotient group*

$$\mathcal{MCG}(F) := \mathrm{Homeo}(F) \big/ \mathrm{Homeo}_0(F).$$

[a] With composition $\circ$ as group operation.

Notice that the mapping class group is a subgroup of the larger group consisting of all homeomorphisms of F modulo isotopy. Notice also that the composition of any two orientation reversing homeomorphisms is orientation preserving. We now describe a set of generators for $\mathcal{MCG}(F)$ known as *Dehn twists.*

Figure 2.11: Dehn twist.

Definition 2.3.5 (Dehn twist) *Let F be a closed orientable surface. Let c be a simple closed curve (the embedding of a circle) in F and consider an annulus $U(c)$ one of whose two boundary components is c. Identifying $U(c)$ with the annulus $\{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$, a* ***Dehn twist*** *$\tau_c : F \to F$ along the curve c is defined as the homeomorphism given by*

$$r \cdot e^{i\phi} \mapsto r \cdot e^{i(\phi+2\pi(r-1))}$$

inside th annulus $U(c)$, and by the identity outside.

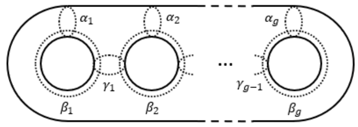

Figure 2.12: Sketch of the generators of the mapping class group of a closed orientable surface of genus g.

A practical way to think of a Dehn twist (see Figure 2.11) is as the result of *stretching around*; i.e. the result of cutting F along c, giving a one whole turn twist to one of the ends in one of the possible two directions and then gluing the ends back together. Different choices of $U(c)$ or of the curve c within its isotopy class yield isotopic twists. Choosing the opposite twist direction gives the inverse element in $\mathcal{MCG}(F)$. These twists give the desired set of generators.

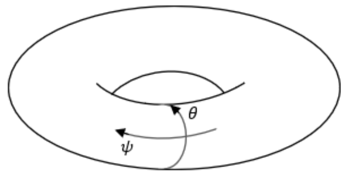

Figure 2.13: Meridian and longitude on a torus.

Theorem 2.3.2 *Given a closed orientable surface F_g of genus g, the mapping class group $\mathcal{MCG}(F_g)$ is generated by the $3g-1$ Dehn twists along the curves $\alpha_i, \beta_j, \gamma_k$, $1 \leq i, j \leq g$, $1 \leq k \leq g-1$, shown in Figure 2.12.*

Having presented this machinery, consider now the case relevant to us consisting in a 2-torus $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$. We are going to study its mapping class group.

First, pick two generators of its fundamental group[12] $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$ as follows. Regard T^2 as the boundary of a solid torus $D^2 \times \mathbb{S}^1$ as in Figure 2.13. Then, denoting by θ and ψ the standard angle coordinates on $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$, the curves μ and λ determined respectively by the equations $\psi = 0$ and $\theta = 0$ are called the **meridian** and **longitude**. These curves play different roles, since μ bounds a disc in $D^2 \times \mathbb{S}^1$ while λ does not and can thus cannot be contracted to a trivial curve (it is not isotopic to the identity). Meridian and longitude constitute then a set of generators for $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$, as desired (See Figure 2.14). An orientation on the torus is given by the choice of basis $(\partial/\partial\psi, \partial/\partial\theta)$ in the tangent space.

12: Recall that the **fundamental group** of a topological space X is the group of equivalence classes under homotopy of the loops in the space, relative to a base point. The group operation is given by composition of loops, "travelling twice as fast". For a path-connected space, the base point makes no difference up to isomorphism and we write $\pi_1(X)$.
The fundamental group records information about the basic shape or holes of the topological space. Moreover, it is the first and simplest homotopy group. The fundamental group is a **homotopy invariant**: homotopy equivalent path-connected spaces have isomorphic fundamental groups:

$$X \simeq Y \Rightarrow \pi_1(X) \cong \pi_1(Y).$$

The abelianization of the fundamental group can be identified with the first homology group of the space.

We are now ready to explicitly describe the mapping class group of the torus. It goes as follows. Observe first that the π_1 functor[13] transforms a homeomorphism f of T^2 into a group automorhism f_* of $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$ and in particular converts a homeomorphism of T^2 isotopic to identity into the identity map on $\pi_1(T^2)$. Now, the automorphisms of $\mathbb{Z} \oplus \mathbb{Z}$ are extremely simple since they can be described by integral 2×2-matrices invertible over the integers, which means that its determinant must be ± 1 to avoid any fraction when inverting elements.[14] Among these, the matrices with unit determinant associated to automorphisms f_* are in one-to-one correspondence with the orientation preserving homeomorphisms f, thus giving a well-defined homomorphism

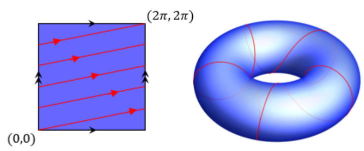

Figure 2.14: A torus knot. Curves on a torus are defined by pairs (a, b) corresponding to the number of meridian and longitude turns, respectively.

$$\Pi : \mathcal{MCG}(T^2) \to SL(2, \mathbb{Z})$$

where $SL(2,\mathbb{Z})$ is the group 2×2-matrices over the integers, with unit determinant. Each matrix $A \in SL(2,\mathbb{Z})$ can be given by a product of matrices of the form

$$\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}$$

since any element in $SL(2,\mathbb{Z})$ can be reduced to the identity by performing elementary transformations on its rows and columns. The orientation preserving homeomorohisms corresponding to these matrices are the twists along the curves μ and λ described before, showing that Π is surjective. Furthermore, it can actually be proven that:

Theorem 2.3.3 *The map $\mathcal{MCG}(T^2) \to SL(2,\mathbb{Z})$ is an isomorphism.*

13: If $f : X \to Y$ is a continuous map, $x_0 \in X$ and $y_0 \in Y$ with $f(x_0) = y_0$, then every loop in X with base point x_0 can be composed with f to yield a loop in Y with base point y_0. This operation is compatible with the homotopy equivalence relation and with composition of loops. The resulting group homomorphism, called the **induced homomorphism**, is written as $\pi_1(f)$ or $f_* : \pi_1(X,x_0) \to \pi_1(Y,y_0)$. The necessary compatibility conditions allow one to state that π_1 is actually a functor

$$\begin{aligned} \pi_1 : \mathbf{Top}_* &\to \mathbf{Grp} \\ (X,x_0) &\mapsto \pi_1(X,x_0) \end{aligned}$$

from the category of topological spaces together with a base point to the category of groups.
Last to mention, the fundamental group functor takes products to products. That is, if X and Y are path-connected, then

$$\pi_1(X\times Y) \cong \pi_1(X)\times\pi_1(Y).$$

14: We have:

$$\begin{pmatrix} -q & s \\ p & r \end{pmatrix}^{-1} = \frac{-1}{qr+ps}\begin{pmatrix} r & -s \\ -p & -q \end{pmatrix}.$$

With all these results at hand, one can proceed to describe the 3-manifolds of Heegaard genus 1, such as the torus we are interested in. Indeed, as has been done before, consider a manifold M obtained by the process of gluing two solid tori by an orientation reversing homeomorphism of their boundaries $f : T^2 \to T^2$. Choosing the meridian-longitude basis on each torus, (μ_1,λ_1) and (μ_2,λ_2), the matrix corresponding to the homeomorphism f has the form

$$A = \begin{pmatrix} -q & s \\ p & r \end{pmatrix} \quad \text{with} \quad qr+ps = 1.$$

Observe that the image of the meridian μ_1 on the first torus is then isotopic to the curve $-q\cdot\mu_2 + p\cdot\lambda_2$ on the second torus, winding $-q$ times in the θ_2-direction and p times in the ψ_2-direction.

We finally reach the last statement claimed in the previous section:

Lemma 2.3.4 *The image of the meridian μ_1 completely determines the manifold M.*

This means that the manifold M is completely determined by just two integer numbers p and q.[15] Such manifold is what we call a **lens space**, denoted $L(p,q)$, which will be used in the next section. The reason for their relevance will be clear in a moment. Notice that the previous unit determinant condition $qr+ps=1$ on the matrix A tells us that p and q must be relatively prime numbers.

15: This fact will be very important for the surgeries we will describe next.

Now, it must be pointed out the fact that different pairs (p,q) may give the same lens space $L(p,q)$ up to homeomorphism. The reason for this lies in an ambiguity in the choice of basis curves on T^2 and in the non-uniqueness of longitude choice.

- Concerning the first case, the meridians μ_1 and μ_2 are uniquely determined[16] by the condition of bounding a 2-disc. Changing the orientation of μ_1 entails a change in the orientation of λ_1 as well, replacing thus A by $-A$. Therefore, one may assume $p \geq 0$.

- As for the second reason, notice that any curve of the form $n\cdot\mu_1 + \lambda_1$ maps to λ_1 by n Dehn twists along μ_1, thus being as good as the latter as candidate choice for a longitude.

16: Up to isotopy and change of orientation

Now, for $p = 0$ one may assume that $A = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and so the corresponding lens space $L(0,1)$ is $\mathbb{S}^1 \times \mathbb{S}^2$. Suppose then that $p > 0$, thus being able reduce the cases by making q non-negative and less than p, i.e. $0 \leq q \leq p-1$. If $p = 1$ then necessarily $q = 0$, so one can assume that $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and thus $L(1,0)$ is the three-sphere[17] $\mathbb{S}^3$. Hence, in any other case $p \geq 2$ and $1 \leq q \leq p-1$, and so the following result holds.

17: Which is a manifold of Heegard genus 0.

Theorem 2.3.5 *Any 3-dimensional manifold of genus 1 is either* $\mathbb{S}^1 \times \mathbb{S}^2$ *or a lens space* $L(p,q)$ *with* p *and* q *relatively prime,* $p \geq 2$*, and* $1 \leq q \leq p-1$.

Last, the construction of lens spaces can be generalized to give:

Definition 2.3.6 (Seifert manifold) *Consider the surface* $F = \mathbb{S}^2 \backslash \mathrm{int}(D_1^2 \cup \cdots \cup D_n^2)$ *consisting of a 2-sphere after removing the interior of* n *disjoint discs. Then the product* $F \times \mathbb{S}^1$ *will be a compact orientable 3-manifold with* n *tori* $(\partial D_i^2) \times \mathbb{S}^1$, $i = 1, \ldots, n$, *as boundary. Its fundamental group* $F \times \mathbb{S}^1$ *has a presentation in terms of generators and relations as*

$$\langle x_1, \ldots, x_n, h \mid hx_i = x_i h, \quad x_1 \ldots x_n = 1 \rangle,$$

the generators x_i *representing the curves* ∂D_i^2 *oriented as the boundary curves of* F*. Now, gluing in* n *solid tori so that the meridian of the i-th solid torus is glued to a curve on* $(\partial D_i^2) \times \mathbb{S}^1$ *isotopic to* $a_i \cdot x_i + b_i \cdot h$ *—with* (a_i, b_i) *pairs of relatively prime integers,* $a_i \geq 2$*—, the obtained closed manifold is called the* ***Seifert manifold*** $M((a_1,b_1), \ldots, (a_n,b_n))$ *with* n *singular fibers.*

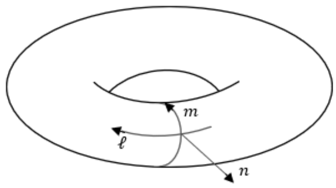

Figure 2.15: Orienting the boundary $\partial K \cong T^2$ of the tubular neighborhood.

2.4 Surgery on Links in $\mathbb{S}^3$

With this knowledge on the mapping class group of the torus, we turn now to surgery on links on $M = \mathbb{S}^3$, where a curve on $\partial K \cong T^2$ will as well be given —up to isotopy— by a pair (p,q) of relatively prime integers. The construction is detailed in the following, relying on the results in the last section through the homeomorphism between tubular neighborhoods and tori.[18]

18: Since the tubular neighborhood is an embedded torus, hence in particular homeomorphic.

Observe first that the knot complement K has integral homology groups[19] $H_0(K) = H_1(K) = \mathbb{Z}$ and $H_i(K) = 0$ if i ≥ 2. Then, any **meridian** of $N(k)$ —the image of the torus meridian through the embedding $N(k) \cong T^2$— is a generator of $H_1(K)$, a curve on ∂K we will denote by m. Concerning the **longitude**, there is —up to isotopy— a unique one[20] determined by the condition that it be homologically trivial in K, a curve on ∂K we will denote ℓ. Thus, we have a basis $\{m, \ell\}$ for $H_1(\partial K)$ that is unique modulo isotopy and reversing the orientations of m and ℓ.

19: We refer the reader to Appendix C on Homology and Cohomology Theory.

20: Called *canonical* longitude.

As for the orientations, chose first the standard orientation on $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$, thus inducing an orientation on K. Then, directions on m and ℓ are chosen so that the triple $\langle m, \ell, n \rangle$ has the positive orientation, where n denotes the vector normal to ∂K pointing inwards to the knot complement K (see Figure 2.15).

Notice that, in this way, any simple curve c on ∂K will now be —up to isomorphism— of the form

$$c = p \cdot m + q \cdot \ell,$$

i.e. completely determined by a pair (p, q), which may be conveniently thought of[a] as a reduced fraction $p/q \in \mathbb{Q} \cup \{\infty\}$, the so called **surgery coefficient** of K.

[a] Recall that p and q are relatively prime.

This means that we have a one-to-one correspondence between reduced fractions p/q and isotopy classes of non-trivial simple closed curves on the "torus" ∂K, completed with $1/0 = \infty$ corresponding to the meridian m. The surgery along such curves is called p/q-**surgery**, or **rational surgery**, and in particular 1/0-surgery on any knot $k \subset \mathbb{S}^3$ gives $\mathbb{S}^3$ again.

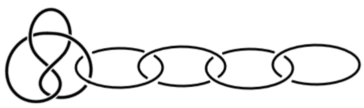

Figure 2.16: Link with an integer framing. In general, framings are given by rational numbers a_i/b_i.

Definition 2.4.1 *A surgery is said to be **integral** if* $q = \pm 1$.

Rational and integral surgeries along a link $\mathcal{L} \subset M$ are defined in a similar way, by taking rational (integral) surgeries along each link component. In the case of links in the 3-sphere $\mathbb{S}^3$ there is a canonical choice —as mentioned before—of the longitudes ℓ_i, which are taken to be null-homologous in the knot complement K. The key theorem then reads that *integral* surgeries suffice.

Theorem 2.4.1 (Lickorish and Wallace) *Every closed orientable 3-manifold M can be obtained from $\mathbb{S}^3$ by an integral surgery on a link $\mathcal{L} \subset \mathbb{S}^3$.*

Conclusion

Any closed orientable 3-manifold can be given by an integral surgery along a link $\mathcal{L} \subset \mathbb{S}^3$. The result depends both on the link $\mathcal{L}$ *and* on the chosen simple closed curves in the boundary $\partial N(k)$ of each link component k, these being uniquely determined by reduced fractions p/q, including the case 1/0.

This leads to the following definition.

Definition 2.4.2 *A **framed link** is a link $\mathcal{L}$ with a **framing**, i.e. a choice of one such fraction for each link component.*

Thus, any closed orientable 3-manifold can be specified by a framed link, as in Figure 2.16. And, since an integral surgery corresponds to a link framed by integers, the previous theorem guarantees us that it suffices to consider only integer-framed links. The precise reduction to the integer case is given by the following proposition, by means of a so-called *continued fraction decomposition.*[21]

21: **Continued fraction decomposition**, defined inductively: consider a real number r. Let $i = \lfloor r \rfloor$ be the integer part of r and let $f = r - i$ be the fractional part of r. Then the continued fraction decomposition of r is $[i, a_1, a_2, \dots]$, where $[a_1, a_2, \dots]$ is the continued fraction decomposition of $1/f$. The algorithm stops when $f = 0$ (corresponding to r being an integer number). This procedure yields the expression:

$$r = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

If r is rational, the sequence is finite.

Proposition 2.4.2 *Let k be a knot in $\mathbb{S}^3$. Then, the 3-manifold obtained by a rational p/q-surgery on k can also be obtained by an integral surgery on the link consisting of k together with a chain of unknots with integer framings determined by a continued fraction decomposition $p/q = [x_1, \dots, x_n]$, as shown in Figure 2.17*

Figure 2.17: Equivalence of a rational p/q-surgery on a trefoil and an integral surgery on a link composed by a chain of unknots knotted to the trefoil and framed by a continued fraction decomposition of p/q.

The appearance of continued fraction decompositions into the picture will be clear in the following, when studying surgeries of lens spaces and Seifert manifolds.

2.5 Surgery decription of Lens spaces and Seifert manifolds

Recall how we introduced the notion of a lens space $L(p, q)$ when considering the mapping class group of a torus, where we saw that a manifold obtained by surgery on a torus was determined by the image of the meridian and thus by only two coprime integers p and q.

Let now $p \geq 2$ and consider first the lens space $L(p, 1)$, obtained by gluing two solid tori along their boundaries by the homeomorphism

$$\begin{pmatrix} -1 & 0 \\ p & 1 \end{pmatrix}$$

matching the meridian μ_1 of the first torus to the curve $-\mu_2 + p \cdot \lambda_2$ on the second.

To visualize the gluing, turn the second solid torus inside out and view it as a being a trivial knot exterior —so that it complements the first torus in $\mathbb{S}^3$. Then, the meridian μ_1 will now be matched with the curve $\ell - p \cdot m$ on ∂K. Thus $L(p, 1)$ has surgery description given by a trivial knot framed by $-p$.

Any lens space $L(p, q)$ will similarly be given by a rational surgery on a $(-p/q)$-framed unknot. The construction, however, is slightly more involved. One first replaces one of the solid tori $\mathbb{S}^1 \times D^2$ by $\mathbb{S}^1 \times \Theta^2$ with Θ^2 being an annulus (so we have removed a core torus from inside the torus). Following the previous construction for $L(p, 1)$ give then a manifold with boundary consisting of a torus (the one corresponding to the part missing from the annulus: $(\mathbb{S}^1 \times D^2) \backslash (\mathbb{S}^1 \times \Theta^2)$. Schematically (Figure 2.18), we have a $(-p)$-surgery described by an unknot with $-p$ framing, but with a missing torus yet to be "filled in".[22]

22: The Lens space $L(p, 1)$ can be obtained from it by gluing in a solid torus by the homeomorphism

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

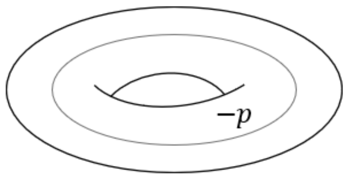

Figure 2.18: Sketch of the obtained manifold with boundary a torus.

The same procedure is carried out with any integer q coprime to p. The two resulting surgered solid tori as just described are then glued together along their boundaries (the missing tori) by the homeomorphism

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

obtaining $\mathbb{S}^3$ surgered along the link depicted in Figure 2.19.

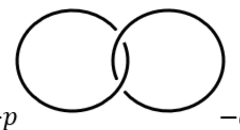

Figure 2.19: Surgery description of $L(pq - 1, q)$ with framings $-p$ and $-q$.

To see the lens space to which this picture corresponds, we observe that on the other hand the resulting gluing homeomorphism is given by the composition

$$\begin{pmatrix} -1 & 0 \\ p & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ q & 1 \end{pmatrix} = \begin{pmatrix} -q & -1 \\ pq-1 & p \end{pmatrix},$$

giving a matrix which tells us —by looking at the first column determining the image of the meridian— that the previous framed link represents the Lens space

$$L(pq-1, q).$$

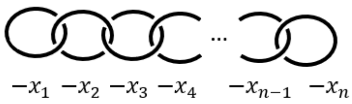

Figure 2.20: Surgery description of a Lens space $L(p,q)$, where $p/q = [x_1, \ldots, x_n]$ is a continued fraction decomposition.

This brings us to the following theorem.

Theorem 2.5.1 *Any lens space $L(p,q)$ has a surgery description as in Figure 2.20, where $p/q = [x_1, \ldots, x_n]$ is a continued fraction decomposition of the form*

$$[x_1, \ldots, x_n] = x_1 - \cfrac{1}{x_2 - \cfrac{1}{\cdots - \cfrac{1}{x_n}}}$$

Notice the change in sign with respect to the usual definition for a continued fraction decomposition. The following proof reveals how continued fractions come into play.

Proof. To produce the previous link, it is enough to repeat the construction for $L(pq-1,q)$ sufficiently many times. The only thing which remains to check is that, if $p/q = [x_1, \ldots, x_n]$, then

$$\begin{pmatrix} -1 & 0 \\ x_1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ x_2 & 1 \end{pmatrix} \cdots \begin{pmatrix} -1 & 0 \\ x_n & 1 \end{pmatrix} = \begin{pmatrix} -q & s \\ p & r \end{pmatrix}$$

for some r and s (the ones such that $qr + ps = 1$). Clearly, this is true when $n = 1$ and $n = 2$ because $\frac{p}{1} = [p]$ and[23] $\frac{pq-1}{q} = p - \frac{1}{q} = [p, q]$, respectively. Proceed then by induction and suppose that $p'/q' = [x_2, \ldots, x_n]$. Compute then

23: Notice that we already computed the corresponding matrix when obtaining $L(pq-1,q)$.

$$\begin{pmatrix} -1 & 0 \\ x_1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -q' & s' \\ p' & r' \end{pmatrix} = \begin{pmatrix} -p' & -r' \\ x_1 p' - q' & x_1 r' + s' \end{pmatrix}$$

to see from the first column that

$$\frac{x_1 p' - q'}{p'} = x_1 - \frac{q'}{p'} = x_1 - \frac{1}{[x_2, \ldots, x_n]} = [x_1, \ldots, x_n].$$

Finally, every rational number has a finite continued fraction of the desired form, so we are done. □

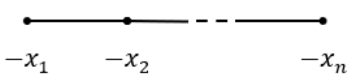

Figure 2.21: Graph description of a chain of unknots framed by integers, corresponding to the continued fraction decomposition of a rational surgery coefficient.

The link in Figure 2.20 is typically simplified and drawn as the weighted graph in Figure 2.21, vertices corresponding to unknots and edges connecting two vertices corresponding to linked unknots.

Similarly, Seifert manifolds $M((a_1, b_1), \ldots, (a_n, b_n))$ have rational surgery descriptions as in Figure 2.22, which can be described as in Figure 2.23 with the continued fraction decompositions $a_i/b_i = [x_{i_1}, \ldots, x_{i_{m_i}}]$.

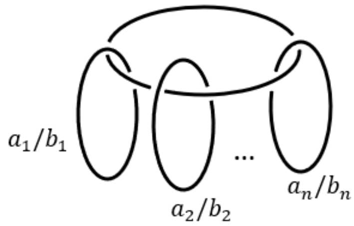

Figure 2.22: Surgery description of a Seifert manifold $M((a_1, b_1), \ldots, (a_n, b_n))$.

2.6 The Linking number

Let us now introduce a topological invariant of knots which will appear latter on in the next chapter. Wee provide two equivalent definitions which are most relevant to us.

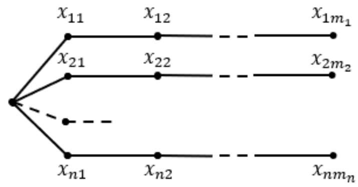

Figure 2.23: Surgery description of a Seifert manifold $M((a_1, b_1), \ldots, (a_n, b_n))$ given by a plumbing graph.

Definition 2.6.1 *Given two disjoint oriented knots L_1 and L_2 in $\mathbb{S}^3$ or $\mathbb{R}^3$, their* ***linking number*** *$\ell k(L_1, L_2)$ is defined in either of the equivalent forms:*

(1) Considering a regular projection of $L_1 \cup L_2$, counting as shown in Figure 2.24 for each point where L_1 crosses under L_2 and summing over all such crossings to give $\ell k(L_1, L_2)$.
(2) In terms of homology groups, denoting by $[L_1]$ the homology class of L_1 in $H_1(\mathbb{S}^3 \backslash L_2)$ and by $[m]$ the homology class of a meridian m of L_2 generating the group $H_1(\mathbb{S}^3 \backslash L_2) = \mathbb{Z}$, as the solution to the equation $[L_1] = \ell k(L_1, L_2) \cdot m$, with the choice of orientation on m as shown in Figure 2.15.

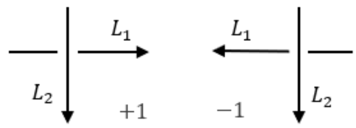

Figure 2.24: Assignment of under-crossings defining the linking number.

Observe that the following symmetries hold: $\ell k(L_1, L_2) = \ell k(L_2, L_1)$ and $\ell k(-L_1, L_2) = -\ell k(L_1, L_2)$, where $-L_1$ is L_1 with the opposite orientation.

The linking number allows one to describe easily the canonical meridian-longitude pair (m, ℓ) for a knot $k \subset \mathbb{S}^3$ in the following way. Recall when we saw that m and ℓ are simple closed curves on the surface ∂K such that $[m] \in H_1(K) = \mathbb{Z}$ is a generator and ℓ is *a* longitude such that it is null-homologous in the knot complement: $0 = [\ell] \in H_1(K)$. The second definition of the linking number then tells us that $[\ell] = 0$ is equivalent to $\ell k(\ell, k) = 0$. Last, recall how orientations for m and ℓ were chosen so that $\ell k(k, m) = +1$, assuming the orientations of k and ℓ to be consistent.

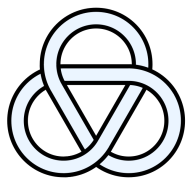

Figure 2.25: A trefoil knot along with its natural parallel longitude. The corresponding linking number is $\ell k(k, \ell) = -3$.

Example 2.6.1 Consider the trefoil knot. Observe that the natural choice of ℓ as a longitude running "parallel" to k as seen in Figure 2.25 yields $\ell k(k, \ell) = -3$, so it does not give the canonical longitude —which involves instead several twists.

Remark

In this way, the integral n-framing of a knot k is equivalent to the choice of a longitude ℓ turning around k in such a way that

$$\ell k(\ell, k) = n.$$

Thus, another way of representing a framed knot is depicting it as a closed band, one of whose boundary components represents the knot itself and the other one the chosen longitude (see Figure 2.25).

2.7 Kirby moves

Last, let us address the natural question of determining when two framed links in $\mathbb{S}^3$ give rise to the same 3-manifold under integral surgery. The answer is given by the following two elementary operations which leave the resulting 3-manifold unchanged.

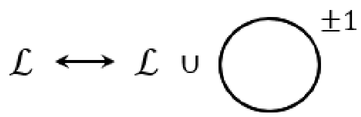

Figure 2.26: Kirby move K1.

Definition 2.7.1 *Let $\mathcal{L} \subset \mathbb{S}^3$ be a framed link, defining a closed orientable 3-manifold by integral surgery on it. The* ***Kirby moves*** *on the link and its framing are defined as:*

K1. *Add or delete any unknotted circle with framing ± 1 (see Figure 2.26).*

K2. *Slide one component of the link $\mathcal{L}$ over another.*
This goes as follows. Consider two link components L_1 and L_2 respectively framed by integers n_1 and n_2 and let L_2' be a longitude defining the framing n_2 of L_2 —i.e. $\ell k(L_2, L_2') = n_2$. The pair $L_1 \cup L_2$ is then replaced by $L_\# \cup L_2$ where $L_\# = L_1 \#_b L_2'$ with b being any band connecting L_1 to L_2' (see Figure 2.27). It is said that L_1 was slid over L_2 and denote the new component $L_\# = L_1 + L_2$.

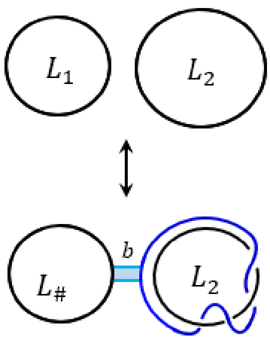

Figure 2.27: Kirby move K2.

Remark 2.7.1 Observe that all framings are preserved but the one for L_1, whose modification $L_\#$ has framing

$$n_1 + n_2 + 2\ell k(L_1, L_2).$$

An orientation for both L_1 and L_2 must be given in order to compute $\ell k(L_1, L_2)$. This is done in such a way that they together define an orientation on $L_\#$, which dependes on how the band b is glued in. Finally, the answer to our question:

Theorem 2.7.1 (Kirby) *The closed oriented manifolds obtained by integral surgery on framed links $\mathcal{L}$ and $\mathcal{L}'$ are homeomorphic by an orientation preserving homeomorphism if and only if $\mathcal{L}'$ can be obtained from $\mathcal{L}$ by a sequence of moves of types K1 and K2.*

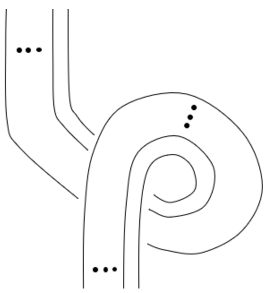

Figure 2.28: Sketch of a full right twist.

For computational purposes, it is handy to have the following result, where a full twist means the one given by Figure 2.28.

Proposition 2.7.2 *An unknot with framing ± 1 can always be moved away from the rest of the link $\mathcal{L}$ with the effect of giving all arcs going through the unknot a full left/right twist and changing the framings by adding ∓ 1 to each arc, assuming they represent different comoponents of $\mathcal{L}$ (see Figure 2.29 below).*[a]

[a] In general, the framing changes according to the rule $n_1 + n_2 + 2\ell k(L_1, L_2)$ seen before.

See Figure 2.30 and Figure 2.31 for the cases with 1 and 2 arcs. In Figure 2.31, the framings increase by ± 1 if the arcs belong to different components of $\mathcal{L}$, but change by either 0 or ± 4 if they belong to the same component.

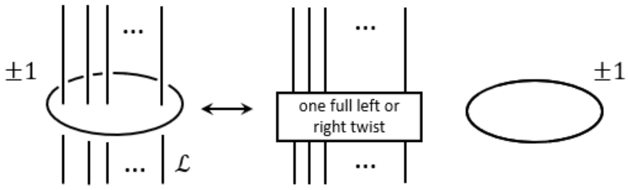

Figure 2.29: Reversible operation on an unknot with framing ±1.

Definition 2.7.2 *The operation shown in Figure 2.29 along with discarding the unknotted, unlinked component is known as* ***blow-down****, and the converse operation as* ***blow-up****.*

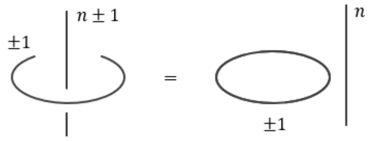

Figure 2.30: Operation for 1 arc passing through the ±1 framed unknot.

The following final useful result allows one to further simplify these framing diagrams when special required conditions are met, as described below.

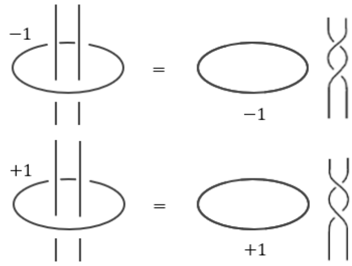

Figure 2.31: Operation for 2 arcs passing through the ±1 framed unknot, showing the corresponding full twist.

Proposition 2.7.3 *Given a framed link $\mathcal{L}$ with a zero-framed unknot component L_0 linking just one other component L_1 geometrically once, then removing $L_0 \cup L_1$ away from the link $\mathcal{L}$ doesn't change framings and cancells out.*

Similarly, one can work with the earlier described graphs and define corresponding moves on them. For instance, the blow up and blow down moves for the equivalent plumbing graphs are shown in Figure 2.32 when considering (a) two arcs and (b) one single arc going through the unknot with framing -1.

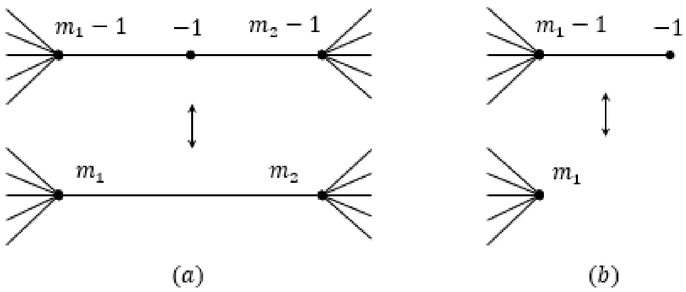

Figure 2.32: Neumann moves (a) and (b) for plumbing graphs, corresponding to Kirby moves K1 and K2.

These notions are used for instance in [GM19] when considering negative definite plumbing graphs to obtain specific invariants of plumbed knot complements.

Chern-Simons Theory | 3

It is shown that $2 + 1$ dimensional quantum Yang-Mills theory, with an action consisting purely of the Chern-Simons term, is exactly soluble and gives a natural framework for understanding the Jones polynomial of knot theory in three dimensional terms. In this version, the Jones polynomial can be generalized from $\mathbb{S}^3$ to arbitrary three manifolds, giving invariants of three manifolds that are computable from a surgery presentation. These results shed a surprising new light on conformal field theory in $1 + 1$ dimensions.

Edward Witten [Wit89]

In theoretical physics, Quantum Field Theory (QFT) is an extremely successful theoretical framework combining both Special Relativity and Quantum Mechanics. While yet not fully understood and unable to describe Gravity in a consistent manner, some of its predictions have been experimentally tested to agree with an incredibly high accuracy, higher than any other theoretical prediction in physics. The final great unification fitting Gravity into this picture is being studied by different candidate theories such as Quantum Loop Gravity or Twistor Theory, for instance, related to the String Theory world of theoretical physics.

Quantum Field Theory is important because it enables the construction of physical models of subatomic particles and quasiparticles in different fields such as particle physics and condensed matter physics. It regards particles as excited states (quanta) of their fundamental underlying quantum field, whose interactions are described by coupling or interaction terms in the Lagrangian of the theory[1] . Typically, QFTs are defined by giving their Lagrangian or directly by the corresponding action, from where the partition function of the theory is obtained.

1: These interactions are the ones famously represented by Feynman diagrams in the perturbation theory of quantum mechanics (see Figure 3.1).

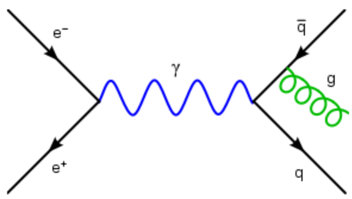

Figure 3.1: Feynman diagram for gluon radiation.

Here we present a famous QFT known as Chern-Simons theory (mainly following [Wit89; PK16; Gra]), which has turned out to be very attractive due to its interesting features. Given that its defining action does not depend on the metric of spacetime, the theory shows to own interesting topological properties. It is in fact the main example of a Topological Quantum Field Theory as defined in the first chapter. It further describes topological order in the so-called fractional quantum Hall effect in condensed matter physics, enables the description of topological insulators and is a key mathematical object in the theory of Topological Quantum Computers (TQC), where the most simple anyonic model[2] is described by a $SU(2)$ Chern-Simons theory. Moreover, its dynamics on the 2-dimensional boundary of 3-manifolds reveals a strong relation to Conformal Field Theory, specifically to the Wess-Zumino-Witten theory. In mathematics, it is closely related to the theory of quantum groups, Khovanov homology, and the theory of knot and 3-manifold invariants such as the Jones polynomial, as shown by

2: The Yang-Lee-Fibonacci model.

Edward Witten in [Wit89], and which constitutes a field of current research.

It is in this last framework where our project is carried out, yet without losing our interest in the physical features of the theory. In fact, we end this chapter with some comments on the relation between Chern-Simons theory and both the Quantum Hall Effect and Quantum Gravity.

3.1 Abelian Chern-Simons Theory

We start by presenting the abelian Chern-Simons theory, which recovers some weak topological invariants such as the linking number encountered in Chapter 2.

First, we introduce some geometrical background. We start with an oriented 3-manifold M representing spacetime and further a mathematical object confering some structure: a compact, simple Lie group G (also called **gauge group**) with the corresponding Lie algebra $\mathfrak{g}$.[3] The term *abelian*, then, refers to the commutativity of the gauge group G in the theory. In physics, one calls a **gauge theory** a field theory with some symmetry –given by the Lie group– in a way that the Lagrangian of the theory remains invariant under local transformations carried out by smooth operations composing the Lie group, known as **gauge transformations**.[4] The symmetry naturally extends to the dynamics of the theory. The term *gauge* refers to the mathematical formalism to regulate redundancies in degrees of freedom of the Lagrangian of the theory. Chern-Simons theory is indeed a gauge theory, meaning that a classical configuration on a given manifold M with gauge group G can mathematically be described by a G**-bundle**[5] on M, typically denoted $E \to M$. The sections of E are precisely the gauge transformations; i.e. smooth maps

3: We refer the reader to Appendix B for a review on Lie groups and Lie algebras.

4: The Standar Model of particle physics, for example, is a non-abelian gauge theory with symmetry group $U(1) \times SU(2) \times SU(3)$.

5: To simplify, one may think of it as the trivial bundle $E = M \times G$, although the theory works in the general setting.

$$\begin{aligned} g : M &\to E \\ x &\mapsto (x, g_x) \end{aligned}$$

where $g_x \in G$. Abusing notation one may simply write $g(x) = g_x$ thinking of g as a map $M \to G$ producing spacetime-dependent elements of G. For g_x lying close to the identity element of G, one considers instead infinitesimal gauge transformations (the generators of the gauge group by exponentiation) and views $g(x)$ as a $\mathfrak{g}$-valued 0-form, a spacetime-dependent Lie algebra generator.

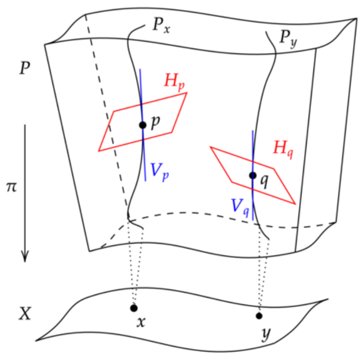

Figure 3.2: A connection can equivalently be described by the choice of a horizontal subspace $H_p \subset T_pP$ for each tangent space to the principal bundle P. A way to think of a connection form ω of the principal bundle is viewing it as projection operator onto the tangent bundle TP. The kernel is then given by the horizontal subspaces for the associated Ehresmann connection. Compatibility with the right group action of G on P is required.

We also introduce into the game a **principal connection** A on E, which may be roughly regarded as a $\mathfrak{g}$-valued 1-form on M. In local coordinates, it reads $A = A_\mu dx^\mu$ with the $A_\mu(x)$ lying in the Lie algebra called the **gauge field**. Each of these can be expanded in a basis T_a of $\mathfrak{g}$ as $A_\mu(x) = A_\mu^a(x)T_a$, with $a = 1, \ldots, \mathrm{dim}G$. Under gauge transformations, it is known that the gauge field transforms as

$$A_\mu \longrightarrow A'_\mu = g^{-1}A_\mu g + g^{-1}\partial_\mu g, \quad g = g(x) \in E.$$

For the infinitesimal analogue of this law dealing with the Lie algebra instead, one introduces the **covariant derivative** D acting on $\mathfrak{g}$-valued differential forms ω by $D_\mu \omega = \partial_\mu \omega + [A_\mu, \omega]$, with $[\cdot, \cdot]$ the Lie bracket.

With this at hand, a gauge field will transform under an infinitesimal or *local* gauge transformation $\epsilon(x) \in \Omega^0(M, \mathfrak{g})$[6] as

$$A_\mu \longrightarrow A'_\mu = A_\mu + D_\mu \epsilon.$$

Notice that D_μ is kind of a *twisted* version of ∂_μ ruled by $[A_\mu, \cdot]$. A way to measure the strength of the twisting effect produced by A_μ is to observe how much D_μ fails to commute with itself[7] . This gives rise to the definition of the **curvature** of the connection (also called gauge field strength), given by

$$F_{\mu\nu} := [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \qquad \equiv \qquad F = dA + A \wedge A.$$

For abelian G, all commutators vanish and so $D_\mu = \partial_\mu$. The curvature $F_{\mu\nu}$ reduces then to the familiar Maxwell field strength tensor. In the next section, however, we will consider the case of non-abelian Chern-Simons theory. The connection A is said to be **flat** if $F = 0$.

Finally, we can present the abelian Chern-Simons theory [PK16] with gauge group[8] $U(1)$, which is the theory is described by the action

Abelian Chern-Simons action

$$S = \frac{\kappa}{4\pi} \int_M A \wedge dA$$

where κ is a coupling parameter or *Chern-Simons level* and 4π is a factor given by convention. Observe the absence of metric dependence in the action, as opposed to what is common in other field theories. It shows gauge invariance under transformations of the form $A \to A + d\Lambda$ with Λ a zero-form[9] . Physicists may rewrite this action as

$$S = \frac{\kappa}{4\pi} \int_M d^3x \; \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho,$$

analogous to the photon field in electrodynamics. Recall that $\epsilon^{\mu\nu\rho}$ is the totally antisymmetric Levi-Civita symbol. Despite its simplicity, this theory already captures some topological invariants, as we shall see in the following. Given the action S, the partition function is the functional integral given by[10]

Partition function

$$Z(M) = \int \mathcal{D}A e^{iS[A]},$$

where $\mathcal{D}A$ is the functional measure. The integration is thus carried out over all possible gauge connections modulo gauge transformations. Actually, the classical phase space of the theory is given by the moduli space of *flat* connections (modulo gauge transformations), since the equations of motion following from the action are

$$\delta S = 0 \Rightarrow \frac{\kappa}{4\pi} \epsilon^{\mu\nu\rho} F_{\nu\rho} = 0 \Rightarrow F = 0,$$

6: Here $\Omega^0(M, \mathfrak{g})$ denotes the set of $\mathfrak{g}$-valued 0-forms on M.

7: For we expect twist-free derivatives to commute.

8: The Lie group $U(n)$ is the group of unitary $n \times n$ matrices, which in the $n = 1$ case reduces to the group of unit complex numbers $e^{i\psi}$, each given by a phase ψ. The symmetry then corresponds to invariance under change of phase.

9: Indeed, observe that $A' \wedge dA'$ with $A' = A + d\Lambda$ expands as

$$A \wedge dA + d\Lambda \wedge dA + (A + d\Lambda) \wedge d^2\Lambda,$$

where the term $d^2\Lambda$ vanishes and the second one can be written as an exact form $d\Lambda \wedge dA = d(\Lambda dA)$ (recall $\Lambda \in \mathcal{C}^\infty(M)$ is a zero-form). Hence, the integral after the change of variables splits into two terms: one giving the abelian Chern-Simons action and the other being an integral of an exact form over a closed manifold M. By Stokes theorem

$$\int_M d\omega = \int_{\partial M} \omega$$

with $\partial M = \emptyset$ and $\omega = \Lambda dA$, the second term vanishes.

10: While in classical mechanics the trajectory followed by a particle is determined by the path minimizing the action functional $S[A]$, in quantum mechanics one considers the contributions of all the paths connecting the two points, averaged by the probability function $e^{iS/\hbar}$. This is called the partition function or **Feynman path integral**. In the present discussion, however, the term $\hbar$ will be absorbed into the action functional.
Concerning the correspondence principle, observe that in the classical limit $\hbar \to 0$ large contributions to the integral will tend to be cancelled because of the rapid oscillations in $e^{iS/\hbar}$. Nevertheless, these cancellations wont happen at the critical points of S, so the main contributions to the Feynman path integral will come from the classical trajectories of the particle.

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ in this abelian situation, hence the boring no dynamics in the theory with zero curvature. However, a more interesting theory can be obtained by adding some coupling terms into the theory, as we shall explore in a later section.

Now, the observables of this theory respecting the symmetries of gauge invariance and metric independence are the Wilson loop or knot operators given by the holonomy[11] of A around the closed loop determined by a knot K as

$$W(K) = \exp\left(n \oint_K A\right)$$

where $n \in \mathbb{Z}$ is said to measure the *charge* of the knot. The quantum information of a theory is obtained through the expectation value of its observables, which in this case are given by

$$\begin{aligned}\langle \exp\left(n \oint_K A\right)\rangle = \langle W(K)\rangle &= \frac{1}{Z}\int \mathcal{D}A\; W(K) e^{iS[A]} \\ &= \exp\left(\frac{n^2}{2}\langle \oint_K A_\mu(x)dx^\mu \oint_K A_\nu(y)dy^\nu\rangle\right).\end{aligned}$$

11: In differential geometry, the notion of **holonomy** of a connection A on a smooth manifold M is a consequence of the curvature of the connection. It measures the loss of geometrical information when being transported through parallel transport (see Figure 3.3).

In the case of a two-component link $\mathcal{L}$ with components K_1 and K_2, the corresponding observable has an expectation value

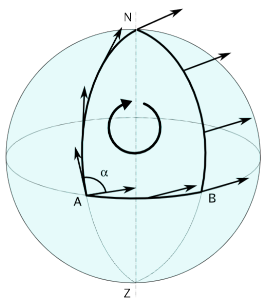

Figure 3.3: Parallel transport of a vector v along a closed piecewise smooth path γ on a sphere, yielding a vector $\mathcal{P}_\gamma(v)$. The corresponding element of the holonomy group is the rotation of v into $\mathcal{P}_\gamma(v)$ by an angle α.

$$\langle W(\mathcal{L})\rangle = \langle \exp\left(\oint_{K_1} A\right)\exp\left(\oint_{K_2} A\right)\rangle$$

whose exponential form can be expressed in terms of two-point functions $\langle A_\mu(x)A_\nu(y)\rangle$. Choosing the Lorentz gauge $\partial_\mu A^\mu = 0$, the explicit form of these for the Chern-Simons action is

$$\langle A_\mu(x)A_\nu(y)\rangle = \frac{i}{\kappa}\epsilon_{\mu\nu\rho}\frac{(x-y)^\rho}{|x-y|^3}.$$

This finally gives

$$\langle \oint_{K_1} A \oint_{K_2} A\rangle = \frac{4\pi i}{\kappa} L(K_1, K_2)$$

where

$$L(K_1, K_2) = \frac{1}{4\pi}\oint_{K_1}\oint_{K_2} dx^\mu dy^\nu \epsilon_{\mu\nu\rho}\frac{(x-y)^\rho}{|x-y|^3}$$

turns out to indeed define the linking number of the two knots, a topological invariant we introduced in Chapter 2. A self-linking number or *framing* can also be defined by choosing a framing of the knot K, giving a displaced knot K_f with shifted coordinates $y^\mu(s) = x^\mu(s) + \epsilon n^\mu(s)$ where s parametrizes K, $\epsilon \to 0$ and $n^\mu(s)$ is the unit vector field which is normal to the curve at s:

$$SL(K) = \lim_{\epsilon\to 0}\frac{1}{4\pi}\oint_K\oint_{K_f} dx^\mu dy^\nu \epsilon_{\mu\nu\rho}\frac{(x-y)^\rho}{|x-y|^3}.$$

In general, for a link $\mathcal{L} = \sqcup_{i=1}^N K_i$, the expectation value of the Wilson loop will be

$$\langle W(\mathcal{L})\rangle = \exp\left(\frac{2\pi i}{\kappa}\left\{\sum_{\ell=1}^N n_\ell^2 SL(K_\ell) + \sum_{\ell\neq m} n_\ell n_m L(K_\ell, K_m)\right\}\right)$$

Recall that there exists a canonical frame in $\mathbb{S}^3$ where $SL(K)$ is zero, the 0-framing for knots described in Chapter 2. This canonical framing is unchanged under Reidemeister moves so it can be used to construct ambient isotopy invariants. On the other hand, the braiding does not preserve the frame and extra correction factors must be added. However, the relating framing factors are exactly known. In our project, we will deal with 0-framed knots.

3.2 Non-abelian Chern-Simons Theory

The name "Chern-Simons theory", however, mainly refers to the more interesting non-abelian case refering to the non-commutativity of its gauge group G. We start with its definition.[12]

Definition 3.2.1 *Consider given a closed 3-manifold M and a compact (non-abelian) semisimple Lie group G. Let E be a G-bundle and on E place a connection $A = A_\mu dx^\mu = (A^a_\mu T_a)dx^\mu$, a field which we may view as a Lie algebra valued one-form. Then, the (non-abelian)* ***Chern-Simons theory*** *in three dimensions is the one given by the action*[a]

$$S_{CS} = \frac{\kappa}{4\pi}\int_M \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)$$

where κ is the level or coupling constant.

[a] The symbol Tr here denotes an invariant bilinear form on the Lie algebra $\mathfrak{g}$ of G, which is a multiple of the Killing form. The normalization condition is given by $\mathrm{Tr}(T_aT_b) = \frac{1}{2}\delta_{ab}$.

12: Notice the resemblance to other **index theorems** in differential geometry —such as the Atiyah-Singer theorem or the Gauss-Bonnet theorem

$$2\pi\chi(\Sigma) = \int_\Sigma \mathcal{K}\, d\mu_\Sigma,$$

where Σ is an embedded closed surface in $\mathbb{R}^3$, $\mathcal{K}:\Sigma\to\mathbb{R}$ its gaussian curvature, μ_Σ the measure induced by the riemannian metric and $\chi(\Sigma)$ its Euler characteristic–, all of which appearing as an equality involving the integral of a differential form on the one side and a specific topological invariant on the other.

Compare the this expression to the abelian case, where the additional cubic term was missing. The choice of the famous Chern-Simons three-form for the definition of S_{CS} is naturally given by the aim of formulating a generally covariant theory through a metric-independent Lagrangian, so that all observables be topological invariants[13] . Physicists may write the CS action in the equivalent form

$$S_{CS} = \frac{\kappa}{4\pi}\int_M d^3x\, \epsilon^{\mu\nu\rho}\ \mathrm{Tr}\left(A_\mu(\partial_\nu A_\rho - \partial_\rho A_\nu) + \frac{2}{3}A_\mu[A_\nu, A_\rho]\right).$$

The equations of motion are obtained from the variation δS_{CS} produced by a field variation δA, yielding

$$\delta S_{CS} = 0 \Rightarrow F^a_{\mu\nu} = 0.$$

Thus, the phase space of the theory is the space of flat connections on the G-bundle E.

To impose the theory to be gauge invariant[14] , means imposing gauge invariance on the quantity with physical meaning[15] e^{iS}. This results in the condition that the phase term $e^{i2\pi\kappa}$ equals one, from where the quantization condition $\kappa\in\mathbb{Z}$ is obtained.

13: This is to be put into contrast with other theories, such as the standard **Yang-Mills theory**

$$S_{YM} = \int_M \sqrt{g}g^{\mu\rho}g^{\nu\sigma}\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma}),$$

depending on a choice of metric $g_{\mu\nu}$.

14: Under infinitessimal gauge transformations

$$A_\mu \to A_\mu + D_\mu\epsilon,$$

with ϵ a generator of the gauge group and corresponding covariant derivative

$$D_\mu = \partial_\mu + [A_\mu, \cdot].$$

15: The one actually appearing in the partition function

$$Z = \int \mathcal{D}A\ e^{iS}.$$

Consistency of QFT does not require the action of the theory to be single valued, but only the term e^{iS}, as in Dirac's famous work on magnetic monopoles for instance.

One can proceed with perturbation theory (as is common in QFT) by rescaling $A_\mu \to \lambda A_\mu$ and redefining $\kappa = 4\pi/\lambda^2$, yielding

$$S_{CS} = \int_M d^3x\, \epsilon^{\mu\nu\rho} \mathrm{Tr}\left(A_\mu \partial_\nu A_\rho + \lambda \frac{2}{3} A_\mu A_\nu A_\rho \right),$$

and expanding this expression in λ. Notice that large κ means here weak coupling, since $\kappa \propto \frac{1}{\lambda^2}$.

Concerning observables, in QFT one wants them to satisfy gauge invariance as well. Again, the Wilson loops from QCD give a natural family of observables which are further metric independent, thus keeping general covariance. A Wilson loop operator of a link $\mathcal{L}$ will now be a functional of the connection A consisting of the product of the path-ordered holonomies[16] of A around the curve defined by each knot component K_i, yielding a group element of G defined up to conjugacy, and subsequently taking its trace Tr_{R_i} in the irreducible representation R_i with which each K_i is decorated:

16: Path-ordering P being necessary for a consistent definition, since holonomies yield elements of the gauge group, which in this case is non-commutative.

$$W_{R_1,\ldots,R_s}(\mathcal{L}) = \prod_{i=1}^{s} \mathrm{Tr}_{R_i} \left(P \exp \oint_{K_i} A \right).$$

The obtained observables are thus the vacuum expectation values given by the Feynman path integrals[17] [18]

17: Although not explicitly written, notice the dependence on A of both integral factors. Recall that integration is carried out over the space of all connections modulo gauge transformations $\mathcal{A}/\mathcal{G}$.

18: As mentioned earlier, the path integral formulation of QT comes from the notion that particles don't follow the definite paths governed by Newton's laws but rather follow "all the possible paths", to be summed with a given assigned probability each. These paths in spacetime can be quite irregular and even knotted, with any number of loops and zigzags, although these complicated ones turn out to be less likely.

Chern-Simons Feynman path integral

$$V_{R_1,\ldots,R_s}(\mathcal{L}) = \langle W_{R_1,\ldots,R_s}(\mathcal{L}) \rangle = \frac{1}{Z} \int_{\mathcal{A}/\mathcal{G}} \mathcal{D}A\, e^{iS_{CS}} W_{R_1,\ldots,R_s}(\mathcal{L})$$

which requires the link to be framed. Here the normalization factor Z is the partition function, corresponding to the integral computed in the absence of Wilson loops. As we shall see, these expressions indeed give invariants of framed links as expected from the topological invariance. For convenience, we will work with the unnormalized integral $Z \cdot V_{R_1,\ldots,R_s}(\mathcal{L})$ and denote it $Z(M;\mathcal{L})$ for short. Observe that the orientation of the knots K_i give the direction in which a particle moves around that loop, with charge corresponding to the representation R_i. Since changing orientation $K_i \to -K_i$ is equivalent to conjugation of $R_i \to \bar{R}_i$, **charge conjugation**[19] leaves $Z(M;\mathcal{L})$ invariant and hence the Chern-Simons action as well. Notice also that it is through the Wilson loop operators that knots have come to play a role.

19: Meaning the operation of both $K_i \to -K_i$ and $R_i \to \bar{R}_i$ for all $i = 1, \ldots, s$.

In the coming section we will show how these invariants are in the case of links in $\mathbb{S}^3$ precisely the ones appearing in the Jones theory and its generalizations, which further yield invariants of three-manifolds—knot complements— through surgery on links, as seen in Chapter 2. This will be done by choosing the gauge group $SU(N)$ and the R_i all being its N-dimensional representation, yielding in this case the two variable generalization of the Jones polynomial, where the variables N and κ are analytically continued to complex values, such as[20]

20: Definition of q may vary in sign or a power of two depending on conventions.

$$q = \exp\left(\frac{2\pi i}{N + \kappa} \right).$$

However, other Lie groups (or corresponding Lie algebras) can be chosen. In fact, in the section on Quantum Gravity we will deal with the Lie algebra $\mathfrak{sl}_2$, the complexification of $\mathfrak{su}_2$. And in our project in the next chapter we will not only comment on results concerning $\mathfrak{sl}_2$ and $\mathfrak{sl}_{N+1}$ in general, but will also work with the Lie algebras $\mathfrak{so}_{2N}$.

3.3 Chern-Simons and the Jones Polynomial

We present here the main ideas shown by Witten in [Wit89] about the connnection between the Chern-Simons field theory and the the Jones polynomial invariant of knot theory. We use notions on TQFTs introduced in Chapter 1. Invariants of 3-manifolds can then obtained by arguments concerning surgeries on links, as described in Chapter 2. We also refer the reader to [PK16].

We will follow an approach in the framework of canonical quantization[21] in which a manifold M containing a link $\mathcal{L}$ is sliced into many pieces, each appearing locally as $\Sigma \times R$ with Σ a two-dimensional Riemann surface. On each of these pieces, the action S_{CS} in the gauge $A_0 = 0$ yields the classical solution of zero curvature $F_{\mu\nu} = 0$ corresponding to a flat connection. This means that the physical space is the moduli space of flat connections on Σ modulo gauge transformations, which —nicely enough— has a finite volume. Quantization after imposing such constraint produces a finite dimensional Hilbert space $\mathcal{H}_\Sigma$ with states related to the correlation functions of the Wess-Zumino-Novikov-Witten Conformal Field Theory (CFT) in dimension two.

Applied to Chern-Simons theory, this yields a TQFT[22] given by the Feynman path integral[23] $Z(\Sigma \times R; \mathcal{L})$ sending $\Sigma \times R$ to a vector $|\psi\rangle \in \mathcal{H}_\Sigma$ that can be expanded in the basis states of the Hilbert space (the conformal blocks in CFT). Now, a key step in the reasoning we will follow relies on the fact that the dimension of $\mathcal{H}_\Sigma$ depends on Σ, the number of punctures we have on the boundary Σ due to Wilson loops and the choice of representations of the gauge group with which these loops are decorated. In the case we will deal with, CFT tells us that the dimension must be exactly two, and this will be crucial.

Indeed, consider a three sphere $\mathbb{S}^3$ as our closed manifold M and a link $\mathcal{L}$ sitting inside, decorated with the fundamental representation R of the gauge group $SU(N)$ of our CS theory. Now, as described above, slice $\mathbb{S}^3$ into two pieces, say M_L and M_R, with boundary $\mathbb{S}^2$ each but with opposite orientations, cutting the link $\mathcal{L}$ inside it in such a way that the boundaries $\mathbb{S}^2$ are left with four punctures or marked points each (see Figure 3.4 and Figure 3.5). The TQFT sends the two $\mathbb{S}^2$ boundaries of M_L and M_R with opposite orientations to dual Hilbert spaces $\mathcal{H}_L = \mathcal{H}_R^*$. CFT then tells us that these must be two-dimensional. And this will be the key fact to exploit to obtain the skein relations defining the Jones polynomial. As we will see, the Chern-Simons TQFT sends M_R to a vector $|\psi\rangle \in \mathcal{H}_R$ and M_L to a dual vector $\langle\chi| \in \mathcal{H}_L$, the Jones polynomial skein relation emerging from the pairing $\langle\chi|\psi\rangle$.

Indeed, let us show how this happens. As discussed in Chapter 1, a TQFT will send a closed manifold such as $\mathbb{S}^3$ to a linear map $\Gamma : \mathbb{K} \to \mathbb{K}$, yielding an invariant $\gamma = \Gamma(1)$ which will be our polynomial in the

21: In the formalism of canonical quantization, the mathematical description of classical mechanics is upgraded to the one of quantum mechanics by replacing objects such as the manifolds representing phase spaces, fields on them and the Poisson bracket, by the usual Hilbert spaces, operators and the commutator bracket, respectively.

22: To view the **Chern-Simons** partition function as a **TQFT**, one formulates the theory to allow manifolds with boundary. Given a manifold M with boundary ∂M and a connection $\alpha \in \Omega^1(\partial M, \mathfrak{g})$, denote $\mathcal{A}_\alpha$ for the space of fields $A \in \mathcal{A}$ such that they restrict to $A|_{\partial M} = \alpha$. Then

$$Z(M)_\alpha = \int_{\mathcal{A}_\alpha/\mathcal{G}} \mathcal{D}A e^{iS_{CS}}$$

and is viewed as a function of α on $\Omega^1(\partial M, \mathfrak{g})$. In [Koh02] it is described in detail how to define the **quantum Hilbert space** $Z_{\partial M}$ of functions on $\Omega^1(\partial M, \mathfrak{g})$ where $Z(M)_\alpha$ lives in and satisfying the TQFT axioms such as

- (orientation) $Z_{(\partial M)^*} = Z^*_{\partial M}$.
- (multiplicativity) For a disjoint boundary $\partial M = N_1 \sqcup N_2$,

$$Z_{N_1 \sqcup N_2} = Z_{N_1} \otimes Z_{N_2}.$$

- (gluing) For a decomposition $M = M_1 \cup M_2$ with boundaries $\partial M_1 = (\partial M_2)^*$,

$$Z(M) = \langle Z(M_1), Z(M_2)\rangle.$$

23: To be normalized by $Z(\mathbb{S}^3; \mathcal{L})$.

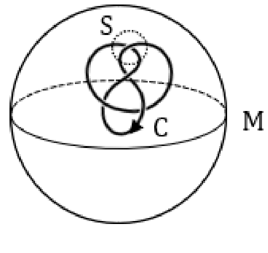

Figure 3.4: A manifold M with a link consisting of one knot or curve C sitting inside. Around an inconvenient crossing, a small sphere S is considered, cutting M into the simple interior piece and the more complicated exterior piece.

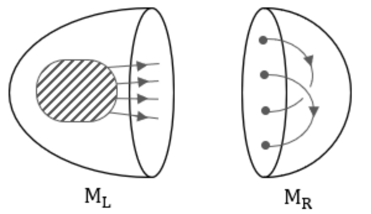

Figure 3.5: The cutting of M into the two pieces is shown more explicitly in a schematic way: the complicated piece on the left and the simple one on the right.

variable q depending on both N and κ. This linear map can alternatively be computed in terms of the decomposition given by M_L and M_R, according to the axioms of TQFTs. That is, being M_L a cobordism with empty in-boundary and out-boundary Σ^* the sphere $\mathbb{S}^2$ with opposite orientation, it is sent by the TQFT functor to a linear map[24] $\psi \in \mathbb{K} \otimes \mathcal{H}_L \cong \mathrm{Hom}(\mathbb{K}, \mathcal{H}_L) \cong \mathcal{H}_L$ (by linearity, $\phi(1)$ determines the morphism), and likewise M_R with boundary Σ is sent to a linear map $\chi \in \mathcal{H}_R \otimes \mathbb{K} = \mathcal{H}_L^* \otimes \mathbb{K} = \mathrm{Hom}(\mathcal{H}_L, \mathbb{K}) = \mathcal{H}_L^*$. Then, the map Γ can be expressed as the composition of maps $\Gamma : \mathbb{K} \xrightarrow{\psi} \mathcal{H}_L \xrightarrow{\chi} \mathbb{K}$ corresponding to the decomposition $M = M_L \cup_\Sigma M_R$, which is given, denoting these mutually dual vectors in the bra-ket notation as $\langle\chi|$ and $|\psi\rangle$, by the pairing $\langle\chi|\psi\rangle =: \gamma$ (as stated in [Wit88b]).

24: Recall the vector space isomorphisms $U^* \otimes V \cong \mathrm{Hom}(U, V)$ and $U \otimes V \cong V \otimes U$ and the fact that $\mathbb{K}^* \cong \mathbb{K}$.

25: It will correpond to an overcrossing in the simple piece of link inside M_R.

Denoting our link by $\mathcal{L}_+$ instead for convenience[25] , we have just seen that the Chern-Simons TQFT given by the Feynman path integral gives us the knot invariant

$$V_R(\mathcal{L}_+) = \langle\chi|\psi\rangle.$$

It is now where we use the two-dimensionality of the Hilbert space to evaluate this invariant. Indeed, this information tells us that any state $|\psi\rangle$ can be expressed as a linear combination of precisely two independent states $|\psi_1\rangle$ and $|\psi_2\rangle$, condition which we may write as

$$|\psi\rangle + \alpha|\psi_1\rangle + \beta|\psi_2\rangle = 0,$$

for some complex scalars α and β.

Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be the vectors correspondingly assigned by the TQFT two some other three-balls[26] X_1 and X_2 with the same boundary $\mathbb{S}^2$ as M_R, but with different strand structure in them as shown in Figure 3.6. These *linearly independent* configuration differ in that strands overcross, undercross or not cross at all. When gluing, this yields the same manifold M as before but with different links $\mathcal{L}_+$, $\mathcal{L}_-$ and $\mathcal{L}_0$ differing only in that particular crossing. By gluing M_L, the linear independence condition gives the following relation between the corresponding link invariants

26: They must be three-balls, since they have the two-sphere $\mathbb{S}^2$ as boundary.

$$\langle\chi|\psi\rangle + \alpha\langle\chi|\psi_1\rangle + \beta\langle\chi|\psi_2\rangle = 0,$$

which constitutes thus a recursion relation between these three link invariants, what we call a skein relation.

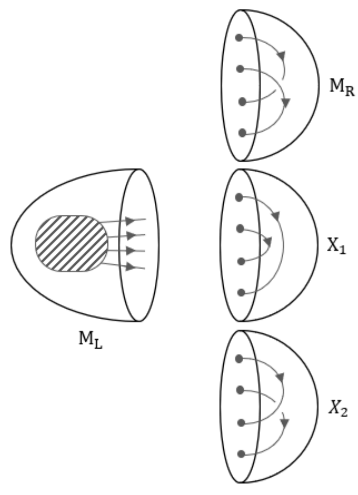

Figure 3.6: The skein relation will follow from considering the replacement of M_R with other *linearly independent* substitutes X_1, X_2.

Remark 3.3.1 It can be shown that the CS invariant for a link $\mathcal{L}$ consisting of different knot components is given by the product of the knot invariants corresponding to each component separately (slice M separating the different components).

It is yet left to show that this indeed corresponds to the Jones polynomial skein relation. What we need to do is to figure out what the coefficients α and β are. For this purpose one uses results from CFT. Specifically, in the Hilbert space of four-point conformal blocks, the vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ are obtained from $|\psi\rangle$ through a braiding operator B producing half-twists among the strands by interchanging two of the marked points. In Figure 3.6 the two bottom punctures are flipped when moving from M_R to X_1 and again to X_2, each time in the same direction. In other words,

$$|\psi_1\rangle = B|\psi\rangle, \quad |\psi_2\rangle = B^2|\psi\rangle.$$

We may use now a classic result from linear algebra[27] which states that for an operator B over a commutative ring (such as $\mathbb{C}$ in our case) satisfies its own characteristic equation. Recall that on a two-dimensional vectors space (as is the case we are dealing with) the characteristic equation is

27: Cayley-Hamilton theorem.

$$\det(B - \lambda \mathbb{1}) \equiv \lambda^2 - \mathrm{Tr}(B)\lambda + \det(B) = 0,$$

where $\mathrm{Tr}(B) = \lambda_1 + \lambda_2$ and $\det(B) = \lambda_1\lambda_2$. Cayley-Hamilton's theorem then tells us that

$$B^2 - \mathrm{Tr}(B)B + \det(B) = 0.$$

Applying $|\psi\rangle$ to this equation and recalling the expressions $|\psi_1\rangle = B|\psi\rangle$ and $|\psi_2\rangle = B^2|\psi\rangle$, we get

$$|\psi_2\rangle - \mathrm{Tr}(B)|\psi_1\rangle + \det(B)|\psi\rangle = 0.$$

CFT gives us then the eigenvalues of this braiding operator:

$$\lambda_k = \pm \exp\left(i\pi(4h_R - h_{E_k})\right), \quad k = 1, 2,$$

where a framing factor is taken into account and where h_R and h_{E_k} are conformal weights corresponding to representations R and E_k (each E_k being the irreducible representation appearing in the decomposition of $R \otimes R = E_1 \oplus E_2$ as symmetric or antisymmetric parts, to which the signs $\pm$ correspond, respectively). Explicitly,

$$h_R = \frac{N^2 - 1}{2N(N+\kappa)}, \quad h_{E_1} = \frac{N^2 + N - 2}{N(N+\kappa)}, \quad h_{E_2} = \frac{N^2 - N - 2}{N(N+\kappa)},$$

which yields

$$\lambda_k = \pm \exp\left(i\pi \frac{N \mp 1}{N+\kappa}\right).$$

Using the definition of the parameter $q = \exp\left(\frac{2\pi i}{N+\kappa}\right)$, this means

$$\lambda_1 = q^{\frac{N-1}{2}}, \quad \lambda_2 = -q^{\frac{N+1}{2}}.$$

and so with $\mathrm{Tr}(B) = \lambda_1 + \lambda_2$ and $\det(B) = \lambda_1\lambda_2$ the above expression $|\psi_2\rangle - \mathrm{Tr}(B)|\psi_1\rangle + \det(B)|\psi\rangle = 0$ after hitting it by $\langle\chi|$ turns into

$$\langle\chi|\psi_2\rangle - (q^{\frac{N-1}{2}} - q^{\frac{N+1}{2}})\langle\chi|\psi_1\rangle - q^N\langle\chi|\psi\rangle = 0$$

Recalling $V_R(\mathcal{L}_+) = \langle\chi|\psi\rangle$, $V_R(\mathcal{L}_0) = \langle\chi|\psi_1\rangle$ and $V_R(\mathcal{L}_-) = \langle\chi|\psi_2\rangle$, we finally obtain[28]

28: We have now seen a way of computing the Jones polynomial in quantum theory using CS theory, which demands the knot to be a path in a three-dimensional spacetime (two space dimensions plus one time dimension) rather than the four-dimensional one describing real world in GR. Since the 1980s, attempts to generalize the Jones polynomial have lead to the concept of **Khovanov homology**, where a knot becomes a physical object in four-dimensional spacetime. It was Sergei Gukov, Albert Schwarz and Cumrun Vafa who recently developed a quantum interpretation of Khovanov homology, connecting it closely to most innovative ideas on QFT and String Theory.

Skein relation for the (generalized) Jones polynomial

$$q^{\frac{N}{2}} V_R(\mathcal{L}_+) - q^{-\frac{N}{2}} V_R(\mathcal{L}_-) = \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) V_R(\mathcal{L}_0),$$

which precisely is the skein relation for the Jones polynomial when $N = 2$ and replacing $t = -q^{\frac{1}{2}}$ and it is the HOMFLY-PT polynomial skein relation

$$w^{-\frac{1}{2}}q^{-\frac{1}{2}}P_+(w,q) - w^{\frac{1}{2}}q^{\frac{1}{2}}P_-(w,q) = \left(q^{-\frac{1}{2}} - q^{\frac{1}{2}}\right)P_0(w,q)$$

for $w = -q^{-(N+1)}$. In the continuum limit $N \to 0$, this recursion recovers the skein relation for the Alexander polynomial. Given a skein relation, one can compute the polynomial invariant for any link and knot by recursively using it and the multiplicative property[29] . Some normalization must be given, typically by giving a value to the unknot U, such as the quantum integer:

29: The invariant for two disconnected component links is the product of their corresponding invariants.

$$V_R(U) = [N] \equiv \frac{q^N - q^{-N}}{q^1 - q^{-1}}.$$

Hence, we have seen how Chern-Simons theory gives invariants of arbitrary links in $\mathbb{S}^3$ through the expectation value of Wilson loop operators. These extend to invariants for closed oriented three-manifold M by means of Dehn surgery, since any such manifold can be reduced to $\mathbb{S}^3$ through a surgery description on a link.[30] Thus, the classification of three-dimensional manifolds is closely related to the classification of framed links. The remaining step understanding how these invariants are transformed under surgery is detailed in [Wit89]. In [PK16] it is shown how these invariants for 3-manifolds are given by the Chern-Simons partition function, up to some factor.

30: As seen in Chapter 2. The extension requires some further arguments, but the underlying notions are the ones presented there.

3.4 Chern-Simons and Quantum Hall Effect

We now move on to considerations Physics. Recall that the Chern-Simons theory on its own presents no dynamics, since the equations of motion are given by the condition of zero curvature. However, the Chern-Simons term can be coupled together with other theories to obtain a theory with dynamics where the CS term plays an interesting role. Here we discuss the so-called *Quantum Hall Effect* (QHE) in the context of condensed matter physics and its connection to Chern-Simons theory.[31]

31: We refer the reader to [Ton16] for a complete treatment on the Quantum Hall Effect.

Recall first the classical Hall effect in which electrons are restricted to move in the (x, y)-plane in the presence of a magentic field $\mathbf{B}$ towards the z-direction. The **Hall effect** then describes that a current I made to flow in the x-direction will induce a voltage V_H in the y-direction (see Figure 3.7). This effect can be explained through the Drude model[32]

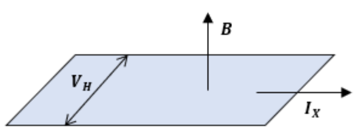

Figure 3.7: Classical Hall effect.

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E} - e\mathbf{v} \times \mathbf{B} - \frac{m\mathbf{v}}{\tau},$$

32: Here m is the mass of the particles, e its charge and τ a friction term known as *scattering time.*
In the previous sections we worked in units where $e = \hbar = 1$. In the present discussion, however, we will write these parameters explicitly for clarity.

whose equilibrium solutions, after rewriting the velocity in terms of the current density $\mathrm{J} = -ne\mathrm{v}$ with n the density of charge carriers, must then satisfy

$$\begin{pmatrix} 1 & \omega_B\tau \\ -\omega_B\tau & 1 \end{pmatrix}\mathbf{J} = \frac{e^2 n\tau}{m}\mathbf{E}$$

where $\omega_B = \frac{eB}{m}$ is the cyclotron frequency. This can in turn be expressed as $\mathbf{J} = \sigma\mathbf{E}$ with[33]

33: The structure of the matrix follows from rotational invariance.

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{yy} \end{pmatrix}$$

the conductivity tensor, which for the Drude model takes the form

$$\sigma = \frac{\sigma_{DC}}{1+\omega_B^2\tau^2} \begin{pmatrix} 1 & -\omega_B\tau \\ \omega_B\tau & 1 \end{pmatrix}$$

with $\sigma_{DC} = \frac{ne^2\tau}{m}$ the DC conductivity in the absence of a magnetic field. The off-diagonal terms in the conductivity tensor are then the responsible for the Hall effect.

Now, concerning the resistivity tensor

$$\rho = \sigma^{-1} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{yy} \end{pmatrix}$$

with, respectively, ρ_{xx} and ρ_{xy} the transverse and longitudinal resistivities one measures experimentally, the classical prediction (see Figure 3.8) is that they be given by

$$\rho_{xx} = \frac{m}{ne^2\tau} \quad \text{and} \quad \rho_{xy} = \frac{B}{ne}.$$

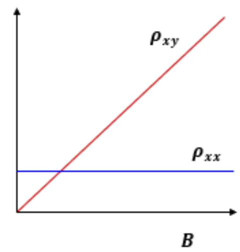

Figure 3.8: Plot for the longitudinal ρ_{xx} and transverse ρ_{xy} resistivities against the magnetic field intensity, as predicted classically by the Hall effect.

However, at low temperatures and strong magnetic fields, quantum effects emerge and one experimentally stumbles with the **Integer** and **Fractional Quantum Hall Effects** (see Figure 3.9). One of the most striking features here are the plateaux on which the transverse resistivity ρ_{xy} sits in some ranges of magnetic field before jumping to the contiguous plateau. Measurements carried out to an extraordinary accuracy show that the transverse resistivity is taking the values

$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu}$$

with $\nu \in \mathbb{Z}$ for the Integer QHE and $\nu \in \mathbb{Q}$ for the Fractional QHE. The latter appears when impurities are decreased, causing the integer plateaux to be less prominent while emerging other plateaux at fractional values.

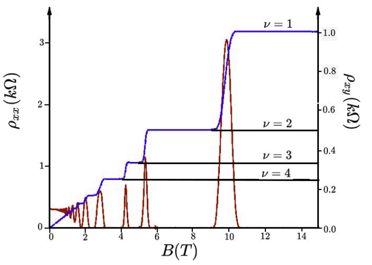

Figure 3.9: Plot for the longitudinal ρ_{xx} and transverse ρ_{xy} resistivities against the magnetic field intensity, showing the Integer Quantum Hall Effect.

Notice the relevance of these results, since one typically expects quantum effects to be observed at the microscopic level whereas these phenomena are clearly macroscopic. Interestingly, these effects turn out to be well explained by the Chern-Simons theory, as we shall comment briefly in the following.

The key in the describing model lies in coupling the electromagnetic gauge potential[34] A_μ to the dynamical degrees of freedom through the appropriate current J_μ by including a term[35]

$$S_A = \int d^3x\ J^\mu A_\mu$$

in the action of the theory. Gauge invariance of S_A under transformations $A_\mu \to A_\mu + \partial_\mu\omega$ is guaranteed by the conservation of current $\partial_\mu J^\mu = 0$.[36] An effective field theory is given by the abelian Chern-

34: Recall that the electromagnetic potential $A_\mu = (A_0, \mathbf{A})$ contains the information about the electric and magnetic fields

$$\mathbf{E} = -\frac{1}{c}\nabla A_0 - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

35: the measure d^3x corresponds to dealing with currents living in a $d = 2+1$ dimensional slice of spacetime.

36: Recall that $J_\mu = (\rho, \mathbf{J})$ in $c = 1$ units, so that the familiar continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \mathbf{J} = 0$$

is recovered.

Simons action

$$S_{\text{eff}}[A_\mu] = \frac{\kappa}{4\pi}\int d^3x\ \epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho,$$

which together with the coupling term yields the arising current through the functional derivative[37]

$$J_i = \langle J_i(x)\rangle = \frac{\delta S_{\text{eff}}[A]}{\delta A_i} = -\frac{\kappa}{2\pi}\epsilon_{ij}E_i$$

from where the Hall conductivity

$$\sigma_{xy} = \frac{\kappa}{2\pi} = \frac{e^2}{2\pi\hbar}\nu$$

can be read after identifying $\kappa = e^2\nu/\hbar$, thus recovering the $\nu \in \mathbb{Z}$ Landau levels in the Integer QHE.[38]

37: Given a functional $F[f]$ depending on fields $f(x)$, the **functional derivative** $\frac{\delta F}{\delta f}$ is defined as

$$\frac{\delta F}{\delta f(x)} = \lim_{\epsilon\to 0}\frac{F[f(x')+\epsilon\delta(x-x')]-F[f(x')]}{\epsilon},$$

in analogy to the familiar derivative of a function. Here, instead, the functional derivative tells how the number returned by the functional $F[f(x)]$ changes when slightly varying the fed function $f(x)$ by a delta distribution. **Hamilton's principle of least action** then reads

$$\frac{\delta S}{\delta x(t)} = 0$$

for $S[x(t)]$ the action functional.

38: Recall the quantization condition $\kappa \in \mathbb{Z}$ previously seen for the Chern-Simons theory in units where $e = \hbar = 1$, which justifies the fact $\nu \in \mathbb{Z}$.

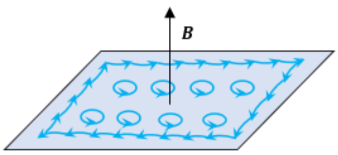

Figure 3.10: Quantum Hall Effect describing *topological insulators*, materials behaving as insulators in their interior (with electrons moving in localized closed orbits) while containing surface conducting states along the edges.

Regarding the Fractional QHE, one can describe the $\nu = 1/m$ Laughlin states by considering the mixed Chern-Simons effective action

$$S_{\text{eff}}[a;A] = \frac{e^2}{\hbar}\int d^3x\left(\frac{1}{2\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho - \frac{m}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho + \dots\right)$$

involving the additional $U(1)$ gauge field a_μ. The previous arguments lead to $m \in \mathbb{Z}$ and by removing the dynamical variable a_μ by means of the solution $a_\mu = A_\mu/m$ to its equation of motion, gives

$$S_{\text{eff}}[A] = \frac{e^2}{2\pi}\int d^3x\frac{1}{4\pi m}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho.$$

From here, the fractional Hall conductivity

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar}\frac{1}{m}$$

is obtained, as is expected for the Laughlin state. This can be generalized to other filling fractions other than $1/m$, as discussed in [Ton16].

To end this section, let us comment that the Quantum Hall Effect plays a role in —to give an example— describing a topological state of quantum matter in two dimensions concerning edge states in topological insulator thin films (see Figure 3.10). We refer the interested reader to [ZLS15] for more information.

3.5 Chern-Simons and Quantum Gravity

Last, let us briefly comment the relation between Chern-Simons theory and $(2+1)$-dimensional gravity.[39] We refer the interested reader to Edward Witten's work [Wit88a] and the more recent work [Guk05] by Sergei Gukov for the *quantum* aspects of this gravitational theory.

39: We follow [Ren].

Start by recalling the non-abelian Chern-Simons in local coordinates

$$S_{CS} = \frac{\kappa}{4\pi}\int_M dx^3\ \epsilon^{\mu\nu\rho}\ \text{Tr}\left(A_\mu(\partial_\nu A_\rho - \partial_\rho A_\nu) + \frac{2}{3}A_\mu[A_\nu, A_\rho]\right),$$

where $A = A_\mu dx^\mu$, with $\mu = 1, 2, 3$, was the connection and $A_\mu = A_\mu^a T_a$ was expanded in the basis T_a, with $a = 1, \ldots, \dim(G)$, generating the corresponding Lie algebra with corresponding Lie group G. Here, the gauge group $G = SL(2, \mathbb{C})$ is chosen, viewed as a complexification of the $SU(2)$ Lie group considered before.[40] Taking the trace and using the Killing metric form[41] $\mathcal{K}_{ab} = R\eta_{ab}$ with $\eta_{ab} = \mathrm{diag}(-1, +1, +1)$ the Minkowski metric, the CS action takes the form

40: This choice will appear as well in the coming chapter, when invariants corresponding to the Lie algebra choice $\mathfrak{sl}_2$ will be considered.

41: The **Killing form** is $\mathcal{K}$ with components $\mathcal{K}_{ab} = \mathrm{Tr}(T_a T_b)$.

$$S_{CS} = \frac{\kappa R}{4\pi} \int_M d^3x \, \left(A_\mu^a \partial_\nu A_{a\rho} + \frac{1}{3} f_{abc} A_\mu^a A_\nu^b A_\rho^c \right) \epsilon^{\mu\nu\rho},$$

where f_{abc} are the structure constants of the Lie algebra given by $[T_a, T_b] = f_{ab}{}^c \, T_c$, up to contractions with the metric. We will come to this expression for the non-abelian CS theory later on.

On the other hand, Gravity is described as a field theory through the Einstein-Hilbert action

$$S_{EH}[g_{\mu\nu}] = \kappa \int_M d^{(D+1)}x \, \sqrt{-g}(R - 2\Lambda)$$

where M is a $(D+1)$-dimensional Lorentzian manifold, g denotes the determinant of the metric $g_{\mu\nu}$, R is the scalar curvature and the cosmological constant Λ has been added.[42] In the present case, we are interested in $D = 2$. The equations of motion are obtained from Hamilton's principle of least action $\frac{\delta S}{\delta g_{\mu\nu}} = 0$, yielding the famous Einstein equations

42: Usual gravity is recovered when $d = 3$ and $\kappa = \frac{c^4}{16\pi G}$.

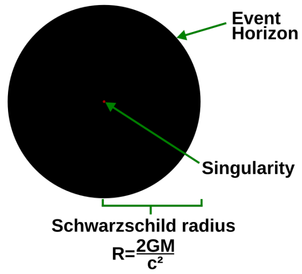

Figure 3.11: Diagram of a Black Hole, delimited by its event horizon at a distance given by Schwarzschild's radius. The information paradox at the singularity may be solved by means of a *Fuzzball*.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0.$$

It can be seen, [Fec06], that the Einstein-Hilbert action can be written in the form

$$\begin{aligned} S_{\text{Cartan}}[e, \omega] = &- \frac{1}{8\pi G} \int_M d^3x \, \left\{ e^a \wedge \left(d\omega_a + \frac{1}{2} \epsilon_{abc} \, \omega^b \wedge \omega^c \right) \right\} \\ &+ \frac{1}{8\pi G} \int_M d^3x \, \left(\frac{\Lambda}{3!} \epsilon_{abc} \, e^a \wedge e^b \wedge e^c \right), \end{aligned}$$

where e^a are the so-called *dreibein* and ω^a the components of the spin connection. Restricting to the case of negative cosmological constant $\Lambda = -1/\ell^2$ —since it is the only one where black hole[43] type solutions appear— and defining the variables

43: Fuzzballs are theorized by some superstring theory scientists to be the true quantum description of black holes.
Fuzzball theory removes the singularity at the heart of a black hole by replacing the whole interior region within its event horizon by a ball of strings, considered as the ultimate building blocks of matter and energy. These are thought of as bundles of energy constantly vibrating in complex ways in the three physical dimensions of space and further in extra, compact dimensions in the *quantum* or *spacetime foam*.

$$A^{\pm a} = \omega^a \pm \frac{1}{\ell} e^a,$$

the action takes the following form

$$\begin{aligned} S = &- \frac{\ell}{32\pi G} \int_M \left(A^{+a} \wedge dA_a^+ + \frac{1}{3} \epsilon_{abc} \, A^{+a} \wedge A^{+b} \wedge A^{+c} \right) \\ &+ \frac{\ell}{32\pi G} \int_M \left(A^{-a} \wedge dA_a^- + \frac{1}{3} \epsilon_{abc} \, A^{-a} \wedge A^{-b} \wedge A^{-c} \right) \\ &- \frac{\ell}{32\pi G} \int_M d \left(A^{+a} \wedge A_a^- \right). \end{aligned}$$

Comparing this expression with the one for the Chern-Simons action S_{CS} found before, we see that taking[44]

44: In particular, this means that the chosen Lie algebra is determined by the commutation relations

$$[T_a, T_b] = \epsilon_{ab}{}^c T_c,$$

which are precisely satisfied by the (rescaled) $\mathfrak{sl}_2$ Lie algebra generators.

$$\kappa R = -\frac{\ell}{8G} \quad \text{and} \quad f_{abc} = \epsilon_{abc},$$

the Einstein-Hilbert action describing Gravity can be expressed in terms of two copies of the Chern-Simons action

$$S_{EH} = S_{CS}[A^+] - S_{CS}[A^-] + B.T.$$

plus a boundary term that vanishes for suitable boundary conditions.[45]

45: According to Stokes theorem, the integral of an exact form can be expressed as an integral over the manifold's boundary

$$\int_M d\xi = \int_{\partial M} \xi,$$

which vanishes for instance if $\partial M = \emptyset$.

Quantum Knot Invariants | 4

Resurgence is the process of recovering non-perturbative features of a function from its asymptotic (perturbative) expansion. This is very useful in quantum mechanics and quantum field theory.

We are interested in applying resurgence analysis to the Chern-Simons functional. This was done for closed 3-manifolds, and we will show how the same techniques can be used for knot complements.

Sergei Gukov and Ciprian Manolescu [GM19]

In their recent work [GM19], S. Gukov and C. Manolescu theorized about the existence of a two-variable series, denoted $F_K(x, q)$ for every knot K, which may be viewed as an analytical continuation of the colored Jones polynomial*. Here, the large color R-matrix approach discussed in [Par20] is followed to study F_K for some links and extend these results to other Lie algebras different from the building block $\mathfrak{sl}_2$, such as $\mathfrak{so}(2n)$ for small $n \in \mathbb{N}$.

By means of the theory of quantum groups [Kas95], we first derive the R-matrix for $\mathfrak{sl}_2$ by working with the symmetric representation and a suitable basis. This R-matrix is in turn implemented in a `Mathematica` notebook, where the F_K series for different links in its braid descriptions can be obtained. The Jones polynomial is shown to recover by setting $x = q^2$. Having understood this case, we present how to extend these results for other Lie algebras such as the first $\mathfrak{so}(2n)$, $n = 2, 3, 4 \ldots$, corresponding to the D_n family of Dynkin diagrams. The main difficulty here being to find a suitable description for the symmetric representation of $\mathfrak{so}(8)$ and its quantum version.

4.1 Braiding and Large Color R-matrix

Let us start by introducing the F_K series and how it may be obtained. The story begins with the invariant $\hat{Z}_a(Y; z, n, q)$ for knot complements[1] $Y = \hat{Y} \backslash \nu K$ introduced by S. Gukov and C. Manolescu in [GM19], which consists in a two-variable series in z and q depending on a choice of a relative Spin^c structure $a \in \text{Spin}^c(Y, \partial Y)$ and a parameter $n \in \mathbb{Z}$. It turns out that in the case where the weakly negative definite plumbed manifold Y is the complement of a knot in an integral homology sphere $\hat{Y}$, all the different $\hat{Z}_a(Y; z, n, q)$ are given by a single two-variable series

$$F_K(x, q) := \hat{Z}_0(Y; x^{1/2}, n, q) \in 2^{-c} q^{\Delta} \mathbb{Z}[x^{1/2}, x^{-1/2}][q, q^{-1}]]$$

1: These knot complements are represented by plumbing graphs with one distinguished vertex, as the ones described in the end of Chapter 2.

* The *colored* Jones polynomial is the analogue of the Jones polynomial of a knot when decorated or *colored* by choosing the n-dimensional representation of $\mathfrak{sl}_2 := \mathfrak{sl}(2, \mathbb{C})$ (instead of the fundamental one, where elements of $\mathfrak{sl}_2$ act naturally as 2 by 2 matrices, i.e. $n = 2$). The *reduced* colored Jones polynomials, denoted $J_K(n; q)$, are the ones normalized so that they yield 1 for the unknot.

where $c \in \mathbb{Z}_+$, $\Delta \in \mathbb{Q}$ and $\mathbb{Z}[x^{1/2}, x^{-1/2}][q, q^{-1}]]$ denotes the ring of Laurent power series in q with coefficients in the polynomial ring $\mathbb{Z}[x^{1/2}, x^{-1/2}]$. It is in [Par20] where S. Park proposed a way to actually compute the $F_K(x, q)$ series for a wide class of knots[2] by using the so-called "Large Color R-matrix".

2: At least, bigger than the previously known ones, since F_K had only been obtained for torus knots and —in an experimental way— the figure-eight knot.

To understand the idea, let us introduce some concepts (we refer the reader to [PK16; KRT]). Start by considering what is called a **braid**,[3] a mathematical object which is going to be very useful for our purposes. A braid consists in n vertical (possibly crossing) strings between a lower and an upper horizontal bar.[4] When there is no crossing at all (see Figure 4.1), one calls it the **trivial braid** or identity braid e. When there is a single overcrossing of the i-th string with the $(i+1)$-th string (see Figure 4.2), we denote the braid as σ_i —the undercrossing being denoted σ_i^{-1}. One then realizes that these elements form a group under composition of these generators σ_i, which is called the **braid group** B_n and has neutral element e.

3: The name *braid* comes from the familiar notion of hair braids.

4: The n here is different from the one appearing in $\hat{Z}_a(Y; z, n, q)$.

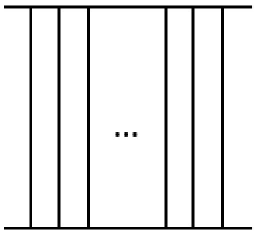

Figure 4.1: Trivial braid.

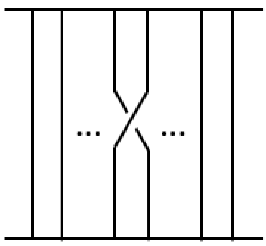

Figure 4.2: Generator σ_i of the braid group B_n.

Definition 4.1.1 (Braid group) *Given an integer $n \geq 3$, the **braid group** with n strands is the group B_n generated by $n-1$ elements $\sigma_1, \ldots, \sigma_{n-1}$ and relations*

$$\begin{aligned} &\sigma_i\sigma_j = \sigma_j\sigma_i && \text{if } |i-j| > 1, \\ &\sigma_i\sigma_{i+1}\sigma_i, = \sigma_{i+1}\sigma_i\sigma_{i+1} && \text{for } 1 \leq i, j \leq n-1. \end{aligned}$$

A general element of the braid group can be seen in solid lines in Figure 4.3. Observe in the dotted lines how a knot or link can be obtained by identifying opposite ends of braid. This process is known as **closure of braids**. The key statement is that *any* link can be obtained in such a way. This theorem was proven by Alexander. Although the mapping of braids to knots is not one to one, there is a set of moves known as **Markov moves** under which braids can be transformed while still keeping invariant the corresponding knot obtained by braid closure. These are just two moves, given by

$$\mathrm{I} : AB \longleftrightarrow BA, \qquad \mathrm{II} : A \longleftrightarrow A\sigma_n^{\pm}$$

for $A, B \in B_n$ and $A\sigma_n^{\pm} \in B_{n+1}$, and sketched in Figure 4.4 and Figure 4.5. The steps to follow to obtain a topological invariant for knots and links are then:

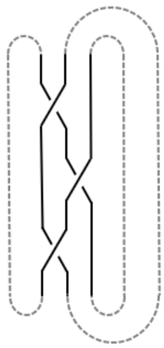

Figure 4.3: A knot or link may be obtained from a braid by closing it, i.e. identifying opposite end strands.

- Construct a braid representation $\rho : B_n \to \mathrm{Aut}(V)$.
- Find a **trace**[5] operation ρ such that it is invariant under these Markov moves, defining thus a knot invariant.

Here is where R-matrices come into play. Recall its definition.

5: The term *trace* refers to an operation satisfying the usual cyclic relation for matrix traces $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ —motivated here by demanding invariance under Markov move I—, although further conditions on this operation may be required.

Definition 4.1.2 *Let V be a vector space over $\mathbb{K}$. The **Yang-Baxter equation** is the following equation for a linear automorphism c of $V \otimes V$:*

$$(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c),$$

*which holds in the automorphism group of $V \otimes V \otimes V$. A solution is called an R-**matrix**.*

In a basis $\{v_i\}$ of V, such an automorphism c may be given by

$$c(v_i \otimes v_j) = \sum_{k,\ell} c_{ij}^{k\ell} v_k \otimes v_\ell$$

for some scalars $\{c_{ij}^{k\ell}\}$. The Yang-Baxter equation takes then the form

$$\sum_{p,q,y} c_{ij}^{pq} c_{qk}^{yn} c_{py}^{\ell m} = \sum_{y,q,r} c_{jk}^{q,r} c_{iq}^{\ell y} c_{yr}^{mn}, \quad \forall i,j,k,\ell,m,n.$$

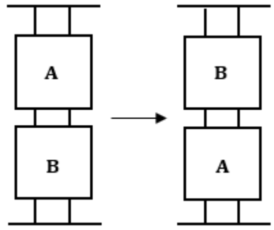

Figure 4.4: Markov move I.

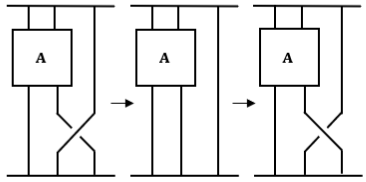

Figure 4.5: Markov move II.

Solving these equations is a highly non-trivial problem. Nevertheless, great progress in finding solutions to the Yang-Baxter equation has been made in the last decades. Some of them, for instance, appear in the context of the theory of Quantum Groups. This is in fact the underlying mathematical structure on which the Large Color R-matrix is defined, the choice of vector space V being given by a representation of the selected quantum group.

Now, these solutions to the Yang-Baxter equation play a role here in the discussion on braids because of the following result stating that R-matrices give rise to representations of the braid groups B_n, as desired.

Proposition 4.1.1 *Let V be a vector space and $c \in \mathrm{Aut}(V \otimes V)$ a solution to the Yang-Baxter equation, i.e. an R-matrix. Then, for any $n > 0$, there exists a unique homomorphism $\rho_n^c : B_n \to \mathrm{Aut}(V^{\otimes n})$ such that $\rho_n^c(\sigma_i) = c_i$ for $i = 1, \dots, n-1$, where the c_i are the linear automorphisms of the n-fold tensor power $V^{\otimes n}$ given by*

$$c_i = \begin{cases} c \otimes \mathrm{id}_{V^{\otimes(n-2)}} & \text{if } i = 1, \\ \mathrm{id}_{V^{\otimes(i-1)}} \otimes c \otimes \mathrm{id}_{V^{\otimes(n-i-1)}} & \text{if } 1 < i < n-1, , \\ \mathrm{id}_{V^{\otimes(n-2)}} \otimes c & \text{if } i = n-1. \end{cases}$$

This representation allows us to think of a given braid in the following way. Slice the braid $\beta \in B_n$ horizontally into finitely many blocks in such a way that each block has a single over or under crossing (see Figure 4.6). Assign to each block the automorphism of the n-fold tensor $V^{\otimes n}$ consisting of id_V for each vertical strand and an R-matrix (or its inverse R^{-1}) for each overcrossing (or undercrossing). Composing from bottom to top the different operators corresponding to each sliced block, one obtains an automorphism $\rho_\beta := \rho_n^R(\beta) \in \mathrm{Aut}(V^{\otimes n})$ for the braid.[6] The knot invariant is then obtained by taking its trace.

6: The representation corresponding to Figure 4.6 is the map $\rho_\beta : V^{\otimes 4} \to V^{\otimes 4}$ given by

$$\begin{array}{c} V \otimes V \otimes V \otimes V \\ \uparrow {\scriptstyle \mathrm{id}_V \otimes R \otimes \mathrm{id}_V} \\ V \otimes V \otimes V \otimes V \\ \uparrow {\scriptstyle R^{-1} \otimes \mathrm{id}_V \otimes \mathrm{id}_V} \\ V \otimes V \otimes V \otimes V \\ \uparrow {\scriptstyle \mathrm{id}_V \otimes R \otimes \mathrm{id}_V} \\ V \otimes V \otimes V \otimes V \end{array}$$

composed from bottom to top.

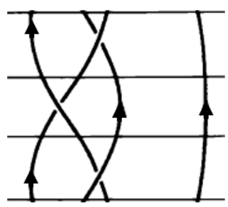

Figure 4.6: A braid $\beta \in B_4$ sliced in three blocks each containing an over or undercrossing.

In general, the trace of an endomorphism is defined as follows (see [Loc02]), where the evaluation and coevaluation maps —e_V, i_V— and the bidual isomorphism —δ_V— appearing in the definition are provided by the *ribbon category*, which is a braided category with some additional structure.

Definition 4.1.3 (Trace) *Let V be an object in a ribbon category $\mathcal{C}$ and f an endomorpshism of V. Then, the **trace** of f, denoted $\mathrm{Tr}(f) \in \mathrm{End}(\mathbf{1}) \cong \mathbb{K}$, is defined as the composition*

$$\mathbf{1} \xrightarrow{i_V} V \otimes V^* \xrightarrow{f \otimes \mathrm{id}_{V^*}} V \otimes V^* \xrightarrow{\delta_V \otimes \mathrm{id}_{V^*}} V^{**} \otimes V^* \xrightarrow{e_{V^*}} \mathbf{1}.$$

This generalizes in the following way. Given a braid $\beta \in B_n$ with n strands, let the corresponding automorphism $f = \rho_\beta : V^{\otimes n} \to V^{\otimes n}$ be given on a basis $\{v_i\}$ of V by

$$v_{i_1} \otimes \cdots \otimes v_{i_n} \mapsto \sum_{k_1,\ldots,k_n} f^{k_1,\ldots,k_n}_{i_1,\ldots,i_n} v_{k_1} \otimes \cdots \otimes v_{k_n}.$$

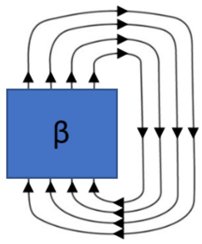

Figure 4.7: The closure of a braid β gives a knot, whose invariant is obtained by the tracing operation.

Now consider the closure of this braid (as depicted in Figure 4.7) and read it from bottom to top, starting with no strands at all —situation to which we assign the ground field $\mathbb{K}$. To each strand pointing upwards assign the vector space V, and to each strand pointing downwards assign its dual V^*. Keep reading upwards till encountering the n bottom lines from right to left —to which we assign coevaluations $i_V : \mathbb{K} \to V \otimes V^*$ given by $1 \mapsto \sum_i v_i \otimes v^i$, where $\{v^i\}$ is the basis of V^* dual to $\{v_i\}$. For the n strands one gets then a map

$$\begin{aligned} i_{V^{\otimes n}} &: \mathbb{K} \to V^{\otimes n} \otimes (V^*)^{\otimes n} \\ & 1 \mapsto \sum_{i_1,\ldots,i_n} v_{i_1} \otimes \cdots \otimes v_{i_n} \otimes v^{i_1} \otimes \cdots \otimes v^{i_n} \end{aligned}$$

creating n left upward strands and n right downward strands out of nothing, corresponding to $V^{\otimes n}$, $(V^*)^{\otimes n}$ and $\mathbb{K}$, respectively. Procede reading upwards and encounter the braid β and n downward strands on its right hand side. To this we assign the operator

$$\begin{aligned} f \otimes \mathrm{id}_{(V^*)^{\otimes n}} &: V^{\otimes n} \otimes (V^*)^{\otimes n} \to V^{\otimes n} \otimes (V^*)^{\otimes n} \\ & \sum_{i_1,\ldots,i_n} v_{i_1} \otimes \cdots \otimes v_{i_n} \otimes v^{i_1} \otimes \cdots \otimes v^{i_n} \mapsto \\ & \mapsto \sum_{i_1,\ldots,i_n} \sum_{k_1,\ldots,k_n} f^{k_1,\ldots,k_n}_{i_1,\ldots,i_n} v_{k_1} \otimes \cdots \otimes v_{k_n} \otimes v^{i_1} \otimes \cdots \otimes v^{i_n} \end{aligned}$$

where recall we are using the notation $f = \rho_\beta$. Finally, read the top part of the braid closure, assigning evaluation maps $e_V : V \otimes V^* \to \mathbb{K}$ given by $v_i \otimes v^j \mapsto v^j(v_i) = \delta^j_i$, where the dual isomorphism $V \to V^{**}$ is implicitly used. For n strands we get

$$\begin{aligned} e_{V^{\otimes n}} &: V^{\otimes n} \otimes (V^*)^{\otimes n} \to \mathbb{K} \\ & \sum_{i_1,\ldots,i_n} \sum_{k_1,\ldots,k_n} f^{k_1,\ldots,k_n}_{i_1,\ldots,i_n} v_{k_1} \otimes \cdots \otimes v_{k_n} \otimes v^{i_1} \otimes \cdots \otimes v^{i_n} \mapsto \\ & \mapsto \sum_{i_1,\ldots,i_n} \sum_{k_1,\ldots,k_n} f^{k_1,\ldots,k_n}_{i_1,\ldots,i_n} v^{i_1}(v_{k_1}) \otimes \cdots \otimes v^{i_n}(v_{k_n}) \\ & = \sum_{i_1,\ldots,i_n} \sum_{k_1,\ldots,k_n} f^{k_1,\ldots,k_n}_{i_1,\ldots,i_n} \delta^{i_1}_{k_1} \otimes \cdots \otimes \delta^{i_n}_{k_n} = \sum_{i_1,\ldots,i_n} f^{i_1,\ldots,i_n}_{i_1,\ldots,i_n} = \mathrm{Tr}(f) \end{aligned}$$

annihilating all the n upward strands with the n downward strands. The trace is then this state sum operation of *creation*, braid action and subsequent *annihilation* given by the linear operator

$$\mathbb{K} \xrightarrow{i_{V^{\otimes n}}} V^{\otimes n} \otimes (V^*)^{\otimes n} \xrightarrow{f \otimes \mathrm{id}_{(V^*)^{\otimes n}}} V^{\otimes n} \otimes (V^*)^{\otimes n} \xrightarrow{e_{V^{\otimes n}}} \mathbb{K}$$

defining thus a scalar $\mathrm{Tr}(f) \in \mathbb{K}$.

In the context of Quantum Groups, there is an analogue called **quan-**

tum trace differing by some factor of q,

$$\mathrm{Tr}_q(f) = \mathrm{Tr}(q^{2\rho} f),$$

given by quantum versions for the evaluation and coevaluation maps.

For example, the unreduced V_n-colored[7] Jones polynomial of a knot K when presented as the closure of a braid β can be recovered by

$$\tilde{J}_K(n;q) = q^{-\frac{n^2-1}{4}w(\beta)}\mathrm{Tr}_q(\rho_\beta)$$

where $w(\beta)$ is the *writhe*[8] of the braid. In the case being considered here, however, the framing factor wont be relevant, since only 0-framed knots are considered.

To obtain reduced invariants, the trace is carried out over the $i_2, \dots, i_n$ leaving the first strand open (Figure 4.8). The choice of i_1 is not relevant when choosing an irreduccible representation, since Schur's lemma guarantees that the resulting map $V_n \to V_n$ be central, i.e. a constant multiple of the identity: $C \cdot \mathrm{id}$ for some $C \in \mathbb{K}$. The factor C is then the reduced V_n-colored Jones polynomial $J_K(n;q)$.

7: Where V_n is the irreducible n-dimensional representation of the quantum group $U_q(\mathfrak{sl}_2)$ —the notation n being arbitrary and different from the one in B_n, and will actually be denoted r in coming pages.

8: A topological invariant of a knot given by the number of positive crossings minus the number of negative crossings.

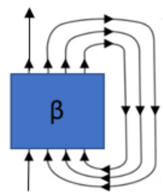

Figure 4.8: Right-closure of a braid with the leftmost strand open.

Notice how in this whole construction of the trace operation underlying the invariant, the building block is precisely the R-matrix representing the crossings in the braiding. The one used to compute the F_K invariant series, and which recovers the Jones polynomials, is a special R-matrix —called **Large Color R-matrix**— given by considering infinite-dimensional modules[9] of the chosen quantum group (e.g. $U_q(\mathfrak{sl}_2)$ for the $\mathfrak{sl}_2$ Lie algebra) appearing in the large color limit of V_n. This ends up giving a two-variable series in q and $x := q^n$, which is precisely the $F_K(x,q)$ invariant.[10] For details, we refer the reader to [Par20], where the Large Color R-matrix is defined for the special case $U_q(\mathfrak{sl}_2)$.

9: Verma modules.

10: Although $n \in \mathbb{Z}_+$, after this change the variable x is considered to be generic.

To deal with the general case, one obtains the Large Color R-matrix by taking the n-dimensional irreducible representation of the quantum group taken into consideration, which (from now on and for convenience, using the notation r instead of n) will be the r-th symmetric representation defined over the fundamental representation, and subsequently replacing $q^r \mapsto x$. This ends up yielding an R-matrix in two variables, $R(x,q)$, which is then employed in the state sum operation of tracing, giving the $F_K(x,q)$ invariant.

4.2 The $\mathfrak{sl}_2$ case

The Lie algebra[11] $\mathfrak{sl}(2,\mathbb{C})$ —or $\mathfrak{sl}_2$ for short— owes its name to its corresponding Lie group $SL(2,\mathbb{C})$, the special linear group consisting of invertible matrices with unit determinant.[12] Given a Lie group G, one can obtain a corresponding Lie algebra $\mathfrak{g}$ as the tangent space to G at the identity element e of the group. This construction deals with the so-called left invariant vector fields and involves a certain amount of calculations. There is, however, a shortcut argument in the case of matrix Lie groups[13] that shows how this can be achieved more directly.

11: We refer the reader to Appendix B for a review on Lie algebras and their Representation Theory, containing a large amount of the notions used throughout this chapter.

12: Observe that $\mathfrak{sl}(2,\mathbb{C})$ is the **complexification** of $\mathfrak{su}(2)$ —the Lie algebra of 2×2 traceless skew-Hermitian matrices corresponding to the Lie group $SU(2)$ of unitary matrices with unit determinant we used when introducing the non-abelian Chern-Simons theory—, meaning that

$$\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{su}(2) \oplus i\mathfrak{su}(2)$$

given by the Lie algebra isomorphism

$$\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{su}(2) \to \mathfrak{sl}(2,\mathbb{C})$$
$$1 \otimes X_1 + i \otimes X_2 \mapsto X_1 + \left(\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}\right) X_2$$

for $X_1, X_2 \in \mathfrak{su}(2)$ any two Lie algebra generators.

13: That is, closed subgroups of the general linear group $GL(n,\mathbb{C})$.

It is well-known that there exists a map, the **exponential map**[14]

$$\begin{aligned}\exp : \mathfrak{g} &\to G \\ A &\mapsto \exp(A),\end{aligned}$$

allowing to recover the local group structure of the Lie group from its Lie algebra.[15] The idea is to figure out through this exponential map what the conditions imposing the Lie algebra structure must be, given the ones for the Lie group. For instance, consider any *small* element ϵA in $\mathfrak{sl}_2$ and obtain a corresponding element in $SL(2,\mathbb{C})$ by exponentiating it, which will be close to the identity element of the group. We want to impose that it indeed belongs to the Lie group for small $\epsilon > 0$, so we only have to keep track of the first order terms in ϵ. Hence, we directly simplify the exponential to this order, writing it as $\mathbb{1} + \epsilon A$. Imposing that this element belongs to the Lie group $SL(2,\mathbb{C})$,

$$1 = \det(\mathbb{1} + \epsilon A) = \det \begin{pmatrix} 1+\epsilon a & \epsilon b \\ \epsilon c & 1+\epsilon d \end{pmatrix} = 1 + \epsilon \mathrm{Tr}(A) + \mathcal{O}(\epsilon^2),$$

and looking at the first order term, we see that the condition to be satisfied for A is clearly $\mathrm{Tr}(A) = 0$.[16] Thus, $\mathfrak{sl}_2$ consists of the vector space over $\mathbb{C}$ of 2×2 traceless matrices together with the commutator as Lie bracket. The standard notations and generators for $\mathfrak{sl}_2$ are

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which satisfy the commutation relations

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h.$$

Observe, thus, that we have three generators e, f and h of $\mathfrak{sl}_2$ *as a vector space*, while only two —say e and f— suffice to generate it as a *Lie algebra*: $h = [e,f]$.

14: In the case of matrix Lie groups, this map is precisely the **exponential power series**

$$e^{sX} = \sum_{k=0}^{\infty} \frac{s^k}{k!} X^k.$$

15: [See Appendix B.1]. The exponential map is a **local diffeomorphism**: there exists $U \subset T_e G$ an open set containing zero such that the restriction

$$\exp|_U : U \to \exp(U) \subset G$$

is bijective, smooth and with smooth inverse $\log = \exp|_U^{-1}$.
The Lie group is completely recovered when exp is surjective (e.g. if G is compact).

16: This method is equivalent to applying the **Regular Value Theorem** showing that $\mathfrak{g} = T_e G$ is given by the kernel of the differential of the function f —defining the Lie group as a closed manifold by the condition $f(\cdot) = 0$— evaluated at the identity $e := \mathrm{Id}$, that is:
A path $\gamma(t) \in G$ such that $\gamma(0) = \mathrm{Id}$ and $\gamma'(0) = A \in T_e G = \mathfrak{g}$, is necessarily of the form $\gamma(t) = \mathrm{Id} + tA + \mathcal{O}(t^2)$. Then, for $G = SL(2,\mathbb{C})$ given by the condition $f(X) = \det(X) - 1 = 0$ for $X \in G$, differentiating $f(\gamma(t)) = \det(\mathrm{Id} + tA + \mathcal{O}(t^2)) - 1$ at $t = 0$ —corresponding to the identity $\gamma(0) = \mathrm{Id}$— precisely gives the condition of traceless matrices.

Symmetric Representation

One always has the fundamental representation[17] V_2 given by acting simply by matrix multiplication. We wish to deal with the more interesting r-th symmetric representations, which will give us an asymptotic behavior for large r. These are defined as

$$\mathrm{Sym}^r V_2 := \{(v_1, \dots, v_r) \in V_2^r\}/\mathcal{S}_r.$$

Typically one works with the symmetric tensor product representation, where the r-th vector space V_r is given by symmetrizing the r-th tensor product of copies of V_2 and the action follows from the algebra comultiplication map $\Delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ given by $\Delta(x) \cdot v_1 \otimes v_2 = x \cdot v_1 \otimes v_2 + v_1 \otimes x \cdot v_2$, extended iteratively as $\Delta^k = (\Delta^{k-1} \otimes \mathbb{1}) \circ \Delta$, obtaining

$$\Delta^r(x) \cdot v_1 \otimes \cdots \otimes v_r = x \cdot v_1 \otimes \cdots \otimes v_r + \cdots + v_1 \otimes \cdots \otimes x \cdot v_r.$$

Notice that elements acting on symmetric tensors under index permutation will give the same coefficients. As for the case of $\mathfrak{sl}_2$, we will be interested in finding the highest weight vector of this representation

17: Usually one refers to a **representation** $\rho : \mathfrak{g} \to \mathrm{End}(V)$ of a Lie algebra $\mathfrak{g}$ both as being the morphism ρ and the vector space V. And typically one omits the notation $\rho(x)v$ (note that $\rho(x) \in \mathrm{End}(V)$ is an endomorphism, so it may act on a vector $v \in V$) and denotes simply $x \cdot v$, being the action understood by the context. Notice, as well, that a representation may be viewed as a **$\mathfrak{g}$-module** whose action is given by

$$\begin{aligned}\mathfrak{g} \times V &\to V \\ (x,v) &\mapsto x \cdot v\end{aligned}$$

and the corresponding (unique) highest weight. This requires that such vector v be annihilated[18] by the lowering operator e and at the same time be an eigenvector of h. In this very case, the vector $w = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in V_2$ is trivially an eigenvector for h on the fundamental representation and is as well annihilated by e. This highest weight vector with highest weight $\lambda = 1$ extends naturally to a highest weight vector for the (symmetric) tensor product representation to the vector $w_r = w \otimes \cdots \otimes w \in V_r$ with highest weight $\lambda = r$, since it trivially satisfies the required conditions just by looking at the expressions:

18: The highest weight vector v sits "at the bottom of the ladder" defined by the raising action f and lowering e (also known as **ladder operators**). At the bottom, e lowers v acting by zero, annihilating the vector. Then, repeated action of f on v ascends through the *ladder* yielding a full set of vectors forming a basis for the considered representation.

$$\begin{aligned} \Delta^r(e) \cdot w_r &= e \cdot w \otimes \cdots \otimes w + \cdots + w \otimes \cdots \otimes e \cdot w = 0 \\ \Delta^r(h) \cdot w_r &= h \cdot w \otimes \cdots \otimes w + \cdots + w \otimes \cdots \otimes h \cdot w = r \cdot w_r \end{aligned}$$

We are about to introduce the polynomial representation. As will be shown, such representation is isomorphic to the symmetric representation. This is done by invoking the Highest Weight Theorem, which amounts to showing that it is an irreducible representation whose highest weight vector has the same highest weight r.

Polynomial Representation

Beyond the fundamental representation where $SL(2,\mathbb{C})$ acts naturally by matrix multiplication on vectors $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbb{C}^2$, one can let $SL(2,\mathbb{C})$ act also on $\mathbb{C}[x,y]$ as[19]

19: Observe the need of the inverse to indeed define a group action:

$$\begin{aligned} (BA) \cdot p(x,y) &= p((BA)^{-1} \cdot (x,y)) \\ &= p(A^{-1}B^{-1} \cdot (x,y)) \\ &= B \cdot p(A^{-1} \cdot (x,y)) \\ &= B \cdot (A \cdot p(x,y)). \end{aligned}$$

$$A \cdot p(x,y) = p(A^{-1} \cdot (x,y)),$$

where $p(x,y) \in \mathbb{C}[x,y]$, $A \in SL(2,\mathbb{C})$ and $A^{-1} \cdot (x,y)$ is the standard representation acting by matrix multiplication $A^{-1}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. One can then use the derived action:

Definition 4.2.1 *Given a Lie group G with Lie algebra $\mathfrak{g}$ and a smooth G-action on a set V, the corresponding **derived action** of $\mathfrak{g}$ on V is given by*

$$X \cdot v := \frac{d}{dt}(\exp(tX) \cdot v)\Big|_{t=0}$$

where $X \in \mathfrak{g}$ and $v \in V$.

This indeed defines a Lie algebra action and, for V a complex vector space, it allows us to obtain a Lie algebra representation of $\mathfrak{g}$ from a Lie group representation of G.

For instance, we may see how the thus induced action acts on $\mathbb{C}[x,y]$ by letting the generators of $\mathfrak{sl}_2$ act on the degree-r monomials $x^{r-k}y^k$. We obtain:

$$\begin{aligned} e \cdot x^{r-k}y^k &= \frac{d}{dt}\left(\exp(te) \cdot x^{r-k}y^k\right)\Big|_{t=0} = \frac{d}{dt}\left((x-ty)^{r-k}y^k\right)\Big|_{t=0} \\ &= -(r-k)x^{r-k-1}y^{k+1} = -y\frac{\partial}{\partial x}\left(x^{r-k}y^k\right) \end{aligned}$$

where we have used that the inverse of $\exp(tX)$ is $\exp(-tX)$ and the fact that $e^2 = 0 \Rightarrow \exp(te) = \mathbb{1} + te$. Similarly, for f and h we obtain

$$
\begin{aligned}
f \cdot x^{r-k}y^k &= \frac{d}{dt}\left(\exp(tf)\cdot x^{r-k}y^k\right)\Big|_{t=0} = \frac{d}{dt}\left(x^{r-k}(-tx+y)^k\right)\Big|_{t=0} \\
&= -kx^{r-k+1}y^{k-1} = -x\frac{\partial}{\partial y}\left(x^{r-k}y^k\right)
\end{aligned}
$$

and, with $\exp(th) = \cosh(t)\mathbb{1} + \sinh(t)h$,[20]

$$
\begin{aligned}
h \cdot x^{r-k}y^k &= \frac{d}{dt}\left(\exp(th)\cdot x^{r-k}y^k\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(((\cosh t - \sinh t)x)^{r-k}((\cosh t + \sinh t)y)^k\right)\Big|_{t=0} \\
&= -(r-k)x^{r-k}y^k + kx^{r-k}y^k = \left[-x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right]\left(x^{r-k}y^k\right)
\end{aligned}
$$

20: Indeed, by using $h^2 = \mathbb{1}$ and thus $h^{2k} = \mathbb{1}$ and $h^{2k+1} = h$ for $k \geq 0$, we have

$$
\begin{aligned}
\exp(th) &= \sum_{k=0}^{\infty}\frac{t^k}{k!}h^k \\
&= \sum_{k=0}^{\infty}\frac{t^{2k}}{(2k)!}\mathbb{1} + \sum_{k=0}^{\infty}\frac{t^{2k+1}}{(2k+1)!}h \\
&= \cosh(t)\mathbb{1} + \sinh(t)h.
\end{aligned}
$$

These generators indeed satisfy the commutation relations for $\mathfrak{sl}_2$. For convenience, however, we prefer to use these other slightly modified generators which still satisfy the commutation relations[21]

21: All signs have been reversed and the swap $e \leftrightarrow f$ has been applied. This is a symmetry of the commutation relations.

$$
e = x\frac{\partial}{\partial y}, \quad f = y\frac{\partial}{\partial x}, \quad h = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.
$$

Observe that $V := \mathbb{C}[x,y]$ splits now in the different subspaces V_r of degree-r polynomials

$$
\mathbb{C}[x,y] = \bigoplus_{r\geq 0} V_r,
$$

where every V_r is irreducible. Indeed, we first observe that $\{x^{r-k}y^k\}_{k=0}^r$ is a basis of V_r (with $\dim(V_r) = r+1$) and the generators act on this basis elements as

$$
e \cdot x^m y^n = nx^{m+1}y^{n-1}, \quad f \cdot x^m y^n = mx^{m-1}y^{n+1}, \quad h \cdot x^m y^n = (m-n)x^m y^n
$$

and thus (setting $r = m+n$) the V_r are invariant subspaces:

$$
e \cdot V_r \subset V_r, \quad f \cdot V_r \subset V_r, \quad h \cdot V_r \subset V_r, \qquad r \geq 0.
$$

Further, let $W \subset V_r$ be a non-zero invariant subspace under the action of the generators and $p_r \in W \subset V_r$ a non-zero degree-r polynomial $p_r(x,y) = \sum_{k=0}^r c_k x^{r-k}y^k$. Let it act by e exactly r times such that we are only left with $e^r \cdot p_r = r!c_r x^k = cx^r \in W \subset V_r$, concluding that $x^r \in W$. Let now f act on this element to iteratively obtain $f \cdot x^r = rx^{r-1}y \in W$, $f^2 \cdot x^r = r(r-1)x^{r-2}y^2 \in W, \ldots, f^r \cdot x^r = r!y^r \in W$. Thus, the $\{f^k \cdot x^r\}_{k=0}^r \subset W$ and, given the obtained expressions, they are known to span all of V_r. Hence, $W = V_r$ and V_r is irreducible.

Definition 4.2.2 *Given a representation V of a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and $\lambda : \mathfrak{h} \to \mathbb{C}$ a linear functional on its Cartan subalgebra $\mathfrak{h}$, define the **weight space** of V with weight λ as the subspace*

$$
V[\lambda] := \{v \in V \mid \forall h \in \mathfrak{h}, \quad h \cdot v = \lambda(h)v\}.
$$

*A **weight** of V is then a linear functional λ such that $V[\lambda]$ is nonzero, and nonzero elements in $V[\lambda]$ are called **weight vectors**. Thus,*

weight vectors are simultaneous eigenvectors for the action of the elements of $\mathfrak{h}$ with eigenvalue λ.[a]
Further, a weight $\lambda \in \mathfrak{h}^*$ is called a ***highest weight*** *if it is higher than every other weight of V. In turn, V is called a* ***highest weight module*** *if it is generated by a weight vector $v \in V[\lambda]$ that is annihilated by all positive roots in $\mathfrak{g}$. Then, v is called a* ***highest weight vector*** *and λ is the highest weight.*

[a] Notice that simultaneous diagonalization is guaranteed by being $\mathfrak{h}$ composed of commutative elements, by definition.

Here, a weight $\lambda \in \mathfrak{h}^*$ is said to be *higher* than another weight $\eta \in \mathfrak{h}^*$ if their difference $\lambda - \eta$ can be expressed as a linear combination of positive roots with non-negative real coefficients.[22] Having defined these concepts, we can now state the following important result.[23]

22: See Appendix B.5 for a review on Root systems.

23: For this theorem, recall that an element $\lambda \in \mathfrak{h}^*$ is said to be **dominant** if

$$(\lambda, \alpha) \geq 0$$

for all positive roots $\alpha \in R_+$, and is said to be **integral** if

$$\frac{2(\lambda, \alpha)}{(\alpha, \alpha)}$$

is an integer number for each root $\alpha \in R$. A **dominant integer** element is one that is both dominant and integral.

Theorem 4.2.1 (Highest Weight Theorem) *Let V be a finite dimensional irreducible representation of a semisimple complex Lie algebra $\mathfrak{g}$. Then,*

(1) V has a unique highest weight.
(2) The highest weight is a dominant integral element.
(3) Two finite-dimensional irreducible representations with the same highest weight are isomorphic.
(4) Every dominant integral element is the highest weight of an irreducible representation.

In our case, we immediately see that $x^r \in V_r$ is the highest weight vector for each irreducible representation V_r, since on the one hand the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_2$ is generated by h and we have $h \cdot x^r = rx^r$ (also showing that the highest weight is $\lambda = r$),[24] and on the other hand the positive root space is generated by e and we have $e \cdot x^r = 0$.

24: In an abuse of notation, we mean here that $\lambda \in \mathfrak{h}^*$ is given by sending the unique generator h of $\mathfrak{h}$ to the complex number r, i.e. the linear functional

$$\begin{aligned} \lambda : \mathfrak{h} &\to \mathbb{C} \\ h &\mapsto r \end{aligned}$$

Observe that the polynomial representation has given us finite dimensional irreducible representations V_r with the same highest weight r as in the symmetric representation. By the Highest Weight Theorem we then conclude that they are isomorphic representations and so we can equivalently work with the former one instead.

Further, the highest weight vector x^r generates all of V_r under the action of f yielding the basis $\{x^r, f \cdot x^r, \ldots, f^k \cdot x^r = \frac{r!}{(r-k)!} x^{r-k} y^k, \ldots, f^r \cdot x^r\}$. Using the renormalized basis $\{x^r, \ldots, \frac{(r-k)!}{r!} f^k \cdot x^r, \ldots, \frac{1}{r!} f^r \cdot x^r\}$ we have for $k = 0, \ldots, r$, that the generators act on this basis vectors as[25]

25: We introduce here the notation

$$v^k := \frac{(r-k)!}{r!} f^k \cdot x^r = x^{r-k} y^k.$$

$$\begin{cases} h \cdot v^k &= (r-2k) v^k \\ e \cdot v^k &= k v^{k-1} \\ f \cdot v^k &= (r-k) v^{k+1} \end{cases}$$

obtained by using simply the commutation relations. One can check that indeed $[e, f] \cdot v^k = ((k+1)(r-k) - (r-k+1)k) v^k = (r-2k) v^k = h \cdot v^k$. The situation for each V_r can be described as a ladder of weight spaces

$$V_r[-r] \underset{f}{\overset{e}{\rightleftarrows}} \cdots \underset{f}{\overset{e}{\rightleftarrows}} V_r[r-2k] \underset{f}{\overset{e}{\rightleftarrows}} V_r[r-2k+2] \underset{f}{\overset{e}{\rightleftarrows}} \cdots \underset{f}{\overset{e}{\rightleftarrows}} V_r[r].$$

The representation is the same for the universal enveloping algebra $U(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$.[26]

26: See Appendix B.3 for the definition of the universal enveloping algebra.

Quantum Enveloping Algebra of $\mathfrak{sl}_2$

This picture extends to the quantum enveloping algebra[27] $U_q = U_q(\mathfrak{sl}_2)$, a one-parameter deformation of $U(\mathfrak{sl}_2)$, the universal enveloping algebra of $\mathfrak{sl}_2$ (see [Kas95], Chapter VI). We transfer to the world of quantum groups since the R-matrix approach provides us invariants of links. This is actually the main example of a quantum group and it presents properties similar to the ones in $U(\mathfrak{sl}_2)$ when the parameter q is not a root of unity. The definition goes as follows:

27: QEA for short.

Definition 4.2.3 *Define $U_q = U_q(\mathfrak{sl}_2)$ as the algebra generated by the variables E, F, K, K^{-1} with the relations*

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1, \\ KEK^{-1} &= q^2E, \quad KFK^{-1} = q^{-2}F, \\ [E,F] &= \frac{K-K^{-1}}{q-q^{-1}}. \end{aligned}$$

Here the parameter q is an invertible element of the ground field $\mathbb{C}$ different from 1 and -1 so that the above fraction is well defined. Also, K plays a similar role to the usual Cartan subalgebra generator H in the form $e^{hH/2}$, up to possible $\pm$ signs and factors of 2 due to possible conventions.[28] Identifying q with $e^{h/2}$, the usual universal enveloping algebra $U(\mathfrak{sl}_2)$ is recovered from $U_q(\mathfrak{sl}_2)$ when $q \to 1$.

28: Unfortunately, there are several different conventions throughout the literature on Quantum Groups.

It is useful to define further the so-called **quantum integer** for any integer n, given by the geometric sum[29]

$$[n] := \frac{q^n - q^{-n}}{q-q^{-1}} = q^{-n+1} + q^{-n+3} + \cdots + q^{n-3} + q^{n-1}.$$

29: Again, up to conventions. Other references may use

$$\frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q-q^{-1}}$$

or even

$$\frac{1-q^n}{1-q}.$$

One may explicitly show the dependence on q by denoting the quantum integer as $[n]_q$.

This is clearly a q-deformation of n, recovered when setting $q \to 1$. One also has the following combinatorial q-analogues. For any non-negative integer k, set $[k]! = [1][2]\cdots[k]$ if $k > 0$ and $[0]! = 1$. Then, for $0 \le k \le n$, the q-binomial is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}.$$

Alternatively, one may work instead with the similar and closely related Hopf algebra $U_h(\mathfrak{sl}_2)$ defined as follows.[30]

30: As done before, we omit here the details on the Hopf algebra structure (comultiplication, antipode, etc.), unnecessary for our discussion. We refer the reader to [Kas95].

Definition 4.2.4 *Define $U_h = U_h(\mathfrak{sl}_2)$ to be the algebra generated by the three variables X, Y, H and relations*

$$\begin{aligned} [H,X] &= 2X, \quad [H,Y] = -2Y, \\ [X,Y] &= \frac{e^{hH/2} - e^{-hH/2}}{e^{h/2} - e^{-h/2}}. \end{aligned}$$

Both U_h and U_q are indeed related through the following proposition, and so we may talk about one or the other indistinctly.

Proposition 4.2.2 *There exists an injective map of Hopf algebras*

$$\iota : U_q \longrightarrow U_h$$

such that

$$\iota(E) = Xe^{hH/4},\ \iota(F) = e^{-hH/4}Y,\ \iota(K) = e^{hH/2},\ \iota(K^{-1}) = e^{-hH/2}$$

and $\iota(q) = e^{h/2}$. *The ground field on which* U_q *is defined is the field of fractions of the algebra of complex formal series, denoted* $\mathbb{C}(q)$.

As mentioned, this allows us to identify U_q with the subalgebra of U_h generated by $q = e^{h/2}$, $E = Xe^{hH/4}$, $F = e^{-hH/4}Y$, $K = e^{hH/2}$ and $K^{-1} = e^{-hH/2}$. With this at hand, the following expression for an R-matrix can be obtained.[31]

31: See [Kas95], Chapter XVII.

Theorem 4.2.3 *The element*

$$R = \sum_{\ell \geq 0} \frac{(q-q^{-1})^\ell}{[\ell]!} q^{-\ell(\ell+1)/2} e^{\frac{h}{2}\left(\frac{H\otimes H}{2} + \frac{1}{2}(\ell H \otimes \mathbb{1} - \mathbb{1} \otimes \ell H)\right)} (X^\ell \otimes Y^\ell)$$

is a universal R*-matrix for* $U_h(\mathfrak{sl}_2)$.

Notice that the infinite sum is actually finite for a representation whose ladder is finite (at least semi-infinite), since $X^L \equiv 0$ or $Y^L \equiv 0$ for some $\ell = L$, truncating thus the sum and yielding polynomials instead of infinite series. Now, given this expression, one chooses one such representation and picks up a basis in order to explicitly obtain the corresponding entries for this R-matrix, which in turn enables the computation of the desired link invariants.[32]

32: An appopriate choice of basis, as the one shown below, yields simpler expressions for the R-matrix, which considerably ease the implementation to a `Mathematica` code.

Inspired by the above symmetric representation in the classical case, we again have analogue irreducible $(r+1)$-dimensional representations V_r of $U_h(\mathfrak{sl}_2)$ with the ladder picture of weight spaces

$$V_r[-r] \underset{Y}{\overset{X}{\leftrightharpoons}} \cdots \underset{Y}{\overset{X}{\leftrightharpoons}} V_r[r-2k] \underset{Y}{\overset{X}{\leftrightharpoons}} V_r[r-2k+2] \underset{Y}{\overset{X}{\leftrightharpoons}} \cdots \underset{Y}{\overset{X}{\leftrightharpoons}} V_r[r].$$

where $V_r[n]$ is the q^n-eigenspace of V_r under the action of q^H, although now

$$[X, Y]v = [r-2k]v$$

for $v \in V_r[r-2k]$. Replacing the notation $x^{r-k}y^k$ by the basis elements $|r;k\rangle \in V_r$, we get an analogous representation

$$\begin{cases} q^H|r;k\rangle &= q^{r-2k}|r;k\rangle \\ X|r;k\rangle &= [k]|r;k-1\rangle \\ Y|r;k\rangle &= [r-k]|r;k+1\rangle \end{cases}$$

Where we indeed have[33]

$$[X,Y]|r;k\rangle = ([k+1][r-k] - [r-k+1][k])|r;k\rangle = [r-2k]|r;k\rangle$$

33: It follows from the identity

$$[a][b] - [a+c][b-c] = [c][a-b+c].$$

Given this *quantum* representation and using the ket notation for basis vectors, we derive the following explicit expression for the $\mathfrak{sl}_2$ R-matrix in terms of q-Pochhammer symbols.[34]

34: Recall the definition of the q-**Pochhammer symbol**

$$(x;q)_k := \prod_{j=0}^{k-1} (1 - xq^j).$$

Proposition 4.2.4 *In this basis* $\{|r;i,j\rangle := |r;i\rangle \otimes |r;j\rangle\}_{i,j=0}^{r}$*, the* R*-matrix on* $V_r \otimes V_r$ *has the following form*

$$R|r;i_1,j_1\rangle = q^{r^2/4} \sum_{j_2=0}^{i_1} \delta_{i_1+j_1}^{i_2+j_2} x^{-\frac{j_1+j_2}{2}} q^{j_1 j_2} \frac{(q;q)_{i_1} \cdot (x^{-1}q^{j_1};q)_{i_1-j_2}}{(q;q)_{i_1-j_2} \cdot (q;q)_{j_2}} |r;j_2,i_2\rangle$$

where we have redefined $q \mapsto q^{1/2}$ *and hence* $x := q^r \mapsto x^{1/2}$.

The reason for the redefinition of the variable q is that it leads to this simpler R-matrix, more handy for our computations. We now give the proof of this result.[35]

35: For clarity throughout the proof, we recall here the employed quantum representation:

$$\begin{cases} q^H|r;k\rangle &= q^{r-2k}|r;k\rangle \\ X|r;k\rangle &= [k]|r;k-1\rangle \\ Y|r;k\rangle &= [r-k]|r;k+1\rangle \end{cases}$$

Proof. As mentioned earlier, notice first that the infinite sum over $\ell \geq 0$ truncates at some point where the ladder ends, yielding a polynomial rather than an infinite series. Indeed, we have $X^\ell|r;k\rangle = \frac{[\ell]!}{[k-\ell]!}|r;k-\ell\rangle$ and so after applying X precisely k times we run out of basis elements and the action of X^{k+1} on $|r;k\rangle$ must then be zero. With this formula at hand, we may now write

$$(X^\ell \otimes Y^\ell)|r;i,j\rangle = \frac{[i]!}{[i-\ell]!}\frac{[r-j]!}{[r-j-\ell]!}|r;i-\ell,j+\ell\rangle.$$

Recalling now the action of q^H on this eigenbasis yielding the weights q^{r-2k}, the R-matrix

$$R_h = \sum_{\ell\geq 0} \frac{(q-q^{-1})^\ell}{[\ell]!} q^{-\ell(\ell+1)/2} e^{\frac{h}{2}\left(\frac{H\otimes H}{2} + \frac{1}{2}(\ell H \otimes \mathbb{1} - \mathbb{1}\otimes \ell H)\right)}(X^\ell \otimes Y^\ell),$$

after replacing $e^{h/2}$ by q, yields

$$R_h|r;i_1,j_1\rangle = \sum_{\ell=0}^{i} \frac{(q-q^{-1})^\ell}{[\ell]!} q^{-\ell(\ell+1)/2} q^{\varphi} \frac{[i]!}{[i-\ell]!}\frac{[r-j]!}{[r-j-\ell]!}|r;i-\ell,j+\ell\rangle$$

where $\varphi = \frac{1}{2}(r-2(i-\ell))(r-2(j+\ell)) + \frac{1}{2}(\ell(r-2(i-\ell)) - \ell(r-2(j+\ell))) = \frac{r^2}{2} - r(j+i) + 2(i-\ell)j + 2(i-\ell)\ell - \ell(i-j-2\ell)$. The first quantum factorials here can be grouped into a q-binomial and we may then rewrite the remaining fraction of factorials as

$$\begin{aligned}
\frac{[r-j]!}{[r-j-\ell]!} = \prod_{k=0}^{\ell-1} \frac{q^{r-j-k} - q^{-(r-j-k)}}{q-q^{-1}} &= \frac{\prod_{k=0}^{\ell-1} q^{r-j}q^{-k}\left(1-q^{-2(r-j-k)}\right)}{(q-q^{-1})^\ell} \\
&= \frac{q^{(r-j)\ell}q^{-\sum_{k=0}^{\ell-1}k}}{(q-q^{-1})^\ell} \prod_{k=0}^{\ell-1}\left(1-x^{-2}q^{2(j+k)}\right) \\
&= \frac{q^{(r-j)\ell}q^{-\ell(\ell-1)/2}}{(q-q^{-1})^\ell} \prod_{k=0}^{\ell-1}\left(1-x^{-2}q^{2(j+k)}\right)
\end{aligned}$$

where $x = q^r$. Notice how the denominator will cancel out the first factor in the R-matrix. Since $\ell(\ell+1) + \ell(\ell-1) = 2\ell^2$, we get

$$\sum_{\ell\geq 0} \begin{bmatrix} i \\ \ell \end{bmatrix} \left(\prod_{k=0}^{\ell-1}\left(1-x^{-2}q^{2(j+k)}\right)\right) q^{(r-j)\ell} q^{-\ell^2} q^{r^2/2} x^{-(j+i)} q^{\phi} |r;i-\ell,j+\ell\rangle$$

where $\phi = 2(i-\ell)j + 2(i-\ell)\ell + \ell(j-i+2\ell)$. Rearranging terms, it becomes

$$q^{\frac{r^2}{2}} \sum_{\ell\geq 0} \begin{bmatrix} i \\ \ell \end{bmatrix} \left(\prod_{k=0}^{\ell-1}\left(1-x^{-2}q^{2(j+k)}\right)\right) x^{-((j-\ell)+j)} q^{2(i-\ell)j+(i-\ell)\ell} |r; i-\ell, j+\ell\rangle.$$

We now apply the change $q \mapsto q^{1/2}$ and thus, correspondingly, $x \mapsto x^{1/2}$. At this stage we realize that the product becomes simply the q-Pochhammer $(x^{-1}q^j;q)_\ell$. Also, the quantum integer becomes[36]

$$[n]' = \frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}} \quad \text{and thus} \quad [n]'! = \frac{q^{-n(n+1)/4}}{(q^{1/2}-q^{-1/2})^n}(-1)^n \cdot (q;q)_n.$$

Therefore also

$$\begin{bmatrix} i \\ \ell \end{bmatrix}' = \frac{[i]'!}{[\ell]'![i-\ell]'!} = \frac{q^{-(i-\ell)\ell/2}(q;q)_i}{(q;q)_\ell (q;q)_{i-\ell}}.$$

With these changes, we get

$$q^{\frac{r^2}{2}} \sum_{\ell=0}^{i} x^{-\frac{(i-\ell)+j}{2}} q^{(i-\ell)j} q^{\frac{(i-\ell)\ell}{2}} q^{-\frac{(i-\ell)\ell}{2}} \frac{(q;q)_i (x^{-1}q^j;q)_\ell}{(q;q)_\ell (q;q)_{i-\ell}} |r; i-\ell, j+\ell\rangle$$

Thus, by defining $i_1 := i$, $j_1 := j$ and $i_2 := j_1+\ell$, $j_2 := i_1-\ell$, we finally obtain the expression

$$R|r;i_1,j_1\rangle = q^{r^2/4}\sum_{j_2=0}^{i_1} \delta^{i_2+j_2}_{i_1+j_1} x^{-\frac{j_1+j_2}{2}} q^{j_1j_2} \frac{(q;q)_{i_1}\cdot(x^{-1}q^{j_1};q)_{i_1-j_2}}{(q;q)_{i_1-j_2}\cdot(q;q)_{j_2}}|r;j_2,i_2\rangle,$$

as desired. □

36: Indeed, concerning the quantum factorial we have

$$\begin{aligned}[n]'! &= \prod_{k=1}^{n} \frac{q^{k/2}-q^{-k/2}}{q^{1/2}-q^{-1/2}} \\ &= \prod_{k=1}^{n} \frac{q^{-k/2}(1-q^k)(-1)}{q^{1/2}-q^{-1/2}} \\ &= \frac{q^{-\frac{1}{2}\sum_{k=1}^{n} k}(-1)^n}{(q^{1/2}-q^{-1/2})^n}\prod_{k=1}^{n}(1-q^k) \\ &= \frac{q^{-n(n+1)/4}}{(q^{1/2}-q^{-1/2})^n}(-1)^n \cdot (q;q)_n.\end{aligned}$$

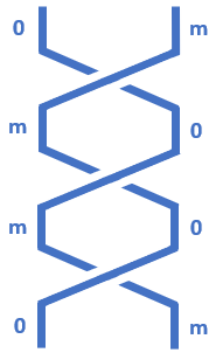

Figure 4.9: The braid description of a right-handed Trefoil knot, decorated with the dummy indices involved in the state sum.

One can implement this R-matrix to a `Mathematica` notebook and compute knot invariants such as the Jones polynomial or the $F_K(x,q)$ invariant. Consider for instance the case of the right-handed Trefoil knot (see Figure 4.9). One can use the fact that there is a delta $\delta^{i_2+j_2}_{i_1+j_1}$ in the R-matrix giving the conservation equation $i_1+j_1 = i_2+j_2$. All other cases simply don't contribute to the state sum. One then assigns integer labels on each strand and obtains the entries for each R-matrix involved in the state sum computation. One does so by assigning the label 0 to the open left strand both at the bottom and at the top of the braid (recall that they are identified when closing the braid) and dummy labels m to the other strands, always respecting the conservation equation. These dummy variables are the ones over which the state sum is carried out, after composing the corresponding R-matrices and the eventual inverses. In the case of the right-handed Trefoil, we only have to compose `RMat[x, q, 0, m, m, 0]`, `RMat[x, q, m, 0, m, 0]` and `RMat[x, q, m, 0, 0, m]` and sum over m from 0 to i.[37] In the case of the n-th colored Jones polynomial, we obtain:

37: See Appendix D for more details.

n	$J_n(q)$
0	1
1	$-\frac{1}{q^4}+\frac{1}{q^3}+\frac{1}{q}$
2	$\frac{1}{q^{11}}-\frac{1}{q^{10}}-\frac{1}{q^9}+\frac{1}{q^8}-\frac{1}{q^7}+\frac{1}{q^5}+\frac{1}{q^2}$
3	$-\frac{1}{q^{21}}+\frac{1}{q^{20}}+\frac{1}{q^{19}}-\frac{1}{q^{17}}+\frac{1}{q^{15}}-\frac{1}{q^{14}}-\frac{1}{q^{13}}+\frac{1}{q^{11}}-\frac{1}{q^{10}}+\frac{1}{q^7}+\frac{1}{q^3}$

Table 4.1: The values for the first n-colored Jones polynomials.

As for the $F_K(x,q)$ invariant, one can fix an order and expand up to that power. For example, expanding up to sixth order yields the expression

$$F_K(x,q) = 1 + qx - (-1+q)q^2x^2 - q^3(-1+q+q^2)x^3 \\ - q^4(-1+q+q^2)x^4 + q^5(1-q-q^2+q^5)x^5 + \mathcal{O}(x^6)$$

4.3 Procedure for other semisimple Lie algebras

To obtain the R-matrix for any other semisimple Lie algebra, we need to introduce Serre's presentation, to be then extended to the quantum version.[38] Recall first that any semisimple Lie algebra can be characterized by its Cartan matrix $A = (a_{ij})_{1\leq i,j\leq n}$. This matrix consists of non-positive integers on the off-diagonal entries and has $a_{ii} = 2$ on the diagonal. Recall also that the diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_n)$ —whose entries are the (renormalized) root lengths: 1, 2 or 3— is such that the matrix DA is symmetric positive definite (in particular, invertible). We have the following theorem:

38: Although different in form, the expressions given here are equivalent to the ones presented in the end of Appendix B, where Serre's relations are given in terms of the adjoint maps. This is done for convenience and clarity, since it shows more explicitly the relation with the quantum version.

Theorem 4.3.1 (Chevalley-Serre) *The enveloping algebra $U(\mathfrak{g})$ of a complex semisimple Lie algebra with Cartan matrix $A = (a_{ij})_{1\leq i,j\leq n}$ is isomorphic to the algebra generated by $\{X_i, Y_i, H_i\}_{1\leq i\leq n}$ and the relations*

$$[H_i, H_j] = 0, \quad [X_i, Y_j] = \delta_{ij}H_i, \\ [H_i, X_j] = a_{ij}X_j, \quad [H_i, Y_j] = -a_{ij}Y_j,$$

and if $i \neq j$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} X_i^k X_j X_i^{1-a_{ij}-k} = 0$$

and [a]

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} Y_i^k Y_j Y_i^{1-a_{ij}-k} = 0.$$

[a] These are equivalent to $\operatorname{ad}(X_i)^{1-a_{ij}}(X_j) = 0$ and $\operatorname{ad}(Y_i)^{1-a_{ij}}(Y_j) = 0$.

Then, the q-analogue is given by the following definition.

Definition 4.3.1 *Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan matrix $A = (a_{ij})_{1\leq i,j\leq n}$ and diagonal matrix of root lengths $D = \operatorname{diag}(d_1, \dots, d_n)$, $d_i = \frac{1}{2}(\alpha_i, \alpha_i)$. Then, the algebra $U_h(\mathfrak{g})$ is the algebra generated by $\{X_i, Y_i, H_i\}_{1\leq i,j\leq n}$ and the relations*

$$[H_i, H_j] = 0, \quad [X_i, Y_j] = \delta_{ij}\frac{e^{hd_iH_i/2} - e^{-hd_iH_i/2}}{e^{hd_i/2} - e^{-hd_i/2}}, \\ [H_i, X_j] = a_{ij}X_j, \quad [H_i, Y_j] = -a_{ij}Y_j,$$

and if $i \neq j$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix} X_i^k X_j X_i^{1-a_{ij}-k} = 0$$

and

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix} Y_i^k Y_j Y_i^{1-a_{ij}-k} = 0,$$

where $q_i = e^{hd_i/2}$.

The third and fourth relations imply that for all i, j and any complex number $\lambda \in \mathbb{C}$ we have[39]

$$e^{\lambda H_i} X_j = e^{\lambda a_{ij}} X_j e^{\lambda H_i}, \quad e^{\lambda H_i} Y_j = e^{-\lambda a_{ij}} Y_j e^{\lambda H_i}.$$

Setting h to zero (which is equivalent to $q \to 1$), one recovers the enveloping algebra of $U(\mathfrak{g})$ in Serre's presentation. To put it another way, there is an isomoprhism of algebras

$$U_h(\mathfrak{g})/hU_h(\mathfrak{g}) \cong U(\mathfrak{g}).$$

Concerning the existence of a universal R-matrix for $U_h(\mathfrak{g})$, Drinfeld proved that such is of the form[40]

$$R_h = \sum_{\ell \in \mathbb{N}^n} e^{h\left(\frac{t_0}{2} + \frac{1}{4}(H_\ell \otimes 1 - 1 \otimes H_\ell)\right)} P_\ell$$

where (1) $H_\ell = \sum_{1 \leq \ell \leq n} \ell_i H_i$ for $\ell = (\ell_1, \cdots, \ell_n)$, (2) t_0 is the element

$$t_0 = \sum_{1 \leq i,j \leq n} (DA)^{-1}_{ij} H_i \otimes H_j$$

of $\mathfrak{g} \otimes \mathfrak{g}$, and (3) P_ℓ is a polynomial in the variables $X_i \otimes 1$ and $1 \otimes Y_j$ (homogeneous of degree ℓ_i in $X_i \otimes 1$ and $1 \otimes Y_i$). These polynomials P_ℓ can be determined inductively on ℓ with $P_0 = 1 \otimes 1$ and using the relation $\Delta^{op}(a) = R\Delta(a)R^{-1}$ in the definition of the universal R-matrix of a topological quasi-bialgebra A.[41]

There is actually a more explicit formula in the literature, given by A.N. Kirillov and N. Reshetikhin [KR90] and rewritten by M. Rosso [Ros91], nicely given as a multiplicative analogue of the $\mathfrak{sl}_2$ case.[42]

Theorem 4.3.2 *The universal R-matrix for $U_h(\mathfrak{g})$ is of the form*

$$R = \prod_{\alpha \in \Delta_+} \left(\sum_{\ell=0}^{\infty} \frac{(1-q_\alpha^{-2})^\ell}{[\ell]_\alpha!} q_\alpha^{\frac{\ell(\ell-1)}{2}} (E_\alpha^\ell \otimes F_\alpha^\ell) \right) \exp\left(\frac{h}{2} \sum_{i,j} B_{ij} H_i \otimes H_j \right)$$

where the product is carried out over the total order of positive roots Δ_+ determined by the reduced decomposition of the longest element w_0 in the Weyl group; i.e. for[a] *$w_0 = r_{i_1} \cdots r_{i_\nu}$ (with $\nu = |\Delta_+|$ and $r_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$, $i, j = 1, \ldots, n$, the reflection automorphisms)*

39: Indeed, rewriting the commutations relation $[H_i, X_j] = a_{ij} X_j$ as

$$H_i X_j = X_j (H_i + a_{ij} \mathbb{1})$$

we have

$$H_i^k X_j = X_j (H_i + a_{ij} \mathbb{1})^k$$

and thus

$$\begin{aligned} e^{\lambda H_i} X_j &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} H_i^k X_j \\ &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} X_j (H_i + a_{ij}\mathbb{1})^k \\ &= X_j e^{\lambda(H_i + a_{ij}\mathbb{1})} \\ &= e^{\lambda a_{ij}} X_j e^{\lambda H_i}. \end{aligned}$$

40: As stated in [Kas95].

41: Recall that $\Delta^{op} = \tau \circ \Delta$, where $\tau : A \otimes B \to B \otimes A$ is the twist map.

42: We warn the reader to pay careful attention to convention differences when comparing in the literature.

and $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ the simple roots, the order on Δ_+ is given by

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = r_{i_1}(\alpha_{i_2}), \quad \ldots \quad , \quad \beta_\nu = r_{i_1} \cdots r_{i_{\nu-1}}(\alpha_{i_\nu}).$$

The E_α and F_α are then given by

$$E_{\beta_s} = T_{i_1} \cdots T_{i_{s-1}}(X_{i_s}), \quad F_{\beta_s} = T_{i_1} \cdots T_{i_{s-1}}(Y_{i_s}), \quad s = 1, \ldots, \nu,$$

where the T_i are the automorphisms uniquely determined by

$$T_i X_i = -Y_i e^{-\frac{h d_i H_i}{2}}, \quad T_i X_j = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} \, q_i^s \, X_i^{(-a_{ij}-s)} X_j X_i^{(s)} \quad i \neq j$$

$$T_i Y_i = -e^{\frac{h d_i H_i}{2}} X_i, \quad T_i Y_j = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} \, q_i^{-s} \, Y_i^{(s)} Y_j Y_i^{(-a_{ij}-s)} \quad i \neq j$$

$$T_i e^{\frac{h d_j H_j}{2}} = e^{\frac{h(d_j H_j - a_{ij} d_i H_i)}{2}}.$$

Finally, q_α is $q^{d_i} = e^{\frac{h d_i}{2}}$ for each α in the orbit of α_i under the action of the Weyl group and $(B_{ij})_{1 \leq i,j \leq n}$ is the inverse matrix of $DA = (d_i a_{ij})_{1 \leq i,j \leq n}$.

[a] The result is independent of the choice of reduced tuple $(i_1, \ldots, i_\nu)$ of integers $1 \leq i_s \leq n$.

Notice the importance of the order in Δ_+ since — in general — different generators wont commute. One can check that the $\mathfrak{sl}_2$ R-matrix indeed recovers from this expression.[43]

4.4 The $\mathfrak{so}_{2n}$ family

Now, having studied the $\mathfrak{sl}_2$ case, the proposed project aims to extend the same procedure of computing $F_K(x, q)$ to other semisimple Lie algebras, obtaining the corresponding invariants.

The family A_n in Cartan's classification (see Figure 4.10) has already been studied by Angus Gruen.[44] The choice for this work has been an element of the family D_n corresponding to the Lie algebras $\mathfrak{so}(2n, \mathbb{C})$. The Lie algebra $\mathfrak{so}(m, \mathbb{C})$ —$\mathfrak{so}_m$ for short— is the one associated to the special orthogonal group $SO(m, \mathbb{C}) = \{M \in GL(m, \mathbb{C}) \mid M^t M = \mathbb{1}, \ \det M = 1\}$. The description for $\mathfrak{so}(m, \mathbb{C})$ is obtained was shown shown for $SL(2, \mathbb{C})$.[45] In this case,

$$\mathbb{1} = (\mathbb{1} + \epsilon A)^t (\mathbb{1} + \epsilon A) = \mathbb{1} + \epsilon (A^t + A) + \mathcal{O}(\epsilon^2)$$

gives the condition $A + A^t = 0$ and

$$1 = \det(\mathbb{1} + \epsilon A) = 1 + \epsilon \operatorname{Tr}(A) + \mathcal{O}(\epsilon^2)$$

gives $\operatorname{Tr}(A) = 0$, although this condition is included in the other one and is therefore redundant.[46] Thus, the corresponding Lie algebra is given by the skew-symmetric matrices

$$\mathfrak{so}(m, \mathbb{C}) = \{A \in \mathfrak{gl}(m, \mathbb{C}) \mid A + A^t = 0\}.$$

The classification in terms of Dynkin diagrams for these Lie algebras fall into two families: B_n for odd $m = 2n+1$ and D_n for even $m = 2n$.

43: This is done by matching the different notations and the following considerations. First, the T_i automorphisms don't apply since $n = 1$ and $a_{ij} = 2$, and so $E_\alpha = X$, $F_\alpha = Y$. Further, the exponential term reads now $e^{\frac{h}{4} H \otimes H}$ since $d = 1$ and so $B = 1/2$. This also means $q_\alpha = q = e^{h/2}$ and then $[\ell]_\alpha! = [\ell]!$ and $(1 - q^{-2})^\ell$ can be rewritten as $q^{-\ell}(q - q^{-1})^\ell$ yielding an extra factor $q^{-\ell}$. To move $e^{\frac{h}{4} H \otimes H}$ to the left, one takes $\lambda = h/4$ in the expression

$$(X^\ell \otimes Y^\ell) e^{\lambda H \otimes H} = \\ = e^{\lambda (H \otimes H + 2\ell H \otimes \mathbb{1} - \mathbb{1} \otimes 2\ell H)} e^{-4\lambda \ell^2}.$$

This expression is obtained by rewritting $[H, X] = 2X$ and $[H, Y] = -2Y$ as

$$XH = (H - 2\mathbb{1})X, \quad YH = (H + 2\mathbb{1})Y$$

and using these in

$$\begin{aligned} X^\ell H^k &= X^{\ell-1}(XH) H^{k-1} \\ &= X^{\ell-1}(H - 2\mathbb{1}) X H^{k-1} \\ &= \cdots = X^{\ell-1}(H - 2\mathbb{1})^k X \\ &= \cdots = (H - 2\mathbb{1})^k X^\ell \end{aligned}$$

and

$$\begin{aligned} Y^\ell H^k &= \cdots = Y^{\ell-1}(H + 2\mathbb{1})^k Y \\ &= \cdots = (H + 2\mathbb{1})^k Y^\ell \end{aligned}$$

to obtain

$$\begin{aligned} &(X^\ell \otimes Y^\ell) e^{\lambda H \otimes H} = \\ &= \sum_{k \geq 0} \frac{\lambda^k}{k!} (X^\ell H^k) \otimes (Y^\ell H^k) \\ &= \sum_{k \geq 0} \frac{\lambda^k}{k!} ((H - 2\ell \mathbb{1})^k X^\ell) \otimes ((H + 2\ell \mathbb{1})^k Y^\ell) \\ &= e^{\lambda (H - 2\ell \mathbb{1}) \otimes (H + 2\ell \mathbb{1})} (X^\ell \otimes Y^\ell). \end{aligned}$$

Rearranging the factors of $q = e^{h/2}$, one then recovers the $\mathfrak{sl}_2$ R-matrix up to notation conventions.

44: Work about to publish.

45: Again, the proposed method is equivalent to applying the **Regular Value Theorem**:
Feed the path $\gamma(t) = \mathrm{Id} + At + \mathcal{O}(t^2)$ into the functions defining the Lie group. In this case, the second function $\det(X) - 1 = 0$ gives redundant information already contained in proceeding with the first ones $(\gamma(t))^t(\gamma(t)) - \mathrm{Id} = 0$. Differentiating $(\mathrm{Id} + At + \mathcal{O}(t^2))^t (\mathrm{Id} + At + \mathcal{O}(t^2)) - \mathrm{Id} = t(A + A^t) + \mathcal{O}(t^2)$ at $t = 0$ yields the condition $A + A^t = 0$.

46: $A + A^t = 0$ means $a_{kk} + a_{kk} = 0$ on the diagonal and thus $a_{kk} = 0$ so that $\operatorname{Tr}(A)$ is eventually satisfied.

Here we deal with the latter case, mainly focusing on $\mathfrak{so}(8,\mathbb{C})$ as the first interesting case.[47] The reason for this will soon be clear.

47: Also, to be accurate, the family D_n in Cartan's classification starts with $n=4$, corresponding to $\mathfrak{so}_8$.

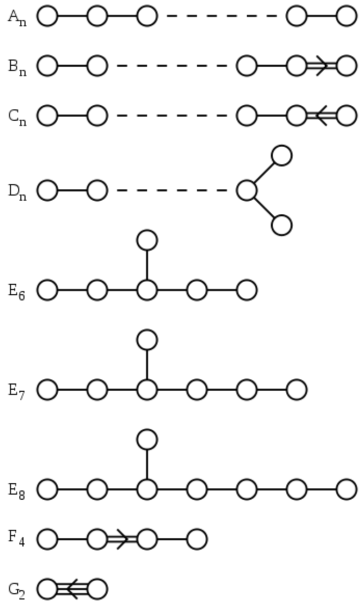

Figure 4.10: Cartan's classification of semisimple Lie algebras in terms of their Dynkin diagrams, where A_n and D_n correspond to the Lie algebras $\mathfrak{sl}_n$ and $\mathfrak{so}_{2n}$, respectively.

The $\mathfrak{so}_4$ case

Being $\mathfrak{so}_2 \cong \mathbb{C}$ too simple, we start with $\mathfrak{so}_4$, which can be easily worked out through the Lie algebra isomorphism $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ —what in particular shows that it is semisimple.

Isomorphism $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$

The isomorphism is explicitly given by

$$\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} \mapsto \begin{pmatrix} i(c-d) & (a-f)+i(e+b) \\ -(a-f)+i(e+b) & -i(c-d) \end{pmatrix} \oplus \begin{pmatrix} i(c+d) & (a+f)+i(e-b) \\ -(a+f)+i(e-b) & -i(c+d) \end{pmatrix}.$$

Hence, we can make use of all our results on $\mathfrak{sl}_2$ with the following notation for the generators:

$$\begin{cases} e_1 = e \oplus 0, & e_2 = 0 \oplus e \\ f_1 = f \oplus 0, & f_2 = 0 \oplus f \\ h_1 = h \oplus 0, & h_2 = 0 \oplus h \end{cases}$$

where e, f, h are the usual generators of $\mathfrak{sl}_2$ with the corresponding commutation relations. By using these, it is immediate that the generators e_k, f_k, h_k $(k=1,2)$ satisfy Serre's relations determining the semisimple Lie algebra[48]

$$[h_1, h_2] = 0, \quad [h_i, e_j] = 2\delta_{ij} e_j, \quad [h_i, f_j] = -2\delta_{ij} f_j$$
$$[e_i, f_j] = \delta_{ij} h_j, \quad [e_1, e_2] = [f_1, f_2] = 0$$

48: The Cartan matrix for $\mathfrak{so}_4$ is then given by

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

where the Lie bracket is defined component-wise:

$$[x_1 \oplus y_1, x_2 \oplus y_2] := [x_1, x_2] \oplus [y_1, y_2].$$

Now, let $\{v^k\}_{k=0}^r$ be the eigenbasis we used for the r-th symmetric representation V_r of $\mathfrak{sl}_2$. Then, $\{v^{k_1} \oplus v^{k_2}\}_{k_1,k_2=0}^r$ is an eigenbasis for the representation $V_r \oplus V_r$ of $\mathfrak{so}_4$. The highest weight vector will now be $v^r \oplus v^r$, being v^r the highest weight vector of $\mathfrak{sl}_2$ with highest weight r. Observe that the highest weight remains the same, which thus shows we are indeed still considering the r-th symmetric representation. The action of the generators on this basis is given by

$$\begin{cases} e_j \cdot v^{k_1} \oplus v^{k_2} = k_j \cdot v^{k_1 - \delta_{1j}} \oplus v^{k_2 - \delta_{2j}} \\ f_j \cdot v^{k_1} \oplus v^{k_2} = (r - k_j) \cdot v^{k_1 + \delta_{1j}} \oplus v^{k_2 + \delta_{2j}} \\ h_j \cdot v^{k_1} \oplus v^{k_2} = (r - 2k_j) \cdot v^{k_1} \oplus v^{k_2} \end{cases}$$

The quantum version reads then

$$\begin{cases} X_j |r; (k_1, k_2)\rangle = [k_j] \cdot |r; (k_1 - \delta_{1j}, k_2 - \delta_{2j})\rangle \\ Y_j |r; (k_1, k_2)\rangle = [r - k_j] \cdot |r; (k_1 + \delta_{1j}, k_2 + \delta_{2j})\rangle \\ q^{H_j} |r; (k_1, k_2)\rangle = q^{r - 2k_j} \cdot |r; (k_1, k_2)\rangle \end{cases}$$

where $|r;(k_1,k_2)\rangle := v^{k_1} \oplus v^{k_2}$. Now, going back to the last theorem in the previous section and specifying to $\mathfrak{so}_4$, we get that: $d_1 = d_2 = 1$ and so all the q_α reduce to $q = e^{\frac{h}{2}}$. There are only two positive roots $\Delta_+ = \{\alpha_1, \alpha_2\}$ corresponding each copy of the $\mathfrak{sl}_2$ root system.[49] The longest element of the Weyl group is just a reflection $w_0 = r_2$ sending $r_2(\alpha_1) = \alpha_2 - a_{12}\alpha_1 = \alpha_2$ since $a_{12} = 0$, and the automorphism T_2 simply acts as the identity. The order of the positive roots doesn't matter because of the commutativity between both e_1 and f_1 with both e_2 and f_2. We further use the fact that each h_k commutes with the other $e_{k'}, f_{k'}$. Finally, the matrix B is simply $\mathrm{diag}(1/2, 1/2)$. Thus, using all these observations showing that the R-matrix splits in the product of two $\mathfrak{sl}_2$ R-matrices and the fact that the general R-matrix recovers the $\mathfrak{sl}_2$ case, the R-matrix for $\mathfrak{so}_4$ is then given by

49: See Appendix B.5 on root systems for the case of reducible root systems $R = R_1 \sqcup R_2$ corresponding to semisimple Lie algebras $\mathfrak{g}_1 \oplus \mathfrak{g}_2$.

$$
\begin{aligned}
R &= \exp\left(\frac{h}{2}\sum_{i,j} B_{ij} h_i \otimes h_j\right) \prod_{k=1}^{2} \left(\sum_{\ell \geq 0} \frac{(q_k - q_k^{-1})^\ell}{[\ell]_k!} q_k^{\frac{\ell(\ell-1)}{2}} (e_k^\ell \otimes f_k^\ell) \right) \\
&= e^{\frac{h}{4} h_1 \otimes h_1} \left(\sum_{\ell \geq 0} \frac{(q - q^{-1})^\ell}{[\ell]!} q^{\frac{\ell(\ell-1)}{2}} (e_1^\ell \otimes f_1^\ell) \right) \cdot \\
&\quad \cdot e^{\frac{h}{4} h_2 \otimes h_2} \left(\sum_{s \geq 0} \frac{(q - q^{-1})^s}{[s]!} q^{\frac{s(s-1)}{2}} (e_1^s \otimes f_1^s) \right) =: R^1 \cdot R^2.
\end{aligned}
$$

Finally, the explicit expression in the basis[50] $|r;(i_1,i_2),(j_1,j_2)\rangle := (v^{i_1} \oplus v^{i_2}) \otimes (v^{j_1} \oplus v^{j_2})$ of $(V_r \oplus V_r) \otimes (V_r \oplus V_r)$ is given by

50: Warning: notice the change of notation with respect to the $\mathfrak{sl}_2$ case, where the indices i_2 and j_2 where used differently.

$$
\begin{aligned}
R\ &|r;(i_1,i_2),(j_1,j_2)\rangle = R^1 R^2\ |r;(i_1,i_2),(j_1,j_2)\rangle \\
&= R^1 \cdot \sum_{\ell_2 \geq 0} R(\ell_2; i_2, j_2) |r;(i_1, i_2 - \ell_2),(j_1, j_2 + \ell_2)\rangle \\
&= \sum_{\ell_2 \geq 0} R(\ell_2; i_2, j_2) \cdot R^1\ |r;(i_1, i_2 - \ell_2),(j_1, j_2 + \ell_2)\rangle \\
&= \prod_{k=1}^{2} \left(\sum_{\ell_k \geq 0} R(\ell_k; i_k, j_k) \right) |r;(i_1 - \ell_1, i_2 - \ell_2),(j_1 + \ell_1, j_2 + \ell_2)\rangle
\end{aligned}
$$

where again the redefinition $q \mapsto q^{1/2}$ ha been applied and where we have defined the function

$$
R(\ell; i, j) := q^{\frac{r^2}{4}} \theta[i - \ell] x^{-\frac{(i-\ell)+j}{2}} q^{(i-\ell)j} \frac{(q;q)_i \cdot (x^{-1} q^j; q)_\ell}{(q;q)_\ell \cdot (q;q)_{i-\ell}}
$$

coming from the $\mathfrak{sl}_2$ case, where $\theta[n]$ is the discrete Heaviside function, introduced to limit the range of the sum up to $\ell = i$.

The $\mathfrak{so}_6$ case

In this case we have the Lie algebra isomorphism $\mathfrak{so}_6 \cong \mathfrak{sl}_4$.

Isomorphism $\mathfrak{sl}_4 \cong \mathfrak{so}_6$

The Lie algebra isomorphism here is more involved. It comes from considering the 6-dimensional vector space $\bigwedge^2 \mathbb{C}^4$ and defining a symmetric bilinear form $B : \bigwedge^2 \mathbb{C}^4 \times \bigwedge^2 \mathbb{C}^4 \to \mathbb{C}$ as follows. Considering $u_1 \wedge v_1, u_2 \wedge v_2 \in \bigwedge^2 \mathbb{C}^4$ and using the fact that $\bigwedge^4 \mathbb{C}^4 \cong \mathbb{C}$ is generated by $e_1 \wedge e_2 \wedge e_3 \wedge e_4$ with $\{e_i\}_{i=1}^4$ the canonical basis, $B(u_1 \wedge v_1, u_2 \wedge v_2)$ can be defined to be the unique scalar for which it holds

$$u_1 \wedge v_1 \wedge u_2 \wedge v_2 = B(u_1 \wedge v_1, u_2 \wedge v_2)\; e_1 \wedge e_2 \wedge e_3 \wedge e_4.$$

One then checks that this defines a symmetric bilinear form which is non-degenerate (the corresponding matrix in the basis $\{e_i \wedge e_j\}_{i<j}$ has non-zero determinant), thus defining an isomorphism $w_2 \mapsto (w_1 \mapsto B(w_1, w_2))$ for $w_1, w_2 \in \bigwedge^2 \mathbb{C}^4$. The action of $\mathfrak{sl}_2$ on $\bigwedge^2 \mathbb{C}^4$ (inherited from the tensor representation $\mathbb{C}^4 \otimes \mathbb{C}^4$, determined in the usual way from the standard action on $\mathbb{C}^4$ acting by matrix multiplication) then defines a morphism of algebras

$$\mathfrak{sl}_4 \to \mathfrak{so}_6$$

which is injective. Being both $\mathfrak{sl}_4$ and $\mathfrak{so}_6$ 15-dimensional, we actually have an isomorphism.

Observe as well that they indeed have the same Dynkin diagram A_4. The work of obtaining $F_K(x, q)$ with the family $\mathfrak{sl}_N$ has already been studied by Angus Gruen in a work yet to be published. Hence the reasons for skipping this case.

The $\mathfrak{so}_8$ case

We naturally reach the $\mathfrak{so}_8$ case as the first non-trivial one. Moreover, it presents an interesting three-fold symmetry to be observed through its Dynkin diagram (Figure 4.11), or more specifically through the Cartan matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix},$$

where an explicit common behavior with respect to the second row and column is manifest.

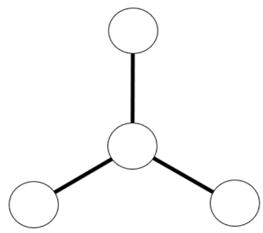

Figure 4.11: Dynkin diagram D_4 for the semisimple Lie algebra $\mathfrak{so}_8$.

Let us begin by obtaining a set of Serre generators $\{X_i, Y_i, H_i\}_{i=1}^4$ for this semisimple Lie algebra (in fact *simple*). Recall that

$$\mathfrak{so}_8 = \{A \in \mathfrak{gl}(8, \mathbb{C}) \mid A + A^t = 0\}.$$

The natural basis to choose for the underlying vector space is $\ell_{ij} := e_{ij} - e_{ji}$ for $0 \leq i + 1 \leq j \leq 8$ (the e_{ij} being the canonical basis for $\mathfrak{gl}(m, \mathbb{C})$: all entries 0 but the (i, j)-th entry being 1), consisting of $\sum_{k=1}^{8-1} k = \frac{(8-1)8}{2} = 28$ vector space generators. The elements $h_k = \ell_{2k-1,2k}$ with $k = 1, \ldots, 4$ are seen to generate the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{so}_8$,[51] the commutator given by

$$[\ell_{ij}, \ell_{k\ell}] = \delta_{jk}\ell_{i\ell} + \delta_{\ell i}\ell_{jk} - \delta_{j\ell}\ell_{ik} - \delta_{ki}\ell_{j\ell}$$

51: It indeed is a toral subalgebra (commutative and consisting of semisimple elements) and coincides with its centralizer:

$$C(\mathfrak{h}) = \{x \in \mathfrak{g} \mid [x, \mathfrak{h}] = 0\} = \mathfrak{h}.$$

following from the trivial relation $e_{ij}e_{k\ell} = \delta_{jk}e_{i\ell}$. In matrix form:

$$h_1 = \begin{pmatrix} 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$h_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad h_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

Notice that all $[\ell_{ij}, \ell_{k\ell}]$ vanish but the ones obtained from the symmetry $\ell_{ij} = -\ell_{ji}$ and $[\ell_{kj}, \ell_{k\ell}] = -\ell_{j\ell}$, $j \neq k \neq \ell$. It will be convenient to define the index set

$$\Lambda := \{(i,j) \mid 0 \leq i+1 \leq j \leq 8\} \backslash \{(2k-1, 2k)\}_{k=1}^{4}$$

corresponding to the indices for the generators ℓ_{ij} different from the four $h_k = \ell_{2k-1,2k}$ ones.[52] Now, Serre's theorem tells us that $\mathfrak{so}_8$ *as a Lie algebra* is generated by only 12 generators $\{X_i, Y_i, H_i\}_{i=1}^{4}$, all other vector space generators being obtained through the Lie bracket $[\cdot,\cdot]$. To obtain these generators, we impose Serre's relations starting with the $[H_i, X_j] = a_{ij}X_j$ and $[H_i, Y_j] = -a_{ij}Y_j$ ones. Such H_i generators will be linear combinations of the h_k's we just presented (thus still generating the Cartan subalgebra $\mathfrak{h}$). A general element of the Lie algebra can be written in our chosen basis as $\sum_{1\leq i+1\leq j\leq 8}\beta^{ij}\ell_{ij}$ for some coefficients β^{ij}. Among these we know that at least four will vanish in the commutator with the h_k's (the h_k's themselves). So the index set in the sum can be constrained to Λ. Thus, the general expression for the commutator to impose and determine the parameters α^k and β^{ij} determining the Lie algebra generators X_i, Y_i and H_i is

52: All the generators ℓ_{ij} are of the form

$$\ell_{25} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with a $+1$ at position (i,j) and a -1 at position (j,i).

$$\left[\sum_{k=1}^{4}\alpha^k h_k, \sum_{(i,j)\in\Lambda}\beta^{ij}\ell_{ij}\right]$$

which has been computed to yield[53]

$$\sum_{k=1}^{4}\alpha^k\left\{\sum_{j=2k+1}^{8}(\beta^{2k,j}\ell_{2k-1,j} - \beta^{2k-1,j}\ell_{2k,j}) + \sum_{i=1}^{2(k-1)}(\beta^{i,2k}\ell_{i,2k-1} - \beta^{i,2k-1}\ell_{i,2k})\right\}.$$

53: This can be shown by computing the different commutators $[h_k, \ell_{ij}]$ and observing that there are some invariant subspaces of the form $\{\ell_{2k-1,j}, \ell_{2k,j}\}$ and $\{\ell_{i,2k-1} - \ell_{i,2k}\}$. The inspiration for this comes from the previous study of the $\mathfrak{so}_4$ Lie algebra.

At this stage, it should already be clear that this description for the Lie algebra $\mathfrak{so}_8$ is not nice at all for computational purposes. Anyways, one can go ahead and impose the special cases when $a_{ij} = 0$ in the Cartan matrix —which yield the conditions $\alpha^i = \pm\alpha^j$ — or the diagonal elements $a_{ii} = 2$ —which fix some of the β^{ij}'s. This long procedure finally allows us to write the X_i and Y_i with only six remaining free parameters[54] and all having a form similar to

54: These should then be fixed by the remaining Serre relations.

$$\begin{aligned} X_4^{\pm} = \beta^{13}(\ell_{13} \mp i\ell_{23} \mp i\ell_{14} - \ell_{24}) + \beta^{15}(\ell_{15} \mp i\ell_{25} \mp i\ell_{16} - \ell_{26}) + \\ + \beta^{17}(\ell_{17} \mp i\ell_{27} \pm i\ell_{18} + \ell_{28}) + \beta^{35}(\ell_{35} \mp i\ell_{45} \mp i\ell_{36} - \ell_{46}) + \\ + \beta^{37}(\ell_{37} \mp i\ell_{47} \pm i\ell_{38} + \ell_{48}) + \beta^{57}(\ell_{57} \mp i\ell_{67} \pm i\ell_{58} + \ell_{68}) \end{aligned}$$

where $X_4^+ = X_4$ and $X_4^- = Y_4$. This clearly demands for an alternative description.

Such alternative basis for the Lie algebra is obtained through a change of basis with the help of a `Mathematica` notebook.[55] The idea is that the h_k's all commute with each other so one can simultaneously diagonalize them. Indeed, the eigenvectors are easily seen to be $\{\pm i e_{2k-1} + e_{2k}\}_{k=1}^{4}$ with e_i the canonical basis vectors. Applying this change of basis[56] to a general element of $\mathfrak{so}_8$ expressed as

$$\begin{pmatrix} 0 & \ell_{12} & \ell_{13} & \ell_{14} & \ell_{15} & \ell_{16} & \ell_{17} & \ell_{18} \\ -\ell_{12} & 0 & \ell_{23} & \ell_{24} & \ell_{25} & \ell_{26} & \ell_{27} & \ell_{28} \\ -\ell_{13} & -\ell_{23} & 0 & \ell_{34} & \ell_{35} & \ell_{36} & \ell_{37} & \ell_{38} \\ -\ell_{14} & -\ell_{24} & -\ell_{34} & 0 & \ell_{45} & \ell_{46} & \ell_{47} & \ell_{48} \\ -\ell_{15} & -\ell_{25} & -\ell_{35} & -\ell_{45} & 0 & \ell_{56} & \ell_{57} & \ell_{58} \\ -\ell_{16} & -\ell_{26} & -\ell_{36} & -\ell_{46} & -\ell_{56} & 0 & \ell_{67} & \ell_{68} \\ -\ell_{17} & -\ell_{27} & -\ell_{37} & -\ell_{47} & -\ell_{57} & -\ell_{67} & 0 & \ell_{78} \\ -\ell_{18} & -\ell_{28} & -\ell_{38} & -\ell_{48} & -\ell_{58} & -\ell_{68} & -\ell_{78} & 0 \end{pmatrix}$$

one can see by inspection —checking the related pairs of entries— that a general element of the Lie algebra in this basis is given by

$$\mathcal{B} = \begin{pmatrix} H1 & 0 & B1 & B2 & B5 & B6 & B9 & B10 \\ 0 & -H1 & B3 & B4 & B7 & B8 & B11 & B12 \\ -B4 & -B2 & H2 & 0 & B13 & B14 & B17 & B18 \\ -B3 & -B1 & 0 & -H2 & B15 & B16 & B19 & B20 \\ -B8 & -B6 & -B16 & -B14 & H3 & 0 & B21 & B22 \\ -B7 & -B5 & -B15 & -B13 & 0 & -H3 & B23 & B24 \\ -B12 & -B10 & -B20 & -B18 & -B24 & -B22 & H4 & 0 \\ -B11 & -B9 & -B19 & -B17 & -B23 & -B21 & 0 & -H4 \end{pmatrix}$$

Notice that these new generators "Bk" are no longer anti-symmetric elements.[57] However, they still generate the same Lie algebra as $\mathfrak{so}_8$ up to isomorphism, since they will satisfy the same Serre relations which completely determine any semisimple Lie algebra. Notice also that the elements we have denoted by $H1, \ldots, H4$ constitute a basis for the Cartan subalgebra and they act now diagonally![58]

55: See Appendix D for some lines of the code.

56: The change of basis matrix thus given by

$$\begin{pmatrix} +i & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +i & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +i & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & +i & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

57: For example, $B1$ is given by

$$B1 = \begin{pmatrix} 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

58: For example, $H2$ is given by

$$H2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Inspired by the similar description of this Lie algebra and its root system given in [Kir] for $\mathfrak{so}_{2n}$ (see the appendix therein), we guess that the H_k generators are given by the following linear combinations: $H_1 = H1 - H2$, $H_2 = H2 - H3$, $H_3 = H3 - H4$, $H_4 = H3 + H4$. With these, we check the output of the commutators $[H_k, \mathcal{B}]$ for $k = 1, \ldots, 4$ and see which elements fit the a_{ij} coefficients for the $\mathfrak{so}_8$ Cartan matrix in Serre's relations for all the H_k simultaneously. This finally leads to the following Serre generators

$$H_1 = \begin{pmatrix} +1 &&&&&&& \\ & -1 &&&&&& \\ && -1 &&&&& \\ &&& +1 &&&& \\ &&&& 0 &&& \\ &&&&& 0 && \\ &&&&&& 0 & \\ &&&&&&& 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 &&&&&&& \\ & 0 &&&&&& \\ && +1 &&&&& \\ &&& -1 &&&& \\ &&&& -1 &&& \\ &&&&& +1 && \\ &&&&&& 0 & \\ &&&&&&& 0 \end{pmatrix},$$

$$H_3 = \begin{pmatrix} 0 &&&&&&& \\ & 0 &&&&&& \\ && 0 &&&&& \\ &&& 0 &&&& \\ &&&& +1 &&& \\ &&&&& -1 && \\ &&&&&& -1 & \\ &&&&&&& +1 \end{pmatrix}, \quad H_4 = \begin{pmatrix} 0 &&&&&&& \\ & 0 &&&&&& \\ && 0 &&&&& \\ &&& 0 &&&& \\ &&&& +1 &&& \\ &&&&& -1 && \\ &&&&&& +1 & \\ &&&&&&& -1 \end{pmatrix},$$

$$X_1 = \begin{pmatrix} 0 & 0 & +i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +i & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +i & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_3 = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&+i&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&-i&0&0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&+i \\ 0&0&0&0&-i&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix},$$

$$X_4 = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&+i \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&-i&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \quad Y_4 = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&+i&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&-i&0&0&0 \end{pmatrix},$$

which are checked to indeed satisfy all other Serre relations. Notice the radical simplifications this represents with respect to the previous approach. Observe further that the pairs X_k, Y_k are all hermitian conjugates. The fact of obtaining generators H_k of the Cartan subalgebra acting **diagonally** has been essential, and will further be extremely helpful when working with representations.

Now one aims to find a nice description for the r-th symmetric representation[59]

$$V^r := \mathrm{Sym}^r V_8 = \{(v_1, \dots, v_r) \in V_8^r\}/\mathcal{S}_r,$$

whose dimension is[60]

$$\dim V^r = \binom{8-1+r}{r}.$$

59: Where V_8 denotes the fundamental representation of $\mathfrak{so}_8$ given by matrix multiplication

60: Indeed, the problem reduces to counting the number of different ways to assign r index numbers between 1 and 8. If each number k appears n_k times, the constraint is

$$n_1 + \cdots + n_8 = r$$

with $n_k \geq 0$. This is the same as the number of combinations of typing exactly $8-1$ + signs between a row of r 1's. The number of 1's between each + sign corresponds to n_k each. This fills a total of $8-1+r$ entries out of which $8-1$ have to be chosen to be + signs:

$$\binom{8-1+r}{8-1} = \binom{8-1+r}{r}.$$

This is equivalent to considering symmetric tensor products. As will be shown in a moment, we have seen that this representation can again be obtained by the polynomial representation, now slightly more involved.

Polynomial Representation

Recall the polynomial representation we introduced for the $\mathfrak{sl}_2$ case. In the case of $\mathfrak{so}_8$, we search for generators acting on the vector space of polynomials in eight variables with total degree r, denoted $V_r \subset \mathbb{C}[x_1, \dots, x_8]$.[61] By acting on a general element $q_r(x) := \prod_{i=1}^8 x_i^{\alpha_i}$, with $\sum_{i=1}^8 \alpha_i = r$, through the derived action, one obtains[62]

$$\begin{aligned} X_1 \cdot q_r(x) &= \frac{d}{dt}\left(\exp(tX_1) \cdot \prod_{i=1}^8 x_i^{\alpha_i} \right)\Bigg|_{t=0} \\ &= \frac{d}{dt}\left((x_1 - tix_3)^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} (itx_2 + x_4)^{\alpha_4} x_5^{\alpha_5} \cdots x_8^{\alpha_8}\right)\Bigg|_{t=0} \\ &= -ix_3\alpha_1 \, x_1^{\alpha_1 - 1} \prod_{i=2}^8 x_i^{\alpha_i} + ix_2\alpha_4 \, x_1^{\alpha_1} \cdots x_3^{\alpha_3} x_4^{\alpha_4 - 1} \prod_{i=5}^8 x_i^{\alpha_i} \\ &= -\left(ix_3 \frac{\partial}{\partial x_1} - ix_2 \frac{\partial}{\partial x_4} \right) q_r(x) \end{aligned}$$

61: Notice again that

$$\mathbb{C}[x_1, \dots, x_8] = \bigoplus_{r=0}^{\infty} V_r.$$

62: Observe that the matrix E_1^2 vanishes and recall $\exp(tX)^{-1} = \exp(-tX)$.

and similarly for the other generators.[63] As done in the $\mathfrak{sl}_2$ case and as a matter of preference, one can choose to switch the roles of the X_k's and Y_k's while flipping the signs and changing the H_k's by a sign as well,[64] finally obtaining the generators:

63: Observe that the resulting expression is given by

$$X_k = -\sum_{i,j=1}^{8} M_{ij}^k \, x_j \frac{\partial}{\partial x_i},$$

where M_{ij}^k are the entries of X_k in matrix form.

64: This is a symmetry of Serre's relations, so the new generators still satisfy the desired commutation relations.

$$
\begin{aligned}
X_1 &= i\left(x_4\frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial x_3}\right), &\quad Y_1 &= i\left(x_3\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_4}\right)\\
X_2 &= i\left(x_6\frac{\partial}{\partial x_4} - x_3\frac{\partial}{\partial x_5}\right), &\quad Y_2 &= i\left(x_5\frac{\partial}{\partial x_3} - x_4\frac{\partial}{\partial x_6}\right)\\
X_3 &= i\left(x_8\frac{\partial}{\partial x_6} - x_5\frac{\partial}{\partial x_7}\right), &\quad Y_3 &= i\left(x_7\frac{\partial}{\partial x_5} - x_6\frac{\partial}{\partial x_8}\right)\\
X_4 &= i\left(x_7\frac{\partial}{\partial x_6} - x_5\frac{\partial}{\partial x_8}\right), &\quad Y_4 &= i\left(x_8\frac{\partial}{\partial x_5} - x_6\frac{\partial}{\partial x_7}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
H_1 &= x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} - x_3\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_4},\\
H_2 &= x_3\frac{\partial}{\partial x_3} - x_4\frac{\partial}{\partial x_4} - x_5\frac{\partial}{\partial x_5} + x_6\frac{\partial}{\partial x_6},\\
H_3 &= x_5\frac{\partial}{\partial x_5} - x_6\frac{\partial}{\partial x_6} - x_7\frac{\partial}{\partial x_7} + x_8\frac{\partial}{\partial x_8},\\
H_4 &= x_5\frac{\partial}{\partial x_5} - x_6\frac{\partial}{\partial x_6} + x_7\frac{\partial}{\partial x_7} - x_8\frac{\partial}{\partial x_8}.
\end{aligned}
$$

One then wishes to find a suitable assignment for the α_i's to obtain a basis for V_r in which the generators act by changing some of the labels by ± 1 at most.[65] By following the actions of the X_i and Y_i generators on the labels, where each $x_j\frac{\partial}{\partial x_i}$ reduces α_i by 1 and increases α_j by 1, we observe the following chain:

65: This will be clear in a moment.

$$1 \longleftrightarrow 3 \longleftrightarrow 5 \longleftrightarrow \{7,8\} \longleftrightarrow 6 \longleftrightarrow 4 \longleftrightarrow 2,$$

where the left and right arrows correspond to X_i and Y_i actions, respectively. From this diagram, we obtain the suitable basis of degree-r polynomials[66]

66: At a first stage, the obtained basis consisted of eight variables a_i showing a closer relation to the above mentioned chain. However, only seven variables where expected, since the condition of the exponents summing up to r was directly met by construction. Such reduction was indeed found, yielding the presented result.

$$p_r(x) = x_1^{a_1}x_2^{r-a_2}x_3^{a_3-a_1}x_4^{a_2-a_4}x_5^{a_5-a_3}x_6^{a_4-a_6-a_7}x_7^{a_7-a_5}x_8^{a_6}.$$

With the notation $p_r(x) \equiv |r;a\rangle$, for $a = (a_1,\ldots,a_7)$ subject to the constraints

$$
\begin{aligned}
0 \le \boldsymbol{a_1, a_2} \le r, \quad a_1 \le \boldsymbol{a_3} \le a_3 + r, \quad a_2 - r \le \boldsymbol{a_4} \le a_2,\\
a_3 \le \boldsymbol{a_5} \le a_3 + r, \quad a_5 \le \boldsymbol{a_7} \le a_5 + r, \quad 0 \le \boldsymbol{a_6} \le a_4 - a_7,
\end{aligned}
$$

the action of the H_k's on this basis yields the eigenvalues

$$
\begin{array}{llllll}
H_1: & +2(a_1+a_2) & -1(a_3+a_4) & & & -r\\
H_2: & -1(a_1+a_2) & +2(a_3+a_4) & -1(a_5+a_6) & -1(a_7) & \\
H_3: & & -1(a_3+a_4) & +2(a_5+a_6) & & \\
H_4: & & -1(a_3+a_4) & & +2(a_7) &
\end{array}
$$

Notice how the symmetry in the Cartan matrix[67] is recovered in the eigenvalues of this representation. The action of the remaining generators is given by[68]

67: Recall that the Cartan matrix for $\mathfrak{so}_8$ is
$$\begin{pmatrix} 2 & -1 & 0 & 0\\ -1 & 2 & -1 & -1\\ 0 & -1 & 2 & 0\\ 0 & -1 & 0 & 2\end{pmatrix}.$$

68: We denote $|r;a_i+1\rangle$ for the vector with tuple $(a_1,\ldots,a_i+1,\ldots,a_7)$.

$$
\begin{aligned}
X_1|r;a\rangle &= i((r-a_2)|r;a_2+1\rangle - (a_3-a_1)|r;a_1+1\rangle)\\
X_2|r;a\rangle &= i((a_2-a_4)|r;a_4+1\rangle - (a_5-a_3)|r;a_3+1\rangle)\\
X_3|r;a\rangle &= i((a_4-a_6-a_7)|r;a_6+1\rangle - (a_7-a_5)|r;a_5+1\rangle)\\
X_4|r;a\rangle &= i((a_4-a_6-a_7)|r;a_7+1\rangle - a_6|r;a_5+1,a_6-1,a_7+1\rangle)
\end{aligned}
$$

$$
\begin{aligned}
Y_1|r;a\rangle &= i(a_1|r;a_1-1\rangle - (a_2-a_4)|r;a_2-1\rangle) \\
Y_2|r;a\rangle &= i((a_3-a_1)|r;a_3-1\rangle - (a_4-a_6-a_7)|r;a_4-1\rangle) \\
Y_3|r;a\rangle &= i((a_5-a_3)|r;a_5-1\rangle - a_6|r;a_6-1\rangle) \\
Y_4|r;a\rangle &= i((a_5-a_3)|r;a_5-1,a_6+1,a_7-1\rangle - (a_7-a_5)|r;a_7-1\rangle)
\end{aligned}
$$

where we see that only ± 1 jumps in the seven-dimensional lattice given by the a_i parameters occur. The X_i's and Y_i's act here, respectively, almost as **raising** and **lowering** operators in this lattice, each raising or lowering in two directions at the same time. The *almost* refers to the second term in X_4 and the first in Y_4, where instead two entries are raised and one lowered, and viceversa. We will call them raising and lowering operators anyway. This is by far a much better description than the one we first encountered and shows the desired features as we move forward.[69]

69: These generators have been checked to indeed define the Lie algebra $\mathfrak{so}_8$ by checking one by one each of the Serre relations with the help of a `Mathematica` notebook (see Appendix D.3).

For the R-matrix we should compute ℓ-th powers of these generators. They yield formulas of the form

$$X_1^\ell|r;a\rangle = (i)^\ell \sum_{k=0}^{\ell} \binom{\ell}{k}(-1)^k \tfrac{(r-a_2)!}{(r-(a_2+k))!} \tfrac{(a_3-a_1)!}{(a_3-(a_1+\ell-k))!}|r;a_1+\ell-k,a_2+k\rangle.$$

However, since these expressions are getting complicated enough to handle with in the expression for the R-matrix, we leave them aside for the time being. The idea is that these will be implemented in a `Mathematica` notebook after having obtained the corresponding quantum representation for the R-matrix.

Nevertheless, let us stop for a moment and show that the obtained polynomial representation is indeed the desired r-th symmetric representation.[70] We first observe that the highest weight vector is x_1^r, with highest weight[71] $\lambda = (r,0,0,0)$. Indeed, we see that the coefficients in the above expressions for the action of the raising operators X_i's correspond to the exponents of the x_i's from 2 to 8 in the general expression for the basis vectors. Thus, for $p_r(x) = x_1^r$ all exponents other than the power of x_1 vanish, and hence all coefficients in the action for the X_i's. Therefore, x_1^r is annihilated by all the raising operators $X_i \in \mathfrak{n}_+$. By repeated action of the Y_i's one obtains all the other basis vectors. This will be clear when we argue the irreducibility of this representation.

70: It should be clear that it is so when choosing the basis $\prod_{i=1}^8 x_i^{\alpha_i}$ with $\sum_{i=1}^8 \alpha_i = r$, since it is completely analogous to the symmetric tensor representation construction. However, we show here that the different choice of labeling a_i still gives the symmetric representation.

71: Each entry respectively corresponding to the eigenvalue of $H_1,\dots,H_4$. These values for λ can be obtained from the eigenvalues shown above by plugging the corresponding values for a_i's in the case $p_r(x) = x_1^r$.

On the other hand, we have that the highest weight for the fundamental representation is clearly the first canonical vector e_1 (since it is annihilated by all the matrices X_i and is an eigenvector for all the H_i's) with eigenvalue $\lambda = (1,0,0,0)$. Thus, for the symmetric tensor representation with the action of an element $x \in \mathfrak{g}$ on a tensor vector being

$$\Delta^r(x)\cdot v_1\otimes\cdots\otimes v_r = x\cdot v_1\otimes\cdots\otimes v_r + \cdots + v_1\otimes\cdots\otimes x\cdot v_r,$$

the highest weight vector is non other than $e_1\otimes\cdots\otimes e_1$ with highest weight $\lambda = (r,0,0,0)$ given by the sum of each of the r terms with eigenvalues 1 or 0, respectively.

Thus, both the polynomial and the symmetric representation have the same highest weight. Now, if the former one is indeed irreducible, we would be done since the Highest Weight Theorem tells us they are isomorphic representations. Suppose it were not irreducible. Then the polynomial representation would contain the symmetric representation as a subrepresentation[72] and would thus necessarily have a strictly greater dimension than this. However, the dimensions are the same. Indeed, the dimension of the polynomial representation is given by the number of assignments for the a_i's respecting the constraints that were given previously, so it can be computed as the iterated sum

$$\sum_{a_1=0}^{r}\sum_{a_2=0}^{r}\sum_{a_3=a_1}^{a_1+r}\sum_{a_4=a_2-r}^{a_2}\sum_{a_5=a_3}^{a_3+r}\sum_{a_7=a_5}^{a_5+r}\sum_{a_6=0}^{a_4-a_7} 1$$

which has been implemented into a `Mathematica` notebook to give the expression

$$\frac{5040+13068r+13132r^2+6769r^3+1960r^4+322r^5+28r^6+r^7}{5040} = \binom{8-1+r}{r},$$

which precisely is the dimension of the symmetric representation. Therefore, the polynomial representation is irreducible and, having the same highest weight, it is isomoprhic to the symmetric representation.

72: Indeed, one of its irreducible subrepresentations should contain the highest weight vector, and thus by the Highest Weight Theorem be isomorphic to the symmetric representation.

Having seen this, we aim now to obtain the positive roots of $\mathfrak{so}_8$ and figure out their order, as explained in the theorem giving the general R-matrix.

Using the already mentioned results in [Kir] (see the appendix therein), we notice that the Cartan subalgebra of $\mathfrak{so}_8$ in this description is given by $\mathfrak{h} = \{\operatorname{diag}(x_1, -x_1, \ldots, x_4, -x_4)\}$. Define the following basis $\{e_i\}_{i=1}^4$ of $\mathfrak{h}^*$ by

$$e_i : \operatorname{diag}(x_1, -x_1, \ldots, x_4, -x_4) \mapsto x_i$$

with bilinear form $(e_i, e_j) = \delta_{ij}$. With this notation, the root system is then $\Delta = \{\pm e_i \pm e_j \mid i \neq j\}$, where the signs are chosen independently. The set of positive roots is $\Delta_+ = \{e_i \pm e_j \mid i < j\}$ with a total of $4(4-1) = 12$ positive roots. Among these, the fundamental roots are $\Pi = \{\alpha_1, \ldots, \alpha_4\}$ with

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \alpha_3 = e_3 - e_4, \quad \alpha_4 = e_3 + e_4.$$

One can thus write all the positive roots in terms of positive sums of simple roots, and the explicit relations can be seen to be given by

$$\begin{aligned}
e_1 - e_2 &= \alpha_1 & e_1 + e_2 &= \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4\\
e_1 - e_3 &= \alpha_1 + \alpha_2 & e_1 + e_3 &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\\
e_1 - e_4 &= \alpha_1 + \alpha_2 + \alpha_3 & e_1 + e_4 &= \alpha_1 + \alpha_2 + \alpha_4\\
e_2 - e_3 &= \alpha_2 & e_2 + e_3 &= \alpha_2 + \alpha_3 + \alpha_4\\
e_2 - e_4 &= \alpha_2 + \alpha_3 & e_2 + e_4 &= \alpha_2 + \alpha_4\\
e_3 - e_4 &= \alpha_3 & e_3 + e_4 &= \alpha_4.
\end{aligned}$$

Next, one needs to figure out the order of the positive roots. This can be done as it was stated in the theorem, i.e. in terms of the longest element in the Weyl group $\mathcal{W}$ generated by the four reflections $r_1, \ldots, r_4$ given by $r_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$, with a_{ij} the Cartan matrix of $\mathfrak{so}_8$.[73] We will

73: The reflections inherit thus the symmetry in the Cartan matrix.

need the following table, which can be computed easily:

$$\begin{array}{llll} r_1(\alpha_1) = -\alpha_1, & r_1(\alpha_2) = \alpha_2 + \alpha_1, & r_1(\alpha_3) = \alpha_3, & r_1(\alpha_4) = \alpha_4 \\ r_2(\alpha_1) = \alpha_1 + \alpha_2, & r_2(\alpha_2) = -\alpha_2, & r_2(\alpha_3) = \alpha_3 + \alpha_2, & r_2(\alpha_4) = \alpha_4 + \alpha_2 \\ r_3(\alpha_1) = \alpha_1, & r_3(\alpha_2) = \alpha_2 + \alpha_3, & r_3(\alpha_3) = -\alpha_3, & r_3(\alpha_4) = \alpha_4 \\ r_4(\alpha_1) = \alpha_1, & r_4(\alpha_2) = \alpha_2 + \alpha_4, & r_4(\alpha_3) = \alpha_3, & r_4(\alpha_4) = -\alpha_4 \end{array}$$

Now, from [SST16] we obtain an expression for the longest element of the Weyl group for the family D_n, explicitly given by

$$\begin{aligned} w_0 =& (r_1 r_2 \cdots r_{n-1} r_n r_{n-2} r_{n-1} \cdots r_2 r_1)(r_2 \cdots r_{n-1} r_n r_{n-2} r_{n-1} r_2) \cdots \\ & \cdots (r_{n-2} r_{n-1} r_n r_{n-2}) r_{n-1} r_n. \end{aligned}$$

In our case D_4, the longest element is thus given by[74]

$$w_0 = (r_1 r_2 r_3 r_4 r_2 r_1)(r_2 r_3 r_4 r_2) r_3 r_4.$$

Notice that the longest element has indeed length 12, the number of positive roots. With this at hand, we can finally give an order for the positive roots, which we will denote as β_i with i specifying the order:[75]

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = r_{i_1}(\alpha_{i_2}), \quad \ldots \quad , \quad \beta_{12} = r_{i_1} \cdots r_{i_{11}}(\alpha_{i_{12}})$$

with $(i_1, \ldots, i_{12}) = (1,2,3,4,2,1,2,3,4,2,3,4)$. Carrying out the somehow tedious but easy iterative process of feeding the α_i's to the sequences of r_j's and using the table above, one obtains

$$\begin{array}{ll} \beta_1 = \alpha_1 = e_1 - e_2, & \beta_2 = \alpha_1 + \alpha_2 = e_1 - e_3, \\ \beta_3 = \alpha_1 + \alpha_2 + \alpha_3 = e_1 - e_4, & \beta_4 = \alpha_1 + \alpha_2 + \alpha_4 = e_1 + e_4, \\ \beta_5 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = e_1 + e_3, & \beta_6 = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = e_1 + e_2, \\ \beta_7 = \alpha_2 = e_2 - e_3, & \beta_8 = \alpha_2 + \alpha_4 = e_2 + e_4, \\ \beta_9 = \alpha_2 + \alpha_3 = e_2 - e_4, & \beta_{10} = \alpha_2 + \alpha_3 + \alpha_4 = e_2 + e_3, \\ \beta_{11} = \alpha_3 = e_3 - e_4, & \beta_{12} = \alpha_4 = e_3 + e_4. \end{array}$$

Also importantly, one needs to keep track of the orbits to which the β_i's belong.[76] These are given by

$$\begin{cases} \beta_1, \beta_6 & \text{belong to the } \alpha_1 \text{ orbit} \\ \beta_2, \beta_5, \beta_7, \beta_{10} & \text{belong to the } \alpha_2 \text{ orbit} \\ \beta_3, \beta_8, \beta_{11} & \text{belong to the } \alpha_3 \text{ orbit} \\ \beta_4, \beta_9, \beta_{12} & \text{belong to the } \alpha_4 \text{ orbit} \end{cases}$$

Finally, we give here the action of the four automorphisms[77] $T_1, \ldots, T_4$ that were described in the theorem giving the R-matrix, acting on the generators X_i, Y_i and on elements of the form $e^{\frac{h}{2}H_i}$. Notice that we are using here the (normalized) lengths $d_i = 1$ for the simple roots in D_4.

	X_1	Y_1	$e^{\frac{h}{2}H_1}$
T_1	$-Y_1 e^{-\frac{h}{2}H_1}$	$-e^{-\frac{h}{2}H_1} X_1$	$e^{\frac{h}{2}H_1}$
T_2	$q^2 X_1 X_2 - X_2 X_1$	$q^{-2} Y_2 Y_1 - Y_1 Y_2$	$e^{\frac{h}{2}H_2} e^{\frac{h}{2}H_1}$
T_3	X_1	Y_1	$e^{\frac{h}{2}H_1}$
T_4	X_1	Y_1	$e^{\frac{h}{2}H_1}$

Table 4.2: Action of the T_i automorphisms on the generators X_1, Y_1 and $e^{\frac{h}{2}H_1}$.

74: This is just *one* possible reduced expression for w_0 the longest element in the Weyl group, but there might exist other reduced expressions (all of them with the same length, of course). However, one only needs to fix one such reduced expression to work with, since the computations wont depend on this choice.

75: That is, $\beta_i < \beta_j$ for $i < j$.

76: The **orbits** are the ones given by the action of the Weyl group.

77: It is enough to give this tables, since by definition of algebra homomorphisms they satisfy linearity and

$$T(AB) = T(A) \cdot T(B),$$

and so

$$T(A^{-1}) = T(A)^{-1}, \quad T(A^k) = T(A)^k.$$

	X_2	Y_2	$e^{\frac{h}{2}H_2}$
T_1	$q^2X_2X_1 - X_1X_2$	$q^{-2}Y_1Y_2 - Y_2Y_1$	$e^{\frac{h}{2}H_1}e^{\frac{h}{2}H_2}$
T_2	$-Y_2e^{-\frac{h}{2}H_2}$	$-e^{\frac{h}{2}H_2}X_2$	$e^{-\frac{h}{2}H_2}$
T_3	$q^2X_2X_3 - X_3X_2$	$q^{-2}Y_3Y_2 - Y_2Y_3$	$e^{\frac{h}{2}H_3}e^{\frac{h}{2}H_2}$
T_4	$q^2X_2X_4 - X_4X_2$	$q^{-2}Y_4Y_2 - Y_2Y_4$	$e^{\frac{h}{2}H_4}e^{\frac{h}{2}H_2}$

Table 4.3: Action of the T_i automorphisms on the generators X_2, Y_2 and $e^{\frac{h}{2}H_2}$.

	X_3	Y_3	$e^{\frac{h}{2}H_3}$
T_1	X_3	Y_3	$e^{\frac{h}{2}H_3}$
T_2	$q^2X_3X_2 - X_2X_3$	$q^{-2}Y_2Y_3 - Y_3Y_2$	$e^{\frac{h}{2}H_2}e^{\frac{h}{2}H_3}$
T_3	$-Y_3e^{-\frac{h}{2}H_3}$	$-e^{\frac{h}{2}H_3}X_3$	$e^{-\frac{h}{2}H_3}$
T_4	X_3	Y_3	$e^{\frac{h}{2}H_3}$

Table 4.4: Action of the T_i automorphisms on the generators X_3, Y_3 and $e^{\frac{h}{2}H_3}$.

	X_4	Y_4	$e^{\frac{h}{2}H_4}$
T_1	X_4	Y_4	$e^{\frac{h}{2}H_4}$
T_2	$q^2X_4X_2 - X_2X_4$	$q^{-2}Y_2Y_4 - Y_4Y_2$	$e^{\frac{h}{2}H_2}e^{\frac{h}{2}H_4}$
T_3	X_4	Y_4	$e^{\frac{h}{2}H_4}$
T_4	$-Y_4e^{-\frac{h}{2}H_4}$	$-e^{\frac{h}{2}H_4}X_4$	$e^{-\frac{h}{2}H_4}$

Table 4.5: Action of the T_i automorphisms on the generators X_4, Y_4 and $e^{\frac{h}{2}H_4}$.

We would then have to compute the expressions

$$E_{\beta_s} = T_{i_1}\cdots T_{i_{s-1}}(X_{i_s}), \quad F_{\beta_s} = T_{i_1}\cdots T_{i_{s-1}}(Y_{i_s}), \quad s = 1,\ldots,\nu,$$

but these rapidly become very involved for increasing s. Instead, they are left to be implemented in a `Mathematica` notebook. Notice how apparent is that the interactions come from the second triple of generators $\{X_2, Y_2, H_2\}$, since its the one with which all terms interact as shown in the $\mathfrak{so}_8$ Cartan matrix.

4.5 Future steps

Last, it remains to find the quantum analogue of the found representation. This work is still in progress. We outline here a few aspects that have been studied so far.

First to mention, the natural guess for a quantum analogue for our polynomial representation would be to replace all coefficients by quantum integers:[78]

78: This guess comes from replacing the derivatives acting on the polynomials and using instead **quantum derivatives**

$$\partial_i^q \cdot p(x) := \frac{p(qx_i) - p(q^{-1}x_i)}{q - q^{-1}},$$

which return the quantum integer $[n]$ when acting on polynomials x_i^n.

$$\begin{aligned}
X_1|r;a\rangle &= i([r-a_2]|r;a_2+1\rangle - [a_3-a_1]|r;a_1+1\rangle)\\
X_2|r;a\rangle &= i([a_2-a_4]|r;a_4+1\rangle - [a_5-a_3]|r;a_3+1\rangle)\\
X_3|r;a\rangle &= i([a_4-a_6-a_7]|r;a_6+1\rangle - [a_7-a_5]|r;a_5+1\rangle)\\
X_4|r;a\rangle &= i([a_4-a_6-a_7]|r;a_7+1\rangle - [a_6]|r;a_5+1,a_6-1,a_7+1\rangle)
\end{aligned}$$

$$\begin{aligned}
Y_1|r;a\rangle &= i([a_1]|r;a_1-1\rangle - [a_2-a_4]|r;a_2-1\rangle)\\
Y_2|r;a\rangle &= i([a_3-a_1]|r;a_3-1\rangle - [a_4-a_6-a_7]|r;a_4-1\rangle)\\
Y_3|r;a\rangle &= i([a_5-a_3]|r;a_5-1\rangle - [a_6]|r;a_6-1\rangle)\\
Y_4|r;a\rangle &= i([a_5-a_3]|r;a_5-1,a_6+1,a_7-1\rangle - [a_7-a_5]|r;a_7-1\rangle)
\end{aligned}$$

with $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ the quantum integer, and the H_i generators being replaced by

$$\begin{array}{llllll}
q^{H_1} = & q^{+2(a_1+a_2)} & q^{-1(a_3+a_4)} & & & q^{r} \\
q^{H_2} = & q^{-1(a_1+a_2)} & q^{+2(a_3+a_4)} & q^{-1(a_5+a_6)} & q^{-1(a_7)} & \\
q^{H_3} = & & q^{-1(a_3+a_4)} & q^{+2(a_5+a_6)} & & \\
q^{H_4} = & & q^{-1(a_3+a_4)} & & q^{+2(a_7)} &
\end{array}$$

where we recall that $q = e^{\frac{h}{2}}$, so $q^{H_i} = e^{\frac{h}{2}H_i}$.

However, these have been seen not to satisfy all of the commutation relations in Serre's presentation. Hence, some slight modification should be found. In order to obtain such one, a `Mathematica` notebook has been used to implement the action of these generators on basis vectors $|r; a_1, \ldots, a_7\rangle$ and gain some insight in what is impeding them to satisfy Serre's relations.[79] The hope here is to identify some minor corrections that can be applied on the coefficients by adding factors of q (e.g. $q^{a_3-2a_7}$) fixing these inconveniences, while still recovering the polynomial representation when setting $q \to 1$.

79: See Appendix D.4 for some lines of code in `Mathematica`.

In this direction, some progress has been made, although yet incomplete. For example, regarding the commutation relation

$$X_2X_1^2 - [2]X_1X_2X_1 + X_1^2X_2 = 0$$

with the quantum integer [2] being $(q + q^{-1})$, the left hand side was yielding instead

$$\begin{aligned}
&+ \frac{iq^{1-2a_2-a_3-a_5-2r}(q^{2a_3} - q^{2a_5})(-q^{2a_2} + q^{2r})(-q^{2+2a_2} + q^{2r})}{(-1+q)(1+q)^3}|r; a_2+2, a_3+1\rangle \\
&+ \frac{iq^{1-a_1-2a_2-a_3-a_4-r}(q^{2a_3} - q^{2a_1})(q^{2a_4} - q^{1+2a_2})(q^{2r} - q^{2a_2})}{(q-1)(1+q)^2}|r; a_1+1, a_2+1, a_4+1\rangle \\
&+ \frac{iq^{1-a_1-a_2-2a_3-a_5-r}(q^{1+2a_3} - q^{2a_1})(q^{2a_5} - q^{2a_3})(q^{2r} - q^{2a_2})}{(q-1)(1+q)^2}|r; a_1+1, a_2+1, a_3+1\rangle \\
&+ \frac{iq^{1-2a_1-a_2-2a_3-a_4}(q^{2a_1} - q^{2a_3})(q^{2a_3} - q^{2+2a_1})(q^{2a_4} - q^{2a_2})}{(q-1)(1+q)^3}|r; a_1+2, a_4+1\rangle
\end{aligned}$$

when acting on a general basis element $|r; a_1, \ldots, a_7\rangle$. For each one of them, by looking at each contribution of the different $X_2X_1^2$, $X_1X_2X_1$ and $X_1^2X_2$ separately, we have found the following correction[80]

80: Here it can be useful to look at the relation

$$[n+2] - [2][n+1] + [n] = 0$$

to cancel terms between the three contributions. This is obtained from

$$[a][b] - [a+c][b-c] = [c][a-b+c]$$

by setting $a = n+1$, $b = 2$, $c = 1$ and rearranging terms to the right hand side.

$$X_1|r; a\rangle = i(q^{a_3+3a_1}[r-a_2]|r; a_2+1\rangle - q^{a_4+a_2}[a_3-a_1]|r; a_1+1\rangle)$$

on X_1 while leaving X_2 the same.[81] Using this generators simplifies the above non-vanishing terms by killing all of them but one.

81: Being X_2 the interaction term among $X_1, \ldots, X_4$, it is convenient to leave it as simple as possible.

$$\frac{iq^{2+4a_1-a_3+a_4-a_5-r}(1+q^2)(q^{2a_3} - q^{2a_5})(q^{2r} - q^{2a_2})}{q^2-1}|r; a_1+1, a_2+1, a_3+1\rangle$$

still to be analyzed.

On the other hand, forgetting about these additional prefactors and concerning the commutation relations of the form $[X_i, Y_i] = \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}$, we have seen analytically —by letting $[X_1, Y_1]$ act on an arbitrary vector $|r; a\rangle$— that[82]

82: Again, we have used here the identity

$$[a][b] - [a+c][b-c] = [c][a-b+c].$$

$$[X_1, Y_1]|r; a\rangle = ([2a_1 - a_3] + [2a_2 - a_4 - r])|r; a\rangle$$

instead of the initial guess

$$[X_1, Y_1]|r; a\rangle = [2(a_1 + a_2) - (a_3 + a_4) - r]|r; a\rangle.$$

Looking then at the commutation relations

$$[X_1, Y_1]|r;a\rangle = ([2a_1 - a_3] + [2a_2 - a_4 - r])|r;a\rangle$$

Looking at

$$[H_i, X_j] = a_{ij}X_j \quad \rightsquigarrow \quad q^{H_i}X_j = q^{a_{ij}}X_jq^{H_i}$$

we have then found that the quantum analogue for the Cartan subalgebra generators reads instead[83]

$$\begin{aligned}
q^{H_1} &= q^{(a_1)-(a_3-a_1)} &&+ q^{-(r-a_2)+(a_2-a_4)} \\
q^{H_2} &= q^{(a_3-a_1)-(a_5-a_3)} &&+ q^{-(a_2-a_4)+(a_4-a_6-a_7)} \\
q^{H_3} &= q^{(a_5-a_3)-(a_7-a_5)} &&+ q^{-(a_4-a_6-a_7)+(a_6)} \\
q^{H_4} &= q^{(a_5-a_3)+(a_6)} &&+ q^{-(a_2-a_6-a_7)+(a_7-a_5)}
\end{aligned}$$

83: As opposed to the natural guess mentioned above.

whose exponents are to be compared with the coefficients of the X_i and Y_i generators. For instance, regarding q^{H_1}, the first power comes from the a_1 coefficients in X_1 and Y_1 added together, while the second power comes from the a_2 coefficients in X_1 and Y_1 added together as well but with an overall minus sign instead.

As can be seen, some progress in this direction leading to a quantum representation has been made, although yet incomplete. Once such representation be obtained, the implementation of the T_i automorphisms acting on the set of generators will be possible, leading to the final form for the desired $\mathfrak{so}_8$ Large Color R-matrix.

Appendix

A

Smooth Manifolds

Smooth manifolds are important mathematical objects in physics, constituting the basic structure for several theories such as Classical Mechanics, General Relativity and Yang-Mills theory. The development of a calculus on smooth manifolds is what is known as Differential Geometry. Here we give a brief review on some relevant concepts involving smooth manifolds [Grà; Bai; BM94].

A.1 Definition

We start with some basic concepts leading to the formal definition of a smooth manifold while still understanding the intuition and motivating ideas.

Definition A.1.1 *A **topological space** is a set X together with a collection $\mathcal{T}$ of subsets of X, called **open sets**, satisfying:*

1) The empty set $\emptyset$ and X are open.
2) If $U, V \subseteq X$ are open, so is $U \cap V$.
3) If the sets $U_\alpha \subseteq X$ are open, so is the union $\bigcup U_\alpha$.

The collection $\mathcal{T}$ of open sets is called the **topology** of X. A **neighborhood** of a point $x \in X$ is an open set $U \in \mathcal{T}$ containing it. The complements of open sets are called **closed** sets.[1]

Having a topology allows us to define the notion of *continuity*,[2] roughly understood as sending nearby points to nearby points.

Definition A.1.2 (Continuity) *Given two topological spaces X and Y, a function $f : X \to Y$ between them is **continuous** if for any open set $U \subset Y$ its inverse image $f^{-1}(Y) \subset X$ is also open.*

The idea of a manifold is that it can be covered by patches, called *charts*, each one looking like $\mathbb{R}^m$, just as with the Earth globe,[3] such that they are 'glued' with some notion of continuity.

Definition A.1.3 (Chart) *Let M be a topological space. An m-dimensional **chart** of M is a pair (U, φ) where $U \subset M$ is an open subset and $\varphi : U \to \varphi(U) \subset \mathbb{R}^m$ is a homeomorphism with some open set of $\mathbb{R}^m$.*

1: The index α in the third item runs in some (non-necessarily countable) set Λ, whereas the second item applies only for *finite* intersections of open sets. Using set theory, one can define the closed sets in a similar way as the open ones by replacing arbitrary unions and finite intersections with arbitrary intersections and finite unions, respectively.

2: The following definition is equivalent to the familiar epsilon-delta one in the particular case of a function $f : \mathbb{R}^n \to \mathbb{R}^m$: it is continuous if for all $x \in \mathbb{R}^n$ and for all $\epsilon > 0$, there $\exists \delta > 0$ such that $||y - x|| < \delta \implies ||f(y) - f(x)|| < \epsilon$.

3: In fact, we use some related terminology, such as *chart* and *atlas* which remind us about these cartographic ideas.

Recall that in topology a homeomorphism is a continuous function between topological spaces that has a continuous inverse function.

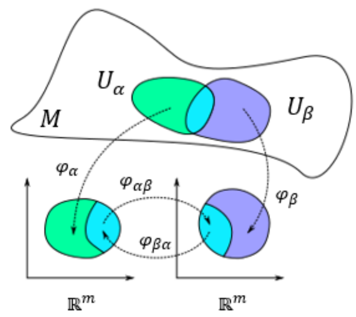

Figure A.1: Sketch of the notion of a smooth manifold.

Definition A.1.4 (Atlas) *An m-dimensional **atlas** in M is a set of m-dimensional charts $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ such that their domains cover M and any two charts are **compatible**. That is:*

- *$M = \bigcup_{\alpha \in A} U_\alpha$ and*
- *for all $\alpha, \beta \in A$ the **transition functions***

$$\varphi_{\alpha\beta} := \varphi_\beta \circ \varphi_\alpha^{-1}|_{\varphi_\alpha(U_\alpha \cap U_\beta)} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$

are smooth.

Observe that the transition functions are defined between open sets of $\mathbb{R}^m$. Now, we finally reach the desired definition.

Definition A.1.5 (Smooth manifold) *An m-dimensional **smooth manifold** M is a topological space equipped with a maximal atlas.*

Maximality here means that no further charts can be added to the given atlas without losing compatibility. The term *smooth* may be omitted when already understood by the context (and further, unless otherwise stated, the manifolds will be assumed to be Hausdorff and paracompact).

Intuitively, what this means is that every point in M lives in an open set U_α that looks like $\mathbb{R}^m$ and one can patch the whole manifold out of such pieces that look like $\mathbb{R}^m$.

A.2 Smooth maps and diffeomorphisms

The smooth manifold structure further enables the definition of *smooth functions* on M without ambiguity.

Definition A.2.1 (Smooth map) *A continuous map $f : M \to N$ between two manifolds is a **smooth map** if for all charts (U, φ) and (V, ψ) of M and N respectively, the composition*

$$\psi \circ f \circ \varphi^{-1}|_{\varphi(U \cap f^{-1}(V))} : \varphi(U \cap f^{-1}(V)) \longrightarrow \psi(V)$$

is smooth as a map from $\mathbb{R}^m$ to $\mathbb{R}^n$.

An isomorphic smooth map (i.e. with smooth inverse) is called a **diffeomorphism**. When $N = \mathbb{R}$, we have the notion of a smooth function.

Definition A.2.2 *Given an m-manifold M, a function $f : M \to \mathbb{R}$ is said to be **smooth** if $f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \subset \mathbb{R}^m \to \mathbb{R}$ is smooth for any chart $(U_\alpha, \varphi_\alpha)$.*

The set of all smooth functions is denoted as $\mathcal{C}^\infty(M)$.

The unambiguity comes from the fact that for charts defined on overlapping domains we can write

$$f \circ \varphi_\beta^{-1} = (f \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \varphi_\beta^{-1}),$$

where the second term in parenthesis on the right hand side is a transition function and hence smooth, so that smoothness on the overlapping is well defined.

A.3 Tangent space

Another elementary notion is the one of **tangent space**. The tangent space to a manifold M at a point $p \in M$, denoted $\mathbf{T_pM}$, is defined as the vector space of equivalence classes of tangent paths at the point. To put it precisely:

Definition A.3.1 (Tangent space) *Given an m-manifold M, a **path** is a smooth map $\gamma : I \to M$ defined in a non-degenerate interval[a] I. Given the set*

$$\mathcal{C}_{M,p} = \{\gamma : I \to M \mid \gamma \text{ smooth path st. } \gamma(0) = p\}$$

*two paths $\gamma_1, \gamma_2 \in \mathcal{C}_{M,p}$ are **tangent** (at 0) whenever there exists a chart (U, φ) of M at p such that[b] $D(\varphi \circ \gamma_1)(0) = D(\varphi \circ \gamma_2)(0)$. In this way, tangency defines an equivalence relation $\sim$ whose equivalence classes are the **tangent vectors** at p, denoted $[\gamma]_p$. The **tangent space** is then the quocient $T_pM := \mathcal{C}_{M,p}/\sim$. The bijective map $\boldsymbol{\theta}_{\varphi,p} : T_pM \to \mathbb{R}^m$ sending $[\gamma]_p \mapsto D(\varphi \circ \gamma)(0)$ enables to transport the vector space structure of $\mathbb{R}^m$ to T_pM.*

[a] We will consider open intervals I containing 0.
[b] $\hat{\gamma} = \varphi \circ \gamma : I \subset \mathbb{R} \to \mathbb{R}^m$ is called the **local expression** of γ in the chart (U, φ) and D is the usual differential operator in $\mathbb{R}^n$. Thus, two paths are tangent if their local expressions are so.

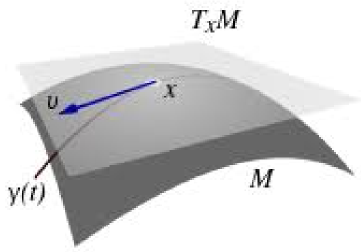

Figure A.2: Tangent space.

Alternatively, there is a theorem stating the equivalence (isomorphism) between tangency classes of paths and punctual derivations, which allows us to identify the tangent space T_pM with the $\mathbb{R}$-vector space of punctual derivations $\mathcal{D}_p(M)$. Thus, we may view tangent vectors as punctual derivations $\delta : \mathcal{C}^\infty(M) \to \mathbb{R}$, i.e. such that

$$\delta(fg) = \delta(f)g(p) + f(p)\delta(g)$$

for any $f, g \in \mathcal{C}^\infty(M)$ smooth functions. In this perspective, it is common when working in a given chart $\varphi = (x^1, \ldots, x^m)$ to write tangent vectors in the basis of the so-called **coordinate tangent vectors** at the given point: $\partial/\partial x^i|_p$. That is,

$$u_p = u^i \left.\frac{\partial}{\partial x^i}\right|_p \in T_pM.$$

A.4 Orientation

One can further endow manifolds with a structure of *orientation*.

Definition A.4.1 (Orientation) *An* ***orientation*** *of the real vector space $\mathbb{R}^n$ is a choice of sign ($+$ or $-$) for every ordered basis, such that two ordered basis have the same sign iff the linear transformation from one to the other has positive determinant.*

In other words, an orientation is a choice of an ordered basis, which we assign to be positive (the sign of the other ordered bases are then uniquely determined).

Notice therefore that $\mathbb{R}^n$ admits only two possible orientations. One naturally assigns a $+$ sign to the canonical basis

$$((1,0,\dots,0),(0,1,0,\dots,0),\dots,(0,\dots,0,1))$$

of $\mathbb{R}^n$, and calls it the **standard orientation**.[4]

4: The standard orientation of $\{0\}$ being the $+$ sign.

The notion of **orientation of a manifold M** is then given by a smooth choice of orientations of the tangent spaces T_pM, meaning that the transition functions $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ are orientation preserving.[5]

5: Orientation preserving means that its differential has positive determinant.

Definition A.4.2 (Orientable manifold) *A manifold M is said to be* ***orientable*** *if it admits an orientation.*

Not all manifolds, however, admit an orientation.[6]

6: The Klein bottle, for instance, is non-orientable.

Remark A.4.1 In Homology Theory[a] language, a closed m-manifold M is said to be **orientable** if $H_m(M;\mathbb{Z}) = \mathbb{Z}$. An **orientation** is the choice of a generator $[M]$ in $\mathbb{Z}$, the fundamental class of M. An oriented manifold is one equipped with a choice of orientation. When M is a compact m-manifold with boundary, it is said to be orientable if $H_m(M,\partial M;\mathbb{Z}) = \mathbb{Z}$, and an orientation is given by a choice of generator $[M,\partial M]$ in $\mathbb{Z}$. Finally, a smooth manifold M is said to be orientable if and only if the restriction of its tangent bundle to every smooth curve is trivial. Last to mention, one can show that the boundary of an orientable manifold is orientable.

[a] See Appendix C on Homology and Cohomology Theory.

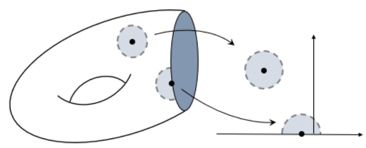

Figure A.3: Manifold with boundary.

A.5 Manifolds with boundary

The previous concepts can be extended to manifolds with boundary by allowing the charts $\varphi : U \longrightarrow \mathbb{R}^m$ have as image open subsets of $\mathbb{H}^m := \{(x_1,\dots,x_m) \mid x_m \geq 0\}$, known as the **half-plane**.

A point lies in the boundary of the manifold M if it is mapped by a chart to a point in $\partial\mathbb{H}^m := \{(x_1,\dots,x_m) \mid x_m = 0\}$. The set of boundary points forming the boundary of M is denoted by ∂M. It turns out to be a submanifold of M without boundary and with codimension one: $\dim\partial M = \dim M - 1$. A compact[7] manifold without boundary, $\partial M = \emptyset$, is called **closed**.

7: See next section for the notion of *compactness*.

There is a notion of how the orientation of ∂M with respect to M is.

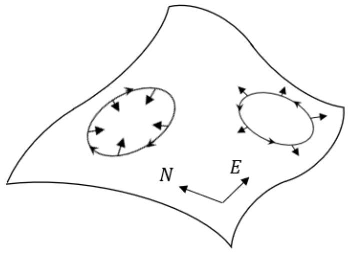

Figure A.4: A two-dimensional manifold with positive basis (N, E) whose positive normal vectors point outwards for counterclockwise-oriented $\mathbb{S}^1$ and inwards for clockwise-oriented $\mathbb{S}^1$.

Definition A.5.1 *Given Σ a closed submanifold of M of codimension one, let $p \in \Sigma$ and $(v_1, \ldots, v_{m-1})$ be a given positive basis for $T_p\Sigma$. Then, a vector $w \in T_pM$ is said to be* ***positive normal*** *if $(v_1, \ldots, v_{m-1}, w)$ is a positive basis for T_pM.*

The different connected components of the boundary of a manifold can thus be classified in two classes.

Definition A.5.2 *Given a manifold M with boundary, a connected component of ∂M is called an* ***in-boundary*** *if its positive normal vectors point inwards, and is called an* ***out-boundary*** *if its positive normal vectors point outwards.*

The definition does indeed not depend on the choice of positive normal vector. In this way, every boundary ∂M of an oriented manifold M is given by a disjoint union of in-boundaries and out-boundaries.

A.6 Compactness

Last, there is an important concept capturing the idea of whether a given manifold extends indefinitely in all directions like $\mathbb{R}^3$ or does not like $\mathbb{S}^3$; it is *compact*.

Definition A.6.1 (Compactness) *A topological space X is said to be* ***compact*** *if for every cover $\bigcup_\alpha U_\alpha$ of X by open sets U_α there is a finite collection $U_{\alpha_1}, \ldots, U_{\alpha_r}$ that still covers X.*

For manifolds, one may as well use the equivalent definition:

Definition A.6.2 (Compact manifold) *A manifold M is* ***compact*** *if and only if every sequence in M has a convergent subsequence.*

Finally, a basic theorem says that a subset of $\mathbb{R}^m$ is compact if and only if it is closed and fits inside a ball of sufficiently large radius.

B

Lie Algebras and Representation Theory

Lie groups and Lie algebras are useful and interesting mathematical objects with many physical applications, with a special emphasis in quantum physics. As an example, the Standard Model of particle physics –describing the fundamental building blocks of the universe we live in– is encapsulated in the *representations* of the Lie group $SU(3) \times SU(2) \times U(1)$, which can in turn be studied in terms of the representations of the associated Lie algebras $\mathfrak{su}(3)$, $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$.

In the following, we present a brief introduction to the theory of Lie algebras, root systems and their representations [Kir; Hum72].

B.1 Lie groups and Lie algebras

We start with the definition of a Lie group and proceed to show how the corresponding Lie algebra is obtained. We then focus on the classification of finite-dimensional semisimple complex Lie algebras.[1]

1: Recall that a **group** is a set with a group operation $\bullet$ such that it is associative, has a neutral element and every element has an inverse (and may be also commutative, what we call an abelian group)

Definition B.1.1 *A* ***Lie group*** $(G, \bullet)$ *is a group which further has a structure of smooth manifold and such that the multiplication and inverse maps are both smooth:*

$$\begin{aligned} \mu : G \times G &\longrightarrow G & \qquad i : G &\to G \\ (g_1, g_2) &\mapsto g_1 \bullet g_2 & g &\mapsto g^{-1} \end{aligned}$$

The *inverse map* i is well defined since G is a group. To be a smooth manifold is a much stronger condition than being just a group, since one is giving a topology on it and a smooth structure. Here, the manifold $G \times G$ inherits the smooth atlas from G.[2]

2: Some examples of Lie groups are:

- The 1-sphere $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ with $\bullet$ the multiplication on $\mathbb{C}$. This (commutative) Lie group $(\mathbb{S}^1, \bullet)$ is usually called $U(1)$.
- The **general linear group** $GL(n, \mathbb{R}) = \{\text{linear } \varphi : \mathbb{R}^n \to \mathbb{R}^n \mid , \det\varphi \neq 0\}$ with $\bullet$ the composition of maps $\circ$.

There is a special map which constitutes the main tool to define the Lie algebra corresponding to a Lie group.

Definition B.1.2 *Given a Lie group* $(G, \bullet)$ *and any element* $g \in G$*, define the* ***left translation*** *with respect to* g *as the map*

$$\begin{aligned} \ell_g : G &\to G \\ h &\mapsto g \bullet h \end{aligned}$$

Each smooth map ℓ_g is clearly bijective and thus a diffeomorphism[3] on G. Through ℓ_g one can now push forward any vector field[4] X on

3: Although not a *group* isomorphism.

4: Given a manifold M, a **vector field** in M is a section of the tangent bundle $\tau_M : TM \to M$, that is, a map $X : M \to TM$ such that $\tau_M \circ X = Id_M$. Thus, at each point $p \in M$ it gives a tangent vector $X(p) \equiv X_p \in T_pM$.

G to another vector field given at any point $h \in G$ by

$$(\ell_{g*}X)_{gh} := \ell_{g*}(X_h).$$

There are special vector fields which remain the same under this procedure and are therefore more interesting.

Definition B.1.3 *Given a Lie group* $(G, \bullet)$*, a vector field* X *on* G *is called* ***left-invariant*** *if for any* $g \in G$ *the vector field equation*

$$\ell_{g*}X = X$$

is satisfied or, equivalently, $\forall f \in \mathcal{C}^\infty(G)$ *it holds*

$$X(f \circ \ell_g) = (Xf) \circ \ell_g.$$

The set of all left-invariant vector fields on a Lie group G is denoted $L(G)$ and is a subset of $\Gamma(TG)$, the set of vector fields on the manifold G (sections of the tangent bundle). We are about to see that these are in fact *Lie algebras*, i.e. vector spaces equipped with a specific additional operation.

Definition B.1.4 *A* ***Lie algebra*** $(\mathfrak{g}, +, \bullet, [\![\cdot,\cdot]\!])$ *is a* $\mathbb{K}$*-vector space* $(\mathfrak{g}, +, \bullet)$ *equipped with a map* $[\![\cdot,\cdot]\!] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$*, called* ***Lie bracket****, which is bilinear, antisymmetric and satisfies the Jacobi identity:*

$$[\![x, [\![y, z]\!]]\!] + [\![y, [\![z, x]\!]]\!] + [\![z, [\![x, y]\!]]\!] = 0.$$

We write here $[\![\cdot,\cdot]\!]$ for the Lie bracket to distinguish it from the usual commutator in differential geometry, but it is typically denoted as $[\cdot,\cdot]$. Also, one usually denotes simply $\mathfrak{g}$ to refer to the whole Lie algebra. With this at hand, we can now state the following theorem.

Theorem B.1.1 $(L(G), [\cdot,\cdot])$ *is a Lie subalgebra of* $(\Gamma(TG), [\cdot,\cdot])$.

That is,[5] the bracket $[\cdot,\cdot] : L(G) \times L(G) \to L(G)$ really maps to $L(G)$; i.e. the image does indeed give left-invariant vector fields. Further and most important, we have the following theorem which enables us to identify the Lie algebra $\mathfrak{g}$ of a group G with its tangent space at the identity T_eG.

5: Recall that a **subalgebra** is a subset of an algebra which is closed under all its operations and defined with the induced operations.
In the case of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$, a subspace $\mathfrak{h} \subset \mathfrak{g}$ is called a **Lie subalgebra** if it is closed under the Lie bracket: $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. A subspace $\mathfrak{h} \subset \mathfrak{g}$ is called an **ideal** if $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.

Theorem B.1.2 *There is a Lie algebra isomorphism* $L(G) \cong T_eG$*, where* e *is the neutral element in* G*.*

Proof. The linear isomorphism $j : T_eG \xrightarrow{\sim} L(G)$ is defined by sending every tangent vector $A \in T_eG$ to the left-invariant vector field $j(A) \in L(G)$ given at each $g \in G$ by the pushforward $j(A)_g := \ell_{g*}A \in T_gG$. To obtain a Lie algebra isomorphism, define the bracket $[\![\cdot,\cdot]\!] : T_eG \times T_eG \to T_eG$ by precisely imposing the condition $j([\![A, B]\!]) = [j(A), j(B)]$, i.e.

$$[\![A, B]\!] := j^{-1}[j(A), j(B)]$$

for any $A, B \in T_eG$. □

Exponential map

One may go back and recover part of the Lie group from its Lie algebra through the so-called *exponential map*, as we briefly outline here. This involves the following concepts.

Definition B.1.5 *Let M be a smooth manifold and X be a smooth vector field on M. Then a smooth curve $\gamma : (a, b) \to M$ is called an* ***integral curve*** *if*

$$\forall \lambda \in (a, b) : \quad \gamma'(\lambda) = X_{\gamma(\lambda)},$$

where $\gamma'(\lambda)$ is the vector tangent to the curve γ at the point $\gamma(\lambda)$.

There is a unique integral curve γ of X through each point of the manifold.[6] Such a curve is called **complete** if its domain (a, b) can be extended to all of $\mathbb{R}$.[7] It turns out that every left-invariant vector field on a Lie group G is complete. This enables the definition of the exponential map.

6: By the local existence and uniqueness of solutions to an ODE
7: On a compact manifold, for example, every vector field is complete.

Definition B.1.6 *Let $(G, \bullet)$ be a Lie group. Given $A \in T_eG$, define the uniquely determined left-invariant vector field X^A given at each $g \in G$ by $X^A_g := \ell_{g*}A$, and let $\gamma^A : \mathbb{R} \to G$ be the integral curve of X^A through the point $\gamma^A(0) = e$. Then, the* ***exponential map*** *is given by*

$$\begin{aligned} \exp : T_eG &\to G \\ A &\mapsto \gamma^A(1) \end{aligned}$$

The name of this map is justified because it satisfies similar properties to the familiar ones for usual exponentials and it further recovers the exponential power series when dealing with matrix Lie groups. The exponential map satisfies some interesting properties.

Theorem B.1.3 *Let $(G, \bullet)$ be a Lie group. Then:*

(1) The exponential map $\exp$ *is a local diffeomorphism: there exists $U \subseteq T_eG$ an open set containing zero such that the restricted map*

$$\exp|_U : U \to \exp(U) \subseteq G$$

is bijective, smooth and with smooth inverse $\log := \exp|_U^{-1}$.

(2) If G is compact, then $\exp$ *is surjective and thus* $\exp(T_eG) = G$.

Choosing a basis $A_1, \ldots, A_\nu$ (with $\nu = \dim G$) of T_eG hence provides a parametrization of the Lie group G through the exponential map. The basis elements are then called the **(infinitesimal) generators** of the Lie group.

Definition B.1.7 *A* ***one-parameter subgroup*** *of a Lie group $(G, \bullet)$ is a Lie group homomorphism; that is to say, a smooth map $\xi : (\mathbb{R}, +) \to (G, \bullet)$ for which $\xi(\lambda_1 + \lambda_2) = \xi(\lambda_1) \bullet \xi(\lambda 2)$.*

Theorem B.1.4 *Let $(G, \bullet)$ be a Lie group. Then:*

(i) The map $\xi^A : \mathbb{R} \to G$, $\xi^A(\lambda) := \exp(\lambda A)$, is a one-parameter subgroup of G for any $A \in T_eG$. In particular, $(\xi^A(\lambda))^{-1} = \xi^A(-\lambda)$.

(ii) Any one-parameter subgroup of G is of this form.

This means that for any element $A = \sum_{i=1}^{\nu} \lambda^i A_i \in T_eG$ of the Lie algebra, the corresponding element $\exp(A)$ of the Lie group can be expressed as $\prod_{i=1}^{\nu} \exp(\lambda_i A_i)$. Finally, the following theorem states that the exponential map behaves well with respect to smooth group homomorphisms.

$$\begin{tikzcd} G \arrow[r, "f"] & H \\ T_eG \arrow[u, "\exp"] \arrow[r, "f_*"] & T_eH \arrow[u, "\exp"'] \end{tikzcd}$$

Figure B.1: Commutative diagram for the exponential map exp and a given smooth group homomorphism f.

Theorem B.1.5 *Let $(G, \bullet)$ and $(H, *)$ be Lie groups and $f : G \to H$ both a smooth map and group homomorphism:*

$$f(g_1 \bullet g_2) = f(g_1) * f(g_2), \quad \forall g_1, g_2 \in G.$$

Then, the diagram in Figure B.1 commutes.

B.2 Representations of Lie algebras

In some sense, one can translate the abstract Lie algebra $\mathfrak{g}$ into a more familiar matrix Lie algebra $\mathfrak{gl}(V)$ through what is known as a *representation*.[8]

8: Recall that the **general linear algebra** $\mathfrak{gl}(V)$ consisting of linear endomorphisms $V \to V$ is the Lie algebra of the **general linear group** $GL(n, \mathbb{K})$ of invertible matrices with matrix multiplication, where n is the dimension of V over the field $\mathbb{K}$.

Definition B.2.1 *A **representation** of a Lie algebra $\mathfrak{g}$ is a vector space V together with a morphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$.*

Representations are often required to be $\mathbb{K}$-linear, with $\mathbb{K}$ the field over which V is defined. Being ρ a morphism of Lie algebras, in particular it has to respect the Lie bracket:

$$\rho([\![a, b]\!]) = [\rho(a), \rho(b)] \equiv \rho(a) \circ \rho(b) - \rho(b) \circ \rho(a), \quad \forall a, b \in \mathfrak{g}.$$

Notice that the commutator $[\cdot, \cdot] : \mathfrak{gl}(V) \times \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ is anti-symmetric and satisfies the Jacobi identity.

Example B.2.1 Some examples:

- Abstractly, the **special linear** algebra $\mathfrak{sl}(2, \mathbb{C})$ is generated by elements X_1, X_2 and X_3 with relations

$$[\![X_1, X_2]\!] = 2X_2, \ [\![X_1, X_3]\!] = -2X_3, \ [\![X_2, X_3]\!] = X_1.$$

  However, $\mathfrak{sl}(2, \mathbb{C})$ is usually identified with the Lie algebra consisting of traceless 2×2 matrices due to the representation $\rho : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(\mathbb{C}^2)$ sending

$$\rho(X_1) = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}, \ \rho(X_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \rho(X_3) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

  usually denoted h, e and f, respectively.

- A representation $\rho : \mathfrak{so}(3,\mathbb{R}) \to \mathfrak{gl}(\mathbb{R}^3)$ for the three-dimensional **special orthogonal** algebra $\mathfrak{so}(3,\mathbb{R})$ is given by

$$\rho(X_1) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \rho(X_2) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \rho(X_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

identifying it with the Lie algebra of skew-symmetric matrices. Another representation is given by $\rho_{\text{spin}} : \mathfrak{so}(3,\mathbb{R}) \to \mathfrak{gl}(\mathbb{C}^2)$ sending

$$\rho_{\text{spin}}(X_1) = \frac{1}{2}\sigma_1, \quad \rho_{\text{spin}}(X_2) = \frac{1}{2}\sigma_2, \quad \rho_{\text{spin}}(X_3) = \frac{1}{2}\sigma_3$$

with σ_i the usual Pauli matrices.
- There is always the **trivial representation** on any vector space

$$\begin{aligned} \rho_{\text{triv}} : \mathfrak{g} &\to \mathfrak{gl}(V) \\ x &\mapsto \mathbf{0}_{\mathfrak{gl}(V)} \end{aligned}$$

although not very interesting.
- Every Lie algebra has a non-trivial representation, the **adjoint representation**

$$\begin{aligned} \text{ad} : \mathfrak{g} &\to \mathfrak{gl}(V) \\ x &\mapsto [\![x, \cdot]\!] \end{aligned}$$

where ad stands for *adjoint*.

So there might be lots of representations for a given Lie algebra. The interesting ones will be presented later.

Notice that, given a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ of a real Lie algebra $\mathfrak{g}$, one may extend it to a representation for its complexification $\mathfrak{g}_{\mathbb{C}}$ by $\rho(x+iy) = \rho(x) + i\rho(y)$.[9] Given two representations V and W of $\mathfrak{g}$, one can also define their **direct sum** $V \oplus W$ and **tensor representation** $V \otimes W$ by their action[10]

$$\rho(x)(v \oplus w) = \rho(x)v \oplus \rho(x)w, \quad \rho(x)(v \otimes w) = \rho(x)v \otimes w + v \otimes \rho(x)w$$

for $x \in \mathfrak{g}$ and $v \in V, w \in W$. To define the **dual representation** V^* one requires the natural pairing $V \otimes V^* \to \mathbb{C}$ to be a morphism of representations. Taking the tensor representation and considering $\mathbb{C}$ as the trivial representation, one gets $\langle \rho(x)v, v^* \rangle + \langle v, \rho(x)v^* \rangle = 0$, from where $\rho_{V^*}(x) = -(\rho_V(x))^t$ is obtained. This extends to a canonical structure for a representation on any tensor space $V^{\otimes k} \otimes (V^*)^{\otimes \ell}$. Further:

Definition B.2.2 *Given a representation V of $\mathfrak{g}$, a* ***subrepresentation*** *is a vector subspace $W \subset V$ which is stable under the action:*

$$\rho(x)W \subset W, \quad \forall x \in \mathfrak{g}.$$

Given a subrepresentation, the quotient space V/W has a canonical structure of a representation, called **quotient representation**.

In order to classify the different existing representations, one defines the building blocks consisting of the most simple ones, known as *irreducible* representations.[11]

9: Given a real Lie algebra $\mathfrak{g}$, its **complexification** is the complex Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$ with the obvious Lie bracket.
For example: $(\mathfrak{su}(2))_{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C})$.

10: In the case of the tensor representation, the natural guess

$$\rho(x)(v \otimes w) = \rho(x)v \otimes \rho(x)w$$

doesn't define a Lie algebra representation. The given one is derived from the Lie group tensor representation using the Leibniz rule, which provides the sum.

11: Irreducible representations are important in physics. For instance, physicists may say sentences like "a particle is an irreducible representation of its underlying symmetry group".

Definition B.2.3 *A non-zero representation V of $\mathfrak{g}$ is called* ***irreducible*** *or* ***simple*** *if it has no subrepresentations other than 0 and V. Otherwise, V is called* ***reducible****.*

Reducible representations may or may not split into irreducible representations.[12]

12: Hence the name *reducible*, referring to the possibility of being reduced to an *irreducible* one.

Definition B.2.4 *A representation V is said to be* ***completely reducible*** *or* ***semisimple*** *if it splits as a direct sum of irreducible representations:*

$$V \cong \bigoplus V_i, \quad V_i \textit{ irreducible.}$$

The problem of whether a representation is completely reducible is equivalent to the one of diagonalizability. Now an important lemma.

Lemma B.2.1 (Schur's Lemma) *Given an irreducible complex representation V of $\mathfrak{g}$, any intertwining operator $V \to V$ is constant:*

$$\mathrm{Hom}_{\mathfrak{g}}(V, V) = \mathbb{C}\ \mathrm{id}.$$

Recall that an **intertwining operator** is a linear map $V \to W$ commuting with the action of $\mathfrak{g}$. As a corollary, any irreducible complex representation of a *commutative* Lie algebra $\mathfrak{g}$ is one-dimensional.[13]

13: In the following, we will simplify notation and denote xv for $x \in \mathfrak{g}$ acting on $v \in V$ instead of $\rho(x)v$.

Example B.2.2 The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ introduced before turns out to be the building block for all finite dimensional semisimple complex Lie algebras. The following results are then the basis for the analysis of more complicated Lie algebras.

One can show that any representation of $\mathfrak{sl}(2, \mathbb{C})$ is semisimple. One then aims to classify all its irreducible representations. Recall first that a basis for $\mathfrak{sl}(2, \mathbb{C})$ is given by the generators e, f, h and relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h,$$

constituting a simple Lie algebra.

One then procedes and diagonalizes h. Given a representation V, a vector $v \in V$ is called **vector of weight** $\lambda \in \mathbb{C}$ if $hv = \lambda v$. The subspace of eigenvectors of weight λ is then denoted $V[\lambda]$. A lemma then shows that

$$eV[\lambda] \subset V[\lambda + 2], \quad fV[\lambda] \subset V[\lambda - 2].$$

This leads to a theorem stating that every finite-dimensional representation V of $\mathfrak{sl}(2, \mathbb{C})$ splits into these eigenspaces

$$V = \bigoplus_{\lambda} V[\lambda]$$

called the **weight decomposition** of V. One then focuses on V irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ and defines a **highest weight** of V to be a weight λ which is maximal with respect to the real part $\mathrm{Re}(\lambda) \geq \mathrm{Re}(\lambda')$. Vectors $v \in V[\lambda]$ are then called **highest weight vectors**. There is a lemma stating that any highest weight vector $v \in V[\lambda]$ is annihilated by e

(i.e. $ev = 0$) and that it defines a basis

$$v^k := \frac{f^k}{k!}v, \quad k \geq 0$$

on which the generators act as

$$hv^k = (\lambda - 2k)v^k, \quad fv^k = (k+1)v^{k+1}, \quad ev^k = (\lambda - k + 1)v^{k-1}.$$

For any $n \geq 0$, this defines then an **irreducible representation** V_n **with highest weight** n and basis $v^0, \dots, v^n$. Any finite-dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ is isomorphic to one such V_n and for any two $n \neq m$ the representations V_n, V_m are non-isomorphic. Finally, any finite-dimensional complex representation V of $\mathfrak{sl}(2,\mathbb{C})$ admits an integral weight decomposition

$$V = \bigoplus_{n\in\mathbb{Z}} V[n]$$

with $\dim V[n] = \dim V[-n]$ and isomorphisms

$$e^n : V[n] \to V[-n], \quad f^n : V[-n] \to V[n]$$

for $n \geq 0$.

B.3 Structure Theory of Lie algebras

Quantum Enveloping Algebra

To begin with, notice that there is no multilication in a Lie algebra $\mathfrak{g}$, i.e. xy is not defined for any $x, y \in \mathfrak{g}$. The product in $\mathfrak{gl}(V)$, however, is well-defined. One may then pick a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ and consider the algebra generated by products of $\rho(x)$ with $x \in \mathfrak{g}$.[14]

14: Th following notion of a universal enveloping algebra (UEA) finds an analogue in the realms of quantum groups, where the **quantum universal enveloping algebra** $U_q(\mathfrak{g})$ is defined as a q-deformation of the UEA.

Definition B.3.1 *Given a Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$, the* ***universal enveloping algebra*** *of $\mathfrak{g}$, denoted $U(\mathfrak{g})$, is the associative algebra with unit over $\mathbb{K}$ and generators $i(x)$, $x \in \mathfrak{g}$, subject to $i(x+y) = i(x) + i(y)$, $i(cx) = ci(x)$, $c \in \mathbb{K}$, and*

$$i(x)i(y) - i(y)i(x) = i([x,y]).$$

It turns out that $i : \mathfrak{g} \to U(\mathfrak{g})$ is injective and so $\mathfrak{g}$ can be seen as a subspace of $U(\mathfrak{g})$ and denote $x \in U(\mathfrak{g})$ instead of $i(x)$. The first two relations in the definition correspond to the tensor algebra

$$T\mathfrak{g} = \bigoplus_{n\geq 0} \mathfrak{g}^{\otimes n},$$

so one can describe the universal enveloping algebra as the quotient

$$U(\mathfrak{g}) = T\mathfrak{g} \big/ (xy - yx - [x,y]), \quad x, y \in \mathfrak{g}.$$

Interestingly, representations of $\mathfrak{g}$ and of $U(\mathfrak{g})$-modules are equivalent.

Semisimple Lie algebras and Killing form

Moving now to *semisimple* Lie algebras, recall that an **ideal** of $\mathfrak{g}$ is a vector subspace $\mathfrak{h} \subset \mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$. Recall also that, given the *derived series* of ideals $D^{i+1}\mathfrak{g} = [D^i\mathfrak{g}, D^i\mathfrak{g}]$ with

$$\mathfrak{g} = D^0\mathfrak{g} \supseteq D^1\mathfrak{g} \supseteq \cdots \supseteq D^k\mathfrak{g} \supseteq \cdots,$$

we say $\mathfrak{g}$ is **solvable** if $D^n\mathfrak{g} = 0$ for large enough n. Then:

Definition B.3.2 *A Lie algebra $\mathfrak{g}$ is called **simple** if it is not abelian and contains no non-trivial ideals (i.e. other than 0 and $\mathfrak{g}$). A Lie algebra is called **semisimple** if it contains no non-zero solvable ideals.*

One then shows that any simple Lie algebra is semisimple. Further, considering the **radical** of $\mathfrak{g}$, denoted $\mathrm{rad}(\mathfrak{g})$, the unique solvable ideal containing no other solvable ideal, one has a stronger result:

Theorem B.3.1 (Levi theorem) *Every Lie algebra can be decomposed as a direct sum*

$$\mathfrak{g} = \mathrm{rad}(\mathfrak{g}) \oplus \mathfrak{g}_{ss}$$

where $\mathfrak{g}_{ss}$ is a semisimple subalgebra in $\mathfrak{g}$.

Given a representation V of $\mathfrak{g}$, one may define a symmetric invariant bilinear form on $\mathfrak{g}$ by

$$B_V(x, y) = \mathrm{Tr}(\rho(x)\rho(y)).$$

An interesting case is when one takes the adjoint representation.

Definition B.3.3 *The **Killing form** $\mathcal{K} : \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$ is the symmetric invariant bilinear form given by $\mathcal{K}(x, y) = \mathrm{Tr}(\mathrm{ad}x\,\mathrm{ad}y)$.*

The following shows how this is related to semisimplicity of $\mathfrak{g}$.

Theorem B.3.2 (Cartan's criterions) *A Lie algebra $\mathfrak{g}$ is solvable iff $\mathcal{K}([\mathfrak{g}, \mathfrak{g}], \mathfrak{g}) = 0$. A Lie algebra $\mathfrak{g}$ is semisimple iff the Killing form is non-degenerate.*

B.4 Complex Semisimple Lie Algebras

Throughout this section $\mathfrak{g}$ will be a finite-dimensional semisimple Lie algebra and it will be complex unless otherwise stated.

Theorem B.4.1 *Given a semisimple Lie algebra $\mathfrak{g}$ and an ideal $I \subset \mathfrak{g}$, there exists and ideal $I' \subset \mathfrak{g}$ such that $\mathfrak{g} = I \oplus I'$.*

Thus, a Lie algebra $\mathfrak{g}$ is semisimple iff $\mathfrak{g} = \mathfrak{g}_1 \oplus\cdots\oplus \mathfrak{g}_k$, for $\mathfrak{g}_i$ simple Lie algebras. And the ideals in $\mathfrak{g}$ are of the form $I = \bigoplus_{i\in J} \mathfrak{g}_i$ for some $J \subset \{1, \dots, k\}$. Further, given a non-trivial irreducible representation V of a semisimple Lie algebra $\mathfrak{g}$, there exists a central element $C_V \in Z(U(\mathfrak{g}))$, called the **Casimir element**,[15] acting by a non-zero constant in V.[16]

15: The **Casimir element** determined by the non-defenerate invariant symmetric bilinear form $B_V(x, y) = \mathrm{Tr}(\rho(x), \rho(y))$ is given by

$$C_{B_V} = \sum x_i x^i \in U(\mathfrak{g})$$

where x_i is a basis of $\mathfrak{g}$ and x^i the dual basis with respect to B_V.

16: Precisely, the constant is

$$\frac{\dim\mathfrak{g}}{\dim V}.$$

The Casimir element plays a relevant role in proving the following important theorem.

Theorem B.4.2 *Any complex finite-dimensional representation of a semisimple Lie algebra $\mathfrak{g}$ is completely reducible.*

There is a well-known decomposition for a given semisimple Lie algebra known as *root decomposition*. It requires some previous definitions.[17]

17: Recall first that an element $x \in \mathfrak{g}$ is said to be **semisimple** [**nilpotent**] if $\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g}$ is a semisimple [nilpotent] operator. A **toral subalgebra** $\mathfrak{h} \subset \mathfrak{g}$ is then a commutative subalgebra consisting of semisimple elements.

Definition B.4.1 *Given a semisimple complex Lie algebra $\mathfrak{g}$, a* ***Cartan subalgebra*** *$\mathfrak{h} \subset \mathfrak{g}$ is a toral subalgebra coinciding with its centralizer*

$$C(\mathfrak{h}) := \{x \mid [x, \mathfrak{h}] = 0\} = \mathfrak{h}.$$

There always exists a Cartan subalgebra in every complex semisimple Lie algebra, since any maximal toral subalgebra is such. We finally reach the root decomposition and root systems.

Theorem B.4.3 *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Then:*

- *The Lie algebra decomposes into*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

called the ***root decomposition****, where*

$$\begin{aligned} \mathfrak{g}_\alpha &= \{x \mid [h, x] = \langle \alpha, h \rangle x \quad \forall h \in \mathfrak{h}\} \\ R &= \{\alpha \in \mathfrak{h}^* \backslash \{0\} \mid \mathfrak{g}_\alpha \neq 0\} \end{aligned}$$

are the ***root subspaces*** *$\mathfrak{g}_\alpha$ and the* ***root system*** *R of $\mathfrak{g}$.*
- *We have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, with $\mathfrak{g}_0 = \mathfrak{h}$.*
- *If $\alpha + \beta \neq 0$, then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal with respect to the Killing form: $\mathcal{K}(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.*
- *For any α, the pairing $\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \to \mathbb{C}$ given by the Killing form $\mathcal{K}$ is non-degenerate, and in particular the restriction of $\mathcal{K}$ to $\mathfrak{h}$ is so.*

Theorem B.4.4 *Consider $\mathfrak{g} = \bigoplus_{i=1}^n \mathfrak{g}_i$ with $\mathfrak{g}_1, \dots, \mathfrak{g}_n$ simple Lie algebras. Then:*

- *If $\mathfrak{h}_i \subset \mathfrak{g}_i$ are Cartan subalgebras of $\mathfrak{g}_i$ with corresponding root systems $R_i \subset \mathfrak{h}_i^*$, then $\mathfrak{h} = \bigoplus_{i=1}^n \mathfrak{h}_i$ is a Cartan subalgebra in $\mathfrak{g}$ with corresponding root system $R = \bigsqcup_{i=1}^n R_i$.*
- *Each Cartan subalgebra in $\mathfrak{g}$ is of the form $\mathfrak{h} = \bigoplus \mathfrak{h}_i$ with $\mathfrak{h}_i \subset \mathfrak{g}_i$ Cartan subalgebras of $\mathfrak{g}$.*

Denoting by $(\ ,\ )$ the non-degenerate invariant bilinear form on $\mathfrak{g}$, since its restriction to $\mathfrak{h}$ is non-degenerate, notice that it defines a linear isomorphism $\mathfrak{h} \to \mathfrak{h}^*$ and a non-degenerate bilinear form on $\mathfrak{h}^*$, also denoted $(\ ,\ )$. Explicitly, denoting H_α for the element in $\mathfrak{h}$ corresponding to $\alpha \in \mathfrak{h}^*$, then

$$(\alpha, \beta) = \langle H_\alpha, \beta \rangle = (H_\alpha, H_\beta)$$

for any $\alpha, \beta \in \mathfrak{h}^*$. With this definition for H_α, for any $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ it holds $[e_\alpha, f_\alpha] = (e_\alpha, f_\alpha) H_\alpha$, analogous to the $\mathfrak{sl}(2, \mathbb{C})$ case. Some other analogies[18] to the $\mathfrak{sl}(2, \mathbb{C})$ case bring us to:

18: First, for $\alpha \in R$, we have $(\alpha, \alpha) = (H_\alpha, H_\alpha) \neq 0$. Then, given $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $(e_\alpha, f_\alpha) = \frac{2}{(\alpha,\alpha)}$ and defining

$$h_\alpha := \frac{2H_\alpha}{(\alpha, \alpha)},$$

we have $\langle h_\alpha, \alpha \rangle = 2$ integer and the elements $e_\alpha, f_\alpha, h_\alpha$ satisfy the same commutation relations as in $\mathfrak{sl}(2, \mathbb{C})$. Denote this subalgebra as $\mathfrak{sl}(2, \mathbb{C})_\alpha \subset \mathfrak{g}$. Finally, the definition of $\mathfrak{h}_\alpha$ is independent of the choice of non-degenerate invariant bilinear form $(\ ,\ )$.

Theorem B.4.5 *Consider a complex semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$ and root decomposition $\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$. Pick $(\ ,\ )$ a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$. Then:*

(1) The root system R spans $\mathfrak{h}^*$ as a vector space, while elements h_α for $\alpha \in R$ span $\mathfrak{h}$ as a vector space.
(2) Each root subspace $\mathfrak{g}_\alpha$ for $\alpha \in R$ is one-dimensional.
(3) The number

$$\langle h_\alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$$

is integer for any two roots $\alpha, \beta \in R$.
*(4) Defining the **reflection operators** $r_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ for a given root $\alpha \in R$ by*

$$r_\alpha(\lambda) = \lambda - \langle h_\alpha, \lambda \rangle \alpha = \lambda - \frac{2(\alpha, \lambda)}{(\alpha, \alpha)} \alpha,$$

for any root $\beta \in R$ the element $r_\alpha(\beta)$ is also a root.[a]
(5) The only multiples of a root $\alpha \in R$ that are also roots are $\pm\alpha$.
(6) For any two roots $\alpha, \beta \neq \pm\alpha$, the subspace

$$V = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\beta + k\alpha}$$

is an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})_\alpha$.
(7) We have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ for any two roots $\alpha, \beta \in R$ such that $\alpha + \beta \in R$, i.e. is also a root.

[a] In particular, for $\alpha \in R$ then $-\alpha = r_\alpha(\alpha) \in R$ is also a root.

B.5 Root systems

The last theorem shows that the set of roots R of a semisimple complex Lie algebra fits in what is known as an *abstract root system*. These have been widely studied and give important results for the study of Lie algebras.

Definition B.5.1 *An **abstract root system** is a finite set of non-zero elements $R \subset E \backslash \{0\}$ in an Euclidean space E with inner product $(\ ,\ )$, such that:*

(R1) The set R generates E as a vector space.
(R2) For any two roots $\alpha, \beta \in R$, the following number is integer

$$n_{\alpha\beta} = \frac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}.$$

(R3) Given the map $r_\alpha : E \to E$ defined by

$$r_\alpha(\lambda) = \lambda - \frac{2(\alpha,\lambda)}{(\alpha,\alpha)}\alpha,$$

for any roots $\alpha, \beta \in R$ the image $r_\alpha(\beta)$ is again a root.

If R moreover satisfies the property

(R4) If both $\alpha, c\alpha \in R$, then necessarily $c = \pm 1$.

*then it is called a **reduced root system**.*

The number $r = \dim E$ is called the **rank** of the root system.[19] Now:

19: Notice that conditions (R2) and (R3) tell us that r_α is nothing but a **reflection** around the hyperplane $L_\alpha = \{\lambda \in E \mid (\alpha,\lambda) = 0\}$. Notice also that these reflections are linear in λ but not in α.

Theorem B.5.1 *The set of roots $R \subset \mathfrak{h}^*_{\mathbb{R}} \backslash \{0\}$ corresponding to the root decomposition of a semisimple complex Lie algebra $\mathfrak{g}$ is a reduced root system.*

Weyl group

Most of the important information about the root system is contained in the numbers $n_{\alpha\beta}$ rather than in the inner products between roots. This motivates the next definition.

Definition B.5.2 *Given two root systems $R_1 \subset E_1$ and $R_2 \subset E_2$, an **isomorphism** between them $\varphi : R_1 \to R_2$ is a vector space isomoprhism $\varphi : E_1 \to E_2$ such that $\varphi(R_1) = R_2$ and preserves the numbers $n_{\varphi(\alpha)\varphi(\beta)} = n_{\alpha\beta}$ for any $\alpha, \beta \in R_1$.*

Among these, the reflections r_α are special automorphisms of the root system R constituting a group.[20] We give it a name.

20: The group operation being the composition of reflections.

Definition B.5.3 *The subgroup of $GL(E)$ generated by the reflections r_α for $\alpha \in R$ is called the **Weyl group** of the root system R and is denoted $\mathcal{W}$.*

It satisfies several properties such as:

Lemma B.5.2 *The Weyl group $\mathcal{W}$ constitutes a finite subgroup of the orthogonal group $O(E)$ and it leaves the root system R invariant. Further, for any $w \in \mathcal{W}$ and $\alpha \in R$, we have $r_{w(\alpha)} = w r_\alpha w^{-1}$.*

Pairs of roots

The following theorem will be important for the classification of all root systems, which in turn leads to a classification of all semisimple Lie algebras. Throughout R will be a *reduced* root system.

Theorem B.5.3 *Given two roots $\alpha, \beta \in R$, $\alpha \neq c\beta$ and $|\alpha| \geq |\beta|$, with angle φ between them, then one among Table B.1 must hold.*

Table B.1: Exhaustive table of possible relations between pairs of roots.

φ	$\lvert\alpha\rvert/\lvert\beta\rvert$	$n_{\alpha\beta}$	$n_{\beta\alpha}$
$\pi/2$	-	0	0
$2\pi/3$	1	-1	-1
$\pi/3$	1	1	1
$3\pi/3$	$\sqrt{2}$	-2	-1
$\pi/4$	$\sqrt{2}$	2	1
$5\pi/6$	$\sqrt{3}$	3	1
$\pi/6$	$\sqrt{3}$	-3	-1

This result comes from the strong restrictions (R2) and (R3) in the definition of an abstract root system.[21] Last, the following lemma concerning pairs of roots is used later on.

21: Indeed, since $(\alpha, \beta) = |\alpha||\beta| \cos\varphi$, one sees that $n_{\alpha\beta} = 2\frac{|\alpha|}{|\beta|}\cos\varphi$ so that $n_{\alpha\beta}n_{\beta\alpha} = 4\cos^2\varphi$. Along with the fact $n_{\alpha\beta}n_{\beta\alpha} \in Z$, this means that $n_{\alpha\beta}n_{\beta\alpha} \in \{0, 1, 2, 3\}$, which leads to the statement of the theorem.

Lemma B.5.4 *For any two roots $\alpha, \beta \in R$ with (α, β) and $\alpha \neq c\beta$, their sum $\alpha + \beta \in R$.*

Positive and simple roots

To find a set of *generating roots* for R, one chooses a so-called **regular** element $t \in E$, i.e. one such that for any root $\alpha \in R$ it holds $(\alpha, t) \neq 0$. Given such an element, one can decompose R as a **polarization** $R = R_+ \sqcup R_-$ with

$$R_+ = \{\alpha \in R \mid (\alpha, t) > 0\} \quad \text{and} \quad R_- = \{\alpha \in R \mid (\alpha, t) < 0\}$$

the sets of **positive** and **negative** roots, respectively.[22] A positive root $\alpha \in R_+$ is then called **simple** if it cannot be expressed as a sum of two positive roots. The set of simple roots is denoted $\Pi \subset R_+$. Some important results follow.

22: Sometimes the set of roots is denoted Δ and the positive and negative roots Δ_+ and Δ_-, respectively. Also, the simple roots are sometimes called *fundamental* roots.

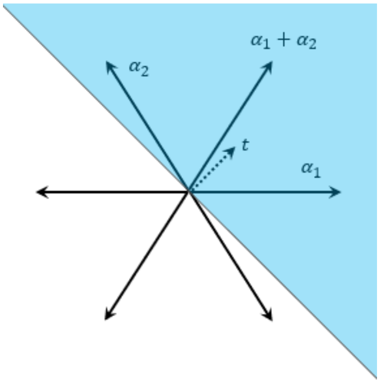

Figure B.2: Example given by the root system A_2 consisting of the six shown vectors on the plane $\mathbb{R}^2$. A polarization is given by the choice of a regular element t, so that the positive roots are α_1, α_2 and $\alpha_1 + \alpha_2$, the first two ones being simple.

Lemma B.5.5 *Every positive root can be expressed as a sum of simple roots. And for any $\alpha, \beta \in R_+$ that are simple, $(\alpha, \beta) \leq 0$.*

Theorem B.5.6 *The simple roots Π of a root system $R = R_+ \sqcup R_- \subset E$ form a basis of the vector space E.*

Corollary B.5.7 *Every root $\alpha \in R$ can be expressed as a linear combination of simple roots with integer coefficients*

$$\alpha = \sum_{i=1}^{r} n_i \alpha_i, \quad n_i \in \mathbb{Z}$$

where $\{\alpha_1, \ldots, \alpha_r\} = \Pi$ is the set of simple roots. For positive roots $\alpha \in R_+$ all $n_i \geq 0$ and for negative roots $\alpha \in R_-$ all $n_i \leq 0$.

The **height** of positive roots $\alpha \in R_+$ is then defined by setting $\mathrm{ht}\left(\sum n_i \alpha_i\right) = \sum n_i \in \mathbb{Z}_+$. In particular, $\mathrm{ht}(\alpha_i) = 1$. Further:

Proposition B.5.8 *Given two polarizations $R = R_+ \sqcup R_- = R'_+ \sqcup R'_-$ of the same root system with corresponding sets of simple roots Π and Π', there exists an element $w \in \mathcal{W}$ sending $w(\Pi) = \Pi'$.*

Simple reflections

Now, the simple roots Π don't only generate E as a vector space, but in some sense generate the whole Weyl group $\mathcal{W}$ and recover the root system R by action of $\mathcal{W}$ on them.

Theorem B.5.9 *Given a reduced root system R with polarization $R = R_+ \sqcup R_-$ and $\Pi = \{\alpha_1, \dots, \alpha_r\}$ the set of simple roots, then:*

- *The **simple reflections** $r_i := r_{\alpha_i}$ generate $\mathcal{W}$.*
- *The simple roots generate the root system by action of the Weyl group: $\mathcal{W}(\Pi) = R$. That is, every root $\alpha \in R$ can be expressed as $w(\alpha_i)$ for some $w \in \mathcal{W}$ and $\alpha_i \in \Pi$.*

One can then define a *length* on the elements w of the Weyl group and show that it coincides with the number ℓ of elements in a **reduced expression** $w = r_{i_1} \cdots r_{i_\ell}$ in terms of simple reflections. The length then defines an order on the elements of the Weyl group and one can show that there is a maximal element w_0, which is called the **longest element** in the Weyl group $\mathcal{W}$.

Dynkin diagrams

One then realizes with the previous results that the classification of root systems is equivalent to that of possible sets of simple roots. In order to procede with the latter, one further notices that given two root systems $R_1 \subset E_1$ and $R_2 \subset E_2$, a larger root system $R_1 \sqcup R_2 \subset E_1 \oplus E_2$ can be defined with the inner product on $E_1 \oplus E_2$ defined so that $E_1 \perp E_2$. Thus, root systems may be decomposed into smaller ones, with corresponding smaller sets of simple roots to be then classified.

Definition B.5.4 *A root system is said to be **reducible** if it can be decomposed as $R_1 \sqcup R_2$ with $R_1 \perp R_2$ root systems, and is said to be **irreducible** otherwise.*

A lemma then states that given a reducible root system $R = R_1 \sqcup R_2$ with a given polarization, the set of simple roots can be decomposed as $\Pi = \Pi_1 \sqcup \Pi_2$ with $\Pi_i := \Pi \cap R_i$ the set of simple roots for each R_i. And conversely, if $\Pi = \Pi_1 \sqcup \Pi_2$ with $\Pi_1 \perp \Pi_2$, then $R = R_1 \sqcup R_2$ defines a root system, where R_i is the root system generated by each Π_i. This extends to show that any root system R can be decomposed as $R_1 \sqcup \cdots \sqcup R_n$ with R_i all mutually orthogonal irreducible root systems, and thus it suffices to study the simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\}$ corresponding to irreducible root systems only. To begin with, an efficient and compact way to describe the relative position of the simple roots is through the so-called *Cartan matrix*.

Definition B.5.5 *Given a set of simple roots $\Pi \subset R$, the **Cartan matrix** A is the $r \times r$ matrix with entries*

$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}.$$

Notice that $a_{ij} = n_{\alpha_j \alpha_i}$ as defined previously. One can show that $a_{ii} = 2$ on the diagonal and $a_{ij} \in \mathbb{Z}, a_{ij} \leq 0$, non-positive integers on the off-diagonal.[23] Further, the Cartan matrix contains information about the lengths of simple roots.[24] All this information encapsulated

23: For example, the Cartan matrix for the root system A_n is given by

$$A = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & & \ddots & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}.$$

Notice from the definition, however, that the Cartan matrix need **not** be symmetric. Notice also that the off-diagonal entries can only be $0, -1, -2$ or -3.

24: If φ is the angle between simple roots and $\varphi \neq \pi/2$, then

$$\frac{|\alpha_i|^2}{|\alpha_j|^2} = \frac{a_{ji}}{a_{ij}}.$$

in the Cartan matrix can be presented in a graphical way by means of the well-known *Dynkin diagram*, obtained as follows.

Definition B.5.6 *Given a set of simple roots Π of a root system R, the **Dynkin diagram** of Π is the graph*

- *with a vertex v_i for each simple root α_i,*
- *with edges between them for each pair $\alpha_i \neq \alpha_j$ and drawn with $0, 1, 2, 3$ lines depending on whether the angle between α_i and α_j is $\varphi = \pi/2, 2\pi/3, 3\pi/4, 5\pi/6$, respectively,*
- *and oriented for each pair of non-orthogonal distinct simple roots with $|\alpha_i| \neq |\alpha_j|$ by drawing the corresponding edge with an arrow pointing towards the shorter root.*

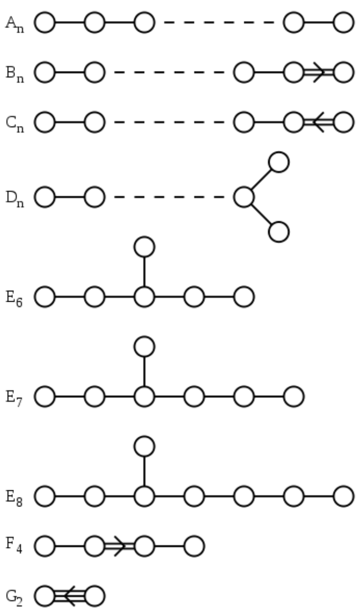

Figure B.3: Complete classification of finite-dimensional complex semisimple Lie algebras in terms of Dynkin diagrams: the four classical Lie algebra series A_n for $n \geq 1$, B_n for $n \geq 2$, C_n for $n \geq 3$, D_n for $n \geq 4$, plus the diagrams E_6, E_7, E_8, F_4 and G_2.

This diagrams satisfy:

Theorem B.5.10 *Given a set of simple roots Π of a reduced root system R, the following properties hold:*

(1) The Dynkin diagram of Π is a connected graph iff R is irreducible.
(2) The Dynkin diagram uniquely determines the Cartan matrix.
(3) The root system R is uniquely determined by the Dynkin diagram, up to isomorphism.

Any reduced irreducible root system has a Dynkin diagram isomorphic to one of Figure B.3. This classification extends to the one for complex semisimple Lie algebras.

B.6 Classification of complex semisimple Lie algebras

Theorem B.6.1 *Consider a semisimple complex Lie algebra $\mathfrak{g}$ with root system $R \subset \mathfrak{h}^*$ equipped with a non-degenerate invariant symmetric bilinear form $(\ ,\ )$. Choose a polarization $R = R_+ \sqcup R_-$ of R and let $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be the corresponding set of simple roots. Then:*

1) The Lie algebra decomposes as a vector space in

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

with $\mathfrak{n}_\pm \subset \mathfrak{g}$ being the subalgebras

$$\mathfrak{n}_\pm = \bigoplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha .$$

2) Choose $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$ such that $(e_i, f_i) = \frac{2}{(\alpha_i, \alpha_i)}$ and let $h_i := h_{\alpha_i} \in \mathfrak{h}$. Then, $e_1, \ldots, e_r$ generate $\mathfrak{n}_+$, $f_1, \ldots, f_r$ generate $\mathfrak{n}_-$ and $h_1, \ldots, h_r$ form a basis of $\mathfrak{h}$. All together, $\{e_i, f_i, h_i\}_{i=1,\ldots,r}$ generate $\mathfrak{g}$.

3) The elements e_i, f_i and h_i satisfy the so-called ***Serre relations***

$$\begin{aligned} &[h_i, h_j] = 0 \qquad &[h_i, e_j] = a_{ij} e_j \qquad &(\mathrm{ad} e_i)^{1-a_{ij}} e_j = 0 \\ &[e_i, f_j] = \delta_{ij} \qquad &[h_i, f_j] = -a_{ij} f_j \qquad &(\mathrm{ad} f_i)^{1-a_{ij}} f_j = 0 \end{aligned}$$

where a_{ij} are the entries of the Cartan matrix.

These relations completely define the Lie algebra.

Theorem B.6.2 *Consider a reduced irreducible root system with polarization $R = R_+ \sqcup R_-$ and simple roots $\Pi = \{\alpha_1, \ldots, \alpha\}$. Then, the complex Lie algebra given by generators e_i, f_i, h_i and the Serre relations is a finite-dimensional semisimple Lie algebra with root system R, denoted $\mathfrak{g}(R)$.*

As a corollary, any semisimple Lie algebra $\mathfrak{g}$ with root system R is isomorphic to the one given by Serre's presentation, i.e. $\mathfrak{g} \cong \mathfrak{g}(R)$. Further, there is a natural bijection between the set of isomorphism classes of reduced root systems and the set of isomorphism classes of finite-dimensional complex semisimple Lie algebras, and in particular the Lie algebra is simple iff the root system is irreducible. This finally leads to the desired result.

Theorem B.6.3 *All simple finite-dimensional complex Lie algebras are classified by the Dynkin diagrams in Figure B.3.*

C

Homology and Cohomology Theory

Homology theory first arouse as a rigorous method to mathematically define holes in a manifold, and is in general a way of relating sequences of algebraic objects with other mathematical objects of different kind.

We give here a brief review on Homology and Cohomology Theory, following [Sav12]. We start by introducing the structure of a CW-complex, which plays a role in the definition of homology groups. Then, after defining the concept of homology theory and presenting the Mayer-Vietoris exact sequence, we move on to cellular homology as one of the most common examples of homology theories. We end by defining the corresponding concept of cohomology theory.

C.1 CW-complexes

First, as mentioned, we define the notion of a CW-complex.

Definition C.1.1 *We call a topological space X a **CW-complex** if it can be written as a union*

$$X = \bigcup_{q=0}^{\infty} X^{(q)}$$

*where the 0-skeleton $X^{(0)}$ is a countable (maybe finite) discrete set of points, and inductively each $(q+1)$-skeleton $X^{(q+1)}$ is obtained from the q-**skeleton** $X^{(q)}$ by attaching $(q+1)$-cells. To be more precise, for every q there is a set $\{e_j \mid j \in J_{q+1}\}$ where:*

(1) Each e_j is a subset of $X^{(q+1)}$ such that if $e'_j = e_j \cap X^{(q)}$, then $e_j \backslash e'_j$ is disjoint from $e_i \backslash e'_i$ for all $j, i \in J_{q+1}$ with $j \neq i$.

*(2) For each $j \in J_{q+1}$, there exists a **characteristic map***

$$g_j : (D^{q+1}, \partial D^{q+1}) \longrightarrow (X^{(q+1)}, X^{(q)})$$

such that g_j is a quotient map from D^{q+1} to e_j mapping $D^{q+1} \backslash \partial D^{q+1}$ homeomorphically onto $e_j \backslash e'_j$.

(3) A subset of X is closed iff its intersection with each skeleton $X^{(q)}$ is closed.

*The $e_j \backslash e'_j$ are called $(q+1)$-**cells**. We say that the CW-complex is **regular** when all characteristic maps g_j are embeddings.*

See Figure C.1 for an intuitively view of a CW-complex, although the general notion is more abstract.

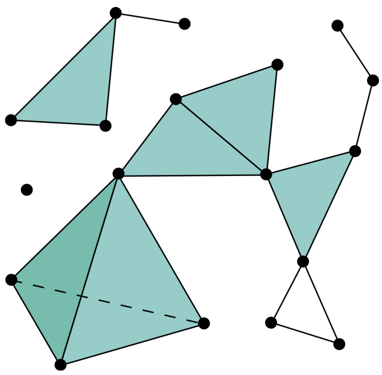

Figure C.1: A simplicial 3-complex as an example of a CW-complex, where dots, edges and faces correspond to 1-cells, 2-cells and 3-cells respectively.

C.2 Homology Theory

We turn now to the abstract definition of a homology theory. This notion enables the distinction of mathematical objects by examining their different "kinds of holes", an obstruction to shrinking an object (cycles or loops, for instance) within a space which is measured by the corresponding homology groups.

Definition C.2.1 *Given a commutative ring with unit, R, we refer to a* ***homology theory*** *as a functor from the category consisting of pairs of spaces (objects) and continuous maps (homomorphisms) to the category of graded R-modules and graded homomorphisms. In other words, for each pair (X, A) with $A \subset X$, we have an R-module*

$$H_*(X, A; R) = \bigoplus_{q=0}^{\infty} H_q(X, A; R)$$

and for each map $f : (X, A) \to (Y, B)$ we have homomorphisms $f_* : H_q(X, A; R) \to H_q(Y, B; R)$ for every q, satisfying the condition $(f \circ g)_* = f_* \circ g_*$. One often simplifies notation and denotes $H_q(X, A)$ for $H_q(X, A; R)$ and $H_q(X)$ for $H_q(X, \emptyset)$, the ring R being clear from the context. Further, the so-called ***Eilenberg-Steenrod axioms*** *must be met:*

(1) If $f, g : (X, A) \to (Y, B)$ are homotopic then $f_* = g_*$.
(2) For each pair (X, A) and each q there exist (boundary) homomorphisms $\partial : H_q(X, A) \to H_{q-1}(A)$ fitting into a long exact sequence

$$\cdots \to H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial} H_{q-1}(A) \to \cdots \to H_0(X, A) \to 0,$$

with $i : A \to X$ and $j : (X, \emptyset) \to (X, A)$ the inclussion maps.
(3) For a given open subspace U of X with closure contained in the interior of A, the inclusion map $j : (X \backslash U, A \backslash U) \to (X, A)$ induces maps $j_* : H_q(X \backslash U, A \backslash U) \to H_q(X, A)$ which are isomorphisms for all q.
(4) If P is a space consisting of one point, then $H_0(P) = R$ and $H_q(P) = 0$ for $q \geq 1$.

These can be stated as "homotopy invariance", "long exact sequence", "excision" and "coefficient module" axioms, respectively.

According to the fourth axiom, we call R the **coefficient ring** for the homology theory. The $H_q(X, A)$ are often called **homology groups**, although they actually are modules. However, one commonly has $R = \mathbb{Z}$ or $R = \mathbb{Z}/n\mathbb{Z}$, and the corresponding homology modules are thus abelian groups.[1]

1: Remark: For a fixed ring R, all the standard ways to define homology groups yield the same results when X is a simplicial or a CW-complex and A is a subcomplex. We will work with cellular homology (defined below), which along with singular and simplicial homologies constitute the most common homology theories.

The Eilenberg-Steenrod axioms appearing in the definition of a homology theory yield implicitly a powerful tool for homology computations: the **Mayer-Vietoris exact sequence**. Let us introduce it for CW-complexes. Given A and B subcomplexes of a CW-complex X, such that $X = A \cup B$, there exist homomorphisms $\partial : H_q(X) \to H_{q-1}(A \cap B)$

fitting into a long exact sequence

$$\cdots \to H_q(A \cap B) \xrightarrow{(i_*, -j_*)} H_q(A) \oplus H_q(B) \xrightarrow{I_* + J_*} H_q(X) \xrightarrow{\partial} H_{q-1}(A \cap B) \to \ldots$$

being $i : A \cap B \to A$, $j : A \cap B \to B$, $I : A \to X$ and $J : B \to X$ the corresponding inclusion maps.

One important consequence following from the Mayer-Vietoris sequence and the homology axioms is that given a CW-complex K and a subcomplex L (possibly empty) such that the dimension of the cell $K \backslash L$ is less than or equal to n, then the homology groups $H_q(K, L)$ vanish for all $q > n$. Also, for $q = 0$ one has $H_0(K) = \bigoplus_{i=1}^{p} R$, with p the number of path components of K.

As we will see, a cohomology theory is defined in a similar way, with cohomological analogues of the Eilenberg-Steenrod axioms and the Mayer-Vietoris exact sequence.

C.3 Cellular Homology

Let us give now an example of homology theory in detail.

Start with a CW-complex X and a commutative ring with unit R. For each q, denote $C_q(X, R)$ the free R-module with basis the q-cells. Proceed then and define the **boundary homomorphism** $\partial_{q+1} : C_{q+1}(X, R) \to C_q(X, R)$ as follows. To begin with, fix an orientation on D^{q+1}, determining thus as well an orientation on the q-sphere ∂D^{q+1}. Given then a $(q+1)$-cell $c \in C_{q+1}(X, R)$, look at how the characteristic map g of c takes ∂D^{q+1} to $X^{(q)}$. Now, for each $e_k \in X^{(q)}$ fix a point z_k in $c_k = e_k \backslash e'_k$. It turns out that g is homotopic to a map such that for each k, the preimage of z_k is a finite set of points $p_{k,1}, \ldots, p_{k,n_k}$. Further, one can also show that g takes a neighborhood of each $p_{k,j}$ homeomorphically to a neighborhood of z_k.[2] Define now $\epsilon_{k,j} = \pm 1$ for each j with $q \leq j \leq n_k$ corresponding to whether g restricted to the neighborhood of $p_{k,j}$ is orientation preserving or reversing, and denote

$$\epsilon_k = \sum_{j=1}^{n_k} \epsilon_{k,j} \quad \text{to write} \quad \partial_{q+1}(c) = \sum_{k=1}^{\infty} \epsilon_k c_k$$

where only finitely many ϵ_k are non-zero. This defines the boundary map $c \mapsto \partial_{q+1}(c)$ on the generators. Extending to the whole free R-module $C_{q+1}(X, R)$ by linearity, one obtains ∂_{q+1} as desired.

2: By compactness, the preimage of z_k is empty for all but finitely many k.

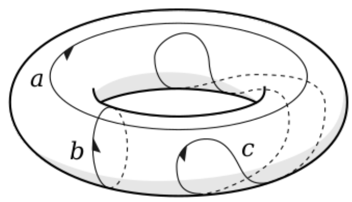

Figure C.2: The torus $\mathbb{T}^2$ has cycles which cannot be continuously deformed into each other. For example, none of the cycles a, b or c can be deformed into one another. The two first ones cannot be shrunk to a point whereas cycle c can; it is homologous to zero.

There is an alternative way to describe the numbers ϵ_k by considering the quotient space $X^{(q)}/X^{(q-1)}$, which is homeomorphic to union of q-dimensional spheres at a common point (one for each q-cell $c_k = e_k \backslash e'_k$). Then, given a $(q+1)$-cell c, its characteristic map $g : (D^{q+1}, \partial D^{q+1}) \to (X^{(q+1)}, X^{(q)})$ induces the maps

$$\varphi_k : \partial D^{q+1} \longrightarrow X^{(q)} \longrightarrow X^{(q)}/X^{(q-1)} \longrightarrow \mathbb{S}^q,$$

where for the defining k the last arrow sends the sphere $\mathbb{S}^q$ corresponding to the cell c_k identically to itself, while contracting all other spheres to a point. The **degree** of φ_k is ϵ_k. This way of describing ϵ_k ensures that $\partial_{q+1}(c)$ is well-defined.

Next, it can be shown that the special property $\partial_q\partial_{q+1} = 0$ holds. The reason comes from the fact that, algebraically, the q-sphere ∂D^{q+1} acts as if it were a regular CW-complex with a q-cell for each preimage point of a z_k. Being ∂D^{q+1} a manifold, the boundaries of these q-cells form a set of $(q-1)$-cells, each of which appearing as part of the boundary of two q-cells with opposite orientations. In this way, when applying ∂_q to $\partial_{q+1}(c)$ adding up the images of the boundaries of the q-cells in $C_{q-1}(X, R)$, all pairs with opposite signs cancel out yielding 0.

With this property at hand, one can define the homology groups as follows. Denote any element of $C_q(X, R)$ as a formal finite sum $\sum r_k c_k$ (each c_k being a q-cell), called a **q-chain**, and then form a sequence of R-modules

$$\cdots \to C_{q+1}(X, R) \xrightarrow{\partial_{q+1}} C_q(X, R) \xrightarrow{\partial_q} C_{q-1}(X, R) \cdots \to C_0(X, R) \to 0,$$

called a **chain complex** for having $\partial_q\partial_{q+1} = 0$ for all q. By this very same property, the image of ∂_{q+1} is clearly contained in the kernel of ∂_q for each q. It is when the image of ∂_{q+1} equals the kernel of ∂_q for each q that the sequence is called **exact**. However, in general it is not, and the measure of its deviation from exactness is given by the **cellular homology groups**

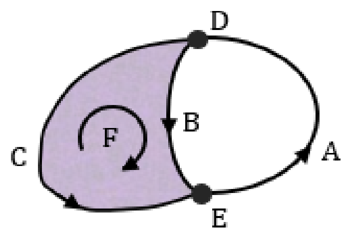

Figure C.3: An example of a cellular subdivision with homology groups $H_0 \cong H_1 \cong \mathbb{Z}$ and $H_i \cong 0$ for $i \geq 2$. Some examples are given by the 1-cycle $A+B$ since $\partial(A+B) = D-E+E-D = 0$ and the 1-boundary $B-C = \partial F$. Further, $A+B$ and $A+C$ are homologous 1-chains since $A+B = A+C+\partial F$. Last, D and $A+C$ are generators for H_0 and H_1, respectively.

$$H_q(X; R) := {}^{\ker(\partial_q)}\!\big/_{\operatorname{im}(\partial_{q+1})}.$$

The standard terminology is to call elements of $\ker(\partial_q)$ as **cycles** and elements of $\operatorname{im}(\partial_{q+1})$ as **boundaries**. Explicitly, an element of $H_q(X; R)$ is then a coset $a_q + \partial_{q+1}(C_{q+1}(X, R))$ with $\partial_q a_q = 0$, usually denoted $[a_q]$. Notice that $[a_q] = [a'_q]$ if and only if $a_q = a'_q + \partial_{q+1}(b_{q+1})$, i.e. differ by a boundary of some $(q+1)$-chain b_{q+1}.

Finally, to complete the definition of H_* as a homology theory one defines the morphisms f_* associated by the functor to every continuous map $f : X \to Y$. This is done by first defining $C_q(f) : C_q(X, R) \to C_q(Y, R)$. The Cellular Approximation Theorem allows us to change f within its homology class in order to have $f(X^{(q)}) \subset Y^{(q)}$ for all q. In this way, $C_q(f)(c) \in C_q(Y, R)$ can be defined similarly to how it was done above for $\partial_q(c)$. The morphism f_* is then defined by sending the homology classes to one another: $f_*([c]) = [C_q(f)(c)]$.

It can be proven that this is well-defined, that it satisfies the Eilenberg-Steenrod axioms and that $H_*(X; R)$ does not depend on the choice of CW-complex structure for X.

For the case of a subcomplex A of X, one can define **relative homology groups** $H_q(X, A; R)$ by setting $C_q(X, A; R) = {}^{C_q(X; R)}\!\big/_{C_q(A; R)}$ and observing that ∂_q induces a corresponding boundary map $\partial_q : C_q(X, A; R) \to C_{q-1}(X, A; R)$, producing a chain complex $C_*(X, A; R)$. The homology of such chain complex defines then the homology groups $H_q(X, A; R)$, whose elements can be represented each by a q-chain whose boundary lies in A. The long exact sequence of the second Eilenberg-Steenrod axiom turns out to be a purely algebraic consequence of the existence of short exact sequences

$$0 \to C_q(A; R) \to C_q(X; R) \to C_q(X, A; R) \to 0.$$

C.4 Cohomology of Spaces

Once a cellular (or simplicial or singular) homology is defined, cohomology can be defined algebraically.

Definition C.4.1 *Let A and B be R-modules. Then, for every R-module homomorphism $\varphi : A \to B$, there is another one*

$$\varphi^* : \mathrm{Hom}(B, R) \longrightarrow \mathrm{Hom}(A, R)$$

*given by $\varphi^*(\alpha) = \alpha \circ \varphi$. Being clear that $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$, one defines the **coboundary homomorphism** by $\delta_q := \partial_q^*$ so that $\delta_{q+1}\delta_q = \partial_{q+1}^*\partial_q^* = (\partial_q\partial_{q+1})^* = 0^* = 0$. Thus, using $C^q(X; R)$ to denote $\mathrm{Hom}(C_q(X), R)$ for short, one has a **cochain complex***

$$0 \to C^0(X; R) \to \cdots \to C^{q-1}(X; R) \xrightarrow{\delta_q} C^q(X; R) \xrightarrow{\delta_{q+1}} C^{q+1}(X; R) \to \ldots$$

*whose deviation from exactness is measured by the **cohomology groups***

$$H^q(X; R) = {\ker(\delta_{q+1})}\big/{\mathrm{im}(\delta_q)}.$$

In this setting, every continuous map $f : X \to Y$ induces a homomorphism $f^* : H^q(Y; R) \to H^q(X; R)$ with the property $(f \circ g)^* = g^* \circ f^*$. Moreover, there are as well corresponding versions of the Eilenberg-Steenrod axioms and the Mayer-Vietoris exact sequence for cohomology.

When $R = \mathbb{F}$ is a field, it can be proven that $\mathrm{Hom}(H_q(X; \mathbb{F}), \mathbb{F}) \cong H^q(X; \mathbb{F})$, the dual vector space of $H_q(X; \mathbb{F})$. This means that $H^q(X; \mathbb{F})$ and $H_q(X; \mathbb{F})$ are vector spaces of the same rank, yet not existing any natural isomorphism between them.

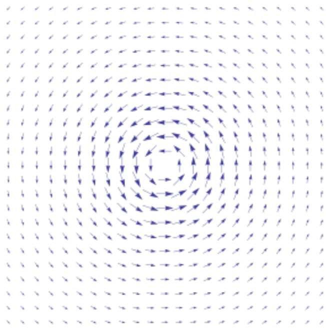

Figure C.4: Example given by a vector field that corresponds to a closed but not exact differential form on the punctured plane, the De Rham cohomology of this space hence being nontrivial.

C.5 De Rham Cohomology

As an example, we present here the well-known De Rham cohomology appearing in algebraic topology.

Recall first that given a smooth m-manifold M, a **differential p-form** on M may be compactly expressed in local coordinates as

$$\sum_{I'} f_I \, \mathrm{d}x_I,$$

where $I = (i_1, \ldots, i_p)$, $f_I = f_{i_1,\ldots,i_p}$ are $\mathcal{C}^\infty$-functions, $\mathrm{d}x_I = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}$ and the summation is carried out over the I such that $1 \leq i_1 \leq \cdots \leq i_p \leq m$. The $\mathbb{R}$-vector space of differential p-forms on M is then denoted $\Omega^p(M)$ and the direct sum

$$\Omega(M) = \bigoplus_{p=0}^{m} \Omega^p(M)$$

is the space of differential forms on M. As *coboundary map* one has the **exterior derivative** $\mathrm{d} : \Omega \to \Omega$, defined to be the unique $\mathbb{R}$-linear map sending p-forms to $(p+1)$-forms (antiderivation) and satisfying:

- At the level of smooth functions $f \in \Omega^0(M)$, it is given by

$$\mathrm{d}f = \sum_{k=1}^{m} \frac{\partial f}{\partial x_k} \mathrm{d}x_k.$$

- It satisfies the property $\mathrm{d} \circ \mathrm{d} = 0$.
- Inductively, given a p-form α and any differential form β in $\Omega(M)$, their exterior product $\alpha \wedge \beta$ satisfies

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \mathrm{d}\beta.$$

Moreover, there is a cochain complex

$$0 \to \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^m(M) \to 0.$$

Recalling that a differential p-form ω is **closed** if $\mathrm{d}\omega = 0$ and that it is **exact** if there exists a differential $(p-1)$-form η such that $\omega = \mathrm{d}\eta$, notice that all exact forms are closed but not vice versa. It is the study of this "vice versa" what is called **De Rham Cohomology**, after the inventor of differential forms.

Recall that closed 1-forms on a manifold are automatically exact if certain *topological* conditions hold, for instance that the manifold be simply connected [BM94]. Intuitively, if a manifold is not simply connected, it has some sort of "holes" in it (e.g. the punctured plane $\mathbb{R}^2 \setminus \{0\}$). One might call these 1-holes because they impede closed 1-forms *wrapping around them* from being exact. Further, they prevent certain 1-dimensional objects —paths namely— from being homotopic. There are, however, various sorts of holes besides the 1-holes.[3] In some sense, Cohomology is thus the study of holes by algebraic methods. De Rham cohomology groups can be described as[4]

3: For example, the space $\mathbb{R}^3 \setminus \{0\}$ clearly has some sort of hole in it, but it is simply connected. One may then call it a 2-hole instead of a 1-hole, since it prevents some 2-dimensional surfaces from being deformed into one another (think for instance of the upper and lower hemispheres of the unit sphere sitting in this punctured space).

4: In this definition, the dimension of the cohomology group $H^p(M)$ may then be regarded as the number of "p-holes" in M.

Definition C.5.1 *Let $Z^p(M)$ be the vector space of closed p-forms on M and $B^p(M) \subseteq Z^p(M)$ the vector subspace of exact p-forms. Then, a measure of the amount of existing closed forms that are not exact is given by the quotient space*

$$H^p(M) := Z^p(M) \big/ B^p(M),$$

*called the **p-th de Rham cohomology group of** M.*

An element of $H^p(M)$ is thus an equivalence class of closed p-forms, where two closed forms ω and ω' are equivalent —*cohomologous*— if they differ by an exact p-form. Each equivalence or *cohomology* class is then denoted

$$[\omega] := \{\omega' \mid \exists \eta \in \Omega^p(M) \quad \text{st} \quad \omega - \omega' = \mathrm{d}\eta\}.$$

One can show that the pullback of a closed form is closed and, similarly, that the pullback of an exact form is exact. Last to mention, a classical theorem by De Rham asserts that singular and De Rham cohomologies are actually isomorphic.

D

Mathematica code

D.1 Large Color R-matrix in action

We gather here some lines of the `Mathematica` code for the implementation of the colored Jones polynomial and the $F_K(x,q)$ invariant for the right-handed Trefoil. In the former case, an extra overall factor of q^{-i} appears from setting $x = q^i$.

```
(* Define R-matrix function and its inverse *)

RMat[x_, q_, i1_, j1_, i2_, j2_] :=
  DiscreteDelta[i1 + j1 - i2 - j2] x^(-((j1 + j2)/2)) q^(j1*j2)
  (QPochhammer[q, q, i1] QPochhammer[x^-1 q^j1, q, i1 - j2])
  /(QPochhammer[q, q, j2] QPochhammer[q, q, i1 - j2]);
RMatinv[x_, q_, i1_, j1_, i2_, j2_] :=
  DiscreteDelta[i1 + j1 - i2 - j2] x^((i1 + i2)/2) q^(-i1*i2)
  (QPochhammer[q^-1, q^-1, j1] QPochhammer[x q^-i1, q^-1, j1-i2])
  /(QPochhammer[q^-1, q^-1, i2] QPochhammer[q^-1, q^-1, j1-i2]);

(* Compute the i-th colored Jones polynomial (x=q^i) *)
Do[Print[q^-i (Sum[ q^{-m} RMat[q^i, q, 0, m, m, 0] RMat[q^i, q, m
    , 0, m, 0] RMat[q^i, q, m, 0, 0, m], {m, 0, i}] //
    FunctionExpand) // Simplify // Expand], {i, 0, 4}]

(* Check it agrees with the expected result: *)
Do[Print[ColouredJones[Knot[3, 1], n][q]], {n, 0, 4}]

(* Compute FK (x,q) up to order O(x^6) *)
Collect[(Sum[ q^{-m} RMat[x, q, 0, m, m, 0] RMat[x, q, m, 0, m, 0]
     RMat[x, q, m, 0, 0, m], {m, 0, 5}] /. x -> q^-2 x^-1 //
    FunctionExpand) + O[x]^6, x, Simplify]
```

D.2 Finding suitable generators for $\mathfrak{so}_8$

He give here some of the lines of code used to obtain the new set of generators for $\mathfrak{so}_8$.

```
(* Define general element of so(8,C) and the hk's*)
(testelem = {{0, a12, a13, a14, a15, a16, a17, a18}, {-a12, 0, a23
    , a24, a25, a26, a27, a28}, {-a13, -a23, 0, a34, a35, a36,
    a37, a38}, {-a14, -a24, -a34, 0, a45, a46, a47, a48}, {-a15,
    -a25, -a35, -a45, 0, a56, a57, a58}, {-a16, -a26, -a36, -a46,
     -a56, 0, a67, a68}, {-a17, -a27, -a37, -a47, -a57, -a67, 0,
    a78}, {-a18, -a28, -a38, -a48, -a58, -a68, -a78, 0}}) //
    MatrixForm
```

```

(h1 = {{0, 1, 0, 0, 0, 0, 0, 0}, {-1, 0, 0, 0, 0, 0, 0, 0}, {0, 0,
     0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0,
     0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0,
    0}, {0, 0, 0, 0, 0, 0, 0, 0}}) // MatrixForm;
(h2 = {{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0,
    0, 1, 0, 0, 0, 0}, {0, 0, -1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0,
     0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0,
    0}, {0, 0, 0, 0, 0, 0, 0, 0}}) // MatrixForm;
(h3 = {{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0,
    0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0,
    1, 0, 0}, {0, 0, 0, 0, -1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0,
    0}, {0, 0, 0, 0, 0, 0, 0, 0}}) // MatrixForm;
(h4 = {{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0,
    0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0,
    0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 1},
     {0, 0, 0, 0, 0, 0, -1, 0}}) // MatrixForm;

(* Apply change of basis *)
(changebasis = Transpose[{{i, 1, 0, 0, 0, 0, 0, 0}, {-i, 1, 0, 0,
    0, 0, 0, 0}, {0, 0, i, 1, 0, 0, 0, 0}, {0, 0, -i, 1, 0, 0, 0,
     0}, {0, 0, 0, 0, i, 1, 0, 0}, {0, 0, 0, 0, -i, 1, 0, 0}, {0,
     0, 0, 0, 0, 0, i, 1}, {0, 0, 0, 0, 0, 0, -i, 1}}]) //
    MatrixForm;

(testmat = 2 Inverse[changebasis].testelem.changebasis) //
  Expand // MatrixForm
```

The factor of two here is to ease the identification of corresponding pairs. These are guessed by inspection and checked with lines like the following one, giving true whenever the matching is correct.

```
testmat[[1, 3]] == -testmat[[4, 2]] // Simplify;
```

The new general element in this basis is then defined (`test2`), along with the new H generators (`HC`) whose linear combinations end up giving the Cartan subalgebra generators.

```
(* Define general element test2 and HCk's *)
(test2 = {{H1, 0, X1, X2, X5, X6, X9, X10}, {0, -H1, X3, X4, X7,
    X8, X11, X12}, {-X4, -X2, H2, 0, X13, X14, X17, X18}, {-X3, -
    X1, 0, -H2, X15, X16, X19, X20}, {-X8, -X6, -X16, -X14, H3,
    0, X21, X22}, {-X7, -X5, -X15, -X13, 0, -H3, X23, X24}, {-X12
    , -X10, -X20, -X18, -X24, -X22, H4, 0}, {-X11, -X9, -X19, -
    X17, -X23, -X21, 0, -H4}}) // MatrixForm

(HC1 = {{1, 0, 0, 0, 0, 0, 0, 0}, {0, -1, 0, 0, 0, 0, 0, 0}, {0,
    0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0,
    0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0,
    0}, {0, 0, 0, 0, 0, 0, 0, 0}}) // MatrixForm;
(HC2 = {{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0,
     1, 0, 0, 0, 0, 0}, {0, 0, 0, -1, 0, 0, 0, 0}, {0, 0, 0, 0,
    0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0,
    0}, {0, 0, 0, 0, 0, 0, 0, 0}}) // MatrixForm;
(HC3 = {{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0,
     0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 1,
     0, 0, 0}, {0, 0, 0, 0, 0, -1, 0, 0}, {0, 0, 0, 0, 0, 0, 0,
    0}, {0, 0, 0, 0, 0, 0, 0, 0}}) // MatrixForm;
(HC4 = {{0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0,
```

```
    0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0,
    0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 1,
   0}, {0, 0, 0, 0, 0, 0, 0, -1}}) // MatrixForm;
```

We then obtain a general matrix giving the commutator of linear combinations of `HC`'s with the general elements `test2`. By inspection, one finds the suitable coefficients and the generators E_k and F_k.[1]

1: The notation E_k, F_k instead of X_k, Y_k for the generators has been used here.

```
(* General comutator *)
(lincomb = (g1*HC1 + g2*HC2 + g3*HC3 + g4*HC4).test2 - test2.(g1*
    HC1 + g2*HC2 + g3*HC3 + g4*HC4)) // MatrixForm // Simplify

(lincomb /. {g1 -> 1, g2 -> -1, g3 -> 0, g4 -> 0}) // MatrixForm;
(lincomb /. {g1 -> 0, g2 -> 1, g3 -> -1, g4 -> 0}) // MatrixForm;
(lincomb /. {g1 -> 0, g2 -> 0, g3 -> 1, g4 -> -1}) // MatrixForm;
(lincomb /. {g1 -> 0, g2 -> 0, g3 -> 1, g4 -> +1}) // MatrixForm;

(* Lie algebra generators *)
HH1 = (HC1 - HC2);    HH2 = (HC2 - HC3);    HH3 = (HC3 - HC4);
    HH4 = (HC3 + HC4);
E1 = I (test2 - (test2 /. {X1  -> 0})) /. {X1  -> 1};
F1 = I (test2 - (test2 /. {X4  -> 0})) /. {X4  -> 1};
E2 = I (test2 - (test2 /. {X13 -> 0})) /. {X13 -> 1};
F2 = I (test2 - (test2 /. {X16 -> 0})) /. {X16 -> 1};
E3 = I (test2 - (test2 /. {X21 -> 0})) /. {X21 -> 1};
F3 = I (test2 - (test2 /. {X24 -> 0})) /. {X24 -> 1};
E4 = I (test2 - (test2 /. {X22 -> 0})) /. {X22 -> 1};
F4 = I (test2 - (test2 /. {X23 -> 0})) /. {X23 -> 1};
```

One finally checks all Serre relations for these matrix generators with true or false booleans, like in the following examples:

```
(* Check commutation relations [Hi, Hj] = 0 *)
HH1.HH2 - HH2.HH1 == Zero // Simplify;

(* Check commutation relations [Hi,Ej] = Aij Ej *)
HH2.E3 - E3.HH2 == -E3 // Simplify;

(* Check commutation relations [Hi,Fj] = -Aij Fj *)
HH4.F3 - F3.HH4 == Zero // Simplify;

(* Check commutation relations [Ei,Fj] = delta_ij H_j *)
E1.F4 - F4.E1 == Zero // Simplify;
```

D.3 Polynomial representation

Here we check that the obtained *polynomial* representation in the chosen a_i basis indeed satisfies each and every Serre relation defining the $\mathfrak{so}_8$ Lie algebra.

We first define the generators through the following functions acting on the basis vectors implemented as lists

`{f, lst}` := `{f, {r, a1, a2, a3, a4, a5, a6, a7}}`

where the extra parameter `f` in front allows dealing with different coefficients when composing different generators.

```
E1 := Function[{f, lst}, {{i f (lst[[1]] - lst[[3]]), lst + Table[
      Boole[j == 3], {j, 1, 8}]}, {-i f(lst[[4]] - lst[[2]]), lst
     + Table[Boole[j == 2], {j, 1, 8}]}}]
```

and similar for the remaining E_k's and all the F_k's.[2] The H_k's are simpler. For example:

```
H1 := Function[{f, lst}, {f (2 lst[[2]] + 2 lst[[3]] - lst[[4]] -
     lst[[5]] - lst[[1]]), lst}]
```

Then, all of the commutation relations are checked one by one. For example, to check

$$E_2 E_1^2 - 2E_1 E_2 E_1 + E_1^2 E_2 = 0,$$

the following line has been implemented.

```
DeleteCases[
   condense[
    Join[
     Flatten[
       E2 @@@ Flatten[
         E1 @@@ E1 @@ {1, {r, a1, a2, a3, a4, a5, a6, a7}}, 1],
     1],
     Flatten[
       E1 @@@ Flatten[
         E2 @@@ E1 @@ {-2, {r, a1, a2, a3, a4, a5, a6, a7}}, 1],
     1],
     Flatten[
       E1 @@@ Flatten[
         E1 @@@ E2 @@ {1, {r, a1, a2, a3, a4, a5, a6, a7}}, 1],
     1]
    ]
   ] // Simplify, x_ /; x[[1]] == 0] === {};
```

Here the extra function `condense` has been defined in order to group together contributions coming from different generators but corresponding to the same basis vector.

```
condense[lst_] := (
  conds = {};
  For[j = 1; t = False, j <= Length[lst], j++,
   For[k = 1; aux = 0, k <= Length[conds], k++,
    t = (lst[[j, 2]] === conds[[k, 2]]);
    If[t, conds[[k, 1]] = conds[[k, 1]] + lst[[j, 1]] ];
    If[t, Break[]]
    ];
   If[t, t =! t, conds = Join[conds, {lst[[i]]}]]
  ];
  conds)
```

D.4 Quantum representation

Concerning the search for the quantum representation, we give here some of the lines that have been used. While the `condense` function remains the same as in the previous section, the quantum integer function `qint` must be defined. This is done through either of the

2: Notice that we are using again the notation E_k, F_k instead of X_k, Y_k.

following two functions, which are useful to define together to allow their use indistinctly.

```
qint[n_, q_] := ( (q^{n} - q^{-n})/(q^{1} - q^{-1}) )
qint[n_] := ( (q^{n} - q^{-n})/(q^{1} - q^{-1}) )
```

Then, some quantum modifications of the generators in the previous section are implemented. For example:

```
E1 := Function[{f, lst}, {{i f q^{lst[[4]] + 3 lst[[2]]} qint[lst
    [[1]] - lst[[3]]], lst + Table[ Boole[j == 3], {j, 1, 8}]},
    {-i f q^{lst[[5]] + lst[[3]]} qint[lst[[4]] - lst[[2]]], lst
    + Table[Boole[j == 2], {j, 1, 8}]}}]
```

Also, we have the q^{H_i} generators, such as:

```
qH1 := Function[{f, lst}, {q^(f (2 lst[[2]] - lst[[4]])) + q^(f (2
     lst[[3]] - lst[[5]] - lst[[1]])), lst}]
```

Then, Serre's commutation relations are analyzed to study where they fail to be fulfilled. For instance, for

$$[E_1, F_1] = \frac{q^{H_1} - q^{-H_1}}{q - q^{-1}}$$

we have

```
Expand[Flatten[
 DeleteCases[
  condense[
   Join[
    Flatten[E1 @@@ F1 @@ {1, {r, a1, a2, a3, a4, a5, a6, a7}}, 1],
    Flatten[F1 @@@ E1 @@ {-1, {r, a1, a2, a3, a4, a5, a6, a7}},1]
   ]
  ],
 x_ /; x[[1]] == 0], 1] [[1 ;; 1]]
 == ((qH1 @@ {1, {r, a1, a2, a3, a4, a5, a6, a7}})[[1 ;; 1]]
 - (qH1 @@ {-1, {r, a1, a2, a3, a4, a5, a6, a7}})[[1 ;; 1]])
 /(q - q^(-1))
] // Simplify;
```

and for

$$E_2 E_1^2 - \left(q + q^{-1}\right) E_1 E_2 E_1 + E_1^2 E_2 = 0,$$

we have

```
DeleteCases[
 condense[Join[
    Flatten[E2 @@@
      Flatten[E1 @@@
        E1 @@ {1, {r, a1, a2, a3, a4, a5, a6, a7}}, 1], 1],
    Flatten[E1 @@@
      Flatten[E2 @@@
        E1 @@ {-(q + q^{-1}), {r, a1, a2, a3, a4, a5, a6, a7}},
        1], 1],
    Flatten[E1 @@@
      Flatten[E1 @@@
        E2 @@ {1, {r, a1, a2, a3, a4, a5, a6, a7}}, 1], 1]]
 ] //Simplify, x_ /; x[[1]] == {0}
]
```

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