Isogeny graphs of superspecial abelian varieties (Theory and Applications of Supersingular Curves and Supersingular Abelian Varieties)

JORDAN, Bruce W.

RIMS Kôkyûroku Bessatsu
B90 (2022), 131–144

Isogeny graphs of superspecial abelian varieties

By

Bruce W. Jordan*

Abstract

We define three different isogeny graphs of principally polarized superspecial abelian varieties, prove foundational results on them, and explain their role in number theory and geometry. This is background to joint work with Yevgeny Zaytman on properties of these isogeny graphs for dimension $g > 1$, especially the result that they are connected, but not in general Ramanujan.

§ 1. Introduction

A superspecial abelian variety $A/\overline{\mathbb{F}}_p$ of dimension g is by definition isomorphic to a product of g supersingular elliptic curves. There is in fact only *one* superspecial abelian variety of dimension $g > 1$: Fix a supersingular elliptic curve $E/\overline{\mathbb{F}}_p$ with $\mathcal{O} = \mathcal{O}_E = \mathrm{End}(E)$ a maximal order in the rational definite quaternion algebra $\mathbb{H}_p$ ramified at p.

Theorem 1.1. (Deligne, Ogus [22], Shioda [27]) *Suppose $A/\overline{\mathbb{F}}_p$ is a superspecial abelian variety with* $\dim A = g > 1$. *Then $A \cong E^g$.*

If $A = E^g$, then

$$\mathrm{End}(A) = \mathrm{Mat}_{g\times g}(\mathcal{O}) \subseteq \mathrm{End}^0(A) = \mathrm{End}(A) \otimes \mathbb{Q} = \mathrm{Mat}_{g\times g}(\mathbb{H}_p).$$

The theory of superspecial abelian varieties thus bifurcates: for dimension $g = 1$ there are *many* superspecial abelian varieties (= supersingular elliptic curves) each with *one* principal polarization, whereas for dimension $g > 1$ there is *one* superspecial abelian variety with *many* principal polarizations.

Received December 31, 2020. Revised May 8, 2021.
*Baruch College, The City University of New York, One Bernard Baruch Way, New York, NY 10010 USA.
e-mail: `bruce.jordan@baruch.cuny.edu`

Let $\mathscr{A} = (A = E^g, \lambda)$ be a principally polarized superspecial abelian variety of dimension g over $\overline{\mathbb{F}}_p$ with $\overline{\mathbb{F}}_p$-isomorphism class $[\mathscr{A}]$. The principal polarization λ is an isomorphism from A to $\hat{A} = \mathrm{Pic}^0(A)$ satisfying the conditions of Definition 2.1. The number of such isomorphism classes $[\mathscr{A}]$ is finite and denoted $h = h_g(p)$; we shall see that h is a type of class number. Set

$$\begin{aligned} \mathsf{SP}_g(p)_0 &= \{\overline{\mathbb{F}}_p\text{-isomorphism classes } [\mathscr{A}]\} \\ &= \{[\mathscr{A}_1], \ldots, [\mathscr{A}_h]\} \text{ with } \mathscr{A}_j = (A_j, \lambda_j) \text{ and } A_j = E^g \text{ if } g > 1. \end{aligned} \tag{1.1}$$

So, for example,

$$\mathsf{SP}_1(p)_0 = \{\text{supersingular } j\text{-invariants in characteristic } p\} \text{ and}$$
$$\#\,\mathsf{SP}_1(p)_0 = h_1(p) = h(\mathbb{H}_p), \text{ the class number of the quaternion algebra } \mathbb{H}_p.$$

A principal polarization λ of an abelian variety $A/\overline{\mathbb{F}}_p$ defines the Weil pairing on $A[\ell]$, $\ell \neq p$ prime: $\langle\ ,\ \rangle_{\lambda,\ell} : A[\ell] \times A[\ell] \to \mu_\ell$. Put

$$\mathrm{Iso}_\ell(\mathscr{A}) = \{\text{maximal isotropic subgroups } C \subseteq A[\ell]\};\ \#\,\mathrm{Iso}_\ell(\mathscr{A}) = \prod_{k=1}^{g} (\ell^k + 1). \tag{1.2}$$

Proposition 1.2. (cf. [21, §23], [24, p. 36]). *Suppose $\ell \neq p$, $\mathscr{A} = (A, \lambda)$ is a principally polarized abelian variety over $\overline{\mathbb{F}}_p$, and $C \subseteq A[\ell]$. Let $\psi_C : A \to A/C =: A'$. Then there is a principal polarization λ' on A' so that $\psi_C^* \lambda' = \ell\lambda$ if and only if $C \in \mathrm{Iso}_\ell(\mathscr{A})$. In this case write $\mathscr{A}' = (A', \lambda') = \mathscr{A}/C$. If $[\mathscr{A}] \in \mathsf{SP}_g(p)_0$, then $[\mathscr{A}'] \in \mathsf{SP}_g(p)_0$.*

If $[\mathscr{A}] \in \mathsf{SP}_g(p)_0$ and $C \in \mathrm{Iso}_\ell(\mathscr{A})$, then $\mathscr{A} \to \mathscr{A}' = \mathscr{A}/C$ is an $(\ell)^g$-isogeny. Such $(\ell)^g$-isogenies induce correspondences from the finite set $\mathsf{SP}_g(p)_0$ to itself. These correspondences can be used to define various graphs—in this paper we define *three* $(\ell)^g$-isogeny graphs: the big isogeny graph $Gr_g(\ell, p)$, the little isogeny graph $gr_g(\ell, p)$, and the enhanced isogeny graph $\widetilde{gr}_g(\ell, p)$. In this introduction we content ourselves with defining the simplest of the three, the big isogeny graph $Gr := Gr_g(\ell, p)$:

Definition 1.3. The vertices of the graph $Gr = Gr_g(\ell, p)$ are $\mathrm{Ver}(Gr) = \mathsf{SP}_g(p)_0$, so $h = h_g(p) = \#\,\mathrm{Ver}(Gr)$. The (directed) edges of the graph Gr connecting the vertex $[\mathscr{A}_i] \in \mathsf{SP}_g(p)_0$ to the vertex $[\mathscr{A}_j] \in \mathsf{SP}_g(p)_0$ are

$$\mathrm{Ed}(Gr)_{ij} = \{C \in \mathrm{Iso}_\ell(\mathscr{A}_i) \mid [\mathscr{A}_i/C] = [\mathscr{A}_j]\}.$$

The adjacency matrix $\mathrm{Ad}(Gr)_{ij} = \#\,\mathrm{Ed}(Gr)_{ij}$ is a constant row-sum matrix by (1.2):

$$\sum_{j=1}^{h} \#\,\mathrm{Ed}(Gr)_{ij} = \prod_{k=1}^{g} (\ell^k + 1). \tag{1.3}$$

Yevgeny Zaytman and I spoke at the conference on these isogeny graphs and our results in [16], focusing on the theorem:

Theorem 1.4. ([16, §8]) *The isogeny graphs $Gr_g(\ell, p)$, $gr_g(\ell, p)$, and $\widetilde{gr}_g(\ell, p)$ are connected. If $g > 1$, the regular graph $Gr_g(\ell, p)$ is in general not Ramanujan.*

One ingredient of our proof is strong approximation for the quaternionic unitary group. The quaternionic unitary group has previously been applied to moduli of abelian varieties in characteristic p; see Ekedahl/Oort [23, §7, esp. Lemma 7.9], Chai/Oort [4, Prop. 4.3], and Chai [3, Prop. 1]. In my lecture and here I treat the general background and broader context of the isogeny graphs. Zaytman [28] will explain the proof of Theorem 1.4. A common notation is shared between the two papers. Full proofs and references for the results considered here can be found in [16].

My task of explaining isogeny graphs in arithmetic geometry is complicated by their opaque history: The subject is certainly over 75 years old, dating back at least to Brandt [1] from 1943. During this time, our graphs appear in disguises and in variations: the big, little, and enhanced isogeny graphs all are there. So the broader contexts and work done in other settings are not readily accessible. Let me give a personal example: The work on these graphs I have used most from graduate student days to the present is the 1979 paper [18] of Kurihara entitled *On some examples of equations defining Shimura curves and the Mumford uniformization.* Who would guess that this had anything to do with isogeny graphs? In fact, the word "isogeny" does not appear in the entire paper.

It is perhaps helpful to list in chronological order the four lives of our isogeny graphs, with **A**, **B**, and **C** subsequently playing a role in our story:

A. 1943 – : Brandt matrices. In this first appearance there are no graphs, no elliptic curves, and no abelian varieties—only the Brandt matrices which are the adjacency matrices of $Gr_g(\ell, p)$ and the weighted adjacency matrices of $gr_g(\ell, p)$. The major theorem was the trace formula. Brandt [1] defined the matrices for $g = 1$, primarily treating definite quaternion algebras over $\mathbb{Q}$. Eichler then introduces strong approximation and develops the theory for higher weight and totally real fields in case $g = 1$, including the trace formula. Shimura [26] laid the foundations to generalize to $g > 1$ and the quaternionic unitary group; Brandt matrices in this setting were defined in the 1980's by Hashimoto, Ibukiyama, Ihara, and Shimizu—see [8]. Gross's algebraic modular forms [7] subsequently provided a more general context for these matrices.

B. 1976 – : Shimura curves and $\mathbf{g = 1}$. In this incarnation the graphs appear, but not from isogenies of supersingular elliptic curves. Rather they arise from the bad reduction of Shimura curves in the work of Čerednik and Drinfeld. The explicit graphs are deduced from the results of Čerednik and Drinfeld in [12], the jacobian of the graph $\widetilde{gr}_1(\ell, p)$ is computed in [13], and the integral Hodge theory of the graphs $gr_1(\ell, p)$, $\widetilde{gr}_1(\ell, p)$ with applications to congruences between newforms and old forms is in [11].

In §5.1 of this paper and [16, §9] we uniformize $gr_1(\ell, p)$ and $\widetilde{gr}_1(\ell, p)$ as quotients of the tree $\Delta = \Delta_\ell$ for $\mathrm{SL}_2(\mathbb{Q}_\ell)$ using Kurihara [18]. The question of how the Čerednik-Drinfeld results generalize to the higher-dimensional case $g > 1$ remains a magnet for research.

C. 1988 – : LPS graphs; Ramanujan graphs and complexes. In an influential paper, Lubotzky, Phillips, and Sarnak [19] construct families of Cayley graphs from the Hamilton quaternions and show that they are Ramanujan. This work made the Ramanujan property a central focus. These LPS graphs are shown to be explicit covers of the little isogeny graph $gr_1(\ell, 2)$ in [14, §3]. A higher-dimensional Ramanujan complex was first constructed in [15]. But the full flowering of Ramanujan complexes is due to the work over the last 15 years of Alex Lubotzky and Winnie Li, together with their students, collaborators, and colleagues. Obviously this thread begs for a notion of isogeny complex to generalize isogeny graph; this is the subject of [17].

D. 2011 – and 1986 – : Applications of isogenies. In this current optic, applications are found for isogenies and isogeny graphs. This begins with Mestre's 1986 "méthode des graphes" [20] for computing Hecke operators. Then another completely different application is introduced with Jao and de Feo's 2011 proposal [10] for an isogeny-based key exchange using supersingular elliptic curves.

§ 2. Polarizations of superspecial abelian varieties

Let X be an abelian variety over a field k of dimension g; the dual abelian variety $\hat{X} = \mathrm{Pic}^0(X)$ is defined over k. A homomorphism $\tau : X \to \hat{X}$ is symmetric if $\hat{\tau} = \tau$, where we identify $X = \hat{\hat{X}}$ via the canonical isomorphism

$$\kappa_X : X \xrightarrow{\simeq} \hat{\hat{X}} \text{ of [6, Thm. 7.9], for example.} \tag{2.1}$$

Let $\mathcal{P}$ be the Poincaré line bundle on $X \times \hat{X}$.

Definition 2.1. (cf. [6, Cor. 11.5, Defn. 11.6]) A polarization of an abelian variety X over a field k is a symmetric isogeny $\lambda : X \to \hat{X}$ over k such that the line bundle $(\mathrm{id}_X, \lambda)^*\mathcal{P}$ is ample. The degree $\deg(\lambda)$ of the polarization λ is the degree of the isogeny λ, i.e., $\#\ker(\lambda)$. The degree $\deg(\lambda)$ is always a square by the Riemann-Roch theorem, see [21, §16]. It is convenient to define the reduced degree $\mathrm{rdeg}(\lambda)$ of the polarization λ to be $\mathrm{rdeg}(\lambda) = \sqrt{\deg(\lambda)}$. A polarization of degree 1 is a principal polarization.

Remark 2.2. For a polarization λ we have $\mathrm{rdeg}(n\lambda) = n^g \mathrm{rdeg}(\lambda)$. For an isogeny $f : X \to X'$ and polarization λ' on X', $\deg f^*(\lambda') = \deg(f)^2 \deg \lambda'$.

Many of the results in this section can be found in the paper [9] by Ibukiyama, Katsura, and Oort and many were known to Serre.

Let $\mathbb{H}$ be a positive definite quaternion algebra over $\mathbb{Q}$ with a maximal order $\mathcal{O}_{\mathbb{H}}$, main involution $x \mapsto \overline{x}$, and reduced norm $\mathrm{Nm}_{\mathbb{H}/\mathbb{Q}}(x) = \mathrm{Nm}(x) = x\overline{x}$. The reduced norm $\mathrm{Nm} : \mathrm{Mat}_{g\times g}(\mathbb{H}) \to \mathbb{Q}$ is the multiplicative polynomial of degree $2g$ generalizing the reduced norm $\mathrm{Nm} : \mathbb{H} \to \mathbb{Q}$. Put

$$\mathrm{SL}_g(\mathcal{O}_{\mathbb{H}}) = \{M \in \mathrm{Mat}_{g\times g}(\mathcal{O}_{\mathbb{H}}) \mid \mathrm{Nm}(M) = 1\}.$$

A matrix $H \in \mathrm{Mat}_{g\times g}(\mathbb{H})$ is hermitian if $H = H^\dagger := \overline{H}^t$. Set

$$\mathscr{H}_g(\mathcal{O}_{\mathbb{H}}) = \{H \in \mathrm{Mat}_{g\times g}(\mathcal{O}_{\mathbb{H}}) \mid H \text{ is positive-definite hermitian}\}. \tag{2.2}$$

The "Haupt norm" HNm of Braun-Koecher [2, Chap. 2, §4] (see also [21, Thm. 6 and proof, §21]) gives a map $\mathrm{HNm} : \mathscr{H}_g(\mathcal{O}_{\mathbb{H}}) \to \mathbb{N}$. For an integer $d \geq 1$ put

$$\mathscr{H}_{g,d}(\mathcal{O}_{\mathbb{H}}) = \{H \in \mathscr{H}_g(\mathcal{O}_{\mathbb{H}}) \mid \mathrm{HNm}(H) = d\}. \tag{2.3}$$

Definition 2.3. Let $\mathbb{H}$ be a definite quaternion division algebra over $\mathbb{Q}$ with maximal order $\mathcal{O}_{\mathbb{H}}$. Set $\mathcal{O}_{\hat{\mathbb{H}}} = \mathcal{O}_{\mathbb{H}} \otimes \hat{\mathbb{Z}}$, the profinite completion of $\mathcal{O}_{\mathbb{H}}$, and $\hat{\mathbb{H}} = \mathcal{O}_{\hat{\mathbb{H}}} \otimes \mathbb{Q}$.

There is a right action of $M \in \mathrm{SL}_g(\mathcal{O}_{\mathbb{H}})$ on hermitian $H \in \mathscr{H}_{g,d}(\mathcal{O}_{\mathbb{H}})$:

$$H \cdot M := M^\dagger H M. \tag{2.4}$$

Set $\overline{\mathscr{H}_{g,d}}(\mathcal{O}_{\mathbb{H}}) := \mathscr{H}_{g,d}(\mathcal{O}_{\mathbb{H}})/\mathrm{SL}_g(\mathcal{O}_{\mathbb{H}})$ with $[H] \in \overline{\mathscr{H}_{g,d}}(\mathcal{O}_{\mathbb{H}})$ the class defined by $H \in \mathscr{H}_{g,d}(\mathcal{O}_{\mathbb{H}})$.

If B is an algebra with anti-involution with fixed ring R, set

$$\begin{aligned} \mathrm{U}_g(B) &= \{M \in \mathrm{Mat}_{g\times g}(B) \mid M^\dagger M = \mathrm{Id}_{g\times g}\}, \\ \mathrm{GU}_g(B) &= \{M \in \mathrm{Mat}_{g\times g}(B) \mid M^\dagger M = \lambda\, \mathrm{Id}_{g\times g} \text{ with } \lambda \in R^\times\}. \end{aligned}$$

We will have $B = \mathcal{O} = \mathcal{O}_E$, $\mathcal{O}_{\mathbb{H}}$, $\mathcal{O}_{\mathbb{H}}[1/\ell]$, $\mathbb{H}$, $\mathcal{O}_{\hat{\mathbb{H}}}$, and $\hat{\mathbb{H}}$ in the course of this paper.

We now consider polarizations on $A/\overline{\mathbb{F}}_p = E^g/\overline{\mathbb{F}}_p$ with $\mathrm{End}(E) = \mathcal{O}_E = \mathcal{O} \subseteq \mathbb{H}_p$. For $\mathscr{A} = (A, \lambda)$ a polarized superspecial abelian variety over $\overline{\mathbb{F}}_p$, let $[\mathscr{A}]$ denote the isomorphism class defined by $\mathscr{A}$ over $\overline{\mathbb{F}}_p$.

Let λ_0 be the standard product polarization of A; the polarization λ_0 is principal. For $H \in \mathscr{H}_{g,d}(\mathcal{O}) \subseteq \mathrm{Mat}_{g\times g}(\mathcal{O})$, let λ_H be the polarization with

$$\lambda_H : A \xrightarrow{H} A \xrightarrow{\lambda_0} \hat{A}.$$

If $H \in \mathscr{H}_{g,d}(\mathcal{O})$, then λ_H is a polarization of $A = E^g$ with reduced degree $\mathrm{rdeg}(\lambda_H) = d$.

Theorem 2.4. *If $g > 1$, there are one-to-one correspondences induced by associating the polarization λ_H to the hermitian matrix H:*

(a) *polarizations λ of $A = E^g$ with* $\mathrm{rdeg}(\lambda) = d \longleftrightarrow \mathscr{H}_{g,d}(\mathcal{O})$ *and*

(b) *isomorphism classes $[\mathscr{A} = (A = E^g, \lambda)]$ with* $\mathrm{rdeg}(\lambda) = d \longleftrightarrow \overline{\mathscr{H}_{g,d}}(\mathcal{O})$.

For $H \in \mathscr{H}_{g,d}(\mathcal{O})$ with $g > 1$, we denote by $\mathscr{A}(H) = (A = E^g, \lambda_H)$ the associated polarized superspecial abelian variety. Theorem 2.4 allows us to describe the isomorphism classes $\mathsf{SP}_g(p)_0$ of principally polarized superspecial abelian varieties $[\mathscr{A}]$ over $\overline{\mathbb{F}}_p$ as in (1.1).

Proposition 2.5. *If $g > 1$, there is a bijection*

$$\overline{\mathscr{H}_{g,1}}(\mathcal{O}) \leftrightarrow \mathsf{SP}_g(p)_0$$

associating $[H] \in \overline{\mathscr{H}_{g,1}}(\mathcal{O})$ to $[\mathscr{A}(H)] \in \mathsf{SP}_g(p)_0$.

As discussed in §1, the theory of superspecial abelian varieties bifurcates into the cases $g > 1$ and $g = 1$. In spite of this we are able to give a uniform treatment of the principally polarized isomorphism classes in Theorem 2.9 below by shifting from the hermitian matrices for $g > 1$ in Proposition 2.5 to a notion of hermitian modules.

Let H_0 be the hermitian form on $\mathbb{H}^g$ given by $\mathrm{Id}_{g\times g}$. Let $L \subseteq \mathbb{H}^g$ be a finitely generated right $\mathcal{O}_{\mathbb{H}}$-module such that $L \otimes \mathbb{Q} = \mathbb{H}^g$. We say that L is principally polarized if there exists $c \in \mathbb{Q}^\times$ such that $cH_0|_L$ is $\mathcal{O}_{\mathbb{H}}$-valued and unimodular. We define the dual of L to be $\hat{L} = c^{-1}L$.

We can classify principally polarized right $\mathcal{O}_{\mathbb{H}}$-modules. For $M \in \mathrm{GU}_g(\hat{\mathbb{H}})$, denote by $[M]$ the coset containing M in $\mathrm{GU}_g(\hat{\mathbb{H}})/\mathrm{GU}_g(\mathcal{O}_{\hat{\mathbb{H}}})$ and define the principally polarized right $\mathcal{O}_{\mathbb{H}}$-module $\gamma(M) := M\mathcal{O}_{\hat{\mathbb{H}}}^g \cap \mathbb{H}^g$.

Theorem 2.6. *A one-to-one correspondence*

$$\{\textit{principally polarized right } \mathcal{O}_{\mathbb{H}}\textit{-modules}\} \leftrightarrow \mathrm{GU}_g(\hat{\mathbb{H}})/\mathrm{GU}_g(\mathcal{O}_{\hat{\mathbb{H}}})$$

is given by $\mathrm{GU}_g(\hat{\mathbb{H}})/\mathrm{GU}_g(\mathcal{O}_{\hat{\mathbb{H}}}) \ni M \leftrightarrow \gamma(M) = M\mathcal{O}_{\hat{\mathbb{H}}}^g \cap \mathbb{H}^g$.

Finally we define the classes of $\mathrm{GU}_g(\mathcal{O}_{\mathbb{H}})$, denoted $\mathscr{P}_g(\mathcal{O}_{\mathbb{H}})$, as the equivalence classes $[L]$ of principally polarized right $\mathcal{O}_{\mathbb{H}}$-submodules L up to left multiplication by $\mathrm{GU}_g(\mathbb{H})$:

$$\mathscr{P}_g(\mathcal{O}_{\mathbb{H}}) = \mathrm{GU}_g(\mathbb{H})\backslash\mathrm{GU}_g(\hat{\mathbb{H}})/\mathrm{GU}_g(\mathcal{O}_{\hat{\mathbb{H}}}). \tag{2.5}$$

The set $\mathscr{P}_g(\mathcal{O}_{\mathbb{H}})$ is finite. We define the class number $h_g(\mathbb{H})$ of $\mathrm{GU}_g(\mathcal{O}_{\mathbb{H}})$ by

$$h_g(\mathbb{H}) = \#\mathscr{P}_g(\mathcal{O}_{\mathbb{H}}); \tag{2.6}$$

it is independent of the choice of maximal order $\mathcal{O}_{\mathbb{H}}$.

Remark 2.7. In case $g = 1$, (2.5) becomes $\mathscr{P}_1(\mathcal{O}_{\mathbb{H}}) = \mathbb{H}^\times\backslash\hat{\mathbb{H}}^\times/\mathcal{O}_{\hat{\mathbb{H}}}$. Hence $\mathscr{P}_1(\mathcal{O}_{\mathbb{H}})$ is the usual ideal classes of $\mathcal{O}_{\mathbb{H}}$ and $h_1(\mathbb{H}) = \#\mathscr{P}_1(\mathcal{O}_{\mathbb{H}})$ is the usual class number $h(\mathbb{H})$ of the quaternion algebra $\mathbb{H}$. In particular $\mathscr{P}_1(\mathcal{O})$ is in one-to-one correspondence with $\mathsf{SP}_1(p)_0$.

For $g > 1$ we can relate principally polarized right $\mathcal{O}_{\mathbb{H}}$-modules to hermitian matrices using strong approximation—see [16, §2]—obtaining:

Theorem 2.8. *If $g > 1$, the set $\overline{\mathscr{H}_{g,1}}(\mathcal{O}_{\mathbb{H}})$ is in one-to-one correspondence with $\mathscr{P}_g(\mathcal{O}_{\mathbb{H}})$.*

We thus obtain the following description of $\mathsf{SP}_g(p)_0$.

Theorem 2.9. *We have one-to-one correspondences:*

(a) *For $g \geq 1$, $\mathsf{SP}_g(p)_0 \longleftrightarrow \mathscr{P}_g(\mathcal{O}) = \mathrm{GU}_g(\mathbb{H}_p)\backslash\mathrm{GU}_g(\hat{\mathbb{H}}_p)/\mathrm{GU}_g(\mathcal{O}_{\hat{\mathbb{H}}_p})$.*

(b) *For $g > 1$, $\mathsf{SP}_g(p)_0 \longleftrightarrow \mathscr{P}_g(\mathcal{O}) \longleftrightarrow \overline{\mathscr{H}_{g,1}}(\mathcal{O})$.*

In particular, with $h_g(p) := \#\,\mathsf{SP}_g(p)_0$ as in §1, we have $h_g(p) = h_g(\mathbb{H}_p)$.

§ 3. Brandt matrices

Let $h = h_g(\mathbb{H})$ and $\mathscr{P}_g(\mathcal{O}_{\mathbb{H}}) = \{[L_1], \dots, [L_h]\}$ with $[L_i]$ the class defined by the principally polarized right $\mathcal{O}_{\mathbb{H}}$-module L_i as in (2.5). For $1 \leq j \leq h$, set

$$e_g(j) = \#\{U \in \mathrm{GU}_g(\mathbb{H}) \mid L_j = UL_j\}.$$

Definition 3.1. For $n \geq 1$ define the *Brandt matrix* $B_g(n) \in \mathrm{Mat}_{h\times h}(\mathbb{Z})$ by

$$B_g(n)_{ij} = \frac{\#\{U \in \mathrm{GU}_g(\mathbb{H}) \mid [L_i : UL_j] = n^{2g}\}}{e_g(j)}$$

and define $B_g(0)_{ij} = 1/e_g(j)$.

Suppose $g = 1$. The class number $h_1(\mathbb{H})$ in (2.6) is the usual class number $h = h(\mathbb{H})$ of the quaternion algebra $\mathbb{H}$ by Remark 2.7. The principally polarized right $\mathcal{O}_{\mathbb{H}}$-modules $L_1, L_2, \dots, L_h$ can be identified with representatives $I_1, I_2, \dots, I_h$ for the right $\mathcal{O}_{\mathbb{H}}$-ideal classes. The norm $\mathrm{Nm}(I)$ of a (right or left) fractional $\mathcal{O}_{\mathbb{H}}$-ideal is the positive rational number generating the fractional ideal of $\mathbb{Q}$ generated by $\{\mathrm{Nm}(\alpha) \mid \alpha \in I\}$. Let $\mathcal{O}_i$ be the left order of the right $\mathcal{O}_{\mathbb{H}}$-ideal I_i. Then $e(i) = e_1(i) = \#\mathcal{O}_i^\times$. We thus have

$$B(n)_{ij} = B_1(n)_{ij} = \frac{\#\{\lambda \in I_iI_j^{-1} \mid \mathrm{Nm}(\lambda) = n\,\mathrm{Nm}(I_iI_j^{-1})\}}{e(j)}$$

and $B(0)_{ij} = 1/e(j)$, which is precisely the classical definition of Brandt matrices for a rational definite quaternion algebra.

The Brandt matrices $B_g(n)$ for a maximal order $\mathcal{O}_{\mathbb{H}} \subseteq \mathbb{H}$ are amenable to machine computation, although the memory requirements rapidly grow with n and especially g so that few examples are accessible with $g = 3$. We had no computations finish for $g \geq 4$. As an example, take $\mathbb{H} = \mathbb{H}_5$, the rational definite quaternion algebra of discriminant 5. The first class numbers of $\mathbb{H}_5$ are: $h_1(\mathbb{H}_5) = 1$, $h_2(\mathbb{H}_5) = 2$, $h_3(\mathbb{H}_5) = 3$. The Brandt matrix $B_g(\ell)$ has constant row-sum $\prod_{k=1}^{g}(1+\ell^k)$. Brandt matrices for $\mathbb{H}_5$ with $g = 1$, 2, 3 are given in Table 1, where ? means the computation did not finish.

	$\boldsymbol{B_g(2)}$	$\boldsymbol{B_g(3)}$	$\boldsymbol{B_g(7)}$	$\boldsymbol{B_g(11)}$
$\boldsymbol{g=1}$	$[3]$	$[4]$	$[8]$	$[12]$
$\boldsymbol{g=2}$	$\begin{bmatrix} 12 & 3 \\ 10 & 5 \end{bmatrix}$	$\begin{bmatrix} 34 & 6 \\ 20 & 20 \end{bmatrix}$	$\begin{bmatrix} 322 & 78 \\ 260 & 140 \end{bmatrix}$	$\begin{bmatrix} 1164 & 300 \\ 1000 & 464 \end{bmatrix}$
$\boldsymbol{g=3}$	$\begin{bmatrix} 54 & 27 & 54 \\ 30 & 15 & 90 \\ 14 & 21 & 100 \end{bmatrix}$	$\begin{bmatrix} 292 & 180 & 648 \\ 200 & 200 & 720 \\ 168 & 168 & 784 \end{bmatrix}$	?	?

Table 1. Brandt matrices $B_g(\ell)$ for $\mathbb{H}_5$

§ 4. The big, little, and enhanced isogeny graphs

We will consider $(\ell)^g$-isogenies of principally polarized superspecial abelian varieties in charteristic p with $p \neq \ell$; see Section 1 for the definitions. As discussed there, there are *three* natural graphs constructed from superspecial abelian variety isogenies—the big isogeny graph $Gr_g(\ell, p)$, the little isogeny graph $gr_g(\ell, p)$, and the enhanced isogeny graph $\widetilde{gr}_g(\ell, p)$. The different graphs arise depending on how isogenies and polarizations are identified. Big, little, and enhanced isogeny graphs have subtly different properties, so we need to be careful with the definitions...

Definition 4.1. A graph Gr has a set of vertices $\mathrm{Ver(Gr)} = \{v_1, \ldots, v_s\}$ and a set of (directed) edges Ed(Gr). An edge $e \in \mathrm{Ed(Gr)}$ has initial vertex $o(e)$ and terminal vertex $t(e)$. For $v_i, v_j \in \mathrm{Ver(Gr)}$, put

$$\mathrm{Ed(Gr)}_{ij} = \{e \in \mathrm{Ed(Gr)} \mid o(e) = v_i \text{ and } t(e) = v_j\}.$$

The adjacency matrix $\mathrm{Ad}(\mathrm{Gr}) \in \mathrm{Mat}_{s\times s}(\mathbb{Z})$ of Gr is defined as

$$\mathrm{Ad}(\mathrm{Gr})_{ij} = \#\,\mathrm{Ed}(\mathrm{Gr})_{ij}.$$

We place no further restrictions on our definition of a graph. Serre [25] requires graphs to be graphs with opposites: every directed edge $e \in \mathrm{Ed}(\mathrm{Gr})$ has an opposite edge $\overline{e} \in \mathrm{Ed}(\mathrm{Gr})$ with $\overline{\overline{e}} = e$. An edge e with $\overline{e} = e$ is called a half-edge. Serre forbids half-edges; we will call a graph satisfying his requirements a graph without half-edges. Kurihara [18] relaxes Serre's definition to allow half-edges giving the notion of a graph with half-edges. (A graph with half-edges may have $\emptyset$ as its set of half-edges, so every graph without half-edges is a graph with half-edges.) Following [18], if Gr is a graph with half-edges, Gr^* is the graph with the half-edges removed.

A graph with weights is a graph Gr with opposites together with a weight function $\mathrm{w} : \mathrm{Ver}(\mathrm{Gr}) \cup \mathrm{Ed}(\mathrm{Gr}) \to \mathbb{N}$ satisfying $\mathrm{w}(e) = \mathrm{w}(\overline{e})$ and $\mathrm{w}(e) \mid \mathrm{w}(o(e))$ for each edge e. The weighted adjacency matrix $\mathrm{Ad}_{\mathrm{w}}(\mathrm{Gr})$ of a graph with weights Gr is

$$\mathrm{Ad}_{\mathrm{w}}(\mathrm{Gr})_{ij} = \sum_{e \in \mathrm{Ed}(\mathrm{Gr})_{ij}} \frac{\mathrm{w}(v_i)}{\mathrm{w}(e)}. \tag{4.1}$$

Following [18, §3], a graph with lengths is a graph Gr with opposites together with a length function $f : \mathrm{Ed}(\mathrm{Gr}) \to \mathbb{N}$ satisfying $f(e) = f(\overline{e})$ for $e \in \mathrm{Ed}(\mathrm{Gr})$. A graph with weights determines a graph with lengths by taking the length of an edge to be its weight. If Gr is a graph with weights or lengths, then Gr^* inherits weights or lengths, respectively, from Gr.

§ 4.1. The big isogeny graph $Gr := Gr_g(\ell, p)$

The big isogeny graph $Gr = Gr_g(\ell, p)$ was defined in Definition 1.3; this is the usual "isogeny graph". In particular, $\mathrm{Ver}(Gr) = \mathsf{SP}_g(p)_0$, so $\#\,\mathrm{Ver}(Gr) = h = h_g(p)$. We have

$$\mathrm{Ed}(Gr)_{ij} = \{C \in \mathrm{Iso}_\ell(\mathscr{A}_i) \mid [\mathscr{A}_i/C] = [\mathscr{A}_j]\}$$

with $\mathrm{Iso}_\ell(\mathscr{A})$ as in (1.2). The adjacency matrix $\mathrm{Ad}(Gr)$ is a constant row-sum matrix as in (1.3):

$$\sum_{j=1}^{h} \mathrm{Ad}(Gr)_{ij} = \prod_{k=1}^{g} (\ell^k + 1).$$

It is in fact a familiar matrix:

Theorem 4.2. *Let $B_g(\ell)$ be the Brandt matrix for $\mathcal{O} \subseteq \mathbb{H}_p$. Then $\mathrm{Ad}(Gr_g(\ell, p)) = B_g(\ell)$.*

The adjacency matrix $\mathrm{Ad}(Gr) = B_g(\ell)$ is not in general symmetric, so Gr cannot be a graph with opposites. In particular, taking the dual isogeny does *not* give a well-defined involution on $\mathrm{Ed}(Gr)$, so Gr is *not* a graph with opposites via dual isogenies.

§ 4.2. The little isogeny graph $gr := gr_g(\ell, p)$

The little $(\ell)^g$-isogeny graph $gr = gr_g(\ell, p)$ has vertices

$$\mathrm{Ver}(gr) = \mathsf{SP}_g(p)_0,$$

so $\mathrm{Ver}(gr) = \mathrm{Ver}(Gr)$ and $\#\,\mathrm{Ver}(gr) = h = h_g(p)$. If $[\mathscr{A}] \in \mathsf{SP}_g(p)_0$ and $C, C' \in \mathrm{Iso}_\ell(\mathscr{A})$, say $C \sim C'$ if there exists $\alpha \in \mathrm{Aut}(\mathscr{A})$ such that $\alpha C = C'$. The class $[C] \in \mathrm{iso}_\ell(\mathscr{A}) := \mathrm{Iso}_\ell(\mathscr{A})/\sim$ is defined by $C \in \mathrm{Iso}_\ell(\mathscr{A})$. We put

$$\mathrm{Ed}(gr)_{ij} = \{[C] \in \mathrm{iso}_\ell(\mathscr{A}_i) \mid [\mathscr{A}_i/C] = [\mathscr{A}_j]\}.$$

Unlike the big isogeny graph, the little isogeny graph gr *is* a graph with opposites: the dual isogeny gives a well-defined involution on $\mathrm{Ed}(gr)$. In general we have edges $e \in \mathrm{Ed}(gr)$ with $\overline{e} = e$, so gr is a graph with half-edges. Beyond this, gr is a graph with weights: set $\mathrm{w}([\mathscr{A}]) = \#\,\mathrm{Aut}(\mathscr{A})$ and $\mathrm{w}([C]) = \#\,\mathrm{Aut}(A, \lambda, C)$ for a vertex corresponding to $[\mathscr{A} = (A, \lambda)] \in \mathsf{SP}_g(p)_0$ and the edge emanating from that vertex corresponding to $[C] \in \mathrm{iso}_\ell(\mathscr{A})$. The weighted adjacency matrix (4.1) of the little isogeny graph is the Brandt matrix:

Theorem 4.3. $\mathrm{Ad}_{\mathrm{w}}(gr_g(\ell, p)) = \mathrm{Ad}(Gr_g(\ell, p)) = B_g(\ell)$.

§ 4.3. The enhanced isogeny graph $\widetilde{gr} := \widetilde{gr}_g(\ell, p)$

Recall the notation (1.1):

$$\mathsf{SP}_g(p)_0 = \{[\mathscr{A}_1], \ldots, [\mathscr{A}_h]\} =: \{v_1, \ldots, v_h\}.$$

Suppose $[\mathscr{A} = (A, \lambda)] \in \mathsf{SP}_g(p)_0$. Let $\ell\mathscr{A} := (A, \ell\lambda)$, a g-dimensional superspecial abelian variety with ℓ times a principal polarization (which we call an $[\ell]$-polarization of type g in [16]). Set

$$\mathsf{SP}_g(p)_g = \{[\ell\mathscr{A}_1], \ldots, [\ell\mathscr{A}_h]\} =: \{v_{h+1}, \ldots, v_{2h}\}.$$

Define the $[\ell]$-dual $\hat{\mathscr{A}} = (\hat{A}, [\lambda])$ of $[\mathscr{A}] \in \mathsf{SP}_g(p)_0 \coprod \mathsf{SP}_g(p)_g$ by requiring that the composition

$$[\lambda] \circ \lambda : A \xrightarrow{\lambda} \hat{A} \xrightarrow{[\lambda]} A$$

from A to itself is multiplication by ℓ. This $[\ell]$-dual construction interchanges type 0 (principal polarizations) and type g: If $[\mathscr{A}] \in \mathsf{SP}_g(p)_0$, then $[\hat{\mathscr{A}}] \in \mathsf{SP}_g(p)_g$; and if $[\mathscr{A}] \in \mathsf{SP}_g(p)_g$, then $[\hat{\mathscr{A}}] \in \mathsf{SP}_g(p)_0$.

We can now define the enhanced isogeny graph $\widetilde{gr} := \widetilde{gr}_g(\ell, p)$:

Definition 4.4. The vertices of $\widetilde{gr} = \widetilde{gr}_g(\ell, p)$ are

$$\mathrm{Ver}(\widetilde{gr}) = \mathsf{SP}_g(p)_0 \coprod \mathsf{SP}_g(p)_g = \{v_1, \ldots, v_h\} \coprod \{v_{h+1}, \ldots, v_{2h}\}.$$

Hence $\#\,\mathrm{Ver}(\widetilde{gr}) = 2h = 2h_g(p)$.

The edges connecting the vertex $v_{h+i} = [\ell \mathscr{A}_i] \in \mathsf{SP}_g(p)_g$ to the vertex $v_j = [\mathscr{A}_j] \in \mathsf{SP}_g(p)_0$ are

$$\mathrm{Ed}(\widetilde{gr})_{h+i,j} = \{[C] \in \mathrm{iso}_\ell(\mathscr{A}_i) \mid [\mathscr{A}_i/C] = [\mathscr{A}_j]\}$$

with $\mathrm{iso}_\ell(\mathscr{A})$ as in §4.2. For $v_i = [\mathscr{A}_i] \in \mathsf{SP}_g(p)_0$ and $v_{h+j} = [\ell \mathscr{A}_j] \in \mathsf{SP}_g(p)_g$,

$$\mathrm{Ed}(\widetilde{gr})_{i,h+j} = \{[\hat{C}] \in \mathrm{iso}_\ell(\hat{\mathscr{A}_i}) \mid [\hat{\mathscr{A}_i}/\hat{C}] = [\hat{\mathscr{A}_j}]\}.$$

In case $1 \le i,j \le h$ or $h+1 \le i,j \le 2h$, $\mathrm{Ed}(\widetilde{gr})_{ij} = \emptyset$.

The enhanced isogeny graph $\widetilde{gr}$ is a graph with opposites: If $e \in \mathrm{Ed}(\widetilde{gr})_{ij}$, the opposite edge $\overline{e} \in \mathrm{Ed}(\widetilde{gr})_{ji}$ is the equivalence class of the dual isogeny. We never have $\overline{e} = e$, so $\widetilde{gr}$ is a graph without half-edges. The graph $\widetilde{gr}$ is a graph with weights: define w as the order of automorphism group as for gr.

Theorem 4.5. (a) *The enhanced isogeny graph $\widetilde{gr}$ is the bipartite double cover of the little isogeny graph gr with inherited weights.*

(b) *Let $Ad = \mathrm{Ad}(gr)$ and $Ad_w = \mathrm{Ad}_{\mathrm{w}}(gr) = \mathrm{Ad}(Gr)$. Then*

$$\mathrm{Ad}(\widetilde{gr}) = \begin{bmatrix} 0 & Ad \\ Ad & 0 \end{bmatrix} \quad \textit{and} \quad \mathrm{Ad}_{\mathrm{w}}(\widetilde{gr}) = \begin{bmatrix} 0 & Ad_w \\ Ad_w & 0 \end{bmatrix} = \begin{bmatrix} 0 & B_g(\ell) \\ B_g(\ell) & 0 \end{bmatrix}.$$

Let $\iota : \widetilde{gr} \to \widetilde{gr}$ be the involution defined on vertices by $\iota([\mathscr{A}]) = [\hat{\mathscr{A}}]$ and on edges such that if $e \in \mathrm{Ed}(\widetilde{gr})_{ij}$ corresponds to the class $[C]$, then $\iota(e) \in \mathrm{Ed}(\widetilde{gr})_{i+h,j+h}$ (where the indices are added $\bmod 2h$) also corresponds to the class $[C]$. Then ι fixes no vertices and no edges of $\widetilde{gr}$ and $\widetilde{gr}/\iota = gr$.

§ 5. ℓ-adic uniformization of isogeny graphs

In this section we give the uniformization of the isogeny graphs $gr_1(\ell, p)$ and $\widetilde{gr}_1(\ell, p)$ by the Bruhat-Tits building $\Delta = \Delta_\ell$ of $\mathrm{SL}_2(\mathbb{Q}_\ell)$, which is an $(\ell+1)$-regular tree. We then use this uniformization to relate $gr_1(\ell, p)$ and $\widetilde{gr}_1(\ell, p)$ to the bad reduction of Shimura curves.

§ 5.1. ℓ-adic uniformization of isogeny graphs for $g = 1$

The quaternion algebra $\mathbb{H}(p) := \mathbb{H}_p$ is split at the prime ℓ, so

$$\Gamma_0 := \mathcal{O}[1/\ell]^\times \hookrightarrow \mathrm{GL}_2(\mathbb{Q}_\ell) \cong (\mathbb{H}(p) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times.$$

Set $\Gamma_1 := \{\gamma \in \Gamma_0 \mid \mathrm{Nm}_{\mathbb{H}(p)/\mathbb{Q}}(\gamma) = 1\}$. Then $\Gamma_1\backslash\Delta$ and $\Gamma_0\backslash\Delta$ are finite graphs with weights defined by the orders of the stabilizer subgroups for the action on Δ. Kurihara [18] shows the following:

Theorem 5.1 (Kurihara). *Let $B_1(\ell)$ the Brandt matrix at ℓ for the maximal order $\mathcal{O} \subseteq \mathbb{H}(p) = \mathbb{H}_p$.*

(a) $\mathrm{Ad}_{\mathrm{w}}(\Gamma_0\backslash\Delta) = B_1(\ell)$.

(b) *The graph with weights $\Gamma_1\backslash\Delta$ is the bipartite double cover of $\Gamma_0\backslash\Delta$.*

Proof. (a): [18, p. 294].
(b): [18, p. 296]. ☐

In [16, §9], we prove the following:

Theorem 5.2.

(a) $gr_1(\ell,p) \cong \Gamma_0\backslash\Delta$ *as graphs with weights.*

(b) $\widetilde{gr}_1(\ell,p) \cong \Gamma_1\backslash\Delta$ *as graphs with weights.*

Note that the big isogeny graph $Gr_1(\ell,p)$ is *not* ℓ-adically uniformized since, as we saw in Section 4.1, it is not even a graph with opposites.

Theorem 5.2 in turn will show that our isogeny graphs $gr_1(\ell,p)$ and $\widetilde{gr}_1(\ell,p)$ arise from the bad reduction of Shimura curves, which we now explain. Let B be the indefinite rational quaternion division algebra with $\mathrm{Disc}\, B = \ell p$. Let $V_B/\mathbb{Q}$ be the Shimura curve parametrizing principally polarized abelian surfaces with QM (quaternionic multiplication) by a maximal order $\mathcal{M} \subseteq B$. There is then a model $M_B/\mathbb{Z}$ of $V_B/\mathbb{Q}$ constructed as a coarse moduli scheme by Drinfeld [5]; see also [12]. Let $\mathscr{L}/\mathbb{Z}_\ell$ be the ℓ-adic upper half-plane. The dual graph $G(\mathscr{L}/\mathbb{Z}_\ell)$ of its special fiber is canonically $\Delta = \Delta_\ell$. For $\Gamma \subseteq \mathrm{PGL}_2(\mathbb{Q}_\ell)$ a discrete, cocompact subgroup, the quotient $\Gamma\backslash\mathscr{L}$ is the formal completion of a scheme $\mathscr{L}_\Gamma/\mathbb{Z}_\ell$ along its closed fiber. We have that $\mathscr{L}_\Gamma/\mathbb{Z}_\ell$ is an *admissible curve* in the sense of [12, Defn. 3.1]. Its dual graph $G(\mathscr{L}_\Gamma/\mathbb{Z}_\ell)$ as in [12, Defn. 3.2] is a graph with lengths and $G(\mathscr{L}_\Gamma/\mathbb{Z}_\ell) \simeq (\Gamma\backslash\Delta)^*$, see [18, Prop. 3.2].

For the formulation below, see [12].

Theorem 5.3 (Čerednik, Drinfeld). *Let w_ℓ be the Atkin-Lehner involution at ℓ of M_B. Let $\overline{\Gamma}_0$ be the image of $\Gamma_0 \subseteq \mathrm{GL}_2(\mathbb{Q}_\ell)$ in $\mathrm{PGL}_2(\mathbb{Q}_\ell)$ and similarly for $\overline{\Gamma}_1$. Let $\mathfrak{O}$ be the ring of integers in the unramified quadratic extension of $\mathbb{Q}_\ell$.*

(a) *The scheme $M_B \times \mathbb{Z}_\ell$ is the twist of $\mathscr{L}_{\overline{\Gamma}_1}/\mathbb{Z}_\ell$ given by the 1-cocycle*

$$\chi \in H^1(\mathrm{Gal}(\mathfrak{O}/\mathbb{Z}_\ell), \mathrm{Aut}(\mathscr{L}_{\overline{\Gamma}_1} \times \mathfrak{O}/\mathfrak{O})), \text{ where } \chi : \mathrm{Frob}_\ell \mapsto w_\ell :$$
$$M_B \times \mathbb{Z}_\ell = (\mathscr{L}_{\overline{\Gamma}_1})^\chi.$$

(b) $(M_B/w_\ell) \times \mathbb{Z}_\ell = \mathscr{L}_{\overline{\Gamma}_0}/\mathbb{Z}_\ell$.

Corollary 5.4.

(a) $G(M_B \times \mathbb{Z}_\ell) = \Gamma_1 \backslash \Delta = \widetilde{gr}_1(\ell, p)$ *as graphs with lengths.*

(b) $G((M_B/w_\ell) \times \mathbb{Z}_\ell) = (\Gamma_0 \backslash \Delta)^* = gr_1(\ell, p)^*$ *as graphs with lengths with* $(\Gamma_0\backslash\Delta)^*$, $gr_1(\ell, p)^*$ *as in Definition* 4.1.

§ 5.2. ℓ-adic uniformization of isogeny graphs for $g > 1$

We would like to generalize Theorem 5.2 to $g \geq 1$. Recall that $A = E^g$, $\mathcal{O} = \mathrm{End}(E)$, and $\mathrm{End}(A) = \mathrm{Mat}_{g\times g}(\mathcal{O})$. Let $\mathcal{B}_g$ be the Bruhat-Tits building for $\mathrm{Sp}_{2g}(\mathbb{Q}_\ell)$ with $\mathcal{SS}_g$ its special 1-skeleton of $\mathcal{B}_g$: the vertices of $\mathcal{SS}_g$ are the special vertices of $\mathcal{B}_g$ and its edges are the edges of $\mathcal{B}_g$ between special vertices.

We prove the following theorem in [16, §9].

Theorem 5.5. *The groups* $\mathrm{GU}_g(\mathcal{O}[1/\ell])$, $\mathrm{U}_g(\mathcal{O}[1/\ell])$ *are as in Definition* 2.3.

(a) $gr_g(\ell, p) = \mathrm{GU}_g(\mathcal{O}[1/\ell])\backslash\mathcal{SS}_g$ *as graphs with weights.*

(b) $\widetilde{gr}_g(\ell, p) = \mathrm{U}_g(\mathcal{O}[1/\ell])\backslash\mathcal{SS}_g$ *as graphs with weights.*

In case $g = 1$, $\mathrm{Sp}_2(\mathbb{Q}_\ell) = \mathrm{SL}_2(\mathbb{Q}_\ell)$, $\mathcal{SS}_1 = \Delta_\ell$, $\mathrm{U}_1(\mathcal{O}[1/\ell]) = \Gamma_1$, $\mathrm{GU}_1(\mathcal{O}[1/\ell]) = \Gamma_0$, and we recover Theorem 5.2: $gr_1(\ell, p) = \Gamma_0\backslash\Delta_\ell$, $\widetilde{gr}_1(\ell, p) = \Gamma_1\backslash\Delta_\ell$. As remarked in Section 1, there is great interest in generalizing Theorem 5.3 to $g > 1$.

References

[1] Brandt, H., Zur Zahlentheorie der Quaternionen, *Jber. Deutsch. Math.-Verein.* **53**(1943): 23–57.
[2] Braun, H. and Koecher, M., *Jordan-Algebren*, Springer-Verlag, Berlin New York, 1966.
[3] Chai, C. L., Every ordinary symplectic isogeny class in positive characteristic is dense in the moduli, *Invent. Math.* **121(3)** (1995): 439–479.
[4] Chai, C. L. and Oort, F., Monodromy and irreducibility of leaves, *Ann. of Math.* (2) **173(3)**(2011): 1359–1396.
[5] Drinfeld, V., Coverings of p-adic symmetric regions (in Russian), *Funkcional. Anal. i Priložen* **10(2)** (1976): 29–40. *Functional Anal. Appl.* **10** (1976): 107–115.
[6] van der Geer, G. and Moonen, B., *Abelian Varieties*, `http://page.mi.fu-berlin.de/elenalavanda/BMoonen.pdf`.
[7] Gross, B., Algebraic modular forms, *Israel J. Math.* **113** (1999): 61–93.
[8] Hashimoto, K., On Brandt matrices associated with the positive definite quaternion Hermitian forms, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **27(1)**(1980): 227–245.
[9] Ibukiyama, T., Katsura, T., and Oort, F., Supersingular curves of genus two and class numbers, *Compositio Math.* **57(2)** (1986): 127–152.

[10] Jao, D. and De Feo, L., Towards quantum-resistant cryptosystems from supersingular elliptic curve isogenies. *Post-quantum cryptography*, *Lecture Notes in Comput. Sci.*, Springer, Heidelberg, **7071** (2011): 19–34.

[11] Jordan, B. and Livné, R., Integral Hodge theory and congruences between modular forms, *Duke Math. J.* **80(2)** (1995): 419–484.

[12] Jordan, B. and Livné, R., Local diophantine properties of Shimura curves, *Math. Ann.* **270(2)** (1985): 235–248.

[13] Jordan, B. and Livné, R., On the Néron model of jacobians of Shimura curves, *Compositio Math.* **60(2)** (1986): 227–236.

[14] Jordan, B. and Livné, R., Ramanujan local systems on graphs, *Topology* **36(5)** (1997): 1007–1024.

[15] Jordan, B. and Livné, R., The Ramanujan property for regular cubical complexes, *Duke Math. J.* **105(1)** (2000): 85–103.

[16] Jordan, B. and Zaytman, Y., Isogeny graphs of superspecial abelian varieties and Brandt matrices, `arXiv:2005.09031v4`.

[17] Jordan, B. and Zaytman, Y., Isogeny complexes of superspecial abelian varieties, in preparation.

[18] Kurihara, A., On some examples of equations defining Shimura curves and the Mumford uniformization, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **25**(1979): 277–300.

[19] Lubotzky, A., Phillips, R., and Sarnak, P., Ramanujan graphs, *Combinatorica* **8(3)**(1988): 261–277.

[20] Mestre, J.-F., La méthode des graphes. Exemples et applications, *Proceedings of the international conference on class numbers and fundamental units of algebraic number fields* (*Katata, 1986*), Nagoya Univ., Nagoya, (1986): 217–242.

[21] Mumford, D., *Abelian Varieties*, Hindustan Book Agency, New Delhi, 2008.

[22] Ogus, A., Supersingular $K3$ crystals, *Journées de Géométrie Algébrique de Rennes* (*Rennes, 1978*), *Vol. II*, *Astérique* **64**, Soc. Math. France, Paris (1979): 3 – 86.

[23] Oort, F., A stratification of a moduli space of abelian varieties, *Moduli of abelian varieties* (*Texel Island, 1999*), *Progr. Math.* **195**, Birkhäuser, Basel, (2001): 345–416.

[24] Oort, F., Which abelian surfaces are products of elliptic curves?, *Math. Ann.* **214** (1975): 35–47.

[25] Serre, J. P., *Trees*, Springer-Verlag, 2003.

[26] Shimura, G., Arithmetic of alternating forms and quaternion hermitian forms, *J. Math. Soc. Japan* **15** (1963): 33–65.

[27] Shioda, T., Supersingular $K3$ surfaces, *Algebraic geometry* (*Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978*), *Lecture Notes in Math.* **732**, Springer, Berlin (1979): 564 – 591.

[28] Zaytman, Y., Proving connectedness of isogeny graphs with strong approximation, this volume.