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Ito, Kazuhiro; Ito, Tetsushi; Koshikawa, Teruhisa

Forum of Mathematics, Sigma (2021), **Vol. 9**:e29 1–70
doi:10.1017/fms.2021.24

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

CM liftings of $K3$ surfaces over finite fields and their applications to the Tate conjecture

Kazuhiro Ito[1], Tetsushi Ito[2] and Teruhisa Koshikawa[3]

[1]Laboratoire de Mathématiques d'Orsay, Université Paris-Saclay, 91405, Orsay, France;
E-mail: kazuhiro.ito@universite-paris-saclay.fr.
[2]Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan; E-mail: tetsushi@math.kyoto-u.ac.jp.
[3]Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan; E-mail: teruhisa@kurims.kyoto-u.ac.jp.

Received: 3 March 2020; **Revised:** 26 January 2021; **Accepted:** 1 March 2021

2020 Mathematics Subject Classification: *Primary* – 11G18; *Secondary* – 11G15, 14G35, 14J28

Abstract
We give applications of integral canonical models of orthogonal Shimura varieties and the Kuga-Satake morphism to the arithmetic of $K3$ surfaces over finite fields. We prove that every $K3$ surface of finite height over a finite field admits a characteristic 0 lifting whose generic fibre is a $K3$ surface with complex multiplication. Combined with the results of Mukai and Buskin, we prove the Tate conjecture for the square of a $K3$ surface over a finite field. To obtain these results, we construct an analogue of Kisin's algebraic group for a $K3$ surface of finite height and construct characteristic 0 liftings of the $K3$ surface preserving the action of tori in the algebraic group. We obtain these results for $K3$ surfaces over finite fields of any characteristics, including those of characteristic 2 or 3.

1. Introduction

The integral canonical models of orthogonal Shimura varieties and the Kuga-Satake morphism have applications to the arithmetic of $K3$ surfaces over finite fields. For example, Madapusi Pera used it to prove the Tate conjecture for divisors on $K3$ surfaces over finitely generated fields [49]. (See [38] for the case of characteristic 2.)

The aim of this article is to give further applications. More specifically, we shall prove the following results:

(1) (see Theorem 1.1) Every $K3$ surface X of finite height over a finite field $\mathbb{F}_q$ with q elements admits a complex multiplication (CM) lifting after replacing $\mathbb{F}_q$ by its finite extension (i.e., it admits a characteristic 0 lifting whose generic fibre has complex multiplication).
(2) (see Theorem 1.5) The Tate conjecture holds for algebraic cycles of codimension 2 on the square $X \times X$ of any $K3$ surface X (of any height) over $\mathbb{F}_q$.

These results are consequences of our results on characteristic 0 liftings of $K3$ surfaces; see Theorem 1.7.

Our strategy of the proof is as follows. Let $(X, \mathscr{L})$ be a quasi-polarised $K3$ surface of finite height over $\overline{\mathbb{F}}_q$. Here $\mathscr{L}$ is a (primitive) line bundle on X that is big and nef. We shall attach an algebraic group I over $\mathbb{Q}$ to each polarised $K3$ surface $(X, \mathscr{L})$ of finite height over $\overline{\mathbb{F}}_q$, which is an analogue of the algebraic group attached by Kisin to each mod p point on the integral canonical model of a Shimura variety of

Hodge type [41]. Then, for each maximal torus $T \subset I$ over $\mathbb{Q}$, we shall construct a characteristic 0 lifting of the quasi-polarised $K3$ surface $(X, \mathscr{L})$ such that the action of each element of $T(\mathbb{Q})$ on the singular cohomology of the generic fibre preserves the $\mathbb{Q}$-Hodge structure. We use integral canonical models of Shimura varieties to control the rationality of the action of $T(\mathbb{Q})$. From these results, the result (1) follows by comparing the rank of the algebraic group I and the general spin group attached to the orthogonal Shimura variety. Combined with the results of Mukai and Buskin on the Hodge conjecture for products of $K3$ surfaces, we shall prove that the action of every element of $T(\mathbb{Q})$ is induced by an algebraic cycle of codimension 2 on $X \times X$. Applying this result for several maximal tori $T \subset I$, the result (2) follows.

Note that we do not impose any conditions on the characteristic of the base field. Thus, the main results of this article are valid over finite fields of any characteristics, including those of characteristic 2 or 3. To overcome certain technical difficulties, we essentially use the integral comparison theorems of Bhatt-Morrow-Scholze [6], at least in small characteristics. (Note that, when the characteristic is greater than or equal to 5, we can avoid most of the technical difficulties. Instead, we can use the results of Nygaard-Ogus to obtain the main results of this article. See Subsection 1.4.)

In the course of writing this article, we found an error in the proof of the étaleness of the Kuga-Satake morphism in characteristic 2, which was also used in the proof of the Tate conjecture for $K3$ surfaces in characteristic 2 [38]. We correct it using our results on F-crystals on orthogonal Shimura varieties, which depend on the integral comparison theorem of Bhatt-Morrow-Scholze [6]; see Remark 6.9 for details. (See also Remark 6.10.)

In the rest of the Introduction, we shall first give precise statements on our results on CM liftings and the Tate conjecture; see Theorem 1.1 and Theorem 1.5. Then we explain our results on characteristic 0 liftings (see Theorem 1.7) and how to obtain (1) and (2) from them.

1.1. CM liftings of $K3$ surfaces of finite height over finite fields

First we state our results on CM liftings.

Recall that a projective smooth surface X over a field is called a $K3$ *surface* if its canonical bundle is trivial and it satisfies $H^1(X, \mathcal{O}_X) = 0$. More generally, an algebraic space $\mathscr{X}$ over a scheme S is a $K3$ surface over S if $\mathscr{X} \to S$ is proper, smooth and every geometric fibre is a $K3$ surface.

We say that a projective $K3$ surface Y over $\mathbb{C}$ has *complex multiplication* if the Mumford-Tate group associated with the singular cohomology $H^2_B(Y, \mathbb{Q})$ is commutative; see Subsection 9.1. We say that a $K3$ surface Y over a number field F has CM if $Y_{\mathbb{C}}$ has CM for every embedding $F \hookrightarrow \mathbb{C}$.

We fix a prime number p and a power q of p. Let X be a $K3$ surface over $\mathbb{F}_q$. We say that X admits a *CM lifting* if there exist a number field F, a finite place v of F with residue field $\mathbb{F}_q$ and a $K3$ surface $\mathscr{X}$ over the localisation $\mathscr{O}_{F,(v)}$ of the ring of integers $\mathscr{O}_F$ of F at v such that the special fibre $\mathscr{X}_{\mathbb{F}_q}$ is isomorphic to X and the generic fibre $\mathscr{X}_F$ is a $K3$ surface with CM. The height h of the formal Brauer group of X is called the *height* of X; it satisfies $1 \leq h \leq 10$ or $h = \infty$. When $1 \leq h \leq 10$ (respectively $h = \infty$), we say X is *of finite height* (respectively *supersingular*).

Here is the first main theorem of this article.

Theorem 1.1 (see Corollary 9.10)**.** *Let X be a $K3$ surface over $\mathbb{F}_q$. If X is of finite height, then there is a positive integer $m \geq 1$ such that $X_{\mathbb{F}_{q^m}} := X \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \mathbb{F}_{q^m}$ admits a CM lifting.*

Remark 1.2. After we completed the first draft of this article, the authors learned that Yang also proved the above theorem under the additional conditions that $p \geq 5$ and X admits a quasi-polarisation whose degree is not divisible by p; see [74, Theorem 1.6]. Under these assumptions, our method (or a simplified version presented in Subsection 1.5) and Yang's method share several ingredients, but there is one difference; Yang used Kisin's result [41, Theorem 0.4] on the CM liftings, up to isogeny, of closed points of the special fibre of the integral canonical model of a Shimura variety of Hodge type, whereas we give a refinement of Kisin's result (or argument) itself; see Theorem 1.7 for details.

Remark 1.3. Deuring proved that every elliptic curve over a finite field admits a characteristic 0 lifting whose generic fibre is an elliptic curve with CM; see [17, Theorem 1.7.4.6]. Theorem 1.1 is an analogue of this result for $K3$ surfaces of finite height. It is an interesting question to ask whether Theorem 1.1 holds also for supersingular $K3$ surfaces over finite fields. Our methods in this article cannot be applied to supersingular $K3$ surfaces.

Remark 1.4. We also have similar results on the existence of quasi-canonical liftings (in the sense of Nygaard-Ogus) of $K3$ surfaces of finite height over a finite field; see Corollary 9.11.

1.2. The Tate conjecture for the squares of $K3$ surfaces over finite fields

Next we state our results on the Tate conjecture. (For the statement of the Tate conjecture, see [55, Conjecture 0.1], [69, Section 1], [70, Conjecture 1.1] for example.)

As the second main theorem of this article, we shall prove the Tate conjecture for the square of a $K3$ surface over a finite field.

Theorem 1.5 (see Theorem 10.1). *Let X be a $K3$ surface (of any height) over $\mathbb{F}_q$. We put $X \times X := X \times_{\mathrm{Spec}\,\mathbb{F}_q} X$ and $X_{\overline{\mathbb{F}}_q} \times X_{\overline{\mathbb{F}}_q} := X_{\overline{\mathbb{F}}_q} \times_{\mathrm{Spec}\,\overline{\mathbb{F}}_q} X_{\overline{\mathbb{F}}_q}$. Then, for every i, the ℓ-adic cycle class map*

$$\mathrm{cl}^i_\ell \colon Z^i(X \times X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to H^{2i}_{\text{ét}}(X_{\overline{\mathbb{F}}_q} \times X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)}$$

is surjective for every prime number $\ell \neq p$. Moreover, for every i, the crystalline cycle class map

$$\mathrm{cl}^i_{\mathrm{cris}} \colon Z^i(X \times X) \otimes_{\mathbb{Z}} \mathbb{Q}_p \to H^{2i}_{\mathrm{cris}}((X \times X)/W(\mathbb{F}_q))^{\varphi = p^i} \otimes_{\mathbb{Z}} \mathbb{Q}$$

is surjective.

Here $Z^i(X \times X)$ denotes the group of algebraic cycles of codimension i on $X \times X$, and $W(\mathbb{F}_q)$ is the ring of Witt vectors of $\mathbb{F}_q$. The map φ denotes the action of the absolute Frobenius endomorphism on the crystalline cohomology.

Remark 1.6. Theorem 1.5 was previously known to hold for some $K3$ surfaces.

(1) Theorem 1.5 obviously holds for any $i \notin \{1, 2, 3\}$.
(2) The surjectivity of cl^1_ℓ and cl^3_ℓ follows from the Tate conjecture for X [18, 38, 49, 52]; see also Lemma 10.9.
(3) Theorem 1.5 holds when X is supersingular. In fact, the Tate conjecture for X implies the Picard number of $X_{\overline{\mathbb{F}}_q}$ is 22; see Lemma 10.6. Then the Tate conjecture for the square $X \times X$ follows by the Künneth formula; see Lemma 10.7 and Remark 10.8.
(4) Zarhin proved the Tate conjecture for $X \times X$ when X is an ordinary $K3$ surface; see [77, Corollary 6.1.2]. Here a $K3$ surface X is called *ordinary* if it is of height 1. (More generally, Zarhin proved the Tate conjecture for any power $X \times \cdots \times X$ of an ordinary $K3$ surface X.)
(5) Yu-Yui proved the Tate conjecture for $X \times X$ when X satisfies some conditions on the characteristic polynomial of the Frobenius morphism; see [75, Lemma 3.5, Corollary 3.6].

In the cases studied by Zarhin and Yu-Yui, it turns out that all of the Tate cycles of codimension 2 on $X \times X$ are spanned by the classes of the cycles of the form $X \times \{x_0\}$, $\{x_0\} \times X$ and $D_1 \times D_2$ and the classes of the graphs of powers of the Frobenius morphism on X. Here x_0 is a closed point on X, and D_1 and D_2 are divisors on X. In general, there are Tate classes on $X \times X$ that are not spanned by these classes. Therefore, in order to prove Theorem 1.5 in full generality, we shall prove the algebraicity of Tate cycles on $X \times X$ that are not spanned by Tate cycles considered by Zarhin and Yu-Yui. We shall prove it by constructing characteristic 0 liftings and applying the results of Mukai and Buskin on the Hodge conjecture.

https://doi.org/10.1017/fms.2021.24 Published online by Cambridge University Press

1.3. Construction of characteristic 0 *liftings preserving the action of tori*

Here we explain our results on the construction of characteristic 0 liftings of $K3$ surfaces.

Let X be a $K3$ surface over $\mathbb{F}_q$ and $\mathscr{L}$ a line bundle on X defined over $\mathbb{F}_q$ that gives a primitive quasi-polarisation. Assume that X is of finite height. After replacing $\mathbb{F}_q$ by a finite extension of it, the *Kuga-Satake abelian variety* A associated with $(X, \mathscr{L})$ is defined over $\mathbb{F}_q$. (Precisely, we shall use the Kuga-Satake abelian variety introduced by Madapusi Pera in [49, 50], which has dimension 2^{21}; it is larger than the dimension of the classical Kuga-Satake abelian variety. See Subsection 4.3.)

We have an action of a general spin group, denoted by $\mathrm{GSpin}(L_{\mathbb{Q}})$ in this article, on the cohomology of X and A. We put $G := \mathrm{GSpin}(L_{\mathbb{Q}})$ in this section. We do not recall the precise definition of G here. Instead, we give some of its properties:

- For every prime number $\ell \neq p$, the group of $\mathbb{Q}_\ell$-valued points $G(\mathbb{Q}_\ell)$ acts on the primitive part

$$P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1)) := \mathrm{ch}_\ell(\mathscr{L})^\perp \subset H^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$$

  of the ℓ-adic cohomology of X and the ℓ-adic cohomology

$$H^1_{\text{ét}}(A_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$$

  of A.
- There is a $G(\mathbb{Q}_\ell)$-equivariant $\mathbb{Q}_\ell$-linear map

$$P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1)) \to \mathrm{End}_{\mathbb{Q}_\ell}(H^1_{\text{ét}}(A_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)^\vee),$$

  where $()^\vee$ denotes the $\mathbb{Q}_\ell$-linear dual.
- There is an element $\mathrm{Frob}_q \in G(\mathbb{Q}_\ell)$ whose action on $P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$ (respectively $H^1_{\text{ét}}(A_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$) coincides with the action of the geometric Frobenius morphism on the ℓ-adic cohomology of X (respectively A).

Following Kisin [41], we attach an algebraic group I over $\mathbb{Q}$ to the quasi-polarised $K3$ surface $(X, \mathscr{L})$; see Definition 8.1. Instead of giving the precise definition here, we give its properties:

- The group of $\mathbb{Q}$-valued points $I(\mathbb{Q})$ is considered as a subgroup of the multiplicative group of the endomorphism algebra of $A_{\overline{\mathbb{F}}_q}$ tensored with $\mathbb{Q}$:

$$I(\mathbb{Q}) \subset (\mathrm{End}_{\overline{\mathbb{F}}_q}(A_{\overline{\mathbb{F}}_q}) \otimes_{\mathbb{Z}} \mathbb{Q})^\times.$$

- For every prime number $\ell \neq p$, there is an embedding $I_{\mathbb{Q}_\ell} \hookrightarrow G_{\mathbb{Q}_\ell}$ and an element of $G(\mathbb{Q}_\ell)$ is in $I(\mathbb{Q}_\ell)$ if and only if it commutes with Frob_q^m for a sufficiently divisible $m \geq 1$.
- The algebraic groups G and I have the same rank.

The existence of an algebraic group I over $\mathbb{Q}$ that satisfies these properties is not obvious; it is considered as Kisin's group-theoretic interpretation and generalisation of Tate's original proof of the Tate conjecture for endomorphisms of abelian varieties over finite fields.

As the third main theorem of this article, we shall construct a characteristic 0 lifting of a quasi-polarised $K3$ surface of finite height preserving the action of a maximal torus of the algebraic group I.

Theorem 1.7 (see Theorem 9.7)**.** *Let $T \subset I$ be a maximal torus over $\mathbb{Q}$. Then there exist a finite extension K of $W(\overline{\mathbb{F}}_q)[1/p]$ and a quasi-polarised $K3$ surface $(\mathfrak{X}, \mathcal{L})$ over $\mathcal{O}_K$ such that the special fibre $(\mathfrak{X}_{\overline{\mathbb{F}}_q}, \mathcal{L}_{\overline{\mathbb{F}}_q})$ is isomorphic to $(X_{\overline{\mathbb{F}}_q}, \mathscr{L}_{\overline{\mathbb{F}}_q})$ and, for every embedding $K \hookrightarrow \mathbb{C}$, the quasi-polarised $K3$ surface $(\mathfrak{X}_{\mathbb{C}}, \mathcal{L}_{\mathbb{C}})$ satisfies the following properties:*

(1) *The $K3$ surface $\mathcal{X}_{\mathbb{C}}$ has CM.*
(2) *There is a homomorphism of algebraic groups over $\mathbb{Q}$,*

$$T \to \mathrm{SO}(P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))).$$

Here $P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))$ is the primitive part of the Betti cohomology of $\mathcal{X}_{\mathbb{C}}$.
(3) *For every $\ell \neq p$, the action of $T(\mathbb{Q}_\ell)$ on $P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1)) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ is identified with the action of $T(\mathbb{Q}_\ell)$ on $P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$ via the canonical isomorphisms*

$$P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1)) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong P^2_{\text{ét}}(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}_\ell(1)) \cong P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$$

(using the embedding $K \hookrightarrow \mathbb{C}$, we consider K as a subfield of $\mathbb{C}$).
(4) *The action of every element of $T(\mathbb{Q})$ on $P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))$ preserves the $\mathbb{Q}$-Hodge structure on it.*

Remark 1.8. It is known that every $K3$ surface with CM is defined over a number field; see Proposition 9.1 and Remark 9.2. Therefore, Theorem 1.7 implies Theorem 1.1.

Remark 1.9. Our construction of characteristic 0 liftings relies on the theory of integral canonical models of Shimura varieties of Hodge type developed by Milne, Vasiu, Kisin, and Kim-Madapusi Pera. In particular, we need an explicit description of the completion at a closed point of the special fibre of the integral canonical model of a Shimura variety of Hodge type given by Kisin when $p \geq 3$ [40] and by Kim-Madapusi Pera when $p = 2$ [38]. (When $p \geq 5$, we can also use the results of Nygaard-Ogus [55] to obtain necessary results on characteristic 0 liftings; see Subsection 1.4 and Remark 7.2.)

Remark 1.10. When X is ordinary, Theorem 1.7 was essentially proved by Nygaard in [54], although the algebraic group I did not appear there. When X is ordinary, the canonical lifting of X is a CM lifting. On the other hand, when the height of X is finite and $p \geq 5$, Nygaard-Ogus proved that X admits quasi-canonical liftings [55]. But a quasi-canonical lifting is not necessarily a CM lifting.

Remark 1.11. As suggested by the referee, our methods should be able to be applied to more general mod p points on orthogonal Shimura varieties (not only to $K3$ surfaces over $\overline{\mathbb{F}}_q$ of finite height) in order to show that they admit CM liftings. See also Remark 9.8.

1.4. Remarks on the characteristic and the Kuga-Satake morphism

In this article, we do not put any restrictions on the characteristic p. There are several technical difficulties in small characteristics. But, when $p \geq 5$ and p does not divide the degree of the quasi-polarisation, most of the technical difficulties disappear and the proofs of the main theorems can be considerably simplified. See Subsection 1.5 for some details.

We construct characteristic 0 liftings of quasi-polarised $K3$ surfaces corresponding to characteristic 0 liftings of formal Brauer groups in Section 7. Our construction depends on the calculations of F-crystals in Section 6, which in turn depend on the integral comparison theorems of Bhatt-Morrow-Scholze [6]. When $p \geq 5$, we can avoid them. Instead, we can use the results of Nygaard-Ogus [55] to obtain necessary results on liftings of $K3$ surfaces; see Remark 7.2.

Our notation on the Shimura varieties is slightly complicated because we use the Kuga-Satake morphism introduced by Madapusi Pera in [49, 38], which is denoted by

$$\mathrm{KS}\colon M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Z}_{(p)}} \to Z_{\mathrm{K}^p_0}(\Lambda).$$

(See Subsection 5.2.) To define $Z_{\mathrm{K}^p_0}(\Lambda)$, we embed the Shimura variety, which is the target of the classical Kuga-Satake morphism, into a larger Shimura variety and put additional structures (called Λ-*structures*); see Definition 4.3.

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We use the morphism KS to avoid certain technical difficulties that arise when p divides the degree of the quasi-polarisation. (The same technique was used by Madapusi Pera in [49, 50, 38]. See also Remark 4.1.)

When p does not divide the degree of the quasi-polarisation, we can avoid it and directly work with the classical Kuga-Satake morphism into the integral canonical model of the (smaller) Shimura variety.

Remark 1.12. In the course of writing this article, we found some issues on the proof of the étaleness of the Kuga-Satake morphism. The étaleness was used in our construction of characteristic 0 liftings in Section 7. It was also used by Madapusi Pera in his proof of the Tate conjecture for $K3$ surfaces [49, 38]. We can avoid these issues using our results in Section 6; see Remark 6.9 and Remark 6.10 for details. After we communicated the first draft of this article to Madapusi Pera, he found a somewhat different argument; see [51].

1.5. Outline of the proofs of the main theorems

We shall prove Theorem 1.1 and Theorem 1.7 at the same time. Then, combined with the results of Mukai and Buskin, we shall prove Theorem 1.5.

Proofs of Theorem 1.1 and Theorem 1.7 (when $p \geq 5$)
In order to simplify the exposition, we first explain the proof of Theorem 1.7 when $p \geq 5$ using the results of Nygaard-Ogus. In the following argument, we replace $\mathbb{F}_q$ by a sufficiently large finite extension of it. We put $W := W(\mathbb{F}_q)$.

Let $\widehat{\mathrm{Br}} := \widehat{\mathrm{Br}}(X)$ be the formal Brauer group associated with X. First we shall show that $I_{\mathbb{Q}_p}$ acts on $\widehat{\mathrm{Br}}$, up to isogeny. Then we take a finite totally ramified extension E of $W[1/p]$ and a one-dimensional smooth formal group $\mathcal{G}$ over $\mathcal{O}_E$ lifting $\widehat{\mathrm{Br}}$ such that the action of $I_{\mathbb{Q}_p}$ on $\widehat{\mathrm{Br}}$ lifts to an action of $I_{\mathbb{Q}_p}$ on $\mathcal{G}$, up to isogeny.

Let $P^2_{\mathrm{cris}}(X/W)$ denote the primitive part of $H^2_{\mathrm{cris}}(X/W)$. The lifting $\mathcal{G}$ defines filtrations on $P^2_{\mathrm{cris}}(X/W) \otimes_W E$ and $H^1_{\mathrm{cris}}(A/W) \otimes_W E$ as follows. The Kuga-Satake construction gives embeddings that are homomorphisms of F-isocrystals after inverting p:

$$\mathbb{D}(\widehat{\mathrm{Br}})(1) \subset P^2_{\mathrm{cris}}(X/W)(1) \subset \widetilde{L}_{\mathrm{cris}} \subset \mathrm{End}_W(H^1_{\mathrm{cris}}(A/W)^\vee).$$

(Here $\mathbb{D}(\widehat{\mathrm{Br}})$ is the Dieudonné module of $\widehat{\mathrm{Br}}$ considered as a connected p-divisible group. For the W-module $\widetilde{L}_{\mathrm{cris}}$, see Subsection 4.6.) The lifting $\mathcal{G}$ defines a filtration on $\mathbb{D}(\widehat{\mathrm{Br}})(1) \otimes_W E$:

$$\mathrm{Fil}^1(\mathcal{G}) \subset \mathbb{D}(\widehat{\mathrm{Br}})(1) \otimes_W E.$$

Thus, it gives the filtration on $P^2_{\mathrm{cris}}(X/W)(1) \otimes_W E$:

$$\mathrm{Fil}^1(\mathcal{G}) \subset \mathrm{Fil}^1(\mathcal{G})^\perp \subset P^2_{\mathrm{cris}}(X/W)(1) \otimes_W E.$$

Take a generator $e \in \mathrm{Fil}^1(\mathcal{G})$ and write

$$\mathrm{Fil}^1 := \mathrm{Im}(e) \subset H^1_{\mathrm{cris}}(A/W) \otimes_W E.$$

It gives a filtration on $H^1_{\mathrm{cris}}(A/W) \otimes_W E$ that does not depend on the choice of e.

When $p \geq 5$, the results of Nygaard-Ogus [55] imply the existence of a lifting $(\mathcal{X}, \mathcal{L})$ over $\mathcal{O}_E$ corresponding to the filtration defined as above.

We shall show that, for every embedding $E \hookrightarrow \mathbb{C}$, the action of an element of $T(\mathbb{Q})$ on $P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))$ preserves the $\mathbb{Q}$-Hodge structure. To show this, we note that each element of $T(\mathbb{Q})$ can be considered as an element of $(\mathrm{End}_{\mathbb{F}_q}(A) \otimes_{\mathbb{Z}} \mathbb{Q})^\times$. Because its action preserves the filtration on $H^1_{\mathrm{cris}}(A/W) \otimes_W E$, it lifts to an element of $(\mathrm{End}_{\mathbb{C}}(\mathcal{A}_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{Q})^\times$, where $\mathcal{A}_{\mathbb{C}}$ is the Kuga-Satake abelian variety over $\mathbb{C}$ associated with $(\mathcal{X}_{\mathbb{C}}, \mathcal{L}_{\mathbb{C}})$. In particular, it preserves the Hodge structure on the singular cohomology $H^1_B(\mathcal{A}_{\mathbb{C}}, \mathbb{Q})$. Because we have a $T(\mathbb{Q})$-equivariant embedding respecting the $\mathbb{Q}$-Hodge structures

$$P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1)) \hookrightarrow \mathrm{End}_{\mathbb{C}}(H^1_B(\mathcal{A}_{\mathbb{C}}, \mathbb{Q})^{\vee}),$$

the action of each element of $T(\mathbb{Q})$ on $P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))$ preserves the $\mathbb{Q}$-Hodge structure on it.

Because the algebraic groups G and I have the same rank, we conclude that the Mumford-Tate group of $P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))$ is commutative. Thus, $\mathcal{X}_{\mathbb{C}}$ is a $K3$ surface with CM. Consequently, the quasi-polarised $K3$ surface $(\mathcal{X}_E, \mathcal{L}_E)$ is defined over a number field, and Theorem 1.1 and Theorem 1.7 are proved when $p \geq 5$.

Proofs of Theorem 1.1 and Theorem 1.7 (for $p = 2$ or 3)

When $p = 2$ or 3, we cannot use the results of Nygaard-Ogus to construct liftings of $K3$ surfaces. Instead, we use p-adic Hodge theory and the étaleness of the Kuga-Satake morphism to construct liftings. (The following argument works for any p, including $p \geq 5$.)

Using the p-adic Tate module of $\mathcal{G}$, we first construct a $\mathbb{Z}_p[\mathrm{Gal}(\overline{E}/E)]$-module $\widetilde{L}_p$ such that $\widetilde{L}_p[1/p]$ is a crystalline $\mathrm{Gal}(\overline{E}/E)$-representation whose Hodge-Tate weights are in $\{-1, 0, 1\}$; see Lemma 7.3. On the other hand, we can show that the filtration $\mathrm{Fil}^1 \subset H^1_{\mathrm{cris}}(A/W) \otimes_W E$ gives the structure of a weakly admissible filtered φ-module on $H^1_{\mathrm{cris}}(A/W)[1/p]$. It corresponds to a crystalline representation of $\mathrm{Gal}(\overline{E}/E)$, which is denoted by $H_{\text{ét},\mathbb{Q}_p}$.

Next we find a $\mathrm{Gal}(\overline{E}/E)$-stable $\mathbb{Z}_p$-lattice in $H_{\text{ét},\mathbb{Q}_p}$ as follows. We can show that there is an embedding of $\mathrm{Gal}(\overline{E}/E)$-representations

$$\widetilde{L}_p[1/p] \hookrightarrow \mathrm{End}_{\mathbb{Q}_p}(H_{\text{ét},\mathbb{Q}_p}).$$

Moreover, it can be shown that there is an isomorphism of $\mathbb{Q}_p$-vector spaces

$$\mathrm{Cl}(\widetilde{L}_p[1/p]) \cong H_{\text{ét},\mathbb{Q}_p}$$

such that the actions of $\mathrm{Gal}(\overline{E}/E)$ on $\widetilde{L}_p$ and $H_{\text{ét},\mathbb{Q}_p}$ factor through a homomorphism

$$\mathrm{Gal}(\overline{E}/E) \to \mathrm{GSpin}(\widetilde{L}_p)(\mathbb{Z}_p) \subset \mathrm{Cl}(\widetilde{L}_p)^{\times}.$$

Here $\mathrm{Cl}(\widetilde{L}_p)^{\times}$ acts on $\mathrm{Cl}(\widetilde{L}_p[1/p]) \cong H_{\text{ét},\mathbb{Q}_p}$ by the left multiplication. We take a $\mathrm{Gal}(\overline{E}/E)$-stable $\mathbb{Z}_p$-lattice in $H_{\text{ét},\mathbb{Q}_p}$ corresponding to $\mathrm{Cl}(\widetilde{L}_p) \subset \mathrm{Cl}(\widetilde{L}_p[1/p])$. Then we take a p-divisible group $\mathcal{H}$ over $\mathcal{O}_E$ corresponding to it.

Let K be the composite of E and $W(\overline{\mathbb{F}}_q)[1/p]$. We can show that the p-divisible group $\mathcal{H}_{\mathcal{O}_K}$ satisfies a certain technical condition, called 'adaptedness'. Then we can find an appropriate $\mathcal{O}_K$-valued point $\widetilde{s}$ of the integral canonical model of the Shimura variety lifting the $\overline{\mathbb{F}}_q$-valued point associated with $(X, \mathscr{L})$. (Precisely, $\widetilde{s}$ is an $\mathcal{O}_K$-valued point of the target $Z_{\mathrm{K}^p_0}(\Lambda)$ of the Kuga-Satake morphism.) It gives rise to an abelian scheme $\mathcal{A}$ over $\mathcal{O}_K$ lifting A whose associated p-divisible group $\mathcal{A}[p^{\infty}]$ is $\mathcal{H}_{\mathcal{O}_K}$.

By the étaleness of the Kuga-Satake morphism, we obtain a quasi-polarised $K3$ surface $(\mathcal{X}, \mathcal{L})$ over $\mathcal{O}_K$ corresponding to $\widetilde{s}$.

The rest of the argument is the same as before.

Proof of Theorem 1.5

Fix a prime number $\ell \neq p$. By the Künneth formula, we have

$$\begin{aligned} &H^4_{\text{ét}}(X_{\overline{\mathbb{F}}_q} \times X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(2)) \\ \cong &\bigoplus_{(i,j)=(0,4),(2,2)(4,0)} H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} H^j_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(2). \end{aligned}$$

It is enough to show every element fixed by Frob_q in the component of type $(2,2)$ is spanned by the classes of algebraic cycles of codimension 2 on $X \times X$. By the Poincaré duality, such an element can be considered as an endomorphism of $H^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$ commuting with Frob_q.

Thus, we consider the action of $I(\mathbb{Q}_\ell)$ on $P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$. It can be shown that, after replacing $\mathbb{F}_q$ by a finite extension of it, there exist finitely many maximal tori $T_1, \dots, T_n \subset I$ over $\mathbb{Q}$ such that the $\mathbb{Q}_\ell$-vector space of endomorphisms on $P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$ commuting with Frob_q is spanned by the images of $T_1(\mathbb{Q}), \dots, T_n(\mathbb{Q})$.

Therefore, it is enough to show that, for each i, the action of every element of $T_i(\mathbb{Q})$ on $P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$ comes from an algebraic cycle of codimension 2 on $X \times X$. It can be proved by combining Theorem 1.7 with the results of Mukai and Buskin on the Hodge conjecture for certain Hodge cycles on the product of two $K3$ surfaces over $\mathbb{C}$.

1.6. *Outline of this article*

As explained in Subsection 1.4, most of the technical difficulties can be avoided when $p \geq 5$ and p does not divide the degree of the quasi-polarisation. Readers who are mainly interested in the applications of the integral canonical models of orthogonal Shimura varieties to CM liftings and the Tate conjecture may skip earlier sections in the first reading and may go directly to Section 7.

The organisation of this article is as follows.

In Section 2, we recall basic results on Clifford algebras and general spin groups. In Section 3, we recall basic results on Breuil-Kisin modules and integral p-adic Hodge theory. Then, in Section 4 and Section 5, we fix notation and recall necessary results on the integral canonical models of orthogonal Shimura varieties and the Kuga-Satake morphism used in this article.

In Section 6, we compare F-crystals on Shimura varieties and the crystalline cohomology of $K3$ surfaces. We essentially use the integral comparison theorems of Bhatt-Morrow-Scholze [6]. We also explain how to avoid some issues on the proof of the étaleness of the Kuga-Satake morphism; see Remark 6.9 and Remark 6.10.

In Section 7, we construct a characteristic 0 lifting of the $\overline{\mathbb{F}}_q$-valued point of the Shimura variety corresponding to characteristic 0 liftings of formal Brauer group of the $K3$ surface. We construct such liftings using our results on F-crystals in Section 6 and p-adic Hodge theory.

In Section 8, we define and study an analogue of Kisin's algebraic group associated with a quasi-polarised $K3$ surface of finite height over a finite field. We also study its action on the formal Brauer group.

In Section 9, we combine our results in Section 7 and Section 8 to construct a characteristic 0 lifting of a $K3$ surface of finite height over $\overline{\mathbb{F}}_q$ preserving the action of a maximal torus of I. Then we prove Theorem 1.1 and Theorem 1.7. In Section 10, combined with the results of Mukai and Buskin, we prove Theorem 1.5.

Finally, in Section 11, we give necessary results on the compatibility of several comparison isomorphisms in p-adic Hodge theory used in this article.

1.7. *Notation*

Throughout this article, we fix a prime number p and we let q be a power of p. Let $\mathbb{F}_q$ denote a finite field with q elements and $\overline{\mathbb{F}}_q$ an algebraic closure of $\mathbb{F}_q$.

For a perfect field k of characteristic $p > 0$, the ring of Witt vectors of k is denoted by $W(k)$. The Frobenius automorphism of $W(k)$ is denoted by $\sigma\colon W(k) \to W(k)$. If the field k is clear from the context, we omit k and simply write W.

A *quadratic space* over a commutative ring R means a free R-module M of finite rank equipped with a quadratic form Q. We equip M with a symmetric bilinear pairing $(\ ,\ )$ defined by $(x,y) = Q(x+y) - Q(x) - Q(y)$ for $x, y \in M$. For a module M over a commutative ring R equipped with a

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symmetric bilinear form $(\ ,\ )$, we say M (or the bilinear form $(\ ,\ )$) is *even* if, for every $x \in M$, we have $(x, x) = 2a$ for some $a \in R$.

The base change of a module or a scheme is denoted by a subscript. For example, for a module M over a commutative ring R and an R-algebra R', the tensor product $M \otimes_R R'$ is denoted by $M_{R'}$. For a scheme (or an algebraic space) X over R, the base change $X \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, R'$ is denoted by $X_{R'}$. We use similar notation for the base change of group schemes, p-divisible groups, line bundles, morphisms between them, etc. For a homomorphism $f\colon M \longrightarrow N$ of R-modules, the base change $f_{R'}\colon M_{R'} \longrightarrow N_{R'}$ is also denoted by the same notation f if there is no possibility of confusion. For an element $x \in M$, the R-submodule of M generated by x is denoted by $\langle x \rangle$. The dual of M as an R-module is denoted by $M^\vee := \mathrm{Hom}_R(M, R)$.

2. Clifford algebras and general spin groups

In this section, we introduce notation on quadratic spaces and Clifford algebras that will be used in this article. Our basic references are [3], [50, Section 1].

2.1. *Embeddings of lattices*

A quadratic space $U := \mathbb{Z}x \oplus \mathbb{Z}y$ whose associated bilinear form is given by $(x, x) = (y, y) = 0$ and $(x, y) = 1$ is called the *hyperbolic plane*. The $K3$ *lattice* Λ_{K3} is defined by

$$\Lambda_{K3} := {E_8}^{\oplus 2} \oplus U^{\oplus 3},$$

which is a quadratic space over $\mathbb{Z}$. It is unimodular and its signature is $(19, 3)$.

We fix a positive integer $d > 0$. Let L denote the orthogonal complement of $x - dy$ in Λ_{K3}, where $x - dy$ is considered as an element in the third U. Hence, L is equal to

$${E_8}^{\oplus 2} \oplus U^{\oplus 2} \oplus \langle x + dy \rangle,$$

and its signature is $(19, 2)$.

The following result is well known.

Lemma 2.1. *Let p be a prime number. There is a quadratic space $\widetilde{L}$ of rank 22 over $\mathbb{Z}$ satisfying the following properties:*

(1) *Its signature is* $(20, 2)$.
(2) *$\widetilde{L}$ is self-dual at p (i.e., the discriminant of $\widetilde{L}$ is not divisible by p).*
(3) *There is an embedding $L \hookrightarrow \widetilde{L}$ as quadratic spaces that sends L onto a direct summand of $\widetilde{L}$ as a $\mathbb{Z}$-module.*

Proof. This result was proved in [50, Lemma 6.8] when $p > 2$. Here we briefly give a proof that is valid for every prime number p. We consider a quadratic space $L' := \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$ such that the associated bilinear form is given by $(v_1, v_1) = 2d$, $(v_2, v_2) = 2p$ and $(v_1, v_2) = 1$. We put

$$\widetilde{L} := {E_8}^{\oplus 2} \oplus U^{\oplus 2} \oplus L'.$$

This is of signature $(20, 2)$ and self-dual at p. We have an embedding of quadratic spaces

$$L = {E_8}^{\oplus 2} \oplus U^{\oplus 2} \oplus \langle x + dy \rangle \hookrightarrow \widetilde{L}$$

that is the identity on ${E_8}^{\oplus 2} \oplus U^{\oplus 2}$ and sends $x + dy$ to v_1. □

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2.2. Clifford algebras and general spin groups

In the rest of this article, we fix an embedding of quadratic spaces $L \subset \widetilde{L}$ as in Lemma 2.1.

Let $\mathrm{Cl} := \mathrm{Cl}(\widetilde{L})$ denote the *Clifford algebra* over $\mathbb{Z}$ associated with the quadratic space $(\widetilde{L}, q_{\widetilde{L}})$. There is an embedding of $\mathbb{Z}$-modules $\widetilde{L} \hookrightarrow \mathrm{Cl}$ that is universal for morphisms $f\colon \widetilde{L} \to R$ of $\mathbb{Z}$-modules into an associative $\mathbb{Z}$-algebra R such that $f(v)^2 = q_{\widetilde{L}}(v)$ for every $v \in \widetilde{L}$. The algebra Cl has a $\mathbb{Z}/2\mathbb{Z}$-grading structure $\mathrm{Cl} := \mathrm{Cl}^+ \oplus \mathrm{Cl}^-$, where Cl^+ is a subalgebra of Cl. The quadratic space $\widetilde{L}$ is naturally embedded into Cl^-.

Let $\mathbb{Z}_{(p)}$ be the localisation of $\mathbb{Z}$ at p. We define the *general spin group* $\widetilde{G} := \mathrm{GSpin}(\widetilde{L}_{\mathbb{Z}_{(p)}})$ over $\mathbb{Z}_{(p)}$ by

$$\widetilde{G}(R) := \{\, g \in (\mathrm{Cl}^+_R)^\times \mid g\widetilde{L}_R g^{-1} = \widetilde{L}_R \text{ in } \mathrm{Cl}^-_R \,\}$$

for every $\mathbb{Z}_{(p)}$-algebra R. Because $\widetilde{L}$ is self-dual at p, the group scheme $\widetilde{G}$ is a reductive group scheme over $\mathbb{Z}_{(p)}$.

The *special orthogonal group* $\widetilde{G}_0 := \mathrm{SO}(\widetilde{L}_{\mathbb{Z}_{(p)}})$ is a reductive group scheme over $\mathbb{Z}_{(p)}$ whose generic fibre $\widetilde{G}_{0,\mathbb{Q}} := \widetilde{G}_0 \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$ is $\mathrm{SO}(\widetilde{L}_{\mathbb{Q}})$.

We have the canonical morphism $\widetilde{G} \to \widetilde{G}_0$ defined by $g \mapsto (v \mapsto gvg^{-1})$ whose kernel is the multiplicative group $\mathbb{G}_{m,\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$. We have the following exact sequence of group schemes over $\mathbb{Z}_{(p)}$:

$$1 \to \mathbb{G}_{m,\mathbb{Z}_{(p)}} \to \widetilde{G} = \mathrm{GSpin}(\widetilde{L}_{\mathbb{Z}_{(p)}}) \to \widetilde{G}_0 = \mathrm{SO}(\widetilde{L}_{\mathbb{Z}_{(p)}}) \to 1.$$

2.3. Representations of general spin groups and Hodge tensors

We define a $\mathbb{Z}$-module H by $H := \mathrm{Cl}$. We consider $H_{\mathbb{Z}_{(p)}}$ as a $\widetilde{G}$-representation over $\mathbb{Z}_{(p)}$ by the left multiplication. We have a closed embedding of group schemes over $\mathbb{Z}_{(p)}$:

$$\widetilde{G} \hookrightarrow \mathrm{GL}(H_{\mathbb{Z}_{(p)}}).$$

The representation $H_{\mathbb{Z}_{(p)}}$ has a natural $\mathbb{Z}/2\mathbb{Z}$-grading structure, which is preserved by the action of $\widetilde{G}$. Let $p^+\colon H_{\mathbb{Z}_{(p)}} \to H^+_{\mathbb{Z}_{(p)}} \hookrightarrow H_{\mathbb{Z}_{(p)}}$ (respectively $p^-\colon H_{\mathbb{Z}_{(p)}} \to H^-_{\mathbb{Z}_{(p)}} \hookrightarrow H_{\mathbb{Z}_{(p)}}$) denote the idempotent corresponding to $H^+_{\mathbb{Z}_{(p)}}$ (respectively $H^-_{\mathbb{Z}_{(p)}}$).

The representation $H_{\mathbb{Z}_{(p)}}$ is equipped with a right action of $\mathrm{Cl}_{\mathbb{Z}_{(p)}}$ given by the right multiplication, which commutes with the action of $\widetilde{G}$. We fix a $\mathbb{Z}_{(p)}$-basis $\{e_i\}_{1\le i\le 2^{22}}$ for $\mathrm{Cl}_{\mathbb{Z}_{(p)}}$ and let $r_{e_i}\colon H_{\mathbb{Z}_{(p)}} \to H_{\mathbb{Z}_{(p)}}$ denote the endomorphism defined by $x \mapsto xe_i$.

We regard $\widetilde{L}_{\mathbb{Z}_{(p)}}$ as a $\widetilde{G}$-representation via the canonical homomorphism $\widetilde{G} \to \widetilde{G}_0$ as above. Then the injective homomorphism

$$i\colon \widetilde{L}_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{End}_{\mathbb{Z}_{(p)}}(H_{\mathbb{Z}_{(p)}})$$

defined by $v \mapsto (h \mapsto vh)$ is $\widetilde{G}$-equivariant. The cokernel of this homomorphism i is torsion-free as a $\mathbb{Z}_{(p)}$-module.

As in [50, Section 1], we define a nondegenerate symmetric bilinear form [,] on $\mathrm{End}_{\mathbb{Q}}(H_{\mathbb{Q}})$ by

$$[g_1, g_2] := 2^{-21} \cdot \mathrm{Tr}(g_1 \circ g_2)$$

for $g_1, g_2 \in \mathrm{End}_{\mathbb{Q}}(H_{\mathbb{Q}})$. Then the embedding $i \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$ is an isometry. By [50, Lemma 1.4], there is a unique idempotent $\pi\colon \mathrm{End}_{\mathbb{Q}}(H_{\mathbb{Q}}) \to \mathrm{End}_{\mathbb{Q}}(H_{\mathbb{Q}})$ with the following properties:

(1) The image of π is $i(\widetilde{L}_{\mathbb{Q}})$.
(2) The kernel of π is the orthogonal complement $i(\widetilde{L}_{\mathbb{Q}})^\perp$ of $i(\widetilde{L}_{\mathbb{Q}})$ in $\mathrm{End}_{\mathbb{Q}}(H_{\mathbb{Q}})$ with respect to the bilinear pairing [,].

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(3) $\widetilde{G}_{\mathbb{Q}}$ is the stabiliser of the $\mathbb{Z}/2\mathbb{Z}$-grading structure, the right action of $\mathrm{Cl}_{\mathbb{Z}_{(p)}}$ and the idempotent π; that is, the stabiliser of $p^{\pm}$, $\{r_{e_i}\}_{1\leq i\leq 2^{22}}$ and π.

As in [40, (1.3.1)], let $H^{\otimes}_{\mathbb{Z}_{(p)}}$ denote the direct sum of all $\mathbb{Z}_{(p)}$-modules obtained from $H_{\mathbb{Z}_{(p)}}$ by taking tensor products, duals, symmetric powers and exterior powers. (In fact, symmetric powers and exterior powers are unnecessary; see [26].) By [40, Proposition 1.3.2], the group scheme $\widetilde{G}$ over $\mathbb{Z}_{(p)}$ is the stabiliser of a finite collection of tensors

$$\{s_\alpha\} \subset H^{\otimes}_{\mathbb{Z}_{(p)}}.$$

(See also [38, Lemma 4.7].)

In the rest of this article, we fix such tensors $\{s_\alpha\}$. We may assume that $\{s_\alpha\}$ includes the tensors $\{s_\beta\}$ corresponding to $p^{\pm}$, $\{r_{e_i}\}_{1\leq i\leq 2^{22}}$ and the endomorphism π', where we put $\pi' := p^n\pi$ for a sufficiently large n such that π' maps $\mathrm{End}_{\mathbb{Z}_{(p)}}(H_{\mathbb{Z}_{(p)}})$ into itself.

2.4. *Filtrations on Clifford algebras defined by isotropic elements*

In this subsection, let F be a field of characteristic 0.

Take a nonzero element $e \in \widetilde{L}_F$ satisfying $(e, e) = 0$. We consider an endomorphism

$$i(e) := (i \otimes_{\mathbb{Z}_{(p)}} F)(e) \in \mathrm{End}_F(H_F)$$

that is the image of e under the embedding

$$i \otimes_{\mathbb{Z}_{(p)}} F \colon \widetilde{L}_F \hookrightarrow \mathrm{End}_F(H_F).$$

Let $i(e)(H_F)$ denote the image of the endomorphism $i(e)\colon H_F \to H_F$.

We define a decreasing filtration $\{\mathrm{Fil}^i(\widetilde{L}_F)\}_i$ (respectively $\{\mathrm{Fil}^i(H_F)\}_i$) on $\widetilde{L}_F$ (respectively H_F) by

$$\mathrm{Fil}^i(\widetilde{L}_F) := \begin{cases} 0 & i \geq 2, \\ \langle e\rangle & i = 1, \\ \langle e\rangle^{\perp} & i = 0, \\ \widetilde{L}_F & i \leq -1, \end{cases} \qquad \mathrm{Fil}^i(H_F) := \begin{cases} 0 & i \geq 1, \\ i(e)(H_F) & i = 0, \\ H_F & i \leq -1. \end{cases}$$

The following proposition immediately follows from the results of [50, Section 1]; see especially [50, 1.9].

Proposition 2.2.

(1) *The dimension of* $\mathrm{Fil}^0(H_F)$ *as an* F*-vector space is* 2^{21}.
(2) *We define a decreasing filtration* $\{\mathrm{Fil}^i(\mathrm{End}_F(H_F))\}_i$ *on* $\mathrm{End}_F(H_F)$ *by*

$$\mathrm{Fil}^i(\mathrm{End}_F(H_F)) = \{\, g \in \mathrm{End}_F(H_F) \mid g(\mathrm{Fil}^j(H_F)) \subset \mathrm{Fil}^{i+j}(H_F) \textit{ for every } j \,\}.$$

Then the homomorphism $i \otimes_{\mathbb{Z}_{(p)}} F$ *preserves the filtrations.*
(3) *The* $\mathbb{Z}/2\mathbb{Z}$*-grading structure and the right action of* $\mathrm{Cl}_{\mathbb{Z}_{(p)}}$ *preserve the filtration* $\{\mathrm{Fil}^i(H_F)\}_i$ *on* H_F*. The endomorphism* π *of* $\mathrm{End}_F(H_F)$ *preserves the filtration* $\{\mathrm{Fil}^i(\mathrm{End}_F(H_F))\}_i$ *on* $\mathrm{End}_F(H_F)$.

Proof. We sketch the proof for the convenience of the reader. We first note that there is a decomposition

$$\widetilde{L}_F = \langle e\rangle \oplus \langle f\rangle \oplus (\langle e\rangle \oplus \langle f\rangle)^{\perp}$$

such that $(e, f) = 1$ and $(f, f) = 0$. Let $v_3, \ldots, v_{22}$ be an orthogonal basis for $(\langle e\rangle \oplus \langle f\rangle)^{\perp}$. Then we have

$$H_F = \bigoplus_{a_j\in\{0,1\}} \langle e^{a_1} f^{a_2} v_3^{a_3} \ldots v_{22}^{a_{22}}\rangle$$

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by [11, §9.3, Théorème 1].

(1) Because $e^2 = 2^{-1}(e, e) = 0$ in the Clifford algebra $H_F = \mathrm{Cl}_F$, we have

$$\mathrm{Fil}^0(H_F) = \bigoplus_{a_j \in \{0,1\}} \langle e f^{a_2} v_3^{a_3} \dots v_{22}^{a_{22}} \rangle.$$

Hence, its dimension as an F-vector space is 2^{21}.

(2) The assertion immediately follows from the description of the filtrations.

(3) It is clear that the $\mathbb{Z}/2\mathbb{Z}$-grading structure and the right action of $\mathrm{Cl}_{\mathbb{Z}_{(p)}}$ preserve the filtration on H_F. We shall show that the idempotent π preserves the filtration on $\mathrm{End}_F(H_F)$.

We recall an explicit description of π from [50, Section 1]. We put $\beta_j := (v_j, v_j)^{-1}$ for $3 \le j \le 22$. Then π sends $g \in \mathrm{End}_F(H_F)$ to the element $\pi(g) \in \mathrm{End}_F(H_F)$ given by

$$\pi(g) = [g, i(f)]i(e) + [g, i(e)]i(f) + \sum_{3 \le j \le 22} \beta_j [g, i(v_j)] i(v_j).$$

Recall that we have $[g_1, g_2] = 2^{-21} \cdot \mathrm{Tr}(g_1 \circ g_2)$ for $g_1, g_2 \in \mathrm{End}_{\mathbb{Q}}(H_{\mathbb{Q}})$, and the embedding $i \otimes_{\mathbb{Z}_{(p)}} F$ preserves filtrations by (2). Hence, it is enough to show the following assertions to prove that π preserves the filtration on $\mathrm{End}_F(H_F)$:

- $\mathrm{Tr}(g \circ i(e)) = 0$ for every $g \in \mathrm{Fil}^0(\mathrm{End}_F(H_F))$.
- $\mathrm{Tr}(g \circ i(e)) = 0$ and $\mathrm{Tr}(g \circ i(v_j)) = 0$ for every $g \in \mathrm{Fil}^1(\mathrm{End}_F(H_F))$ and every $3 \le j \le 22$.

These assertions can be checked by using the explicit basis for H_F as above. □

3. Breuil-Kisin modules

In this section, we introduce some notation and recall some well-known results on Breuil-Kisin modules and integral p-adic Hodge theory, which play important roles in this work. The reader who is familiar with them may skip this section.

3.1. Preliminaries

In this section, we fix a perfect field k of characteristic $p > 0$ and an algebraic closure $\overline{k}$ of k. To simplify the notation, we put $W := W(k)$.

Let K be a finite totally ramified extension of $W[1/p]$. Let $\overline{K}$ be an algebraic closure of K. We fix a uniformiser ϖ of K and a system $\{\varpi^{1/p^n}\}_{n \ge 0} \subset \overline{K}$ of p^nth roots of ϖ such that $(\varpi^{1/p^{n+1}})^p = \varpi^{1/p^n}$. Let $E(u) \in W[u]$ denote the (monic) Eisenstein polynomial of ϖ (i.e., it is the monic minimal polynomial of ϖ over $W[1/p]$).

Let C denote the completion of $\overline{K}$. Let $\mathcal{O}_C$ denote the ring of integers of C. We put

$$\mathcal{O}_C^\flat := \varprojlim_{x \mapsto x^p} \mathcal{O}_C/p,$$

which is a perfect $\mathbb{F}_p$-algebra. It is an integral domain and the field of fractions is denoted by $C^\flat$. Write

$$A_{\mathrm{inf}} := W(\mathcal{O}_C^\flat)$$

for the ring of Witt vectors of $\mathcal{O}_C^\flat$. Let $\varphi\colon A_{\mathrm{inf}} \to A_{\mathrm{inf}}$ denote the automorphism induced by the Frobenius and the functoriality of Witt vectors. There is a unique $\mathrm{Gal}(\overline{K}/K)$-equivariant surjection

$$\theta\colon A_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_C$$

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such that its reduction modulo p is the first projection $\mathcal{O}_C^\flat \to \mathcal{O}_C/p$. We also have a surjection $A_{\mathrm{inf}} \twoheadrightarrow W(\overline{k})$ induced by the natural surjection $\mathcal{O}_C^\flat \to \overline{k}$.

We put

$$\mathfrak{S} := W[[u]].$$

The ring $\mathfrak{S}$ admits a Frobenius endomorphism φ that acts on W as the canonical Frobenius σ and sends u to u^p.

We put $\varpi^\flat := (\varpi^{1/p^n} \mod p)_{n\geq 0} \in \mathcal{O}_C^\flat$. We have a W-linear homomorphism

$$\mathfrak{S} \longrightarrow A_{\mathrm{inf}}$$

that sends u to the Teichmüller lift $[\varpi^\flat]$ of $\varpi^\flat$. This homomorphism is compatible with the Frobenius endomorphisms. Note that the composite

$$\mathfrak{S} \longrightarrow A_{\mathrm{inf}} \longrightarrow \mathcal{O}_C$$

is a W-linear homomorphism given by $u \mapsto \varpi$.

We also need p-adic period rings B_{dR}, B_{dR}^+, B_{cris} and B_{cris}^+ associated with C defined by Fontaine. For example, see [6, Subsection 3.3] for the definition and basic properties of these rings.

3.2. *Breuil-Kisin modules and crystalline Galois representations*

In this subsection, we recall some basic results on Breuil-Kisin modules and crystalline Galois representations from [40, (1.2)].

For an $\mathfrak{S}$-module $\mathfrak{M}$, we put $\varphi^* \mathfrak{M} := \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. A *Breuil-Kisin module* (over $\mathcal{O}_K$ with respect to $\{\varpi^{1/p^n}\}_{n\geq 0}$) is a free $\mathfrak{S}$-module $\mathfrak{M}$ of finite rank equipped with an isomorphism of $\mathfrak{S}[1/E(u)]$-modules

$$1 \otimes \varphi\colon (\varphi^*\mathfrak{M})[1/E(u)] \cong \mathfrak{M}[1/E(u)].$$

We say that a Breuil-Kisin module $\mathfrak{M}$ is *effective* if the equipped isomorphism $1 \otimes \varphi$ of $\mathfrak{M}$ is induced by a homomorphism

$$1 \otimes \varphi\colon \varphi^*\mathfrak{M} \longrightarrow \mathfrak{M}.$$

We say that an effective Breuil-Kisin module $\mathfrak{M}$ is of height $\leq h$ if the cokernel of $1 \otimes \varphi$ is killed by $E(u)^h$.

For a Breuil-Kisin module $\mathfrak{M}$, we define a decreasing filtration $\{\mathrm{Fil}^i(\varphi^* \mathfrak{M})\}_i$ on $\varphi^* \mathfrak{M}$ by

$$\mathrm{Fil}^i(\varphi^* \mathfrak{M}) := \{\, x \in \varphi^* \mathfrak{M} \mid (1 \otimes \varphi)(x) \in E(u)^i \mathfrak{M} \,\}$$

for every $i \in \mathbb{Z}$.

We say that a $\mathbb{Z}_p[\mathrm{Gal}(\overline{K}/K)]$-module N is a $\mathrm{Gal}(\overline{K}/K)$*-stable* $\mathbb{Z}_p$*-lattice in a crystalline representation* if N is a free $\mathbb{Z}_p$-module of finite rank and $N[1/p]$ is a crystalline $\mathrm{Gal}(\overline{K}/K)$-representation. Kisin constructed a *covariant* fully faithful tensor functor

$$N \mapsto \mathfrak{M}(N)$$

from the category of $\mathrm{Gal}(\overline{K}/K)$-stable $\mathbb{Z}_p$-lattices in crystalline representations to the category of Breuil-Kisin modules (over $\mathcal{O}_K$ with respect to $\{\varpi^{1/p^n}\}_{n\geq 0}$); see [40, Theorem 1.2.1].

In the rest of this subsection, we fix a $\mathrm{Gal}(\overline{K}/K)$-stable $\mathbb{Z}_p$-lattice N in a crystalline representation. We put

$$\mathfrak{M}_{\mathrm{dR}}(N) := \varphi^* \mathfrak{M}(N) \otimes_{\mathfrak{S}} \mathcal{O}_K,$$

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where $\mathfrak{S} \to \mathcal{O}_K$ is a W-linear homomorphism given by $u \mapsto \varpi$. The K-vector subspace of $\mathfrak{M}_{\mathrm{dR}}(N)[1/p]$ generated by the image of $\mathrm{Fil}^i(\varphi^* \mathfrak{M}(N))$ is denoted by $\mathrm{Fil}^i(\mathfrak{M}_{\mathrm{dR}}(N)[1/p])$. We have a canonical isomorphism

$$D_{\mathrm{dR}}(N[1/p]) := (N \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}})^{\mathrm{Gal}(\overline{K}/K)} \cong \mathfrak{M}_{\mathrm{dR}}(N)[1/p],$$

which maps $\mathrm{Fil}^i(\mathfrak{M}_{\mathrm{dR}}(N)[1/p])$ onto $\mathrm{Fil}^i(D_{\mathrm{dR}}(N[1/p]))$; see [40, Theorem 1.2.1]. We define a decreasing filtration $\{\mathrm{Fil}^i(\mathfrak{M}_{\mathrm{dR}}(N))\}_i$ on $\mathfrak{M}_{\mathrm{dR}}(N)$ by taking intersection

$$\mathrm{Fil}^i(\mathfrak{M}_{\mathrm{dR}}(N)) := \mathrm{Fil}^i(\mathfrak{M}_{\mathrm{dR}}(N)[1/p]) \cap \mathfrak{M}_{\mathrm{dR}}(N).$$

Note that, by the construction, the quotient

$$\mathrm{Gr}^i(\mathfrak{M}_{\mathrm{dR}}(N)) := \mathrm{Fil}^i(\mathfrak{M}_{\mathrm{dR}}(N))/\mathrm{Fil}^{i+1}(\mathfrak{M}_{\mathrm{dR}}(N))$$

is a free $\mathcal{O}_K$-module of finite rank for every $i \in \mathbb{Z}$.

Lemma 3.1. *Assume that $\mathfrak{M}(N)$ is effective. Then the image of* $\mathrm{Fil}^1(\varphi^* \mathfrak{M}(N))$ *under the surjection $\varphi^* \mathfrak{M}(N) \twoheadrightarrow \mathfrak{M}_{\mathrm{dR}}(N)$ coincides with* $\mathrm{Fil}^1(\mathfrak{M}_{\mathrm{dR}}(N))$.

Proof. Let Fil' be the image of $\mathrm{Fil}^1(\varphi^* \mathfrak{M}(N))$ under the surjection $\varphi^* \mathfrak{M}(N) \twoheadrightarrow \mathfrak{M}_{\mathrm{dR}}(N)$. It suffices to show that the cokernel of the inclusion $\mathrm{Fil}' \hookrightarrow \mathfrak{M}_{\mathrm{dR}}(N)$ is p-torsion-free. Because

$$E(u)\varphi^* \mathfrak{M}(N) \subset \mathrm{Fil}^1(\varphi^* \mathfrak{M}(N)),$$

the inverse image of Fil' under $\varphi^* \mathfrak{M}(N) \twoheadrightarrow \mathfrak{M}_{\mathrm{dR}}(N)$ is $\mathrm{Fil}^1(\varphi^* \mathfrak{M}(N))$. Hence, the assertion follows from the fact that the cokernel of $\mathrm{Fil}^1(\varphi^* \mathfrak{M}(N)) \hookrightarrow \varphi^* \mathfrak{M}(N)$ is p-torsion-free. □

We put

$$\mathfrak{M}_{\mathrm{cris}}(N) := \varphi^* \mathfrak{M}(N) \otimes_{\mathfrak{S}} W,$$

where $\mathfrak{S} \to W$ is a W-linear homomorphism given by $u \mapsto 0$. The Frobenius $1 \otimes \varphi$ of $\mathfrak{M}(N)$ defines a σ-semilinear endomorphism of $\mathfrak{M}_{\mathrm{cris}}(N)[1/p]$, which makes $\mathfrak{M}_{\mathrm{cris}}(N)[1/p]$ a φ-module. We have a canonical isomorphism of φ-modules

$$D_{\mathrm{cris}}(N[1/p]) := (N \otimes_{\mathbb{Z}_p} B_{\mathrm{cris}})^{\mathrm{Gal}(\overline{K}/K)} \cong \mathfrak{M}_{\mathrm{cris}}(N)[1/p];$$

see [40, Theorem 1.2.1].

Let N be a $\mathrm{Gal}(\overline{K}/K)$-stable free $\mathbb{Z}_p$-module of finite rank in a crystalline representation and f a $\mathrm{Gal}(\overline{K}/K)$-equivariant endomorphism of N. The endomorphism of the Breuil-Kisin module $\mathfrak{M}(N)$ induced by f is denoted by $\mathfrak{M}(f)$. The endomorphism of $\mathfrak{M}_{\mathrm{dR}}(N)$ (respectively $\mathfrak{M}_{\mathrm{cris}}(N)$) induced by f is denoted by $\mathfrak{M}_{\mathrm{dR}}(f)$ (respectively $\mathfrak{M}_{\mathrm{cris}}(f)$).

3.3. Breuil-Kisin modules and p-divisible groups

Let $\mathscr{G}$ be a p-divisible group over $\mathcal{O}_K$. Let

$$T_p\mathscr{G} := \varprojlim_n \mathscr{G}[p^n](\overline{K})$$

denote the p-adic Tate module of $\mathscr{G}$, which is a free $\mathbb{Z}_p$-module of finite rank and admits a continuous action of $\mathrm{Gal}(\overline{K}/K)$.

For the base change $\mathscr{G}_k$ of $\mathscr{G}$, we have a (contravariant) crystal $\mathbb{D}(\mathscr{G}_k)$ over $\mathrm{CRIS}(k/\mathbb{Z}_p)$; see [4, Définition 3.3.6]. Here $\mathrm{CRIS}(k/\mathbb{Z}_p)$ is the (absolute) crystalline site of k. Its value

$$\mathbb{D}(\mathscr{G}_k)(W) := \mathbb{D}(\mathscr{G}_k)_{W \twoheadrightarrow k}$$

in $(\mathrm{Spec}\, k \hookrightarrow \mathrm{Spec}\, W)$ is an F-crystal.

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We have a crystal $\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})$ over $\mathrm{CRIS}((\mathscr{O}_K/p)/\mathbb{Z}_p)$ for the base change $\mathscr{G}_{\mathscr{O}_K/p}$ of $\mathscr{G}$. Its value

$$\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(\mathscr{O}_K) := \mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})_{\mathscr{O}_K \twoheadrightarrow \mathscr{O}_K/p}$$

in $(\operatorname{Spec} \mathscr{O}_K/p \hookrightarrow \operatorname{Spec} \mathscr{O}_K)$ is a free $\mathscr{O}_K$-module of finite rank and admits a Hodge filtration

$$\mathrm{Fil}^1 \mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(\mathscr{O}_K) \hookrightarrow \mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(\mathscr{O}_K).$$

By [5, Proposition 3.14], there is an isomorphism over K:

$$\mathbb{D}(\mathscr{G}_k)(W) \otimes_W K \cong \mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(\mathscr{O}_K) \otimes_{\mathscr{O}_K} K.$$

Using this isomorphism, we consider $\mathbb{D}(\mathscr{G}_k)(W)[1/p]$ as a filtered φ-module. (For example, see [39, 1.1.3] for the notion of filtered φ-modules.)

In [29, Theorem 7], Faltings constructed a $\mathrm{Gal}(\overline{K}/K)$-equivariant isomorphism

$$T_p\mathscr{G}[1/p] \cong \mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathbb{D}(\mathscr{G}_k)(W)[1/p], B^+_{\mathrm{cris}}).$$

(See also [31, Subsection 5.2, Remarque 2] and Subsection 11.4.) It induces an isomorphism of filtered φ-modules

$$c_{\mathscr{G}} \colon D_{\mathrm{cris}}((T_p\mathscr{G})^\vee[1/p]) \cong \mathbb{D}(\mathscr{G}_k)(W)[1/p].$$

There are integral refinements of them:

(1) The composite

$$\mathfrak{M}_{\mathrm{cris}}((T_p\mathscr{G})^\vee)[1/p] \cong D_{\mathrm{cris}}((T_p\mathscr{G})^\vee[1/p]) \overset{c_{\mathscr{G}}}{\cong} \mathbb{D}(\mathscr{G}_k)(W)[1/p]$$

maps $\mathfrak{M}_{\mathrm{cris}}((T_p\mathscr{G})^\vee)$ onto $\mathbb{D}(\mathscr{G}_k)(W)$.

(2) The composite

$$\mathfrak{M}_{\mathrm{dR}}((T_p\mathscr{G})^\vee)[1/p] \cong D_{\mathrm{dR}}((T_p\mathscr{G})^\vee[1/p]) \overset{c_{\mathscr{G}}}{\cong} \mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(\mathscr{O}_K) \otimes_{\mathscr{O}_K} K$$

maps $\mathfrak{M}_{\mathrm{dR}}((T_p\mathscr{G})^\vee)$ onto $\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(\mathscr{O}_K)$ and maps $\mathrm{Fil}^1(\mathfrak{M}_{\mathrm{dR}}((T_p\mathscr{G})^\vee))$ onto $\mathrm{Fil}^1 \mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(\mathscr{O}_K)$.

These results were proved by Kisin when $p > 2$; see [40, Theorem 1.4.2]. The general case follows from the results of Lau [44, 45] as explained in [38, Theorem 2.12]. (See also Remark 11.11.)

3.4. *Integral p-adic Hodge theory*

In this subsection, we recall integral comparison theorems proved by Bhatt-Morrow-Scholze [6]. (See also Subsections 11.1 and 11.2.) Although the results of [6] can be applied in more general cases (including the semistable case [16]), we only recall their results for $K3$ surfaces with good reduction for simplicity.

Let $\mathscr{X}$ be a $K3$ surface over $\mathscr{O}_K$. It is possibly an algebraic space, not necessarily a scheme. We remark that the generic fibre $\mathscr{X}_K$ and the special fibre $\mathscr{X}_k$ are both schemes because a smooth proper algebraic space of dimension 2 over a field is a scheme. We refer to Subsection 11.2 for details on how to apply the results of [6] to the proper smooth algebraic space $\mathscr{X}$ over $\mathscr{O}_K$. We will freely use GAGA results implicitly below.

By [6, Theorem 14.6 (i)], we have the following B_{cris}-linear isomorphism

$$c_{\mathrm{cris},\mathscr{X}} \colon H^2_{\mathrm{cris}}(\mathscr{X}_k/W) \otimes_W B_{\mathrm{cris}} \xrightarrow{\cong} H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{cris}},$$

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which is compatible with the action of $\mathrm{Gal}(\overline{K}/K)$ and the Frobenius endomorphisms. By its construction and [6, Theorem 13.1], this is compatible with the following filtered B_{dR}-linear isomorphism constructed in [62, Theorem 8.4],

$$c_{\mathrm{dR},\mathscr{X}_K} : H^2_{\mathrm{dR}}(\mathscr{X}_K/K) \otimes_K B_{\mathrm{dR}} \xrightarrow{\cong} H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}},$$

which is $\mathrm{Gal}(\overline{K}/K)$-equivariant. More precisely, $c_{\mathrm{cris},\mathscr{X}}$ and $c_{\mathrm{dR},\mathscr{X}_K}$ are compatible via the Berthelot-Ogus isomorphism

$$H^2_{\mathrm{cris}}(\mathscr{X}_k/W) \otimes_W K \cong H^2_{\mathrm{dR}}(\mathscr{X}_K/K);$$

see Proposition 11.5.

The isomorphism $c_{\mathrm{dR},\mathscr{X}_K}$ induces an isomorphism

$$D_{\mathrm{dR}}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p)[1/p]) \cong H^2_{\mathrm{dR}}(\mathscr{X}_K/K),$$

which is also denoted by $c_{\mathrm{dR},\mathscr{X}_K}$. Similarly, the isomorphism $c_{\mathrm{cris},\mathscr{X}}$ induces an isomorphism

$$D_{\mathrm{cris}}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p)[1/p]) \cong H^2_{\mathrm{cris}}(\mathscr{X}_k/W)[1/p],$$

which is also denoted by $c_{\mathrm{cris},\mathscr{X}}$.

Let $\mathfrak{M}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p))$ denote the Breuil-Kisin module (over $\mathscr{O}_K$ with respect to $\{\varpi^{1/p^n}\}_{n\geq 0}$) associated with $H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p)$. We have the composite of the following filtered isomorphisms:

$$\begin{aligned}\mathfrak{M}_{\mathrm{dR}}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p))[1/p] &\cong D_{\mathrm{dR}}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p)[1/p]) \\ &\cong H^2_{\mathrm{dR}}(\mathscr{X}_K/K).\end{aligned}$$

We also have the composite of the following isomorphisms of φ-modules:

$$\begin{aligned}\mathfrak{M}_{\mathrm{cris}}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p))[1/p] &\cong D_{\mathrm{cris}}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p)[1/p]) \\ &\cong H^2_{\mathrm{cris}}(\mathscr{X}_k/W)[1/p].\end{aligned}$$

As in the case of p-divisible groups, there are integral refinements of them.

Theorem 3.2 (Bhatt-Morrow-Scholze [6])**.**

(1) *The isomorphism*

$$\mathfrak{M}_{\mathrm{dR}}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p))[1/p] \cong H^2_{\mathrm{dR}}(\mathscr{X}_K/K)$$

maps $\mathfrak{M}_{\mathrm{dR}}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p))$ *isomorphically onto* $H^2_{\mathrm{dR}}(\mathscr{X}/\mathscr{O}_K)$.

(2) *The isomorphism*

$$\mathfrak{M}_{\mathrm{cris}}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p))[1/p] \cong H^2_{\mathrm{cris}}(\mathscr{X}_k/W)[1/p]$$

maps $\mathfrak{M}_{\mathrm{cris}}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p))$ *isomorphically onto* $H^2_{\mathrm{cris}}(\mathscr{X}_k/W)$.

Proof. (1) Because this part was not stated explicitly in [6], we shall explain how to deduce it from the results of [6].

First recall that

$$\varphi^* \mathfrak{M}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p)) \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$$

naturally becomes a *Breuil-Kisin-Fargues module* in the sense of [6, Definition 4.22]. Under Fargues's equivalence [6, Theorem 4.28], this corresponds to a B_{dR}^+-lattice

$$\begin{aligned} D_{\mathrm{dR}}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}},\mathbb{Z}_p)[1/p]) \otimes_K B_{\mathrm{dR}}^+ &= (H^2_{\text{ét}}(\mathscr{X}_{\overline{K}},\mathbb{Z}_p)\otimes_{\mathbb{Z}_p} B_{\mathrm{dR}})^{\mathrm{Gal}(\overline{K}/K)} \otimes_K B_{\mathrm{dR}}^+ \\ &\subset H^2_{\text{ét}}(\mathscr{X}_{\overline{K}},\mathbb{Z}_p)\otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}. \end{aligned}$$

Note that the following isomorphism

$$\begin{aligned} D_{\mathrm{dR}}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}},\mathbb{Z}_p)[1/p]) &\cong (\varphi^*\mathfrak{M}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}},\mathbb{Z}_p))\otimes_{\mathfrak{S}} C)^{\mathrm{Gal}(\overline{K}/K)} \\ &\cong \mathfrak{M}_{\mathrm{dR}}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}},\mathbb{Z}_p))[1/p] \end{aligned}$$

obtained by the reduction modulo ξ is the same as the one given before.

The above Breuil-Kisin-Fargues module is realised as the A_{inf}-valued cohomology constructed in [6]. In [6], Bhatt-Morrow-Scholze constructed the A_{inf}-valued cohomology $H^2_{A_{\mathrm{inf}}}(\mathscr{X})$ and showed that other cohomology theories can be obtained from it. (Here, $H^2_{A_{\mathrm{inf}}}(\mathscr{X})$ denotes the A_{inf}-cohomology of the completion of $\mathscr{X}_{\mathscr{O}_C}$.) In our case, $H^2_{A_{\mathrm{inf}}}(\mathscr{X})$ is a free A_{inf}-module of rank 22. By [6, Theorem 14.3], $H^2_{A_{\mathrm{inf}}}(\mathscr{X})$ is equipped with an A_{inf}-linear Frobenius isomorphism

$$1\otimes\varphi\colon (A_{\mathrm{inf}}\otimes_{\varphi,A_{\mathrm{inf}}} H^2_{A_{\mathrm{inf}}}(\mathscr{X}))[1/\varphi(\xi)] \cong H^2_{A_{\mathrm{inf}}}(\mathscr{X})[1/\varphi(\xi)],$$

where ξ is a generator of the kernel of the surjection $\theta\colon A_{\mathrm{inf}}\twoheadrightarrow \mathscr{O}_C$. Hence, it is a Breuil-Kisin-Fargues module. By [6, Theorem 14.6], it is isomorphic to the Breuil-Kisin-Fargues module associated with $H^2_{\text{ét}}(\mathscr{X}_{\overline{K}},\mathbb{Z}_p)$ using $c_{\mathrm{dR},\mathscr{X}_K}$. Indeed, by [6, Theorem 14.3(iv)], there is an isomorphism (actually, there are two constructions giving the same map; see [16, Proposition 6.8]):

$$H^2_{A_{\mathrm{inf}}}(\mathscr{X})\otimes_{A_{\mathrm{inf}}} B_{\mathrm{dR}} \cong H^2_{\text{ét}}(\mathscr{X}_{\overline{K}},\mathbb{Z}_p)\otimes_{\mathbb{Z}_p} B_{\mathrm{dR}},$$

and $H^2_{A_{\mathrm{inf}}}(\mathscr{X})$ is determined, via Fargues's equivalence, by a B_{dR}^+-lattice

$$c_{\mathrm{dR},\mathscr{X}_K}(H^2_{\mathrm{dR}}(\mathscr{X}_K/K)\otimes_K B_{\mathrm{dR}}^+) \subset H^2_{\text{ét}}(\mathscr{X}_{\overline{K}},\mathbb{Z}_p)\otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}^+,$$

which is exactly

$$D_{\mathrm{dR}}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}},\mathbb{Z}_p)[1/p])\otimes_K B_{\mathrm{dR}}^+.$$

Therefore, there is an isomorphism

$$\varphi^*\mathfrak{M}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}},\mathbb{Z}_p))\otimes_{\mathfrak{S}} A_{\mathrm{inf}} \cong H^2_{A_{\mathrm{inf}}}(\mathscr{X})$$

making the following diagram commutative:

$$\begin{array}{ccc} H^2_{\mathrm{dR}}(\mathscr{X}_K/K)\otimes_K C & \xrightarrow{\ \cong\ } & H^2_{A_{\mathrm{inf}}}(\mathscr{X})\otimes_{A_{\mathrm{inf}}} C \\ \cong\big\downarrow & & \big\uparrow\cong \\ D_{\mathrm{dR}}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}},\mathbb{Z}_p)[1/p])\otimes_K C & \xrightarrow{\ \cong\ } & \mathfrak{M}_{\mathrm{dR}}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}},\mathbb{Z}_p))\otimes_{\mathscr{O}_K} C \end{array}$$

and $\mathfrak{M}_{\mathrm{dR}}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}},\mathbb{Z}_p))\otimes_{\mathscr{O}_K}\mathscr{O}_C$ is identified with $H^2_{A_{\mathrm{inf}}}(\mathscr{X})\otimes_{A_{\mathrm{inf}}}\mathscr{O}_C$. The inverse of the composite of the left vertical arrow and the bottom arrow is the map appearing in the statement of (1).

By [6, Theorem 14.3(ii)], we have an isomorphism

$$H^2_{A_{\mathrm{inf}}}(\mathscr{X})\otimes_{A_{\mathrm{inf}}}\mathscr{O}_C \cong H^2_{\mathrm{dR}}(\mathscr{X}/\mathscr{O}_K)\otimes_{\mathscr{O}_K}\mathscr{O}_C.$$

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It is compatible with the top arrow in the above diagram; this is essentially checked in the proof of [6, Theorem 14.1]. See also [16, Proposition 5.41, Theorem 6.6].

To complete the proof of (1), it now suffices to remark that two $\mathscr{O}_K$-lattices in a K-vector space are the same if and only if the corresponding two $\mathscr{O}_C$-lattices are the same in the C-vector space obtained by extension of scalars.

(2) See [6, Theorem 14.6(iii)]. We briefly recall the arguments for the convenience of the reader. By [6, Theorem 14.5(i), Theorem 12.1], there is an isomorphism

$$H^2_{A_{\mathrm{inf}}}(\mathscr{X}) \otimes_{A_{\mathrm{inf}}} B_{\mathrm{cris}} \cong H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{cris}}$$

that underlies the following isomorphism used in the proof of (1):

$$H^2_{A_{\mathrm{inf}}}(\mathscr{X}) \otimes_{A_{\mathrm{inf}}} B_{\mathrm{dR}} \cong H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}.$$

This induces an isomorphism

$$H^2_{A_{\mathrm{inf}}}(\mathscr{X}) \otimes_{A_{\mathrm{inf}}} B^+_{\mathrm{cris}} \cong D_{\mathrm{cris}}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p)[1/p]) \otimes_{W[1/p]} B^+_{\mathrm{cris}}$$

such that the composite

$$\begin{aligned} \varphi^* \mathfrak{M}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p)) \otimes_{\mathfrak{S}} B^+_{\mathrm{cris}} &\cong H^2_{A_{\mathrm{inf}}}(\mathscr{X}) \otimes_{A_{\mathrm{inf}}} B^+_{\mathrm{cris}} \\ &\cong D_{\mathrm{cris}}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p)[1/p]) \otimes_{W[1/p]} B^+_{\mathrm{cris}} \end{aligned}$$

is the canonical map. Namely, the map obtained by the specialisation

$$\begin{aligned} D_{\mathrm{cris}}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p)[1/p]) &\cong (\mathfrak{M}_{\mathrm{cris}}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p)) \otimes_W W(\overline{k})[1/p])^{\mathrm{Gal}(\overline{K}/K)} \\ &\cong \mathfrak{M}_{\mathrm{cris}}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p))[1/p] \end{aligned}$$

is equal to the canonical map given before.

Moreover, by the construction of $c_{\mathrm{cris},\mathscr{X}}$, the following isomorphism [6, Theorem 14.3(i)]

$$H^2_{A_{\mathrm{inf}}}(\mathscr{X}) \otimes_{A_{\mathrm{inf}}} W(\overline{k}) \cong H^2_{\mathrm{cris}}(\mathscr{X}_k/W) \otimes_W W(\overline{k})$$

induces $c_{\mathrm{cris},\mathscr{X}}$ by the specialisation

$$\begin{aligned} D_{\mathrm{cris}}(H^2_{\text{ét}}(\mathscr{X}_{\overline{K}}, \mathbb{Z}_p)[1/p]) &\cong (H^2_{\mathrm{cris}}(\mathscr{X}_k/W) \otimes_W W(\overline{k})[1/p])^{\mathrm{Gal}(\overline{K}/K)} \\ &\cong H^2_{\mathrm{cris}}(\mathscr{X}_k/W)[1/p]. \end{aligned}$$

Now a reasoning similar to that used at the end of the proof of (1) implies (2). □

4. Shimura varieties

In this section, we recall basic results on Shimura varieties and their integral models associated with general spin groups and special orthogonal groups. We follow Madapusi Pera's paper [50] for orthogonal Shimura varieties. (For integral models of more general Shimura varieties of abelian type, see Kisin's paper [40]. For the construction of 2-adic integral canonical models, see also [38].)

In this section, we use the same notation as in previous sections. Recall that we fixed the embedding of quadratic spaces $L \subset \widetilde{L}$ as in Lemma 2.1.

4.1. *Orthogonal Shimura varieties over* $\mathbb{Q}$

Let $X_{\widetilde{L}}$ denote the symmetric domain of oriented negative definite planes in $\widetilde{L}_{\mathbb{R}}$. We have Shimura data $(\widetilde{G}_{0,\mathbb{Q}}, X_{\widetilde{L}})$ and $(\widetilde{G}_{\mathbb{Q}}, X_{\widetilde{L}})$. Each of them has the reflex field $\mathbb{Q}$; see [1, Appendix 1, Lemma] for example. The canonical homomorphism $\widetilde{G} \to \widetilde{G}_0$ induces a morphism of Shimura data $(\widetilde{G}_{\mathbb{Q}}, X_{\widetilde{L}}) \to (\widetilde{G}_{0,\mathbb{Q}}, X_{\widetilde{L}})$.

We put $\widetilde{\mathrm{K}}_{0,p} := \widetilde{G}_0(\mathbb{Z}_p)$ (respectively $\widetilde{\mathrm{K}}_p := \widetilde{G}(\mathbb{Z}_p)$), which is a hyperspecial subgroup. Let $\widetilde{\mathrm{K}}_0^p \subset \widetilde{G}_0(\mathbb{A}_f^p)$ (respectively $\widetilde{\mathrm{K}}^p \subset \widetilde{G}(\mathbb{A}_f^p)$) be an open compact subgroup and $\widetilde{\mathrm{K}}_0 := \widetilde{\mathrm{K}}_{0,p}\widetilde{\mathrm{K}}_0^p \subset \widetilde{G}_0(\mathbb{A}_f)$ (respectively $\widetilde{\mathrm{K}} := \widetilde{\mathrm{K}}_p\widetilde{\mathrm{K}}^p \subset \widetilde{G}(\mathbb{A}_f)$). We have the Shimura varieties

$$\mathrm{Sh}_{\widetilde{\mathrm{K}}_0} := \mathrm{Sh}_{\widetilde{\mathrm{K}}_0}(\widetilde{G}_{0,\mathbb{Q}}, X_{\widetilde{L}}) \quad \text{and} \quad \mathrm{Sh}_{\widetilde{\mathrm{K}}} := \mathrm{Sh}_{\widetilde{\mathrm{K}}}(\widetilde{G}_{\mathbb{Q}}, X_{\widetilde{L}})$$

associated with the Shimura data $(\widetilde{G}_{0,\mathbb{Q}}, X_{\widetilde{L}})$ and $(\widetilde{G}_{\mathbb{Q}}, X_{\widetilde{L}})$, respectively. We assume that $\widetilde{\mathrm{K}}_0^p$ and $\widetilde{\mathrm{K}}^p$ are small enough so that $\mathrm{Sh}_{\widetilde{\mathrm{K}}_0}$ and $\mathrm{Sh}_{\widetilde{\mathrm{K}}}$ are smooth quasi-projective schemes over $\mathbb{Q}$. Moreover, we assume that the image of $\widetilde{\mathrm{K}}^p$ under the homomorphism $\widetilde{G} \to \widetilde{G}_0$ is $\widetilde{\mathrm{K}}_0^p$. Then we have a finite étale morphism over $\mathbb{Q}$:

$$\mathrm{Sh}_{\widetilde{\mathrm{K}}} \to \mathrm{Sh}_{\widetilde{\mathrm{K}}_0}.$$

We also consider the reductive group $\mathrm{SO}(L_{\mathbb{Q}})$ over $\mathbb{Q}$. Let X_L denote the symmetric domain of oriented negative definite planes in $L_{\mathbb{R}}$. We have a Shimura datum $(\mathrm{SO}(L_{\mathbb{Q}}), X_L)$ and a morphism of Shimura data: $(\mathrm{SO}(L_{\mathbb{Q}}), X_L) \to (\widetilde{G}_{0,\mathbb{Q}}, X_{\widetilde{L}})$. Let $\mathrm{K}_{0,p} \subset \mathrm{SO}(L_{\mathbb{Q}})(\mathbb{Q}_p)$ be the maximal subgroup that stabilises $L_{\mathbb{Z}_p}$ and acts on $L^{\vee}_{\mathbb{Z}_p}/L_{\mathbb{Z}_p}$ trivially. Let $\mathrm{K}_0^p \subset \mathrm{SO}(L_{\mathbb{Q}})(\mathbb{A}_f^p)$ be an open compact subgroup that stabilises $L_{\widehat{\mathbb{Z}}^p}$ and acts on $L^{\vee}_{\widehat{\mathbb{Z}}^p}/L_{\widehat{\mathbb{Z}}^p}$ trivially. We assume that K_0^p is small enough so that it is contained in $\widetilde{\mathrm{K}}_0^p$ and the associated Shimura variety $\mathrm{Sh}_{\mathrm{K}_0}(\mathrm{SO}(L_{\mathbb{Q}}), X_L)$ is a smooth quasi-projective variety over $\mathbb{Q}$, where $\mathrm{K}_0 := \mathrm{K}_{0,p}\,\mathrm{K}_0^p$. Note that we have $\mathrm{K}_{0,p} \subset \widetilde{\mathrm{K}}_{0,p}$; see the proof of [50, Lemma 2.6]. Hence, we have a morphism of Shimura varieties over $\mathbb{Q}$:

$$\mathrm{Sh}_{\mathrm{K}_0}(\mathrm{SO}(L_{\mathbb{Q}}), X_L) \to \mathrm{Sh}_{\widetilde{\mathrm{K}}_0}.$$

4.2. *Symplectic embeddings of general spin groups*

By [50, Lemma 3.6], there is a nondegenerate alternating bilinear form

$$\psi\colon H \times H \to \mathbb{Z}$$

satisfying the following properties:

(1) The left multiplication induces a closed embedding of algebraic groups over $\mathbb{Q}$,

$$\widetilde{G}_{\mathbb{Q}} \hookrightarrow \mathrm{GSp} := \mathrm{GSp}(H_{\mathbb{Q}}, \psi_{\mathbb{Q}}).$$

(2) The left multiplication induces a morphism of Shimura data

$$(\widetilde{G}_{\mathbb{Q}}, X_{\widetilde{L}}) \to (\mathrm{GSp}, S^{\pm}),$$

where $S^{\pm}$ denotes the Siegel double spaces associated with the symplectic space $(H_{\mathbb{Q}}, \psi)$.

Let $\mathrm{K}'^p \subset \mathrm{GSp}(\mathbb{A}_f^p)$ be an open compact subgroup containing the image of $\widetilde{\mathrm{K}}^p$. Let $\mathrm{K}'_p \subset \mathrm{GSp}(\mathbb{Q}_p)$ be the stabiliser of $H_{\mathbb{Z}_p}$. We put $\mathrm{K}' := \mathrm{K}'_p\mathrm{K}'^p$. After replacing K'^p and $\widetilde{\mathrm{K}}^p$ by their open compact subgroups, we may assume that the associated Shimura variety $\mathrm{Sh}_{\mathrm{K}'}(\mathrm{GSp}, S^{\pm})$ is a smooth quasi-projective scheme over $\mathbb{Q}$ and the morphism of Shimura data $(\widetilde{G}_{\mathbb{Q}}, X_{\widetilde{L}}) \to (\mathrm{GSp}, S^{\pm})$ induces the

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following morphism of Shimura varieties over $\mathbb{Q}$:

$$\mathrm{Sh}_{\widetilde{\mathrm{K}}} \to \mathrm{Sh}_{\mathrm{K}'}(\mathrm{GSp}, S^{\pm}).$$

Let us summarise our situation by the following commutative diagram of algebraic groups over $\mathbb{Q}$:

$$\begin{array}{ccccc} \mathrm{GSpin}(L_{\mathbb{Q}}) & \hookrightarrow & \widetilde{G}_{\mathbb{Q}} & \hookrightarrow & \mathrm{GSp} \\ \downarrow & & \downarrow & & \\ \mathrm{SO}(L_{\mathbb{Q}}) & \hookrightarrow & \widetilde{G}_{0,\mathbb{Q}}. & & \end{array}$$

We also have the corresponding diagram of Shimura varieties over $\mathbb{Q}$:

$$\begin{array}{ccccc} & & \mathrm{Sh}_{\widetilde{\mathrm{K}}} = \mathrm{Sh}_{\widetilde{\mathrm{K}}}(\widetilde{G}_{\mathbb{Q}}, X_{\widetilde{L}}) & \longrightarrow & \mathrm{Sh}_{\mathrm{K}'}(\mathrm{GSp}, S^{\pm}) \\ & & \downarrow & & \\ \mathrm{Sh}_{\mathrm{K}_0}(\mathrm{SO}(L_{\mathbb{Q}}), X_L) & \longrightarrow & \mathrm{Sh}_{\widetilde{\mathrm{K}}_0} = \mathrm{Sh}_{\widetilde{\mathrm{K}}_0}(\widetilde{G}_{0,\mathbb{Q}}, X_{\widetilde{L}}). & & \end{array}$$

4.3. *Integral canonical models and the Kuga-Satake abelian scheme*

Because $\widetilde{\mathrm{K}}_p \subset \widetilde{G}(\mathbb{Q}_p)$ and $\widetilde{\mathrm{K}}_{0,p} \subset \widetilde{G}_0(\mathbb{Q}_p)$ are hyperspecial subgroups, the Shimura varieties $\mathrm{Sh}_{\widetilde{\mathrm{K}}}$ and $\mathrm{Sh}_{\widetilde{\mathrm{K}}_0}$ admit the integral canonical models $\mathscr{S}_{\widetilde{\mathrm{K}}}$ and $\mathscr{S}_{\widetilde{\mathrm{K}}_0}$ over $\mathbb{Z}_{(p)}$, respectively. (This result is proved by Kisin when $p \neq 2$ [40] and by Kim-Madapusi Pera when $p = 2$ [38]. The integral canonical models are characterised by the extension properties. See [40] for details.)

By the construction of $\mathscr{S}_{\widetilde{\mathrm{K}}_0}$, the morphism $\mathrm{Sh}_{\widetilde{\mathrm{K}}} \to \mathrm{Sh}_{\widetilde{\mathrm{K}}_0}$ extends to a finite étale morphism $\mathscr{S}_{\widetilde{\mathrm{K}}} \to \mathscr{S}_{\widetilde{\mathrm{K}}_0}$ over $\mathbb{Z}_{(p)}$.

Let $m := |H^{\vee}_{\mathbb{Z}}/H_{\mathbb{Z}}|$ denote the discriminant of $H_{\mathbb{Z}}$. We put

$$g := (\dim_{\mathbb{Q}} H_{\mathbb{Q}})/2 = 2^{21}.$$

Let $\mathscr{A} := \mathscr{A}_{g,m,\mathrm{K}'}$ be the moduli space over $\mathbb{Z}_{(p)}$ of triples $(A, \lambda, \epsilon'^p)$ consisting of an abelian scheme A of dimension g, a polarisation $\lambda \colon A \to A^*$ of degree m and a K'^p-level structure ϵ'^p. For a sufficiently small K'^p, this is represented by a quasi-projective scheme over $\mathbb{Z}_{(p)}$. We have a canonical open and closed immersion

$$\mathrm{Sh}_{\mathrm{K}'}(\mathrm{GSp}, S^{\pm}) \hookrightarrow \mathscr{A}_{\mathbb{Q}}$$

over $\mathbb{Q}$. Hence, we have a morphism $\mathrm{Sh}_{\widetilde{\mathrm{K}}} \to \mathscr{A}_{\mathbb{Q}}$ over $\mathbb{Q}$. By the construction of $\mathscr{S}_{\widetilde{\mathrm{K}}}$, this morphism extends to a morphism $\mathscr{S}_{\widetilde{\mathrm{K}}} \to \mathscr{A}$ over $\mathbb{Z}_{(p)}$; see [40, (2.3.3)], [38, Section 4.4].

In summary, we have the following diagram of schemes over $\mathbb{Z}_{(p)}$:

$$\begin{array}{ccc} \mathscr{S}_{\widetilde{\mathrm{K}}} & \longrightarrow & \mathscr{A} \\ \downarrow & & \\ \mathscr{S}_{\widetilde{\mathrm{K}}_0}. & & \end{array}$$

Let $\mathcal{A}_{\mathscr{S}_{\widetilde{\mathrm{K}}}} \to \mathscr{S}_{\widetilde{\mathrm{K}}}$ be the abelian scheme corresponding to the morphism $\mathscr{S}_{\widetilde{\mathrm{K}}} \to \mathscr{A}$. The abelian scheme $\mathcal{A}_{\mathscr{S}_{\widetilde{\mathrm{K}}}}$ is called the *Kuga-Satake abelian scheme*. We often drop the subscript $\mathscr{S}_{\widetilde{\mathrm{K}}}$ in the notation. For every $\mathscr{S}_{\widetilde{\mathrm{K}}}$-scheme S, the pullback of $\mathcal{A}_{\mathscr{S}_{\widetilde{\mathrm{K}}}}$ to S is denoted by $\mathcal{A}_S$.

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Remark 4.1. If the discriminant of the quadratic space L is divisible by p, we do not yet have a satisfactory theory of integral canonical models of the Shimura varieties $\mathrm{Sh}_{\mathrm{K}}(\mathrm{GSpin}(L_{\mathbb{Q}}), X_L)$ and $\mathrm{Sh}_{\mathrm{K}_0}(\mathrm{SO}(L_{\mathbb{Q}}), X_L)$ associated with L. (The open compact subgroup $\mathrm{K}_{0,p} \subset \mathrm{SO}(L_{\mathbb{Q}})(\mathbb{Q}_p)$ may not be hyperspecial.) Following Madapusi Pera [49, 50, 38], we embed L into $\widetilde{L}$ whose discriminant is not divisible by p as in Lemma 2.1 and use the integral canonical models $\mathscr{S}_{\widetilde{\mathrm{K}}}$ and $\mathscr{S}_{\widetilde{\mathrm{K}}_0}$ associated with $\widetilde{L}$.

4.4. Local systems on Shimura varieties

In this subsection, we introduce some (complex analytic, ℓ-adic and p-adic) local systems on orthogonal Shimura varieties. For details, see [50, 38].

The $\widetilde{G}$-representation $\widetilde{L}_{\mathbb{Z}_{(p)}}$ and the $\widetilde{G}$-equivariant embedding

$$i\colon \widetilde{L}_{\mathbb{Z}_{(p)}} \hookrightarrow \mathrm{End}_{\mathbb{Z}_{(p)}}(H_{\mathbb{Z}_{(p)}})$$

induce the following objects:

- A $\mathbb{Q}$-local system $\widetilde{\mathbb{V}}_B$ over the complex analytic space $\mathrm{Sh}^{\mathrm{an}}_{\widetilde{\mathrm{K}},\mathbb{C}}$ and an embedding of $\mathbb{Q}$-local systems:

$$i_B\colon \widetilde{\mathbb{V}}_B \hookrightarrow \underline{\mathrm{End}}(H_B^{\vee}).$$

  Here H_B is the relative first singular cohomology with coefficients in $\mathbb{Q}$ of $\mathcal{A}^{\mathrm{an}}_{\mathrm{Sh}_{\widetilde{\mathrm{K}},\mathbb{C}}}$ over $\mathrm{Sh}^{\mathrm{an}}_{\widetilde{\mathrm{K}},\mathbb{C}}$, and $H_B^{\vee}$ is its dual.
- An $\mathbb{A}_f^p$-local system $\widetilde{\mathbb{V}}^p$ over the integral canonical model $\mathscr{S}_{\widetilde{\mathrm{K}}}$ and an embedding of $\mathbb{A}_f^p$-local systems:

$$i^p\colon \widetilde{\mathbb{V}}^p \hookrightarrow \underline{\mathrm{End}}(V^p\mathcal{A}).$$

  Here we put

$$V^p\mathcal{A} := (\varprojlim_{p\nmid n} \mathcal{A}[n]) \otimes_{\mathbb{Z}} \mathbb{Q}$$

  and consider it as an $\mathbb{A}_f^p$-local system over $\mathscr{S}_{\widetilde{\mathrm{K}}}$.
- A $\mathbb{Z}_p$-local system $\widetilde{\mathbb{L}}_p$ over the Shimura variety $\mathscr{S}_{\widetilde{\mathrm{K}},\mathbb{Q}} = \mathrm{Sh}_{\widetilde{\mathrm{K}}}$ and an embedding of $\mathbb{Z}_p$-local systems:

$$i_p\colon \widetilde{\mathbb{L}}_p \hookrightarrow \underline{\mathrm{End}}(T_p\mathcal{A}_{\mathrm{Sh}_{\widetilde{\mathrm{K}}}}).$$

  Here $T_p\mathcal{A}_{\mathrm{Sh}_{\widetilde{\mathrm{K}}}}$ is the p-adic Tate module of $\mathcal{A}_{\mathrm{Sh}_{\widetilde{\mathrm{K}}}}$ over $\mathrm{Sh}_{\widetilde{\mathrm{K}}}$.

4.5. Hodge tensors

Recall that we fixed tensors $\{s_\alpha\} \subset H^{\otimes}_{\mathbb{Z}_{(p)}}$ defining the closed embedding $\widetilde{G} \hookrightarrow \mathrm{GL}(H_{\mathbb{Z}_{(p)}})$ over $\mathbb{Z}_{(p)}$; see Subsection 2.3. The tensors $\{s_\alpha\} \subset H^{\otimes}_{\mathbb{Z}_{(p)}}$ give rise to global sections $\{s_{\alpha,B}\}$ of $H_B^{\otimes}$, global sections $\{s_\alpha^p\}$ of $(V^p\mathcal{A})^{\otimes}$ and global sections $\{s_{\alpha,p}\}$ of $(T_p\mathcal{A})^{\otimes}$.

We recall properties of these tensors. (See [41, (1.3.6)], [38, Proposition 4.10] for details.)

(1) Let k be a field of characteristic 0. For every $x \in \mathscr{S}_{\widetilde{\mathrm{K}},\mathbb{Q}}(k)$ and a geometric point $\overline{x} \in \mathscr{S}_{\widetilde{\mathrm{K}},\mathbb{Q}}(\overline{k})$ above x, the stalk $\widetilde{\mathbb{L}}_{p,\overline{x}}$ at $\overline{x}$ is equipped with an even perfect bilinear form $(\ ,\ )$ over $\mathbb{Z}_p$. The bilinear form is $\mathrm{Gal}(\overline{k}/k)$-invariant; that is, we have

$$(gy_1, gy_2) = (y_1, y_2)$$

for every $y_1, y_2 \in \widetilde{\mathbb{L}}_{p,\overline{x}}$ and every $g \in \mathrm{Gal}(\overline{k}/k)$.

We identify $T_p(\mathcal{A}_{\overline{x}})$ with $H^1_{\text{ét}}(\mathcal{A}_{\overline{x}}, \mathbb{Z}_p)^\vee$. The $\text{Gal}(\overline{k}/k)$-module $\widetilde{\mathbb{L}}_{p,\overline{x}}$ and the homomorphism $i_{p,\overline{x}}$ are characterised by the property that there is an isomorphism of $\mathbb{Z}_p$-modules

$$H_{\mathbb{Z}_p} \cong H^1_{\text{ét}}(\mathcal{A}_{\overline{x}}, \mathbb{Z}_p)^\vee$$

that carries $\{s_\alpha\}$ to $\{s_{\alpha,p,\overline{x}}\}$ and induces the commutative diagram

$$\begin{array}{ccc} \widetilde{L}_{\mathbb{Z}_p} & \xrightarrow{\ i\ } & \text{End}_{\mathbb{Z}_p}(H_{\mathbb{Z}_p}) \\ \cong \downarrow & & \downarrow \cong \\ \widetilde{\mathbb{L}}_{p,\overline{x}} & \xrightarrow{i_{p,\overline{x}}} & \text{End}_{\mathbb{Z}_p}(H^1_{\text{ét}}(\mathcal{A}_{\overline{x}}, \mathbb{Z}_p)^\vee), \end{array}$$

where $\widetilde{L}_{\mathbb{Z}_p} \cong \widetilde{\mathbb{L}}_{p,\overline{x}}$ is an isometry over $\mathbb{Z}_p$. (We will often drop the subscript $\overline{x}$ of $i_{p,\overline{x}}$.)

(2) Let k a field of characteristic 0 or p. For every $x \in \mathcal{S}_{\widetilde{\mathrm{K}}}(k)$ and a geometric point $\overline{x} \in \mathcal{S}_{\widetilde{\mathrm{K}}}(\overline{k})$ above x, the stalk $\widetilde{\mathbb{V}}^p_{\overline{x}}$ at $\overline{x}$ has a bilinear form (,) over $\mathbb{A}^p_f$ satisfying the same property as above with $\mathbb{Z}_p$ replaced by $\mathbb{A}^p_f$.

(3) For every $x \in \mathcal{S}_{\widetilde{\mathrm{K}}}(\mathbb{C})$, the stalk $\widetilde{\mathbb{V}}_{B,x}$ at x has a bilinear form (,) over $\mathbb{Q}$ satisfying the same property as above with $\mathbb{Z}_p$ replaced by $\mathbb{Q}$.

4.6. *F-crystals and Breuil-Kisin modules*

In this subsection, let k be a perfect field of characteristic $p > 0$ and $x \in \mathcal{S}_{\widetilde{\mathrm{K}},\mathbb{F}_p}(k)$. We recall basic results on F-crystals attached to $x \in \mathcal{S}_{\widetilde{\mathrm{K}},\mathbb{F}_p}(k)$. (For details, see [40, 41, 50, 38].)

The $\widetilde{G}$-representation $\widetilde{L}_{\mathbb{Z}_{(p)}}$ and the $\widetilde{G}$-equivariant embedding

$$i\colon \widetilde{L}_{\mathbb{Z}_{(p)}} \hookrightarrow \text{End}_{\mathbb{Z}_{(p)}}(H_{\mathbb{Z}_{(p)}})$$

induce a free W-module $\widetilde{L}_{\text{cris},x}$ of finite rank and an embedding

$$i_{\text{cris}}\colon \widetilde{L}_{\text{cris},x} \hookrightarrow \text{End}_W(H^1_{\text{cris}}(\mathcal{A}_x/W)^\vee).$$

The $W[1/p]$-vector space $\widetilde{L}_{\text{cris},x}[1/p]$ has the structure of an F-isocrystal. Namely, it is equipped with a Frobenius automorphism φ. The embedding i_{cris} becomes an embedding of F-isocrystals after inverting p.

There is an even perfect bilinear form on $\widetilde{L}_{\text{cris},x}$. When p is inverted, we have

$$(\varphi(y_1), \varphi(y_2)) = \sigma(y_1, y_2)$$

for every $y_1, y_2 \in \widetilde{L}_{\text{cris},x}[1/p]$.

We recall the definition of $\widetilde{L}_{\text{cris},x}$ and i_{cris} when k is a finite field $\mathbb{F}_q$ or $\overline{\mathbb{F}}_q$. Take a lift $\widetilde{x} \in \mathcal{S}_{\widetilde{\mathrm{K}}}(\mathcal{O}_K)$ of x, where K is a finite totally ramified extension of $W[1/p]$ and $\mathcal{O}_K$ is its valuation ring. Note that such a lift exists by [40, Proposition 2.3.5], [38, Proposition 4.6]. Let $\eta = \text{Spec}\, K$ denote the generic point of $\text{Spec}\, \mathcal{O}_K$ and $\overline{\eta}$ a geometric point above η.

We fix a uniformiser ϖ of K and a system $\{\varpi^{1/p^n}\}_{n\geq 0}$ of p^nth roots of ϖ such that $(\varpi^{1/p^{n+1}})^p = \varpi^{1/p^n}$. Let

$$\mathfrak{M}(H^1_{\text{ét}}(\mathcal{A}_{\overline{\eta}}, \mathbb{Z}_p))$$

be the Breuil-Kisin module (over $\mathcal{O}_K$ with respect to $\{\varpi^{1/p^n}\}_{n\geq 0}$) associated with the $\text{Gal}(\overline{K}/K)$-stable $\mathbb{Z}_p$-lattice $H^1_{\text{ét}}(\mathcal{A}_{\overline{\eta}}, \mathbb{Z}_p)$ in a crystalline representation. (For the definition of Breuil-Kisin modules, see Subsection 3.2. See also [40, (1.2)].)

Because $\mathfrak{M}(-)$ is a tensor functor, the tensors $\{s_{\alpha,p,\overline{\eta}}\}$ in $(T_p\mathcal{A}_{\overline{\eta}})^{\otimes}$ give rise to Frobenius invariant tensors $\{\mathfrak{M}(s_{\alpha,p,\overline{\eta}})\}$ in $\mathfrak{M}(H^1_{\text{ét}}(\mathcal{A}_{\overline{\eta}},\mathbb{Z}_p))^{\otimes}$ and Frobenius invariant tensors $\{\mathfrak{M}_{\text{cris}}(s_{\alpha,p,\overline{\eta}})\}$ in $\mathfrak{M}_{\text{cris}}(H^1_{\text{ét}}(\mathcal{A}_{\overline{\eta}},\mathbb{Z}_p))^{\otimes}$. By [40, Corollary 1.4.3], [38, Theorem 2.12], we have a canonical isomorphism

$$\mathfrak{M}_{\text{cris}}(H^1_{\text{ét}}(\mathcal{A}_{\overline{\eta}},\mathbb{Z}_p)) \cong H^1_{\text{cris}}(\mathcal{A}_x/W).$$

(See also Proposition 11.12.) Using this canonical isomorphism, the tensors $\{\mathfrak{M}_{\text{cris}}(s_{\alpha,p,\overline{\eta}})\}$ give rise to Frobenius invariant tensors $\{s_{\alpha,\text{cris},x}\}$ in $H^1_{\text{cris}}(\mathcal{A}_x/W)^{\otimes}$. The tensors $\{s_{\alpha,\text{cris},x}\}$ do not depend on the choice of the lift $\widetilde{x} \in \mathcal{S}_{\widetilde{K}}(\mathcal{O}_K)$ of x; see [41, Proposition 1.3.9]. (See also the last paragraph in the proof of [38, Proposition 4.6].)

Kisin proved the following results.

Proposition 4.2 (Kisin [40, Proposition 1.3.4, Corollary 1.3.5])**.** *There is an isomorphism of $\mathfrak{S}$-modules*

$$H^1_{\text{ét}}(\mathcal{A}_{\overline{\eta}},\mathbb{Z}_p)^{\vee} \otimes_{\mathbb{Z}_p} \mathfrak{S} \cong \mathfrak{M}(H^1_{\text{ét}}(\mathcal{A}_{\overline{\eta}},\mathbb{Z}_p)^{\vee})$$

that carries $\{s_{\alpha,p,\overline{\eta}}\}$ to $\{\mathfrak{M}(s_{\alpha,p,\overline{\eta}})\}$ and induces the following commutative diagram:

$$\begin{array}{ccc}
\widetilde{\mathbb{L}}_{p,\overline{\eta}} \otimes_{\mathbb{Z}_p} \mathfrak{S} & \xrightarrow{\ i_p\ } & \mathrm{End}_{\mathfrak{S}}(H^1_{\text{ét}}(\mathcal{A}_{\overline{\eta}},\mathbb{Z}_p)^{\vee} \otimes_{\mathbb{Z}_p} \mathfrak{S}) \\
{\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\
\mathfrak{M}(\widetilde{\mathbb{L}}_{p,\overline{\eta}}) & \xrightarrow{\ \mathfrak{M}(i_p)\ } & \mathrm{End}_{\mathfrak{S}}(\mathfrak{M}(H^1_{\text{ét}}(\mathcal{A}_{\overline{\eta}},\mathbb{Z}_p)^{\vee})),
\end{array}$$

where $\widetilde{\mathbb{L}}_{p,\overline{\eta}} \otimes_{\mathbb{Z}_p} \mathfrak{S} \cong \mathfrak{M}(\widetilde{\mathbb{L}}_{p,\overline{\eta}})$ is an isometry over $\mathfrak{S}$.

Proof. The assertion follows from [40, Proposition 1.3.4, Corollary 1.3.5]. Precisely, the statements in [40, Proposition 1.3.4, Corollary 1.3.5] do not claim the existence of the commutative diagram above, but the same argument works. □

We define

$$\widetilde{L}_{\text{cris},x} := \mathfrak{M}_{\text{cris}}(\widetilde{\mathbb{L}}_{p,\overline{\eta}}) \quad \text{and} \quad i_{\text{cris}} := \mathfrak{M}_{\text{cris}}(i_p).$$

The even perfect bilinear form on $\widetilde{\mathbb{L}}_{p,\overline{\eta}}$ induces an even perfect bilinear form on $\widetilde{L}_{\text{cris},x}$. By (1) in Subsection 4.5 and Proposition 4.2, there is an isomorphism of W-modules

$$H_W \cong H^1_{\text{cris}}(\mathcal{A}_x/W)^{\vee}$$

that carries $\{s_\alpha\}$ to $\{s_{\alpha,\text{cris},x}\}$ and induces the following commutative diagram:

$$\begin{array}{ccc}
\widetilde{L}_W & \xrightarrow{\ \ i\ \ } & \mathrm{End}_W(H_W) \\
{\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\
\widetilde{L}_{\text{cris},x} & \xrightarrow{\ i_{\text{cris}}\ } & \mathrm{End}_W(H^1_{\text{cris}}(\mathcal{A}_x/W)^{\vee}),
\end{array}$$

where $\widetilde{L}_W \cong \widetilde{L}_{\text{cris},x}$ is an isometry over W. It follows that $\widetilde{L}_{\text{cris},x}$ and i_{cris} do not depend on the choice of the lift $\widetilde{x} \in \mathcal{S}_{\widetilde{K}}(\mathcal{O}_K)$ of x.

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4.7. Λ-structures for integral canonical models

Recall that we have fixed an embedding of quadratic spaces $L \hookrightarrow \widetilde{L}$. Let $\Lambda := L^{\perp} \subset \widetilde{L}$ denote the orthogonal complement of L in $\widetilde{L}$ and $\iota\colon \Lambda \hookrightarrow \widetilde{L}$ the natural inclusion.

We recall the definition of Λ-structures from [50].

Definition 4.3 (see [50, Definition 6.11])**.** A Λ-*structure* for an $\mathscr{S}_{\widetilde{\mathrm{K}}}$-scheme S is a homomorphism of $\mathbb{Z}_{(p)}$-modules

$$\iota_S\colon \Lambda_{\mathbb{Z}_{(p)}} \to \mathrm{End}_S(\mathcal{A}_S)_{\mathbb{Z}_{(p)}}$$

satisfying the following properties:

- For any algebraically closed field $\overline{K}$ of characteristic 0 and $x \in S(\overline{K})$, there is an isometry $\iota_p\colon \Lambda_{\mathbb{Z}_p} \to \widetilde{\mathbb{L}}_{p,x}$ over $\mathbb{Z}_p$ such that the composite
$$\Lambda_{\mathbb{Z}_p} \xrightarrow{\iota_p} \widetilde{\mathbb{L}}_{p,x} \xrightarrow{i_p} \mathrm{End}_{\mathbb{Z}_p}(T_p\mathcal{A}_x)$$
coincides with the map induced by ι_S.
- For any perfect field k of characteristic p and $x \in S(k)$, there is an isometry $\iota_{\mathrm{cris}}\colon \Lambda_W \to \widetilde{L}_{\mathrm{cris},x}$ over W such that the composite
$$\Lambda_W \xrightarrow{\iota_{\mathrm{cris}}} \widetilde{L}_{\mathrm{cris},x} \xrightarrow{i_{\mathrm{cris}}} \mathrm{End}_W(H^1_{\mathrm{cris}}(\mathcal{A}_x/W)^{\vee})$$
coincides with the map induced by ι_S.

It turns out that these conditions imply the following:

- By [50, Corollary 5.22], for every geometric point $x \to S$, there is an isometry $\iota^p\colon \Lambda_{\mathbb{A}^p_f} \to \widetilde{\mathbb{V}}^p_x$ over $\mathbb{A}^p_f$ such that the composite
$$\Lambda_{\mathbb{A}^p_f} \xrightarrow{\iota^p} \widetilde{\mathbb{V}}^p_x \xrightarrow{i^p} \mathrm{End}_{\mathbb{A}^p_f}(V^p(\mathcal{A}_x))$$
coincides with the map induced by ι_S.
- For every $\mathbb{C}$-valued point $x \in S(\mathbb{C})$, there is an isometry $\iota_B\colon \Lambda_{\mathbb{Q}} \to \widetilde{\mathbb{V}}_{B,x}$ over $\mathbb{Q}$ such that the composite
$$\Lambda_{\mathbb{Q}} \xrightarrow{\iota_B} \widetilde{\mathbb{V}}_{B,x} \xrightarrow{i_B} \mathrm{End}_{\mathbb{Q}}(H^1_B(\mathcal{A}_x,\mathbb{Q})^{\vee})$$
coincides with the map induced by ι_S.

We recall the definition of a K^p-level structure. Here $\mathrm{K}^p \subset \mathrm{GSpin}(L_{\mathbb{Q}})(\mathbb{A}^p_f)$ is an open compact subgroup whose image under the homomorphism

$$\mathrm{GSpin}(L_{\mathbb{Q}})(\mathbb{A}^p_f) \to \mathrm{SO}(L_{\mathbb{Q}})(\mathbb{A}^p_f)$$

is K^p_0.

Let S be an $\mathscr{S}_{\widetilde{\mathrm{K}}}$-scheme. For simplicity, we assume that S is locally Noetherian and connected. Let ϵ' be the corresponding K'^p-level structure on $\mathcal{A}_S$; as in [40, (3.2.4)], for a geometric point $s \to S$, the K'^p-level structure ϵ' is induced by a $\pi_1(S,s)$-invariant $\widetilde{\mathrm{K}}^p$-orbit $\widetilde{\epsilon}$ of an isometry $H_{\mathbb{A}^p_f} \cong V^p(\mathcal{A}_s)$ over $\mathbb{A}^p_f$ that carries $\{s_\alpha\}$ to $\{s^p_{\alpha,s}\}$ and carries $\widetilde{L}_{\mathbb{A}^p_f}$ to $\widetilde{\mathbb{V}}^p_s$. Here $\pi_1(S,s)$ denotes the étale fundamental group of S, and the Tate module $V^p(\mathcal{A}_s)$ over $\mathbb{A}^p_f$ has a natural action of $\pi_1(S,s)$.

Definition 4.4. Let S be a locally Noetherian connected scheme over $\mathscr{S}_{\widetilde{\mathrm{K}}}$. Let $s \to S$ be a geometric point. A K^p-*level structure* on (S,ι_S) is a $\pi_1(S,s)$-invariant K^p-orbit ϵ_ι of an isometry of

$\mathbb{A}^p_f$-modules

$$H_{\mathbb{A}^p_f} \cong V^p(\mathcal{A}_s)$$

satisfying the following properties:

(1) It carries $\{s_\alpha\}$ to $\{s^p_{\alpha,s}\}$.
(2) The following diagram is commutative:

$$\begin{array}{ccccc}
\Lambda_{\mathbb{A}^p_f} & \xrightarrow{\iota} & \widetilde{L}_{\mathbb{A}^p_f} & \xrightarrow{i} & \mathrm{End}_{\mathbb{A}^p_f}(H_{\mathbb{A}^p_f}) \\
& \searrow^{\iota^p} & \downarrow \cong & & \downarrow \cong \\
& & \widetilde{\mathbb{V}}^p_s & \xrightarrow{i^p} & \mathrm{End}_{\mathbb{A}^p_f}(V^p(\mathcal{A}_s)).
\end{array}$$

(3) The K^p-orbit ϵ_ι induces the $\widetilde{\mathrm{K}}^p$-orbit $\widetilde{\epsilon}$ on S.

Definition 4.5. Let $Z_{\mathrm{K}^p}(\Lambda)$ be the functor on $\mathcal{S}_{\widetilde{\mathrm{K}}}$-schemes defined by

$$Z_{\mathrm{K}^p}(\Lambda)(S) := \{\, (\iota_S, \epsilon_\iota) \mid \iota_S \text{ is a } \Lambda\text{-structure and } \epsilon_\iota \text{ is a } \mathrm{K}^p\text{-level structure on } (S, \iota_S) \,\}$$

for an $\mathcal{S}_{\widetilde{\mathrm{K}}}$-scheme S.

Similarly, we can define a Λ-structure $\iota_{0,S}$ for an $\mathcal{S}_{\widetilde{\mathrm{K}}_0}$-scheme S, a K^p_0-level structure on $(S, \iota_{0,S})$ and a functor $Z_{\mathrm{K}^p_0}(\Lambda)$. (See [50, Definition 6.11] for details.)

The following result was proved by Madapusi Pera.

Proposition 4.6 (Madapusi Pera [50]). *The functor $Z_{\mathrm{K}^p}(\Lambda)$ (respectively $Z_{\mathrm{K}^p_0}(\Lambda)$) is represented by a scheme that is finite and unramified over $\mathcal{S}_{\widetilde{\mathrm{K}}}$ (respectively $\mathcal{S}_{\widetilde{\mathrm{K}}_0}$). Moreover, there is a natural morphism*

$$Z_{\mathrm{K}^p}(\Lambda) \to Z_{\mathrm{K}^p_0}(\Lambda),$$

which is finite and étale.

Proof. See [50, Proposition 6.13]. □

5. Moduli spaces of $K3$ surfaces and the Kuga-Satake morphism

In this section, we recall definitions and basic properties of the moduli space of $K3$ surfaces. Then we recall definitions and basic results on the Kuga-Satake morphism over $\mathbb{Z}_{(p)}$ introduced by Madapusi Pera [49, 38].

5.1. *Moduli spaces of $K3$ surfaces*

Recall that we say $f\colon \mathcal{X} \to S$ is a $K3$ *surface over* S if S is a scheme, $\mathcal{X}$ is an algebraic space and f is a proper smooth morphism whose geometric fibres are $K3$ surfaces.

A *quasi-polarisation* of $f\colon \mathcal{X} \to S$ is a section $\xi \in \underline{\mathrm{Pic}}(\mathcal{X}/S)(S)$ of the relative Picard functor whose fibre $\xi(s)$ at every geometric point $s \to S$ is a line bundle on the $K3$ surface $\mathcal{X}_s$ that is nef and big. We say $\xi \in \underline{\mathrm{Pic}}(\mathcal{X}/S)(S)$ is *primitive* if, for every geometric point $s \to S$, the cokernel of the inclusion $\langle \xi(s) \rangle \hookrightarrow \mathrm{Pic}(\mathcal{X}_s)$ is torsion-free. We say ξ has degree $2d$ if, for every geometric point $s \to S$, we have $(\xi(s), \xi(s)) = 2d$, where $(\ ,\ )$ denotes the intersection pairing on $\mathcal{X}_s$. We say a pair $(f\colon \mathcal{X} \to S, \xi)$ is a *quasi-polarised* $K3$ *surface over* S *of degree* $2d$ if $f\colon \mathcal{X} \to S$ is a $K3$ surface over S and $\xi \in \underline{\mathrm{Pic}}(\mathcal{X}/S)(S)$ is a primitive quasi-polarisation of degree $2d$.

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Let M_{2d} be the moduli functor that sends a scheme S to the groupoid consists of quasi-polarised $K3$ surfaces over S of degree $2d$. The moduli functor M_{2d} is a Deligne-Mumford stack of finite type over $\mathbb{Z}$; see [60, Theorem 4.3.4] and [52, Proposition 2.1].

We put $M_{2d,\mathbb{Z}_{(p)}} := M_{2d} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Let S be an $M_{2d,\mathbb{Z}_{(p)}}$-scheme. For the quasi-polarised $K3$ surface $(f\colon \mathscr{X} \to S, \xi)$ associated with the structure morphism $S \to M_{2d,\mathbb{Z}_{(p)}}$ and a prime number $\ell \neq p$, we equip $R^2 f_* \mathbb{Z}_\ell(1)$ with the *negative* of the cup product pairing. Let

$$P^2 f_* \mathbb{Z}_\ell(1) := \mathrm{ch}_\ell(\xi)^\perp \subset R^2 f_* \mathbb{Z}_\ell(1)$$

denote the orthogonal complement of the ℓ-adic Chern class $\mathrm{ch}_\ell(\xi) \in R^2 f_* \mathbb{Z}_\ell(1)(S)$ with respect to the pairing. We set

$$P^2 f_* \widehat{\mathbb{Z}}^p(1) := \prod_{\ell \neq p} P^2 f_* \mathbb{Z}_\ell(1).$$

The stalk of $P^2 f_* \mathbb{Z}_\ell(1)$ (respectively $P^2 f_* \widehat{\mathbb{Z}}^p(1)$) at a geometric point $s \to S$ will be denoted by $P^2_{\text{ét}}(\mathscr{X}_s, \mathbb{Z}_\ell(1))$ (respectively $P^2_{\text{ét}}(\mathscr{X}_s, \widehat{\mathbb{Z}}^p(1))$).

Let $M^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$ denote the smooth locus of $M_{2d,\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$. Madapusi Pera constructed a twofold finite étale cover $\widetilde{M}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}} \to M^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$ parametrising orientations of $P^2 f_* \widehat{\mathbb{Z}}^p(1)$ that satisfies the following property. For every morphism $S \to \widetilde{M}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$, there is a natural isometry of $\widehat{\mathbb{Z}}^p$-local systems on S

$$\nu\colon \underline{\det L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p} \cong \det P^2 f_* \widehat{\mathbb{Z}}^p(1)$$

such that, for every $s \in S(\mathbb{C})$, the isometry ν restricts to an isometry over $\mathbb{Z}$,

$$\nu_s\colon\ \det L \cong \det P^2_B(\mathscr{X}_s, \mathbb{Z}(1)),$$

under the canonical isomorphism

$$P^2_B(\mathscr{X}_s, \mathbb{Z}(1)) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p \cong P^2_{\text{ét}}(\mathscr{X}_s, \widehat{\mathbb{Z}}^p(1)),$$

where we put $P^2_B(\mathscr{X}_s, \mathbb{Z}(1)) := \mathrm{ch}_B(\xi(s))^\perp \subset H^2_B(\mathscr{X}_s, \mathbb{Z}(1))$. See [49, Section 5] for details.

For an open compact subgroup $\mathrm{K}^p_0 \subset \mathrm{SO}(L_{\mathbb{Q}})(\mathbb{A}^p_f)$ as in Subsection 4.1, we recall the notion of (oriented) K^p_0-level structures from [49, Section 3]. For simplicity, we only consider the case S is a locally Noetherian connected $\widetilde{M}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$-scheme. Let $s \to S$ be a geometric point and $\pi_1(S, s)$ the étale fundamental group of S. A K^p_0-*level structure* on $(f\colon \mathscr{X} \to S, \xi)$ is a $\pi_1(S, s)$-invariant K^p_0-orbit η of an isometry over $\widehat{\mathbb{Z}}^p$,

$$\Lambda_{K3} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p \cong H^2_{\text{ét}}(\mathscr{X}_s, \widehat{\mathbb{Z}}^p(1)),$$

that carries $e - df$ to $\mathrm{ch}_{\widehat{\mathbb{Z}}^p}(\xi(s))$ such that the induced isometry

$$\det L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p \cong \det P^2_{\text{ét}}(\mathscr{X}_s, \widehat{\mathbb{Z}}^p(1))$$

coincides with ν_s. Here the étale cohomology $H^2_{\text{ét}}(\mathscr{X}_s, \widehat{\mathbb{Z}}^p(1))$ has a natural action of $\pi_1(S, s)$, and $\Lambda_{K3} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p$ has a natural action of K^p_0; see [50, Lemma 2.6].

Let $M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Z}_{(p)}}$ be the moduli functor over $\widetilde{M}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$ that sends an $\widetilde{M}^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$-scheme S to the set of (oriented) K^p_0-level structures on the quasi-polarised $K3$ surface $(f\colon \mathscr{X} \to S, \xi)$.

The following result is well known.

Proposition 5.1. *If $\mathrm{K}^p_0 \subset \mathrm{SO}(L_{\mathbb{Q}})(\mathbb{A}^p_f)$ is small enough, $M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Z}_{(p)}}$ is an algebraic space over $\mathbb{Z}_{(p)}$ that is finite, étale and faithfully flat over $M^{\mathrm{sm}}_{2d,\mathbb{Z}_{(p)}}$.*

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Proof. This result was essentially proved by Rizov [60, Theorem 6.2.2], Maulik [52, Proposition 2.8] and Madapusi Pera [49, Proposition 3.11]. Note that their proofs work in every characteristic p, including $p = 2$. Their proofs rely on the injectivity of the map

$$\mathrm{Aut}(X) \to \mathrm{GL}(H^2_{\text{ét}}(X, \mathbb{Q}_\ell))$$

for every $\ell \neq p$, where X is a $K3$ surface over an algebraically closed field of characteristic $p > 0$. The injectivity was proved by Ogus when $p > 2$; see [56, Corollary 2.5]. (Precisely, Ogus proved it for the crystalline cohomology. The injectivity for the ℓ-adic cohomology follows from Ogus's results; see [60, Proposition 3.4.2].) Recently, Keum proved that the injectivity holds also when $p = 2$; see [37, Theorem 1.4]. □

We will assume that an open compact subgroup $\mathrm{K}_0^p \subset \mathrm{SO}(L_{\mathbb{Q}})(\mathbb{A}_f^p)$ as in Subsection 4.1 is small enough so that Proposition 5.1 can be applied.

5.2. *The Kuga-Satake morphism*

Rizov and Madapusi Pera defined the following étale morphism over $\mathbb{Q}$:

$$M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Q}} \to \mathrm{Sh}_{\mathrm{K}_0}(\mathrm{SO}(L_{\mathbb{Q}}), X_L).$$

It is called the *Kuga-Satake morphism* over $\mathbb{Q}$. (See [61, Theorem 3.9.1], [49, Corollary 5.4] for details.)

Because $M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Z}_{(p)}}$ is smooth over $\mathbb{Z}_{(p)}$, the composite

$$M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Q}} \to \mathrm{Sh}_{\mathrm{K}_0}(\mathrm{SO}(L_{\mathbb{Q}}), X_L) \to \mathrm{Sh}_{\widetilde{\mathrm{K}}_0}$$

extends to a morphism over $\mathbb{Z}_{(p)}$,

$$M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Z}_{(p)}} \to \mathscr{S}_{\widetilde{\mathrm{K}}_0},$$

by the extension properties of the integral canonical model $\mathscr{S}_{\widetilde{\mathrm{K}}_0}$; see [40, (2.3.7)], [49, Proposition 5.7].

The following results are proved by Madapusi Pera.

Proposition 5.2 (Madapusi Pera). *There is a natural étale $\mathscr{S}_{\widetilde{\mathrm{K}}_0}$-morphism*

$$\mathrm{KS}\colon M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Z}_{(p)}} \to Z_{\mathrm{K}_0^p}(\Lambda).$$

Proof. The morphism $\mathrm{Sh}_{\mathrm{K}_0}(\mathrm{SO}(L_{\mathbb{Q}}), X_L) \to \mathrm{Sh}_{\widetilde{\mathrm{K}}_0}$ factors through the generic fibre $Z_{\mathrm{K}_0^p}(\Lambda)_{\mathbb{Q}}$ of $Z_{\mathrm{K}_0^p}(\Lambda)$; see [50, 6.15] for details. Hence, we have a morphism over $\mathbb{Q}$,

$$M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Q}} \to Z_{\mathrm{K}_0^p}(\Lambda)_{\mathbb{Q}}.$$

This morphism extends to a $\mathscr{S}_{\widetilde{\mathrm{K}}_0}$-morphism

$$\mathrm{KS}\colon M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Z}_{(p)}} \to Z_{\mathrm{K}_0^p}(\Lambda)$$

by [30, Chapter I, Proposition 2.7].

For the étaleness of the morphism KS, see the proof of [38, Proposition A.12]. See also Remark 6.9. □

One usually calls the morphism KS in Proposition 5.2 a period map and calls a morphism from the moduli space of $K3$ surfaces to the moduli space of abelian varieties a Kuga-Satake morphism. In this article, following Madapusi Pera, we call the morphism KS a *Kuga-Satake morphism* for convenience.

https://doi.org/10.1017/fms.2021.24 Published online by Cambridge University Press

Summarising the above, we have the following commutative diagram of algebraic spaces over $\mathbb{Z}_{(p)}$:

$$\begin{array}{ccccc} & & Z_{\mathrm{K}^p}(\Lambda) & \longrightarrow & \mathscr{S}_{\widetilde{\mathrm{K}}} \\ & & \downarrow & & \downarrow \\ M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Z}_{(p)}} & \xrightarrow{\mathrm{KS}} & Z_{\mathrm{K}_0^p}(\Lambda) & \longrightarrow & \mathscr{S}_{\widetilde{\mathrm{K}}_0}. \end{array}$$

Here $\mathscr{S}_{\widetilde{\mathrm{K}}}$ (respectively $\mathscr{S}_{\widetilde{\mathrm{K}}_0}$) is the integral canonical model of the Shimura variety associated with $\widetilde{G} = \mathrm{GSpin}(\widetilde{L}_{\mathbb{Z}_{(p)}})$ (respectively $\widetilde{G}_0 = \mathrm{SO}(\widetilde{L}_{\mathbb{Z}_{(p)}})$), and $Z_{\mathrm{K}^p}(\Lambda)$ (respectively $Z_{\mathrm{K}_0^p}(\Lambda)$) is the scheme over $\mathscr{S}_{\widetilde{\mathrm{K}}}$ (respectively $\mathscr{S}_{\widetilde{\mathrm{K}}_0}$) as in Definition 4.5 and Proposition 4.6.

6. F-crystals on Shimura varieties

In this section, let k be a perfect field of characteristic $p > 0$. We shall study the W-module $\widetilde{L}_{\mathrm{cris},s}$ associated with a k-valued point $s \in \mathscr{S}_{\widetilde{\mathrm{K}}}(k)$ of the integral canonical model $\mathscr{S}_{\widetilde{\mathrm{K}}}$.

We use the same notation as in Subsection 3.1. In particular, $W := W(k)$, $\mathfrak{S} := W[[u]]$, K is a finite totally ramified extension of $W[1/p]$ and $\overline{K}$ is an algebraic closure of K. We fix a uniformiser ϖ of K and a system $\{\varpi^{1/p^n}\}_{n\geq 0}$ of p^nth roots of ϖ such that $(\varpi^{1/p^{n+1}})^p = \varpi^{1/p^n}$. Let $E(u) \in W[u]$ denote the (monic) Eisenstein polynomial of ϖ.

6.1. The primitive cohomology of quasi-polarised $K3$ surfaces

Let $t \in M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Z}_{(p)}}(\mathscr{O}_K)$ be an $\mathscr{O}_K$-valued point and $(\mathscr{Y}, \xi)$ the quasi-polarised $K3$ surface over $\mathscr{O}_K$ of degree $2d$ associated with t. We assume that ξ is a line bundle.

The orthogonal complement with respect to the cup product of the first Chern class of ξ in the de Rham cohomology is denoted by

$$P^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K) := \mathrm{ch}_{\mathrm{dR}}(\xi)^{\perp} \subset H^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K).$$

We define $P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}}, \mathbb{Z}_p(1))$ and $P^2_{\mathrm{cris}}(\mathscr{Y}_k/W)$ similarly.

We shall recall some well-known properties of the de Rham cohomology of $K3$ surfaces; see [24] for details. The de Rham cohomology $H^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K)$ admits the Hodge filtration $\mathrm{Fil}^i_{\mathrm{Hdg}}$:

$$0 = \mathrm{Fil}^3_{\mathrm{Hdg}} \subset \mathrm{Fil}^2_{\mathrm{Hdg}} \subset \mathrm{Fil}^1_{\mathrm{Hdg}} \subset \mathrm{Fil}^0_{\mathrm{Hdg}} = H^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K).$$

The graded piece $\mathrm{Gr}^i_{\mathrm{Hdg}} := \mathrm{Fil}^i_{\mathrm{Hdg}}/\mathrm{Fil}^{i+1}_{\mathrm{Hdg}}$ is a free $\mathscr{O}_K$-module for every i. The Hodge filtration $\mathrm{Fil}^i_{\mathrm{Hdg}}$ on $H^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K)$ is mapped onto the Hodge filtration $\overline{\mathrm{Fil}}^i_{\mathrm{Hdg}}$ on $H^2_{\mathrm{dR}}(\mathscr{Y}_k/k)$ under the canonical isomorphism

$$H^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K) \otimes_{\mathscr{O}_K} k \cong H^2_{\mathrm{dR}}(\mathscr{Y}_k/k).$$

Moreover, $\mathrm{Fil}^2_{\mathrm{Hdg}}$ is the orthogonal complement of $\mathrm{Fil}^1_{\mathrm{Hdg}}$ with respect to the cup product on $H^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K)$. In other words, the cup product induces an isomorphism of $\mathscr{O}_K$-modules

$$\mathrm{Gr}^2_{\mathrm{Hdg}} \cong (\mathrm{Gr}^0_{\mathrm{Hdg}})^{\vee}.$$

We define a decreasing filtration $\mathrm{Fil}^i_{\mathrm{pr}}$ on $P^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K)$ by

$$\mathrm{Fil}^i_{\mathrm{pr}} := \mathrm{Fil}^i_{\mathrm{Hdg}} \cap P^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K).$$

We put $\mathrm{Gr}^i_{\mathrm{pr}} := \mathrm{Fil}^i_{\mathrm{pr}}/\mathrm{Fil}^{i+1}_{\mathrm{pr}}$.

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Lemma 6.1. *The natural homomorphisms* $\mathrm{Gr}^0_{pr} \to \mathrm{Gr}^0_{\mathrm{Hdg}}$ *and* $\mathrm{Gr}^2_{pr} \to \mathrm{Gr}^2_{\mathrm{Hdg}}$ *are isomorphisms. In particular, the cup product induces an isomorphism of* $\mathscr{O}_K$*-modules:*

$$\mathrm{Gr}^2_{pr} \cong (\mathrm{Gr}^0_{pr})^\vee.$$

Proof. Because the first Chern class $\mathrm{ch}_{\mathrm{dR}}(\xi)$ is contained in $\mathrm{Fil}^1_{\mathrm{Hdg}}$, we have $\mathrm{Fil}^2_{\mathrm{Hdg}} \subset P^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K)$ and $\mathrm{Gr}^2_{\mathrm{pr}} \cong \mathrm{Gr}^2_{\mathrm{Hdg}}$.

We shall show $\mathrm{Gr}^0_{\mathrm{pr}} \cong \mathrm{Gr}^0_{\mathrm{Hdg}}$. By the definition of $\mathrm{Fil}^i_{\mathrm{pr}}$, the homomorphism $\mathrm{Gr}^0_{\mathrm{pr}} \to \mathrm{Gr}^0_{\mathrm{Hdg}}$ is injective. To prove the surjectivity, it suffices to prove that it is surjective after taking the reduction modulo ϖ. Because both $\mathrm{Gr}^0_{\mathrm{pr}}$ and $\mathrm{Gr}^0_{\mathrm{Hdg}}$ are free $\mathscr{O}_K$-modules of rank 1, it suffices to show that $\mathrm{Gr}^0_{\mathrm{pr}} \otimes_{\mathscr{O}_K} k \to \mathrm{Gr}^0_{\mathrm{Hdg}} \otimes_{\mathscr{O}_K} k$ is injective.

Because ξ is primitive, the cokernel of the map $\langle \mathrm{ch}_{\mathrm{dR}}(\xi)\rangle \hookrightarrow H^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K)$ is p-torsion-free by [56, Corollary 1.4], and the following composite

$$H^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K) \cong H^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K)^\vee \to \langle \mathrm{ch}_{\mathrm{dR}}(\xi)\rangle^\vee$$

is surjective, where the first isomorphism is obtained by the Poincaré duality and the second homomorphism is the restriction map. Thus, we have the following split exact sequence:

$$0 \to P^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K) \to H^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K) \to \langle \mathrm{ch}_{\mathrm{dR}}(\xi)\rangle^\vee \to 0.$$

It follows that

$$P^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K) \otimes_{\mathscr{O}_K} k \cong P^2_{\mathrm{dR}}(\mathscr{Y}_k/k) := \mathrm{ch}_{\mathrm{dR}}(\xi_k)^\perp \subset H^2_{\mathrm{dR}}(\mathscr{Y}_k/k).$$

Hence, it is enough to show that the injection

$$\mathrm{Fil}^1_{\mathrm{pr}} \otimes_{\mathscr{O}_K} k \hookrightarrow \overline{\mathrm{Fil}}^1_{\mathrm{Hdg}} \cap P^2_{\mathrm{dR}}(\mathscr{Y}_k/k)$$

is an isomorphism of k-vector spaces. It suffices to show that the intersection $\overline{\mathrm{Fil}}^1_{\mathrm{Hdg}} \cap P^2_{\mathrm{dR}}(\mathscr{Y}_k/k)$ is a 20-dimensional k-vector space. If the dimension is greater than or equal to 21, we have $\overline{\mathrm{Fil}}^1_{\mathrm{Hdg}} \cap P^2_{\mathrm{dR}}(\mathscr{Y}_k/k) = \overline{\mathrm{Fil}}^1_{\mathrm{Hdg}}$ and

$$\mathrm{ch}_{\mathrm{dR}}(\xi_k) \in (\overline{\mathrm{Fil}}^1_{\mathrm{Hdg}})^\perp = \overline{\mathrm{Fil}}^2_{\mathrm{Hdg}}.$$

Then, by the proof of [56, Proposition 2.2.1], this implies that the versal deformation space of $(\mathscr{Y}_k, \xi_k)$ is not regular. This contradicts our assumption that $(\mathscr{Y}_k, \xi_k)$ lies on the smooth locus $M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Z}_{(p)}}$. □

We consider the Breuil-Kisin module

$$\mathfrak{M}(P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}}, \mathbb{Z}_p))$$

(over $\mathscr{O}_K$ with respect to $\{\varpi^{1/p^n}\}_{n\geq 0}$) associated with $P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}}, \mathbb{Z}_p)$.

We define a decreasing filtration $\mathrm{Fil}^i(\mathfrak{M}_{\mathrm{dR}}(P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}}, \mathbb{Z}_p)))$ on $\mathfrak{M}_{\mathrm{dR}}(P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}}, \mathbb{Z}_p))$ as in Subsection 3.2. Let

$$c_{\mathrm{dR},\mathscr{Y}_K} : D_{\mathrm{dR}}(H^2_{\text{ét}}(\mathscr{Y}_{\overline{K}}, \mathbb{Z}_p)[1/p]) \cong H^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K)[1/p],$$
$$c_{\mathrm{cris},\mathscr{Y}} : D_{\mathrm{cris}}(H^2_{\text{ét}}(\mathscr{Y}_{\overline{K}}, \mathbb{Z}_p)[1/p]) \cong H^2_{\mathrm{cris}}(\mathscr{Y}_k/W)[1/p]$$

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be the isomorphisms in Subsection 3.4. These isomorphisms are compatible with Chern classes, cup products and trace maps; see Subsection 11.1 and Subsection 11.2. Hence, we have canonical isomorphisms of filtered K-modules

$$\mathfrak{M}_{\mathrm{dR}}(P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p))[1/p] \cong D_{\mathrm{dR}}(P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p)[1/p]) \cong P^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K)[1/p].$$

We also have canonical isomorphisms of filtered φ-modules

$$\mathfrak{M}_{\mathrm{cris}}(P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p))[1/p] \cong D_{\mathrm{cris}}(P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p)[1/p]) \cong P^2_{\mathrm{cris}}(\mathscr{Y}_k/W)[1/p].$$

Lemma 6.2. *The canonical isomorphism*

$$\mathfrak{M}_{\mathrm{dR}}(P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p))[1/p] \cong P^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K)[1/p]$$

maps $\mathfrak{M}_{\mathrm{dR}}(P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p))$ *onto* $P^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K)$. *It also maps* $\mathrm{Fil}^i(\mathfrak{M}_{\mathrm{dR}}(P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p)))$ *onto* Fil^i_{pr} *for every* $i \in \mathbb{Z}$.

Proof. We shall show the first assertion. The cup product on $H^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p)$ induces a perfect pairing on the Breuil-Kisin module $\mathfrak{M}(H^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p))$. Because the kernel of a homomorphism of free $\mathfrak{S}$-modules of finite rank is free, the orthogonal complement

$$\mathfrak{M}(\langle \mathrm{ch}_p(\xi)\rangle(-1))^{\perp} \subset \mathfrak{M}(H^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p))$$

is a free $\mathfrak{S}$-module of finite rank. Hence, it is a Breuil-Kisin module (of height ≤ 2). By the characterisation of $\mathfrak{M}(P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p))$ in [6, Theorem 4.4], we have

$$\mathfrak{M}(P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p)) \cong \mathfrak{M}(\langle \mathrm{ch}_p(\xi)\rangle(-1))^{\perp}.$$

By Theorem 3.2, we have $\mathfrak{M}_{\mathrm{dR}}(H^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p)) \cong H^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K)$. Under this isomorphism, the homomorphism $\mathfrak{M}_{\mathrm{dR}}(\langle \mathrm{ch}_p(\xi)\rangle(-1)) \hookrightarrow \mathfrak{M}_{\mathrm{dR}}(H^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p))$ is identified with the inclusion $\langle \mathrm{ch}_{\mathrm{dR}}(\xi)\rangle \hookrightarrow H^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K)$ by Proposition 11.2.

To prove the first assertion, it is enough to show that the cokernel of the injective homomorphism

$$j\colon \varphi^*\mathfrak{M}(\langle \mathrm{ch}_p(\xi)\rangle(-1)) \hookrightarrow \varphi^*\mathfrak{M}(H^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p))$$

is a free $\mathfrak{S}$-module. Indeed, if the cokernel of j is a free $\mathfrak{S}$-module, we have the following split exact sequence of $\mathfrak{S}$-modules:

$$\begin{aligned}0 \to \varphi^*\mathfrak{M}(P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p)) \to \varphi^*\mathfrak{M}(H^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p)) \cong \varphi^*\mathfrak{M}(H^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p))^{\vee}\\ \to \varphi^*\mathfrak{M}(\langle \mathrm{ch}_p(\xi)\rangle(-1))^{\vee} \to 0.\end{aligned}$$

By taking $\otimes_{\mathfrak{S}}\mathscr{O}_K$, we obtain the following exact sequence of $\mathscr{O}_K$-modules:

$$0 \to \mathfrak{M}_{\mathrm{dR}}(P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p)) \to H^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K) \to \langle \mathrm{ch}_{\mathrm{dR}}(\xi)\rangle^{\vee} \to 0.$$

Hence, we have $\mathfrak{M}_{\mathrm{dR}}(P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p)) \cong P^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K)$. We shall show that the cokernel of j is a free $\mathfrak{S}$-module. As above, we have

$$\mathfrak{M}(\langle \mathrm{ch}_p(\xi)\rangle(-1)) \cong \mathfrak{M}(P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p))^{\perp}.$$

Hence, the cokernel of j is a torsion-free $\mathfrak{S}$-module. Therefore, in order to prove that it is a free $\mathfrak{S}$-module, it now suffices to show that the cokernel of $\langle \mathrm{ch}_{\mathrm{dR}}(\xi)\rangle \hookrightarrow H^2_{\mathrm{dR}}(\mathscr{Y}/\mathscr{O}_K)$ is p-torsion-free. This follows from [56, Corollary 1.4].

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The second assertion follows from the fact that $\mathrm{Gr}^i(\mathfrak{M}_{\mathrm{dR}}(P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p)))$ and $\mathrm{Gr}^i_{\mathrm{pr}}$ are free $\mathscr{O}_K$-modules. □

Lemma 6.3. *The canonical isomorphism*

$$\mathfrak{M}_{\mathrm{cris}}(P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p))[1/p] \cong P^2_{\mathrm{cris}}(\mathscr{Y}_k/W)[1/p]$$

maps $\mathfrak{M}_{\mathrm{cris}}(P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p))$ *onto* $P^2_{\mathrm{cris}}(\mathscr{Y}_k/W)$.

Proof. As in the proof of Lemma 6.2, the assertion follows from the fact that the cokernel of j is a free $\mathfrak{S}$-module by using Theorem 3.2 and Corollary 11.6. □

6.2. *F-crystals on Shimura varieties and the cohomology of K3 surfaces*

In this subsection, we assume that k is a finite field $\mathbb{F}_q$ or $\overline{\mathbb{F}}_q$. We shall compare the W-module $\widetilde{L}_{\mathrm{cris},s}$ associated with a k-valued point $s \in \mathscr{S}_{\widetilde{\mathrm{K}}}(k)$ of the integral canonical model $\mathscr{S}_{\widetilde{\mathrm{K}}}$ with the crystalline cohomology of $K3$ surfaces.

We consider the following situation.

(1) Let $s \in M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Z}_{(p)}}(k)$ be a k-valued point and $(X,\mathscr{L})$ the quasi-polarised $K3$ surface over k of degree $2d$ associated with s.
(2) The image of s under the Kuga-Satake morphism KS is denoted by the same notation $s \in Z_{\mathrm{K}^p_0}(\Lambda)(k)$. After replacing k by a finite extension of it, there is a k-valued point of $Z_{\mathrm{K}^p}(\Lambda)$ mapped to s. We fix such a point, and it is also denoted by s. Let $\mathcal{A}_s$ be the Kuga-Satake abelian variety over k associated with the point $s \in Z_{\mathrm{K}^p}(\Lambda)(k)$.
(3) We take an $\mathscr{O}_K$-valued point $t \in M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Z}_{(p)}}(\mathscr{O}_K)$ lifting s. The morphism $Z_{\mathrm{K}^p}(\Lambda) \to Z_{\mathrm{K}^p_0}(\Lambda)$ is étale by Proposition 4.6. Hence, the image of the $\mathscr{O}_K$-valued point t under the Kuga-Satake morphism lifts to a unique $\mathscr{O}_K$-valued point of $Z_{\mathrm{K}^p}(\Lambda)$ lifting $s \in Z_{\mathrm{K}^p}(\Lambda)(k)$. It is also denoted by $t \in Z_{\mathrm{K}^p}(\Lambda)(\mathscr{O}_K)$.
(4) Let $(\mathscr{Y},\xi)$ be the quasi-polarised $K3$ surface over $\mathscr{O}_K$ of degree $2d$ associated with t. We assume that ξ is a line bundle.
(5) Let $\bar{t}$ be a geometric point of $Z_{\mathrm{K}^p}(\Lambda)$ above the generic point of $t \in Z_{\mathrm{K}^p}(\Lambda)(\mathscr{O}_K)$.

Then the $\mathrm{Gal}(\overline{K}/K)$-stable $\mathbb{Z}_p$-lattice in a crystalline representation

$$\widetilde{L}'_p := \widetilde{\mathbb{L}}_{p,\bar{t}}$$

satisfies the following properties:

- $\widetilde{L}'_p$ admits an even perfect bilinear form $(\ ,\ )$ over $\mathbb{Z}_p$ that is compatible with the action of $\mathrm{Gal}(\overline{K}/K)$.
- There is a $\mathrm{Gal}(\overline{K}/K)$-equivariant homomorphism $\iota_p\colon \Lambda_{\mathbb{Z}_p} \to \widetilde{L}'_p$ preserving the bilinear forms. The cokernel of ι_p is a free $\mathbb{Z}_p$-module.
- There is a $\mathrm{Gal}(\overline{K}/K)$-equivariant isometry

$$P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}},\mathbb{Z}_p(1)) \cong \iota_p(\Lambda_{\mathbb{Z}_p})^{\perp}.$$

  See [49, Proposition 5.6].
- There is an isometry $\mathfrak{M}_{\mathrm{cris}}(\widetilde{L}'_p) \cong \widetilde{L}_{\mathrm{cris},s}$ that becomes an isomorphism of F-isocrystals after inverting p. (See Subsection 4.6.)

We use the following notation on twists of φ-modules. Let (N,φ) be a pair of a free W-module of finite rank and a σ-linear map φ on $N[1/p]$. The pair $(N,p^{-i}\varphi)$ is denoted by $N(i)$.

We put

$$\widetilde{L}_{\mathrm{cris}} := \widetilde{L}_{\mathrm{cris},s}.$$

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The Frobenius automorphism of $\widetilde{L}_{\mathrm{cris}}(-1)[1/p]$ maps the W-module $\widetilde{L}_{\mathrm{cris}}(-1)$ into itself. Therefore, $\widetilde{L}_{\mathrm{cris}}(-1)$ is an F-crystal.

We now prove the following proposition, which plays a crucial role in this article.

Proposition 6.4. *We have an isomorphism of F-crystals*

$$P^2_{\mathrm{cris}}(X/W) \cong \iota_{\mathrm{cris}}(\Lambda_W)^{\perp}(-1) \subset \widetilde{L}_{\mathrm{cris}}(-1).$$

Proof. This proposition was proved in [49, Corollary 5.14] when $p \neq 2$. Here we give a proof using the integral comparison theorem of Bhatt-Morrow-Scholze [6]. (Our proof works for any p, including $p = 2$.)

By $P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}}, \mathbb{Z}_p) \cong \iota_p(\Lambda_{\mathbb{Z}_p})(-1)^{\perp}$, we have

$$\mathfrak{M}(P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}}, \mathbb{Z}_p)) \cong \mathfrak{M}(\iota_p(\Lambda_{\mathbb{Z}_p})(-1))^{\perp}.$$

(See also the proof of Lemma 6.2.) Moreover, we have $\mathfrak{M}_{\mathrm{cris}}(P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}}, \mathbb{Z}_p)) \cong P^2_{\mathrm{cris}}(X/W)$ by Lemma 6.3. Thus, as in the proof of Lemma 6.2, the proposition follows from Proposition 6.5. □

Proposition 6.5. *The cokernel of the homomorphism*

$$\mathfrak{M}(\iota_p)\colon \mathfrak{M}(\Lambda_{\mathbb{Z}_p}) \to \mathfrak{M}(\widetilde{L}'_p)$$

is a free $\mathfrak{S}$-module.

Proof. Because the cokernel of ι_p is a free $\mathbb{Z}_p$-module, it follows that

$$\mathfrak{M}(\iota_p(\Lambda_{\mathbb{Z}_p})) \cong \mathfrak{M}(\iota_p(\Lambda_{\mathbb{Z}_p})^{\perp\perp}) \cong \mathfrak{M}(\iota_p(\Lambda_{\mathbb{Z}_p})^{\perp})^{\perp}.$$

Let $\mathfrak{M}'$ be the cokernel of the homomorphism $\mathfrak{M}(\iota_p(\Lambda_{\mathbb{Z}_p})(-1)^{\perp}) \to \mathfrak{M}(\widetilde{L}'_p(-1))$. It suffices to prove that $\mathfrak{M}'$ is a free $\mathfrak{S}$-module.

The cokernel $\mathfrak{M}'$ is a torsion-free $\mathfrak{S}$-module. Note that $\mathfrak{M}(\widetilde{L}'_p(-1))$ is an effective Breuil-Kisin module of height ≤ 2 and it induces a Frobenius $1 \otimes \varphi\colon \varphi^* \mathfrak{M}' \to \mathfrak{M}'$.

It is enough to show that the Frobenius of $\mathfrak{M}'$ satisfies the following properties:

(1) For every $x \in \varphi^* \mathfrak{M}'$, the image $(1 \otimes \varphi)(x)$ is divisible by $E(u)$.
(2) The cokernel of $1 \otimes \varphi\colon \varphi^* \mathfrak{M}' \to \mathfrak{M}'$ is killed by $E(u)$.

In fact, if (1) and (2) are proved, the homomorphism $1/E(u)(1 \otimes \varphi)$ makes $\mathfrak{M}'$ a torsion-free φ-module of height 0 in the sense of [47]. Then, we see that $\mathfrak{M}'$ is a free $\mathfrak{S}$-module by [47, Lemma 2.18 (2)].

We shall prove (1). For simplicity, we put

$$\mathfrak{M} := \mathfrak{M}(\iota_p(\Lambda_{\mathbb{Z}_p})(-1)^{\perp}), \qquad \mathfrak{M}_{\mathrm{dR}} := \mathfrak{M}_{\mathrm{dR}}(\iota_p(\Lambda_{\mathbb{Z}_p})(-1)^{\perp}),$$
$$\widetilde{\mathfrak{M}} := \mathfrak{M}(\widetilde{L}'_p(-1)), \qquad \widetilde{\mathfrak{M}}_{\mathrm{dR}} := \mathfrak{M}_{\mathrm{dR}}(\widetilde{L}'_p(-1)).$$

Recall that we have $P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}}, \mathbb{Z}_p) \cong \iota_p(\Lambda_{\mathbb{Z}_p})(-1)^{\perp}$. Thus, by Lemma 6.1 and Lemma 6.2, the bilinear form on $\mathfrak{M}_{\mathrm{dR}}$ induces $\mathrm{Gr}^2(\mathfrak{M}_{\mathrm{dR}}) \cong \mathrm{Gr}^0(\mathfrak{M}_{\mathrm{dR}})^{\vee}$.

Let us show that the perfect bilinear form on $\widetilde{\mathfrak{M}}_{\mathrm{dR}}$ induces $\mathrm{Gr}^2(\widetilde{\mathfrak{M}}_{\mathrm{dR}}) \cong \mathrm{Gr}^0(\widetilde{\mathfrak{M}}_{\mathrm{dR}})^{\vee}$. Because the cokernel of $\mathrm{Gr}^2(\widetilde{\mathfrak{M}}_{\mathrm{dR}}) \hookrightarrow \widetilde{\mathfrak{M}}_{\mathrm{dR}}$ (respectively $\mathrm{Gr}^0(\widetilde{\mathfrak{M}}_{\mathrm{dR}})^{\vee} \hookrightarrow \widetilde{\mathfrak{M}}^{\vee}_{\mathrm{dR}}$) is p-torsion-free, it suffices to show that we have the desired isomorphism after inverting p. This follows from the isomorphism

$$\widetilde{L}'_p(-1)[1/p] \cong \iota_p(\Lambda_{\mathbb{Z}_p})(-1)[1/p] \oplus P^2_{\text{ét}}(\mathscr{Y}_{\overline{K}}, \mathbb{Z}_p)[1/p].$$

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We obtain the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Gr}^2(\mathfrak{M}_{\mathrm{dR}}) & \xrightarrow{\cong} & \mathrm{Gr}^0(\mathfrak{M}_{\mathrm{dR}})^\vee \\ \downarrow & & \uparrow \\ \mathrm{Gr}^2(\widetilde{\mathfrak{M}}_{\mathrm{dR}}) & \xrightarrow{\cong} & \mathrm{Gr}^0(\widetilde{\mathfrak{M}}_{\mathrm{dR}})^\vee. \end{array}$$

Hence, $\mathrm{Gr}^0(\widetilde{\mathfrak{M}}_{\mathrm{dR}})^\vee \to \mathrm{Gr}^0(\mathfrak{M}_{\mathrm{dR}})^\vee$ is surjective. Because both $\mathrm{Gr}^0(\mathfrak{M}_{\mathrm{dR}})^\vee$ and $\mathrm{Gr}^0(\widetilde{\mathfrak{M}}_{\mathrm{dR}})^\vee$ are free $\mathscr{O}_K$-modules of rank 1, we have $\mathrm{Gr}^0(\mathfrak{M}_{\mathrm{dR}})^\vee \cong \mathrm{Gr}^0(\widetilde{\mathfrak{M}}_{\mathrm{dR}})^\vee$. Therefore, we have

$$\mathrm{Gr}^0(\mathfrak{M}_{\mathrm{dR}}) \cong \mathrm{Gr}^0(\widetilde{\mathfrak{M}}_{\mathrm{dR}}) \quad \text{and} \quad \mathrm{Gr}^2(\mathfrak{M}_{\mathrm{dR}}) \cong \mathrm{Gr}^2(\widetilde{\mathfrak{M}}_{\mathrm{dR}}).$$

By $\mathrm{Gr}^0(\mathfrak{M}_{\mathrm{dR}}) \cong \mathrm{Gr}^0(\widetilde{\mathfrak{M}}_{\mathrm{dR}})$ and Lemma 3.1, we have $\mathrm{Fil}^1(\varphi^*\widetilde{\mathfrak{M}}) + \varphi^*\mathfrak{M} = \varphi^*\widetilde{\mathfrak{M}}$. The assertion (1) follows from this equality.

We shall prove (2). By $\mathrm{Gr}^2(\mathfrak{M}_{\mathrm{dR}}) \cong \mathrm{Gr}^2(\widetilde{\mathfrak{M}}_{\mathrm{dR}})$ and Lemma 3.1, we have

$$\mathrm{Fil}^2(\varphi^*\widetilde{\mathfrak{M}}) \subset \mathrm{Fil}^1(\varphi^*\mathfrak{M}) + E(u)\varphi^*\widetilde{\mathfrak{M}}.$$

We take an element $x \in \widetilde{\mathfrak{M}}$. Because $\widetilde{\mathfrak{M}}$ is of height ≤ 2, there is an element $y \in \varphi^*\widetilde{\mathfrak{M}}$ such that $(1\otimes\varphi)(y) = E(u)^2 x$. By the above inclusion, there are $z \in \mathrm{Fil}^1(\varphi^*\mathfrak{M})$ and $w \in \varphi^*\widetilde{\mathfrak{M}}$ such that $y = z + E(u)w$. We put $(1\otimes\varphi)(z) := E(u)z'$ for $z' \in \mathfrak{M}$. Then we have $E(u)x = z' + (1\otimes\varphi)(w)$. This shows that $E(u)x$ is zero in the cokernel of $\varphi^*\mathfrak{M}' \to \mathfrak{M}'$ and completes the proof of the assertion (2).

The proof of Proposition 6.5 is complete. □

6.3. *Formal Brauer groups*

We consider the situation as in Subsection 6.2.

Let $\widehat{\mathrm{Br}} := \widehat{\mathrm{Br}}(X)$ denote the *formal Brauer group* associated with the $K3$ surface X. Recall that $\widehat{\mathrm{Br}}$ is a one-dimensional smooth formal group scheme pro-representing the functor

$$\Phi^2_X : \mathrm{Art}_k \to (\text{abelian groups})$$

defined by

$$R \mapsto \mathrm{Ker}(H^2_{\text{ét}}(X_R, \mathbb{G}_m) \to H^2_{\text{ét}}(X, \mathbb{G}_m)),$$

where Art_k is the category of local Artinian k-algebras with residue field k and (abelian groups) is the category of abelian groups; see [2, Chapter II, Corollary 2.12]. (For basic properties of the formal Brauer group, see also [46, Section 6].) The *height* h of the $K3$ surface X is defined to be the height of $\widehat{\mathrm{Br}}$. We have $1 \leq h \leq 10$ or $h = \infty$.

There is a natural equivalence from the category of one-dimensional smooth formal group schemes of finite height over k to the category of one-dimensional connected p-divisible groups over k. If the height of X is finite, we identify the formal Brauer group $\widehat{\mathrm{Br}}$ with the corresponding connected p-divisible group over k and let $\widehat{\mathrm{Br}}^*$ denote the Cartier dual of $\widehat{\mathrm{Br}}$.

For a crystal $\mathscr{E}$ over $\mathrm{CRIS}(k/\mathbb{Z}_p)$, the value $\mathscr{E}_{W\twoheadrightarrow k}$ in $(\mathrm{Spec}\,k \hookrightarrow \mathrm{Spec}\,W)$ will be denoted by the same letter $\mathscr{E}$. By [4, (5.3.3.1)], we have a canonical perfect bilinear form

$$\mathbb{D}(\widehat{\mathrm{Br}}^*) \times \mathbb{D}(\widehat{\mathrm{Br}})(-1) \to W(-2).$$

The following Proposition 6.6 and Lemma 6.8 are presumably well known to experts (see [50, 6.27] for the case $p \geq 3$). We include proofs for completeness.

Proposition 6.6. *Assume that the height of X is finite. The following assertions hold:*

(1) *There is an isomorphism of F-crystals*

$$\widetilde{L}_{\mathrm{cris}}(-1) \cong \mathbb{D}(\widehat{\mathrm{Br}}^{*}) \oplus \mathbb{D}(D)(-1) \oplus \mathbb{D}(\widehat{\mathrm{Br}})(-1),$$

where D is an étale p-divisible group over k.

(2) *Under this isomorphism, the bilinear form on $\widetilde{L}_{\mathrm{cris}}(-1)$ is the direct sum of a perfect bilinear form*

$$\mathbb{D}(D)(-1) \times \mathbb{D}(D)(-1) \to W(-2)$$

and the canonical perfect bilinear form $\mathbb{D}(\widehat{\mathrm{Br}}^{*}) \times \mathbb{D}(\widehat{\mathrm{Br}})(-1) \to W(-2)$.

Proof. The breaking points of the Newton polygon of $\widetilde{L}_{\mathrm{cris}}(-1)$ lie on the Hodge polygon of it; see Lemma 6.8 and its proof.

By the Hodge-Newton decomposition [36, Theorem 1.6.1], there is a decomposition as an F-crystal over W,

$$\widetilde{L}_{\mathrm{cris}}(-1) \cong \widetilde{L}_{1-1/h} \oplus \widetilde{L}_{1} \oplus \widetilde{L}_{1+1/h},$$

where $\widetilde{L}_{\lambda}$ is an F-crystal over W that has a single slope λ for each $\lambda \in \{1-1/h, 1, 1+1/h\}$. Via this decomposition, the bilinear form $(\ ,\ )$ is the direct sum of a perfect bilinear form $\widetilde{L}_1 \times \widetilde{L}_1 \to W(-2)$ and a perfect bilinear form $\widetilde{L}_{1-1/h} \times \widetilde{L}_{1+1/h} \to W(-2)$.

Similarly, we have a decomposition

$$P^2_{\mathrm{cris}}(X/W) \cong L_{1-1/h} \oplus L_1 \oplus L_{1+1/h}.$$

By Proposition 6.4, we have $P^2_{\mathrm{cris}}(X/W) = \iota_{\mathrm{cris}}(\Lambda_W)(-1)^{\perp}$. Because $\iota_{\mathrm{cris}}(\Lambda_W)(-1)$ is contained in $\widetilde{L}_1$, we have $L_{1-1/h} = \widetilde{L}_{1-1/h}$ and $L_{1+1/h} = \widetilde{L}_{1+1/h}$. We have a natural isomorphism of F-crystals over W

$$\mathbb{D}(\widehat{\mathrm{Br}}^{*}) \cong L_{1-1/h}.$$

(See [68, Proposition 7] for example. See also Remark 6.7.) Using the perfect bilinear form $\widetilde{L}_{1-1/h} \times \widetilde{L}_{1+1/h} \to W(-2)$, we identify $L_{1+1/h}$ with $L^{\vee}_{1-1/h}(-2) \cong \mathbb{D}(\widehat{\mathrm{Br}})(-1)$.

Because we have $p\widetilde{L}_1 = \varphi(\widetilde{L}_1)$, there is an étale p-divisible group D over k such that $\mathbb{D}(D)(-1) \cong \widetilde{L}_1$. □

Remark 6.7. We assume that the height of X is finite. The proof of [68, Proposition 7] shows that there is a natural isomorphism of F-crystals over W,

$$\mathrm{TC}(\widehat{\mathrm{Br}}) \cong L_{1-1/h}.$$

Here $\mathrm{TC}(\widehat{\mathrm{Br}})$ is the Cartier-Dieudonné module of typical curves of $\widehat{\mathrm{Br}}$; see [2, I, Section 3] for example. The F-crystal $\mathrm{TC}(\widehat{\mathrm{Br}})$ is naturally isomorphic to the F-crystal $\mathbb{D}(\widehat{\mathrm{Br}}^{*})$ by [12, (5.8)], [4, Théorème 4.2.14, (5.3.3.1)].

The following strong divisibility result is used in the proof of Proposition 6.6.

Lemma 6.8. *We have an isomorphism of W-modules*

$$\widetilde{L}_{\mathrm{cris}}(-1)/\varphi(\widetilde{L}_{\mathrm{cris}}(-1)) \cong (W/p)^{\oplus 20} \oplus W/p^2.$$

Proof. We put

$$M := \mathfrak{M}_{\mathrm{cris}}(\widetilde{L}'_p)(-1) \cong \widetilde{L}_{\mathrm{cris}}(-1).$$

The Newton polygon of M has three slopes with multiplicities described as follows:

Slope	$1-1/h$	1	$1+1/h$
Multiplicity	h	$22-2h$	h

The Newton polygon of M is above the Hodge polygon of M and both polygons have the same initial point and endpoint; see [36, Theorem 1.4.1]. We put

$$M' := \{\, x \in M \mid px \in \mathrm{Im}(\varphi) \,\}.$$

Then it is enough to show that there is a surjection $W/p \to M/M'$.

We put $\mathfrak{M} := \mathfrak{M}(\widetilde{L}'_p(-1))$ and define

$$\mathfrak{M}' := \{\, x \in \mathfrak{M} \mid E(u)x \in \mathrm{Im}(1 \otimes \varphi\colon \varphi^*\mathfrak{M} \to \mathfrak{M}) \,\}.$$

Via $\varphi^*\mathfrak{M} \otimes_{\mathfrak{S}} W \cong M$, we have a surjection $(\varphi^*\mathfrak{M}/\varphi^*\mathfrak{M}') \otimes_{\mathfrak{S}} W \to M/M'$. We shall show that $(\varphi^*\mathfrak{M}/\varphi^*\mathfrak{M}') \otimes_{\mathfrak{S}} W \cong W/p$.

Because we have a perfect bilinear form $\mathfrak{M} \times \mathfrak{M} \to \mathfrak{S}(-2)$ over $\mathfrak{S}$ that is compatible with the Frobenius endomorphisms, we have

$$E(u)^2\mathfrak{M} \subset \mathrm{Im}(\varphi^*\mathfrak{M} \to \mathfrak{M}).$$

Hence, $\mathfrak{M}/\mathfrak{M}'$ is killed by $E(u)$ and $\mathfrak{M}/\mathfrak{M}'$ is a finite $\mathfrak{S}/E(u)$-module. Using the perfect bilinear form on $\mathfrak{M}$ again, we see that $\mathfrak{M}/\mathfrak{M}'$ is p-torsion-free. Because $\mathfrak{S}/E(u) \cong \mathcal{O}_K$ is a discrete valuation ring, it follows that $\mathfrak{M}/\mathfrak{M}'$ is a free $\mathfrak{S}/E(u)$-module of finite rank. The dimension of the K-vector space $\mathrm{gr}^i D_{\mathrm{dR}}(\widetilde{L}'_p(-1)[1/p])$ is as follows:

$$\dim_K \mathrm{gr}^i D_{\mathrm{dR}}(\widetilde{L}'_p(-1)[1/p]) = \begin{cases} 0 & i \neq 0,1,2, \\ 20 & i = 1, \\ 1 & i = 0,2. \end{cases}$$

It follows that $\mathfrak{M}/\mathfrak{M}'$ is a free $\mathfrak{S}/E(u)$-module of rank 1 by [39, Lemma 1.2.2]. Therefore, we have

$$(\varphi^*\mathfrak{M}/\varphi^*\mathfrak{M}') \otimes_{\mathfrak{S}} W \cong (\mathfrak{S}/\varphi(E(u))) \otimes_{\mathfrak{S}} W \cong W/p.$$

□

6.4. *Remarks on the étaleness of the Kuga-Satake morphism*

The étaleness of the Kuga-Satake morphism

$$\mathrm{KS}\colon M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Z}_{(p)}} \to Z_{\mathrm{K}^p_0}(\Lambda)$$

plays an important role in this article. It also plays an important role in Madapusi Pera's proof of the Tate conjecture for $K3$ surfaces [49, 38]. The proof of the étaleness was given by Madapusi Pera in [49, Theorem 5.8] when p is odd and in [38, Proposition A.12] when $p = 2$.

In the course of writing this article, we found some issues on the proof of the étaleness of KS. We can avoid these issues using our results in this section. See Remark 6.9 and Remark 6.10 for details. (See also [51], where Madapusi Pera gave a somewhat different argument.)

Remark 6.9. When $p = 2$, the proof of the étaleness of KS in [38, Proposition A.12] seems to rely on an incorrect statement on the relation between endomorphisms of an abelian scheme and the crystalline cohomology. Here we explain how to avoid this issue using Proposition 6.5. First, we briefly recall Madapusi Pera's proof. Let $\widetilde{\mathbb{L}}_{\mathrm{dR}}$ be the filtered vector bundle with integrable connection on $\mathscr{S}_{\widetilde{\mathrm{K}}_0}$ associated with the $\widetilde{G}_0$-representation $\widetilde{L}_{\mathbb{Z}_{(p)}}$. The pullback of $\widetilde{\mathbb{L}}_{\mathrm{dR}}$ by KS is also denoted by the same symbol $\widetilde{\mathbb{L}}_{\mathrm{dR}}$. The Λ-structure for $M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Z}_{(p)}}$ induces a homomorphism of vector bundles

$$\iota_{\mathrm{dR}} \colon \Lambda \otimes_{\mathbb{Z}} \mathcal{O}_{M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Z}_{(p)}}} \to \widetilde{\mathbb{L}}_{\mathrm{dR}}.$$

In order to prove the étaleness of KS (see [49, Theorem 5.8], [38, Proposition A.12]), Madapusi Pera showed the orthogonal complement

$$\mathbb{L}_{\mathrm{dR}} := \iota_{\mathrm{dR}}(\Lambda \otimes_{\mathbb{Z}} \mathcal{O}_{M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Z}_{(p)}}})^{\perp} \subset \widetilde{\mathbb{L}}_{\mathrm{dR}}$$

with respect to the canonical bilinear form on $\widetilde{\mathbb{L}}_{\mathrm{dR}}$ is isomorphic to $\mathbb{P}^2_{\mathrm{dR}}$ (up to twist):

$$\alpha_{\mathrm{dR}} \colon \mathbb{P}^2_{\mathrm{dR}} \cong \mathbb{L}_{\mathrm{dR}}(-1).$$

Here $\mathbb{P}^2_{\mathrm{dR}}$ is the primitive part of the relative de Rham cohomology of the universal family on $M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Z}_{(p)}}$. See [49, Proposition 5.11] for $p \geq 3$ and the proof of [38, Proposition A.12] for $p = 2$. In order to prove that $\mathbb{P}^2_{\mathrm{dR}}$ and $\mathbb{L}_{\mathrm{dR}}(-1)$ are isomorphic, he proved that the cokernel of ι_{dR} is a *vector bundle* on $M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Z}_{(p)}}$; see the proof of [38, Proposition A.12]. To prove this, it suffices to show that the cokernel of ι_{dR} is a free $W(\overline{\mathbb{F}}_q)$-module at every $W(\overline{\mathbb{F}}_q)$-valued point of $M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Z}_{(p)}}$; see the proof of [50, Lemma 6.16 (iv)]. When $p \geq 3$, this freeness property was proved by applying [50, Lemma 6.14] for $e = 1$. However, when $p = 2$, we cannot apply [50, Lemma 6.14] because the statement of [50, Lemma 6.14] is *false* when $e = p-1$. (There are counterexamples when $p = 2$ and $e = 1$; see [5, Example 3.18].) To avoid this issue, we can use Proposition 6.5 to prove the required freeness property. Note that we essentially used the integral comparison theorem of Bhatt-Morrow-Scholze [6] in the proof of this proposition.

Remark 6.10. There is another issue on the integral comparison map used in the proof of the isomorphism $\alpha_{\mathrm{dR}} \colon \mathbb{P}^2_{\mathrm{dR}} \cong \mathbb{L}_{\mathrm{dR}}(-1)$. This issue exists for every p (including odd p). In [49, Theorem 5.8] and [38, Proposition A.12], Madapusi Pera used the integral comparison map for varieties with ordinary reduction proved by Bloch-Kato [9, Theorem 9.6] and the density of ordinary locus in the special fibre of $M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Z}_{(p)}}$; see the proof of [49, Lemma 5.10]. Madapusi Pera used the compatibility between the integral comparison map of Bloch-Kato with other comparison maps used in [49, Section 2]. However, we could not find appropriate references for the compatibility used in [49, 38], at least for small p. We can avoid this issue by using the de Rham comparison map $c_{\mathrm{dR},\mathscr{Y}_K}$ of Scholze [62] and the crystalline comparison map $c_{\mathrm{cris},\mathscr{Y}}$ of Bhatt-Morrow-Scholze [6]. The integral comparison map $c_{\mathrm{cris},\mathscr{Y}}$ is a substitute for the integral comparison map of Bloch-Kato; see also Proposition 6.4 in this article. The results of Blasius-Wintenberger [8], which were used in [49] and [38, Appendix A], also hold using $c_{\mathrm{dR},\mathscr{Y}_K}$ and $c_{\mathrm{cris},\mathscr{Y}}$; see Subsection 11.3. Finally, we remark that the integral comparison map $c_{\mathrm{cris},\mathscr{Y}}$ can be applied to all $K3$ surfaces including nonordinary ones.

7. Construction of liftings of points on orthogonal Shimura varieties

Let $s \in Z_{\mathrm{K}^p}(\Lambda)(\mathbb{F}_q)$ be an $\mathbb{F}_q$-valued point that is the image of the point corresponding to a quasi-polarised $K3$ surface $(X, \mathscr{L})$ of finite height over $\mathbb{F}_q$. Because the Brauer group of $\mathbb{F}_q$ is trivial, $\mathscr{L}$ is a line bundle on X. (See [10, Chapter 8, Section 1, Proposition 4].)

In this section, we shall construct characteristic 0 liftings of the point $s \in Z_{\mathrm{K}^p}(\Lambda)(\mathbb{F}_q)$ over a finite extension of $W(\overline{\mathbb{F}}_q)[1/p]$ corresponding to characteristic 0 liftings of the formal Brauer group $\widehat{\mathrm{Br}}$ of X. We construct such liftings using integral p-adic Hodge theory and our results on F-crystals on Shimura varieties in Section 6.

https://doi.org/10.1017/fms.2021.24 Published online by Cambridge University Press

Note that, when $p \geq 5$, a stronger result can be obtained by the method of Nygaard-Ogus [55]; see Remark 7.2. But, when $p \leq 3$, it seems difficult to apply their methods. (The deformation theory of $K3$ crystals developed in [55] does not work in characteristic $p = 2$ or 3; see [55, p. 498].) The method of this article can be applied to $K3$ surfaces of finite height in any characteristic.

7.1. *Liftings of points with additional properties*

In this subsection, we shall state our results on characteristic 0 liftings of points on $Z_{\mathrm{K}^p}(\Lambda)$ with additional properties.

We put $k := \mathbb{F}_q$, and consider the situation as in Subsection 6.2. We assume that the height h of the $K3$ surface X is finite.

Let E be a finite totally ramified extension of $K_0 = W[1/p]$ and $\mathcal{G}$ a one-dimensional smooth formal group over $\mathcal{O}_E$ whose special fibre is isomorphic to $\widehat{\mathrm{Br}}$. We shall construct a characteristic 0 lifting of the point $s \in Z_{\mathrm{K}^p}(\Lambda)(\mathbb{F}_q)$ over a finite extension of $W(\overline{\mathbb{F}}_q)[1/p]$ corresponding to $\mathcal{G}$.

By Proposition 6.6, we have an embedding $\mathbb{D}(\widehat{\mathrm{Br}}) \hookrightarrow \widetilde{L}_{\mathrm{cris}}$. Let

$$\mathrm{Fil}(\mathcal{G}) \hookrightarrow \mathbb{D}(\widehat{\mathrm{Br}}) \otimes_W E \hookrightarrow \widetilde{L}_{\mathrm{cris}} \otimes_W E$$

be the filtration associated with $\mathcal{G}$; see Subsection 3.3. Take a generator e of $\mathrm{Fil}(\mathcal{G})$ and let $i(e) := (i_{\mathrm{cris}} \otimes_W E)(e)$ denote the image of e under the embedding

$$i_{\mathrm{cris}} \otimes_W E \colon \widetilde{L}_{\mathrm{cris}} \otimes_W E \hookrightarrow \mathrm{End}_E(H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee \otimes_W E).$$

Let $i(e)(H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee \otimes_W E)$ be the image of

$$i(e) \colon H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee \otimes_W E \to H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee \otimes_W E.$$

We define a decreasing filtration on $\widetilde{L}_{\mathrm{cris}} \otimes_W E$ by

$$\mathrm{Fil}^i(\widetilde{L}_{\mathrm{cris}} \otimes_W E) := \begin{cases} 0 & i \geq 2, \\ \mathrm{Fil}(\mathcal{G}) & i = 1, \\ \mathrm{Fil}(\mathcal{G})^\perp & i = 0, \\ \widetilde{L}_{\mathrm{cris}} \otimes_W E & i \leq -1. \end{cases}$$

We also define a decreasing filtration on $H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee \otimes_W E$ by

$$\mathrm{Fil}^i(H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee \otimes_W E) := \begin{cases} 0 & i \geq 1, \\ i(e)(H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee \otimes_W E) & i = 0, \\ H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee \otimes_W E & i \leq -1. \end{cases}$$

The following theorem is the main result of this section.

Theorem 7.1. *Let K be the composite of E and $W(\overline{\mathbb{F}}_q)[1/p]$. There is an $\mathcal{O}_K$-valued point $\widetilde{s} \in Z_{\mathrm{K}^p}(\Lambda)(\mathcal{O}_K)$ satisfying the following properties:*

(1) *$\widetilde{s}$ is a lift of $\overline{s} \in Z_{\mathrm{K}^p}(\Lambda)(\overline{\mathbb{F}}_q)$, where $\overline{s}$ is a geometric point above s.*
(2) *The Hodge filtration on $H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee \otimes_W K$ corresponding to $\mathcal{A}_{\widetilde{s}}$ over $\mathcal{O}_K$ coincides with the filtration defined by*

$$\mathrm{Fil}^i(H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee \otimes_W K) := \mathrm{Fil}^i(H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee \otimes_W E) \otimes_E K.$$

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Remark 7.2. Recall that the $K3$ surface X over $\mathbb{F}_q$ comes from the $\mathbb{F}_q$-valued point $s \in M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Z}_{(p)}}(\mathbb{F}_q)$ satisfying the conditions as in Subsection 6.2. In particular, the Kuga-Satake abelian variety $\mathcal{A}_s$ is defined over $\mathbb{F}_q$. When $p \geq 5$, a result stronger than Theorem 7.1 can be obtained by the method of Nygaard-Ogus in [55]. In fact, when $p \geq 5$, Nygaard-Ogus constructed a characteristic 0 lifting of the $K3$ surface X over $\mathscr{O}_E$ corresponding to the one-dimensional smooth formal group $\mathcal{G}$. By the Kuga-Satake morphism, we find an $\mathscr{O}_E$-valued point $\widetilde{s} \in Z_{\mathrm{K}^p}(\Lambda)(\mathscr{O}_E)$ lifting s such that the Hodge filtration on $H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee \otimes_W E$ corresponding to $\mathcal{A}_{\widetilde{s}}$ over $\mathscr{O}_E$ coincides with the filtration $\{\mathrm{Fil}^i(H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee \otimes_W E)\}_i$ defined as above. Namely, we do not need to take the composite with $W(\overline{\mathbb{F}}_q)[1/p]$ when $p \geq 5$. Currently, we do not know how to obtain an $\mathscr{O}_E$-valued point by the methods of this article.

7.2. *Some lemmas*

In this subsection, we give some lemmas that will be used in the proof of Theorem 7.1.

Let

$$\widetilde{L}_{\mathrm{cris}} \cong \mathbb{D}(\widehat{\mathrm{Br}}^*)(1) \oplus \mathbb{D}(D) \oplus \mathbb{D}(\widehat{\mathrm{Br}})$$

be the decomposition as in Proposition 6.6. Becausee D is an étale p-divisible group over k, it canonically extends over $\mathscr{O}_E$ and the extension is also denoted by D. Let $\mathcal{G}^*$ denote the Cartier dual of $\mathcal{G}$. We define a $\mathrm{Gal}(\overline{E}/E)$-stable $\mathbb{Z}_p$-lattice $\widetilde{L}_p$ in a crystalline representation by

$$\widetilde{L}_p := (T_p\mathcal{G}^*)^\vee(1) \oplus (T_pD)^\vee \oplus (T_p\mathcal{G})^\vee.$$

Lemma 7.3. *The* $\mathrm{Gal}(\overline{E}/E)$*-module* $\widetilde{L}_p$ *satisfies the following properties:*

(1) $\widetilde{L}_p$ *admits an even perfect bilinear form* $(\ ,\ )$ *that is* $\mathrm{Gal}(\overline{E}/E)$*-invariant.*
(2) *There is a* $\mathrm{Gal}(\overline{E}/E)$*-equivariant homomorphism* $\iota_p : \Lambda_{\mathbb{Z}_p} \to \widetilde{L}_p$ *preserving the bilinear forms.*
(3) *There is an isometry* $D_{\mathrm{cris}}(\widetilde{L}_p[1/p]) \cong \widetilde{L}_{\mathrm{cris}}[1/p]$ *of filtered* φ*-modules such that the following composite*

$$\mathfrak{M}_{\mathrm{cris}}(\widetilde{L}_p)[1/p] \cong D_{\mathrm{cris}}(\widetilde{L}_p[1/p]) \cong \widetilde{L}_{\mathrm{cris}}[1/p]$$

maps $\mathfrak{M}_{\mathrm{cris}}(\widetilde{L}_p)$ *isomorphically onto* $\widetilde{L}_{\mathrm{cris}}$.
(4) *The following diagram is commutative:*

$$\begin{array}{ccc} \Lambda_W & \xrightarrow{\iota_p} & \mathfrak{M}_{\mathrm{cris}}(\widetilde{L}_p) \\ & {\scriptstyle \iota_{\mathrm{cris}}} \searrow & \downarrow \cong \\ & & \widetilde{L}_{\mathrm{cris}}. \end{array}$$

Proof. We equip the $\mathbb{Z}_p$-module $(T_p\mathcal{G}^*)^\vee(1) \oplus (T_p\mathcal{G})^\vee$ with a natural bilinear form that is even and perfect. Because D is an étale p-divisible group, the even perfect bilinear form $\mathbb{D}(D) \times \mathbb{D}(D) \to W$ induces an even perfect bilinear form $(T_pD)^\vee \times (T_pD)^\vee \to \mathbb{Z}_p$ that is $\mathrm{Gal}(\overline{E}/E)$-invariant.

Let $\underline{\Lambda}^\vee_{\mathbb{Z}_p}$ be the p-divisible group over $\mathscr{O}_E$ associated with the $\mathbb{Z}_p$-module $\Lambda^\vee_{\mathbb{Z}_p}$ with the trivial $\mathrm{Gal}(\overline{E}/E)$-action. So we have an isomorphism of F-crystals $\mathbb{D}(\underline{\Lambda}^\vee_{\mathbb{Z}_p}) \cong \Lambda_W$.

The image of the homomorphism $\iota_{\mathrm{cris}} \colon \Lambda_W \to \widetilde{L}_{\mathrm{cris}}$ is contained in $\mathbb{D}(D)$; hence, we have a homomorphism $\Lambda_W \to \mathbb{D}(D)$. Therefore, we have a morphism $D \to \underline{\Lambda}^\vee_{\mathbb{Z}_p}$ of étale p-divisible groups over k. This extends over $\mathscr{O}_E$. Then we have a $\mathrm{Gal}(\overline{E}/E)$-equivariant homomorphism $\Lambda_{\mathbb{Z}_p} \to (T_pD)^\vee$. This homomorphism preserves the bilinear forms by construction.

The other properties follow from [38, Theorem 2.12]. See also Subsection 3.3 and Subsection 11.4. ◻

Lemma 7.4. *There is a crystalline* $\mathrm{Gal}(\overline{E}/E)$*-representation* $H_{\text{ét},\mathbb{Q}_p}$ *over* $\mathbb{Q}_p$ *such that*

$$D_{\mathrm{cris}}(H_{\text{ét},\mathbb{Q}_p}) \cong H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee[1/p]$$

as filtered φ*-modules.*

Proof. It is a theorem of Colmez-Fontaine that any weakly admissible filtered φ-module is admissible; see [20, Théorème A]. (Today, there are several alternative proofs of this theorem. For example, see [39, Proposition 2.1.5].) Hence, it is enough to prove that the filtered φ-module $H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee[1/p]$ is weakly admissible.

As in Subsection 4.6, we fix an isomorphism of W-modules

$$H_W \cong H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee$$

that carries $\{s_\alpha\}$ to $\{s_{\alpha,\mathrm{cris},s}\}$ and makes the following diagram commutative:

$$\begin{array}{ccc} \widetilde{L}_W & \xrightarrow{\ i\ } & \mathrm{End}_W(H_W) \\ \cong\downarrow & & \downarrow\cong \\ \widetilde{L}_{\mathrm{cris}} & \xrightarrow{\ i_{\mathrm{cris}}\ } & \mathrm{End}_W(H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee). \end{array}$$

Using this isomorphism, we equip $H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee$ with a right action of the Clifford algebra $\mathrm{Cl}_W := \mathrm{Cl}\otimes_{\mathbb{Z}}W$ using the natural right action of Cl_W on H_W.

The Clifford algebra $\mathrm{Cl}_{\mathbb{F}_p}$ is a central simple algebra over the finite field $\mathbb{F}_p$ by [11, §9.4, Corollaire]. Hence, it is isomorphic to $\mathrm{M}_n(\mathbb{F}_p)$ as an $\mathbb{F}_p$-algebra where $n = 2^{11}$. Then $\mathrm{Cl}_{\mathbb{Z}_p}$ is isomorphic to $\mathrm{M}_n(\mathbb{Z}_p)$ as a $\mathbb{Z}_p$-algebra by [42, Chapter IV, Lemma 5.1.16]. We fix an isomorphism $\mathrm{Cl}_{\mathbb{Z}_p} \cong \mathrm{M}_n(\mathbb{Z}_p)$. Using this isomorphism, we equip $H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee$ with a right $\mathrm{M}_n(W)$-action.

The right $\mathrm{M}_n(\mathbb{Z}_p)$-action on $H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee[1/p]$ is compatible with the Frobenius automorphisms, and Fil^0 is an $\mathrm{M}_n(E)$-submodule of $H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee \otimes_W E$. Therefore, a σ-linear endomorphism $\varphi\otimes\sigma$ on

$$H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee[1/p] \otimes_{\mathrm{M}_n(K_0)} (K_0{}^n)$$

is well defined, and we consider the following E-vector subspace:

$$\mathrm{Fil}^0 \otimes_{\mathrm{M}_n(E)} (E^n) \subset (H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee \otimes_W E) \otimes_{\mathrm{M}_n(E)} (E^n).$$

They determine the structure of a filtered φ-module on

$$B := H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee[1/p] \otimes_{\mathrm{M}_n(K_0)} (K_0{}^n).$$

Because we have an isomorphism of filtered φ-modules

$$B^n \cong H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee[1/p],$$

it suffices to show that B is weakly admissible.

The embedding i_{cris} induces an isomorphism of W-modules

$$\mathrm{Cl}(\widetilde{L}_{\mathrm{cris}}) \cong \mathrm{End}_{\mathrm{Cl}_W}(H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee),$$

which is an isomorphism of filtered φ-modules after inverting p. By the Morita equivalence, we have

$$\mathrm{End}_{\mathrm{Cl}_{K_0}}(H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee[1/p]) \cong \mathrm{End}_{K_0}(B).$$

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Hence, we have an isomorphism $\mathrm{Cl}(\widetilde{L}_{\mathrm{cris}}[1/p]) \cong \mathrm{End}_{K_0}(B)$ of filtered φ-modules. Because $\widetilde{L}_{\mathrm{cris}}[1/p]$ is weakly admissible by Lemma 7.3, the filtered φ-module $\mathrm{Cl}(\widetilde{L}_{\mathrm{cris}}[1/p])$ is weakly admissible. Therefore, $B \otimes_{K_0} B^{\vee} \cong \mathrm{End}_{K_0}(B)$ is also weakly admissible.

In order to show that B is weakly admissible, we have to show $t_H(B') \leq t_N(B')$ for every filtered φ-submodule $B' \subset B$. (For the definition of the functions t_H and t_N, see [39, 1.1.3] for example.) Note that we have $t_H(B) = t_N(B) = -2^{10}$. Because $B \otimes_{K_0} B^{\vee}$ is weakly admissible, we have

$$t_H(B' \otimes B^{\vee}) \leq t_N(B' \otimes B^{\vee}).$$

Each side of the inequality may be computed as

$$\begin{aligned} t_H(B' \otimes B^{\vee}) &= \dim_{K_0}(B')t_H(B') + \dim_{K_0}(B^{\vee})t_H(B^{\vee}) \\ &= \dim_{K_0}(B')t_H(B') - \dim_{K_0}(B)t_H(B) \end{aligned}$$

and

$$\begin{aligned} t_N(B' \otimes B^{\vee}) &= \dim_{K_0}(B')t_N(B') + \dim_{K_0}(B^{\vee})t_N(B^{\vee}) \\ &= \dim_{K_0}(B')t_N(B') - \dim_{K_0}(B)t_N(B). \end{aligned}$$

It follows that $t_H(B') \leq t_N(B')$.

Therefore, B is weakly admissible, and the proof of Lemma 7.4 is complete. □

7.3. *p-Divisible groups adapted to general spin groups*

By Lemma 7.4, there is a crystalline $\mathrm{Gal}(\overline{E}/E)$-representation $H_{\text{ét},\mathbb{Q}_p}$ over $\mathbb{Q}_p$ such that

$$D_{\mathrm{cris}}(H_{\text{ét},\mathbb{Q}_p}) \cong H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^{\vee}[1/p]$$

as filtered φ-modules.

The tensors $\{s_{\beta,\mathrm{cris},s}\} \subset \{s_{\alpha,\mathrm{cris},s}\}$ corresponding to $p^{\pm}$, $\{r_{e_i}\}_{1 \leq i \leq 2^{22}}$, and the endomorphism π' preserve the filtration on $(H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^{\vee} \otimes_W E)^{\otimes}$ by Proposition 2.2. (For the tensors $p^{\pm}$, $\{r_{e_i}\}_{1 \leq i \leq 2^{22}}$ and π', see Subsection 2.3 for details.) Therefore, the tensors $\{s_{\beta,\mathrm{cris},s}\}$ induce tensors $\{s'_{\beta,p}\}$ in $H^{\otimes}_{\text{ét},\mathbb{Q}_p}$. By Proposition 2.2 again, the injection

$$i_{\mathrm{cris}} \otimes_W E \colon \widetilde{L}_{\mathrm{cris}} \otimes_W E \hookrightarrow \mathrm{End}_E(H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^{\vee} \otimes_W E)$$

preserves the filtrations. Therefore, $i_{\mathrm{cris}} \otimes_W E$ induces an inclusion of crystalline $\mathrm{Gal}(\overline{E}/E)$-representations

$$\widetilde{L}_p[1/p] \hookrightarrow \mathrm{End}_{\mathbb{Q}_p}(H_{\text{ét},\mathbb{Q}_p}).$$

(See Subsection 7.2 for the $\mathrm{Gal}(\overline{E}/E)$-stable $\mathbb{Z}_p$-lattice $\widetilde{L}_p$ in a crystalline representation.)

The derived group $\widetilde{G}^{\mathrm{der}}_{\mathbb{Q}_p}$ of $\widetilde{G}_{\mathbb{Q}_p}$ is the spin double cover of $\widetilde{G}_{0,\mathbb{Q}_p} = \mathrm{SO}(\widetilde{L}_{\mathbb{Q}_p})$ and a simply connected semisimple algebraic group over $\mathbb{Q}_p$. So, we have $H^1(\mathbb{Q}_p, \widetilde{G}^{\mathrm{der}}_{\mathbb{Q}_p}) = 1$ by [59, Theorem 6.4]. This implies that $H^1(\mathbb{Q}_p, \widetilde{G}_{\mathbb{Q}_p}) = 1$ and every $\widetilde{G}_{\mathbb{Q}_p}$-torsor over $\mathbb{Q}_p$ is trivial. Hence, there is an isomorphism of $\mathbb{Q}_p$-vector spaces

$$H_{\mathbb{Q}_p} \cong H_{\text{ét},\mathbb{Q}_p}$$

that carries $\{s_\beta\}$ to $\{s'_{\beta,p}\}$ and induces the following commutative diagram:

$$\begin{array}{ccc} \widetilde{L}_{\mathbb{Q}_p} & \xrightarrow{\ i\ } & \mathrm{End}_{\mathbb{Q}_p}(H_{\mathbb{Q}_p}) \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ \widetilde{L}_p[1/p] & \longrightarrow & \mathrm{End}_{\mathbb{Q}_p}(H_{\text{ét},\mathbb{Q}_p}), \end{array}$$

where $\widetilde{L}_{\mathbb{Q}_p} \cong \widetilde{L}_p[1/p]$ is an isometry over $\mathbb{Q}_p$.

Because the tensors $\{s'_{\beta,p}\}$ are fixed by $\mathrm{Gal}(\overline{E}/E)$ and $\widetilde{G}(\mathbb{Q}_p)$ is the stabiliser of the tensors $\{s_\beta\}$, we have the following commutative diagram:

$$\begin{array}{ccccc} & & \mathrm{GSpin}(\widetilde{L}_p)(\mathbb{Q}_p) & \xrightarrow{\ \cong\ } & \widetilde{G}(\mathbb{Q}_p) \\ & \overset{\rho}{\nearrow} & \downarrow & & \downarrow \\ \mathrm{Gal}(\overline{E}/E) & \xrightarrow{\ \rho_0\ } & \mathrm{SO}(\widetilde{L}_p)(\mathbb{Q}_p) & \xrightarrow{\ \cong\ } & \widetilde{G}_0(\mathbb{Q}_p). \end{array}$$

Here ρ_0 is the map corresponding to the action of $\mathrm{Gal}(\overline{E}/E)$ on $\widetilde{L}_p[1/p]$.

Lemma 7.5. *We have* $\rho(\mathrm{Gal}(\overline{E}/E)) \subset \mathrm{GSpin}(\widetilde{L}_p)(\mathbb{Z}_p)$.

Proof. Because we have an exact sequence of group schemes over $\mathbb{Z}_p$,

$$1 \to \mathbb{G}_{m,\mathbb{Z}_p} \to \mathrm{GSpin}(\widetilde{L}_p) \to \mathrm{SO}(\widetilde{L}_p) \to 1,$$

it follows that

$$\mathrm{GSpin}(\widetilde{L}_p)(\mathbb{Z}_p) \to \mathrm{SO}(\widetilde{L}_p)(\mathbb{Z}_p)$$

is surjective by Hilbert's theorem 90 and the smoothness of $\mathbb{G}_{m,\mathbb{Z}_p}$. Moreover, we have $\rho_0(\mathrm{Gal}(\overline{E}/E)) \subset \mathrm{SO}(\widetilde{L}_p)(\mathbb{Z}_p)$. Thus, for every $g \in \mathrm{Gal}(\overline{E}/E)$, there are $g' \in \mathrm{GSpin}(\widetilde{L}_p)(\mathbb{Z}_p)$ and $a \in \mathbb{Q}_p^\times$ such that $\rho(g) = ag'$. Let $\nu\colon \mathrm{GSpin}(\widetilde{L}_p) \to \mathbb{G}_{m,\mathbb{Z}_p}$ be the spinor norm. Because $\mathrm{Gal}(\overline{E}/E)$ is compact, we have $\nu(ag') = \nu(\rho(g)) \in \mathbb{Z}_p^\times$. Hence, we have $a \in \mathbb{Z}_p^\times$ and $\rho(g) = ag' \in \mathrm{GSpin}(\widetilde{L}_p)(\mathbb{Z}_p)$. In conclusion, we have $\rho(\mathrm{Gal}(\overline{E}/E)) \subset \mathrm{GSpin}(\widetilde{L}_p)(\mathbb{Z}_p)$. □

By Lemma 7.5, we see that $\mathrm{Cl}(\widetilde{L}_p)$ is a $\mathrm{Gal}(\overline{E}/E)$-stable $\mathbb{Z}_p$-lattice in the crystalline representation $H_{\text{ét},\mathbb{Q}_p}$, which will be denoted by $H_{\text{ét}}$.

Let K be the composite of E and $W(\overline{\mathbb{F}}_q)[1/p]$. In Proposition 7.7, we shall show that $H_{\text{ét}}$ satisfies properties that should be satisfied when $H_{\text{ét}}$ is the p-adic Tate module of the abelian scheme $\mathcal{A}_{\widetilde{s}}$ associated with a desired lift $\widetilde{s} \in Z_{K^p}(\Lambda)(\mathcal{O}_K)$.

The next lemma will be used in the proof of Proposition 7.7.

Lemma 7.6. *There is an isometry* $\widetilde{L}_p \cong \widetilde{L}_{\mathbb{Z}_p}$ *over* $\mathbb{Z}_p$.

Proof. There is an isometry $\widetilde{L}_{\mathrm{cris}} \cong \widetilde{L}_W$ over W; see Subsection 4.6. We have an isometry $\mathfrak{M}_{\mathrm{cris}}(\widetilde{L}_p) \cong \widetilde{L}_{\mathrm{cris}}$ over W by Lemma 7.3. By using [40, Corollary 1.3.5], we see that there is an isometry $\widetilde{L}_p \otimes_{\mathbb{Z}_p} W \cong \mathfrak{M}_{\mathrm{cris}}(\widetilde{L}_p)$ over W. So, the functor $\mathrm{Isom}(\widetilde{L}_p, \widetilde{L}_{\mathbb{Z}_p})$ on $\mathbb{Z}_p$-algebras that sends a $\mathbb{Z}_p$-algebra R to the set of isometries over R from $(\widetilde{L}_p)_R$ to $\widetilde{L}_R$ is represented by an $\mathrm{O}(\widetilde{L}_{\mathbb{Z}_p})$-torsor, which is also denoted by $\mathrm{Isom}(\widetilde{L}_p, \widetilde{L}_{\mathbb{Z}_p})$. Here $\mathrm{O}(\widetilde{L}_{\mathbb{Z}_p})$ is the orthogonal group over $\mathbb{Z}_p$. This $\mathrm{O}(\widetilde{L}_{\mathbb{Z}_p})$-torsor corresponds to an element $x \in H^1(\mathbb{Z}_p, \mathrm{O}(\widetilde{L}_{\mathbb{Z}_p}))$. Because there is an isometry $\widetilde{L}_p[1/p] \cong \widetilde{L}_{\mathbb{Q}_p}$ over $\mathbb{Q}_p$, it follows that x

comes from an element of $H^1(\mathbb{Z}_p, \mathrm{SO}(\widetilde{L}_{\mathbb{Z}_p}))$. By Lang's theorem and the smoothness of $\mathrm{SO}(\widetilde{L}_{\mathbb{Z}_p})$, we have $H^1(\mathbb{Z}_p, \mathrm{SO}(\widetilde{L}_{\mathbb{Z}_p})) = 1$. Thus, we see that the $\mathrm{O}(\widetilde{L}_{\mathbb{Z}_p})$-torsor $\mathrm{Isom}(\widetilde{L}_p, \widetilde{L}_{\mathbb{Z}_p})$ is trivial. □

Proposition 7.7. *There are* $\mathrm{Gal}(\overline{E}/E)$*-invariant tensors* $\{s_{\alpha,p}\}$ *in* $H_{\text{ét}}^{\otimes}$ *and an isomorphism of* $W(\overline{\mathbb{F}}_q)$*-modules*

$$\Phi\colon \mathfrak{M}_{\mathrm{cris}}(H_{\text{ét}}) \otimes_W W(\overline{\mathbb{F}}_q) \xrightarrow{\cong} H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee \otimes_W W(\overline{\mathbb{F}}_q)$$

satisfying the following properties:

(1) *There is an isomorphism of* $\mathbb{Z}_p$*-modules* $H_{\mathbb{Z}_p} \cong H_{\text{ét}}$ *that carries* $\{s_\alpha\}$ *to* $\{s_{\alpha,p}\}$ *and induces the following commutative diagram:*

$$\begin{array}{ccc} \widetilde{L}_{\mathbb{Z}_p} & \xrightarrow{i} & \mathrm{End}_{\mathbb{Z}_p}(H_{\mathbb{Z}_p}) \\ \cong\downarrow & & \downarrow\cong \\ \widetilde{L}_p & \longrightarrow & \mathrm{End}_{\mathbb{Z}_p}(H_{\text{ét}}), \end{array}$$

where $\widetilde{L}_{\mathbb{Z}_p} \cong \widetilde{L}_p$ *is an isometry over* $\mathbb{Z}_p$*.*

(2) Φ *is an isomorphism of* F*-isocrystals after inverting* p*.*

(3) Φ *carries* $\{\mathfrak{M}_{\mathrm{cris}}(s_{\alpha,p})\}$ *to* $\{s_{\alpha,\mathrm{cris},s}\}$*.*

(4) *The following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{M}_{\mathrm{cris}}(\widetilde{L}_p) \otimes_W W(\overline{\mathbb{F}}_q) & \longrightarrow & \mathrm{End}_{W(\overline{\mathbb{F}}_q)}(\mathfrak{M}_{\mathrm{cris}}(H_{\text{ét}}) \otimes_W W(\overline{\mathbb{F}}_q)) \\ \Vert & & \downarrow\cong \\ \widetilde{L}_{\mathrm{cris}} \otimes_W W(\overline{\mathbb{F}}_q) & \xrightarrow{i_{\mathrm{cris}}} & \mathrm{End}_{W(\overline{\mathbb{F}}_q)}(H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee \otimes_W W(\overline{\mathbb{F}}_q)), \end{array}$$

where we identify $\mathfrak{M}_{\mathrm{cris}}(\widetilde{L}_p) \otimes_W W(\overline{\mathbb{F}}_q)$ *with* $\widetilde{L}_{\mathrm{cris}} \otimes_W W(\overline{\mathbb{F}}_q)$ *using the isomorphism* $\mathfrak{M}_{\mathrm{cris}}(\widetilde{L}_p) \cong \widetilde{L}_{\mathrm{cris}}$ *in Lemma 7.3.*

Proof. There is an isometry $f\colon \widetilde{L}_p \cong \widetilde{L}_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$ by Lemma 7.6. We fix such an isometry. Let $\{f^*(s_\alpha)\}$ be the tensors in $H_{\text{ét}}^{\otimes} = \mathrm{Cl}(\widetilde{L}_p)^{\otimes}$ corresponding to the tensors $\{s_\alpha\}$ under f. Because we have $\rho(\mathrm{Gal}(\overline{E}/E)) \subset \mathrm{GSpin}(\widetilde{L}_p)(\mathbb{Z}_p)$, these tensors are $\mathrm{Gal}(\overline{E}/E)$-invariant. Let $\{\mathfrak{M}(f^*(s_\alpha))\}$ denote the induced tensors in $\mathfrak{M}(H_{\text{ét}})^{\otimes}$.

As in the proofs of [40, Proposition 1.3.4, Corollary 1.3.5], there are an isomorphism of $\mathfrak{S}$-modules

$$\Phi_f : \mathfrak{M}(H_{\text{ét}}) \cong H_{\mathfrak{S}}$$

and an isometry $\phi_f : \mathfrak{M}(\widetilde{L}_p) \cong \widetilde{L}_{\mathfrak{S}}$ such that Φ_f carries $\{\mathfrak{M}(f^*(s_\alpha))\}$ to $\{s_\alpha\}$ and a similar diagram as in Proposition 4.2 commutes. Let

$$\Phi'\colon H_W \cong H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee$$

be an isomorphism as in Subsection 4.6. Let $\phi'\colon \widetilde{L}_W \cong \widetilde{L}_{\mathrm{cris}}$ denote the induced isometry. Consider the composite of the following isomorphisms:

$$\psi\colon \widetilde{L}_{\mathrm{cris}} = \mathfrak{M}_{\mathrm{cris}}(\widetilde{L}_p) \xrightarrow{\phi_f} \widetilde{L}_W \xrightarrow{\phi'} \widetilde{L}_{\mathrm{cris}}.$$

We may assume $\det(\psi) = 1$ as follows: Suppose $\det(\psi) = -1$. We take an isometry $h\colon \widetilde{L}_{\mathbb{Z}_p} \cong \widetilde{L}_{\mathbb{Z}_p}$ with $\det(h) = -1$. (For example, h can be constructed using a decomposition $\widetilde{L}_{\mathbb{Z}_p} \cong U_{\mathbb{Z}_p} \oplus U^{\perp}_{\mathbb{Z}_p}$ of

https://doi.org/10.1017/fms.2021.24 Published online by Cambridge University Press

quadratic spaces.) The isomorphism $H_{\mathfrak{S}} \to H_{\mathfrak{S}}$ induced by this isometry is also denoted by h. Then the composite

$$\mathfrak{M}(H_{\text{ét}}) \xrightarrow{\Phi_f} H_{\mathfrak{S}} \xrightarrow{h} H_{\mathfrak{S}}$$

carries $\{\mathfrak{M}((h \circ f)^*(s_\alpha))\}$ to $\{s_\alpha\}$ and the composite

$$\widetilde{L}_{\text{cris}} = \mathfrak{M}_{\text{cris}}(\widetilde{L}_p) \xrightarrow{h\circ\phi_f} \widetilde{L}_W \xrightarrow{\phi'} \widetilde{L}_{\text{cris}}$$

has determinant 1. Therefore, after replacing $\{f^*(s_\alpha)\}$ by $\{(h \circ f)^*(s_\alpha)\}$ and replacing Φ_f by $h \circ \Phi_f$, we may assume $\det(\psi) = 1$.

Then there is an element $g \in \widetilde{G}(W)$ whose image under the surjection $\widetilde{G}(W) \twoheadrightarrow \widetilde{G}_0(W)$ is $(\phi_f \circ \phi')^{-1} \in \widetilde{G}_0(W)$. After replacing Φ' by $\Phi' \circ g$, we may assume $\psi = \text{id}$.

Under the isomorphism Φ_f, the Frobenius on $\mathfrak{M}_{\text{cris}}(H_{\text{ét}})[1/p]$ has the form of $b\sigma$ for some $b \in \widetilde{G}(K_0)$. Similarly, under the isomorphism Φ', the Frobenius on $H^1_{\text{cris}}(\mathcal{A}_s/W)^\vee[1/p]$ has the form of $b'\sigma$ for some $b' \in \widetilde{G}(K_0)$. Because $\psi = \text{id}$, the elements b and b' have the same image under the surjection $\widetilde{G}(K_0) \twoheadrightarrow \widetilde{G}_0(K_0)$. Hence, there is an element $u \in K_0^\times$ such that $b = ub'$.

We shall show $u \in W^\times$. Because the Hodge-Tate weights of $H_{\text{ét},\mathbb{Q}_p}$ are in $\{0, 1\}$ (i.e., $\text{Gr}^i(D_{\text{dR}}(H_{\text{ét},\mathbb{Q}_p})^\vee) = 0$ if $i \neq 0, 1$), the effective Breuil-Kisin module $\mathfrak{M}(H^\vee_{\text{ét}})$ is of height ≤ 1. Hence, the cokernel of the Frobenius of $\mathfrak{M}_{\text{cris}}(H_{\text{ét}})^\vee$ is killed by p. Moreover, because $\mathcal{A}_s$ is an abelian variety of dimension 2^{21}, the cokernel of the Frobenius of $H^1_{\text{cris}}(\mathcal{A}_s/W)$ is isomorphic to $(W/p)^{\oplus 2^{21}}$ as a W-module. From these facts, we conclude $u \in W^\times$.

Take an element $v \in W(\overline{\mathbb{F}}_q)^\times$ such that $\sigma(v)/v = u$. Then the isomorphism

$$\mathfrak{M}_{\text{cris}}(H_{\text{ét}}) \otimes_W W(\overline{\mathbb{F}}_q) \xrightarrow{\Phi_f} H_{W(\overline{\mathbb{F}}_q)} \xrightarrow{\times v} H_{W(\overline{\mathbb{F}}_q)} \xrightarrow{\Phi'} H^1_{\text{cris}}(\mathcal{A}_s/W)^\vee \otimes_W W(\overline{\mathbb{F}}_q)$$

is an isomorphism of F-isocrystals after inverting p. This isomorphism satisfies the properties of the proposition.

The proof of Proposition 7.7 is complete. □

Recall that there is an equivalence of categories between the category of p-divisible groups over $\mathcal{O}_E$ and the category of $\text{Gal}(\overline{E}/E)$-stable $\mathbb{Z}_p$-lattices in crystalline representations whose Hodge-Tate weights are in $\{0, 1\}$; see [39, Corollary 2.2.6], [48, Theorem 2.2.1]. Therefore, there is a (unique) p-divisible group $\mathcal{H}$ over $\mathcal{O}_E$ whose p-adic Tate module is isomorphic to $H_{\text{ét}}$.

By Proposition 7.7, the base change $\mathcal{H}_{\mathcal{O}_K}$ is a lift of the p-divisible group $\mathcal{A}_{\overline{s}}[p^\infty]$ associated with $\mathcal{A}_{\overline{s}}$. As a corollary of Proposition 7.7, the p-divisible group $\mathcal{H}_{\mathcal{O}_K}$ is $\widetilde{G}$-adapted to $\mathcal{A}_{\overline{s}}[p^\infty]$ in the sense of [38, Definition 3.3].

Corollary 7.8. *The p-divisible group $\mathcal{H}_{\mathcal{O}_K}$ is $\widetilde{G}$-adapted to $\mathcal{A}_{\overline{s}}[p^\infty]$.*

Proof. The assertion follows from Proposition 7.7 and [38, Theorem 2.5]. □

7.4. *Proof of Theorem 7.1*

In this subsection, we shall complete the proof of Theorem 7.1.

By Corollary 7.8, there is a lift $\widetilde{s} \in \mathcal{S}_{\widetilde{\mathrm{K}}}(\mathcal{O}_K)$ of $\overline{s} \in \mathcal{S}_{\widetilde{\mathrm{K}}}(\overline{\mathbb{F}}_q)$ such that the p-divisible group associated with $\mathcal{A}_{\widetilde{s}}$ is isomorphic to $\mathcal{H}_{\mathcal{O}_K}$. See [38, Lemma 3.8] and the proof of [38, Proposition 4.6] for details. Although the residue field is assumed to be finite in [38, Lemma 3.8], the same result holds for $\mathcal{O}_K$ because the p-divisible group $\mathcal{A}_{\overline{s}}[p^\infty]$ is defined over the finite field k.

Proposition 7.7 (4) implies that the 0th piece of the Hodge filtration on $H^1_{\text{cris}}(\mathcal{A}_s/W)^\vee \otimes_W K$ coincides with $\text{Fil}^0(H^1_{\text{cris}}(\mathcal{A}_s/W)^\vee \otimes_W K)$ defined in Subsection 7.1.

To prove Theorem 7.1, it remains to show $\widetilde{s} \in \mathscr{S}_{\widetilde{\mathrm{K}}}(\mathscr{O}_K)$ lifts to an $\mathscr{O}_K$-valued point of $Z_{\mathrm{K}^p}(\Lambda)$. By the construction of $H_{\text{ét}}$, we have a $\mathrm{Gal}(\overline{K}/K)$-equivariant homomorphism

$$\Lambda_{\mathbb{Z}_p} \subset \widetilde{L}_p \to \mathrm{End}_{\mathbb{Z}_p}(H_{\text{ét}}).$$

Therefore, we get $\Lambda_{\mathbb{Z}_{(p)}} \to \mathrm{End}_{\mathscr{O}_K}(\mathcal{H}_{\mathscr{O}_K})$ lifting the map $\Lambda_{\mathbb{Z}_{(p)}} \to \mathrm{End}_{\overline{\mathbb{F}}_q}(\mathcal{A}_{\overline{s}}[p^\infty])$ induced by ι. By the Serre-Tate theorem, the Λ-structure ι lifts to a homomorphism

$$\Lambda_{\mathbb{Z}_{(p)}} \to \mathrm{End}_{\mathscr{O}_K}(\mathcal{A}_{\widetilde{s}})_{\mathbb{Z}_{(p)}}.$$

By construction, this is a Λ-structure for $\widetilde{s}$. The proof of Theorem 7.1 is complete. □

8. Kisin's algebraic groups associated with Kuga-Satake abelian varieties over finite fields

In this section, we attach an algebraic group I over $\mathbb{Q}$ to a quasi-polarised $K3$ surface of finite height over $\overline{\mathbb{F}}_q$. It is a subgroup of the multiplicative group of the endomorphism algebra of the Kuga-Satake abelian variety. Then we study its action on the formal Brauer group of the $K3$ surface.

8.1. Kisin's algebraic groups

Let $s \in Z_{\mathrm{K}^p}(\Lambda)(\mathbb{F}_q)$ be an $\mathbb{F}_q$-valued point and $\overline{s} \in Z_{\mathrm{K}^p}(\Lambda)(\overline{\mathbb{F}}_q)$ a geometric point above s. Let $\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_{\overline{s}})$ denote the algebraic group over $\mathbb{Q}$ defined by

$$\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_{\overline{s}})(R) := (\mathrm{End}_{\overline{\mathbb{F}}_q}(\mathcal{A}_{\overline{s}}) \otimes_{\mathbb{Z}} R)^\times$$

for every $\mathbb{Q}$-algebra R.

After replacing $\mathbb{F}_q$ by a finite extension of it, we may assume that all endomorphisms of $\mathcal{A}_{\overline{s}}$ are defined over $\mathbb{F}_q$. Namely, we have $\mathrm{End}_{\mathbb{F}_q}(\mathcal{A}_s) \cong \mathrm{End}_{\overline{\mathbb{F}}_q}(\mathcal{A}_{\overline{s}})$.

The global sections $\{s_\alpha^p\}$ of $V^p(\mathcal{A})^\otimes$ give rise to global sections $\{s_{\alpha,\ell}\}$ of $V_\ell(\mathcal{A})^\otimes$. The $\mathbb{A}_f^p$-local system $\widetilde{\mathbb{V}}^p$ and the embedding i^p give a $\mathbb{Q}_\ell$-local system $\widetilde{\mathbb{V}}_\ell$ and an embedding $i_\ell \colon \widetilde{\mathbb{V}}_\ell \hookrightarrow \underline{\mathrm{End}}(V_\ell(\mathcal{A}))$ of $\mathbb{Q}_\ell$-local systems for every $\ell \neq p$. The Λ-structure for $s \in Z_{\mathrm{K}^p}(\Lambda)(\mathbb{F}_q)$ gives a homomorphism $\iota_\ell \colon \Lambda_{\mathbb{Q}_\ell} \to \widetilde{\mathbb{V}}_{\ell,\overline{s}}$. We fix an isomorphism of $\mathbb{Q}_\ell$-vector spaces

$$H_{\mathbb{Q}_\ell} \cong V_\ell(\mathcal{A}_{\overline{s}})$$

that carries $\{s_\alpha\}$ to $\{s_{\alpha,\ell,\overline{s}}\}$ and induces the following commutative diagram:

$$\begin{array}{ccccc}
\Lambda_{\mathbb{Q}_\ell} & \longrightarrow & \widetilde{L}_{\mathbb{Q}_\ell} & \xrightarrow{\ i\ } & \mathrm{End}_{\mathbb{Q}_\ell}(H_{\mathbb{Q}_\ell}) \\
& \searrow^{\iota_\ell} & \downarrow \cong & & \downarrow \cong \\
& & \widetilde{\mathbb{V}}_{\ell,\overline{s}} & \xrightarrow{\ i_\ell\ } & \mathrm{End}_{\mathbb{Q}_\ell}(V_\ell(\mathcal{A}_{\overline{s}})),
\end{array}$$

where $\widetilde{L}_{\mathbb{Q}_\ell} \cong \widetilde{\mathbb{V}}_{\ell,\overline{s}}$ is an isometry over $\mathbb{Q}_\ell$.

Let $H_W \cong H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee$ be an isomorphism as in Subsection 4.6. After inverting p and composing an element of $\widetilde{G}(W[1/p])$, we can find an isomorphism

$$H_{W[1/p]} \cong H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee[1/p]$$

that carries $\{s_\alpha\}$ to $\{s_{\alpha,\mathrm{cris},s}\}$ and induces the same diagram as above by [50, Lemma 2.8] and the fact that every $\mathrm{GSpin}(L_{W[1/p]})$-torsor over $W[1/p]$ is trivial (see [59, Theorem 6.4] and the arguments in Subsection 7.3). We fix such an isomorphism.

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Kisin introduced an algebraic group $\widetilde{I}_\ell$ over $\mathbb{Q}_\ell$ for every prime number ℓ (including $\ell = p$) and an algebraic group $\widetilde{I}$ over $\mathbb{Q}$ as follows; see [41, (2.1.2)] for details.

(1) For a prime number $\ell \neq p$, let $\mathrm{Frob}_q \in \mathrm{End}_{\mathbb{Q}_\ell}(H_{\mathbb{Q}_\ell})$ denote the $\mathbb{Q}_\ell$-endomorphism of $H_{\mathbb{Q}_\ell}$ induced by the qth power Frobenius of $\mathcal{A}_{\overline{s}}$. Because Frob_q fixes the tensors $\{s_{\alpha,\ell,\overline{s}}\}$, we have $\mathrm{Frob}_q \in \widetilde{G}(\mathbb{Q}_\ell)$. For every integer $m \geq 1$, we define an algebraic $\mathbb{Q}_\ell$-subgroup $\widetilde{I}_{\ell,m}$ of $\widetilde{G}_{\mathbb{Q}_\ell}$ by

$$\widetilde{I}_{\ell,m}(R) := \{\, g \in \widetilde{G}(R) \mid g\,\mathrm{Frob}_q^m = \mathrm{Frob}_q^m\, g \,\}$$

for every $\mathbb{Q}_\ell$-algebra R. For sufficiently divisible $m \geq 1$, the algebraic group $\widetilde{I}_{\ell,m}$ does not depend on m, and it is denoted by $\widetilde{I}_\ell$.

(2) For $\ell = p$, we define an algebraic group $\widetilde{I}_{p,m}$ over $\mathbb{Q}_p$ by

$$\widetilde{I}_{p,m}(R) := \{\, g \in \widetilde{G}(R \otimes_{\mathbb{Q}_p} W(\mathbb{F}_{q^m})[1/p]) \mid g\varphi = \varphi g \,\}.$$

For sufficiently divisible $m \geq 1$, the algebraic group $\widetilde{I}_{p,m}$ does not depend on m, and it is denoted by $\widetilde{I}_p$.

(3) Let $\widetilde{I} \subset \mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_{\overline{s}})$ be the largest closed $\mathbb{Q}$-subgroup of $\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_{\overline{s}})$ mapped into $\widetilde{I}_\ell$ for every ℓ (including $\ell = p$).

Replacing $\mathbb{F}_q$ by a finite extension of it, we may assume $\widetilde{I}_{\ell,1} = \widetilde{I}_\ell$ and $\widetilde{I}_{p,1} = \widetilde{I}_p$.

For our purpose, we need an algebraic subgroup $I \subset \widetilde{I}$ over $\mathbb{Q}$ defined using the Λ-structure; see Definition 4.3. If L is self-dual at p, it coincides with Kisin's algebraic group associated with an $\mathbb{F}_q$-valued point of the integral canonical model of $\mathrm{Sh}_{\mathrm{K}_0}(\mathrm{SO}(L_{\mathbb{Q}}), X_L)$ taken in a similar way as in Subsection 6.2.

Definition 8.1.

(1) Let $I \subset \widetilde{I}$ be the algebraic subgroup over $\mathbb{Q}$ defined by

$$I(R) := \{\, g \in \widetilde{I}(R) \mid ghg^{-1} = h \text{ in } \mathrm{End}_{\overline{\mathbb{F}}_q}(\mathcal{A}_{\overline{s}})_R \text{ for every } h \in \iota(\Lambda_{\mathbb{Q}}) \,\}$$

for every $\mathbb{Q}$-algebra R.

(2) For a prime number $\ell \neq p$, let $I_\ell \subset \widetilde{I}_\ell$ be the algebraic subgroup over $\mathbb{Q}_\ell$ defined by

$$I_\ell(R) := \{\, g \in \widetilde{I}_\ell(R) \mid ghg^{-1} = h \text{ in } \mathrm{End}_{\mathrm{Frob}_q}(H_{\mathbb{Q}_\ell})_R \text{ for every } h \in i(\Lambda_{\mathbb{Q}_\ell}) \,\}$$

for every $\mathbb{Q}_\ell$-algebra R.

(3) For $\ell = p$, let $I_p \subset \widetilde{I}_p$ be the algebraic subgroup over $\mathbb{Q}_p$ defined in a similar way as above.

As in Kisin's paper [41], we shall prove that the natural map

$$I_{\mathbb{Q}_\ell} \to I_\ell$$

is an isomorphism of algebraic groups over $\mathbb{Q}_\ell$ for every ℓ (including $\ell = p$). Here we prove it for some $\ell \neq p$. The case of general ℓ will be proved later; see Corollary 9.9.

Proposition 8.2.

(1) *For some prime number $\ell \neq p$, the natural map $I_{\mathbb{Q}_\ell} \to I_\ell$ is an isomorphism of algebraic groups over $\mathbb{Q}_\ell$.*

(2) *The algebraic groups I and $\mathrm{GSpin}(L_{\mathbb{Q}})$ over $\mathbb{Q}$ have the same rank. (Recall that the rank of an algebraic group over a field k is the dimension of a maximal k-torus of it.)*

Proof. (1) We fix a prime number $\ell \neq p$ such that $\mathrm{GSpin}(L_{\mathbb{Q}})$ and $\widetilde{G}_{\mathbb{Q}}$ are split at ℓ and all of the eigenvalues of Frob_q acting on $H_{\mathbb{Q}_\ell}$ are contained in $\mathbb{Q}_\ell$. We shall show that the assertion (1) holds for such ℓ. By the proof of [41, Corollary 2.1.7], the homomorphism $\widetilde{I}_{\mathbb{Q}_\ell} \to \widetilde{I}_\ell$ is an isomorphism.

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(Precisely, Kisin proved it in [41] assuming $p \geq 3$ and the restriction of ψ to $H_{\mathbb{Z}_{(p)}}$ is perfect. These assumptions are unnecessary; see the proof of [38, Theorem A.8].) By Tate's theorem, we have

$$\mathrm{End}_{\mathbb{F}_q}(\mathcal{A}_s)_{\mathbb{Q}_\ell} \cong \mathrm{End}_{\mathrm{Frob}_q}(H_{\mathbb{Q}_\ell}).$$

Now, the assertion (1) follows from the definitions of I and I_ℓ.

(2) We follow Kisin's proof of [41, Corollary 2.1.7]. Because Frob_q and I_ℓ act trivially on $i(\Lambda_{\mathbb{Q}_\ell})$, we have $\mathrm{Frob}_q \in \mathrm{GSpin}(L_{\mathbb{Q}_\ell})$ and $I_\ell \subset \mathrm{GSpin}(L_{\mathbb{Q}_\ell})$; see [50, (2.6.1)]. The element $\mathrm{Frob}_q \in \mathrm{GSpin}(L_{\mathbb{Q}_\ell})$ is semisimple because the action of Frob_q on $V_\ell(\mathcal{A}_{\bar{s}})$ is semisimple. Thus, the connected component S of the Zariski closure of the group $\langle \mathrm{Frob}_q \rangle$ generated by Frob_q is a split torus in $\mathrm{GSpin}(L_{\mathbb{Q}_\ell})$ by the hypotheses on ℓ. Because I_ℓ is the same as the centraliser of Frob_q^m in $\mathrm{GSpin}(L_{\mathbb{Q}_\ell})$ for a sufficiently divisible m, it follows that I_ℓ coincides with the centraliser of S. Therefore, I_ℓ contains a split maximal torus of $\mathrm{GSpin}(L_{\mathbb{Q}_\ell})$. (Hence, I_ℓ is a connected split reductive group over $\mathbb{Q}_\ell$.) In particular, the rank of I_ℓ as an algebraic group over $\mathbb{Q}_\ell$ is equal to the rank of $\mathrm{GSpin}(L_{\mathbb{Q}_\ell})$ as an algebraic group over $\mathbb{Q}_\ell$. Because we have $I_{\mathbb{Q}_\ell} \cong I_\ell$, the ranks of the algebraic groups I and $\mathrm{GSpin}(L_{\mathbb{Q}})$ over $\mathbb{Q}$ are equal. □

Remark 8.3. With the results of [38], it should be possible to show that Kisin's result [41, Corollary 2.2.5] on CM liftings of mod p points on integral canonical models of Shimura varieties of Hodge type *up to isogeny* is valid in any characteristic $p > 0$ including $p = 2$. Then this would imply that, for every $p > 0$ including $p = 2$, the map $\widetilde{I}_{\mathbb{Q}_\ell} \to \widetilde{I}_\ell$ (and hence $I_{\mathbb{Q}_\ell} \to I_\ell$) is an isomorphism for every prime number ℓ (including $\ell = p$); see [41, Corollary 2.3.2] for the case $p \geq 3$. In this article, we will prove such a result only under the assumption that $s \in Z_{\mathrm{K}^p}(\Lambda)(\mathbb{F}_q)$ arises from a quasi-polarised $K3$ surface of finite height; see Corollary 9.9.

8.2. *The action of Kisin's groups on the formal Brauer groups of $K3$ surfaces*

We consider the situation as in Section 7 and keep the notation. In particular, we assume that the height h of X is finite. We attach the algebraic group I over $\mathbb{Q}$ to $s \in Z_{\mathrm{K}^p}(\Lambda)(\mathbb{F}_q)$.

As in the proof of Proposition 6.6, we have a decomposition

$$P^2_{\mathrm{cris}}(X/W) \cong L_{1-1/h} \oplus L_1 \oplus L_{1+1/h}.$$

Here, L_λ is an F-crystal over W that has a single slope λ for each $\lambda \in \{1 - 1/h,\ 1,\ 1 + 1/h\}$. Moreover, there is an isomorphism of F-crystals over W:

$$\mathbb{D}(\widehat{\mathrm{Br}}) \cong L_{1+1/h}(1).$$

By Proposition 6.4, the algebraic group I acts on $P^2_{\mathrm{cris}}(X/W)(1)[1/p]$. Hence, we have the following homomorphism of algebraic groups over $\mathbb{Q}_p$:

$$I_{\mathbb{Q}_p} \to \mathrm{Res}_{K_0/\mathbb{Q}_p} \mathrm{GL}(P^2_{\mathrm{cris}}(X/W)(1)[1/p]).$$

For a one-dimensional smooth formal group $\mathscr{G}$ over a ring A, let $\mathrm{Aut}_{\mathbb{Q}_p}(\mathscr{G})$ denote the $\mathbb{Q}_p$-group such that

$$\mathrm{Aut}_{\mathbb{Q}_p}(\mathscr{G})(R) := (\mathrm{End}_A(\mathscr{G}) \otimes_{\mathbb{Z}_p} R)^\times$$

for every $\mathbb{Q}_p$-algebra R.

Lemma 8.4. *There is a homomorphism*

$$I_{\mathbb{Q}_p} \to (\mathrm{Aut}_{\mathbb{Q}_p}(\widehat{\mathrm{Br}}))^{\mathrm{op}}$$

that is compatible with $I_{\mathbb{Q}_p} \to \mathrm{Res}_{K_0/\mathbb{Q}_p} \mathrm{GL}(P^2_{\mathrm{cris}}(X/W)(1)[1/p])$ *via the projection* $P^2_{\mathrm{cris}}(X/W)(1) \to \mathbb{D}(\widehat{\mathrm{Br}})$.

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Proof. We put $V_\lambda := L_{1+\lambda}(1)[1/p]$ for each $\lambda \in \{-1/h, 0, 1/h\}$. For an F-isocrystal M over K_0, let $\mathrm{GL}_\varphi(M)$ denote the algebraic group over $\mathbb{Q}_p$ defined by

$$\mathrm{GL}_\varphi(M)(R) := \{\, g \in \mathrm{GL}_{K_0 \otimes_{\mathbb{Q}_p} R}(M \otimes_{\mathbb{Q}_p} R) \mid g\varphi = \varphi g \,\}$$

for every $\mathbb{Q}_p$-algebra R, where φ is the Frobenius of M. We have an isomorphism of algebraic groups over $\mathbb{Q}_p$:

$$\mathrm{GL}_\varphi(P^2_{\mathrm{cris}}(X/W)(1)[1/p]) \cong \mathrm{GL}_\varphi(V_{-1/h}) \times \mathrm{GL}_\varphi(V_0) \times \mathrm{GL}_\varphi(V_{1/h}).$$

Let $I_{\mathbb{Q}_p} \to \mathrm{GL}_\varphi(V_{1/h})$ be the composite of the map $I_{\mathbb{Q}_p} \to \mathrm{GL}_\varphi(P^2_{\mathrm{cris}}(X/W)(1)[1/p])$ with the projection $\mathrm{GL}_\varphi(P^2_{\mathrm{cris}}(X/W)(1)[1/p]) \to \mathrm{GL}_\varphi(V_{1/h})$. Note that we have an isomorphism $(\mathrm{Aut}_{\mathbb{Q}_p}(\widehat{\mathrm{Br}}))^{\mathrm{op}} \cong \mathrm{GL}_\varphi(V_{1/h})$, which completes the proof. □

9. Lifting of $K3$ surfaces over finite fields with actions of tori

In this section, we prove our main results on CM liftings of $K3$ surfaces of finite height over finite fields using the results of Section 7.

9.1. $K3$ *surfaces with complex multiplication*

In this subsection, we recall the definition and basic properties of $K3$ surfaces with complex multiplication over $\mathbb{C}$.

Let Y be a projective $K3$ surface over $\mathbb{C}$. Let

$$T_Y := \mathrm{Pic}(Y)^{\perp}_{\mathbb{Q}} \subset H^2_B(Y, \mathbb{Q}(1))$$

denote the transcendental part of the singular cohomology, which has the $\mathbb{Q}$-Hodge structure coming from $H^2(Y, \mathbb{Q}(1))$.

Let

$$E_Y := \mathrm{End}_{\mathrm{Hdg}}(T_Y)$$

denote the $\mathbb{Q}$-algebra of $\mathbb{Q}$-linear endomorphisms on T_Y preserving the $\mathbb{Q}$-Hodge structure on it. We say that Y has *complex multiplication (CM)* if E_Y is a CM field and $\dim_{E_Y}(T_Y) = 1$. Here a number field is called CM if it is a purely imaginary quadratic extension of a totally real number field.

Let $\mathrm{MT}(T_Y)$ denote the *Mumford-Tate group* of T_Y. By the definition, it is the smallest algebraic $\mathbb{Q}$-subgroup of $\mathrm{SO}(T_Y)$ such that $h_Y(\mathbb{S}(\mathbb{R})) \subset \mathrm{MT}(T_Y)(\mathbb{R})$, where

$$h_Y : \mathbb{S} \to \mathrm{SO}(T_Y)_{\mathbb{R}}$$

is the homomorphism over $\mathbb{R}$ corresponding to the $\mathbb{Q}$-Hodge structure of T_Y. By the results of Zarhin [76, Section 2], the $K3$ surface Y has CM if and only if the Mumford-Tate group $\mathrm{MT}(T_Y)$ is commutative.

In the rest of this subsection, we fix a $\mathbb{C}$-valued point $t \in M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Q}}(\mathbb{C})$. Let (Y, ξ) be the quasi-polarised $K3$ surface over $\mathbb{C}$ associated with t. The image of t under the Kuga-Satake morphism KS is also denoted by $t \in Z_{\mathrm{K}_0^p}(\Lambda)(\mathbb{C})$.

Proposition 9.1. *Assume that Y is a $K3$ surface with CM over $\mathbb{C}$. Then $t \in M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Q}}(\mathbb{C})$ is defined over a number field.*

Proof. This proposition follows from Rizov's result [61, Corollary 3.9.4] as follows. The image of $t \in Z_{\mathrm{K}_0^p}(\Lambda)(\mathbb{C})$ under the morphism $Z_{\mathrm{K}_0^p}(\Lambda) \to \mathcal{S}_{\widetilde{\mathrm{K}}_0}$ is denoted by the same symbol t. If Y has CM, then the residue field of the image $t \in \mathcal{S}_{\widetilde{\mathrm{K}}_0}(\mathbb{C})$ is a number field by the definition of the canonical

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model $\mathcal{S}_{\widetilde{\mathrm{K}}_0,\mathbb{Q}} = \mathrm{Sh}_{\widetilde{\mathrm{K}}_0}$ over $\mathbb{Q}$. Because the morphism $Z_{\mathrm{K}_0^p}(\Lambda) \to \mathcal{S}_{\widetilde{\mathrm{K}}_0}$ is finite by Proposition 4.6, the residue field of $t \in Z_{\mathrm{K}_0^p}(\Lambda)(\mathbb{C})$ is a number field. Now, the assertion follows from the étaleness of the Kuga-Satake morphism KS (in characteristic 0). □

Remark 9.2. Pjateckiĭ-Šapiro and Šafarevič also showed that every $K3$ surface with CM is defined over a number field; see [58, Theorem 4].

For the quasi-polarised $K3$ surface (Y, ξ) over $\mathbb{C}$, the primitive singular cohomology is defined by

$$P^2_B(Y, \mathbb{Q}(1)) := \mathrm{ch}_B(\xi)^\perp \subset H^2_B(Y, \mathbb{Q}(1)).$$

We fix a $\mathbb{C}$-valued point of $Z_{\mathrm{K}^p}(\Lambda)$ mapped to t, and it is also denoted by $t \in Z_{\mathrm{K}^p}(\Lambda)(\mathbb{C})$. We have the Kuga-Satake abelian variety $\mathcal{A}_t$ over $\mathbb{C}$ corresponding to $t \in Z_{\mathrm{K}^p}(\Lambda)(\mathbb{C})$. As in Subsection 6.2, the stalk

$$\widetilde{V}_t := \widetilde{\mathbb{V}}_{B,t}$$

satisfies the following properties:

- $\widetilde{V}_t$ admits a perfect bilinear form $(\ ,\ )$ over $\mathbb{Q}$ that is a polarisation.
- There is a homomorphism $\iota_B \colon \Lambda_{\mathbb{Q}} \to \widetilde{V}_t$ preserving the bilinear forms and the $\mathbb{Q}$-Hodge structures.
- There is an isometry $P^2_B(Y, \mathbb{Q}(1)) \cong \iota_B(\Lambda_{\mathbb{Q}})^\perp$ over $\mathbb{Q}$ preserving the $\mathbb{Q}$-Hodge structures.
- The following diagram commutes:

$$\begin{array}{ccccc} & & \mathrm{GSpin}(\widetilde{V}_t)_{\mathbb{R}} & \longrightarrow & \mathrm{GL}(H^1_B(\mathcal{A}_t, \mathbb{Q})^\vee)_{\mathbb{R}} \\ & \overset{h}{\nearrow} & \downarrow & & \\ \mathbb{S} & \xrightarrow[h_0]{} & \mathrm{SO}(\widetilde{V}_t)_{\mathbb{R}}, & & \end{array}$$

  where h_0 is the homomorphism of algebraic groups over $\mathbb{R}$ corresponding to the $\mathbb{Q}$-Hodge structure on $\widetilde{V}_t$ and the composite

$$\mathbb{S} \xrightarrow{h} \mathrm{GSpin}(\widetilde{V}_t)_{\mathbb{R}} \to \mathrm{GL}(H^1_B(\mathcal{A}_t, \mathbb{Q})^\vee)_{\mathbb{R}}$$

  corresponds to the $\mathbb{Q}$-Hodge structure on $H^1_B(\mathcal{A}_t, \mathbb{Q})^\vee$.

Proposition 9.3. *The $K3$ surface Y has CM if and only if the Kuga-Satake abelian variety $\mathcal{A}_t$ has CM.*

Proof. This proposition was essentially proved by Tretkoff; see [71, Corollary 3.2]. We give an argument from the point of view of algebraic groups. Because h_0 is the composite of h_Y with the following inclusions,

$$\mathrm{SO}(T_Y)_{\mathbb{R}} \hookrightarrow \mathrm{SO}(P^2_B(Y, \mathbb{Q}(1)))_{\mathbb{R}} \hookrightarrow \mathrm{SO}(\widetilde{V}_t)_{\mathbb{R}},$$

we have $\mathrm{MT}(T_Y) \cong \mathrm{MT}(\widetilde{V}_t)$. We shall show that $\mathrm{MT}(\widetilde{V}_t)$ is commutative if and only if $\mathrm{MT}(H^1_B(\mathcal{A}_t, \mathbb{Q})^\vee)$ is commutative. Because $\mathrm{MT}(H^1_B(\mathcal{A}_t, \mathbb{Q})^\vee)$ is contained in $\mathrm{GSpin}(\widetilde{V}_t)$ and $\mathrm{MT}(\widetilde{V}_t)$ is contained in the image of $\mathrm{MT}(H^1_B(\mathcal{A}_t, \mathbb{Q})^\vee)$, we only have to show $\mathrm{MT}(H^1_B(\mathcal{A}_t, \mathbb{Q})^\vee)$ is commutative if $\mathrm{MT}(\widetilde{V}_t)$ is so.

We assume that $\mathrm{MT}(\widetilde{V}_t)$ is commutative. Then the inverse image of $\mathrm{MT}(\widetilde{V}_t)$ under the homomorphism $\mathrm{GSpin}(\widetilde{V}_t) \to \mathrm{SO}(\widetilde{V}_t)$ is a solvable algebraic group and contains $\mathrm{MT}(H^1_B(\mathcal{A}_t, \mathbb{Q})^\vee)$. Because $\mathrm{MT}(H^1_B(\mathcal{A}_t, \mathbb{Q})^\vee)$ is a reductive group (see [25, Proposition 3.6]), it is commutative. □

Corollary 9.4. *Let F be a field that can be embedded in $\mathbb{C}$. Let Z be a $K3$ surface over F. If $Z \otimes_{F,j} \mathbb{C}$ has CM for an embedding $j \colon F \hookrightarrow \mathbb{C}$, then $Z \otimes_{F,j'} \mathbb{C}$ has CM for every embedding $j' \colon F \hookrightarrow \mathbb{C}$.*

Proof. The assertion follows from Proposition 9.3 and the fact that, for an abelian variety A over $\mathbb{C}$ with CM and every automorphism $f \colon \mathbb{C} \cong \mathbb{C}$, the abelian variety $A \otimes_{\mathbb{C},f} \mathbb{C}$ has CM. □

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Remark 9.5. Let F be a field of characteristic 0 that can be embedded into $\mathbb{C}$ and Z a $K3$ surface over F. We say that Z has CM if $Z \otimes_{F,j} \mathbb{C}$ has CM for some embedding $j\colon F \hookrightarrow \mathbb{C}$, in which case $Z \otimes_{F,j'} \mathbb{C}$ has CM for *every* embedding $j'\colon F \hookrightarrow \mathbb{C}$ by Corollary 9.4.

9.2. A lemma on liftings of formal groups with action of tori

The following result on characteristic 0 liftings of one-dimensional smooth formal groups is presumably well known. (For the definition of $\mathrm{Aut}_{\mathbb{Q}_p}(\mathcal{G}_0)$ and $\mathrm{Aut}_{\mathbb{Q}_p}(\mathcal{G})$, see Subsection 8.2.)

Lemma 9.6. *Let $\mathcal{G}_0$ be a one-dimensional smooth formal group over $\mathbb{F}_q$. Let T_p be an algebraic torus over $\mathbb{Q}_p$ and*

$$\rho\colon T_p \to \mathrm{Aut}_{\mathbb{Q}_p}(\mathcal{G}_0)$$

a homomorphism of algebraic groups over $\mathbb{Q}_p$. Assume that the height of $\mathcal{G}_0$ is finite and the Frobenius Φ of $\mathcal{G}_0$ over $\mathbb{F}_q$ is contained in $\rho(T_p(\mathbb{Q}_p))$. Then, there exist a finite totally ramified extension E of K_0 and a smooth formal group $\mathcal{G}$ over $\mathcal{O}_E$ satisfying the following properties:

(1) *The special fibre of $\mathcal{G}$ is isomorphic to $\mathcal{G}_0$.*
(2) *The homomorphism ρ factors as*

$$T_p \to \mathrm{Aut}_{\mathbb{Q}_p}(\mathcal{G}) \to \mathrm{Aut}_{\mathbb{Q}_p}(\mathcal{G}_0).$$

Proof. We fix an isomorphism $\mathcal{G}_0 \cong \mathrm{Spf}\,\mathbb{F}_q[[x]]$ and consider $\mathcal{G}_0$ as a formal group law in $\mathbb{F}_q[[x,y]]$.

The composite of ρ with the inclusion $\mathrm{Aut}_{\mathbb{Q}_p}(\mathcal{G}_0) \to \mathrm{Aut}_{\mathbb{Q}_p}(\mathcal{G}_{0,\overline{\mathbb{F}}_q})$ is also denoted by ρ. Take a maximal $\mathbb{Q}_p$-torus T'_p of $\mathrm{Aut}_{\mathbb{Q}_p}(\mathcal{G}_{0,\overline{\mathbb{F}}_q})$ containing $\rho(T_p)$. It is well known that $\mathrm{End}_{\overline{\mathbb{F}}_q}(\mathcal{G}_{0,\overline{\mathbb{F}}_q}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a central division algebra over $\mathbb{Q}_p$ and $\mathrm{End}_{\overline{\mathbb{F}}_q}(\mathcal{G}_{0,\overline{\mathbb{F}}_q})$ is the maximal order of it; see [33, Corollary 20.2.14]. Hence, there is a maximal commutative $\mathbb{Q}_p$-subalgebra

$$K' \subset \mathrm{End}_{\overline{\mathbb{F}}_q}(\mathcal{G}_{0,\overline{\mathbb{F}}_q}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

such that $K'^{\times} = T'_p$ as algebraic groups over $\mathbb{Q}_p$, and we have $\mathcal{O}_{K'} \subset \mathrm{End}_{\overline{\mathbb{F}}_q}(\mathcal{G}_{0,\overline{\mathbb{F}}_q})$. Moreover, because $T'_p(\mathbb{Q}_p)$ contains the Frobenius Φ of $\mathcal{G}_0$ over $\mathbb{F}_q$, the endomorphisms in K' commute with Φ. Hence, we have $\mathcal{O}_{K'} \subset \mathrm{End}_{\mathbb{F}_q}(\mathcal{G}_0)$.

We regard the formal group law $\mathcal{G}_0$ as a formal $\mathcal{O}_{K'}$-module over $\mathbb{F}_q$ in the sense of [33, (18.6.1)]. The universal formal $\mathcal{O}_{K'}$-module $\mathcal{G}^{\mathrm{univ}}$ exists and it is a formal $\mathcal{O}_{K'}$-group over a polynomial ring $\mathcal{O}_{K'}[(S_i)_{i\in\mathbb{N}}]$ with infinitely many variables over $\mathcal{O}_{K'}$; see [33, (21.4.8)]. (See also [28, Proposition 1.4]. Beware that $\mathcal{G}^{\mathrm{univ}}$ does not classify isomorphism classes of formal $\mathcal{O}_{K'}$-modules but formal $\mathcal{O}_{K'}$-modules.)

We take a finite totally ramified extension E of K_0 such that E is a K'-algebra. Then there is a formal $\mathcal{O}_{K'}$-module $\mathcal{G} \in \mathcal{O}_E[[x,y]]$ over $\mathcal{O}_E$ whose reduction modulo the maximal ideal of $\mathcal{O}_E$ is equal to $\mathcal{G}_0$ such that the homomorphism $\mathcal{O}_{K'} \to \mathrm{End}_{\mathbb{F}_q}(\mathcal{G}_0)$ factors as

$$\mathcal{O}_{K'} \to \mathrm{End}_{\mathcal{O}_E}(\mathcal{G}) \to \mathrm{End}_{\mathbb{F}_q}(\mathcal{G}_0).$$

(See also [28, Corollary (2) to Proposition 1.4].) Therefore, the homomorphism ρ factors as $T_p \to \mathrm{Aut}_{\mathbb{Q}_p}(\mathcal{G}) \to \mathrm{Aut}_{\mathbb{Q}_p}(\mathcal{G}_0)$. □

9.3. Liftings of $K3$ surfaces over finite fields with actions of tori

Let (Y,ξ) be a quasi-polarised $K3$ surface over a field k of characteristic 0 or p. Let $\overline{k}$ be an algebraic closure of k. For a prime number $\ell \neq p$, the primitive part of the ℓ-adic cohomology is denoted by

$$P^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_\ell(1)) := \mathrm{ch}_\ell(\xi)^{\perp} \subset H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_\ell(1)).$$

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It is equipped with a canonical action of $\mathrm{Gal}(\overline{k}/k)$. When $\overline{k}$ is a subfield of $\mathbb{C}$, we have a canonical isomorphism

$$P^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_\ell(1)) \cong P^2_B(Y_{\mathbb{C}}, \mathbb{Q}(1)) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell.$$

We consider the situation as in Subsection 6.2 and Section 7 and keep the notation. In particular, $(X, \mathscr{L})$ is a quasi-polarised $K3$ surface of finite height over $\mathbb{F}_q$. We attach the algebraic group I over $\mathbb{Q}$ to the $\mathbb{F}_q$-valued point $s \in Z_{\mathrm{K}^p}(\Lambda)(\mathbb{F}_q)$; see Definition 8.1.

The following theorem concerns characteristic 0 liftings of $K3$ surfaces of finite height over finite extensions of $W(\overline{\mathbb{F}}_q)[1/p]$. Because every $K3$ surface with CM is defined over a number field (see Proposition 9.1 and Remark 9.2), Theorem 9.7 implies Theorem 1.1 in the Introduction; see Corollary 9.10.

Theorem 9.7. *Let $T \subset I$ be a maximal torus over $\mathbb{Q}$. Then there exist a finite extension K of $W(\overline{\mathbb{F}}_q)[1/p]$ and a quasi-polarised $K3$ surface $(\mathcal{X}, \mathcal{L})$ over $\mathcal{O}_K$ such that the special fibre $(\mathcal{X}_{\overline{\mathbb{F}}_q}, \mathcal{L}_{\overline{\mathbb{F}}_q})$ is isomorphic to $(X_{\overline{\mathbb{F}}_q}, \mathscr{L}_{\overline{\mathbb{F}}_q})$ and, for every embedding $K \hookrightarrow \mathbb{C}$, the quasi-polarised $K3$ surface $(\mathcal{X}_{\mathbb{C}}, \mathcal{L}_{\mathbb{C}})$ satisfies the following properties:*

(1) *The $K3$ surface $\mathcal{X}_{\mathbb{C}}$ has CM.*
(2) *There is a homomorphism of algebraic groups over $\mathbb{Q}$*

$$T \to \mathrm{SO}(P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))).$$

(3) *For every $\ell \neq p$, the action of $T(\mathbb{Q}_\ell)$ on $P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1)) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ is identified with the action of $T(\mathbb{Q}_\ell)$ on $P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$ via the canonical isomorphisms*

$$P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1)) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong P^2_{\text{ét}}(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}_\ell(1)) \cong P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1)))$$

(using the embedding $K \hookrightarrow \mathbb{C}$, we consider K as a subfield of $\mathbb{C}$).
(4) *The action of every element of $T(\mathbb{Q})$ on $P^2_B(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}(1))$ preserves the $\mathbb{Q}$-Hodge structure on it.*

Proof. Recall that $\widehat{\mathrm{Br}} = \widehat{\mathrm{Br}}(X)$ is the formal Brauer group associated with X. Let $I_{\mathbb{Q}_p} \to (\mathrm{Aut}_{\mathbb{Q}_p}(\widehat{\mathrm{Br}}))^{\mathrm{op}}$ be the homomorphism in Lemma 8.4. This induces a homomorphism

$$T_{\mathbb{Q}_p} \to \mathrm{Aut}_{\mathbb{Q}_p}(\widehat{\mathrm{Br}})$$

of algebraic groups over $\mathbb{Q}_p$.

By Lemma 9.6, there exist a finite totally ramified extension E of K_0 and a one-dimensional smooth formal group $\mathcal{G}$ over $\mathcal{O}_E$ whose special fibre is isomorphic to $\widehat{\mathrm{Br}}$ such that the homomorphism ρ factors as

$$T_{\mathbb{Q}_p} \to \mathrm{Aut}_{\mathbb{Q}_p}(\mathcal{G}) \to \mathrm{Aut}_{\mathbb{Q}_p}(\widehat{\mathrm{Br}}).$$

As in Section 7, we have the filtration associated with $\mathcal{G}$,

$$\mathrm{Fil}^1(\mathcal{G}) \hookrightarrow \mathbb{D}(\widehat{\mathrm{Br}}) \otimes_W E \hookrightarrow \widetilde{L}_{\mathrm{cris}} \otimes_W E.$$

Take a generator e of $\mathrm{Fil}^1(\mathcal{G})$ and let $i(e) := (i_{\mathrm{cris}} \otimes_{K_0} E)(e)$ denote the image of e under the embedding

$$i_{\mathrm{cris}} \otimes_W E \colon \widetilde{L}_{\mathrm{cris}} \otimes_W E \hookrightarrow \mathrm{End}_E(H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee \otimes_W E).$$

By Theorem 7.1, there exist a finite extension K of $W(\overline{\mathbb{F}}_q)[1/p]$ containing E and an $\mathcal{O}_K$-valued point $\widetilde{s} \in Z_{\mathrm{K}^p}(\Lambda)(\mathcal{O}_K)$ such that

- $\widetilde{s}$ is a lift of $\overline{s}$ and
- the 0th piece $\mathrm{Fil}^0_{\widetilde{s}} \subset H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee \otimes_W K \cong H^1_{\mathrm{dR}}(\mathcal{A}_{\widetilde{s}}/K)^\vee$ of the Hodge filtration coincides with the image $i(e)(H^1_{\mathrm{cris}}(\mathcal{A}_s/W)^\vee \otimes_W K)$.

Because the embedding i_{cris} is $I_{\mathbb{Q}_p}$-equivariant and the action of $T_{\mathbb{Q}_p}$ on $\widetilde{L}_{\mathrm{cris}}[1/p]$ preserves $\mathrm{Fil}^1(\mathcal{G})$, we see that the action of T on $H^1_{\mathrm{dR}}(\mathcal{A}_{\widetilde{s}}/K)^\vee$ preserves $\mathrm{Fil}^0_{\widetilde{s}}$. Therefore, the $\mathbb{Q}$-torus T can be considered as a $\mathbb{Q}$-torus in $(\mathrm{End}_{\mathcal{O}_K}(\mathcal{A}_{\widetilde{s}}) \otimes_{\mathbb{Z}} \mathbb{Q})^\times$.

Because the Kuga-Satake morphism is étale (see Proposition 5.2), the $\mathcal{O}_K$-valued point $\widetilde{s} \in Z_{\mathrm{K}^p}(\Lambda)(\mathcal{O}_K)$ can be lifted to an $\mathcal{O}_K$-valued point of $M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Z}_{(p)}}$. This lift is also denoted by $\widetilde{s}$. Let $(\mathfrak{X}, \mathfrak{L})$ be the quasi-polarised $K3$ surface over $\mathcal{O}_K$ corresponding to $\widetilde{s}$. Its special fibre is isomorphic to $(X_{\overline{\mathbb{F}}_q}, \mathscr{L}_{\overline{\mathbb{F}}_q})$.

We choose an embedding $K \hookrightarrow \mathbb{C}$. We shall show that $\mathfrak{X} \otimes_{\mathcal{O}_K} \mathbb{C}$ satisfies the conditions of Theorem 9.7. Let x denote the $\mathbb{C}$-valued point that comes from $\widetilde{s}$ and the embedding $K \hookrightarrow \mathbb{C}$. The algebraic group T over $\mathbb{Q}$ can be considered as a subgroup of $\mathrm{GL}(H^1_B(\mathcal{A}_x, \mathbb{Q})^\vee)$. We fix an isomorphism of $\mathbb{Q}$-vector spaces

$$H_{\mathbb{Q}} \cong H^1_B(\mathcal{A}_x, \mathbb{Q})^\vee$$

that carries $\{s_\alpha\}$ to $\{s_{\alpha,B,x}\}$ and induces the following commutative diagram:

$$\begin{array}{ccccc}
\Lambda_{\mathbb{Q}} & \longrightarrow & \widetilde{L}_{\mathbb{Q}} & \xrightarrow{\ i\ } & \mathrm{End}_{\mathbb{Q}}(H_{\mathbb{Q}}) \\
 & \searrow^{\iota_B} & \downarrow \cong & & \cong \downarrow \\
 & & \widetilde{V}_x & \xrightarrow{\ i_B\ } & \mathrm{End}_{\mathbb{Q}}(H^1_B(\mathcal{A}_x, \mathbb{Q})^\vee).
\end{array}$$

This isomorphism identifies $\mathrm{GSpin}(\widetilde{V}_x)$ with the subgroup of $\mathrm{GL}(H^1_B(\mathcal{A}_x, \mathbb{Q}))^\vee$ defined by $\{s_{\alpha,B,x}\}$. Then T is contained in $\mathrm{GSpin}(\widetilde{V}_x)$. Moreover, because T is contained in I, we see that T is compatible with $\iota_B(\Lambda_{\mathbb{Q}})$. Hence, we have

$$T \hookrightarrow \mathrm{GSpin}(P^2_B(\mathfrak{X}_{\mathbb{C}}, \mathbb{Q}(1))).$$

Composing this inclusion with $\mathrm{GSpin}(P^2_B(\mathfrak{X}_{\mathbb{C}}, \mathbb{Q}(1))) \to \mathrm{SO}(P^2_B(\mathfrak{X}_{\mathbb{C}}, \mathbb{Q}(1)))$, we have a homomorphism of algebraic groups over $\mathbb{Q}$,

$$T \to \mathrm{SO}(P^2_B(\mathfrak{X}_{\mathbb{C}}, \mathbb{Q}(1))).$$

The base change of this homomorphism to $\mathbb{Q}_\ell$ is identified with the homomorphism

$$T_{\mathbb{Q}_\ell} \to \mathrm{SO}(P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1)))$$

via the canonical isomorphism $P^2_B(\mathfrak{X}_{\mathbb{C}}, \mathbb{Q}(1)) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$.

Finally, we shall prove that the $K3$ surface $\mathfrak{X}_{\mathbb{C}}$ has CM. It is enough to show that the Mumford-Tate group of $P^2_B(\mathfrak{X}_{\mathbb{C}}, \mathbb{Q}(1))$ is commutative; see Subsection 9.1. It suffices to prove that the image of $\mathbb{S} \to \mathrm{SO}(P^2_B(\mathfrak{X}_{\mathbb{C}}, \mathbb{R}(1)))$ is contained in the image of $T_{\mathbb{R}}$. To prove this, it suffices to show that the image of

$$\mathbb{S} \to \mathrm{GSpin}(P^2_B(\mathfrak{X}_{\mathbb{C}}, \mathbb{R}(1)))$$

is contained in $T_{\mathbb{R}}$. By Proposition 8.2, it follows that T is a maximal $\mathbb{Q}$-torus of $\mathrm{GSpin}(P^2_B(\mathfrak{X}_{\mathbb{C}}, \mathbb{Q}(1)))$. Therefore, it suffices to show that the image of $\mathbb{S}$ is contained in the centraliser of $T_{\mathbb{R}}$. Because $T(\mathbb{Q})$ is Zariski dense in $T_{\mathbb{R}}$, this follows from the fact that every element of $T(\mathbb{Q})$ comes from an element of $\mathrm{End}_{\mathbb{C}}(\mathcal{A}_x) \otimes_{\mathbb{Z}} \mathbb{Q}$.

The proof of Theorem 9.7 is complete. □

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Remark 9.8. As suggested by the referee, it should be possible to apply our methods to any $\overline{\mathbb{F}}_p$-valued point s of $Z_{\mathrm{K}^p}(\Lambda)$ (or on more general orthogonal Shimura varieties), such that the associated F-crystal $\widetilde{L}_{\mathrm{cris},s}(-1)$ has a decomposition

$$\widetilde{L}_{\mathrm{cris},s}(-1) \cong \widetilde{L}_{1-1/h} \oplus \widetilde{L}_1 \oplus \widetilde{L}_{1+1/h}$$

as in the proof of Proposition 6.6, in order to show that the point s admits CM liftings. However, we do not discuss it in this article.

Using Theorem 9.7, we can show that the assertion of Proposition 8.2 (1) holds for every ℓ (including $\ell = p$).

Corollary 9.9. *For every prime number ℓ (including $\ell = p$), the canonical homomorphism $I_{\mathbb{Q}_\ell} \to I_\ell$ is an isomorphism.*

Proof. The assertion follows from Proposition 8.2 and Theorem 9.7. (See the proof of [41, Corollary 2.3.2] for details.) □

We shall give applications of Theorem 9.7 to CM liftings and quasi-canonical liftings of $K3$ surfaces of finite height over a finite field. For the definition of CM liftings used in this article, see Subsection 1.1. For the definition of quasi-canonical liftings, see [55, Definition 1.5].

Corollary 9.10. *Let X be a $K3$ surface of finite height over $\mathbb{F}_q$. Then there is a positive integer $m \geq 1$ such that $X_{\mathbb{F}_{q^m}}$ admits a CM lifting.*

Proof. After replacing $\mathbb{F}_q$ by its finite extension, we may assume that X comes from an $\mathbb{F}_q$-valued point $s \in M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Z}_{(p)}}(\mathbb{F}_q)$ satisfying the conditions as in Subsection 6.2. By Theorem 9.7 and Proposition 9.1, after replacing $\mathbb{F}_q$ by its finite extension again, there exist a number field F, a finite place v of F with residue field $\mathbb{F}_q$ and a $K3$ surface $\mathscr{X}$ over $\mathscr{O}_{F,(v)}$ whose special fibre $\mathscr{X}_{\mathbb{F}_q}$ is isomorphic to X and generic fibre $\mathscr{X}_F$ is a $K3$ surface with CM over F. □

Corollary 9.11. *Let X be a $K3$ surface of finite height over $\mathbb{F}_q$. Then there is a positive integer $m \geq 1$ such that $X_{\mathbb{F}_{q^m}}$ admits a quasi-canonical lifting.*

Proof. After replacing $\mathbb{F}_q$ by its finite extension, we may assume that X comes from an $\mathbb{F}_q$-valued point $s \in M^{\mathrm{sm}}_{2d,\mathrm{K}^p_0,\mathbb{Z}_{(p)}}(\mathbb{F}_q)$ satisfying the conditions as in Subsection 6.2. Because $\mathrm{Frob}_q^m \in T(\mathbb{Q})$ for a sufficiently divisible $m \geq 1$, the characteristic 0 lifting constructed in Theorem 9.7 is a quasi-canonical lifting in the sense of Nygaard-Ogus. Hence, the assertion follows from Theorem 9.7 and Proposition 9.1. □

Remark 9.12. Corollary 9.11 was previously known when $p \geq 5$ by Nygaard-Ogus [55, Theorem 5.6]. Precisely, when $p \geq 5$, Nygaard-Ogus proved the existence of quasi-canonical liftings without extending the base field $\mathbb{F}_q$. On the other hand, by the methods of this article, it is necessary to take a finite extension $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$ for some $m \geq 1$, and we do not know how to control m.

10. The Tate conjecture for the square of a $K3$ surfaces over finite fields

In this section, combining Theorem 9.7 with the results of Mukai and Buskin on the Hodge conjecture for products of $K3$ surfaces, we shall prove the Tate conjecture for the square of a $K3$ surface over a finite field.

10.1. The statement of the main results

We recall the statement of the Tate conjecture. Let V be a projective smooth variety over $\mathbb{F}_q$. For a prime number $\ell \neq p$, the Tate conjecture for ℓ-adic cohomology states the surjectivity of the ℓ-adic

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cycle class map

$$\mathrm{cl}^i_\ell\colon Z^i(V)\otimes_{\mathbb{Z}}\mathbb{Q}_\ell \to H^{2i}_{\text{ét}}(V_{\overline{\mathbb{F}}_q},\mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)}$$

for every i. Similarly, the Tate conjecture for crystalline cohomology states the surjectivity of the crystalline cycle class map

$$\mathrm{cl}^i_{\mathrm{cris}}\colon Z^i(V)\otimes_{\mathbb{Z}}\mathbb{Q}_p \to H^{2i}_{\mathrm{cris}}(V/W(\mathbb{F}_q))^{\varphi=p^i}\otimes_{\mathbb{Z}}\mathbb{Q}$$

for every i. Here $Z^i(V)$ denotes the group of algebraic cycles of codimension i on V. (See [55, Conjecture 0.1], [69, Section 1], [70, Conjecture 1.1].)

Here is the statement of our results on the Tate conjecture for the square of a $K3$ surface over a finite field. As before, let p be a prime number and q a power of p.

Theorem 10.1. *Let X be a $K3$ surface (of any height) over $\mathbb{F}_q$. We put $X\times X := X\times_{\mathrm{Spec}\,\mathbb{F}_q} X$ and $X_{\overline{\mathbb{F}}_q}\times X_{\overline{\mathbb{F}}_q} := X_{\overline{\mathbb{F}}_q}\times_{\mathrm{Spec}\,\overline{\mathbb{F}}_q} X_{\overline{\mathbb{F}}_q}$.*

(1) *For every prime number $\ell\neq p$, the ℓ-adic cycle class map*

$$\mathrm{cl}^i_\ell\colon Z^i(X\times X)\otimes_{\mathbb{Z}}\mathbb{Q}_\ell \to H^{2i}_{\text{ét}}(X_{\overline{\mathbb{F}}_q}\times X_{\overline{\mathbb{F}}_q},\mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)}$$

is surjective for every i.

(2) *The crystalline cycle class map*

$$\mathrm{cl}^i_{\mathrm{cris}}\colon Z^i(X\times X)\otimes_{\mathbb{Z}}\mathbb{Q}_p \to H^{2i}_{\mathrm{cris}}((X\times X)/W(\mathbb{F}_q))^{\varphi=p^i}\otimes_{\mathbb{Z}}\mathbb{Q}$$

is surjective for every i.

10.2. Previous results on the Tate conjecture

In this subsection, we recall previously known results on the Tate conjecture that will be used to prove Theorem 10.1.

Lemma 10.2. *Let V be a projective smooth variety over $\mathbb{F}_q$. Let ℓ be a prime number different from p. Let i be an integer and $m\geq 1$ a positive integer. If the Tate conjecture holds for algebraic cycles of codimension i on the variety $V_{\mathbb{F}_{q^m}}$ over $\mathbb{F}_{q^m}$, then the Tate conjecture holds for algebraic cycles of codimension i on the variety V over $\mathbb{F}_q$.*

Proof. See [70, Section 2, p.6] for example. □

Lemma 10.3. *Let X be a $K3$ surface over $\mathbb{F}_q$. Let ℓ be a prime number different from p. The ℓ-adic cycle class map cl^i_ℓ for $X\times X$ is surjective for every $i\neq 2$. The same is true for the crystalline cycle class map $\mathrm{cl}^i_{\mathrm{cris}}$ for every $i\neq 2$.*

Proof. It is enough to prove the assertion for $i=1,3$. For $i=1$, the Künneth formula gives an isomorphism:

$$\begin{aligned}&H^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}\times X_{\overline{\mathbb{F}}_q},\mathbb{Q}_\ell(1))\\ &\cong \bigoplus_{(i,j)=(0,2),(2,0)} H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_q},\mathbb{Q}_\ell)\otimes_{\mathbb{Q}_\ell} H^j_{\text{ét}}(X_{\overline{\mathbb{F}}_q},\mathbb{Q}_\ell)\otimes_{\mathbb{Q}_\ell}\mathbb{Q}_\ell(1).\end{aligned}$$

Every element in the left-hand side fixed by $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is written as $\mathrm{pr}_1^*\alpha+\mathrm{pr}_2^*\beta$ for some $\alpha,\beta\in H^2_{\text{ét}}(X\otimes_{\mathbb{F}_q}\overline{\mathbb{F}}_q,\mathbb{Q}_\ell(1))$ fixed by $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, where $\mathrm{pr}_1,\mathrm{pr}_2\colon X\times X\to X$ are the projections. By the Tate conjecture for the $K3$ surface X (see [18, 38, 49, 52]), such elements α and β are $\mathbb{Q}_\ell$-linear combinations of classes of divisors on X. Hence, $\mathrm{pr}_1^*\alpha + \mathrm{pr}_2^*\beta$ is a $\mathbb{Q}_\ell$-linear combination of classes of divisors on $X \times X$. This proves the surjectivity of cl_ℓ^1.

To show the surjectivity of cl_ℓ^3, we take an ample line bundle $\mathscr{L}$ on $X \times X$. By the hard Lefschetz theorem, the cup product with the square $(\mathrm{ch}_\ell(\mathscr{L}))^2$ of the first Chern class induces a $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-equivariant isomorphism:

$$H^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q} \times X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1)) \cong H^6_{\text{ét}}(X_{\overline{\mathbb{F}}_q} \times X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(3)).$$

Taking the $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-invariants and using the surjectivity of cl_ℓ^1, we see that cl_ℓ^3 is surjective.

The proof for the crystalline cohomology is exactly the same. □

Lemma 10.4. *Let X be a $K3$ surface over $\mathbb{F}_q$. Assume that Theorem 10.1 (1) holds for $i = 2$ and for some $\ell \neq p$. Then Theorem 10.1 (1) holds for $i = 2$ and for every $\ell \neq p$, and Theorem 10.1 (2) holds for $i = 2$.*

Proof. Let $N^2(X \times X) := Z^2(X \times X)/\sim_{\mathrm{num}}$ denote the group of numerically equivalent classes of algebraic cycles of codimension 2 on $X \times X$. It is a finitely generated abelian group. Assume that Theorem 10.1 (1) holds for a prime number $\ell_0 \neq p$. Because the action of the geometric Frobenius morphism Frob_q on the ℓ_0-adic cohomology $H^4_{\text{ét}}(X_{\overline{\mathbb{F}}_q} \times X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_{\ell_0}(2))$ is semisimple (see [24, Corollaire 1.10]), the order of the zero at $t = 1$ of the characteristic polynomial $P(t)$ of Frob_q on $H^4_{\text{ét}}(X_{\overline{\mathbb{F}}_q} \times X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_{\ell_0}(2))$ is equal to the rank of $N^2(X \times X)$; see [69, Theorem 2.9]. For any prime number $\ell \neq p$, the following inequality holds (see [69, Proposition 2.8 (iii)]):

$$\mathrm{rank}_{\mathbb{Z}} N^2(X \times X) \leq \dim_{\mathbb{Q}_\ell}(H^4_{\text{ét}}(X_{\overline{\mathbb{F}}_q} \times X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(2))^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)}).$$

Because the characteristic polynomial of Frob_q does not depend on ℓ, the above inequality is an equality for any $\ell \neq p$. Hence, Theorem 10.1 (1) holds for $i = 2$ and for every $\ell \neq p$.

The same argument works also for crystalline cohomology. We put $q = p^r$, $K_0 := W(\mathbb{F}_q)[1/p]$ and

$$H := H^4_{\mathrm{cris}}((X \times X)/W(\mathbb{F}_q)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The rth power φ^r of the absolute Frobenius automorphism acts K_0-linearly on H, and the characteristic polynomial of φ^r on H coincides with the characteristic polynomial of Frob_q on the ℓ-adic cohomology for every $\ell \neq q$. Hence, we have the following equality: $\mathrm{rank}_{\mathbb{Z}} N^2(X \times X) = \dim_{K_0}(H^{\varphi^r = q^2})$. The action of $p^{-2}\varphi$ on the K_0-vector space $H^{\varphi^r = q^2}$ can be considered as a semilinear action of $\mathrm{Gal}(K_0/\mathbb{Q}_p)$ on $H^{\varphi^r = q^2}$. By Hilbert's theorem 90, we have $\dim_{\mathbb{Q}_p}(H^{\varphi = p^2}) = \dim_{K_0}(H^{\varphi^r = q^2})$. Hence, we have $\mathrm{rank}_{\mathbb{Z}} N^2(X \times X) = \dim_{\mathbb{Q}_p}(H^{\varphi = p^2})$. Consequently, Theorem 10.1 (2) holds for $i = 2$. □

Remark 10.5. We can prove the following: If Theorem 10.1 (2) holds for $i = 2$, then Theorem 10.1 (1) holds for $i = 2$ and for every $\ell \neq p$. To prove this, it is enough to show that, for a $K3$ surface X over $\mathbb{F}_q$ with $q = p^r$, the action of φ^r on the crystalline cohomology of X is semisimple. When X is of finite height, it follows from the semisimplicity of Frobenius for the Kuga-Satake abelian variety. (See [49, Theorem 5.17 (3)]. For the case $p = 2$, see also [38, Appendix A].) When X is supersingular, it follows from the Tate conjecture for X; see Lemma 10.6.

The following results on supersingular $K3$ surfaces are well known.

Lemma 10.6. *Let X be a $K3$ surface over an algebraically closed field k of characteristic $p > 0$. Then X is supersingular (i.e., the height of X is ∞) if and only if the rank of the Picard group* $\mathrm{Pic}(X)$ *is 22.*

Proof. See [46, Theorem 4.8] for example. (Precisely, the characteristic p is assumed to be odd in [46, Theorem 4.8]. But the same proof works in the case $p = 2$ because the Tate conjecture for $K3$ surfaces in characteristic 2 is now proved by [38, Theorem A.1]. See also [51].) □

Lemma 10.7. *Let X be a supersingular $K3$ surface over $\mathbb{F}_q$. Then the Tate conjecture for $X \times X$ holds for the ℓ-adic cohomology for every prime number $\ell \neq p$ and for the crystalline cohomology.*

Proof. Fix a prime number $\ell \neq p$. After replacing $\mathbb{F}_q$ by a finite extension of it, we may assume that $H^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$ is spanned by classes of divisors on X defined over $\mathbb{F}_q$ by Lemma 10.6. The Künneth formula implies that $H^4_{\text{ét}}(X_{\overline{\mathbb{F}}_q} \times X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(2))$ is spanned by classes of algebraic cycles of codimension 2 on $X \times X$. Thus, the Tate conjecture holds for $X \times X$. The same proof works for the crystalline cohomology. □

Remark 10.8. By the same argument, we can prove that the Tate conjecture holds for any power $X \times \cdots \times X$ for a supersingular $K3$ surface X over $\mathbb{F}_q$.

10.3. Endomorphisms of the cohomology of a $K3$ surface over a finite field

Let X be a $K3$ surface of finite height over $\mathbb{F}_q$. After replacing $\mathbb{F}_q$ by its finite extension, we may assume that X comes from an $\mathbb{F}_q$-valued point $s \in M^{\text{sm}}_{2d, \mathrm{K}_0^p, \mathbb{Z}_{(p)}}(\mathbb{F}_q)$ satisfying the conditions as in Subsection 6.2. Let I be the algebraic group over $\mathbb{Q}$ associated with $s \in Z_{\mathrm{K}^p}(\Lambda)(\mathbb{F}_q)$; see Definition 8.1.

In this subsection, we fix a prime number $\ell \neq p$. Let

$$V_\ell := \mathrm{ch}_\ell(\mathrm{Pic}(X_{\overline{\mathbb{F}}_q}))^\perp \subset H^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$$

denote the transcendental part of the ℓ-adic cohomology. By the Tate conjecture for X [18, 38, 49, 52], none of the eigenvalues of Frob_q is a root of unity.

Lemma 10.9.

(1) *There is a* $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$*-equivariant isomorphism*

$$H^4_{\text{ét}}(X_{\overline{\mathbb{F}}_q} \times X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(2)) \\ \cong \mathbb{Q}_\ell \oplus (\mathrm{Pic}(X_{\overline{\mathbb{F}}_q})^{\otimes 2} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell) \oplus (\mathrm{Pic}(X_{\overline{\mathbb{F}}_q}) \otimes_{\mathbb{Z}} V_\ell)^{\oplus 2} \oplus \mathrm{End}_{\mathbb{Q}_\ell}(V_\ell).$$

(2) *The Tate conjecture holds for $X \times X$ if and only if the $\mathbb{Q}_\ell$-vector subspace*

$$\mathrm{End}_{\mathrm{Frob}_q}(V_\ell) = \mathrm{End}_{\mathbb{Q}_\ell}(V_\ell)^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)}$$

is spanned by classes of algebraic cycles of codimension 2 *on $X \times X$.*

Proof. (1) We have

$$H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell & i = 0 \\ (\mathrm{Pic}(X_{\overline{\mathbb{F}}_q}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell(-1)) \oplus V_\ell(-1) & i = 2 \\ \mathbb{Q}_\ell(-2) & i = 4 \\ 0 & i \neq 0, 2, 4. \end{cases}$$

By the Poincaré duality, we have isomorphisms $V_\ell \otimes_{\mathbb{Q}_\ell} V_\ell \cong V_\ell^\vee \otimes_{\mathbb{Q}_\ell} V_\ell \cong \mathrm{End}_{\mathbb{Q}_\ell}(V_\ell)$. Hence, the assertion (1) follows by the Künneth formula.

(2) The $\mathbb{Q}_\ell$-vector space $\mathrm{Pic}(X_{\overline{\mathbb{F}}_q}) \otimes_{\mathbb{Z}} V_\ell$ has no nonzero $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-invariants. Hence, the assertion (2) follows. □

Because the action of $I(\mathbb{Q}_\ell)$ on the primitive part $P^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell(1))$ commutes with Frob_q, it also acts on V_ℓ. For a sufficiently divisible $m \geq 1$, the following conditions are satisfied:

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- $I_\ell = I_{\ell,m} = I_{\mathbb{Q}_\ell}$ (for the definition of $I_\ell, I_{\ell,m}$, see Subsection 8.1).
- The image of $I_{\ell,m}$ under the homomorphism $\mathrm{GSpin}(V_\ell) \to \mathrm{SO}(V_\ell)$ is equal to the centraliser $\mathrm{SO}_{\mathrm{Frob}_q^m}(V_\ell)$ of Frob_q^m in $\mathrm{SO}(V_\ell)$.

In the rest of this subsection, we fix an integer $m \geq 1$ satisfying the above conditions.

Let $\mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell)$ denote the set of $\mathbb{Q}_\ell$-linear endomorphisms of V_ℓ commuting with Frob_q^m. Similarly, let $\mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell)$ denote the set of $\overline{\mathbb{Q}}_\ell$-linear endomorphisms of $V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$ commuting with Frob_q^m. We have a map

$$I(\mathbb{Q}_\ell) \longrightarrow \mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell).$$

Similarly, we also have a map $I(\overline{\mathbb{Q}}_\ell) \longrightarrow \mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell)$.

Lemma 10.10. *The $\overline{\mathbb{Q}}_\ell$-vector space $\mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell)$ is spanned by the image of $I(\overline{\mathbb{Q}}_\ell)$.*

Proof. Let $R \subset \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell)$ be the $\overline{\mathbb{Q}}_\ell$-vector subspace generated by the image of $I(\overline{\mathbb{Q}}_\ell)$. Because the action of $I(\overline{\mathbb{Q}}_\ell)$ on $V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$ is semisimple, R is a semisimple $\overline{\mathbb{Q}}_\ell$-subalgebra. Hence, it suffices to prove that every element of $\mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell)$ commutes with every element in the commutant of R.

Because the action of Frob_q^m preserves the bilinear form on V_ℓ, if α is an eigenvalue of Frob_q^m, then α^{-1} is also an eigenvalue of Frob_q^m. Because none of the eigenvalues of Frob_q^m on V_ℓ is a root of unity, we may write $\alpha_1, \alpha_1^{-1}, \ldots, \alpha_r, \alpha_r^{-1} \in \overline{\mathbb{Q}}_\ell$ for the distinct eigenvalues of Frob_q^m. Let $W_1, W_1^-, \ldots, W_r, W_r^-$ denote the eigenspaces of the eigenvalues $\alpha_1, \alpha_1^{-1}, \ldots, \alpha_r, \alpha_r^{-1}$, respectively. Because Frob_q^m acts semisimply on V_ℓ, we have

$$V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell \cong \bigoplus_{i=1}^r (W_i \oplus W_i^-).$$

Hence, we have

$$\mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell) \cong \bigoplus_{i=1}^r (\mathrm{End}_{\overline{\mathbb{Q}}_\ell}(W_i) \oplus \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(W_i^-)),$$
$$\mathrm{SO}_{\mathrm{Frob}_q^m}(V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell) \cong \prod_{i=1}^r \mathrm{GL}(W_i).$$

By Schur's lemma, every element $g \in \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell)$ in the commutant of R is written as $g = \bigoplus_{i=1}^r (g_i \oplus g_i^-)$, where $g_1, g_1^-, \ldots, g_r, g_r^-$ are multiplication by scalars. Hence, g commutes with every element of $\mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell)$. □

Lemma 10.11. *Let G be an algebraic group over an algebraically closed field k of characteristic* 0*. Let V be a finite-dimensional k-vector space and $\rho\colon G \longrightarrow \mathrm{GL}(V)$ a morphism of algebraic groups over k. For any Zariski dense subset $Z \subset G(k)$, we have $\langle \rho(Z) \rangle = \langle \rho(G(k)) \rangle$, where $\langle \rho(Z) \rangle$ (respectively $\langle \rho(G(k)) \rangle$) is the k-vector subspace of* $\mathrm{End}_k(V)$ *spanned by $\rho(Z)$ (respectively $\rho(G(k))$).*

Proof. We put $d := \dim_k \langle \rho(G(k)) \rangle$. Let ψ be the composite of the following maps:

$$G(k)^d := \prod_{i=1}^d G(k) \to \prod_{i=1}^d \mathrm{End}_k(V) \to \wedge^d \mathrm{End}_k(V).$$

If $\dim_k \langle \langle \rho(Z) \rangle < d$, we have $\psi(Z^d) = \{0\}$. Because $Z^d \subset G(k)^d$ is Zariski dense, we have $\psi(G(k)^d) = \{0\}$, which is absurd. The contradiction shows $\dim_k \langle \rho(Z) \rangle = d$. □

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Lemma 10.12.

(1) *As a* $\mathbb{Q}_\ell$*-vector space,* $\mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell)$ *is spanned by the image of semisimple elements in* $I(\mathbb{Q})$*.*
(2) *There exist maximal tori* $T_1, \ldots, T_n \subset I$ *over* $\mathbb{Q}$ *such that the* $\mathbb{Q}_\ell$*-vector space* $\mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell)$ *is spanned by the image of* $T_1(\mathbb{Q}), \ldots, T_n(\mathbb{Q})$*.*

Proof. (1) Because I is a connected reductive algebraic group over $\mathbb{Q}$, the set of semisimple elements in $I(\mathbb{Q})$ is Zariski dense in $I(\overline{\mathbb{Q}}_\ell)$; see [27, Expose XIV, Corollaire 6.4]. By Lemma 10.10 and Lemma 10.11, the $\overline{\mathbb{Q}}_\ell$-vector space $\mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell)$ is spanned by the image of semisimple elements in $I(\mathbb{Q})$. Because $\mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell) = \mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell) \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$, the $\mathbb{Q}_\ell$-vector space $\mathrm{End}_{\mathrm{Frob}_q^m}(V_\ell)$ is also spanned by the image of semisimple elements in $I(\mathbb{Q})$.

(2) The assertion follows from the fact that every semisimple element of $I(\mathbb{Q})$ is contained in a maximal torus of I over $\mathbb{Q}$. □

10.4. The results of Mukai and Buskin

The following theorem will be used in our proof of Theorem 10.1.

Theorem 10.13 (Mukai, Buskin)**.** *Let* T *and* S *be projective* $K3$ *surfaces over* $\mathbb{C}$*. Let* $\psi \colon H^2_B(S, \mathbb{Q}) \cong H^2_B(T, \mathbb{Q})$ *be an isomorphism of* $\mathbb{Q}$*-vector spaces that preserves the cup product pairings and the* $\mathbb{Q}$*-Hodge structure. Let* $[\psi] \in H^4_B(S \times T, \mathbb{Q}(2))$ *be the class corresponding to* ψ *by the Poincaré duality and the Künneth formula. Then* $[\psi]$ *is the class of an algebraic cycle of codimension* 2 *on* $S \times T$*.*

Proof. See [13, Theorem 1.1], [53, Theorem 2]. (See also [35, Corollary 0.4].) □

10.5. Proof of Theorem 10.1

In this subsection, we shall prove Theorem 10.1.

By Lemma 10.7, it is enough to prove Theorem 10.1 for $K3$ surfaces of finite height.

Let X be a $K3$ surface of finite height over $\mathbb{F}_q$. After replacing $\mathbb{F}_q$ by its finite extension, we may assume that X comes from an $\mathbb{F}_q$-valued point $s \in M^{\mathrm{sm}}_{2d,\mathrm{K}_0^p,\mathbb{Z}_{(p)}}(\mathbb{F}_q)$ satisfying the conditions as in Subsection 6.2. Let I be the algebraic group over $\mathbb{Q}$ associated with $s \in Z_{\mathrm{K}^p}(\Lambda)(\mathbb{F}_q)$; see Definition 8.1.

By Lemma 10.4, it is enough prove Theorem 10.1 (1) for a fixed ℓ. We fix a prime number $\ell \neq p$. We take a sufficiently divisible integer $m \geq 1$ as in Subsection 10.3. Replacing $\mathbb{F}_q$ by a finite extension of it (see Lemma 10.2), we may assume $m = 1$.

By Lemma 10.12, there exist maximal tori $T_1, \ldots, T_n \subset I$ over $\mathbb{Q}$ such that $\mathrm{End}_{\mathrm{Frob}_q}(V_\ell)$ is spanned by the image of $T_1(\mathbb{Q}), \ldots, T_n(\mathbb{Q})$. By Lemma 10.3 and Lemma 10.9, it is enough to show that, for every i with $1 \leq i \leq n$, the image of $T_i(\mathbb{Q})$ in $\mathrm{End}_{\mathrm{Frob}_q}(V_\ell)$ is spanned by classes of algebraic cycle of codimension 2 on $X_{\overline{\mathbb{F}}_q} \times X_{\overline{\mathbb{F}}_q}$.

Fix an integer i with $1 \leq i \leq n$. By Theorem 9.7, there exist a finite extension K of $W(\overline{\mathbb{F}}_q)[1/p]$ and a quasi-polarised $K3$ surface $(\mathcal{X}, \mathcal{L})$ over $\mathcal{O}_K$ whose special fibre is isomorphic to $(X_{\overline{\mathbb{F}}_q}, \mathscr{L}_{\overline{\mathbb{F}}_q})$ such that, for any embedding $K \hookrightarrow \mathbb{C}$, there is a homomorphism of algebraic groups over $\mathbb{Q}$,

$$T_i \to \mathrm{SO}(P^2_B(\mathcal{X}_\mathbb{C}, \mathbb{Q}(1))),$$

and the action of every element of $T_i(\mathbb{Q})$ on $P^2_B(\mathcal{X}_\mathbb{C}, \mathbb{Q}(1))$ preserves the $\mathbb{Q}$-Hodge structure on it. We extend the action of $T_i(\mathbb{Q})$ on the primitive part $P^2_B(\mathcal{X}_\mathbb{C}, \mathbb{Q}(1))$ to the full cohomology $H^2_B(\mathcal{X}_\mathbb{C}, \mathbb{Q}(1))$ so that every element of $T_i(\mathbb{Q})$ acts trivially on the first Chern class $\mathrm{ch}_B(\mathcal{L}_\mathbb{C})$. Hence, we have a homomorphism of algebraic groups $T_i \to \mathrm{SO}(H^2_B(\mathcal{X}_\mathbb{C}, \mathbb{Q}(1)))$ over $\mathbb{Q}$ whose image preserves the $\mathbb{Q}$-Hodge structure on $H^2_B(\mathcal{X}_\mathbb{C}, \mathbb{Q}(1))$.

By the results of Mukai and Buskin (see Theorem 10.13), the image of every element of $T_i(\mathbb{Q})$ in $\mathrm{SO}(H^2_B(\mathcal{X}_\mathbb{C}, \mathbb{Q}(1)))$ is a class of an algebraic cycle of codimension 2 on $\mathcal{X}_\mathbb{C} \times \mathcal{X}_\mathbb{C}$.

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Taking the specialisation of algebraic cycles, we conclude that the image of every element of $T_i(\mathbb{Q})$ in $\mathrm{End}_{\mathrm{Frob}_q}(V_\ell)$ is a class of an algebraic cycle of codimension 2 on $X_{\overline{\mathbb{F}}_q} \times X_{\overline{\mathbb{F}}_q}$.

The proof of Theorem 10.1 is complete. □

11. Compatibility of p-adic comparison isomorphisms

Throughout the article, we only use the following types of p-adic period morphisms from the literature:

(1) the de Rham comparison map of Scholze [62],
(2) the crystalline comparison map of Bhatt-Morrow-Scholze [6],
(3) Faltings's comparison map for p-divisible groups over $\mathcal{O}_K$ [29] and
(4) Lau's period morphism in display theory [45].

All of these period morphisms are known to be compatible. The maps (1) and (2) are compatible via the Berthelot-Ogus isomorphism by [6, Theorem 13.1]. For abelian schemes over $\mathcal{O}_K$, (2) and (3) are compatible by [65, Proposition 14.8.3]; see also [63, Proposition 4.15] for the Hodge-Tate counterpart. Finally, (3) and (4) are compatible by [45, Proposition 6.2]. More details are given in the rest of this section.

We also check some basic properties of (1) and (2) to use the results of Blasius-Wintenberger [8].

In this section, we fix a perfect field k of characteristic $p > 0$. We put $W := W(k)$. We fix a finite totally ramified extension K of $W[1/p]$ and an algebraic closure $\overline{K}$ of K. We use the same notation as in Section 3.

11.1. The de Rham comparison map of Scholze

We give some basic properties of the de Rham comparison map constructed by Scholze [62]. (Compare with [73, Theorem A1].)

Let X be a smooth proper variety over K, and let X^{ad} denote the adic space associated with it. By [62, Theorem 8.4], we have an isomorphism

$$H^i_{\mathrm{dR}}(X^{\mathrm{ad}}/K) \otimes_K B_{\mathrm{dR}} \cong H^i_{\text{ét}}(X^{\mathrm{ad}}_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}},$$

where C is the completion of $\overline{K}$. Another construction of the same map is given in [6, Theorem 13.1]. Combined with GAGA results such as [34, Theorem 3.7.2], we get a filtered isomorphism of $\mathrm{Gal}(\overline{K}/K)$-modules

$$c_{\mathrm{dR},X} : H^i_{\mathrm{dR}}(X/K) \otimes_K B_{\mathrm{dR}} \xrightarrow{\cong} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}.$$

By construction, the de Rham comparison map $c_{\mathrm{dR},X}$ is functorial in X with respect to pullback and compatible with cup products.

Remark 11.1. The construction of $c_{\mathrm{dR},X}$ can be generalised to any smooth proper algebraic space X over K. Indeed, [22] supplies a functorial construction of the analytification of X and the corresponding adic space X^{ad}. As before, we have an isomorphism

$$H^i_{\mathrm{dR}}(X^{\mathrm{ad}}/K) \otimes_K B_{\mathrm{dR}} \cong H^i_{\text{ét}}(X^{\mathrm{ad}}_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$$

by [62, Theorem 8.4]. In order to construct $c_{\mathrm{dR},X}$, it is enough to show

$$H^i_{\mathrm{dR}}(X/K) \cong H^i_{\mathrm{dR}}(X^{\mathrm{ad}}/K) \quad \text{and} \quad H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong H^i_{\text{ét}}(X^{\mathrm{ad}}_C, \mathbb{Q}_p).$$

(The analytification induces a morphism of étale topoi.)

- Concerning the isomorphism for de Rham cohomology, by the Hodge-de Rham spectral sequence, it is enough to show $H^i(X, \Omega^j) \cong H^i(X^{\mathrm{ad}}, \Omega^j)$ for every i, j. It follows from GAGA results in [22].

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◦ For the isomorphism for p-adic étale cohomology, we take an étale covering $\mathfrak{U} = \{U_\alpha \to X\}_\alpha$ of X that consists of schemes. We put $\mathfrak{U}_{\overline{K}} := \{(U_\alpha)_{\overline{K}} \to X_{\overline{K}}\}_\alpha$ and $\mathfrak{U}_C^{\mathrm{ad}} := \{(U_\alpha)_C^{\mathrm{ad}} \to X_C^{\mathrm{ad}}\}_\alpha$. The analytification preserves étale coverings. We have the Čech-to-cohomology spectral sequences

$$E_2^{i,j} = \check{H}^i(\mathfrak{U}_{\overline{K}}, \underline{H}^j(\mathbb{Z}/p^n\mathbb{Z})) \Rightarrow H_{\text{ét}}^{i+j}(X_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z}),$$
$$E_2^{i,j} = \check{H}^i(\mathfrak{U}_C^{\mathrm{ad}}, \underline{H}^j(\mathbb{Z}/p^n\mathbb{Z})) \Rightarrow H_{\text{ét}}^{i+j}(X_C^{\mathrm{ad}}, \mathbb{Z}/p^n\mathbb{Z}).$$

(See [66, Tag 03OW].) There is a canonical morphism between these spectral sequences. By Huber's theorem [34, Theorem 3.8.1], it induces isomorphisms between E_2-terms. Therefore, we have an isomorphism

$$H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z}) \cong H_{\text{ét}}^i(X_C^{\mathrm{ad}}, \mathbb{Z}/p^n\mathbb{Z})$$

for every i and n. Taking the projective limit with respect to n and tensoring with $\mathbb{Q}_p$, the required isomorphism for p-adic étale cohomology follows.

Proposition 11.2. *Let $\mathscr{L}$ be a line bundle on X, and let $\mathrm{ch}_{\mathrm{dR}}(\mathscr{L})$ and $\mathrm{ch}_p(\mathscr{L})$ denote the first Chern classes of $\mathscr{L}$ in the de Rham cohomology and the p-adic étale cohomology respectively. The following equality holds:*

$$c_{\mathrm{dR},X}(\mathrm{ch}_{\mathrm{dR}}(\mathscr{L})) = \mathrm{ch}_p(\mathscr{L}) \in H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_p)(1) \subset H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}.$$

Here, $\mathbb{Q}_p(1)$ is naturally embedded into B_{dR}.

Proof. The proof given below shows an analogous statement for any smooth proper rigid analytic variety over K as well.

First, we recall the definition of $\mathrm{ch}_{\mathrm{dR}}(\mathscr{L})$. There are an exact sequence of complex of sheaves on X

$$0 \to (1 \xrightarrow{1 \mapsto 0} \Omega_X^{\geq 1}) \to (\mathcal{O}_X^\times \xrightarrow{d\log} \Omega_X^{\geq 1}) \to \mathcal{O}_X^\times \to 0$$

and an isomorphism

$$(1 \xrightarrow{1 \mapsto 0} \Omega_X^{\geq 1}) \cong (0 \longrightarrow \Omega_X^{\geq 1}).$$

These induce the Chern class map

$$\mathrm{ch}_{\mathrm{dR}} \colon H^1(X, \mathcal{O}_X^\times) \to H_{\mathrm{dR}}^2(X/K).$$

This construction can be analytified.

There is a similar construction on the pro-étale site $X_{\text{proét}}^{\mathrm{ad}}$. Recall that Scholze [64] defined a sheaf $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ of $\mathcal{O}_X$-modules on $X_{\text{proét}}^{\mathrm{ad}}$ with a surjection $\theta \colon \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \to \hat{\mathcal{O}}_X$. In the above construction, we will replace $\mathcal{O}_X^\times \xrightarrow{d\log} \Omega_X^{\geq 1}$ by

$$\mathcal{O}\mathbb{B}_{\mathrm{dR}}^{+\times} \xrightarrow{d\log} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \otimes_{\mathcal{O}_X} \Omega_X^{\geq 1}.$$

Similarly, we replace $(1 \xrightarrow{1 \mapsto 0} \Omega_X^{\geq 1})$ by

$$1 + \operatorname{Ker}\theta \xrightarrow{d\log} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \otimes_{\mathcal{O}_X} \Omega_X^{\geq 1}.$$

Because $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ is complete with respect to $\operatorname{Ker}\theta$, we have the logarithm map

$$\log \colon 1 + \operatorname{Ker}\theta \to \operatorname{Ker}\theta.$$

This finishes the construction on $X_{\text{proét}}^{\mathrm{ad}}$.

On the other hand, $\mathrm{ch}_p(\mathscr{L})$ is defined at the level of the pro-étale site as follows: it is induced by the following exact sequence of sheaves on $X^{\mathrm{ad}}_{\mathrm{pro\acute{e}t}}$

$$0 \to \hat{\mathbb{Z}}_p(1) \to \hat{\mathcal{O}}^{\times}_{X^\flat} = \varprojlim_{x \mapsto x^p} \hat{\mathcal{O}}^{\times}_X \to \hat{\mathcal{O}}^{\times}_X \to 0.$$

The following commutative diagram completes the proof:

$$\begin{array}{ccccc} \hat{\mathbb{Z}}_p(1) & \longrightarrow & \hat{\mathcal{O}}^{\times}_{X^\flat} & \longrightarrow & \hat{\mathcal{O}}^{\times}_X \\ \downarrow & & \downarrow{\scriptstyle 1\otimes[-]} & & \downarrow \\ 1+\mathrm{Ker}\,\theta & \longrightarrow & \mathcal{O}\mathbb{B}^{+\times}_{\mathrm{dR}} & \longrightarrow & \hat{\mathcal{O}}^{\times}_X, \end{array}$$

where $[-]$ denotes the Teichmüller lift. □

Corollary 11.3. *The de Rham comparison map $c_{\mathrm{dR},X}$ is compatible with Chern classes of vector bundles and hence classes of algebraic cycles.*

Proof. This follows from the splitting principle and the previous proposition. (The class of an algebraic cycle is written as a $\mathbb{Q}$-linear combination of Chern classes of vector bundles.) □

Corollary 11.4. *If X is of equidimension d, the de Rham comparison map $c_{\mathrm{dR},X}$ is compatible with trace maps. Therefore, $c_{\mathrm{dR},X}$ is functorial in X with respect to pushforward.*

Proof. This follows from the compatibility with cycle classes of a point. □

11.2. The crystalline comparison map of Bhatt-Morrow-Scholze

Let $\mathscr{X}$ be a smooth proper algebraic space over $\mathscr{O}_K$ with generic fibre X. We assume that the generic fibre X and the special fibre $\mathscr{X}_k$ are both schemes. For example, it is satisfied for smooth proper curves and surfaces over $\mathscr{O}_K$ because a smooth proper algebraic space of dimension ≤ 2 over a field is a scheme.

As in the proof of [19, Theorem 2.4], the formal completion $\mathfrak{X}$ of $\mathscr{X}$ along the special fibre is a formal scheme (rather than a formal algebraic space), and there is an isomorphism $t(\mathfrak{X})_\eta \cong X^{\mathrm{ad}}$ of adic spaces over K, where $t(\mathfrak{X})_\eta$ denotes the adic generic fibre of $\mathfrak{X}$. We remark that the isomorphsim $t(\mathfrak{X})_\eta \cong X^{\mathrm{ad}}$ is functorial in $\mathscr{X}$; this can be seen by an argument similar to [22, Theorem 2.2.3].

We recall the construction of the crystalline comparison map of Bhatt-Morrow-Scholze in [6]. Let A_{cris} denote the p-adic completion of the divided power envelope of A_{inf} with respect to $\mathrm{Ker}\,\theta$.

The absolute crystalline comparison theorem in [6, Theorem 14.3 (iii)] (and [6, Theorem 14.3 (iv)]) gives the following isomorphism:

$$H^i_{\mathrm{cris}}(\mathscr{X}_{\mathscr{O}_C/p}/A_{\mathrm{cris}}) \otimes_{A_{\mathrm{cris}}} B_{\mathrm{cris}} \xrightarrow{\cong} H^i_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{cris}}.$$

By [6, Proposition 13.21] and the base change of crystalline cohomology, we have an isomorphism

$$H^i_{\mathrm{cris}}(\mathscr{X}_{\mathscr{O}_C/p}/A_{\mathrm{cris}})[1/p] \cong H^i_{\mathrm{cris}}(\mathscr{X}_k/W) \otimes_W A_{\mathrm{cris}}[1/p].$$

The above two isomorphisms give us the crystalline comparison map

$$c_{\mathrm{cris},\mathscr{X}} : H^i_{\mathrm{cris}}(\mathscr{X}_k/W) \otimes_W B_{\mathrm{cris}} \xrightarrow{\cong} H^i_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{cris}}.$$

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Proposition 11.5. *The crystalline comparison map $c_{\mathrm{cris},\mathscr{X}}$ is compatible with the de Rham comparison map $c_{\mathrm{dR},X}$ under an isomorphism*

$$H^i_{\mathrm{cris}}(\mathscr{X}_k/W)\otimes_W K\cong H^i_{\mathrm{dR}}(X/K)$$

of Berthelot-Ogus [5].

Proof. By [6, Proposition 13.23, Remark 13.20], we have an isomorphism

$$H^i_{\mathrm{cris}}(\mathscr{X}_{\mathscr{O}_C/p}/A_{\mathrm{cris}})\otimes_{A_{\mathrm{cris}}} B^+_{\mathrm{dR}}\cong H^i_{\mathrm{dR}}(X/K)\otimes_K B^+_{\mathrm{dR}}.$$

It is already shown in [6, Theorem 14.5(i)] that $c_{\mathrm{dR},X}$ is compatible with $c_{\mathrm{cris},\mathscr{X}}$ under this identification.

The above identification induces a K-linear map

$$s_{\mathrm{dR}}\colon H^i_{\mathrm{dR}}(X/K)\longrightarrow H^i_{\mathrm{cris}}(\mathscr{X}_{\mathscr{O}_C/p}/A_{\mathrm{cris}})\otimes_{A_{\mathrm{cris}}} B^+_{\mathrm{dR}}$$

such that the composite of s_{dR} with the following map obtained by the reduction modulo ξ

$$\begin{aligned}H^i_{\mathrm{cris}}(\mathscr{X}_{\mathscr{O}_C/p}/A_{\mathrm{cris}})\otimes_{A_{\mathrm{cris}}} B^+_{\mathrm{dR}}&\longrightarrow H^i_{\mathrm{cris}}(\mathscr{X}_{\mathscr{O}_C/p}/A_{\mathrm{cris}})\otimes_{A_{\mathrm{cris}}} C\\&\cong H^i_{\mathrm{dR}}(X_C/C)\end{aligned}$$

(this has no ambiguity; see the commutative diagram (6.5.7) in [16]) is equal to the canonical map

$$H^i_{\mathrm{dR}}(X/K)\longrightarrow H^i_{\mathrm{dR}}(X_C/C).$$

The K-linear map s_{dR} is characterised as a unique $\mathrm{Gal}(\overline{K}/K)$-equivariant K-linear map satisfying this property.

On the other hand, [6, Proposition 13.21] gives a W-linear map

$$s_{\mathrm{cris}}\colon H^i_{\mathrm{cris}}(\mathscr{X}_k/W)[1/p]\longrightarrow H^i_{\mathrm{cris}}(\mathscr{X}_{\mathscr{O}_C/p}/A_{\mathrm{cris}})[1/p]$$

such that the composite of s_{cris} with the following map obtained by the specialisation $A_{\mathrm{cris}}\longrightarrow W(\overline{k})$

$$\begin{aligned}H^i_{\mathrm{cris}}(\mathscr{X}_{\mathscr{O}_C/p}/A_{\mathrm{cris}})[1/p]&\longrightarrow H^i_{\mathrm{cris}}(\mathscr{X}_{\mathscr{O}_C/p}/A_{\mathrm{cris}})\otimes_{A_{\mathrm{cris}}} W(\overline{k})[1/p]\\&\cong H^i_{\mathrm{cris}}(\mathscr{X}_{\overline{k}}/W(\overline{k}))[1/p]\end{aligned}$$

is equal to the canonical map

$$H^i_{\mathrm{cris}}(\mathscr{X}_k/W)[1/p]\longrightarrow H^i_{\mathrm{cris}}(\mathscr{X}_{\overline{k}}/W(\overline{k}))[1/p].$$

The W-linear map s_{cris} is also characterised as a unique $\mathrm{Gal}(\overline{K}/K)$-equivariant W-linear map satisfying this property, which is φ-equivariant. Because the images of s_{dR} and $s_{\mathrm{cris}}\otimes K$ coincide, the reduction modulo $\mathrm{Ker}\,\theta$ induces an isomorphism

$$\overline{s_{\mathrm{cris}}\otimes K}\colon H^i_{\mathrm{cris}}(\mathscr{X}_k/W)\otimes_W K\cong H^i_{\mathrm{dR}}(X/K),$$

and $c_{\mathrm{cris},\mathscr{X}}$ and $c_{\mathrm{dR},X}$ are compatible under $\overline{s_{\mathrm{cris}}\otimes K}$. So, we need only show that $\overline{s_{\mathrm{cris}}\otimes K}$ is equal to the Berthelot-Ogus isomorphism.

We can finish by recalling the following well-known interpretation of the Berthelot-Ogus isomorphism: Fix a uniformiser ϖ and a system $\{\varpi^{1/p^n}\}_{n\geq 0}\subset\overline{K}$ of p^nth roots of ϖ such that $(\varpi^{1/p^{n+1}})^p=\varpi^{1/p^n}$, and let S_ϖ be the p-adic completion of the PD envelope of the surjection $W[u]\longrightarrow\mathscr{O}_K$ defined by $u\mapsto\varpi$. Then, there is a unique φ-equivariant section

$$s\colon H^i_{\mathrm{cris}}(\mathscr{X}_k/W)[1/p]\longrightarrow H^i_{\mathrm{cris}}(\mathscr{X}_{\mathscr{O}_K/p}/S_\varpi)[1/p],$$

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and $s \otimes K$ induces the Berthelot-Ogus isomorphism. The section s can be constructed in a way parallel to [5] and [6, Proposition 13.21]; see [14, Proposition 5.1] for example. This implies that $s = s_{\mathrm{cris}}$ under an isomorphism

$$H^i_{\mathrm{cris}}(\mathscr{X}_{\mathscr{O}_K/p}/S_\varpi) \otimes_{S_\varpi} A_{\mathrm{cris}}[1/p] \cong H^i_{\mathrm{cris}}(\mathscr{X}_{\mathscr{O}_C/p}/A_{\mathrm{cris}})[1/p]$$

by base change along the embedding $S_\varpi \to A_{\mathrm{cris}}$ defined by $u \mapsto [\varpi^\flat]$, and the specialisation of s is indeed the Berthelot-Ogus isomorphism. □

Corollary 11.6. *The crystalline comparison map $c_{\mathrm{cris},\mathscr{X}}$ is*

- *functorial in $\mathscr{X}$ with respect to pullback and pushforward and*
- *compatible with cup products, Chern classes of vector bundles on $\mathscr{X}$, classes of algebraic cycles on $\mathscr{X}$ and trace maps.*

Proof. The above properties are already checked for the de Rham comparison map $c_{\mathrm{dR},X}$. Therefore, it suffices to show that the Berthelot-Ogus isomorphism satisfies the above properties. This is done in [5] and [32, B. Appendix], at least when $\mathscr{X}$ is a scheme.

Let us briefly explain a variant using the interpretation of the Berthelot-Ogus isomorphism given in the proof of Proposition 11.5, which works in the semistable case as well. The functoriality with respect to pullback and the compatibility with cup products are checked (by the same argument as) in [72, Corollary 4.4.13]. Next, we check the compatibility with the first Chern classes of line bundles. We freely use the notation from the proof of Proposition 11.5. Let $\mathscr{L}$ be a line bundle on $\mathscr{X}$. This gives the first Chern class $\mathrm{ch}_{\mathrm{cris}}(\mathscr{L}/S_\varpi)$ in $H^2_{\mathrm{cris}}(\mathscr{X}_{\mathscr{O}_K/p}/S_\varpi)$ lifting $\mathrm{ch}_{\mathrm{cris}}(\mathscr{L})$. By the characterisation of s, we see that the following equality holds:

$$s(\mathrm{ch}_{\mathrm{cris}}(\mathscr{L})) = \mathrm{ch}_{\mathrm{cris}}(\mathscr{L}/S_\varpi).$$

Because it is well known that $\mathrm{ch}_{\mathrm{cris}}(\mathscr{L}/S_\varpi)$ maps to $\mathrm{ch}_{\mathrm{dR}}(\mathscr{L})$ modulo $\mathrm{Ker}(S_\varpi \to \mathscr{O}_K)$, we conclude that the Berthelot-Ogus isomorphism maps $\mathrm{ch}_{\mathrm{cris}}(\mathscr{L})$ to $\mathrm{ch}_{\mathrm{dR}}(\mathscr{L})$. Now, the other compatibilities and functoriality can be deduced from what we have shown. □

Remark 11.7. In the above proof, we implicitly use that the isomorphism $t(\mathfrak{X})_\eta \cong X^{\mathrm{ad}}$ is compatible with line bundles in the following sense: For a line bundle $\mathscr{L}$ on $\mathscr{X}$, let $\mathfrak{L}$ be the formal completion of $\mathscr{L}$ along the special fibre and $t(\mathfrak{L})_\eta$ the corresponding line bundle on $t(\mathfrak{X})_\eta$. Let $\mathscr{L}_K$ be the restriction of $\mathscr{L}$ to the generic fibre X and $(\mathscr{L}_K)^{\mathrm{ad}}$ its analytification. Then there is an isomorphism of line bundles $t(\mathfrak{L})_\eta \cong (\mathscr{L}_K)^{\mathrm{ad}}$ on $t(\mathfrak{X})_\eta \cong X^{\mathrm{ad}}$.

This can be seen as follows: By the construction of $t(\mathfrak{X})_\eta \cong X^{\mathrm{ad}}$ and étale descent for line bundles on rigid analytic varieties, the above claim reduces to an analogous statement in the case where $\mathscr{X}$ is a scheme of finite type, but not necessarily proper, over $\mathscr{O}_K$: we only have a morphism $t(\mathfrak{X})_\eta \to X^{\mathrm{ad}}$ in this case, and the statement is that the pullback of $(\mathscr{L}_K)^{\mathrm{ad}}$ is isomorphic to $t(\mathfrak{L})_\eta$.

11.3. Remarks on the work of Blasius-Wintenberger on p-adic properties of absolute Hodge cycles

The theory of integral canonical models of Shimura varieties relies on the results of Blasius-Wintenberger [8], where p-adic properties of absolute Hodge cycles were studied.

Technically speaking, one chooses constructions of comparison maps when using their results. In this article, we choose the comparison maps $c_{\mathrm{dR},X}$ and $c_{\mathrm{cris},\mathscr{X}}$. Then the main results of [8] hold as all requirements are checked above. This fact for $c_{\mathrm{dR},X}$ is also used implicitly in [15].

Moreover, to fill in details of the proofs of [49, Proposition 5.3, Proposition 5.6 (4)], one should generalise 'Principle B' in Blasius's paper [8] to smooth algebraic spaces. (Let $\mathcal{X}_{\widetilde{\mathrm{M}}_{2d,\mathbb{Q}}} \to \widetilde{\mathrm{M}}_{2d,\mathbb{Q}}$ be the universal family considered in [49]. This morphism is representable by an algebraic space, not a scheme.) For this purpose, one needs to generalise the arguments in the proof of [8, Theorem 3.1] allowing X in its statement to be an algebraic space. This requires us to check the following:

- Artin's comparison theorem and GAGA for algebraic spaces over $\mathbb{C}$. (See [67, Proposition 3.4.1 (iii)] for the former. GAGA is well known and follows from the standard argument using Chow's lemma.) Artin's comparison theorem for algebraic spaces was also used in the construction of α_ℓ in [49, Proposition 5.6 (1)].
- A functorial construction of pure Hodge structure in the cohomology of smooth proper algebraic spaces over $\mathbb{C}$. (The usual construction works; see [57, Part I, Section 2.5]. Also, we have geometric variations in this setting; see [57, Part IV, Corollary 10.32].)
- A generalisation of Deligne's theorem on the fixed part [23, Théorème 4.1.1] to algebraic spaces over $\mathbb{C}$. (The argument in [23] works using Chow's lemma for algebraic spaces.)
- Existence of a smooth compactification of X. (Use [21] and [7].)
- The de Rham comparison theorem for smooth proper algebraic spaces over a p-adic field. (See Remark 11.1.)

11.4. Comparison isomorphisms for p-divisible groups

In [29], Faltings constructed a comparison map

$$c_{\mathscr{G}}\colon D_{\mathrm{cris}}((T_p\mathscr{G})^\vee[1/p]) \cong \mathbb{D}(\mathscr{G}_k)(W)[1/p]$$

for a p-divisible group $\mathscr{G}$ over $\mathscr{O}_K$. The purpose of this subsection is to show that Faltings's comparison map $c_{\mathscr{G}}$ coincides with the comparison map used by Kim-Madapusi Pera in [38, Theorem 2.12 (2.12.3)]. This fact is used implicitly in the proof of [38, Proposition 3.12] to apply Kisin's result [41, Lemma 1.1.17].

Let $\mathscr{G}$ be a p-divisible group over $\mathscr{O}_K$. For the base change $\mathscr{G}_{\mathscr{O}_K/p}$ of $\mathscr{G}$, we have a (contravariant) crystal $\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})$ over $\mathrm{CRIS}((\mathscr{O}_K/p)/\mathbb{Z}_p)$. Its value

$$\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(A_{\mathrm{cris}}) := \mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})_{A_{\mathrm{cris}}\twoheadrightarrow \mathscr{O}_C/p}$$

in $(\operatorname{Spec}\mathscr{O}_C/p \hookrightarrow \operatorname{Spec} A_{\mathrm{cris}})$ is a free A_{cris}-module of finite rank and equipped with a Frobenius endomorphism φ. We define a filtration

$$\mathrm{Fil}^1\,\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(A_{\mathrm{cris}}) \hookrightarrow \mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(A_{\mathrm{cris}})$$

by the inverse image of the Hodge filtration $\mathrm{Fil}^1\,\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(\mathscr{O}_C)$ under the surjection $\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(A_{\mathrm{cris}}) \twoheadrightarrow \mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(\mathscr{O}_C)$.

In [29], Faltings constructed a period map

$$\mathrm{Per}_{\mathrm{cris},\mathscr{G}}\colon T_p\mathscr{G} \longrightarrow \mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(A_{\mathrm{cris}}), A_{\mathrm{cris}}).$$

Here the right-hand side is the $\mathbb{Z}_p$-module of A_{cris}-linear homomorphisms from $\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(A_{\mathrm{cris}})$ to A_{cris} that commute with Frobenius endomorphisms and map $\mathrm{Fil}^1\,\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(A_{\mathrm{cris}})$ into $\mathrm{Fil}^1(A_{\mathrm{cris}}) := \operatorname{Ker}\theta$. When $p \neq 2$, the period map $\mathrm{Per}_{\mathrm{cris},\mathscr{G}}$ is an isomorphism by [29, Theorem 7] (see also [31, Section 5.2]). When $p = 2$, it is an injection and the cokernel is killed by 2.

By the rigidity of quasi-isogenies, there is a unique quasi-isogeny

$$f \in \mathrm{Hom}_{\mathscr{O}_K/p}(\mathscr{G}_k \otimes_k \mathscr{O}_K/p,\ \mathscr{G}_{\mathscr{O}_K/p}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

lifting the identity of $\mathscr{G}_k$. The quasi-isogeny f induces an isomorphism

$$\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(\mathscr{O}_K)[1/p] \cong \mathbb{D}(\mathscr{G}_k)(W) \otimes_W K,$$

and the Hodge filtration on the left-hand side makes $\mathbb{D}(\mathscr{G}_k)(W)[1/p]$ a filtered φ-module. This isomorphism is equal to the isomorphism given by Berthelot-Ogus [5, Proposition 3.14]; see Remark 11.10.

The quasi-isogeny f also induces an isomorphism

$$\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(A_{\mathrm{cris}}) \otimes_{A_{\mathrm{cris}}} B^+_{\mathrm{cris}} \cong \mathbb{D}(\mathscr{G}_k)(W) \otimes_W B^+_{\mathrm{cris}},$$

which in turn induces a bijection

$$\mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(A_{\mathrm{cris}}), A_{\mathrm{cris}})[1/p] \cong \mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathbb{D}(\mathscr{G}_k)(W)[1/p], B^+_{\mathrm{cris}}).$$

Here the right-hand side is the set of homomorphisms from $\mathbb{D}(\mathscr{G}_k)(W)[1/p]$ to B^+_{cris} that commute with Frobenius endomorphisms and preserving the filtrations after base change to K; we equip $B^+_{\mathrm{cris}} \otimes_{W[1/p]} K$ with a filtration $\mathrm{Ker}(\theta \otimes_W K) \subset B^+_{\mathrm{cris}} \otimes_{W[1/p]} K$. The following composite is $\mathrm{Gal}(\overline{K}/K)$-equivariant:

$$\begin{aligned} T_p\mathscr{G}[1/p] &\cong \mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(A_{\mathrm{cris}}), A_{\mathrm{cris}})[1/p] \\ &\cong \mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathbb{D}(\mathscr{G}_k)(W)[1/p], B^+_{\mathrm{cris}}), \end{aligned}$$

where the first isomorphism is $\mathrm{Per}_{\mathrm{cris},\mathscr{G}}$ with p inverted. It induces an isomorphism of filtered φ-modules

$$c_{\mathscr{G}}\colon D_{\mathrm{cris}}((T_p\mathscr{G})^\vee[1/p]) \cong \mathbb{D}(\mathscr{G}_k)(W)[1/p].$$

Next, we shall recall the construction of the comparison map given in [38, Theorem 2.12 (2.12.2)], which is based on the theory of Dieudonné displays developed by Zink and Lau.

We put $\mathfrak{S} := W[[u]]$. For a Breuil-Kisin module $\mathfrak{M}$ (over $\mathscr{O}_K$ with respect to $\{\varpi^{1/p^n}\}_{n\geq 0}$) of height ≤ 1, there is an $\mathfrak{S}$-linear homomorphism $\phi\colon \mathfrak{M} \to \varphi^*\mathfrak{M}$ such that $\varphi\circ\phi$ is the multiplication by $pE(u)/E(0)$. Let $\mathfrak{M}^t$ be a Breuil-Kisin module such that its underlying $\mathfrak{S}$-module is $\mathfrak{M}^\vee := \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M},\mathfrak{S})$ and its Frobenius $\varphi^*(\mathfrak{M}^\vee) = (\varphi^*\mathfrak{M})^\vee \to \mathfrak{M}^\vee$ is given by $f \mapsto f\circ\phi$.

Lau constructed an equivalence $\mathfrak{M}'$ from the category of p-divisible groups over $\mathscr{O}_K$ to the category of Breuil-Kisin modules (over $\mathscr{O}_K$ with respect to $\{\varpi^{1/p^n}\}_{n\geq 0}$) of height ≤ 1; see [44, Corollary 5.4, Theorem 6.6]. Let $\mathfrak{M}^L$ be the (Cartier) dual of the equivalence of categories constructed by Lau. Namely, we put

$$\mathfrak{M}^L(\mathscr{G}) := \mathfrak{M}'(\mathscr{G})^t.$$

By [45, Proposition 4.1, Proposition 8.5], there is an isomorphism

$$T_p\mathscr{G} \cong \mathrm{Hom}_\varphi(\mathfrak{M}^L(\mathscr{G}), \mathfrak{S}^{\mathrm{nr}}).$$

For the ring $\mathfrak{S}^{\mathrm{nr}}$, see [45, Section 7] for example. It is equipped with a Frobenius endomorphism and there are inclusions $\mathfrak{S} \hookrightarrow \mathfrak{S}^{\mathrm{nr}} \hookrightarrow A_{\mathrm{inf}}$ commuting with the Frobenius endomorphisms.

Let $\mathrm{Fil}^1(S_\varpi)$ be the kernel of the surjection $S_\varpi \to \mathscr{O}_K$. (Recall that S_ϖ is the p-adic completion of the PD envelope of the surjection $W[u] \to \mathscr{O}_K$ defined by $u \mapsto \varpi$.) Let $\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(S_\varpi)$ be the value in $(\mathrm{Spec}\,\mathscr{O}_K/p \hookrightarrow \mathrm{Spec}\,S_\varpi)$. It is equipped with a Frobenius endomorphism φ and a filtration

$$\mathrm{Fil}^1\,\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(S_\varpi) \hookrightarrow \mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(S_\varpi),$$

which is the inverse image of the Hodge filtration $\mathrm{Fil}^1\,\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(\mathscr{O}_K)$ under the surjection $\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(S_\varpi) \twoheadrightarrow \mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(\mathscr{O}_K)$.

Let $c' \in S_\varpi$ be a unique unit that maps to $1 \in W$ with

$$c'\varphi(c'^{-1}) = \varphi(E(u)/E(0)) \in S_\varpi.$$

(For the element $\lambda \in S_\varpi[1/p]$ in [39, Subsection 1.1.1], we have $c' = \varphi(\lambda)$.) By [44, Proposition 7.1], there is a canonical isomorphism

$$\varphi^*\mathfrak{M}'(\mathscr{G}) \otimes_{\mathfrak{S}} S_\varpi \cong \mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(S_\varpi)^\vee.$$

https://doi.org/10.1017/fms.2021.24 Published online by Cambridge University Press

The dual of this isomorphism multiplied by c' is an isomorphism

$$\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(S_\varpi) \cong \varphi^* \mathfrak{M}^L(\mathscr{G}) \otimes_{\mathfrak{S}} S_\varpi$$

that is compatible with Frobenius endomorphisms and maps $\mathrm{Fil}^1\, \mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(S_\varpi)$ onto

$$\{\, x \in \varphi^* \mathfrak{M}^L(\mathscr{G}) \otimes_{\mathfrak{S}} S_\varpi \mid (1 \otimes \varphi)(x) \in \mathrm{Fil}^1(S_\varpi)(\mathfrak{M}^L(\mathscr{G}) \otimes_{\mathfrak{S}} S_\varpi) \,\}.$$

Let $\mathfrak{M}((T_p\mathscr{G})^\vee)$ be the Breuil-Kisin module associated with the dual $(T_p\mathscr{G})^\vee$ of the p-adic Tate module of $\mathscr{G}$ defined by Kisin; see Subsection 3.2. By construction, there is a canonical isomorphism

$$T_p\mathscr{G} \cong \mathrm{Hom}_\varphi(\mathfrak{M}((T_p\mathscr{G})^\vee), \mathfrak{S}^{\mathrm{nr}}).$$

By [38, Theorem 2.12 (2.12.2)] and its proof, there is an isomorphism of Breuil-Kisin modules

$$\mathfrak{M}^L(\mathscr{G}) \cong \mathfrak{M}((T_p\mathscr{G})^\vee)$$

such that, under this isomorphism, the isomorphism $T_p\mathscr{G} \cong \mathrm{Hom}_\varphi(\mathfrak{M}^L(\mathscr{G}), \mathfrak{S}^{\mathrm{nr}})$ is compatible with the above isomorphism.

The comparison isomorphism used by Kim-Madapusi Pera in [38, Theorem 2.12 (2.12.3)] is defined as the composite of the following isomorphisms:

$$\begin{aligned} D_{\mathrm{cris}}((T_p\mathscr{G})^\vee[1/p]) &\cong \mathfrak{M}_{\mathrm{cris}}((T_p\mathscr{G})^\vee)[1/p] \\ &\cong \varphi^* \mathfrak{M}^L(\mathscr{G}) \otimes_{\mathfrak{S}} W[1/p] \\ &\cong \mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(S_\varpi) \otimes_{S_\varpi} S_\varpi/uS_\varpi[1/p] \\ &\cong \mathbb{D}(\mathscr{G}_k)(W)[1/p], \end{aligned}$$

where the last isomorphism is provided by the crystalline property of $\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})$. It is an isomorphism of filtered φ-modules and denoted by $c^L_{\mathscr{G}}$.

Proposition 11.8. *$c^L_{\mathscr{G}}$ coincides with $c_{\mathscr{G}}$.*

Proof. We put $\mathfrak{M}^L_{\mathrm{cris}}(\mathscr{G}) := \varphi^* \mathfrak{M}^L(\mathscr{G}) \otimes_{\mathfrak{S}} W$. There is an S_ϖ-linear homomorphism

$$f_{\mathfrak{M}^L(\mathscr{G})} \colon \mathfrak{M}^L_{\mathrm{cris}}(\mathscr{G}) \otimes_W S_\varpi[1/p] \cong \varphi^* \mathfrak{M}^L(\mathscr{G}) \otimes_{\mathfrak{S}} S_\varpi[1/p]$$

that commutes with Frobenius endomorphisms and lifts the identity of $\mathfrak{M}^L_{\mathrm{cris}}(\mathscr{G})$. In fact, it is characterised by such properties. Translating the construction of the isomorphism

$$D_{\mathrm{cris}}((T_p\mathscr{G})^\vee[1/p]) \cong \mathfrak{M}_{\mathrm{cris}}((T_p\mathscr{G})^\vee)[1/p]$$

in [39, Proposition 2.1.5] in terms of $\mathrm{Gal}(\overline{K}/K)$-representations, we see that the isomorphism $c^L_{\mathscr{G}}$ corresponds to the composite of the following isomorphisms:

$$\begin{aligned} T_p\mathscr{G}[1/p] &\cong \mathrm{Hom}_\varphi(\mathfrak{M}^L(\mathscr{G}), \mathfrak{S}^{\mathrm{nr}})[1/p] \\ &\cong \mathrm{Hom}_{\mathrm{Fil},\varphi}(\varphi^* \mathfrak{M}^L(\mathscr{G}), A_{\mathrm{cris}})[1/p] \\ &\cong \mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathfrak{M}^L_{\mathrm{cris}}(\mathscr{G})[1/p], B^+_{\mathrm{cris}}) \\ &\cong \mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathbb{D}(\mathscr{G}_k)(W)[1/p], B^+_{\mathrm{cris}}). \end{aligned}$$

Here, the second isomorphism is induced by the Frobenius of $\mathfrak{M}^L(\mathscr{G})$ and the injection $\mathfrak{S}^{\mathrm{nr}} \hookrightarrow A_{\mathrm{cris}}$, and the third isomorphism is induced by $f_{\mathfrak{M}^L(\mathscr{G})}$. Using the uniqueness of the isomorphism $f_{\mathfrak{M}^L(\mathscr{G})}$, we may replace the composite of the last two isomorphisms by the following composite:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Fil},\varphi}(\varphi^{*}\mathfrak{M}^{L}(\mathscr{G}),A_{\mathrm{cris}})[1/p] &\cong \mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(S_{\varpi}),A_{\mathrm{cris}})[1/p]\\ &\cong \mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(A_{\mathrm{cris}}),A_{\mathrm{cris}})[1/p]\\ &\cong \mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathbb{D}(\mathscr{G}_{k})(W)[1/p],B^{+}_{\mathrm{cris}}).\end{aligned}$$

Here, the second isomorphism is induced by the base change $\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(S_{\varpi})\otimes_{S_{\varpi}}A_{\mathrm{cris}}\cong\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(A_{\mathrm{cris}})$, and the third isomorphism is induced by the quasi-isogeny f.

Therefore, the assertion follows from the following Proposition 11.9, which was essentially proved by Lau in [45]. □

Proposition 11.9 (Lau). *Let* Per′ *be the following composite:*

$$\begin{aligned}T_p\mathscr{G} &\cong \mathrm{Hom}_{\varphi}(\mathfrak{M}^{L}(\mathscr{G}),\mathfrak{S}^{\mathrm{nr}})\\ &\to \mathrm{Hom}_{\mathrm{Fil},\varphi}(\varphi^{*}\mathfrak{M}^{L}(\mathscr{G}),A_{\mathrm{cris}})\\ &\cong \mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(S_{\varpi}),A_{\mathrm{cris}})\\ &\cong \mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(A_{\mathrm{cris}}),A_{\mathrm{cris}}).\end{aligned}$$

Then, Faltings's period map $\mathrm{Per}_{\mathrm{cris},\mathscr{G}}$ *coincides with* Per′.

Proof. We shall explain how the equality $\mathrm{Per}_{\mathrm{cris},\mathscr{G}}=\mathrm{Per}'$ follows from Lau's result [45, Proposition 6.2]. We freely use the notion of a frame and a window from Lau's papers [44, 45].

We have the following commutative diagram of homomorphisms of rings:

$$\begin{array}{ccccc}\mathfrak{S} & \longrightarrow & \mathfrak{S}^{\mathrm{nr}} & \xrightarrow{\varkappa^{\mathrm{nr}}} & \hat{\mathbb{W}}(\widetilde{\mathscr{O}}_K)\\ & & \downarrow & & \downarrow{\scriptstyle\iota}\\ & & A_{\mathrm{cris}} & \xrightarrow{\varkappa_{\mathrm{cris}}} & \hat{\mathbb{W}}^{+}(\widetilde{\mathscr{O}}_K).\end{array}$$

The above diagram induces the following commutative diagram of homomorphisms of frames:

$$\begin{array}{ccccc}\mathscr{B} & \longrightarrow & \mathscr{B}^{\mathrm{nr}} & \xrightarrow{\varkappa^{\mathrm{nr}}} & \hat{\mathscr{D}}_{\widetilde{\mathscr{O}}_K}\\ & & \downarrow & & \downarrow{\scriptstyle\iota}\\ & & \mathcal{A}_{\mathrm{cris}} & \xrightarrow{\varkappa_{\mathrm{cris}}} & \hat{\mathscr{D}}^{+}_{\widetilde{\mathscr{O}}_K}.\end{array}$$

(For the above two commutative diagrams, see [45].) The homomorphism $\varkappa^{\mathrm{nr}}\colon\mathscr{B}^{\mathrm{nr}}\to\hat{\mathscr{D}}_{\widetilde{\mathscr{O}}_K}$ (respectively $\iota\colon\hat{\mathscr{D}}_{\widetilde{\mathscr{O}}_K}\to\hat{\mathscr{D}}^{+}_{\widetilde{\mathscr{O}}_K}$) is a u-homomorphism (respectively u_0-homomorphism) of frames for a unit $u\in\hat{\mathbb{W}}(\widetilde{\mathscr{O}}_K)$ (respectively $u_0\in\hat{\mathbb{W}}^{+}(\widetilde{\mathscr{O}}_K)$). There is a unique unit $c\in\hat{\mathbb{W}}(\widetilde{\mathscr{O}}_K)$ (respectively $c_0\in\hat{\mathbb{W}}^{+}(\widetilde{\mathscr{O}}_K)$) that maps to $1\in W$ with $c\varphi(c^{-1})=u$ (respectively $c_0\varphi(c_0^{-1})=u_0$).

The Breuil-Kisin module $\mathfrak{M}'(\mathscr{G})$ corresponds to a window over $\mathscr{B}$, and let $\mathfrak{M}'(\mathscr{G})^{\mathrm{nr}}$ be the window over $\mathscr{B}^{\mathrm{nr}}$ obtained by base change. There is a window over $\mathcal{A}_{\mathrm{cris}}$ associated with $\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(A_{\mathrm{cris}})$, which will be denoted by the same notation; see [45, Section 6]. The base change of $\mathfrak{M}'(\mathscr{G})^{\mathrm{nr}}$ to $\mathcal{A}_{\mathrm{cris}}$ is identified with the dual $\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(A_{\mathrm{cris}})^{t}$ by [44, Proposition 7.1]. (For the dual of a window, see [44, Section 2A].)

For a window $\mathscr{P}$, let $T(\mathscr{P})$ denote the module of invariants; see [45, Section 3]. There are natural isomorphisms

$$T(\mathfrak{M}'(\mathscr{G})^{\mathrm{nr}}) \cong \mathrm{Hom}_{\varphi}(\mathfrak{M}^L(\mathscr{G}), \mathfrak{S}^{\mathrm{nr}}),$$
$$T(\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(A_{\mathrm{cris}})^t) \cong \mathrm{Hom}_{\mathrm{Fil},\varphi}(\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(A_{\mathrm{cris}}), A_{\mathrm{cris}}).$$

Under the above isomorphisms, the isomorphism Per' is identified with the following composite:

$$T_p\mathscr{G} \xrightarrow{\cong} T(\mathfrak{M}'(\mathscr{G})^{\mathrm{nr}}) \xrightarrow{x\mapsto c'\otimes x} T(\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(A_{\mathrm{cris}})^t).$$

Here the image of $c' \in S_{\varpi}$ in A_{cris} is denoted by the same letter.

In [45, Proposition 4.1], Lau constructed a period isomorphism

$$\mathrm{Per}_{\mathscr{G}} : T_p\mathscr{G} \cong T(\varkappa^{\mathrm{nr}}_{*}\, \mathfrak{M}'(\mathscr{G})^{\mathrm{nr}}).$$

Here we normalise this period map as in [45, Remark 4.2]. By [45, Proposition 8.5], the homomorphism

$$T(\mathfrak{M}'(\mathscr{G})^{\mathrm{nr}}) \to T(\varkappa^{\mathrm{nr}}_{*}\, \mathfrak{M}'(\mathscr{G})^{\mathrm{nr}})$$

defined by $x \mapsto c \otimes x$ is an isomorphism. The composite of $\mathrm{Per}_{\mathscr{G}}$ with the inverse of the above isomorphism is identified with the isomorphism $T_p\mathscr{G} \cong T(\mathfrak{M}'(\mathscr{G})^{\mathrm{nr}})$ by the definition of $T_p\mathscr{G} \cong \mathrm{Hom}_{\varphi}(\mathfrak{M}^L(\mathscr{G}), \mathfrak{S}^{\mathrm{nr}})$.

By [45, Proposition 6.2], the following diagram commutes:

$$\begin{array}{ccc}
T_p\mathscr{G} & \xrightarrow{\mathrm{Per}_{\mathrm{cris},\mathscr{G}}} & T(\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(A_{\mathrm{cris}})^t) \\
{\scriptstyle\cong}\downarrow & \searrow{\scriptstyle \mathrm{Per}_{\mathscr{G}}} & \\
T(\mathfrak{M}'(\mathscr{G})^{\mathrm{nr}}) & & \downarrow{\scriptstyle x\mapsto 1\otimes x} \\
{\scriptstyle x\mapsto c\otimes x}\downarrow & & \\
T(\varkappa^{\mathrm{nr}}_{*}\, \mathfrak{M}'(\mathscr{G})^{\mathrm{nr}}) & \xrightarrow{x\mapsto c_0\otimes x} & T(\iota_{*}\varkappa^{\mathrm{nr}}_{*}\, \mathfrak{M}'(\mathscr{G})^{\mathrm{nr}}).
\end{array}$$

Let τ denote the right vertical homomorphism, which is an isomorphism; see [45, Proposition 6.2]. Using $\iota(c)c_0 = \varkappa_{\mathrm{cris}}(c')$, we see that the composite of $\mathrm{Per}_{\mathscr{G}}$ and the bottom horizontal arrow is equal to the composite of Per' and τ. Therefore, we have

$$\tau \circ \mathrm{Per}_{\mathrm{cris},\mathscr{G}} = \tau \circ \mathrm{Per}'.$$

Because τ is an isomorphism, the equality $\mathrm{Per}_{\mathrm{cris},\mathscr{G}} = \mathrm{Per}'$ is proved.

The proof of Proposition 11.9 is complete. □

Remark 11.10. The isomorphism $\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(\mathscr{O}_K)[1/p] \cong \mathbb{D}(\mathscr{G}_k)(W) \otimes_W K$ induced by the quasi-isogeny f is equal to the isomorphism given by Berthelot-Ogus [5, Proposition 3.14]. This follows from the fact that there is a unique φ-equivariant isomorphism

$$\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(S_{\varpi}) \otimes_{S_{\varpi}} S_{\varpi}[1/p] \cong \mathbb{D}(\mathscr{G}_k)(W) \otimes_W S_{\varpi}[1/p]$$

that lifts the Berthelot-Ogus isomorphism. This isomorphism can be constructed in a way similar to [5, Proposition 3.14].

Remark 11.11. Given the construction of $c^L_{\mathscr{G}}$ and Proposition 11.8, we can restate [38, Theorem 2.12 (2.12.3), (2.12.4)] using Faltings's comparison map $c_{\mathscr{G}}$ as follows:

(1) The composite

$$\mathfrak{M}_{\mathrm{cris}}((T_p\mathscr{G})^\vee)[1/p] \cong D_{\mathrm{cris}}((T_p\mathscr{G})^\vee[1/p]) \overset{c_{\mathscr{G}}}{\cong} \mathbb{D}(\mathscr{G}_k)(W)[1/p]$$

maps $\mathfrak{M}_{\mathrm{cris}}((T_p\mathscr{G})^\vee)$ onto $\mathbb{D}(\mathscr{G}_k)(W)$.

(2) The composite

$$\mathfrak{M}_{\mathrm{dR}}((T_p\mathscr{G})^\vee)[1/p] \cong D_{\mathrm{dR}}((T_p\mathscr{G})^\vee[1/p]) \overset{c_{\mathscr{G}}}{\cong} \mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(\mathscr{O}_K) \otimes_{\mathscr{O}_K} K$$

maps $\mathfrak{M}_{\mathrm{dR}}((T_p\mathscr{G})^\vee)$ onto $\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(\mathscr{O}_K)$ and maps $\mathrm{Fil}^1(\mathfrak{M}_{\mathrm{dR}}((T_p\mathscr{G})^\vee))$ onto $\mathrm{Fil}^1\,\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(\mathscr{O}_K)$.

Assume that $\mathscr{G}$ is the p-divisible group $\mathcal{B}[p^\infty]$ associated with an abelian scheme $\mathcal{B}$ over $\mathscr{O}_K$. By [4, Théorème 2.5.6, Proposition 3.3.7], there is a natural isomorphism

$$\mathbb{D}(\mathscr{G}_k)(W) \cong H^1_{\mathrm{cris}}(\mathcal{B}_k/W).$$

Proposition 11.12. *Let $\mathcal{B}$ be an abelian scheme over $\mathscr{O}_K$ and $\mathscr{G} := \mathcal{B}[p^\infty]$ the p-divisible group associated with $\mathcal{B}$. Let $T_p\mathscr{G}$ denote the p-adic Tate module of $\mathscr{G}$. Under the isomorphisms $(T_p\mathscr{G})^\vee \cong H^1_{\text{ét}}(\mathcal{B}_{\overline{K}}, \mathbb{Z}_p)$ and $\mathbb{D}(\mathscr{G}_k)(W) \cong H^1_{\mathrm{cris}}(\mathcal{B}_k/W)$, the isomorphism $c_{\mathscr{G}}$ is compatible with the crystalline comparison map $c_{\mathrm{cris},\mathcal{B}}$.*

Proof. By [4, Théorème 2.5.6, Proposition 3.3.7], there is a natural isomorphism

$$\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(A_{\mathrm{cris}}) \cong H^1_{\mathrm{cris}}(\mathcal{B}_{\mathscr{O}_K/p}/A_{\mathrm{cris}}).$$

After inverting p, this isomorphism is identified with the base change of the map $\mathbb{D}(\mathscr{G}_k)(W) \cong H^1_{\mathrm{cris}}(\mathcal{B}_k/W)$ along $W \to A_{\mathrm{cris}}[1/p]$ under the isomorphism

$$\mathbb{D}(\mathscr{G}_{\mathscr{O}_K/p})(A_{\mathrm{cris}})[1/p] \cong \mathbb{D}(\mathscr{G}_k)(W) \otimes_W A_{\mathrm{cris}}[1/p]$$

induced by the quasi-isogeny f and the isomorphism

$$H^1_{\mathrm{cris}}(\mathcal{B}_{\mathscr{O}_K/p}/A_{\mathrm{cris}})[1/p] \cong H^1_{\mathrm{cris}}(\mathcal{B}_k/W) \otimes_W A_{\mathrm{cris}}[1/p]$$

in Subsection 11.2. This follows from the characterisation of the W-linear map s_{cris} in the proof of Proposition 11.5. Now, the assertion follows from [65, Proposition 14.8.3]. (Alternatively, one can use the Hodge-Tate version [63, Proposition 4.15] by checking a certain compatibility, but we omit the details.) □

Acknowledgements. The authors thank Keerthi Madapusi Pera for e-mail correspondence on the proof of the étaleness of the Kuga-Satake morphism. The authors also thank the referee for remarks and comments. The work of the first author was supported by JSPS Research Fellowships for Young Scientists KAKENHI Grant Number 18J22191. The work of the second author was supported by JSPS KAKENHI Grant Numbers 20674001 and 26800013. The work of the third author was supported by JSPS KAKENHI Grant Number 20K14284.

Conflict of Interest: None.

References

[1] André, Y., ‘On the Shafarevich and Tate conjectures for hyper-Kähler varieties’, *Math. Ann.* **305**(2) (1996), 205–248.
[2] Artin, M. and Mazur, B., ‘Formal groups arising from algebraic varieties’, *Ann. Sci. École Norm. Sup. (4)* **10**(1) (1977), 87–131.
[3] Bass, H., ‘Clifford algebras and spinor norms over a commutative ring’, *Amer. J. Math.* **96** (1974), 156–206.
[4] Berthelot, P., Breen, L. and Messing, W., *Théorie de Dieudonné Cristalline. II*, *Lecture Notes in Mathematics*, Vol. **930** (Springer, Berlin, 1982).
[5] Berthelot, P. and Ogus, A., ‘F-isocrystals and de Rham cohomology. I’, *Invent. Math.* **72**(2) (1983), 159–199.

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[6] Bhatt, B., Morrow, M. and Scholze, P., 'Integral p-adic Hodge theory', *Publ. Math. Inst. Hautes Études Sci.* **128** (2018), 219–397.
[7] Bierstone, E. and Milman, P. D., 'Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant', *Invent. Math.* **128**(2) (1997), 207–302.
[8] Blasius, D., 'A p-adic property of Hodge classes on abelian varieties', in *Motives*, *Proc. Sympos. Pure Math.*, Part 2, Vol. **55** (American Mathematical Society, Providence, RI, 1994), 293–308.
[9] Bloch, S. and Kato, K., 'p-adic étale cohomology', *Inst. Hautes Études Sci. Publ. Math.* **63** (1986), 107–152.
[10] Bosch, S., Lütkebohmert, W. and Raynaud, M., *Néron Models*, *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*, Vol. **21**. (Springer, Berlin, 1990).
[11] Bourbaki, N., *Éléments de mathématique. Algèbre*. Chapitre 9, Reprint of the 1959 original (Springer, Berlin, 2007).
[12] Breen, L., 'Rapport sur la théorie de Dieudonné' *Astérisque* **63** (1979), 39–66.
[13] Buskin, N., 'Every rational Hodge isometry between two $K3$ surfaces is algebraic', *J. Reine Angew. Math.* **755** (2019), 127–150.
[14] Cais, B. and Liu, T., 'Breuil-Kisin modules via crystalline cohomology', *Trans. Amer. Math. Soc.* **371**(2) (2019), 1199–1230.
[15] Caraiani, A. and Scholze, P., 'On the generic part of the cohomology of compact unitary Shimura varieties', *Ann. Math. (2)* **186**(3) (2017), 649–766.
[16] Česnavičius, K. and Koshikawa, T., 'The $A_{\inf}$-cohomology in the semistable case', *Compos. Math.* **155**(11) (2019), 2039–2128.
[17] Chai, C.-L., Conrad, B. and Oort, F., *Complex Multiplication and Lifting Problems*, *Mathematical Surveys and Monographs*, Vol. **195** (American Mathematical Society, Providence, RI, 2014).
[18] Charles, F., 'The Tate conjecture for $K3$ surfaces over finite fields', *Invent. Math.* **194**(1) (2013), 119–145.
[19] Chiarellotto, B., Lazda, C. and Liedtke, C., 'A Néron-Ogg-Shafarevich criterion for $K3$ surfaces', *Proc. Lond. Math. Soc. (3)* **119**(2) (2019), 469–514.
[20] Colmez, P. and Fontaine, J.-M., 'Construction des représentations p-adiques semi-stables', *Invent. Math.* **140**(1) (2000), 1–43.
[21] Conrad, B., Lieblich, M. and Olsson, M., 'Nagata compactification for algebraic spaces', *J. Inst. Math. Jussieu* **11**(4) (2012), 747–814.
[22] Conrad, B. and Temkin, M., 'Non-Archimedean analytification of algebraic spaces', *J. Algebraic Geom.* **18**(4) (2009), 731–788.
[23] Deligne, P., 'Théorie de Hodge. II', *Inst. Hautes Études Sci. Publ. Math.* **40** (1971), 5–57.
[24] Deligne, P., 'Relèvement des surfaces $K3$ en caractéristique nulle', in *Algebraic Surfaces*, *Lecture Notes in Mathematics*, Vol. **868**, (Springer, Berlin, 1981), 58–79.
[25] Deligne, P., 'Hodge Cycles on Abelian Varieties', in *Hodge Cycles, Motives, and Shimura Varieties, Lecture Notes in Mathematics*, Vol. **900** (Springer, Berlin, 1982).
[26] Deligne, P., Letter to M. Kisin (cc: A. Vasiu) (August 26, 2011). URL: http://people.\break math.binghamton.edu/adrian/Letter_Deligne.pdf.
[27] Demazure, M. and Grothendieck, A., *Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux*, *Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3), Lecture Notes in Mathematics*, Vol. **152** (Springer, Berlin, 1970).
[28] Drinfeld, V. G., 'Elliptic modules', *Math. USSR-Sb.* **23**(4) (1974), 561–592.
[29] Faltings, G., 'Integral crystalline cohomology over very ramified valuation rings', *J. Amer. Math. Soc.* **12**(1) (1999), 117–144.
[30] Faltings, G. and Chai, C.-L., *Degeneration of Abelian Varieties*, *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*, Vol. **22** (Springer, Berlin, 1990). With an appendix by David Mumford.
[31] Fargues, L., 'La filtration canonique des points de torsion des groupes p-divisibles', *Ann. Sci. Éc. Norm. Supér. (4)* **44**(6) (2011), 905–961. With collaboration of Yichao Tian.
[32] Gillet, H. and Messing, W., 'Cycle classes and Riemann-Roch for crystalline cohomology', *Duke Math. J.* **55**(3) (1987), 501–538.
[33] Hazewinkel, M., *Formal Groups and Applications, corrected reprint of the 1978 original*. (AMS Chelsea Publishing, Providence, RI, 2012).
[34] Huber, R., *Étale Cohomology of Rigid Analytic Varieties and Adic Spaces*, *Aspects of Mathematics*, Vol. E30 (Friedr. Vieweg & Sohn, Braunschweig, Germany, 1996).
[35] Huybrechts, D., 'Motives of isogenous $K3$ surfaces', *Comment. Math. Helv.* **94**(3) (2019), 445–458.
[36] Katz, N. M., 'Slope filtration of F-crystals', *Astérisque* **63** (1979), 113–163.
[37] Keum, J., 'Orders of automorphisms of $K3$ surfaces', *Adv. Math.* **303** (2016), 39–87.
[38] Kim, W. and Madapusi Pera, K., '2-adic integral canonical models', *Forum Math. Sigma* **4** (2016), e28, 34 pp.
[39] Kisin, M., 'Crystalline representations and F-crystals', in *Algebraic Geometry and Number Theory*, *Progr. Math.*, Vol. **253** (Birkhäuser, Boston, 2006), 459–496.
[40] Kisin, M., 'Integral models for Shimura varieties of abelian type', *J. Amer. Math. Soc.* **23**(4) (2010), 967–1012.
[41] Kisin, M., 'Mod p points on Shimura varieties of abelian type', *J. Amer. Math. Soc.* **30**(3) (2017), 819–914.
[42] Knus, M.-A., *Quadratic and Hermitian Forms over Rings*, *Grundlehren der Mathematischen Wissenschaften*, Vol. **294** (Springer, Berlin, 1991). With a foreword by I. Bertuccioni.

[43] Lau, E., 'Frames and finite group schemes over complete regular local rings', *Doc. Math.* **15** (2010), 545–569.
[44] Lau, E., 'Relations between Dieudonné displays and crystalline Dieudonné theory', *Algebra Number Theory* **8**(9) (2014), 2201–2262.
[45] Lau, E., 'Displayed equations for Galois representations', *Nagoya Math. J.* **235** (2019), 86–114.
[46] Liedtke, C., 'Lectures on supersingular $K3$ surfaces and the crystalline Torelli theorem', in *$K3$ Surfaces and Their Moduli*, *Progr. Math.*, Vol. 315 (Birkhäuser/Springer, [Cham], 2016), 171–235.
[47] Liu, T., 'Lattices in filtered (φ, N)-modules', *J. Inst. Math. Jussieu* **11**(3) (2012), 659–693.
[48] Liu, T., 'The correspondence between Barsotti-Tate groups and Kisin modules when $p = 2$', *J. Théor. Nombres Bordeaux*, **25**(3) (2013), 661–676.
[49] Madapusi Pera, K., 'The Tate conjecture for $K3$ surfaces in odd characteristic', *Invent. Math.* **201**(2) (2015), 625–668.
[50] Madapusi Pera, K., 'Integral canonical models for spin Shimura varieties', *Compos. Math.* **152**(4) (2016), 769–824.
[51] Madapusi Pera, K., 'Erratum to appendix to '2-adic integral canonical models'', *Forum Math. Sigma* **8** (2020), e14, 11 pp.
[52] Maulik, D., 'Supersingular $K3$ surfaces for large primes', *Duke Math. J.* **163**(13) (2014), 2357–2425. With an appendix by Andrew Snowden.
[53] Mukai, S., 'Vector bundles on a $K3$ surface', *Proc. ICM, Beijing* **2** (2002), 495–502.
[54] Nygaard, N., 'The Tate conjecture for ordinary $K3$ surfaces over finite fields', *Invent. Math.* **74**(2) (1983), 213–237.
[55] Nygaard, N. and Ogus, A., 'Tate's conjecture for $K3$ surfaces of finite height', *Ann. Math. (2)* **122**(3) (1985), 461–507.
[56] Ogus, A., 'Supersingular $K3$ crystals', *Astérisque* **64** (1979), 3–86.
[57] Peters, C. A. M. and Steenbrink, J. H. M., *Mixed Hodge structures, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*, Vol. **52** (Springer, Berlin, 2008).
[58] Pjateckiĭ-Šapiro, I. I. and Šafarevič, I. R., 'The arithmetic of surfaces of type $K3$', *Trudy Mat. Inst. Steklov.* **132** (1973), 44–54, 264.
[59] Platonov, V. and Rapinchuk, A., *Algebraic Groups and Number Theory*, *Pure and Applied Mathematics*, Vol. **139** (Academic Press, Boston, 1994). Translated from the 1991 Russian original by Rachel Rowen.
[60] Rizov, J., 'Moduli stacks of polarized $K3$ surfaces in mixed characteristic', *Serdica Math. J.* **32**(2–3) (2006), 131–178.
[61] Rizov, J., 'Kuga-Satake abelian varieties of $K3$ surfaces in mixed characteristic', *J. Reine Angew. Math.* **648** (2010), 13–67.
[62] Scholze, P., 'p-adic Hodge theory for rigid-analytic varieties', *Forum Math. Pi* **1** (2013), e1, 77 pp.
[63] Scholze, P., '*Perfectoid spaces: a survey*', in *Current Developments in Mathematics 2012*, (Int. Press, Somerville, MA, 2013), 193–227.
[64] Scholze, P., 'p-adic Hodge theory for rigid-analytic varieties – corrigendum', *Forum Math. Pi* **4** (2016), e6, 4 pp.
[65] Scholze, P., Weinstein, J., *Berkeley Lectures on p-adic Geometry*, *Annals of Mathematics Studies*, Vol. **389** (Princeton University Press, Princeton, NJ, 2020).
[66] The Stacks Project Authors , '*Stacks Project*' (2018). URL: https://stacks.math.columbia.edu.
[67] Sun, S., 'Generic base change, Artin's comparison theorem, and the decomposition theorem for complex Artin stacks', *J. Algebraic Geom.* **26**(3) (2017), 513–555.
[68] Taelman, L., '$K3$ surfaces over finite fields with given L-function', *Algebra Number Theory* **10**(5) (2016), 1133–1146.
[69] Tate, J., 'Conjectures on algebraic cycles in ℓ-adic cohomology', in *Motives, Proc. Sympos. Pure Math., Part 1*, Vol. **55** (American Mathematical Society, Providence, RI, 1994), 71–83.
[70] Totaro, B., 'Recent progress on the Tate conjecture', *Bull. Amer. Math. Soc. (N.S.)* **54**(4) (2017), 575–590.
[71] Tretkoff, P., '$K3$ surfaces with algebraic period ratios have complex multiplication', *Int. J. Number Theory* **11**(5) (2015), 1709–1724.
[72] Tsuji, T., 'p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case', *Invent. Math.* **137**(2) (1999), 233–411.
[73] Tsuji, T., 'Semi-stable conjecture of Fontaine-Jannsen: a survey', Cohomologies p-adiques et applications arithmétiques, II, *Astérisque* **279** (2002), 323–370.
[74] Yang, Z., 'Isogenies between $K3$ Surfaces over $\overline{\mathbb{F}}_p$', *Int. Math. Res. Not.*, arXiv:1810.08546, 2020.
[75] Yu, J.-D. and Yui, N., '$K3$ surfaces of finite height over finite fields', *J. Math. Kyoto Univ.* **48**(3) (2008), 499–519.
[76] Zarhin, Y. G., 'Hodge groups of $K3$ surfaces', *J. Reine Angew. Math.* **341** (1983), 193–220.
[77] Zarhin, Y. G., 'The Tate conjecture for powers of ordinary $K3$ surfaces over finite fields', *J. Algebraic Geom.* **5**(1) (1996), 151–172.

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