W-ALGEBRAS AS COSET VERTEX ALGEBRAS

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Abstract. We prove the long-standing conjecture on the coset construction of the minimal series principal W-algebras of ADE types in full generality. We do this by first establishing Feigin's conjecture on the coset realization of the universal principal W-algebras, which are not necessarily simple. As consequences, the unitarity of the "discrete series" of principal W-algebras is established, a second coset realization of rational and unitary W-algebras of type A and D are given and the rationality of Kazama-Suzuki coset vertex superalgebras is derived.

1. Introduction

Let $\mathfrak{g}$ be a simple Lie algebra. For each nilpotent element $f \in \mathfrak{g}$ and $k \in \mathbb{C}$, one associates the W-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ at level k via quantum Drinfeld-Sokolov reduction [FF90a, KRW03]. In the instance that f is a principal nilpotent element $\mathcal{W}^k(\mathfrak{g}, f)$ is called the universal principal W-algebra of $\mathfrak{g}$ at level k and denoted by $\mathcal{W}^k(\mathfrak{g})$. These W-algebras have appeared prominently in various problems of mathematics and physics as the conformal field theory to higher spin gravity duality [GG11], the AGT correspondence [AGT10, SV13, BFN16], the (quantum) geometric Langlands program [Fre07, Gai16, AFO18, CG17, Gai18, FG18] and integrable systems [B89, D03, DSKV13, BM13].

Let $\mathcal{W}_k(\mathfrak{g})$ be the unique simple graded quotient of $\mathcal{W}^k(\mathfrak{g})$. It has been conjectured in [FKW92] and was proved by the first named author [Ara15a, Ara15b] that $\mathcal{W}_k(\mathfrak{g})$ is rational and lisse for some special values of k. These W-algebras are called the *minimal series principal W-algebras* since in the case that $\mathfrak{g} = \mathfrak{sl}_2$ they are exactly the minimal series Virasoro vertex algebras [BPZ84, BFM, Wan93]. As in the case of the Virasoro algebra, a minimal series principal W-algebra is not necessarily unitary. However, in the case that $\mathfrak{g}$ is simply laced, there exists a sub-series called the *discrete series* which are conjectured to be unitary.

It has been believed in physics since 1988 that the discrete series principal W-algebras can be realized by the coset construction [GKO86] from integrable representations of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ [BBSS88, FL88]. Note that the validity of this belief immediately proves the unitarity of the discrete series of W-algebras. The conjectural character formula of Frenkel, Kac and Wakimoto [FKW92] of minimal series representations of W-algebras that was proved in [Ara07]

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together with the character formula of Kac-Wakimoto [KW90] of branching rules proves the matching of characters, which gives strong evidence to this belief. In fact these character formulas give an even stronger conjecture that all minimal series principal W-algebras of ADE types should be realized by the coset construction if we consider more general representations of $\widehat{\mathfrak{g}}$, namely, admissible representations [KW89].

One of the aims of the present paper is to prove this conjectural realization of the minimal series principal W-algebras in full generality.

1.1. **Main Theorems.** Let us formulate our result more precisely. Let $V_k(\mathfrak{g})$ be the universal affine vertex algebra associated to $\mathfrak{g}$ at level k, and denote by $L_k(\mathfrak{g})$ the unique simple graded quotient of $V_k(\mathfrak{g})$. Suppose that k is an admissible level for $\widehat{\mathfrak{g}}$, that is, $L_k(\mathfrak{g})$ is an admissible representation. Consider the tensor product vertex algebra $L_k(\mathfrak{g})\otimes L_1(\mathfrak{g})$. The invariant subspace $(L_k(\mathfrak{g})\otimes L_1(\mathfrak{g}))^{\mathfrak{g}[t]}$ with respect to the diagonal action of $\mathfrak{g}[t]$ is naturally a vertex subalgebra of $L_k(\mathfrak{g})\otimes L_1(\mathfrak{g})$, consisting of elements whose Fourier modes commute with the diagonal action of $\widehat{\mathfrak{g}}$. This is an example of *coset vertex algebras*.

Main Theorem 1. *Let $\mathfrak{g}$ be simply laced, and let k be an admissible level for $\widehat{\mathfrak{g}}$. Define the rational number ℓ by the formula*

$$\ell + h^\vee = \frac{k+h^\vee}{k+h^\vee+1}, \tag{1}$$

which is a non-degenerate admissible level for $\widehat{\mathfrak{g}}$ so that $\mathcal{W}_\ell(\mathfrak{g})$ is a minimal series W-algebra. We have the vertex algebra isomorphism

$$\mathcal{W}_\ell(\mathfrak{g}) \cong (L_k(\mathfrak{g})\otimes L_1(\mathfrak{g}))^{\mathfrak{g}[t]},$$

and $L_{k+1}(\mathfrak{g})$ and $\mathcal{W}_\ell(\mathfrak{g})$ form a dual pair in $L_k(\mathfrak{g})\otimes L_1(\mathfrak{g})$.

For $\mathfrak{g} = \mathfrak{sl}_2$ and k a non-negative integer, Theorem 1 recovers a celebrated result of Goddard, Kent and Olive [GKO86], which is known as the coset construction (or the *GKO construction*) of the discrete unitary series of the Virasoro algebra. Theorem 1 was extended to the case of an arbitrary admissible level k for $\mathfrak{g} = \mathfrak{sl}_2$ by Kac and Wakimoto [KW90]. For a higher rank $\mathfrak{g}$, Theorem 1 has been proved only in some special cases: $\mathfrak{g} = \mathfrak{sl}_n$ and $k = 1$ by Arakawa, Lam and Yamada [ALY19]; for $\mathfrak{g} = \mathfrak{sl}_3$ and $k \in \mathbb{Z}_{\geqslant 0}$ by Arakawa and Jiang [AJ17].

Theorem 1 realizes an arbitrary minimal series W-algebra of ADE types as the coset $(L_k(\mathfrak{g})\otimes L_1(\mathfrak{g}))^{\mathfrak{g}[t]}$ for some admissible level k. The discrete series W-algebras corresponds to the cases that k is a non-negative integer. As we have already mentioned above, with Main Theorem 1 we are able to prove the unitarity of the discrete series of W-algebras, see Theorem 12.6.

We also note that Theorem 1 is the key starting assumption of the conformal field theory to higher spin gravity correspondence of [GG11].

Since Kac and Wakimoto [KW90] have already confirmed the matching of characters, the essential step in proving Main Theorem 1 is to define the action of $\mathcal{W}^\ell(\mathfrak{g})$ on $(L_k(\mathfrak{g})\otimes L_1(\mathfrak{g}))^{\mathfrak{g}[t]}$, which is highly non-trivial since there is no closed presentation of $\mathcal{W}^\ell(\mathfrak{g})$ by generators and relations (OPEs) for a general $\mathfrak{g}$. We overcome this difficulty by establishing the following assertion that has been conjectured by B. Feigin (cf. [FJMM16]).

Main Theorem 2 (Theorem 8.7)**.** *Let $\mathfrak{g}$ be simply laced, $k+h^\vee \notin \mathbb{Q}_{\leqslant 0}$, and define $\ell \in \mathbb{C}$ by the formula (1). We have the vertex algebra isomorphism*

$$\mathcal{W}^\ell(\mathfrak{g}) \cong (V_k(\mathfrak{g}) \otimes L_1(\mathfrak{g}))^{\mathfrak{g}[t]}.$$

Moreover, $\mathcal{W}^\ell(\mathfrak{g})$ and $V_{k+1}(\mathfrak{g})$ form a dual pair in $V_k(\mathfrak{g})\otimes L_1(\mathfrak{g})$ if k is generic.

The advantage of replacing $\mathcal{W}_\ell(\mathfrak{g})$ by the universal W-algebra $\mathcal{W}^\ell(\mathfrak{g})$ lies in the fact that one can use the description of $\mathcal{W}^\ell(\mathfrak{g})$ in terms of screening operators, at least for a generic ℓ. Using such a description, we are able to establish the statement of Main Theorem 2 for deformable families [CL19] of $\mathcal{W}^\ell(\mathfrak{g})$ and $(V_k(\mathfrak{g})\otimes L_1(\mathfrak{g}))^{\mathfrak{g}[t]}$, see Section 8 for the details. The main tool here is a property of the semi-regular bimodule obtained in [Ara14], see Proposition 3.4.

The second part of our main result is the branching rules, i.e. the decomposition of modules of $V_k(\mathfrak{g})\otimes L_1(\mathfrak{g})$ and $L_k(\mathfrak{g})\otimes L_1(\mathfrak{g})$ into modules of the tensor product of the two commuting subalgebras. For this we need to introduce some notation that is also explained in full detail in the main body of the work.

Let P_+ be the set of dominant weights of $\mathfrak{g}$, Q its root lattice and $\rho^\vee$ half the sum of positive coroots. For $\lambda \in P_+$ define $\mathbb{V}_k(\lambda) := U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]\oplus \mathbb{C}K)} E_\lambda$, where E_λ is the irreducible finite-dimensional $\mathfrak{g}$-module with highest weight λ regarded as a $\mathfrak{g}[t]$-module on which $\mathfrak{g}[t]t$ acts trivially and K acts by multiplication with the level $k \in \mathbb{C}$. Let $\mathbb{L}_k(\lambda)$ be the simple quotient of $\mathbb{V}_k(\lambda)$ and for $m \in \mathbb{N}$, let P_+^m be the set of highest-weights such that $\{\mathbb{L}_m(\lambda) \mid \lambda \in P_+^m\}$ gives the complete set of isomorphism classes of irreducible integrable representation of $\widehat{\mathfrak{g}}$ of level m. We denote by χ_λ the central character associated to the weight λ, see (27) for details. Let $\mathbf{M}_k(\chi_\lambda)$ be the Verma module of $\mathcal{W}^k(\mathfrak{g})$ with highest weight χ_λ (see Section 6) and denote by $\mathbf{L}_k(\chi_\lambda)$ be the unique irreducible (graded) quotient of $\mathbf{M}_k(\chi_\lambda)$.

Main Theorem 3. *Define $\ell \in \mathbb{C}$ by (1), then the following branching rules hold:*

(1) *Let $\mu \in P_+^{p-h^\vee}$, $\nu \in P_+^1$, We have*

$$\mathbb{L}_k(\mu) \otimes \mathbb{L}_1(\nu) \cong \bigoplus_{\substack{\lambda \in P_+^{p+q-h^\vee} \\ \lambda-\mu-\nu \in Q}} \mathbb{L}_{k+1}(\lambda) \otimes \mathbf{L}_\ell(\chi_{\mu-(\ell+h^\vee)\lambda})$$

as $L_{k+1}(\mathfrak{g})\otimes\mathcal{W}_\ell(\mathfrak{g})$-modules.

(2) *Suppose that $k \notin \mathbb{Q}$. For $\lambda, \mu \in P_+$ and $\nu \in P_+^1$, we have*

$$\mathbb{V}_k(\mu)\otimes\mathbb{L}_1(\nu) = \bigoplus_{\substack{\lambda \in P_+ \\ \lambda-\mu-\nu \in Q}} \mathbb{V}_{k+1}(\lambda)\otimes\mathbf{L}_\ell(\chi_{\mu-(\ell+h^\vee)\lambda})$$

as $V_{k+1}\otimes\mathcal{W}^\ell(\mathfrak{g})$-modules.

The generic decomposition is Theorem 11.1 and the one at admissible level is Theorem 12.3.

We note that the $\mathcal{W}^\kappa(\mathfrak{g})$-modules $\mathbf{L}_\ell(\chi_{\mu-(\ell+h^\vee)\lambda})$ that appear in Theorem 11.1 play a crucial role in the quantum geometric Langlands program and gauge theory [CG17, Gai18, FG18].

1.2. **Corollaries.** Since the minimal series W-algebras are rational and lisse [Ara15a, Ara15b], Theorem 1 establishes the rationality of a large class of coset vertex algebras. We are able to derive further rationality statements as Corollaries from Main Theorem 1. It is worth mentioning that the rationality problem is wide open for a general coset vertex algebra.

The first one is Corollary 12.5, saying that

Corollary 1.1. *Let $\mathfrak{g}$ be simply laced, k an admissible number, n a positive integer. Then the coset vertex algebra $(L_k(\mathfrak{g})\otimes L_1(\mathfrak{g})^{\otimes n})^{\mathfrak{g}[t]}$ is rational and lisse. Here $L_1(\mathfrak{g})^{\otimes n}$ denotes the tensor product of n copies of $L_1(\mathfrak{g})$, and $\mathfrak{g}[t]$ acts on $L_k(\mathfrak{g})\otimes L_1(\mathfrak{g})^{\otimes n}$ diagonally. In particular, $(L_m(\mathfrak{g})\otimes L_1(\mathfrak{g})^{\otimes n})^{\mathfrak{g}[t]}$ is rational and lisse for any positive integers m, n.*

Second, we establish level-rank dualities of types A and D. For this let $L_k(\mathfrak{gl}_n) = L_k(\mathfrak{sl}_n)\otimes\mathcal{H}$ be the simple affine vertex algebra associated to $\mathfrak{gl}_n$ at level k, where $\mathcal{H}$ is the rank 1 Heisenberg vertex algebra. The natural embedding $\mathfrak{gl}_n \hookrightarrow \mathfrak{gl}_{n+1}$ gives rise to the vertex algebra embedding $L_k(\mathfrak{gl}_n) \hookrightarrow L_k(\mathfrak{gl}_{n+1})$. The invariant subspace $L_k(\mathfrak{gl}_{n+1})^{\mathfrak{gl}_n[t]}$ is the coset vertex subalgebra of $L_k(\mathfrak{gl}_{n+1})$, consisting of elements whose Fourier modes commute with the action of $L_k(\mathfrak{gl}_n)$. Theorem 13.1 implies:

Corollary 1.2. (Level-rank duality of type A) *For positive integers k, n one has*

$$L_k(\mathfrak{gl}_{n+1})^{\mathfrak{gl}_n[t]} \cong \mathcal{W}_\ell(\mathfrak{gl}_k),$$

where ℓ is the non-degenerate admissible number defined by the formula

$$\ell + k = \frac{k+n}{k+n+1}.$$

In particular, $L_k(\mathfrak{sl}_{n+1})^{\mathfrak{gl}_n[t]} \cong \mathcal{W}_\ell(\mathfrak{sl}_k)$, and is simple, rational and lisse.

In other words, the simple affine vertex algebra $L_k(\mathfrak{gl}_n)$ contains a simple vertex subalgebra isomorphic to

$$\mathcal{W}_{\ell_1}(\mathfrak{gl}_k) \otimes \mathcal{W}_{\ell_2}(\mathfrak{gl}_k) \otimes \cdots \otimes \mathcal{W}_{\ell_n}(\mathfrak{gl}_k)$$

with $\ell_i + k = (k+n-i)/(k+n-i+1)$, which may be regarded as an affine, noncommutative analogue of the Gelfand-Tsetlin subalgebra of $U(\mathfrak{gl}_n)$. Note that iterating this coset construction tells us that $L_k(\mathfrak{sl}_n)^{\mathfrak{gl}_m[t]}$ for positive integers $m < n$ is simple, rational and lisse (Corollary 13.2). Note also that for $n = 1$, $L_k(\mathfrak{sl}_{n+1})^{\mathfrak{gl}_n[t]}$ is the $\mathfrak{sl}_2$-parafermion vertex algebra and Theorem 1.2 has been proved in [ALY19].

There is a similar statement for type D. It is Theorem 13.3 and it implies:

Corollary 1.3. (Level-rank duality of type D) *Let k, n be positive integers and k even. Let $G = \mathbb{Z}/2\mathbb{Z}$ for n odd and $G = \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ for n even and let ω_1 be the first fundamental weight of $\mathfrak{so}_{n+1}$. Then*

$$\Big((L_k(\mathfrak{so}_{n+1}) \oplus \mathbb{L}_n(n\omega_1))^{\mathfrak{so}_n[t]}\Big)^G \cong \mathcal{W}_\ell(\mathfrak{so}_k),$$

where ℓ is the non-degenerate admissible number defined by the formula

$$\ell + k - 2 = \frac{k+n-2}{k+n-1}.$$

In particular, $L_k(\mathfrak{so}_{n+1})^{\mathfrak{so}_n[t]}$ is simple, rational and lisse.

Iterating this coset construction tells us that $L_k(\mathfrak{so}_n)^{\mathfrak{so}_m[t]}$ for positive integers m, n, k such that $2 \leqslant m < n$ and k even is simple, rational and lisse (Corollary 13.4).

Main Theorem 1 has as another Corollary rationality of certain coset vertex superalgebras. The type A case is called Kazama-Suzuki coset in physics and it is our Corollary 14.1. We also have a type D case which is Corollary 14.2. These are important since the corresponding superconformal field theories for the Kazama-Suzuki cosets can be used as building blocks for sigma models in string theory à la Gepner [Gep88], and also its rationality is the starting assumption of the superconformal field theory to higher spin supergravity correspondences of [CHR12, CHR13].

1.3. **Gauge Theory and the quantum geometric Langlands program.** Recently there has been considerable interest in connecting four-dimensional supersymmetric gauge theories, the quantum geometric Langlands program and vertex algebras. On the vertex algebra side, one is interested in certain *master chiral algebras* that serve as a kernel for the quantum geometric Langlands correspondence [Gai18] and at the same time as a corner vertex algebra for the junction of topological Dirichlet boundary conditions in gauge theory [CG17, FG18]. Roughly speaking, physics predicts the existence of vertex algebra extensions of tensor products of vertex algebras associated to $\mathfrak{g}$, the Langlands dual ${}^L\mathfrak{g}$ of $\mathfrak{g}$, and the coupling Ψ. Different such extensions are expected to be related via coset constructions and quantum Drinfeld-Sokolov reductions. These vertex algebra extensions are then expected to imply equivalences of involved vertex tensor categories and also spaces of conformal blocks (twisted D-modules). We refer to [FG18, Gai18] for recent progress in this direction.

We will now explain that our Main Theorem 3 (b) proves two physics conjectures. The gauge theory is specified by a coupling Ψ, a generic complex number, and a compact Lie group G, the gauge group. Let $\mathfrak{g}$ be the Lie algebra of G and assume $\mathfrak{g}$ is simply-laced. Let n be a positive integer and define $k = \Psi - h^\vee$, then the conjectural junction vertex algebra for the Dirichlet boundary conditions $B^D_{n,1}$ and $B^D_{0,1}$ is

$$A^{(n)}[G,\Psi] \cong \bigoplus_{\substack{\lambda\in P_+\\ \lambda\in Q}} \mathbb{V}_k(\lambda)\otimes\mathbb{V}_\ell(\lambda), \qquad\qquad \frac{1}{k+h^\vee}+\frac{1}{\ell+h^\vee}=n;$$

as a module for $V_k(\mathfrak{g})\otimes V_\ell(\mathfrak{g})$. The existence of this simple vertex algebra is presently only established for $\mathfrak{g} = \mathfrak{sl}_2$ and $n = 1, 2$ [CG17, CGL18]. Our main Theorem 3 fits very nicely into this context and confirms two physics predictions of [CG17, Section 2 and 3]: First, one takes the case $n = 1$ and then notices that $V_{k-1}(\mathfrak{g}) \otimes L_1(\mathfrak{g})$ is isomorphic to the $V_k(\mathfrak{g}) \otimes \mathcal{W}^\ell(\mathfrak{g})$-module obtained by replacing the Weyl modules

$\mathbb{V}_\ell(\lambda)$ of $V_\ell(\mathfrak{g})$ in $A^{(1)}[G,\Psi]$ by the corresponding modules of $\mathcal{W}^\ell(\mathfrak{g})$ obtained via quantum Drinfeld-Sokolov reduction. Second, the junction vertex algebra between Neumann and Dirichlet boundary conditions is the affine vertex algebra $V_{\Psi-h^\vee}(\mathfrak{g})$ and the one between Neumann and Neumann boundary conditions is the regular W-algebra of $\mathfrak{g}$. Concatenating two such junction vertex algebras gives an extension of these two vertex algebras that is precisely of the form of Main Theorem 3 (b) and so isomorphic to $V_{\Psi-1-h^\vee}(\mathfrak{g})\otimes L_1(\mathfrak{g})$, confirming another physics prediction.

We note that the relations between vertex algebras, physics, and quantum geometric Langlands is rich and we will continue to prove further vertex algebra statements in this context as e.g. [CGL18]. The methods of our work should be quite helpful for that.

Notation. For a vertex algebra V and a vertex subalgebra $W\subset V$, the *commutant* of W in V, or the coset of V by W, is the vertex subalgebra of V defined by

$$\begin{aligned}\mathrm{Com}(W,V)&=\{v\in V\mid [w_{(m)},v_{(n)}]=0 \text{ for all } m,n\in\mathbb{Z},\ w\in W\}\\&=\{v\in V\mid w_{(n)}v=0 \text{ for all } n\in\mathbb{Z}_{\geqslant 0},\ w\in W\}.\end{aligned}$$

Vertex subalgebras W_1 and W_2 of a vertex algebra V are said to form a *dual pair* if they are mutually commutant, that is,

$$W_2=\mathrm{Com}(W_1,V)\quad\text{and}\quad W_1=\mathrm{Com}(W_2,V).$$

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2. The universal commutant vertex algebra

Let $\mathfrak{g}$ be a simple Lie algebra, $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}$ a triangular decomposition, Δ the set of roots of $\mathfrak{g}$, Δ_+ the set of positive roots of $\mathfrak{g}$, W the Weyl group of $\mathfrak{g}$, Q the root lattice of $\mathfrak{g}$, $Q^\vee$ the coroot lattice of $\mathfrak{g}$, P the weight lattice of $\mathfrak{g}$, $P^\vee$ the coweight lattice of $\mathfrak{g}$, $P_+\subset P$ the set of dominant weights of $\mathfrak{g}$, $P_+^\vee\subset P^\vee$ the set of dominant coweights of $\mathfrak{g}$. For $\lambda\in P_+$, denote by E_λ the irreducible finite-dimensional representation of $\mathfrak{g}$ with highest weight λ.

Let $\widehat{\mathfrak{g}}=\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K$, the affine Kac-Moody algebra associated to $\mathfrak{g}$, $\widehat{\mathfrak{h}}=\mathfrak{h}\oplus\mathbb{C}K$ the Cartan subalgebra of $\widehat{\mathfrak{g}}$, $\widehat{\mathfrak{h}}^*=\mathfrak{h}^*\oplus\mathbb{C}\Lambda_0$ the dual of $\widehat{\mathfrak{h}}$. Set $\widehat{\mathfrak{n}}=\mathfrak{n}+\mathfrak{g}[t]t$, $\widehat{\mathfrak{n}}_-=\mathfrak{n}_-+\mathfrak{g}[t^{-1}]t$, $\widehat{\mathfrak{b}}=\widehat{\mathfrak{h}}\oplus\widehat{\mathfrak{n}}$, so that $\widehat{\mathfrak{g}}=\widehat{\mathfrak{n}}_-\oplus\widehat{\mathfrak{h}}\oplus\widehat{\mathfrak{n}}=\widehat{\mathfrak{n}}_-\oplus\widehat{\mathfrak{b}}$.

Let $\widetilde{\mathfrak{h}}=\widehat{\mathfrak{h}}\oplus\mathbb{C}D$ be the extended Cartan subalgebra ([Kac90]) of $\widehat{\mathfrak{g}}$. Then $\widetilde{\mathfrak{h}}^*=\mathfrak{h}^*\oplus\mathbb{C}\Lambda_0\oplus\mathbb{C}\delta$, where $\Lambda_0(K)=\delta(D)=1$, $\Lambda_0(\mathfrak{h}\oplus\mathbb{C}D)=\delta(\mathfrak{h}\oplus K)=0$. Let $\widehat{\Delta}\subset\widetilde{\mathfrak{h}}^*$ be the set of roots of $\widehat{\mathfrak{g}}$, $\widehat{\Delta}^{re}$, the set of real roots, $\widehat{\Delta}_+$, the set of positive roots, $\widehat{\Delta}^{re}_+=\widehat{\Delta}_+\cap\widehat{\Delta}^{re}$. Let $\widehat{W}=W\ltimes Q^\vee$, the affine Weyl group of $\widehat{\mathfrak{g}}$. For $\mu\in Q^\vee$, the corresponding element in $\widehat{W}$ is denote by t_μ. The dot action of $\widehat{W}$ acts on $\widehat{\mathfrak{h}}^*$ is given by $w\circ\Lambda=w(\Lambda+\hat{\rho})-\hat{\rho}$, where $\hat{\rho}=\rho+h^\vee\Lambda_0$ and ρ is the half sum of positive roots of $\mathfrak{g}$.

Let T be an integral $\mathbb{C}[K]$-domain with the structure map $\tau : \mathbb{C}[K] \longrightarrow T$. Define

$$V_T(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} T,$$

where T is regarded as a $\mathfrak{g}[t]$-module on which $\mathfrak{g}[t]$ acts trivially and K acts as the multiplication by $\tau(K)$. There is a unique vertex algebra structure on $V_T(\mathfrak{g})$ such that $\mathbf{1} = 1 \otimes 1$ is the vacuum vector, $Y(T\mathbf{1}, z) = 1 \otimes T$, $Y(xt^{-1}\mathbf{1}, z) = x(z) := \sum_{n\in\mathbb{Z}}((xt^n)\otimes 1)z^{-n-1}$, $x \in \mathfrak{g}$. $V_T(\mathfrak{g})$ is called the *universal affine vertex algebra associated to* $\mathfrak{g}$ *over* T, cf. [DSK06, CL19].

If $T = \mathbb{C}$ and $\tau(K) = k \in \mathbb{C}$, $V_T(\mathfrak{g})$ is the *universal affine vertex algebra associated to* $\mathfrak{g}$ *at level* k and, is denoted also by $V_k(\mathfrak{g})$. The simple graded quotient of $V_k(\mathfrak{g})$ is denoted by $L_k(\mathfrak{g})$.

A $V_T(\mathfrak{g})$-module is the same as a smooth $U(\widehat{\mathfrak{g}})\otimes T$-module on which K acts as the multiplication by $\tau(K)$. Here a $U(\widehat{\mathfrak{g}})\otimes T$-module M is called smooth if $x(z)$, $x \in \mathfrak{g}$, is a field on M, that is, $x(z)m \in M((z))$ for all $m \in M$.

In the rest of this section we assume that $\tau(K) + h^\vee$ is invertible in T, so that the vertex algebra $V_T(\mathfrak{g})$ is conformal by the Sugawara construction. The field corresponding to the conformal vector $\omega \in V_T(\mathfrak{g})$ is given by

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} := \frac{1}{2(\tau(K)+h^\vee)} \sum_i :x_i(z)x^i(z): .$$

The central charge of $V_T(\mathfrak{g})$ equals to $\tau(K)\dim\mathfrak{g}/(\tau(K)+h^\vee) \in T$.

Set $\mathfrak{h}_T = \mathfrak{h}\otimes T$, $\mathfrak{h}_T^* = \mathrm{Hom}_T(\mathfrak{h}_T, T)$. We regard $\mathfrak{h}^*$ as a subset of $\mathfrak{h}_T^*$ by the natural embedding $\mathfrak{h}^* \hookrightarrow \mathfrak{h}_T^*$, $\lambda \mapsto (h\otimes a \mapsto a\lambda(h))$. For a $V_T(\mathfrak{g})$-module M, $\lambda \in \mathfrak{h}_T^*$, and $\Delta \in T$, put

$$\begin{aligned}
&M^\lambda = \{m \in M \mid hm = \lambda(h)m,\ \forall h \in \mathfrak{h}\},\\
&M_{[\Delta]} = \{m \in M \mid (L_0 - \Delta)^r m = 0,\ r \gg 0\}, \quad M_\Delta = \{m \in M \mid L_0 m = \Delta m\},\\
&M^\lambda_{[\Delta]} = M^\lambda \cap M_{[\Delta]}.
\end{aligned}$$

M is called as a *weight module* if $M = \bigoplus_{\lambda\in\mathfrak{h}_T^*,\ \Delta\in T} M^\lambda_{[\Delta]}$.

The *Kazhdan-Lusztig category* KL_T over T is the category of all $V_T(\mathfrak{g})$-modules M such that (1) M is a weight module and $M^\lambda = 0$ unless $\lambda \in P$, (2) $U(\mathfrak{g}[t])\otimes T.m$ is finitely generated as a T-module for all $m \in M$, (3) M is direct sum of finite-dimensional $\mathfrak{g}$-modules.

For $\lambda \in P_+$, define

$$\mathbb{V}_T(\lambda) := U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}K)} (E_\lambda \otimes T) \in \mathrm{KL}_T,$$

where E_λ is the irreducible finite-dimensional $\mathfrak{g}$-module with highest weight λ regarded as a $\mathfrak{g}[t]$-module on which $\mathfrak{g}[t]t$ acts trivially. We have

$$\mathbb{V}_T(\lambda) = \bigoplus_{d\in\mathbb{Z}_{\geqslant 0}} \mathbb{V}_T(\lambda)_{h_\lambda+d}, \quad \mathbb{V}_T(\lambda)_{h_\lambda} = E_\lambda\otimes T,$$

and each $\mathbb{V}_T(\lambda)_{h_\lambda+d}$ is a free T-module of finite rank, where

$$h_\lambda := \frac{(\lambda+2\rho|\lambda)}{2(\tau(K)+h^\vee)} = \frac{1}{2(\tau(K)+h^\vee)}(|\lambda|^2 + \sum_{\alpha\in\Delta_+}(\lambda|\alpha)) \in T. \tag{2}$$

Note that $V_T(\mathfrak{g}) \cong \mathbb{V}_T(0)$ as an object of KL_T.

If $T=\mathbb{C}$ and $\tau(K)=k\in\mathbb{C}$, we write KL_k for KL_T and $\mathbb{V}_k(\lambda)$ for $\mathbb{V}_T(\lambda)$.

Lemma 2.1. *Suppose that $k+h^\vee \in \mathbb{C}\backslash\mathbb{Q}_{\leqslant 0}$. Then $\mathrm{Hom}_{\widehat{\mathfrak{g}}}(V_k(\mathfrak{g}),M)\cong M_{[0]}$ for any $M\in\mathrm{KL}_k$. In particular $V_k(\mathfrak{g})$ is projective in KL_k.*

Proof. Although this statement is well-known, we include a proof for completeness. The assumption and the second formula of (2) imply that $h_\lambda \notin \mathbb{R}_{<0}$ for $\lambda\in P_+$, and that $h_\lambda = 0$ if and only $\lambda=0$. It follows that $M_{[-n]}=0$ for all $n\in\mathbb{Z}_{>0}$, and thus, L_0 acts as $\Omega/2(k+h^\vee)$ on $M_{[0]}$, where Ω is the Casimir element of $U(\mathfrak{g})$. As Ω acts semisimply on an object of KL_k, it follows that $M_{[0]}=M_0$. Moreover, $M_{[0]}$ is a direct sum of trivial representations of $\mathfrak{g}$. Therefore, we conclude that $M_{[0]} = M_0^{\mathfrak{g}[t]} := M_0\cap M^{\mathfrak{g}[t]}$. Hence $\mathrm{Hom}_{\widehat{\mathfrak{g}}}(V_k(\mathfrak{g}),M)\cong M_0^{\mathfrak{g}[t]} = M_{[0]}$, by the Frobenius reciprocity. The last statement follows from the fact that $M\mapsto M_{[0]}$ is an exact functor from KL_k to the category of $\mathfrak{g}$-modules. □

We now assume that $\tau(K)+h^\vee+1$ is invertible in T as well as $\tau(K)+h^\vee$. For $a\in\mathbb{C}$, let $T+a$ denote the $\mathbb{C}[K]$-algebra T with the structure map $K\mapsto\tau(K)+a$.

Consider the tensor product $V_T(\mathfrak{g})\otimes L_1(\mathfrak{g})$. As $V_T(\mathfrak{g})$ is free over $U(\mathfrak{g}[t^{-1}]t^{-1})$, so is $V_T(\mathfrak{g})\otimes L_1(\mathfrak{g})$. It follows that we have the vertex algebra embedding

$$V_{T+1}(\mathfrak{g}) \hookrightarrow V_T(\mathfrak{g})\otimes L_1(\mathfrak{g}), \quad u\mathbf{1}\mapsto \Delta(u)(\mathbf{1}\otimes\mathbf{1}) \quad (u\in U(\widehat{\mathfrak{g}})),$$

where $\Delta(u)$ denotes the coproduct of $U(\widehat{\mathfrak{g}})$.

The following assertion is clear.

Lemma 2.2. *$V_T(\mathfrak{g})\otimes L_1(\mathfrak{g})$ is an object of KL_{T+1} as a $V_{T+1}(\mathfrak{g})$-module. Moreover, each weight space $(V_T(\mathfrak{g})\otimes L_1(\mathfrak{g}))^\lambda_\Delta$ is a free T-module.*

Define the vertex algebra $\mathcal{C}_T(\mathfrak{g})$ by

$$\mathcal{C}_T(\mathfrak{g}) := \mathrm{Com}(V_{T+1}(\mathfrak{g}), V_T(\mathfrak{g})\otimes L_1(\mathfrak{g})). \tag{3}$$

Note that

$$\mathcal{C}_T(\mathfrak{g}) \cong \mathrm{Hom}_{\mathrm{KL}_{T+1}}(V_{T+1}(\mathfrak{g}), V_T(\mathfrak{g})\otimes L_1(\mathfrak{g})) \cong (V_T(\mathfrak{g})\otimes L_1(\mathfrak{g})))^{\mathfrak{g}[t]}$$

by the Frobenius resprocity, where $(V_T(\mathfrak{g})\otimes L_1(\mathfrak{g})))^{\mathfrak{g}[t]}$ denotes the $\mathfrak{g}[t]$-invariant subspace of $V_T(\mathfrak{g})\otimes L_1(\mathfrak{g})$ with respect to the diagonal action. As $V_T(\mathfrak{g})$, $L_1(\mathfrak{g})$, and $V_{T+1}(\mathfrak{g})$ are conformal, $\mathcal{C}_T(\mathfrak{g})$ is conformal with central charge

$$\frac{\tau(K)\dim\mathfrak{g}}{\tau(K)+h^\vee} + \frac{\dim\mathfrak{g}}{h^\vee+1} - \frac{(\tau(K)+1)\dim\mathfrak{g}}{\tau(K)+h^\vee+1} = \frac{\tau(K)(\tau(K)+2h^\vee+1)\dim\mathfrak{g}}{(h^\vee+1)(\tau(K)+h^\vee)(\tau(K)+h^\vee+1)},$$

which equals to

$$\frac{\tau(K)(\tau(K)+2h^\vee+1)\,\mathrm{rank}\,\mathfrak{g}}{(\tau(K)+h^\vee)(\tau(K)+h^\vee+1)}, \tag{4}$$

in the case that $\mathfrak{g}$ is simply laced.

If $T = \mathbb{C}$ and $\tau(K) = k \in \mathbb{C}$, we write $\mathcal{C}_k(\mathfrak{g})$ for $\mathcal{C}_T(\mathfrak{g})$. As $V_k(\mathfrak{g})\otimes L_1(\mathfrak{g})$ belongs to KL_{k+1}, Lemma 2.1 gives the following assertion.

Lemma 2.3. *Suppose that $k + h^\vee + 1 \notin \mathbb{Q}_{\leqslant 0}$. Then*

$$\mathcal{C}_k(\mathfrak{g}) \xrightarrow{\sim} (V_k(\mathfrak{g})\otimes L_1(\mathfrak{g}))_{[0]}.$$

Throughout this paper we will use the following notation.

Definition 2.4. Let R be the ring of rational functions in $\mathbf{k}$ with poles lying in $\{\mathbb{Q}_{\leqslant 0} - h^\vee\} \cup \{\infty\}$, regarded as a $\mathbb{C}[K]$-algebra by the structure map $\tau(K) = \mathbf{k}$. Let F denote the quotient field of R, which is just field of rational functions $\mathbb{C}(\mathbf{k})$.

Proposition 2.5. *We have $\mathcal{C}_R(\mathfrak{g}) \cong (V_R(\mathfrak{g})\otimes L_1(\mathfrak{g}))_{[0]}$. Therefore $\mathcal{C}_R(\mathfrak{g})$ is a free R-module.*

Proof. Although the statement is essentially proved in [CL19], we include the proof for completeness. By the Frobenius reciprocity we have $\mathcal{C}_R(\mathfrak{g}) \cong (V_R(\mathfrak{g})\otimes L_1(\mathfrak{g}))_0^{\mathfrak{g}[t]}$. Hence it is sufficient to show that $(V_R(\mathfrak{g})\otimes L_1(\mathfrak{g}))_{[0]} = (V_R(\mathfrak{g})\otimes L_1(\mathfrak{g}))_0^{\mathfrak{g}[t]}$. The inclusion $\supset$ is clear. Let $v \in (V_R(\mathfrak{g})\otimes L_1(\mathfrak{g}))_{[0]}$. Then $(L_0 v)\otimes 1 = L_0(v\otimes 1) = 0$ in $(V_R(\mathfrak{g})\otimes L_1(\mathfrak{g}))\otimes_R \mathbb{C}_k = V_k(\mathfrak{g})\otimes L_1(\mathfrak{g})$ for all $k \notin \mathbb{Q}_{\leqslant 0} - h^\vee$ by Lemma 2.1, where $\mathbb{C}_k = R/(\mathbf{k} - k)$. Thus $L_0 v = 0$. Similarly, $\mathfrak{g}[t]v = 0$. □

Observe that KL_F is semisimple (see e.g. [Fie06]). Therefore,

$$\mathcal{C}_F(\mathfrak{g}) \cong (V_F(\mathfrak{g})\otimes L_1(\mathfrak{g}))_{[0]} = (V_F(\mathfrak{g})\otimes L_1(\mathfrak{g}))_0 = (V_F(\mathfrak{g})\otimes L_1(\mathfrak{g}))_0^{\mathfrak{g}[t]}.$$

It follows from Proposition 2.5 that

$$\mathcal{C}_F(\mathfrak{g}) = \mathcal{C}_R(\mathfrak{g})\otimes_R F. \tag{5}$$

Proposition 2.6. *Suppose that $k + h^\vee \notin \mathbb{Q}_{\leqslant 0}$. We have*

$$\mathcal{C}_k(\mathfrak{g}) = \mathcal{C}_R(\mathfrak{g})\otimes_R \mathbb{C}_k.$$

In particular, the character of $\mathcal{C}_k(\mathfrak{g})$ is independent of k and coincides with that of $\mathcal{C}_F(\mathfrak{g})$.

Proof. We have $(V_k(\mathfrak{g})\otimes L_1(\mathfrak{g}))_{[0]} = (V_R(\mathfrak{g})\otimes L_1(\mathfrak{g}))_{[0]}\otimes_R \mathbb{C}_k$. Therefore the assertion follows from Lemma 2.3, Proposition 2.5, and (5). □

3. Wakimoto modules and Screening operators

We continue to assume that T is an integral $\mathbb{C}[K]$-domain with the structure map $\tau : \mathbb{C}[K] \longrightarrow T$ such that $\tau(K) + h^\vee$ is invertible.

For $\lambda \in \mathfrak{h}_T^*$, the *Verma module with highest weight λ over T* is defined as

$$\mathbb{M}_T(\lambda) = U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{b}})} T_\lambda,$$

where T_λ denotes the $\widehat{\mathfrak{b}}$-module T on which $\widehat{\mathfrak{n}}$ acts trivially, $\mathfrak{h}$ acts by the character λ, and K acts as the multiplication by $\tau(K)$. This is an object of the deformed category $\mathcal{O}_T$, which is the category of all $V_T(\mathfrak{g})$-modules such that (1) M is a weight module, (2) $U(\widehat{\mathfrak{b}})\otimes_{\mathbb{C}} T.m$ is finitely generated as a T-module for all $m \in M$.

We have $\mathbb{M}_T(\lambda) = \bigoplus_\Delta \mathbb{M}_T(\lambda)_\Delta$. Let $\mathbb{M}_T(\lambda)^* \in \mathcal{O}_T$ be the contragredient dual $\bigoplus_{\mu,\Delta} \mathrm{Hom}_T(\mathbb{M}_T(\lambda)^\mu_\Delta, T)$ of $\mathbb{M}_T(\lambda)$.

If $T = \mathbb{C}$ and $\tau(K) = k \in \mathbb{C}\backslash\{-h^\vee\}$, we write $\mathbb{M}_k(\lambda)$ for $\mathbb{M}_T(\lambda)$ and $\mathcal{O}_k$ for $\mathcal{O}_T$. Let $\mathbb{L}_k(\lambda) \in \mathcal{O}_k$ be the unique simple quotient of $\mathbb{M}_k(\lambda)$. Note that $L_k(\mathfrak{g}) \cong \mathbb{L}_k(0)$.

We now introduce the Wakimoto modules following [FF90b, Fre05]. Let $M_\mathfrak{g}$ be the $\beta\gamma$-system generated by $a_\alpha(z)$, $a^*_\alpha(z)$, $\alpha \in \Delta$, satisfying the OPEs

$$a_\alpha(z)a^*_\beta(w) \sim \frac{\delta_{\alpha\beta}}{z-w}, \quad a_\alpha(z)a_\beta(w) \sim a^*_\alpha(z)a^*_\beta(w) \sim 0.$$

Let π_T be the Heisenberg vertex algebra over T, which is generated by fields

$$b_i(z) = \sum_{n\in\mathbb{Z}} (b_i)_{(n)} z^{-n-1}, \qquad i = 1, \dots, \mathrm{rank}\,\mathfrak{g},$$

with OPEs

$$b_i(z)b_j(w) \sim \frac{\tau(K)(\alpha_i|\alpha_j)}{(z-w)^2}. \tag{6}$$

Define the vertex algebra

$$\mathbb{W}_T(0) := M_\mathfrak{g} \otimes_\mathbb{C} \pi_{T+h^\vee}.$$

By [Fre05, Theorem 5.1], we have the vertex algebra embedding

$$V_T(\mathfrak{g}) \hookrightarrow \mathbb{W}_T(0). \tag{7}$$

Here, since we work over T we need to replace κ in [Fre05, Theorem 5.1] by

$$(\tau(k) + h^\vee)\kappa_0.$$

More generally, for any $\lambda \in \mathfrak{h}^*_T$, let $\pi_{T,\lambda} = U(\mathcal{H}) \otimes_{\mathcal{H}_{\geqslant 0}} T_\lambda$, where $\mathcal{H} = \mathfrak{h}[t, t^{-1}] \otimes \mathbb{C}K \subset \widehat{\mathfrak{g}}$, $\mathcal{H}_{\geqslant 0} = \mathfrak{h}[t] \oplus \mathbb{C}K \subset \mathcal{H}$, $T_\lambda = T$ on which $\mathfrak{h}[t]t$ acts trivially, $h \in \mathfrak{h}$ acts by multiplication by $\lambda(h)$, and K acts by multiplication by $\tau(K)$. Then $\pi_{T,\lambda}$ is naturally a π_T-module, and thus,

$$\mathbb{W}_T(\lambda) := M_\mathfrak{g} \otimes_\mathbb{C} \pi_{T+h^\vee,\lambda}$$

is a $M_\mathfrak{g} \otimes_\mathbb{C} \pi_{T+h^\vee}$-module, and hence, a $V_T(\mathfrak{g})$-module, which belongs to $\mathcal{O}_T$. $\mathbb{W}_T(\lambda)$ is called the *Wakimoto module with highest weight* λ *over* T. If $T = \mathbb{C}$ and $\tau(K) = k \in \mathbb{C}$, $\mathbb{W}_T(\lambda)$ is the usual Wakimoto module with highest weight λ at level k and is denoted also by $\mathbb{W}_k(\lambda)$. If this is the case $\pi_{T+h^\vee,\lambda}$ is denoted by $\pi_{k+h^\vee,\lambda}$.

Let

$$L\mathfrak{n} := \mathfrak{n}[t, t^{-1}] \subset \widehat{\mathfrak{g}},$$

and let $H^{\frac{\infty}{2}+i}(L\mathfrak{n}, M)$ be the semi-infinite $L\mathfrak{n}$-cohomology with coefficients in a smooth $L\mathfrak{n}$-module M ([Fei84]). Fix a basis $\{x_\alpha \mid \alpha \in \Delta_+\}$ of $\mathfrak{n}$, and let $c^\gamma_{\alpha,\beta}$ be the corresponding structure constant of $\mathfrak{n}$. Denote by $\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{n})$ the fermionic ghost system generated by odd fields $\psi_\alpha(z)$, $\psi^*_\alpha(z)$, $\alpha \in \Delta_+$, with OPEs

$$\psi_\alpha(z)\psi^*_\beta(w) \sim \frac{\delta_{\alpha\beta}}{z-w}, \quad \psi_\alpha(z)\psi_\beta(w) \sim \psi^*_\alpha(z)\psi^*_\beta(w) \sim 0.$$

By definition, $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{n}, M)$ is the cohomology of the complex $(M\otimes \bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{n}), Q^{st}_{(0)})$, where

$$Q^{st}(z) = \sum_{n\in\mathbb{Z}} Q^{st}_{(n)} z^{-n-1} = \sum_{\alpha\in\Delta_+} x_\alpha(z)\psi^*_\alpha(z) - \frac{1}{2}\sum_{\alpha,\beta,\gamma\in\Delta_+} c^\gamma_{\alpha,\beta}\psi^*_\alpha(z)\psi^*_\beta(z)\psi_\gamma(z).$$

If M is a π_T-module, then $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{n}, M)$ naturally a $\pi_{T+h^\vee}$-module by the correspondence

$$b_i(z) \mapsto b_i(z) + \sum_{\alpha\in\Delta_+} (\alpha|\alpha_i) :\psi_\alpha(z)\psi^*_\alpha(z): . \tag{8}$$

By construction, $L\mathfrak{n}$ only acts on the first factor $M_\mathfrak{g}$ of $\mathbb{W}_T(\lambda) = M_\mathfrak{g}\otimes_\mathbb{C}\pi_{T+h^\vee,\lambda}$. Hence

$$H^{\frac{\infty}{2}+i}(L\mathfrak{n}, \mathbb{W}_T(\lambda)) \cong H^{\frac{\infty}{2}+i}(L\mathfrak{n}, M_\mathfrak{g})\otimes\pi_{T+h^\vee,\lambda} \cong \begin{cases}\pi_{T+h^\vee,\lambda} & \text{for } i=0\\ 0 & \text{otherwise,}\end{cases} \tag{9}$$

by [Vor93, Theorem 2.1] since $M_\mathfrak{g}$ is free as $U(\mathfrak{n}[t^{-1}]t^{-1})$ and cofree as $U(\mathfrak{n}[t])$-module. In the case that $\lambda = 0$, (9) gives the isomorphism $H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_T(0)) \cong \pi_{T+h^\vee}$ of vertex algebras, and (9) is an isomorphism as $\pi_{T+h^\vee}$-modules.

Lemma 3.1 ([FF92]). *The above described vertex algebra isomorphism*

$$\pi_{T+h^\vee} \longrightarrow H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_T(0)), \quad b_i(z)\mapsto b_i(z)$$

coincides with the vertex algebra homomorphism $\pi_{T+h^\vee} \longrightarrow H^{\frac{\infty}{2}+i}(L\mathfrak{n}, \mathbb{W}_T(0))$ *induced by the action of* $\pi_T \subset V_T(\mathfrak{g})$ *on* $\mathbb{W}_T(0)$ *by* (8).

Proof. The difference of the two actions of $b_i(z)$ on $H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_T(0))$ is given by

$$A_i(z) = -\sum_{\alpha\in\Delta_+}(\alpha|\alpha_i^\vee) :a_\alpha(z)a^*_\alpha(z): + \sum_{\alpha\in\Delta_+}(\alpha|\alpha_i^\vee) :\psi_\alpha(z)\psi^*_\alpha(z):,$$

see [Fre05, Theorem 4.7]. As $A_i(z)$ commutes with $b_j(z)$ for any j, the corresponding state $A_i = \lim_{z\longrightarrow 0} A_i(z)\mathbf{1}$ belongs to the center $\pi^{\mathfrak{h}[t]}_{T+h^\vee}$ of the vertex algebra $\pi_{T+h^\vee} = H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_T(0))$. On the other hand it is straightforward to see that $\pi^{\mathfrak{h}[t]}_{T+h^\vee}$ is trivial, that is, $\pi^{\mathfrak{h}[t]}_{T+h^\vee} = T$. Hence, $A_i = 0$ in $H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_T(0))$ as A_i has the conformal weight 1. □

Though the following theorem was stated in [FF90b] and proved in [Ara14] for $T = \mathbb{C}$, the same proof applies.

Theorem 3.2. *Assume that* T *is a field. The Wakimoto module* $\mathbb{W}_T(\lambda)$ *is a unique object in* $\mathcal{O}_T$ *satisfying* (9).

For $\lambda\in\mathfrak{h}^*_T$, set

$$\hat{\lambda} = \lambda + \tau(K)\Lambda_0 \in \widehat{\mathfrak{h}}^*_T. \tag{10}$$

Let

$$\Delta(\hat{\lambda}) = \{\alpha\in\widehat{\Delta} \mid \langle\hat{\lambda}+\hat{\rho}, \alpha^\vee\rangle \in \mathbb{Z}\},$$

the set of integral roots with respect to $\hat{\lambda}$.

Proposition 3.3. *Suppose that T is a field, $\lambda \in \mathfrak{h}_T^*$, and suppose that $\widehat{\Delta}(\hat{\lambda}) \cap \{-\alpha + n\delta \mid \alpha \in \Delta_+,\ n \in \mathbb{Z}_{\geqslant 1}\} = \emptyset$. Then*

$$\mathbb{W}_T(\lambda) \cong \mathbb{M}_T(\lambda)^*.$$

Proof. By [Ara04, Theorem 3.1], the assumption implies that $\mathbb{M}_T(\lambda)$ is free over $U(\widehat{\mathfrak{n}} \cap t_\mu(\widehat{\mathfrak{n}}_-))$ for any $\mu \in P_+$, where t_μ denotes a Tits lifting of t_μ. Since $\mathfrak{n}_-[t]t = \varinjlim_{\mu \in P_+} \widehat{\mathfrak{n}}_+ \cap t_\mu(\widehat{\mathfrak{n}}_-)$, this shows that $\mathbb{M}_T(\lambda)$ is cofree over $U(\mathfrak{n}_-[t]t)$, and hence, $\mathbb{M}_T(\lambda)^*$ is cofree over $U(\mathfrak{n}[t^{-1}]t^{-1})$. As $\mathbb{M}_T(\lambda)^*$ is obviously free over $U(\mathfrak{n}[t^{-1}]t^{-1})$, $H^{\frac{\infty}{2}+i}(L\mathfrak{n}, \mathbb{M}_T(\lambda)^*) = 0$ for $i \neq 0$. It follows from the Euler-Poincaré principle that the character of $H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{M}_T(\lambda)^*)$ equals to that of $\pi_{T+h^\vee,\lambda}$. On the other hand, there is an obvious non-zero homomorphism $\pi_{T+h^\vee,\lambda} \longrightarrow H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{M}_T(\lambda)^*)$. Since $\pi_{T+h^\vee,\lambda}$ is simple, we conclude that $H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{M}_T(\lambda)^*) \cong \delta_{i,0}\pi_{T+h^\vee,\lambda}$, and we are done by Theorem 3.2. □

Let $V(\mathfrak{n})$ be the universal affine vertex algebra associated with $\mathfrak{n}$, which can be identified with the vertex subalgebra of $V_T(\mathfrak{g})$ generated by $x_\alpha(z)$, $\alpha \in \Delta_+$. The $L\mathfrak{n}$-action on $M_\mathfrak{g}$ induces the vertex algebra embedding $V(\mathfrak{n}) \hookrightarrow M_\mathfrak{g}$.

There is also a right action $x \mapsto x^R$ of $L\mathfrak{n}$ on $M_\mathfrak{g}$ that commutes with the left action of $L\mathfrak{n}$ ([Fre05]). In fact, as a $U(L\mathfrak{n})$-bimodule $M_\mathfrak{g}$ is isomorphic to the *semi-regular bimodule* [Vor93, Vor99] of $L\mathfrak{n}$, see [Ara14].

Proposition 3.4. (1) *([Ara14, Proposition 2.1]). Let M be a $\mathfrak{n}((t))$-module that is integrable over $\mathfrak{n}[[t]]$. There is a T-linear isomorphism*

$$\Phi : \mathbb{W}_T(\lambda) \otimes_\mathbb{C} M \xrightarrow{\sim} \mathbb{W}_T(\lambda) \otimes_\mathbb{C} M$$

such that

$$\Phi \circ \Delta(x) = (x \otimes 1) \circ \Phi, \quad \Phi \circ (x^R \otimes 1) = (x^R \otimes 1 - 1 \otimes x) \circ \Phi \quad \textit{for } x \in L\mathfrak{n}.$$

Here Δ denotes the coproduct: $\Delta(x) = x \otimes 1 + 1 \otimes x$.

(2) *Let V be a vertex algebra equipped with a vertex algebra homomorphism $V(\mathfrak{n}) \longrightarrow V$, and the induced action of $\mathfrak{n}[[t]]$ on V is integrable. Then the map Φ in (1) for $M = V$ is a vertex algebra isomorphism.*

Proof. (2) The commutative vertex subalgebra of $M_\mathfrak{g}$ generated by $a_\alpha^*(z)$, $\alpha \in \Delta_+$, is naturally identified with functions $\mathbb{C}[J_\infty N]$ on the arc space $J_\infty N$ of the unipotent group N whose Lie algebra is $\mathfrak{n}$. In this identification, the subalgebra $\mathbb{C}[N]$ of $\mathbb{C}[J_\infty N]$ is identified with $\mathbb{C}[(a_\alpha^*)_{(-1)}] = \mathbb{C}[(a_\alpha^*)_{(-1)}]\mathbf{1}$. The arc space $J_\infty N$ is a prounipotent group whose Lie algebra is $J_\infty \mathfrak{n} = \mathfrak{n}[[t]]$. The vertex subalgebra $\mathbb{C}[J_\infty N] \subset M_\mathfrak{g}$ is a $\mathfrak{n}[[t]]$-bi-submodule of $M_\mathfrak{g}$, and the $\mathfrak{n}[[t]]$-bi-submodule structure of $\mathbb{C}[J_\infty N]$ is identical to the one obtained by differentiating the natural $J_\infty N$-bimodule structure of $\mathbb{C}[J_\infty N]$. We have the isomorphism

$$V(\mathfrak{n}) \otimes_\mathbb{C} \mathbb{C}[J_\infty N] \xrightarrow{\sim} M_\mathfrak{g}, \quad u\mathbf{1} \otimes f \mapsto uf,$$

$(u \in U(t^{-1}\mathfrak{n}[t^{-1}]))$ as left $t^{-1}\mathfrak{n}[t^{-1}]$-modules and right $\mathfrak{n}[[t]]$-modules. We have

$$x_\alpha(z) a_\beta^*(w) \sim \frac{1}{z-w}(x_\alpha a_\beta^*)(w) \tag{11}$$

for $\alpha,\beta\in\Delta_+$, where on the right-hand side $x_\alpha\in\mathfrak{n}$ acts on $a^*_\beta=(a^*_\beta)_{(-1)}\mathbf{1}\in\mathbb{C}[N]$ as a left-invariant vector field.

By the assumption the action of $\mathfrak{n}[[t]]$ on V integrates to the action of $J_\infty N$. Let $\phi: V\longrightarrow \mathbb{C}[J_\infty N]\otimes V$ be the corresponding comodule map. Thus, $\phi(v_i)=\sum_j f_{ij}\otimes v_j$ if $\{v_i\}$ is a basis of V and $gv_i=\sum_j f_{ij}(g)v_j$ with $f_{ij}\in\mathbb{C}[J_\infty N]$ for all $g\in J_\infty N$. We have $\phi\circ g=(g\otimes 1)\circ\phi$ for $g\in J_\infty N$. We shall show that ϕ is a vertex algebra homomorphism, that is,

$$\phi((v_i)_{(n)}v_k)=\phi(v_i)_{(n)}\phi(v_k)=\sum_{\substack{j,l\\ r\geqslant 0}}((f_{ij})_{(-r-1)}f_{kl})\otimes((v_k)_{(n+r)}v_l)$$

for all i,k,n. By the definition of ϕ this is equivalent to that

$$g.(v_i)_{(n)}v_k=\sum_{\substack{i,j\\ r\geqslant 0}}(f_{ij})_{(-r-1)}(g)f_{kl}(g)(v_k)_{(n+r)}v_l=\sum_{r\geqslant 0}(f_{ij})_{(-r-1)}(g)(v_k)_{(n+r)}(g.v_k),$$

for $g\in J_\infty N$, or equivalently,

$$\mathrm{Ad}(g)(v_i)_{(n)}=\sum_{\substack{j\\ r\geqslant 0}}(f_{ij})_{(-r-1)}(g)(v_j)_{(n+r)}$$

for $g\in J_\infty N$. By differentiating both sides, it is enough to show that

$$[x_{(m)},(v_i)_{(n)}]=\sum_{\substack{j\\ r\geqslant 0}}(x_{(m)}(f_{ij})_{(-r-1)})(1)(v_j)_{(n+r)} \tag{12}$$

for $x\in\mathfrak{n}$, $m\geqslant 0$, $n\in\mathbb{Z}$, where $(x_{(m)}(f_{ij})_{(-r-1)})(1)$ is the value of $x_{(m)}(f_{ij})_{(-r-1)}\in\mathbb{C}[J_\infty N]$ at the identity. By the commutation formula we have $[x_{(m)},(v_i)_{(n)}]=\sum_{s\geqslant 0}\binom{m}{s}(x_{(s)}v_i)_{(m+n-s)}=\sum_{\substack{s\geqslant 0\\ j}}\binom{m}{s}((x_{(s)}f_{ij})(1)v_j)_{(m+n-s)}$. Hence (12) follows from the fact that

$$(x_{(m)}(f_{ij})_{(-r-1)})(1)=\frac{m}{r}(x_{(m-1)}(f_{ij})_{(-r)})(1)=\cdots=\binom{m}{r}(x_{(m-r)}(f_{ij})_{(-1)})(1)$$

for $m,r\geqslant 0$.

Next set

$$\tilde{\phi}:\mathbb{C}[J_\infty N]\otimes V\longrightarrow\mathbb{C}[J_\infty N]\otimes V,\quad f\otimes v\mapsto (f\otimes 1)\phi(v).$$

Then $\tilde{\phi}$ is a linear isomorphism that satisfies

$$\tilde{\phi}\circ(g\otimes g)=(g\otimes 1)\circ\tilde{\phi}, \tag{13}$$

$$\tilde{\phi}\circ(g^R\otimes 1)=(g^R\otimes g^{-1})\circ\tilde{\phi}, \tag{14}$$

for $g\in J_\infty N$, where g^R denotes the right action $(g^Rf)(a)=f(ga)$. Moreover, $\tilde{\phi}$ is a vertex algebra homomorphism since ϕ is so and $\mathbb{C}[J_\infty N]$ is commutative.

Define the linear isomorphism

$$\Psi: M_{\mathfrak{g}}\otimes V=V(\mathfrak{n})\otimes\mathbb{C}[J_\infty N]\otimes V\xrightarrow{\sim} M_{\mathfrak{g}}\otimes V,\quad u\otimes w\mapsto \Delta(u)(\tilde{\phi}^{-1}(u)),$$

($u\in V(\mathfrak{n})$, $w\in\mathbb{C}[J_\infty N]\otimes V$), where Δ is the coproduct of $U(t^{-1}\mathfrak{n}[t^{-1}])$ (that is identified with $V(\mathfrak{n})$) and $\mathbb{C}[J_\infty N]\otimes V$ is naturally considered as a vertex subalgebra of $M_{\mathfrak{g}}\otimes V$. We claim that Ψ is a vertex algebra homomorphism. To see this, first note that the restrictions of Ψ to vertex subalgebras $V(\mathfrak{n})$, $\mathbb{C}[J_\infty N]\otimes V$ are clearly vertex algebra homomorphism. Therefore it is sufficient to check that Ψ preserves the OPE's between generators of $V(\mathfrak{n})$ and $\mathbb{C}[J_\infty N]\otimes V$. By (13), $\Psi(V)=\tilde{\phi}^{-1}(V)$ is contained in the commutant $(M_\mathfrak{g}\otimes V)^{\mathfrak{n}[t]}$ of vertex subalgebra $\Delta(V(\mathfrak{n}))$ in $M_\mathfrak{g}\otimes V$. Also, since the restriction of Ψ to $\mathbb{C}[J_\infty N]$ is the identify map, we find that (11) is preserved by Ψ. We have shown that Ψ is a vertex algebra isomorphism, and thus. $\tilde{\Phi}:=\Psi^{-1}$ is also a vertex algebra isomorphism.

Since $\tilde{\Phi}\circ\Delta(x)=(x\otimes 1)\circ\tilde{\Phi}$ for $x\in\mathfrak{n}\subset V(\mathfrak{n})$ by definition and $\tilde{\Phi}$ is a vertex algebra homomorphism, we get that $\tilde{\Phi}\circ\Delta(x)=(x\otimes 1)\circ\tilde{\Phi}$ for all $x\in\mathfrak{n}((t))$. Next we show that $\tilde{\Phi}\circ(x^R\otimes 1)=(x^R\otimes 1-1\otimes x)\circ\tilde{\Phi}$ for $x\in\mathfrak{n}((t))$. By the same reasoning as above, it is sufficient to show that the element $a=\tilde{\Phi}(x_\alpha^R\otimes 1)-x_\alpha^R\otimes 1+1\otimes x_\alpha$ is zero for all $\alpha\in\Delta_+$. By (14), we have $a_{(n)}v=0$ for all $n\geqslant 0$, $v\in M_\mathfrak{g}\otimes V$, that is, a belongs to the center of $M_\mathfrak{g}\otimes V$. Since $M_\mathfrak{g}$ is simple, this implies that a belongs to the center of $V\subset M_\mathfrak{g}\otimes V$. On the other hand, we have $x_\alpha^R=x_\alpha+\sum_{\beta>\alpha}(P_{\alpha,\beta})_{(-1)}x_\beta$ for some polynomial $P_{\alpha,\beta}$ in $\mathbb{C}[N]$ of weight $\alpha-\beta$, see [Fre05, Remark 4.4], where we count the weight of a_α^* as $-\alpha$. Also we have $\tilde{\Phi}(x_\beta\otimes 1)=x_\beta\otimes 1-\tilde{\Phi}(1\otimes x_\beta)=x_\beta\otimes 1-\phi(x_\beta)=x_\beta\otimes 1-1\otimes x_\beta-\sum_{\gamma>\beta}R_{\beta,\gamma}\otimes x_\gamma$, where $R_{\beta,\gamma}$ is some polynomial in $\mathbb{C}[N]$ of weight $\beta-\gamma$. It follows that $a\in\mathbb{C}[N]^*V(\mathfrak{n})\otimes V$, where $\mathbb{C}[N]^*$ is the argumentation ideal of $\mathbb{C}[N]$. Therefore, we get that $a=0$.

The assertion is proved by extending $\tilde{\Phi}$ to the vertex algebra isomorphism $\Phi:\mathbb{W}_T(\lambda)\otimes_\mathbb{C}V\xrightarrow{\sim}\mathbb{W}_T(\lambda)\otimes_\mathbb{C}V$ whose restriction to the vertex subalgebra $\pi_{T+h^\vee}$ is the identity map.

(1) Although the assertion was proved in [Ara14], we give a yet another based on the statement (2) we have just proved. As in (2), we obtain a linear isomorphism $\Phi:\mathbb{W}_T(\lambda)\otimes_\mathbb{C}M\xrightarrow{\sim}\mathbb{W}_T(\lambda)\otimes_\mathbb{C}M$. Note that $\mathbb{W}_T(\lambda)\otimes_\mathbb{C}M$ is naturally a module over the vertex algebra $\mathbb{W}_T(\lambda)\otimes_\mathbb{C}V(\mathfrak{n})$, and we have the isomorphism $\Phi:\mathbb{W}_T(\lambda)\otimes_\mathbb{C}V(\mathfrak{n})\xrightarrow{\sim}\mathbb{W}_T(\lambda)\otimes_\mathbb{C}V(\mathfrak{n})$ of vertex algebras obtained in (2). By construction we have $\Phi(u_{(n)})\Phi(v)=\Phi(u_{(n)}v)$ for $u\in\mathbb{W}_T(\lambda)\otimes_\mathbb{C}V(\mathfrak{n})$, and therefore, Φ satisfies the required properties. □

For each $i=1,\dots,\operatorname{rank}\mathfrak{g}$, define an operator $S_i(z):\mathbb{W}_T(\mu)\longrightarrow\mathbb{W}_T(\mu-\alpha_i)$, $\mu\in\mathfrak{h}_T^*$, by

$$S_i(z)=:e_i^R(z):e^{\int-\frac{1}{\tau(K)+h^\vee}b_i(z)dz}::,$$

where

$$\begin{aligned}(15)\quad &:e^{\int-\frac{1}{\tau(K)+h^\vee}b_i(z)dz}:\\ &=T_{-\alpha_i}z^{-\frac{(b_i)_{(0)}}{\tau(K)+h^\vee}}\exp\Big(-\frac{1}{\tau(K)+h^\vee}\sum_{n<0}\frac{(b_i)_{(n)}}{n}z^{-n}\Big)\exp\Big(-\frac{1}{\tau(K)+h^\vee}\sum_{n>0}\frac{(b_i)_{(n)}}{n}z^{-n}\Big).\end{aligned}$$

Here $z^{-\frac{(b_i)_{(0)}}{\tau(K)+h^\vee}} = \exp(-\frac{(b_i)_{(0)}}{\tau(K)+h^\vee}\log z)$ and $T_{-\alpha_i}$ is the translation operator $\pi_{T,\mu} \longrightarrow \pi_{T,\mu-\alpha_i}$ sending the highest weight vector to the highest weight vector and commuting with all $(b_j)_{(n)}$, $n \neq 0$. The residue

$$S_i := \int S_i(z)dz \tag{16}$$

is an intertwining operator between the $\widehat{\mathfrak{g}}$-modules $\mathbb{W}_T(0)$ and $\mathbb{W}_T(-\alpha_i)$ ([Fre05]).

Proposition 3.5 ([FF92])**.** *Let $T = F$ (see Definition 2.4), or $T = \mathbb{C}$ with $\tau(K) = k \notin \mathbb{Q}$. Then there exists a resolution of the $\widehat{\mathfrak{g}}$-module $V_T(\mathfrak{g})$ of the form*

$$0 \longrightarrow V_T(\mathfrak{g}) \longrightarrow C_0 \xrightarrow{d_0} C_1 \longrightarrow \dots \longrightarrow C_{\ell(w_0)} \longrightarrow 0,$$
$$C_i = \bigoplus_{\substack{w\in W\\ \ell(w)=i}} \mathbb{W}_T(w\circ 0),$$

where $\ell(w)$ is the length of $w \in W$, w_0 is the longest element of W, and $d_0 = \oplus_{i=1}^{\mathrm{rank}\,\mathfrak{g}} c_i S_i$ for some $c_i \in \mathbb{C}^*$.

Proof. By Fiebig's equivalence [Fie06], there exists a resolution

$$0 \longrightarrow \mathbb{L}_k(0) \longrightarrow C_0 \xrightarrow{d_0} C_1 \longrightarrow \dots \longrightarrow C_n \longrightarrow 0 \tag{17}$$

such that $C_i = \bigoplus_{\substack{w\in W\\ \ell(w)=i}} \mathbb{M}_T(w\circ 0)^*$, which corresponds to the dual of BGG resolution of the trivial representation of $\mathfrak{g}$. By Proposition 3.3, $\mathbb{M}_T(w\circ 0)^* \cong \mathbb{W}_T(w\circ 0)$. The equality $d_0 = \oplus_{i=1}^{\mathrm{rank}\,\mathfrak{g}} c_i S_i$ follows from the facts that S_i is non-trivial and that $\dim_T \mathrm{Hom}_{\mathcal{O}_T}(\mathbb{M}_T(0)^*, \mathbb{M}_T(-\alpha_i)^*) = 1$. □

4. More on screening operators

In this section we let $T = \mathbb{C}$ with $\tau(K) = k \in \mathbb{C}\backslash\{-h^\vee\}$.

By the formula just before Proposition 7.1 of [Fre05], the construction (16) of the intertwining operator is generalized as follows (see [TK86] for the details): Let $\mu \in \mathfrak{h}^*$ such that

$$(\mu|\alpha_i) + m(k+h^\vee) = \frac{(\alpha_i|\alpha_i)}{2}(n-1). \tag{18}$$

for some $n \in \mathbb{Z}_{\geqslant 0}$ and $m \in \mathbb{Z}$. Note that (18) is equivalent to

$$\langle \hat{\mu} + \hat{\rho}, (\alpha_i + m\delta)^\vee \rangle = n,$$

where $\hat{\mu} = \mu + k\Lambda_0$ as before, see (10). We have

$$S_i(z_1)S_i(z_2)\dots S_i(z_n)|_{\mathbb{W}_k(\mu)}$$
$$= \prod_{i=1}^n z_i^{-\frac{(\mu|\alpha_i)}{k+h^\vee}} \prod_{1\leqslant i<j\leqslant n} (z_i - z_j)^{\frac{(\alpha_i|\alpha_i)}{k+h^\vee}} : S_i(z_1)S_i(z_2)\dots S_i(z_n) : .$$

By setting $z_1 = z$, $z_i = zy_{i-1}$, $i \geqslant 2$, we have

(19)

$$\prod_{i=1}^{n} z_i^{-\frac{(\mu|\alpha_i)}{k+h^\vee}} \prod_{1\leqslant i<j\leqslant n} (z_i - z_j)^{\frac{(\alpha_i|\alpha_i)}{k+h^\vee}}$$

$$= z^{-\frac{n((\mu|\alpha_i)-\frac{(\alpha_i|\alpha_i)}{2}(n-1))}{k+h^\vee}} \prod_{i=1}^{n-1} y_i^{m-\frac{(n-1)(\alpha_i|\alpha_i)}{2(k+h^\vee)}} (1-y_i)^{\frac{(\alpha_i|\alpha_i)}{k+h^\vee}} \prod_{1\leqslant i<j\leqslant n-1} (y_i - y_j)^{\frac{(\alpha_i|\alpha_i)}{k+h^\vee}}$$

$$= z^{nm} \prod_{i=1}^{n-1} y_i^{m-\frac{(n-1)(\alpha_i|\alpha_i)}{2(k+h^\vee)}} (1-y_i)^{\frac{(\alpha_i|\alpha_i)}{k+h^\vee}} \prod_{1\leqslant i<j\leqslant n-1} (y_i - y_j)^{\frac{(\alpha_i|\alpha_i)}{k+h^\vee}}.$$

Let $\mathcal{L}_n^*(\mu, k)$ be the the local system with coefficients in $\mathbb{C}$ associated to the monodromy group of the multi-valued function (19) on the manifold $Y_n = \{(z_1, \dots, z_n) \in (\mathbb{C}^*)^n \mid z_i \neq z_j\}$, and denote by $\mathcal{L}_n(\mu, k)$ the dual local system of $\mathcal{L}_n^*(\mu, k)$ ([AK11]). Then, for an element $\Gamma \in H_n(Y_n, \mathcal{L}_n(\mu, k))$,

(20) $$S_i(n, \Gamma) := \int_\Gamma S_i(z_1)S_i(z_2)\dots S_i(z_n)dz_1\dots dz_n : \mathbb{W}_k(\mu) \longrightarrow \mathbb{W}_k(\mu - n\alpha_i)$$

defines a $\widehat{\mathfrak{g}}$-module homomorphism.

The following statement was proved by Tsuchiya and Kanie in the case of affine $\mathfrak{sl}_2$ (see [TK86, Theorem 0.6]), but the same proof applies.

Theorem 4.1. *Suppose that*

$$\frac{2d(d+1)}{(k+h^\vee)(\alpha_i|\alpha_i)} \notin \mathbb{Z}, \quad \frac{2d(d-n)}{(k+h^\vee)(\alpha_i|\alpha_i)} \notin \mathbb{Z},$$

for all $1 \leqslant d \leqslant n-1$. Then there exits a cycle $\Gamma \in H_n(Y_n, \mathcal{L}_n(\mu, k))$ such that $S_i(n, \Gamma)$ is non-trivial.

Proposition 4.2 ([Fre92])**.** *Let k be generic, $\lambda \in P_+$, $\mu \in P_+^\vee$. There exists a resolution of the $\widehat{\mathfrak{g}}$-module $\mathbb{L}_k(\lambda - (k+h^\vee)\mu)$ of the form*

$$0 \longrightarrow \mathbb{L}_k(\lambda - (k+h^\vee)\mu) \longrightarrow C_0 \xrightarrow{d_0} C_1 \longrightarrow \dots \longrightarrow C_n \longrightarrow 0,$$
$$C_i = \bigoplus_{\substack{w\in W\\ \ell(w)=i}} \mathbb{W}_k(w\circ\lambda - (k+h^\vee)\mu).$$

The map d_0 is given by

$$d_0 = \sum_{i=1}^{\operatorname{rank}\mathfrak{g}} c_i S_i(n_i, \Gamma_i)$$

for some $c_i \in \mathbb{C}$, with $n_i = \langle \lambda + \rho, \alpha_i^\vee \rangle$ and Γ_i is the cycle as in Theorem 4.1.

Proof. We prove in the same manner as Proposition 3.5. Set $\Lambda = \lambda - (k+h^\vee)\mu + k\Lambda_0 \in \widehat{\mathfrak{h}}^*$. Then $\hat{\Delta}(\Lambda) = t_{-\mu}(\Delta) \cong \Delta$. Hence Fiebig's equivalence [Fie06] implies that there exists a resolution

(21) $$0 \longrightarrow \mathbb{L}_k(\lambda - (k+h^\vee)\mu) \longrightarrow C_0 \xrightarrow{d_0} C_1 \longrightarrow \dots \longrightarrow C_n \longrightarrow 0$$

such that

$$C_i = \bigoplus_{\substack{w\in W\\ \ell(w)=i}} \mathbb{M}_k(w\circ\lambda-(k+h^\vee)\mu)^*.$$

On the other hand, Proposition 3.3 gives that

$$\mathbb{M}_k(w\circ\lambda-(k+h^\vee)\mu)^*\cong \mathbb{W}_k(w\circ\lambda-(k+h^\vee)\mu).$$

The second assertion follows from the non-triviality of the map $S_i(n_i,\Gamma_i)$ and the fact that $\mathrm{Hom}_{\widehat{\mathfrak{g}}}(\mathbb{M}_k(\lambda-(k+h^\vee)\mu)^*,\mathbb{M}_k(s_i\circ\lambda-(k+h^\vee)\mu)^*)$ is one-dimensional. □

Corollary 4.3 ([FF92]). *Let k be generic, $\lambda\in P_+$, $\mu\in P_+^\vee$. Then*

$$H^{\frac{\infty}{2}+i}(L\mathfrak{n},\mathbb{L}_k(\lambda-(k+h^\vee)\mu))\cong \bigoplus_{\substack{w\in W\\ \ell(w)=i}} \pi^{k+h^\vee}_{w\circ\lambda-(k+h^\vee)\mu}$$

as modules over the Heisenberg vertex algebra $\pi^{k+h^\vee}$.

Proof. As Wakimoto modules are acyclic with respect to the cohomology functor $H^{\frac{\infty}{2}+i}(L\mathfrak{n},?)$, $H^{\frac{\infty}{2}+i}(L\mathfrak{n},\mathbb{L}_k(\lambda-(k+h^\vee)\mu))$ is the cohomology of the complex obtained by applying the functor $H^{\frac{\infty}{2}+i}(L\mathfrak{n},?)$ to the resolution in Proposition 4.2, whence the statement. □

5. Principal W-algebras and the Miura map

Let T be an integral $\mathbb{C}[K]$-domain with the structure map $\tau:\mathbb{C}[K]\longrightarrow T$.

Let $f\in\mathfrak{n}_-$ be a principal nilpotent element, $\{e,f,h\}$ an $\mathfrak{sl}_2$-triple. Let $H^\bullet_{DS}(?)$ be the Drinfeld-Sokolov reduction cohomology functor associated to f ([FF90a, FBZ04]). By definition,

$$H^\bullet_{DS}(M)=H^{\frac{\infty}{2}+\bullet}(L\mathfrak{n},M\otimes\mathbb{C}_\Psi),$$

where $L\mathfrak{n}$ acts on $M\otimes\mathbb{C}_\Psi$ diagonally and $\mathbb{C}_\Psi$ is the one-dimensional representation of $L\mathfrak{n}$ defined by the character $\Psi: L\mathfrak{n}\ni xt^n\mapsto \delta_{n,-1}(f|x)$. The space

$$\mathcal{W}^T(\mathfrak{g}):=H^0_{DS}(V_T(\mathfrak{g}))$$

is naturally a vertex algebra, and is called the *principal W-algebra associated to $\mathfrak{g}$ over T*. We have

$$H^i_{DS}(V_T(\mathfrak{g}))=0\quad\forall i\neq 0. \tag{22}$$

This was proved in [FBZ04] for $T=\mathbb{C}$, but the same proof applies for the general case. We write $\mathcal{W}^k(\mathfrak{g})$ for $\mathcal{W}^T(\mathfrak{g})$ if $T=\mathbb{C}$ and $\tau(K)=k$.

We have

$$\operatorname{gr}\mathcal{W}^T(\mathfrak{g})\cong\mathbb{C}[J_\infty\mathcal{S}_f]\otimes T \tag{23}$$

as Poisson vertex algebras, where $\operatorname{gr}V$ denotes the graded Poisson vertex algebra associated to the canonical filtration [Li05] of V, $\mathcal{S}_f=f+\mathfrak{g}^e\subset\mathfrak{g}=\mathfrak{g}^*$ is the Kostant slice, $J_\infty X$ is the infinite jet scheme of X. This was proved in [Ara15a] for $T=\mathbb{C}$, but the same proof applies. In particular, $\mathcal{W}^T(\mathfrak{g})$ is free as a T-module.

From the proof of (22), or the fact (23), it follows that for a given $\mathbb{C}[K]$-algebra homomorphism $T_1 \longrightarrow T_2$ we have

$$\mathcal{W}^{T_2}(\mathfrak{g}) = \mathcal{W}^{T_1}(\mathfrak{g}) \otimes_{T_1} T_2.$$

Suppose that $\tau(K)+h^\vee$ is invertible. Then the vertex algebra $\mathcal{W}^T(\mathfrak{g})$ is conformal and its central charge is given by

$$-\frac{((h+1)(\tau(K)+h^\vee)-h^\vee)(r^\vee {}^L h^\vee(\tau(K)+h^\vee)-(h+1))\operatorname{rank}\mathfrak{g}}{\tau(K)+h^\vee}, \tag{24}$$

where h is the Coxeter number of $\mathfrak{g}$, ${}^L h^\vee$ is the dual Coxeter number of the Langlands dual ${}^L\mathfrak{g}$ of $\mathfrak{g}$, and $r^\vee$ is the maximal number of the edges in the Dynkin diagram of $\mathfrak{g}$. We have $\mathcal{W}^T(\mathfrak{g}) = \bigoplus_{\Delta \in \mathbb{Z}_{\geqslant 0}} \mathcal{W}^T(\mathfrak{g})_\Delta$, $\mathcal{W}^T(\mathfrak{g})_0 = T$, and each $\mathcal{W}^T(\mathfrak{g})_\Delta$ is a free T-module of finite rank. Here $\mathcal{W}^T(\mathfrak{g})_\Delta$ is the T-submodule of $\mathcal{W}^T(\mathfrak{g})$ spanned by the vectors of conformal weight Δ.

As explained in [Ara17], there is a vertex algebra embedding

$$\Upsilon : \mathcal{W}^T(\mathfrak{g}) \hookrightarrow \pi_{T+h^\vee}$$

called the *Miura map* ([FF90a, Ara17]). The induced map $\operatorname{gr}\Upsilon : \operatorname{gr}\mathcal{W}^T(\mathfrak{g}) = \mathbb{C}[J_\infty \mathcal{S}_f] \otimes T \longrightarrow \pi_{T+h^\vee} = \mathbb{C}[J_\infty \mathfrak{h}^*] \otimes T$ between associated graded Poisson vertex algebras is an injective homomorphism, and we have

$$\operatorname{gr}(\Upsilon(\mathcal{W}^T(\mathfrak{g}))) = (\operatorname{gr}\Upsilon)(\operatorname{gr}\mathcal{W}^T(\mathfrak{g})) = \mathbb{C}[J_\infty(\mathfrak{h}^*/W)] \otimes T. \tag{25}$$

Lemma 5.1. *Let T_2 be a $\mathbb{C}[K]$-subalgebra of T_1. Then*

$$\Upsilon(\mathcal{W}^{T_2}(\mathfrak{g})) = \Upsilon(\mathcal{W}^{T_1}(\mathfrak{g})) \cap \pi_{T_2+h^\vee}.$$

Proof. Clearly, $\Upsilon(\mathcal{W}^{T_1}(\mathfrak{g})) \subset \Upsilon(\mathcal{W}^T(\mathfrak{g})) \cap \pi_{T_1+h^\vee}$. Thus, it is sufficient to show that $\operatorname{gr}(\Upsilon(\mathcal{W}^{T_1}(\mathfrak{g}))) = \operatorname{gr}(\Upsilon(\mathcal{W}^T(\mathfrak{g})) \cap \pi_{T_1+h^\vee})$. But this follows from (25). □

The Miura map Υ may be described as follows. By applying the functor $H^0_{DS}(?)$ to the embedding (7), we obtain the vertex algebra homomorphism

$$\mathcal{W}^T(\mathfrak{g}) = H^0_{DS}(V_T(\mathfrak{g})) \longrightarrow H^0_{DS}(\mathbb{W}_T(0)) \cong \pi_{T+h^\vee}. \tag{26}$$

Here the last isomorphism follows from the following lemma.

Lemma 5.2 ([FF92])**.** *We have $H^i_{DS}(\mathbb{W}_T(0)) = 0$ for $i \neq 0$ and $H^0_{DS}(\mathbb{W}_T(0)) \cong \pi_{T+h^\vee}$ as vertex algebras. More generally, $H^i_{DS}(\mathbb{W}_T(\lambda)) \cong \delta_{i,0}\pi_{T+h^\vee,\lambda}$ as $\pi_{T+h^\vee}$-modules for any $\lambda \in \mathfrak{h}^*_T$.*

Proof. By applying Proposition 3.4 for $M = \mathbb{C}_\Psi$, we obtain the isomorphism

$$H^i_{DS}(\mathbb{W}_T(\lambda)) \xrightarrow[\sim]{[\Phi]} H^{\frac{\infty}{2}+i}(L\mathfrak{n}, \mathbb{W}_T(\lambda)) \cong \delta_{i,0}\pi_{T+h^\vee,\lambda},$$

where $[\Phi]$ denotes the map induced by the isomorphism $\Phi : \mathbb{W}_T(\lambda) \otimes \mathbb{C}_\Psi \xrightarrow{\sim} \mathbb{W}_T(\lambda) \otimes \mathbb{C}_\Psi$ in Proposition 3.4. □

Proposition 5.3 ([Fre92], see also [Gen17, Lemma 5.1])**.** *The map (26) coincides with the Miura map Υ via the isomorphism $H^0_{DS}(\mathbb{W}_T(0)) \cong \pi_{T+h^\vee}$ in Lemma 5.2.*

Since it is a $L\mathfrak{n}$-module homomorphism, the map $S_i(z) : \mathbb{W}_T(\mu) \longrightarrow \mathbb{W}_T(\mu - \alpha_i)$ induces the linear map

$$S_i^W(z) : \pi_{T+h^\vee,\mu} = H^0_{DS}(\mathbb{W}_T(\mu)) \longrightarrow \pi_{T+h^\vee,\mu-\alpha_i} = H^0_{DS}(\mathbb{W}_T(\mu - \alpha_i))$$

for $\mu \in \mathfrak{h}_T^*$.

Lemma 5.4. *For each $i = 1, \dots, \operatorname{rank}\mathfrak{g}$, we have*

$$S_i^W(z) =: e^{\int -\frac{1}{\tau(K)+h^\vee} b_i(z)dz} :,$$

where the right-hand-side is defined in (15).

Proof. We have used the isomorphism Φ in Proposition 3.4 in the proof of Lemma 5.2, and we have

$$\Phi \circ S_i(z) = (S_i(z) + : e^{\int -\frac{1}{\tau(K)+h^\vee} b_i(z)dz} :) \circ \Phi.$$

This means that under the isomorphisms $H^0_{DS}(\mathbb{W}_T(\mu)) \cong H^{\frac{\infty}{2}+i}(L\mathfrak{n}, \mathbb{W}_T(\mu))$, $S_i^W(z)$ is realized as the operator

$$S_i(z) + : e^{\int -\frac{1}{\tau(K)+h^\vee} b_i(z)dz} :$$

But the weight consideration shows that the first factor $S_i(z)$ is the zero map. □

We now reprove the following well-known fact.

Proposition 5.5 ([FF90a, FBZ04]). *Let $T = F$, or $T = \mathbb{C}$ with $\tau(K) = k \notin \mathbb{Q}$. Then*

$$\Upsilon(\mathcal{W}^T(\mathfrak{g})) = \bigcap_{i=1}^{\operatorname{rank}\mathfrak{g}} (\operatorname{Ker} \int S_i^W(z)dz : \pi_{T+h^\vee} \longrightarrow \pi_{T+h^\vee,-\alpha_i}).$$

Proof. Consider the exact sequence $0 \longrightarrow V_T(\mathfrak{g}) \longrightarrow \mathbb{W}_T(0) \xrightarrow{\oplus_i S_i} \bigoplus_i \mathbb{W}_T(-\alpha_i)$ described in Proposition 3.5. By applying the functor $H^0_{DS}(?)$, we get the exact sequence

$$\mathcal{W}^T(\mathfrak{g}) \xrightarrow{\Upsilon} \pi_{T+h^\vee} \longrightarrow \bigoplus_i \pi_{T+h^\vee,-\alpha_i}.$$

Hence by Lemma 5.4, $\Upsilon(\mathcal{W}^T(\mathfrak{g}))$ equals to the intersection of the kernel of the maps $\int S_i^W(z)dz : \pi_{T+h^\vee} \longrightarrow \pi_{T+h^\vee,-\alpha_i}$. □

Let $\mathfrak{g}^L$ denote the Langlands dual Lie algebra of $\mathfrak{g}$. The Feigin-Frenkel duality states that

$$\mathcal{W}^k(\mathfrak{g}) \cong \mathcal{W}^{{}^L k}({}^L\mathfrak{g})$$

for a generic k, where ${}^L k$ is defined by the formula $r^\vee(k + h^\vee)({}^L k + {}^L h^\vee) = 1$ ([FF91]).

Theorem 5.6 ([FF91, FF96]). *The Feigin-Frenkel duality $\mathcal{W}^k(\mathfrak{g}) \cong \mathcal{W}^{{}^L k}({}^L\mathfrak{g})$ remains valid for all non-critical k.*

Proof. Although the statement follows from [FF96], see [AFO18, Theorem 6.1], we include a proof for completeness.

For $T = R, F$ (see Definition 2.4), let ${}^L T$ denote the $\mathbb{C}[K]$-algebra T with structure map

$$K \mapsto \frac{1}{r^\vee(\tau(K) + h^\vee)} - h^\vee.$$

We have the isomorphism

$$\pi_{T+h^\vee} \xrightarrow{\sim} {}^L\pi_{{}^LT+h^\vee}, \quad b_i(z) \mapsto -r^\vee(\tau(K)+h^\vee)\frac{(\alpha_i|\alpha_i)}{2}{}^L b_i(z),$$

where ${}^L\pi_{{}^LT+h^\vee}$ is the Heisenberg vertex algebra corresponding to ${}^L\mathfrak{g}$. This isomorphism restricts to the the isomorphism

$$\mathcal{W}^F(\mathfrak{g}) \cong \mathcal{W}^{{}^LF}({}^L\mathfrak{g}),$$

see chapter 15 of [FBZ04]. By Lemma 5.1, this further restricts to the isomorphism $\mathcal{W}^R(\mathfrak{g}) \cong \mathcal{W}^{{}^LR}({}^L\mathfrak{g})$. Therefore, we get that $\mathcal{W}^k(\mathfrak{g}) = \mathcal{W}^R(\mathfrak{g})\otimes_R \mathbb{C}_k \cong \mathcal{W}^{{}^LR}({}^L\mathfrak{g})\otimes_R\mathbb{C}_k = \mathcal{W}^{{}^LR}({}^L\mathfrak{g})\otimes_{{}^LR}\mathbb{C}_{{}^Lk} = \mathcal{W}^{{}^Lk}(\mathfrak{g}^L)$. □

6. Miura map and representation theory of W-algebras

Let $\mathrm{Zhu}(V)$ be the Zhu's algebra of a vertex operator algebra V. We have the isomorphism

$$\mathrm{Zhu}(\mathcal{W}^T(\mathfrak{g})) \xrightarrow{\sim} \mathcal{Z}(\mathfrak{g})\otimes T,$$

where $\mathcal{Z}(\mathfrak{g})$ is the center of $U(\mathfrak{g})$. This was proved in [Ara07] for $T = \mathbb{C}$, but the proof given in [Ara15b, Ara17] applies to the general case without any changes. Below we describe the above isomorphism in terms of the Miura map Υ.

Consider the homomorphism of Zhu's algebras

$$\Upsilon_Z : \mathrm{Zhu}(\mathcal{W}^T(\mathfrak{g})) \longrightarrow \mathrm{Zhu}(\pi_{T+h^\vee}) \cong S(\mathfrak{h}_T) := S(\mathfrak{h})\otimes T$$

induced by the Miura map Υ. This further induces the homomorphism

$$\mathrm{gr}\,\Upsilon_Z : \mathrm{gr}\,\mathrm{Zhu}(\mathcal{W}^T(\mathfrak{g})) \longrightarrow \mathrm{gr}\,\mathrm{Zhu}(\pi_{T+h^\vee}) = S(\mathfrak{h}_T),$$

where the associated graded are taken with respect to the filtration defined by Zhu [Zhu96].

Lemma 6.1. (1) *The map* $\mathrm{gr}\,\Upsilon_Z$ *is an injective homomorphism onto the subalgebra* $S(\mathfrak{h}_T)^W = S(\mathfrak{h})^W\otimes T$,
(2) *The map* Υ_Z *is injective.*

Proof. (1) Since both $\mathcal{W}^T(\mathfrak{g})$ and $\pi_{T+h^\vee}$ admits a PBW basis, $\mathrm{gr}\,\Upsilon_Z$ can be identified with the map $R_{\mathcal{W}^T(\mathfrak{g})} \longrightarrow R_{\pi_{T+h^\vee}}$ induced by Υ by [Ara17, Theorem 4.8], where R_V denotes the Zhu's C_2-algebra of a vertex algebra V. By (25), this is an embedding on to $S(\mathfrak{h})^W\otimes T \subset S(\mathfrak{h})\otimes T = R_{\pi_{T+h^\vee}}$. (2) follows from (1). □

For $\lambda \in \mathfrak{h}_T^*$, set

$$\chi_\lambda = (\text{evaluation at } \lambda) \circ \Upsilon_Z : \mathrm{Zhu}(\mathcal{W}^T(\mathfrak{g})) \longrightarrow T. \tag{27}$$

Also, define

$$\chi = T_{\rho-(\tau(K)+h^\vee)\rho^\vee} \circ \Upsilon_Z : \mathrm{Zhu}(\mathcal{W}^T(\mathfrak{g})) \longrightarrow S(\mathfrak{h}_T), \tag{28}$$

where for $\mu \in \mathfrak{h}_T^*$, $T_\mu : S(\mathfrak{h}_T) \xrightarrow{\sim} S(\mathfrak{h}_T)$ is the translation defined by $(T_\mu p)(\lambda) = p(\lambda - \mu)$, $\lambda \in \mathfrak{h}_T^*$.

The following assertion can be regarded as a version of the Harish-Chandra isomorphism.

Proposition 6.2. *We have* $\chi(\mathrm{Zhu}(\mathcal{W}^T(\mathfrak{g}))) \subset S(\mathfrak{h})^W \otimes T$, *and*

$$\chi : \mathrm{Zhu}(\mathcal{W}^T(\mathfrak{g})) \longrightarrow S(\mathfrak{h})^W \otimes T$$

is an algebra isomorphism.

Proof. For $\lambda \in \mathfrak{h}_T^*$, let $\mathbb{M}(\lambda) \longrightarrow \mathbb{W}_T(\lambda)$ be the homomorphism that sends the highest weight vector $v_\lambda \in \mathbb{M}(\lambda)$ to the highest weight vector $|\lambda\rangle$ of $\mathbb{W}_T(\lambda)$. The induced map $H^0_{DS}(\mathbb{M}(\lambda)) \longrightarrow H^0_{DS}(\mathbb{W}_T(\lambda)) = \pi_{T+h^\vee,\lambda}$ sends $[v_\lambda]$ to $[|\lambda\rangle]$, where $[v_\lambda]$ and $[|\lambda\rangle]$ denote the images of v_λ and $|\lambda\rangle$ in $H^0_{DS}(\mathbb{M}(\lambda))$ and $\pi_{T+h^\vee}$, respectively. As $z \in \mathrm{Zhu}(\mathcal{W}^T(\mathfrak{g}))$ acts on $|\lambda\rangle$ as the multiplication by $\chi_\lambda(z)$, it follows from [Ara07, Proposition 9.2.3] that $\chi_\lambda(z) = \tilde{\chi}_{\lambda-(\tau(K)+h^\vee)\rho^\vee}(\lambda)$, where $\tilde{\chi}_\lambda = (\text{evaluation at } \lambda) \circ p$ and $p : \mathcal{Z}(\mathfrak{g}) \otimes T \longrightarrow S(\mathfrak{h}_T)$ is the Harish-Chandra projection with respect to the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$. Hence $\chi_\lambda = \chi_\mu$ if $\lambda + \rho - (\tau(K) + h^\vee)\rho^\vee \in W(\mu + \rho - (\tau(K) + h^\vee)\rho^\vee)$, and therefore, $\chi(\mathrm{Zhu}(\mathcal{W}^T(\mathfrak{g}))) \subset S(\mathfrak{h})^W \otimes T$. Since χ is injective and $\mathrm{gr}\,\mathrm{Zhu}(\mathcal{W}^T(\mathfrak{g})) \cong S(\mathfrak{h})^W \otimes T$ by Lemma 6.1, $\chi : \mathrm{Zhu}(\mathcal{W}^T(\mathfrak{g})) \longrightarrow S(\mathfrak{h}_T)^W$ must be an isomorphism. □

By Proposition 6.2,

$$\chi_\lambda = \chi_\mu \iff \lambda + \rho - (\tau(K) + h^\vee)\rho^\vee \in W(\mu + \rho - (\tau(K) + h^\vee)\rho^\vee) \tag{29}$$

for $\lambda, \mu \in \mathfrak{h}_T^*$.

For $\lambda \in \mathfrak{h}_T^*$, let $\mathbf{M}_T(\chi_\lambda)$ be the Verma module of $\mathcal{W}^T(\mathfrak{g})$ with highest weight χ_λ (see [Ara07, 5.1]). By definition we have

$$\begin{aligned} \mathbf{M}_T(\chi_\lambda) \cong \mathbf{M}_T(\chi_\mu) \quad &\iff \quad \chi_\lambda = \chi_\mu \\ \iff \quad &\lambda + \rho - (\tau(K) + h^\vee)\rho^\vee \in W(\mu + \rho - (\tau(K) + h^\vee)\rho^\vee). \end{aligned}$$

If $\tau(K) + h^\vee$ is invertible then we have

$$\mathrm{ch}\,\mathbf{M}_T(\chi_\lambda) = \frac{q^{h_\lambda}}{\prod_{j=1}^\infty (1-q^j)^{\mathrm{rank}\,\mathfrak{g}}},$$

where

$$h_\lambda = \frac{|\lambda+\rho|^2 - |\rho|^2}{2(\tau(K)+h^\vee)} - \langle \lambda | \rho^\vee \rangle. \tag{30}$$

Here, $\mathrm{ch}\, M = \mathrm{tr}_M\, q^{L_0}$.

If $T=\mathbb{C}$ and $\tau(K)=k$ we write $\mathbf{M}_k(\chi_\lambda)$ for $\mathbf{M}_T(\chi_\lambda)$, and denote by $\mathbf{L}_k(\chi)$ be the unique irreducible (graded) quotient of $\mathbf{M}_k(\chi)$[1].

Let $H_-^{\bullet}(M)$ be the BRST cohomology of the "−"-Drinfeld-Sokolov reduction [FKW92] with coefficient in a $V_k(\mathfrak{g})$-module M.

Theorem 6.3 ([Ara07])**.**

(1) $H_-^i(M)=0$ *for all* $i\neq 0$ *and* $M\in\mathcal{O}_k$.
(2) $H_-^0(\mathbb{M}_k(\lambda))\cong\mathbf{M}_k(\chi_{\lambda+(k+h^\vee)\rho^\vee})$.
(3) $H_-^0(\mathbb{L}_k(\lambda))\cong\begin{cases}\mathbf{L}_k(\chi_{\lambda+(k+h^\vee)\rho^\vee}) & \text{if } \langle\lambda+\rho,\alpha^\vee\rangle\notin\mathbb{N}\text{ for any }\alpha\in\Delta_+\\ 0 & \text{else}\end{cases}$

Theorem 6.4 ([Ara07, Theorem 9.1.4])**.** *Suppose that* $\langle\hat{\lambda}+\hat{\rho},\alpha^\vee\rangle\notin\mathbb{Z}$ *for all* $\alpha\in\{-\beta+n\delta\mid\beta\in\Delta_+,1\leqslant n\leqslant \mathrm{ht}(\beta)\}$. *Then* $H^0_{DS}(\mathbb{L}_k(\lambda))\cong\mathbf{L}_k(\chi_\lambda)$.

7. More on screening operators for W-algebras

In this section we let $T=\mathbb{C}$ with $\tau(K)=k\in\mathbb{C}\backslash\{-h^\vee\}$. As in Section 4, let $\mu\in\mathfrak{h}^*$ such that

$$(\mu|\alpha_i)+m(k+h^\vee)=\frac{(\alpha_i|\alpha_i)}{2}(n-1). \tag{31}$$

for some $n\in\mathbb{Z}_{\geqslant 0}$ and $m\in\mathbb{Z}$. For an element $\Gamma\in H_n(Y_n,\mathcal{L}_n(\mu,k))$, set

$$S_i^W(n,\Gamma):=\int_\Gamma S_i^W(z_1)S_i^W(z_2)\dots S_i^W(z_n)dz_1\dots dz_n:\pi_{k,\mu}\longrightarrow\pi_{k,\mu-n\alpha_i}. \tag{32}$$

The following result was proved in [TK86] for the case that $\mathfrak{g}=\mathfrak{sl}_2$.

Theorem 7.1. *The map* $S_i^W(n,\Gamma)$ *is the homomorphism of* $\mathcal{W}^k(\mathfrak{g})$*-modules obtained from the* $\widehat{\mathfrak{g}}$*-homomorphism* $S_i(n,\Gamma):\mathbb{W}_k(\mu)\longrightarrow\mathbb{W}_k(\mu-n\alpha_i)$ *by applying the functor* $H^0_{DS}(?)$. *Moreover, suppose that*

$$\frac{2d(d+1)}{(k+h^\vee)(\alpha_i|\alpha_i)}\notin\mathbb{Z},\qquad \frac{2d(d-n)}{(k+h^\vee)(\alpha_i|\alpha_i)}\notin\mathbb{Z},$$

for all $1\leqslant d\leqslant n-1$. *Then there exits a cycle* $\Gamma\in H_n(Y_n,\mathcal{L}_n(\mu,k))$ *such that* $S_i^W(n,\Gamma)$ *is non-trivial.*

Proof. The first assertion following from Lemma 5.4. The proof of the non-triviality is the same as that of Theorem 4.1 (and hence as [TK86]). □

Proposition 7.2. *Let* k *be generic,* $\lambda\in P_+$, $\mu\in P_+^\vee$. *Then*

$$\mathbf{L}_k(\chi_{\lambda-(k+h^\vee)\mu})\cong\bigcap_{i=1}^r\mathrm{Ker}\,S_i^W(n_i,\Gamma_i):\pi^{k+h^\vee}_{\lambda-(k+h^\vee)\mu}\longrightarrow\pi^{k+h^\vee}_{\lambda-(k+h^\vee)\mu-n_i\alpha_i}$$

where $n_i=\langle\lambda+\rho,\alpha_i^\vee\rangle$ *and* $\Gamma_i\in H_n(Y_n,\mathcal{L}_n(\lambda-(k+h^\vee)\mu,k))$ *is as in Theorem 7.1.*

[1] We have changed slightly the notation for the parametrization of simple $\mathcal{W}^k(\mathfrak{g})$-modules from [Ara07]. Namely we have $\chi_\lambda=\gamma_{\lambda-(k+h^\vee)\rho^\vee}$, where γ_λ is the central character used in [Ara07].

Proof. By Theorem 6.4, we have $\mathbf{L}_k(\chi_{\lambda-(k+h^\vee)\mu}) \cong H^0_{DS}(\mathbb{L}_k(\lambda-(k+h^\vee)\mu))$. As Wakimoto modules are acyclic with respect to the cohomology functor $H^\bullet_{DS}(?)$, $H^\bullet_{DS}(\mathbb{L}_k(\lambda-(k+h^\vee)\mu))$ is the cohomology of the complex obtained by applying the functor $H^\bullet_{DS}(?)$ to the resolution in Proposition 4.2. Therefore the assertion follows from Theorem 7.1. □

8. Coset construction of universal principal W-algebras

We assume that $\mathfrak{g}$ is simply laced for the rest of the paper. The goal of this section is to provide a proof of Main Theorem 2.

For $m \in \mathbb{N}$, let $P_+^m = \{\lambda \in P_+ \mid \lambda(\theta) \leqslant m\}$. Then, $\{\mathbb{L}_k(\lambda) \mid \lambda \in P_+^m\}$ gives the complete set of isomorphism classes of irreducible integrable representation of $\widehat{\mathfrak{g}}$ of level m.

Recall that, for $\nu \in P_+^1$, the Frenkel-Kac construction [FK81] gives the isomorphism of $\widehat{\mathfrak{g}}$-modules

$$\mathbb{L}_1(\nu) \cong \bigoplus_{\beta\in Q} \pi_{1,\nu+\beta}, \tag{33}$$

where the action of $\widehat{\mathfrak{g}}$ on the right-hand-side is given by

$$h_i(z) \mapsto \alpha_i(z), \quad e_i(z) =: e^{\int \alpha_i(z)dz} :, \quad f_i(z) =: e^{-\int \alpha_i(z)dz} : .$$

Here we denoted by $\alpha_i(z)$ the generator of the Heisenberg vertex algebra π_1 satisfying the OPE

$$\alpha_i(z)\alpha_j(w) \sim \frac{(\alpha_i,\alpha_j)}{(z-w)^2}. \tag{34}$$

Let T be an integral $\mathbb{C}[K]$-domain with the structure map $\tau : \mathbb{C}[K] \longrightarrow T$ such that $\tau(K)+h^\vee$ and $\tau(K)+h^\vee+1$ are invertible. Consider the vertex algebra $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{n}, \mathbb{W}_T(0)\otimes L_1(\mathfrak{g}))$, where $L\mathfrak{n}$ acts on $\mathbb{W}_T(0)\otimes L_1(\mathfrak{g})$ diagonally. For $\lambda \in \mathfrak{h}_T^*$ and $\nu \in P_+^1$, $H^{\frac{\infty}{2}+i}(L\mathfrak{n}, \mathbb{W}_T(\lambda)\otimes \mathbb{L}_1(\nu))$ is naturally a $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{n}, \mathbb{W}_T(0)\otimes L_1(\mathfrak{g}))$-module.

Lemma 8.1. *There is an embedding*

$$\pi_{T+h^\vee}\otimes\pi_1 \hookrightarrow H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_T(0)\otimes L_1(\mathfrak{g}))$$

of vertex algebras, and we have

$$H^{\frac{\infty}{2}+i}(L\mathfrak{n}, \mathbb{W}_T(\lambda)\otimes \mathbb{L}_1(\nu)) \cong \delta_{i,0}\bigoplus_{\beta} \pi_{T+h^\vee,\lambda}\otimes\pi_{1,\nu+\beta}$$

as $\pi_{T+h^\vee}\otimes\pi_1$-modules for $\lambda \in \mathfrak{h}_T^*$, $\nu \in P_+^1$.

Proof. By Proposition 3.4, we have the isomorphism

$$H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_T(0)\otimes L_1(\mathfrak{g})) \overset{[\Phi]}{\underset{\sim}{\longrightarrow}} H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_T(0))\otimes L_1(\mathfrak{g}) \cong \pi_{T+h^\vee}\otimes L_1(\mathfrak{g}) \\ \cong \bigoplus_{\beta\in Q} \pi_{T+h^\vee}\otimes\pi_{1,\beta},$$

of vertex algebras, where $[\Phi]$ denotes the map induced by the isomorphism $\Phi : \mathbb{W}_T(0)\otimes L_1(\mathfrak{g}) \xrightarrow{\sim} \mathbb{W}_T(0)\otimes L_1(\mathfrak{g})$ for $M = L_1(\mathfrak{g})$ in Proposition 3.4. Similarly, we have

$$H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_T(\lambda)\otimes\mathbb{L}_1(\nu)) \overset{[\Phi]}{\xrightarrow{\sim}} H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_T(\lambda))\otimes\mathbb{L}_1(\nu) \cong \pi_{T+h^\vee,\lambda}\otimes\mathbb{L}_1(\nu) \cong \bigoplus_{\beta\in Q} \pi_{T+h^\vee,\lambda}\otimes\pi_{1,\nu+\beta}$$

as $\pi_{T+h^\vee}\otimes\pi_1$-modules. □

The proof of the following assertion is the same as Lemma 3.1.

Lemma 8.2. *The embedding* $\pi_1 \hookrightarrow H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_T(0)\otimes L_1(\mathfrak{g}))$ *in Lemma 8.1 coincides with the vertex algebra homomorphism*

$$\begin{array}{ccc} \pi_1 & \longrightarrow & H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_T(0)\otimes L_1(\mathfrak{g})), \\ \alpha_i(z) & \mapsto & \alpha_i(z) - \sum_{\alpha\in\Delta_+} \alpha(\alpha_i^\vee) : a_\alpha(z)a^*_\alpha(z) : + \sum_{\alpha\in\Delta_+} \alpha(\alpha_i^\vee) : \psi_\alpha(z)\psi^*_\alpha(z) : . \end{array}$$

Here we have omitted the tensor product sign.

Let T' denote the $\mathbb{C}[K]$-algebra T with the structure map

$$K \mapsto \tau'(K) := \frac{\tau(K)+h^\vee}{\tau(K)+h^\vee+1} - h^\vee. \tag{35}$$

We have the vertex algebra isomorohism

$$\begin{array}{ccc} \pi_{T+h^\vee+1}\otimes\pi_{T'} & \xrightarrow{\sim} & \pi_{T+h^\vee}\otimes\pi_1, \\ b_i(z)\otimes 1 & \mapsto & b_i(z)\otimes 1 + 1\otimes\alpha_i(z), \\ 1\otimes b_i(z) & \mapsto & \frac{1}{\tau(K)+h^\vee+1} b_i(z)\otimes 1 - \frac{\tau(K)+h^\vee}{\tau(K)+h^\vee+1} 1\otimes\alpha_i(z). \end{array} \tag{36}$$

Note that, by Lemma 8.2, the map

$$\pi_{T+h^\vee+1} \hookrightarrow \pi_{T+h^\vee+1}\otimes\pi_{T'} \xrightarrow{\sim} \pi_{T+h^\vee}\otimes\pi_1 \hookrightarrow H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_T(0)\otimes L_1(\mathfrak{g})),$$

where the first map is the embedding to the first component, coincides with the vertex algebra homomorphism induced by the diagonal embedding $\pi_{T+1} \hookrightarrow \mathbb{W}_T(0)\otimes L_1(\mathfrak{g})$.

The following assertion follows immediately from Lemma 8.1.

Proposition 8.3. *We have the isomorphism*

$$H^{\frac{\infty}{2}+i}(L\mathfrak{n}, \mathbb{W}_T(\mu)\otimes\mathbb{L}_1(\nu)) \cong \delta_{i,0} \bigoplus_{\substack{\lambda\in\mathfrak{h}^* \\ \lambda-\mu-\nu\in Q}} \pi_{T+h^\vee+1,\lambda} \otimes \pi_{T'+h^\vee,\mu-(\tau'(K)+h^\vee)\lambda}$$

as $\pi_{T+h^\vee+1}\otimes\pi_{T'+h^\vee}$*-modules for* $\lambda \in \mathfrak{h}^*_T$ *and* $\nu \in P^1_+$*. In particular,* $\pi_{T+h^\vee+1}$ *and* $\pi_{T'+h^\vee}$ *form a dual pair in* $H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_T(0)\otimes L_1(\mathfrak{g}))$*.*

Now recall the commutant vertex algebra $\mathcal{C}_T(\mathfrak{g}) = \mathrm{Com}(V_{T+1}(\mathfrak{g}), V_T(\mathfrak{g})\otimes L_1(\mathfrak{g}))$. We define the vertex algebra homomorphism

$$\Psi_T : \mathcal{C}_T(\mathfrak{g}) \longrightarrow \pi_{T'+h^\vee}$$

as follows. First, define the vertex algebra homomorphism

(37)

$$\Psi_{1,T} : \mathcal{C}_T(\mathfrak{g}) = (V_T(\mathfrak{g})\otimes L_1(\mathfrak{g}))^{\mathfrak{g}[t]} \longrightarrow H^{\frac{\infty}{2}+0}(L\mathfrak{n}, V_T(\mathfrak{g})\otimes L_1(\mathfrak{g})), \quad v \mapsto [v\otimes\mathbf{1}].$$

The diagonal embedding $\pi_{T+1} \hookrightarrow V_T(\mathfrak{g}) \otimes L_1(\mathfrak{g})$, gives rise to a vertex algebra embedding $\pi_{T+h^\vee+1} \hookrightarrow H^{\frac{\infty}{2}+0}(L\mathfrak{n}, V_T(\mathfrak{g}) \otimes L_1(\mathfrak{g}))$. By definition, the image of $\psi_{1,T}$ is contained in

$$\mathrm{Com}(\pi_{T+h^\vee+1}, H^{\frac{\infty}{2}+0}(L\mathfrak{n}, V_T(\mathfrak{g}) \otimes L_1(\mathfrak{g}))) \subset H^{\frac{\infty}{2}+0}(L\mathfrak{n}, V_T(\mathfrak{g}) \otimes L_1(\mathfrak{g})).$$

Next let

(38)

$$\psi_{2,T} : \mathrm{Com}(\pi_{T+h^\vee+1}, H^{\frac{\infty}{2}+0}(L\mathfrak{n}, V_T(\mathfrak{g}) \otimes L_1(\mathfrak{g}))) \longrightarrow \mathrm{Com}(\pi_{T+h^\vee+1}, H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_T(0) \otimes L_1(\mathfrak{g}))) \cong \pi_{T'+h^\vee}$$

be the vertex algebra homomorphism induced by the embedding $V_T(\mathfrak{g}) \hookrightarrow \mathbb{W}_T(0)$. We set

$$\psi_T = \psi_{2,T} \circ \psi_{1,T}. \tag{39}$$

If $T = \mathbb{C}$ and $\tau(K) = k$ we write ψ_k for ψ_T and $\psi_{i,k}$ for $\psi_{i,T}$, $i = 1, 2$, and we define the complex number ℓ by the formula (1)

Proposition 8.4. *Suppose that $k + h^\vee \notin \mathbb{Q}_{\leqslant 0}$. The vertex algebra homomorphism*

$$\psi_k : \mathcal{C}_k(\mathfrak{g}) \longrightarrow \pi_{\ell+h^\vee}$$

is injective.

Proof. We show both $\psi_{1,k}$ and $\psi_{2,k}$ are injective.

The vertex algebra $H^{\frac{\infty}{2}+0}(L\mathfrak{n}, V_{k+1}(\mathfrak{g}))$ is naturally conformal. For a $H^{\frac{\infty}{2}+0}(L\mathfrak{n}, V_{k+1}(\mathfrak{g}))$-module M and $\Delta \in \mathbb{C}$, set $M_{[\Delta]} = \{m \in M \mid (L_0 - \Delta)^r m = 0,\ r \gg 0\}$.

Let N be the cokernel of the injection

$$V_{k+1}(\mathfrak{g}) \otimes \mathcal{C}_k(\mathfrak{g}) \hookrightarrow V_k(\mathfrak{g}) \otimes L_1(\mathfrak{g}), \quad u\mathbf{1} \otimes v \mapsto \Delta(u)v, \quad u \in U(\widehat{\mathfrak{g}}).$$

The long exact sequence of the BRST cohomology gives the exact sequence

(40)

$$H^{\frac{\infty}{2}-1}(L\mathfrak{n}, N) \longrightarrow H^{\frac{\infty}{2}+0}(L\mathfrak{n}, V_{k+1}(\mathfrak{g}) \otimes \mathcal{C}_k(\mathfrak{g})) \longrightarrow H^{\frac{\infty}{2}+0}(L\mathfrak{n}, V_k(\mathfrak{g}) \otimes L_1(\mathfrak{g})).$$

Since $N_{[\Delta]} = 0$ for $\Delta \in \mathbb{Z}_{\leqslant 0}$ by Lemma 2.3, we have $H^{\frac{\infty}{2}-1}(L\mathfrak{n}, N)_{[0]} = 0$. Hence (40) restricts to the embedding

$$H^{\frac{\infty}{2}+0}(L\mathfrak{n}, V_{k+1}(\mathfrak{g}) \otimes \mathcal{C}_k(\mathfrak{g}))_{[0]} \hookrightarrow H^{\frac{\infty}{2}+0}(L\mathfrak{n}, V_k(\mathfrak{g}) \otimes L_1(\mathfrak{g}))_{[0]}. \tag{41}$$

On the other hand, $H^{\frac{\infty}{2}+0}(L\mathfrak{n}, V_{k+1}(\mathfrak{g}) \otimes \mathcal{C}_k(\mathfrak{g}))_{[0]} = H^{\frac{\infty}{2}+0}(L\mathfrak{n}, V_{k+1}(\mathfrak{g}))_{[0]} \otimes \mathcal{C}_k(\mathfrak{g}) = \mathbb{C} \otimes \mathcal{C}_k(\mathfrak{g})$. It follows that (41) coincides with $\psi_{1,k}$. In particular, $\psi_{1,k}$ is injective.

Next, let U be the cokernel of the embedding $V_k(\mathfrak{g}) \hookrightarrow \mathbb{W}_k(0)$. This gives rise to the exact sequence

$$H^{\frac{\infty}{2}-1}(L\mathfrak{n}, U \otimes L_1(\mathfrak{g})) \longrightarrow H^{\frac{\infty}{2}+0}(L\mathfrak{n}, V_k(\mathfrak{g}) \otimes L_1(\mathfrak{g})) \longrightarrow H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_k(0) \otimes L_1(\mathfrak{g})).$$

In the same way as above we have $H^{\frac{\infty}{2}-1}(L\mathfrak{n}, U \otimes L_1(\mathfrak{g}))_{[0]} = 0$, and so the above map restricts to the embedding

$$H^{\frac{\infty}{2}+0}(L\mathfrak{n}, V_k(\mathfrak{g}) \otimes L_1(\mathfrak{g}))_{[0]} \hookrightarrow H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_k(0) \otimes L_1(\mathfrak{g}))_{[0]}. \tag{42}$$

Because $\mathrm{Com}(\pi_{T+h^\vee+1}, H^{\frac{\infty}{2}+0}(L\mathfrak{n}, V_T(\mathfrak{g})\otimes L_1(\mathfrak{g}))) \subset H^{\frac{\infty}{2}+0}(L\mathfrak{n}, V_k(\mathfrak{g})\otimes L_1(\mathfrak{g}))_{[0]}$, $\mathrm{Com}(\pi_{T+h^\vee+1}, H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_T(0)\otimes L_1(\mathfrak{g}))) \subset H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_k(0)\otimes L_1(\mathfrak{g}))_{[0]}$ and $\psi_{2,k}$ is the restriction of (42), it follows that $\psi_{2,k}$ is injective as well. □

Recall the $\mathbb{C}[K]$-algebras R and F defined in Section 2 (see Definition 2.4), and the Miura map Υ in Section 6. Also, recall that R' and F' denote the $\mathbb{C}[K]$-algebras obtained by modifying the structure map on R and F using (35).

Theorem 8.5. *The map $\psi_F : \mathcal{C}_F(\mathfrak{g}) \longrightarrow \pi_{F'+h^\vee}$ is an injective vertex algebra homomorphism onto $\Upsilon(\mathcal{W}^{F'}(\mathfrak{g}))$. Therefore, $\mathcal{C}_F(\mathfrak{g}) \cong \mathcal{W}^{F'}(\mathfrak{g})$ as vertex algebras.*

The proof of Theorem 8.5 will be given in Section 10.

Theorem 8.5 together with Proposition 2.6 immediately gives the following.

Corollary 8.6. *Assume that $k + h^\vee \notin \mathbb{Q}_{\leqslant 0}$. Then $\mathrm{ch}\,\mathcal{C}_k(\mathfrak{g}) = \mathrm{ch}\,\mathcal{W}^\ell(\mathfrak{g})$.*

Theorem 8.7.
(1) *The map $\psi_R : \mathcal{C}_R(\mathfrak{g}) \longrightarrow \pi_{R'+h^\vee}$ is an injective vertex algebra homomorphism onto $\Upsilon(\mathcal{W}^{R'}(\mathfrak{g}))$. Therefore, $\mathcal{C}_R(\mathfrak{g}) \cong \mathcal{W}^{R'}(\mathfrak{g})$ as vertex algebras.*
(2) *Assume that $k + h^\vee \notin \mathbb{Q}_{\leqslant 0}$. The map $\psi_k : \mathcal{C}_k(\mathfrak{g}) \longrightarrow \pi_{\ell+h^\vee}$ is an injective vertex algebra homomorphism onto $\Upsilon(\mathcal{W}^\ell(\mathfrak{g}))$. Therefore, $\mathcal{C}_k(\mathfrak{g}) \cong \mathcal{W}^\ell(\mathfrak{g})$ as vertex algebras.*

Proof. Consider the vertex algebra homomorphism $\psi_R : \mathcal{C}_R(\mathfrak{g}) \longrightarrow \pi_{R'}$. By Proposition 2.5 and (5), $\mathcal{C}_R(\mathfrak{g}) \subset \mathcal{C}_F(\mathfrak{g})$, and so ψ_R is the restriction of ψ_F to $\mathcal{C}_R(\mathfrak{g})$. Hence,

$$\psi_R(\mathcal{C}_R(\mathfrak{g})) \subset \psi_F(\mathcal{C}_F(\mathfrak{g})) \cap \pi_{R'} = \Upsilon(\mathcal{W}^{F'}(\mathfrak{g})) \cap \pi_{R'} = \Upsilon(\mathcal{W}^{R'}(\mathfrak{g}))$$

by Lemma 5.1. Since $\mathcal{C}_k(\mathfrak{g}) = \mathcal{C}_R(\mathfrak{g})\otimes_R \mathbb{C}_k$ by Proposition 2.6, this implies that

$$\begin{aligned}\psi_k(\mathcal{C}_k(\mathfrak{g})) &= \mathrm{Im}(\psi_R(\mathcal{C}_R(\mathfrak{g}))\otimes_R \mathbb{C}_k \longrightarrow \pi_{R'+h^\vee}\otimes_R \mathbb{C}_k = \pi_{\ell+h^\vee})\\ &\subset \Upsilon(\mathcal{W}^{R'}(\mathfrak{g}))\otimes_{R'} \mathbb{C}_\ell = \Upsilon(\mathcal{W}^\ell(\mathfrak{g})).\end{aligned}$$

Since $\psi_k(\mathcal{C}_k(\mathfrak{g}))$ and $\Upsilon(\mathcal{W}^\ell(\mathfrak{g}))$ have the same character by Corollary 8.6, we conclude that the natural map $\psi_R(\mathcal{C}_R(\mathfrak{g}))\otimes_R \mathbb{C}_k \longrightarrow \pi_{R'+h^\vee}\otimes_R \mathbb{C}_k$ is injective and that $\psi_k(\mathcal{C}_k(\mathfrak{g})) = \Upsilon(\mathcal{W}^\ell(\mathfrak{g}))$. As $\psi_R(\mathcal{C}_R(\mathfrak{g}))\otimes_R \mathbb{C}_k = \Upsilon(\mathcal{W}^{R'}(\mathfrak{g}))\otimes_{R'} \mathbb{C}_\ell$ for all $k \in \mathrm{Specm}\, R$, we get that $\psi_R(\mathcal{C}_R(\mathfrak{g})) = \Upsilon(\mathcal{W}^{R'}(\mathfrak{g}))$. □

Remark 8.8. By the Feigin-Frenkel duality, Theorem 8.7 implies that

$$(V_k(\mathfrak{g})\otimes L_1(\mathfrak{g}))^{\mathfrak{g}[t]} \cong (V_{-k-2h^\vee-1}(\mathfrak{g})\otimes L_1(\mathfrak{g}))^{\mathfrak{g}[t]}$$

for a generic k.

9. Key Observation

In this section we establish the key fact needed for the proof of Theorem 8.5.

Let T be an integral $\mathbb{C}[K]$-domain with the structure map $\tau : \mathbb{C}[K] \longrightarrow T$ such that $\tau(K) + h^\vee$ is invertible.

Recall the operator $S_i(z) : \mathbb{W}_T(\mu) \longrightarrow \mathbb{W}_T(\mu - \alpha_i)$ defined in Section 3. As $S_i(z)$ is an $\widehat{\mathfrak{h}} \oplus L\mathfrak{n}$-module homomorphism, $S_i(z)\otimes \mathrm{id}_{\mathbb{L}_1(\nu)}$ defines an $\widehat{\mathfrak{h}} \oplus L\mathfrak{n}$-module homomorphism $\mathbb{W}_T(\mu)\otimes\mathbb{L}_1(\nu) \longrightarrow \mathbb{W}_T(\mu-\alpha_i)\otimes\mathbb{L}_1(\nu)$. Therefore, it induces the T-linear map

$$[S_i(z)\otimes \mathrm{id}_{\mathbb{L}_1(\nu)}] : H^{\frac{\infty}{2}+\bullet}(L\mathfrak{n}, \mathbb{W}_T(\mu)\otimes\mathbb{L}_1(\nu)) \longrightarrow H^{\frac{\infty}{2}+\bullet}(L\mathfrak{n}, \mathbb{W}_T(\mu-\alpha_i)\otimes\mathbb{L}_1(\nu)).$$

Proposition 9.1. *Let $\nu\in P^1_+$. Under the isomorphism in Proposition 8.3, we have*

$$[S_i(z)\otimes \mathrm{id}_{\mathbb{L}_1(\nu)}] = -1\otimes S_i^W(z) : \bigoplus_{\substack{\lambda\in\mathfrak{h}^*_T\\ \lambda-\mu-\nu\in Q}} \pi_{T+h^\vee+1,\lambda}\otimes\pi_{T'+h^\vee,\mu-(\tau'(K)+h^\vee)\lambda}$$
$$\longrightarrow \bigoplus_{\substack{\lambda\in\mathfrak{h}^*_T\\ \lambda-\mu-\nu\in Q}} \pi_{T+h^\vee+1,\lambda}\otimes\pi_{T'+h^\vee,\mu-\alpha_i-(\tau'(K)+h^\vee)\lambda}.$$

Proof. We have used the map Φ in Proposition 3.4 to achieve the isomorphism in Proposition 8.3, and so the factor $e_i(z)^R$ of $S_i(z)$ becomes $e_i^R(z)$ minus the action of $e_i(z)$ on $\mathbb{L}_1(\nu)$. Namely,

$$\begin{aligned}\Phi\circ S_i(z)\otimes\mathrm{id}_{\mathbb{L}_1(\nu)} &= (e_i^R(z)\otimes 1 - 1\otimes e_i(z)) : e^{\int -\frac{1}{\tau(k)+h^\vee}b_i(z)}dz : \circ\Phi\\ &= S_i(z)\otimes 1\circ\Phi - : e^{\int -\left(\frac{1}{\tau(K)+h^\vee}b_i(z)-\alpha_i(z)\right)}dz : \circ\Phi.\end{aligned}$$

Here, as before, $b_i(z)$ and $\alpha_i(z)$ are generating fields of $\pi_{T+h^\vee}\subset\mathbb{W}_T(0)$ and $\pi_1\subset L_1(\mathfrak{g})$ described in (6) and (34), respectively, and we have omitted the tensor product sign. The first factor $S_i(z)\otimes 1$ is the zero map by weight consideration. The second factor equals to $-1\otimes S_i^W(z)$ since

$$\begin{aligned}&\frac{1}{\tau(K)+h^\vee}b_i(z)-\alpha_i(z)\\ &=\frac{1}{\tau'(K)+h^\vee}\left(\frac{1}{\tau(K)+h^\vee+1}b_i(z)-\frac{\tau(K)+h^\vee}{\tau(K)+h^\vee+1}\alpha_i(z)\right),\end{aligned}$$

see (36) and Lemma 5.4. This completes the proof. □

10. Proof of Theorem 8.5

Let T be the field F, or $\mathbb{C}$ with the structure map $\tau(K)=k\neq\mathbb{Q}$, so that KL_k is semisimple. We will prove that ψ_T gives the isomorphism

$$\psi_T:\mathcal{C}_T(\mathfrak{g})\xrightarrow{\sim}\Upsilon(\mathcal{W}^{T'}(\mathfrak{g}))\subset\pi_{T'+h^\vee}. \tag{43}$$

Here $\psi_T=\psi_{T,2}\circ\psi_{T,1}$, where $\psi_{T,1}$ and $\psi_{T,2}$ are given by (37) and (38).

First, let us describe the map

$$\psi_{1,T}:\mathcal{C}_T(\mathfrak{g})\longrightarrow\mathrm{Com}(\pi_{T+h^\vee+1}, H^{\frac{\infty}{2}+0}(L\mathfrak{n}, V_T(\mathfrak{g})\otimes L_1(\mathfrak{g}))).$$

As KL_T is semisimple, we have

$$V_T(\mathfrak{g})\otimes L_1(\mathfrak{g})\cong\bigoplus_{\lambda\in P_+}\mathbb{V}_{T+1}(\lambda)\otimes\mathcal{B}_\lambda,\quad \mathcal{B}_\lambda=\mathrm{Hom}_{\widehat{\mathfrak{g}}}(\mathbb{V}_{T+1}(\lambda), V_T(\mathfrak{g})\otimes L_1(\mathfrak{g})).$$

It follows that

$$H^{\frac{\infty}{2}+0}(L\mathfrak{n}, V_T(\mathfrak{g})\otimes L_1(\mathfrak{g}))\cong\bigoplus_{\lambda\in P_+}H^{\frac{\infty}{2}+0}(L\mathfrak{n},\mathbb{V}_{T+1}(\lambda))\otimes\mathcal{B}_\lambda=\bigoplus_{\lambda\in P_+}\pi_{T+h^\vee+1,\lambda}\otimes\mathcal{B}_\lambda.$$

Hence, $\mathrm{Com}(\pi_{T+h^\vee+1}, H^{\frac{\infty}{2}+0}(L\mathfrak{n}, V_T(\mathfrak{g})\otimes L_1(\mathfrak{g}))) = \mathcal{B}_0$. Therefore,

$$\psi_{1,T} : \mathcal{C}_T(\mathfrak{g}) \xrightarrow{\sim} \mathcal{B}_0.$$

Next, by applying the exact functor $?\otimes L_1(\mathfrak{g})$ to the resolution of $V_T(\mathfrak{g})$ described in Proposition 3.5, we obtain the resolution

$$0 \longrightarrow V_T(\mathfrak{g})\otimes L_1(\mathfrak{g}) \longrightarrow C'_0 \xrightarrow{d_0\otimes 1} C'_1 \longrightarrow \dots \longrightarrow C'_n \longrightarrow 0, \tag{44}$$
$$C'_i = \bigoplus_{\substack{w\in W\\ \ell(w)=i}} \mathbb{W}_T(w\circ 0)\otimes L_1(\mathfrak{g})$$

of $V_k(\mathfrak{g})\otimes L_1(\mathfrak{g})$. As each $\mathbb{W}_T(w\circ 0)\otimes L_1(\mathfrak{g})$ is acyclic with respect to $H^{\frac{\infty}{2}+i}(L\mathfrak{n}, ?)$, we can compute the cohomology $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{n}, V_T(\mathfrak{g})\otimes L_1(\mathfrak{g}))$ using the resolution (45). That is, $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{n}, V_T(\mathfrak{g})\otimes L_1(\mathfrak{g}))$ is the cohomology of the induced complex

$$0 \longrightarrow H^{\frac{\infty}{2}+0}(L\mathfrak{n}, C'_0) \longrightarrow H^{\frac{\infty}{2}+0}(L\mathfrak{n}, C'_1) \longrightarrow \dots \longrightarrow H^{\frac{\infty}{2}+0}(L\mathfrak{n}, C'_n) \longrightarrow 0,$$

of $H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_T(0)\otimes L_1(\mathfrak{g}))$-modules. It follows that we have the exact sequence

$$\begin{aligned} 0 \longrightarrow H^{\frac{\infty}{2}+0}(L\mathfrak{n}, V_T(\mathfrak{g})\otimes L_1(\mathfrak{g})) &\xrightarrow{\psi_{2,T}} H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_T(0)\otimes \mathbb{L}_1(\nu)) \\ &\xrightarrow{[d_0\otimes 1]} \bigoplus_i H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_T(-\alpha_i)\otimes \mathbb{L}_1(\nu)), \end{aligned}$$

where $[d_0\otimes 1]$ denotes the map induced by $d_0\otimes 1$. Hence, $\psi_T(\mathcal{C}_T(\mathfrak{g}))$ is the kernel of the map $[d_0\otimes 1]$. It follows from Proposition 9.1 that

$$\psi_T(\mathcal{C}_T(\mathfrak{g})) = \bigcap_i \mathrm{Ker}\Big(\int S_i^W(z)dz : \pi_{T'+h^\vee} \longrightarrow \pi_{T'+h^\vee,-\alpha_i}\Big) = \Upsilon(\mathcal{W}^{T'}(\mathfrak{g}))$$

as required. $\square$.

11. Generic decomposition

Having established Theorem 8.7, we set $T = \mathbb{C}$ with the structure map $\tau(K) = k$ for the rest of the paper. In this section we assume further that $k \notin \mathbb{Q}$, so that KL_k is semisimple and $\mathbb{V}_k(\lambda) = \mathbb{L}_k(\lambda)$ for all $\mu \in P_+$. Also, $\mathcal{W}^\ell(\mathfrak{g}) = \mathcal{W}_\ell(\mathfrak{g})$ by Theorem 6.4, where ℓ is as in (1).

For $\mu \in P_+$ and $\nu \in P_+^1$, we have

$$\mathbb{V}_k(\mu)\otimes \mathbb{L}_1(\nu) = \bigoplus_{\lambda\in P_+} \mathbb{V}_{k+1}(\lambda)\otimes \mathcal{B}_\lambda^{\mu,\nu}, \quad \mathcal{B}_\lambda^{\mu,\nu} = \mathrm{Hom}_{\widehat{\mathfrak{g}}}(\mathbb{V}_{k+1}(\lambda), \mathbb{V}_k(\mu)\otimes \mathbb{L}_1(\nu)).$$

According to Theorem 8.7, each $\mathcal{B}_\lambda^{\mu,\nu}$ is a $\mathcal{W}^\ell(\mathfrak{g})$-module.

Theorem 11.1. *Suppose that $k \notin \mathbb{Q}$. For $\lambda, \mu \in P_+$ and $\nu \in P_+^1$, we have*

$$\mathcal{B}_\lambda^{\mu,\nu} \cong \begin{cases} \mathbf{L}_\ell(\chi_{\mu-(\ell+h^\vee)\lambda}) & \text{if } \lambda-\mu-\nu \in Q, \\ 0 & \text{otherwise,} \end{cases}$$

as $\mathcal{W}^\ell(\mathfrak{g})$-modules.

Proof. The proof is similar to that of Theorem 8.5. By Corollary 4.3, we have

$$H^{\frac{\infty}{2}+i}(L\mathfrak{n}, \mathbb{V}_k(\mu)\otimes\mathbb{L}_1(\nu)) = \bigoplus_{\lambda\in P_+} H^{\frac{\infty}{2}+i}(L\mathfrak{n}, \mathbb{V}_{k+1}(\lambda))\otimes\mathcal{B}_\lambda^{\mu,\nu}$$
$$\cong \bigoplus_{\lambda\in P_+}\left(\bigoplus_{\substack{w\in W\\ \ell(w)=i}} \pi_{k+h^\vee+1, w\circ\lambda}\right)\otimes\mathcal{B}_\lambda^{\mu,\nu}.$$

Thus,

$$\mathcal{B}_\lambda^{\mu,\nu} \cong \mathrm{Hom}_{\pi_{k+h^\vee+1}}(\pi_{k+h^\vee+1,\lambda}, H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{V}_k(\mu)\otimes\mathbb{L}_1(\nu)))$$

as modules over $\mathcal{W}^\ell(\mathfrak{g}) = \mathcal{C}_k(\mathfrak{g})$. On the other hand, by applying the exact functor $?\otimes L_1(\mathfrak{g})$ to the resolution of $\mathbb{V}_k(\mu) = \mathbb{L}_k(\mu)$ described in Proposition 4.2, we obtain the resolution

$$0 \longrightarrow \mathbb{V}_k(\mu)\otimes\mathbb{L}_1(\nu) \longrightarrow C_0' \xrightarrow{d_0\otimes 1} C_1' \longrightarrow \dots \longrightarrow C_n' \longrightarrow 0, \tag{45}$$
$$C_i' = \bigoplus_{\substack{w\in W\\ \ell(w)=i}} \mathbb{W}_k(w\circ\mu)\otimes\mathbb{L}_1(\nu)$$

of $\mathbb{V}_k(\mu)\otimes\mathbb{L}_1(\nu)$. So we can compute $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{n}, \mathbb{V}_k(\mu)\otimes\mathbb{L}_1(\nu))$ using the resolution (45), that is, $H^{\frac{\infty}{2}+\bullet}(L\mathfrak{n}, \mathbb{V}_k(\mu)\otimes\mathbb{L}_1(\nu))$ is the cohomology of the induced complex

$$0 \longrightarrow H^{\frac{\infty}{2}+0}(L\mathfrak{n}, C_0') \longrightarrow H^{\frac{\infty}{2}+0}(L\mathfrak{n}, C_1') \longrightarrow \dots \longrightarrow H^{\frac{\infty}{2}+0}(L\mathfrak{n}, C_n') \longrightarrow 0$$

of $H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_k(0)\otimes L_1(\mathfrak{g}))$-modules. In particular, $H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{V}_k(\mu)\otimes\mathbb{L}_1(\nu))$ is isomorphic to the kernel of the map

$$H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_k(\mu)\otimes\mathbb{L}_1(\nu)) \longrightarrow \bigoplus_{i=1}^{\mathrm{rank}\,\mathfrak{g}} H^{\frac{\infty}{2}+0}(L\mathfrak{n}, \mathbb{W}_k(\mu - n_i\alpha_i)\otimes\mathbb{L}_1(\nu)) \tag{46}$$

induced by $d_0\otimes 1$, where $n_i = \langle \mu+\rho, \alpha_i^\vee\rangle$. Therefore from Propositions 8.3 and 9.1 one finds that for $\lambda \in P_+$

$$\mathcal{B}_\lambda^{\mu,\nu} = \bigcap_{i=1}^{\mathrm{rank}\,\mathfrak{g}} \mathrm{Ker}(S^W(n_i, \Gamma_i) : \pi^{\ell+h^\vee}_{\mu-(\ell+h^\vee)\lambda} \longrightarrow \pi^{\ell+h^\vee}_{\mu-(\ell+h^\vee)\lambda-n_i\alpha_i})$$

if $\lambda - \mu - \nu \in Q$, and $\mathcal{B}_\lambda^{\mu,\nu} = 0$ otherwise. We are done by Proposition 7.2. □

Corollary 11.2. *Suppose that $k \notin \mathbb{Q}$. Then $V_{k+1}(\mathfrak{g})$ and $\mathcal{W}^\ell(\mathfrak{g})$ form a dual pair in $V_k(\mathfrak{g})\otimes L_1(\mathfrak{g})$.*

12. Coset construction of minimal series W-algebras

Recall that the level k is called an *admissible number* for $\widehat{\mathfrak{g}}$ if $L_k(\mathfrak{g})$ is an admissible representation [KW89]. Since we have assumed that $\mathfrak{g}$ is simply laced, this condition is equivalent to that

$$k + h^\vee = \frac{p}{q}, \quad p, q \in \mathbb{Z}_{\geqslant 1},\ (p,q) = 1,\ p \geqslant h^\vee. \tag{47}$$

An admissible number k is called *non-degenerate* ([FKW92], see also [Ara15a]) if the associated variety ([Ara12]) of $L_k(\mathfrak{g})$ is exactly the nilpotent cone of $\mathfrak{g}$, which is equivariant to the condition $q \geqslant h^\vee$ for a simply laced $\mathfrak{g}$. The main result of [Ara15b] states that $\mathcal{W}_k(\mathfrak{g})$ is is rational (and lisse [Ara15a]) if k is an non-degenerate admissible number. The rational W-algebra $\mathcal{W}_{p/q-h^\vee}(\mathfrak{g})$ for $p,q \in \mathbb{Z}_{\geqslant 1}$, $(p,q)=1$, $p,q \geqslant h^\vee$ is called the (p,q)-*minimal series principal* W-*algebra* and denoted also by $\mathcal{W}_{p,q}(\mathfrak{g})$. Note that $\mathcal{W}_{p,q}(\mathfrak{g}) \cong \mathcal{W}_{q,p}(\mathfrak{g})$ by the Feigin-Frenkel duality Theorem 5.6.

If k is an admissible number, then the level ℓ, defined by (1), is given by the formula

$$\ell + h^\vee = \frac{p}{p+q}, \tag{48}$$

so that ℓ is a non-degenerate admissible number and $\mathcal{W}_\ell(\mathfrak{g})$ is the $(p+q,q)$-minimal series principal W-algebra $\mathcal{W}_{p,p+q}(\mathfrak{g})$.

Let k be an admissible level,

$$Pr_k = \{\lambda \in \mathfrak{h}^* \mid \hat{\lambda} := \lambda + k\Lambda_0 \text{ is a principal admissible weight of } \widehat{\mathfrak{g}}\}.$$

Here a weight $\hat{\lambda}$ of $\widehat{\mathfrak{g}}$ is called principal admissible if $\hat{\Delta}(\hat{\lambda}) \cong \hat{\Delta}_{re}$ as root systems ([KW89]). By [Ara16], any $L_k(\mathfrak{g})$-module that belongs to $\mathcal{O}_k$ is completely reducible and $\{\mathbb{L}_k(\lambda) \mid \lambda \in Pr_k\}$ is exactly the set of isomorphism classes of simple $L_k(\mathfrak{g})$-modules that belong to in $\mathcal{O}_k$.

For k an admissible level, $\mu \in Pr_k$, $\nu \in P_1^+$. By [KW90], we have

$$\mathbb{L}_k(\mu) \otimes \mathbb{L}_1(\nu) \cong \bigoplus_{\substack{\lambda \in Pr^{k+1} \\ \lambda-\mu-\nu \in Q}} \mathbb{L}_{k+1}(\lambda) \otimes \mathcal{K}^\lambda_{\mu,\nu}, \tag{49}$$

where

$$\mathcal{K}^\lambda_{\mu,\nu} := \mathrm{Hom}_{\widehat{\mathfrak{g}}}(\mathbb{L}_{k+1}(\lambda), \mathbb{L}_k(\mu) \otimes \mathbb{L}_1(\nu)) = \mathrm{Hom}_{\widehat{\mathfrak{g}}}(\mathbb{V}_{k+1}(\lambda), \mathbb{L}_k(\mu) \otimes \mathbb{L}_1(\nu)).$$

Note that $\mathcal{K}^0_{0,0} = \mathrm{Com}(L_{k+1}(\mathfrak{g}), L_k(\mathfrak{g}) \otimes L_1(\mathfrak{g})) = (L_k(\mathfrak{g}) \otimes L_1(\mathfrak{g}))^{\mathfrak{g}[t]}$. The character of each $\mathcal{K}^\lambda_{\mu,\nu}$ has been computed in [KW90].

Theorem 12.1. *For an admissible level k, we have*

$$\mathrm{Com}(L_{k+1}(\mathfrak{g}), L_k(\mathfrak{g}) \otimes L_1(\mathfrak{g})) \cong \mathcal{W}_\ell(\mathfrak{g})$$

as vertex algebras. In particular, $\mathrm{Com}(L_{k+1}(\mathfrak{g}), L_k(\mathfrak{g}) \otimes L_1(\mathfrak{g}))$ *is simple, rational and lisse.*

Proof. By [CL19], $\mathrm{Com}(L_{k+1}(\mathfrak{g}), L_k(\mathfrak{g}) \otimes L_1(\mathfrak{g}))$ is a quotient vertex algebra of $C_k(\mathfrak{g})$. Indeed, since $V_{k+1}(\mathfrak{g})$ is projective in KL_{k+1} by Lemma 2.1, the surjection

$$V_k(\mathfrak{g}) \otimes L_1(\mathfrak{g}) \longrightarrow L_k(\mathfrak{g}) \otimes L_1(\mathfrak{g})$$

induces the surjection

$$C_k(\mathfrak{g}) = \mathrm{Hom}(V_{k+1}(\mathfrak{g}), V_k(\mathfrak{g}) \otimes L_1(\mathfrak{g})) \longrightarrow \mathrm{Hom}(V_{k+1}(\mathfrak{g}), L_k(\mathfrak{g}) \otimes L_1(\mathfrak{g})).$$

On the other hand, (49) implies that

$$\begin{aligned} \mathrm{Hom}(V_{k+1}(\mathfrak{g}), L_k(\mathfrak{g}) \otimes L_1(\mathfrak{g})) &= \mathrm{Hom}(L_{k+1}(\mathfrak{g}), L_k(\mathfrak{g}) \otimes L_1(\mathfrak{g})) \\ &= \mathrm{Com}(L_{k+1}(\mathfrak{g}), L_k(\mathfrak{g}) \otimes L_1(\mathfrak{g})). \end{aligned}$$

Thus $\mathrm{Com}(L_{k+1}(\mathfrak{g}), L_k(\mathfrak{g})\otimes L_1(\mathfrak{g}))$ is a quotient of $\mathcal{W}^\ell(\mathfrak{g})$ by Theorem 8.7. We are done since it has been already shown in [KW90] that the character of the coset $\mathrm{Com}(L_{k+1}(\mathfrak{g}), L_k(\mathfrak{g})\otimes L_1(\mathfrak{g}))$ coincides with that of $\mathcal{W}_\ell(\mathfrak{g})$. □

Remark 12.2. By Theorem 5.6, Theorem 12.1 realizes *all* the minimal series principal W-algebras for ADE types.

We continue to assume that k and ℓ are admissible levels defined by (47) and (48). Recall that

$$\{\mathbf{L}_\ell(\chi_{\mu-(\ell+h^\vee)(\lambda+\rho^\vee)}) \mid \lambda\in P_+^{p+q-h^\vee},\ \mu\in P_+^{p-h^\vee}\}$$

gives the complete set of isomorphism classes of simple $\mathcal{W}_\ell(\mathfrak{g})$-modules ([FKW92, Ara15b]). Observe that

$$P_+^{p-h^\vee}=Pr_k\cap P,\quad P_+^{p+q-h^\vee}=Pr_{k+1}\cap P.$$

Note that $Pr_k\cap P=Pr_k=P_+^k$ if $k\in\mathbb{Z}_{\geqslant 1}$.

Theorem 12.3. *Let $\mu\in P_+^{p-h^\vee}\subset Pr_k$, $\nu\in P_+^1$. We have*

$$\mathbb{L}_k(\mu)\otimes\mathbb{L}_1(\nu)\cong\bigoplus_{\substack{\lambda\in P_+^{p+q-h^\vee}\\ \lambda-\mu-\nu\in Q}}\mathbb{L}_{k+1}(\lambda)\otimes\mathbf{L}_\ell(\chi_{\mu-(\ell+h^\vee)\lambda})$$

as $L_{k+1}(\mathfrak{g})\otimes\mathcal{W}_\ell(\mathfrak{g})$-modules.

Note that since the coset P/Q can be represented by nonzero elements of P_+^1, any simple $\mathcal{W}_\ell(\mathfrak{g})$-module appears in the decomposition of $\mathbb{L}_k(\mu)\otimes\mathbb{L}_1(\nu)$ for some $\mu\in P_+^{p-h^\vee}$ and $\nu\in P_+^1$.

Proof of Theorem 12.3. By Theorem 6.13 of [Ara14], we have

$$H^{\frac{\infty}{2}+i}(L\mathfrak{n},\mathbb{L}_{k+1}(\lambda))=\bigoplus_{\substack{w\in\widehat{W}(\lambda)\\ \ell^{\frac{\infty}{2}}(w)=i}}\pi_{k+h^\vee+1,w\circ_{k+1}\lambda}.$$

for $\lambda\in Pr_{k+1}\cap P$, where $\ell^{\frac{\infty}{2}}(w)$ is the semi-infinite length of $w\in\widehat{W}(\hat{\mu})$ and

$$w\circ_k\lambda=w\circ\lambda\quad(w\in W),\quad t_\mu\circ_k\lambda=\lambda+(k+h^\vee)\mu\quad(\mu\in Q^\vee).$$

By (49), it follows that

$$H^{\frac{\infty}{2}+i}(L\mathfrak{n},\mathbb{L}_k(\mu)\otimes\mathbb{L}_1(\nu))\cong\bigoplus_{\lambda\in P_+^{p+q-h^\vee}}\bigoplus_{\substack{w\in\widehat{W}(\lambda)\\ \ell^{\frac{\infty}{2}}(w)=i}}\pi_{k+h^\vee+1,w\circ_{k+1}\lambda}\otimes\mathcal{K}^\lambda_{\mu,\nu}.\tag{50}$$

Thus,

$$\mathcal{K}^\lambda_{\mu,\nu}\cong\mathrm{Hom}_{\pi_{k+h^\vee+1}}(\pi_{k+h^\vee+1,\lambda},H^{\frac{\infty}{2}+0}(L\mathfrak{n},\mathbb{L}_k(\mu)\otimes\mathbb{L}_1(\nu)))\tag{51}$$

as $\mathcal{W}^\ell(\mathfrak{g})$-modules.

On the other hand, by [Ara14], there exists a complex

$$C^\bullet:\dots\longrightarrow C^{-1}\longrightarrow C^0\xrightarrow{d_0}C^1\longrightarrow\dots\longrightarrow C^n\longrightarrow\dots$$

in $\mathcal{O}_k$ such that

$$C^i = \bigoplus_{\substack{w\in \widehat{W}(\hat{\mu}) \\ \ell^{\frac{\infty}{2}}(w)=i}} \mathbb{W}_k(w\circ_k \mu), \quad \text{and} \quad H^i(C^\bullet)\cong \delta_{i,0}\mathbb{L}_k(\mu).$$

Thus, $C^\bullet$ is a two-sided resolution of $\mathbb{L}_k(\mu)$. So $C^\bullet\otimes\mathbb{L}_1(\nu)$ is a two-sided resolution of $\mathbb{L}_k(\mu)\otimes\mathbb{L}_1(\nu)$.

Let $C^\bullet_W$ denote the complex $H^{\frac{\infty}{2}+0}(L\mathfrak{n}, C^\bullet\otimes\mathbb{L}_1(\nu))$. Since each $\mathbb{W}_k(w\circ\mu)\otimes\mathbb{L}_1(\nu)$ is acyclic with respect to $H^{\frac{\infty}{2}+0}(L\mathfrak{n}, ?)$, we have

$$H^{\frac{\infty}{2}+i}(L\mathfrak{n}, \mathbb{L}_k(\nu)\otimes\mathbb{L}_1(\nu))\cong H^i(C^\bullet_W). \tag{52}$$

Note that $C^\bullet_W$ is a complex of $H^{\frac{\infty}{2}+i}(L\mathfrak{n}, \mathbb{W}_k(0)\otimes L_1(\mathfrak{g}))$-modules, and hence is a complex of $\mathcal{W}^\ell(\mathfrak{g})$-modules, where $\mathcal{W}^\ell(\mathfrak{g})$ acts via the embedding $\mathcal{W}^\ell(\mathfrak{g})=\mathcal{C}_k(\mathfrak{g})\hookrightarrow \pi_{\ell+h^\vee}=\mathrm{Com}(\pi_{k+h^\vee+1}, H^{\frac{\infty}{2}+i}(L\mathfrak{n}, \mathbb{W}_k(0)\otimes L_1(\mathfrak{g})))$. Therefore, (52) is an isomorphism of $\mathcal{W}^\ell(\mathfrak{g})$-modules.

We have

$$C^i_W = \bigoplus_{\substack{w\in \widehat{W}(\hat{\mu}) \\ \ell^{\frac{\infty}{2}}(w)=i}} \bigoplus_{\substack{\lambda\in\mathfrak{h}^* \\ \lambda-w\circ_k\mu-\nu\in Q}} \pi_{k+h^\vee+1,\lambda}\otimes\pi_{\ell+h^\vee, w\circ_k\mu-(\ell+h^\vee)\lambda} \tag{53}$$

by Proposition 8.3. Therefore, by (51), we find that $\mathcal{K}^\lambda_{\mu,\nu}$ is a subquotient of $\pi_{\ell+h^\vee,\mu-(\ell+h^\vee)\lambda}$ as a $\mathcal{W}^\ell(\mathfrak{g})$-module, as $\hat{\mu}$ is regular dominant. On the other hand, $\mathcal{K}^\lambda_{\mu,\nu}$ is a module over the rational quotient $\mathcal{W}_\ell(\mathfrak{g})$ and $\mathbf{L}_\ell(\chi_{\mu-(\ell+h^\vee)\lambda})$ is the unique simple $\mathcal{W}_\ell(\mathfrak{g})$-module that appears in the local composition factor of $\pi_{\ell+h^\vee,\mu-(\ell+h^\vee)\lambda}$ ([Ara07]). Since the character of $\mathcal{K}^\lambda_{\mu,\nu}$ coincides with that of $\mathbf{L}_\ell(\chi_{\mu-(\ell+h^\vee)\lambda})$ [KW90], we conclude that $\mathcal{K}^\lambda_{\mu,\nu}\cong \mathbf{L}_\ell(\chi_{\mu-(\ell+h^\vee)\rho^\vee})$. □

Corollary 12.4. *For an admissible level k, Then $L_{k+1}(\mathfrak{g})$ and $\mathcal{W}_\ell(\mathfrak{g})$ form a dual pair in $L_k(\mathfrak{g})\otimes L_1(\mathfrak{g})$.*

Let $\mathfrak{g}$ be simply laced as before, and let k be an admissible level for $\widehat{\mathfrak{g}}$ and let $n\in\mathbb{Z}_{>0}$. For $i=0,1,\dots,n-1$ define the rational numbers ℓ_i by the formula

$$\ell_i+h^\vee = \frac{k+i+h^\vee}{k+i+h^\vee+1}, \tag{54}$$

which are non-degenerate admissible levels for $\widehat{\mathfrak{g}}$ so that each $\mathcal{W}_{\ell_i}(\mathfrak{g})$ is a minimal series W-algebra. Then the invariant subspace $(L_k(\mathfrak{g})\otimes L_1(\mathfrak{g})^{\otimes n})^{\mathfrak{g}[t]}$ with respect to the diagonal action of $\mathfrak{g}[t]$ is naturally a vertex subalgebra of $L_k(\mathfrak{g})\otimes L_1(\mathfrak{g})^{\otimes n}$ and by Theorem 12.1 it is a vertex algebra extension of $\mathcal{W}_{\ell_0}(\mathfrak{g})\otimes\mathcal{W}_{\ell_1}(\mathfrak{g})\otimes\ldots\otimes\mathcal{W}_{\ell_{n-1}}(\mathfrak{g})$ and as such rational and lisse as well by [HKL15, Theorem 3.5]. It is also simple by [ACKL17, Lemma 2.1].

Corollary 12.5. *Let $\mathfrak{g}$ be simply laced, k an admissible number, n a positive integer. Then the coset vertex algebra $(L_k(\mathfrak{g})\otimes L_1(\mathfrak{g})^{\otimes n})^{\mathfrak{g}[t]}$ is simple, rational and lisse.*

A vertex operator algebra V is called *strongly unitary* if V is unitary and any positive energy representation of V is unitary in the sense of [DL14].

Theorem 12.6 (Classification of unitary minimal series W-algebras of type ADE)**.** *The (p,q)-minimal series principal W-algebra $\mathcal{W}_{p,q}(\mathfrak{g})$ is strongly unitary if and only if $|p-q|=1$.*

Proof. Suppose that $|p-q|=1$. Since $\mathcal{W}_{p,q}(\mathfrak{g})$ is rational, it is sufficient to show any simple $\mathcal{W}_{p,q}(\mathfrak{g})$-module M is unitary. By Theorem 5.6, we may assume that $p=q-1$. Then by Theorem 12.3,

$$M \cong \mathrm{Hom}_{\widehat{\mathfrak{g}}}(\mathbb{L}_{k+1}(\lambda), \mathbb{L}_k(\mu)\otimes\mathbb{L}_1(\nu))$$

for some $\lambda \in P_+^{k-h^\vee+1}$, $\mu \in P_+^k$, $\nu \in P_+^1$, where $k = p - h^\vee \in \mathbb{Z}_{\geqslant 0}$. Since both $L_{k+1}(\lambda)$ and $\mathbb{L}_k(\mu)\otimes\mathbb{L}_1(\nu)$ are unitary, M is unitary is as well (cf. Corollary 2.8 of [DL14]). Conversely, suppose that $\mathcal{W}_{p,q}(\mathfrak{g})$ is strongly unitary. Since any $\mathcal{W}_{p,q}(\mathfrak{g})$-module is a direct sum of unitary modules of the Virasoro algebra, the lowest L_0-eigenvalue of any simple $\mathcal{W}_{p,q}(\mathfrak{g})$-module is non-negative. Therefore, the *effective central charge* [DM04] of $\mathcal{W}_{p,q}(\mathfrak{g})$ coincides with the central charge of $\mathcal{W}_{p,q}(\mathfrak{g})$, which equals to

$$-\frac{r((h^\vee+1)p - h^\vee q)(h^\vee p - (h^\vee+1)q)}{pq}, \tag{55}$$

where r is the rank of $\mathfrak{g}$. On the other hand, Corollary 3.3 of [DM04] implies that the effective central charge coincides the *growth* ([Kac90, Exercise 13.39]) of the normalized character of $\mathcal{W}_{p,q}(\mathfrak{g})$, which is given by

$$r - \frac{h^\vee}{pq}\dim\mathfrak{g} \tag{56}$$

in Theorem 2.16 of [KW08]. Using the formula $\dim\mathfrak{g} = r(h^\vee+1)$ (note that $\mathfrak{g}$ is simply laced), we find that (55) coincides with (56) if and only if $|p-q|=1$. □

Note that for $\mathfrak{g}=\mathfrak{sl}_2$, the above series is exactly the discrete series [GKO86] of the Virasoro algebra.

13. Level-rank Duality

We derive another coset realization of the rational W-algebras of types A and D. These are called level-rank duality in the physics of conformal field theories, but they have first appeared in work by Igor Frenkel [Fre82] on vertex algebras of type A. There, the decomposition of $L_1(\mathfrak{gl}_{mn})$ into modules of $L_n(\mathfrak{sl}_m)\otimes L_m(\mathfrak{gl}_n)$ was studied. This level-rank duality has then be investigated further in the context of conformal field theory [Wal89, ABI90, NT92]. We can rephrase it as follows. By $\mathcal{H}$ we denote the rank one Heisenberg vertex algebra.

Theorem 13.1. *For positive integers $n,m,\ell \geqslant 2$, one has*

$$\mathrm{Com}\left(L_{m+\ell}(\mathfrak{sl}_n), L_m(\mathfrak{sl}_n)\otimes L_\ell(\mathfrak{sl}_n)\right) \cong \mathrm{Com}\left(L_n(\mathfrak{sl}_m)\otimes L_n(\mathfrak{sl}_\ell)\otimes\mathcal{H}, L_n(\mathfrak{sl}_{m+\ell})\right).$$

Especially it follows that

$$\mathrm{Com}\left(L_n(\mathfrak{sl}_m)\otimes\mathcal{H}, L_n(\mathfrak{sl}_{m+1})\right) \cong \mathcal{W}_k(\mathfrak{sl}_n)$$

at level k satisfying $k+n=(m+n)/(m+n+1)$.

Proof. Let n, m, ℓ be three fixed positive integers. Let $G_m, G_\ell, G_{m+\ell}$ be the vertex algebras of $nm, n\ell, n(m+\ell)$ fermionic ghosts-systems, so that $G_{m+\ell} \cong G_m \otimes G_\ell$. Then G_m is strongly generated by the fields $\psi_{i,j}$ and $\psi^*_{j,i}$ for integers i, j in $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m$ and operator products

$$\psi_{i,j}(z)\psi^*_{k,r}(w) \sim \delta_{j,k}\delta_{i,r}(z-w)^{-1}.$$

G_m has various interesting vertex subalgebras. Its even subalgebra is isomorphic to $L_1(\mathfrak{gl}_{nm})$ and the n^2, respectively m^2, fields of the form

$$\sum_{j=1}^{m} :\psi_{i,j}(z)\psi^*_{j,k}(z): \qquad \text{respectively} \qquad \sum_{i=1}^{n} :\psi_{i,j}(z)\psi^*_{r,i}(z):$$

generate vertex operators algebras isomorphic to $L_m(\mathfrak{gl}_n)$ respectively $L_n(\mathfrak{gl}_m)$. Furthermore $L_m(\mathfrak{sl}_n)$ and $L_n(\mathfrak{gl}_m)$ form a mutually commuting pair in G_m [OS14, Thm. 4.1] (which used [Wal89]). Similarly, one easily realizes $L_{m+\ell}(\mathfrak{gl}_n)$, $L_n(\mathfrak{gl}_m)$, $L_n(\mathfrak{gl}_\ell)$ and $L_n(\mathfrak{gl}_{m+\ell})$ as vertex subalgebras of $G_{m+\ell}$. We then have the following list of mutually commuting pairs:

(1) G_m and G_ℓ in $G_{m+\ell}$
(2) $L_m(\mathfrak{sl}_n)$ and $L_n(\mathfrak{gl}_m)$ in G_m
(3) $L_\ell(\mathfrak{sl}_n)$ and $L_n(\mathfrak{gl}_\ell)$ in G_ℓ
(4) $L_{m+\ell}(\mathfrak{sl}_n)$ and $L_n(\mathfrak{gl}_{m+\ell})$ in $G_{m+\ell}$

We thus have the following level-rank duality of coset vertex algebras

$$\begin{aligned}
\mathrm{Com}\left(L_{m+\ell}(\mathfrak{sl}_n), L_m(\mathfrak{sl}_n)\right. & \left.\otimes L_\ell(\mathfrak{sl}_n)\right) \cong \\
&\cong \mathrm{Com}\left(L_{m+\ell}(\mathfrak{sl}_n), \mathrm{Com}\left(L_n(\mathfrak{gl}_m)\otimes L_n(\mathfrak{gl}_\ell), G_{m+\ell}\right)\right)\\
&\cong \mathrm{Com}\left(L_{m+\ell}(\mathfrak{sl}_n)\otimes L_n(\mathfrak{gl}_m)\otimes L_n(\mathfrak{gl}_\ell), G_{m+\ell}\right)\\
&\cong \mathrm{Com}\left(L_n(\mathfrak{gl}_m)\otimes L_n(\mathfrak{gl}_\ell), \mathrm{Com}\left(L_{m+\ell}(\mathfrak{sl}_n), G_{m+\ell}\right)\right)\\
&\cong \mathrm{Com}\left(L_n(\mathfrak{gl}_m)\otimes L_n(\mathfrak{gl}_\ell), L_n(\mathfrak{gl}_{m+\ell})\right)\\
&\cong \mathrm{Com}\left(L_n(\mathfrak{sl}_m)\otimes L_n(\mathfrak{sl}_\ell)\otimes \mathcal{H}, L_n(\mathfrak{sl}_{m+\ell})\right)
\end{aligned}$$

where here $\mathcal{H}$ denotes the rank one Heisenberg vertex algebra in $L_n(\mathfrak{sl}_{m+\ell})$ commuting with $L_n(\mathfrak{sl}_m)\otimes L_n(\mathfrak{sl}_\ell)$. The case of $\ell = 1$ together with Theorem 12.1 then implies the second statement. □

This Theorem has a nice rationality corollary. Define k_i by $k_i + n = (m+n-i)/(m+n+1-i)$ and let $\mathcal{H}(\ell)$ be the Heisenberg vertex operator algebra of rank ℓ. Then the coset $\mathrm{Com}\left(L_n(\mathfrak{sl}_{m-\ell})\otimes \mathcal{H}(\ell), L_n(\mathfrak{sl}_m)\right)$ is a vertex algebra extension of

$$\mathcal{W}_{k_1}(\mathfrak{sl}_n)\otimes\mathcal{W}_{k_2}(\mathfrak{sl}_n)\otimes\ldots\otimes\mathcal{W}_{k_{\ell-1}}(\mathfrak{sl}_n)\otimes\mathcal{W}_{k_\ell}(\mathfrak{sl}_n)$$

and as such rational and lisse as well by [HKL15, Theorem 3.5]. It is also simple by [ACKL17, Lemma 2.1]. $L_n(\mathfrak{sl}_m)$ contains the rank $m-1$ Heisenberg vertex algebra. The latter extends to the lattice VOA of the lattice $\sqrt{n}A_{m-1}$ and the $\mathcal{H}(\ell)$ subVOA of $L_n(\mathfrak{sl}_m)$ extends to the lattice VOA V_{Λ_ℓ} with Λ_ℓ the orthogonal complement of $\sqrt{n}A_{m-\ell-1}$ in $\sqrt{n}A_{m-1}$. The coset $\mathrm{Com}\left(L_n(\mathfrak{sl}_{m-\ell}), L_n(\mathfrak{sl}_m)\right)$ is thus a vertex algebra extension of $\mathrm{Com}\left(L_n(\mathfrak{sl}_{m-\ell})\otimes\mathcal{H}(\ell), L_n(\mathfrak{sl}_m)\right)\otimes V_{\Lambda_\ell}$ and as such rational, lisse and simple as well. Summarizing:

Corollary 13.2. *Let n, m, ℓ be positive integers, such that $2 \leqslant m - \ell$. Then the cosets* $\mathrm{Com}\,(L_n(\mathfrak{sl}_{m-\ell}) \otimes \mathcal{H}(\ell), L_n(\mathfrak{sl}_m))$ *and* $\mathrm{Com}\,(L_n(\mathfrak{sl}_{m-\ell}), L_n(\mathfrak{sl}_m))$ *are rational, simple and lisse.*

The case of type D is a little bit more complicated. We need two statements. Recall that by a simple current one means an invertible object in the tensor category and by a simple current extension of a vertex algebra, one means a vertex algebra extension by a (abelian) group of simple currents. For lisse vertex algebras, this group must be finite. Let n, m, ℓ be positive integers and n in addition even.

Firstly, [KFPX12, Theorem 7.4] says especially that a simple current extension of $L_n(\mathfrak{so}_m) \otimes L_m(\mathfrak{so}_n)$ embeds conformally in $L_1(\mathfrak{so}_{nm})$. This means that $\mathrm{Com}\,(L_n(\mathfrak{so}_m), L_1(\mathfrak{so}_{nm}))$ is a simple current extension of $L_m(\mathfrak{so}_n)$ and the extension can be read off from [KFPX12, Table 3], see also [JL, Lemmas 4.11 and 4.13].

$$\begin{aligned}
&\mathrm{Com}\,(L_n(\mathfrak{so}_m), L_1(\mathfrak{so}_{nm})) \cong \begin{cases} L_m(\mathfrak{so}_n) & m \text{ odd} \\ L_m(\mathfrak{so}_n) \oplus \mathbb{L}_m(m\omega_1) & m \text{ even,} \end{cases} \\
&\mathrm{Com}\,(L_n(\mathfrak{so}_m), L_1(\mathfrak{so}_{nm}))^{A_{m,n}} \cong L_m(\mathfrak{so}_n), \qquad A_{m,n} := \begin{cases} \{1\} & m \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & m \text{ even,} \end{cases} \\
&\mathrm{Com}\,(L_m(\mathfrak{so}_n), L_1(\mathfrak{so}_{nm})) \cong L_n(\mathfrak{so}_m) \oplus \mathbb{L}_n(n\omega_1).
\end{aligned} \tag{57}$$

with $\omega_0, \omega_1, \dots, \omega_{m/2}$ the fundamental weights of $\mathfrak{so}_m$ in the m even case. Here $A_{m,n}$ is the group of simple currents of this extension. The second statement we need is that $L_1\left(\mathfrak{so}_{n(m+\ell)}\right)$ is a simple current extension of $L_1\left(\mathfrak{so}_{nm}\right) \otimes L_1\left(\mathfrak{so}_{n\ell}\right)$ for $n, m, \ell \in \mathbb{Z}_{>0}$ and n even [AKFPP16, Sec. 6.1]. This means that

$$L_1\left(\mathfrak{so}_{nm}\right) \otimes L_1\left(\mathfrak{so}_{n\ell}\right) \cong L_1\left(\mathfrak{so}_{n(m+\ell)}\right)^{\mathbb{Z}/2\mathbb{Z}}. \tag{58}$$

Level-rank duality for type D is then

Theorem 13.3. *For positive integers n, m, ℓ and n even, let $G = A_{m,n} \times A_{\ell,n} \times \mathbb{Z}/2\mathbb{Z}$ with $A_{m,n} = \mathbb{Z}/2\mathbb{Z}$ for even m and $A_{m,n}$ trivial otherwise. Then*

$$\mathrm{Com}\,(L_{m+\ell}(\mathfrak{so}_n), L_m(\mathfrak{so}_n) \otimes L_\ell(\mathfrak{so}_n)) \cong \mathrm{Com}\,(L_n(\mathfrak{so}_m) \otimes L_n(\mathfrak{so}_\ell), L_n(\mathfrak{so}_{m+\ell}) \oplus \mathbb{L}_n(n\omega_1))^G.$$

Especially it follows that

$$\mathrm{Com}\,(L_n(\mathfrak{so}_m), L_n(\mathfrak{so}_{m+1}) \oplus \mathbb{L}_n(n\omega_1))^{A_{m,n} \times \mathbb{Z}/2\mathbb{Z}} \cong \mathcal{W}_k(\mathfrak{so}_n)$$

at level k satisfying $k + n - 2 = (m + n - 2)/(m + n - 1)$.

Proof. Let $C := \mathrm{Com}\,(L_{m+\ell}(\mathfrak{so}_n), L_m(\mathfrak{so}_n) \otimes L_\ell(\mathfrak{so}_n))$. Using (57) twice, we get

$$\begin{aligned}
C &= \mathrm{Com}\,(L_{m+\ell}(\mathfrak{so}_n), L_m(\mathfrak{so}_n) \otimes L_\ell(\mathfrak{so}_n)) \\
&\cong \mathrm{Com}\left(L_{m+\ell}(\mathfrak{so}_n), \mathrm{Com}\,(L_n(\mathfrak{so}_m), L_1(\mathfrak{so}_{nm}))^{A_{m,n}} \otimes \mathrm{Com}\,(L_n(\mathfrak{so}_\ell), L_1(\mathfrak{so}_{n\ell}))^{A_{\ell,n}}\right) \\
&\cong \mathrm{Com}\,(L_{m+\ell}(\mathfrak{so}_n), \mathrm{Com}\,(L_n(\mathfrak{so}_m), L_1(\mathfrak{so}_{nm})) \otimes \mathrm{Com}\,(L_n(\mathfrak{so}_\ell), L_1(\mathfrak{so}_{n\ell})))^{A_{m,n} \times A_{\ell,n}} \\
&\cong \mathrm{Com}\,(L_{m+\ell}(\mathfrak{so}_n) \otimes L_n(\mathfrak{so}_m) \otimes L_n(\mathfrak{so}_\ell), L_1(\mathfrak{so}_{nm}) \otimes L_1(\mathfrak{so}_{n\ell}))^{A_{m,n} \times A_{\ell,n}}.
\end{aligned}$$

In the third line we used that the actions of $A_{m,n}$ and $L_n(\mathfrak{so}_m)$ commute on $L_1(\mathfrak{so}_{nm})$. Using (58) it follows that

$$C \cong \mathrm{Com}\left(L_{m+\ell}(\mathfrak{so}_n) \otimes L_n(\mathfrak{so}_m) \otimes L_n(\mathfrak{so}_\ell), L_1(\mathfrak{so}_{n(m+\ell)})\right)^{A_{m,n}\times A_{\ell,n}\times \mathbb{Z}/2\mathbb{Z}}.$$

The claim follows, since $\mathrm{Com}\left(L_{m+\ell}(\mathfrak{so}_n), L_1(\mathfrak{so}_{n(m+\ell)})\right) \cong L_n(\mathfrak{so}_{m+\ell}) \oplus \mathbb{L}_n(n\omega_1)$ by (57). □

We have shown that for even n, the coset $\mathrm{Com}\left(L_n(\mathfrak{so}_m), L_n(\mathfrak{so}_{m+1}) \oplus \mathbb{L}_n(n\omega_1)\right)$ has the regular vertex algebra $\mathcal{W}_k(\mathfrak{so}_n)$ as orbifold vertex subalgebra. This in turn means that $\mathrm{Com}\left(L_n(\mathfrak{so}_m), L_n(\mathfrak{so}_{m+1}) \oplus \mathbb{L}_n(n\omega_1)\right)$ is a vertex algebra extension of $\mathcal{W}_k(\mathfrak{so}_n)$. It is regular, since a vertex algebra extension of a regular vertex algebra is regular [HKL15, Theorem 3.5].

The module $\mathbb{L}_n(n\omega_1)$ is a simple current for $L_n(\mathfrak{so}_{m+1})$ so that $L_n(\mathfrak{so}_{m+1}) \oplus \mathbb{L}_n(n\omega_1)$ is a simple current extension of $L_n(\mathfrak{so}_{m+1})$. Conversely, $L_n(\mathfrak{so}_{m+1})$ is a $\mathbb{Z}/2\mathbb{Z}$-orbifold of $L_n(\mathfrak{so}_{m+1}) \oplus \mathbb{L}_n(n\omega_1)$. Here $\mathbb{Z}/2\mathbb{Z}$ leaves $L_n(\mathfrak{so}_{m+1})$ invariant and acts as -1 on $\mathbb{L}_n(n\omega_1)$, hence it especially commutes with the action of $L_n(\mathfrak{so}_m)$ on $L_n(\mathfrak{so}_{m+1}) \oplus \mathbb{L}_n(n\omega_1)$ and thus

$$\begin{aligned}\mathrm{Com}\left(L_n(\mathfrak{so}_m), L_n(\mathfrak{so}_{m+1})\right) &= \mathrm{Com}\left(L_n(\mathfrak{so}_m), \left(L_n(\mathfrak{so}_{m+1}) \oplus \mathbb{L}_n(n\omega_1)\right)^{\mathbb{Z}/2\mathbb{Z}}\right)\\ &= \mathrm{Com}\left(L_n(\mathfrak{so}_m), L_n(\mathfrak{so}_{m+1}) \oplus \mathbb{L}_n(n\omega_1)\right)^{\mathbb{Z}/2\mathbb{Z}}.\end{aligned}$$

By [CM, Theorem 5.24] the orbifold of a regular vertex operator algebra by a finite cyclic group is regular. Hence $\mathrm{Com}\left(L_n(\mathfrak{so}_m), L_n(\mathfrak{so}_{m+1})\right)$ is rational and lisse. It is simple by [ACKL17, Lemma 2.1]. We can even do better by iterating this coset construction. Let $2 \leqslant \ell < m$, then $C_{\ell,m} := \mathrm{Com}\left(L_n(\mathfrak{so}_\ell), L_n(\mathfrak{so}_m)\right)$ is a vertex algebra extension of $C_{\ell,\ell+1} \otimes C_{\ell+1,\ell+2} \otimes \dots \otimes C_{m-2,m-1} \otimes C_{m-1,m}$ and as such rational, lisse and simple as well.

Corollary 13.4. *Let n, m, ℓ be integers, such that n is even and $2 \leqslant \ell < m$. Then $\mathrm{Com}\left(L_n(\mathfrak{so}_\ell), L_n(\mathfrak{so}_m)\right)$ is simple, rational and lisse.*

14. Kazama-Suzuki coset vertex superalgebras

Let n, m be positive integers. In the physics of sigma models for string theories the Kazama-Suzuki [KS89] coset

$$KS(n,m) := \mathrm{Com}\left(L_{n+1}(\mathfrak{sl}_m) \otimes V_{\sqrt{m(m+1)(m+n+1)}\mathbb{Z}}, L_n(\mathfrak{sl}_{m+1}) \otimes G_m\right)$$

is used as a building block for superconformal field theories on Calabi-Yau manifolds. Here G_m is the vertex algebra of m copies of the fermionic ghost-system, and by V_L we mean the rational lattice vertex algebra of the positive definite lattice L. From Theorems 12.1 and 13.1 we know that $L_n(\mathfrak{sl}_{m+1}) \otimes G_m$ is a vertex superalgebra extension of $\mathcal{W}_\ell(\mathfrak{sl}_m) \otimes \mathcal{W}_k(\mathfrak{sl}_n) \otimes L_{n+1}(\mathfrak{sl}_m) \otimes \mathcal{H}(2)$ with $\ell + m = (n+m)/(n+m+1) = k+n$, where $\mathcal{H}(2)$ is a rank two Heisenberg vertex algebra. A short computation reveals that the latter extends to $V_{\sqrt{m(m+1)(m+n+1)}\mathbb{Z}} \otimes V_{\sqrt{mn(m+n+1)}\mathbb{Z}}$ so that we can conclude that $KS(n,m)$ is a vertex superalgebra extension of $\mathcal{W}_\ell(\mathfrak{sl}_m) \otimes \mathcal{W}_k(\mathfrak{sl}_n) \otimes V_{\sqrt{mn(m+n+1)}\mathbb{Z}}$ and hence rational by [HKL15, CKM].

Corollary 14.1. *Let n, m be positive integers. Then the coset $KS(n, m)$ is a simple, rational, and lisse vertex superalgebra.*

Level-rank duality of type D also gives rise to an interesting coset vertex superalgebra. It is well known that for $m \geqslant 3$, $L_1(\mathfrak{so}_m)$ is the even subalgebra of the vertex superalgebra $F(m)$ of m free fermions. It follows that the coset vertex superalgebra $\mathrm{Com}\,(L_{n+1}(\mathfrak{so}_m), L_n(\mathfrak{so}_{m+1}) \otimes F(m))$ is an extension of its even subalgebra $\mathrm{Com}\,(L_{n+1}(\mathfrak{so}_m), L_n(\mathfrak{so}_{m+1}) \otimes L_1(\mathfrak{so}_m))$. But by 12.1 and 13.3 the latter is a vertex algebra extension of $\mathcal{W}_k(\mathfrak{so}_m)^{\mathbb{Z}/2\mathbb{Z}} \otimes \mathcal{W}_\ell(\mathfrak{so}_n)$ if both n and m are even. By [CM], if $\mathcal{V}$ is a rational and lisse vertex algebra and G is an abelian group of automorphisms of $\mathcal{V}$, the orbifold $\mathcal{V}^G$ is also rational and lisse. It follows that $\mathcal{W}_k(\mathfrak{so}_m)^{\mathbb{Z}/2\mathbb{Z}} \otimes \mathcal{W}_\ell(\mathfrak{so}_n)$ is rational and lisse. Finally, since $\mathrm{Com}\,(L_{n+1}(\mathfrak{so}_m), L_n(\mathfrak{so}_{m+1}) \otimes F(m))$ is a vertex superalgebra extension of $\mathcal{W}_k(\mathfrak{so}_m)^{\mathbb{Z}/2\mathbb{Z}} \otimes \mathcal{W}_\ell(\mathfrak{so}_n)$, it is rational and lisse as well by [HKL15, CKM].

Corollary 14.2. *Let n, m be positive even integers. Then the coset*

$$\mathrm{Com}\,(L_{n+1}(\mathfrak{so}_m), L_n(\mathfrak{so}_{m+1}) \otimes F(m))$$

is a simple, rational, and lisse vertex superalgebra.

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