Closed-Form Solutions for Distributionally Robust Inventory Management: Extended Reformulation using Zero-Sum Game

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When only the moments (mean, variance or t-th moment) of the underline distribution are known, many max-min optimization models can be interpreted as zero-sum games, in which the firm chooses actions to maximize her expected profit while Adverse Nature chooses a distribution subject to the moment conditions to minimize the firm's expected profit. We propose a new method to reformulate the zero-sum game as a robust moral hazard model, in which Adverse Nature chooses both the distribution and actions to minimize the firm's expected profit subject to incentive compatibility (IC) constraints. Under quasi-concavity, these IC constraints are replaced by the first-order conditions, which give rise to additional moment constraints and an extended reformulation of the dual problem in a higher dimensional space, facilitating the search for the closed form solution. In the equilibrium, these moment constraints are binding and have zero Lagrangian multipliers. This property enables us to derive closed-form solutions, hitherto unknown, for several distributionally robust inventory models, including the newsvendor problem with mean and t-th moment (for $t > 1$), the capacity planning model with multiple supply sources, and the two-product inventory system with common component.

Key words: Robust optimization, zero-sum games, moment-based ambiguity, inventory management

1. Introduction

In many practical settings, managers often need to make decision without precisely knowing the underline distribution that governs the (random) operating environment. For example, forecasting demand for new products is notoriously difficult (see Chapter 5 in Lilien et al. 2017, page 135 to 168). Small and medium-sized enterprises usually lack the resources needed to collect and analyze data. Environmental factors such as a trade embargo or flood also increase the level of ambiguity experienced by many supply chains. Therefore, ambiguity (which differs from risk) has received increasing research attention from various disciplines.

Researchers have focused on finding robust solutions to address the ambiguity in the planning problems, using only a parsimonious set of information such as the means and variances (or higher moments) to characterize the distributional ambiguity in the operating environment (e.g., Natarajan et al. 2018, Li and Kirshner 2021). In this literature stream, the robust max-min decision rule is the most popular decision rule (see page 1931 in Lu and Shen 2021). While the practical reason is that the worst case performance is of special interest to risk managers (Tang 2006) who may favour a conservative approach, the technical reason is that an ambiguity-based model can often be transformed into a semi-infinite programming (SIP) model using duality, especially when the moments (such as the mean, variance, or t-th moment) are known. The practical relevance and computational tractability make this class of max-min models under moment conditions prevalent in the literature (see Section 2.1.2 of Lu and Shen 2021, and the references therein).

Mamani et al. (2017), for instance, propose and analyze robust optimization models of an inventory problem, where cumulative purchase, inventory, and shortage costs over n periods are minimized for non-identically distributed demand. They apply the Central Limit Theorem to derive the robust counterpart (uncertainty set) for this class of problem. Their method simplifies the original distributional robust problem into a more tractable version such that closed-form ordering quantities can be derived by using standard robust optimization tools. However, their approach fails when n is small, or when there this no uncertainty set that can be used to approximate the distributional robust problem. In this case, we need to address the problem head-on, to find a closed form solution to the semi-infinite programming model.

In a typical max-min optimization problem with known moments, the firm chooses her actions to maximize her worst-case expected profit based on the available information. In supply chains for example, managers often make pricing and production joint decisions, while Adverse Nature chooses an adversarial demand realization to reduce managers' payoff. This max-min problem can be formulated as a zero-sum game, in which the firm chooses an action while Adverse Nature chooses the realized value subject to the constraints on moments. In this game, the firm employs a pure strategy, while Adverse Nature employs a mixed strategy that notably translates into a distribution satisfying the moment conditions. We refer to this class of models as the *zero-sum games under moment conditions*.

The traditional method to solve this class of zero-sum games is based on the firm's perspective and proceeds in two stages. We first compute the firm's worst case expected profit for any given actions and then optimize the actions. When the actions are fixed, the second-stage problem is reformulated as an SIP model using duality. We label this model as the firm's SIP model, in which the second stage SIP model is linear and hence, Karush–Kuhn–Tucker (KKT) conditions are sufficient and necessary. However, the action-dependent SIP model in the second stage often produces a piece-wise or even implicit objective function, rendering the first-stage problem cumbersome or intractable. In Section 2, we thoroughly explain challenges that the traditional method encounters.

We propose a new and efficient method to solve the firm's equilibrium strategy (which is also known as her robust optimal solution). The key feature is that our method is based on Adverse Nature's perspective. The optimization model of Adverse Nature becomes a min-max problem, in which Adverse Nature chooses a distribution by anticipating that the firm will choose actions that are optimal for the chosen distribution. Conceptually, the min-max problem of Adverse Nature is a robust moral hazard model where Adverse Nature is the principal and the firm is the agent. Adverse Nature jointly chooses the distribution and actions to minimize the firm's expected profit subject to the moment constraints and the firm's incentive compatibility (IC) constraints. Moral hazard interpretation is crucial because it not only enables us to apply tools in the economics literature to simplify IC constraints but also broadens our application to moral hazard. With mean-variance (or t-th moment) ambiguity, the IC constraints become additional moment constraints that can be conveniently incorporated into Adverse Nature's SIP model.

These additional moment constraints produce two noteworthy consequences. First, the additional moment constraints effectively integrate the firm's first-order conditions (FOCs) into the min-max version of the zero-sum game, enabling us to obtain closed-form solutions without explicitly deriving the firm's objective function. Consequently, we bypass the second-stage problem where the traditional method encounters significant challenges. Second, *zero* Lagrangian multipliers facilitate the characterization of the equilibrium. We assume that the zero-game has an equilibrium (otherwise, the original max-min problem has no solution). Whenever the zero-game has an equilibrium, which is a saddle point, equality must hold in the well-known min-max inequality, implying that the firm's SIP model and Adverse Nature's SIP model must produce the same optimal objective value. By contrasting the two SIP models of the firm and Adverse Nature, we find that the difference is attributable to the additional moment constraints associated with the firm's IC constraints. Thus, if equality holds in the min-max inequality, these additional moment constraints are binding but the relevant Lagrangian multipliers are zero, enabling us to solve numerous zero-sum games in closed forms that are unavailable in the extant literature.

1.1. Literature Review

Our paper relates to the literature on robust operations management. We refer readers to Lu and Shen (2021) for an updated literature review. The research on optimal ordering decisions with limited information about the demand distribution is inspired by Scarf (1958). In this literature stream, the firm knows only the mean and variance of demand and aims to find an order quantity to maximize her expected profit against the worst possible distribution. With a single supply source, the worst-case distribution is a two-point distribution. The robust optimal order quantity and the resulting profit can be derived in closed forms. Gallego and Moon (1993) provide a more concise proof of Scarf's result by using the Cauchy-Schwartz inequality. Natarajan et al. (2018) examine the impact of an asymmetric demand distribution by using second-order partitioned statistics to measure distributional asymmetry. Das et al. (2021) analyze the impact of heavy-tailed demand distributions by assuming that the newsvendor firm knows only the mean and the t-th moment of the demand distribution (where $t > 1$ is a real number). We refer to this model as the "$1 + t$" model. While Das et al. (2021) encounter challenge in deriving the objective function and are unable to solve the robust inventory level in closed forms, we overcome this challenge. Minimax regret is another robust decision rule commonly used in the literature. Yue et al. (2006) define the value of information (VOI) as the difference between knowing and not knowing the underlying demand distribution, while Perakis and Roels (2008) refer to VOI as regret (which measures forgone profit in the absence of full information on the underlying demand distribution). Yue et al. (2006) focus on mean-variance ambiguity, whereas Perakis and Roels (2008) consider a variety of partial information on the distribution such as its mode, range, mean, and variance.

The robust optimization literature distinguishes between risk aversion and ambiguity aversion. A risk-averse decision maker prefers an order quantity that avoids profit volatility in addition to the expected profit, whereas an ambiguity-averse decision maker does not have complete knowledge of the demand distribution and thus prefers an order quantity that is distributionally robust (Han et al. 2014). Incorporating the variance of the profit, Han et al. (2014) study a distributionally robust newsvendor model by combining both risk aversion and ambiguity aversion. Several recent articles (Yang et al. 2018, Kouvelis et al. 2021, Yang et al. 2021) employ the conditional value at risk as a measure of risk tolerance. With advances in machine learning, several recent articles have proposed data-driven methods to determine the robust order quantity. For instance, Chen and Xie (2021) consider an unknown joint distribution for demand and yield, while He and Lu (2021) consider a price-setting newsvendor firm that is partially informed about the demand distribution and has limited data on a few historical prices. It is possible that multiple decision makers could simultaneously face ambiguity. For instance, Fu et al. (2018) consider an agricultural supply chain where both the upstream and downstream entities face mean-variance ambiguity. Li and Kirshner (2021) label this ambiguous environment "two-sided ambiguity" and consider the contracting issue between the firm and her sales agent.

1.2. Our Contributions

We make the following contributions to the literature.

- We develop a new and efficient method to solve a class of zero-sum games under moment conditions. By solving the zero-sum games, we solve the corresponding max-min optimization models, which are popular in the literature of robust operations management.

- We demonstrate that using min-max inequality and the property of zero Lagrangian multipliers, we can efficiently identify the equilibrium due to the following reasons. First, the IC constraints associated with pure strategy are much simpler than those associated with mixed strategy. Second, our method integrates the firm's FOCs into the new formulation such that we can solve the equilibrium without explicitly deriving the firm's objective function.

- To demonstrate the scalability and efficiency of our method, we solve three application examples and generate new results that are unavailable in the extant literature. In the first example (i.e., the $1+t$ model), the available information includes the mean and the t-th moment. This example considers the impact of heavy (when $1 < t < 2$) or light (when $t > 2$) tail on inventory planning. In the second example, the ex post payoff function is piece-wise linear (with $n \geq 1$ pieces) and concave. This example prescribes how to use multiple supply sources or option contracts to cope with random demand and has broad applications in the electronic appliance, energy, and remanufacturing industries. In the third example, we consider the impact of component commonality on inventory planning.

Table 1 Three Important Examples with Different Levels of Complexity

Model	Demand	Action	Extant Literature
$1+t$	Single-Dimensional		Unable to obtain closed-form solution
n-option	Single-Dimensional	Multi-Dimensional	Unknown
Common Component	Multi-Dimensional		Unknown

The remaining sections are organized as follows. Section 2 introduces the new method. Section 3 solves the $1+t$ newsvendor model. Section 4 investigates a capacity planning model with multiple supply sources under mean-variance conditions. Section 5 analyzes the impact of component commonality. Section 6 discusses a few technical issues and Section 7 concludes the paper. We present all the technical proofs in Online Appendix.

2. The New Method

2.1. Firm's Perspective

Let $\tilde{\theta}$ be a random variable that affects the firm's ex post payoff. In supply chains, this random variable $\tilde{\theta}$ can represent the random demand or yield. We use θ to indicate the realization and $F(\theta)$ to represent the cumulative distribution function. However, the firm does not know the exact functional form of $F(\theta)$ except the mean and variance. Specifically, let $E\left(\tilde{\theta}\right) = \mu > 0$ be the mean and $Var\left(\tilde{\theta}\right) = \sigma^2 > 0$ be the variance of $\tilde{\theta}$. In Section 3, we generalize the analysis by replacing variance with the t-th moment, where $t > 1$. A wide range of max-min optimization models can be abstracted as the following distributionally robust optimization model:

$$Z = \max_{\mathbf{Q} \geq \mathbf{0}} \left\{ \inf_{F \in \Omega} \int_0^\infty Z(\theta|\mathbf{Q})\, dF(\theta) \right\}, \tag{2.1}$$

in which Ω represents the ambiguity set (or the feasible action space of Adverse Nature) and $Z(\theta|\mathbf{Q})$ is the firm's ex post payoff function when the realized random variable is θ and her action vector is $\mathbf{Q} = (Q_1, Q_2, ...Q_n)$, implying that the firm can take multiple actions such as choosing prices or production quantities. The mathematical properties of the ex post payoff function $Z(\theta|\mathbf{Q})$ vary from case to case. On the other hand, the partial information about $F(\theta)$ determines the ambiguity set Ω. With mean and variance, we define the ambiguity set Ω as follows:

$$\Omega = \left\{ F(\theta) \,|\, \int_0^\infty dF(\theta) = 1, \int_0^\infty \theta dF(\theta) = \mu, \int_0^\infty \theta^2 dF(\theta) = \mu^2 + \sigma^2 \right\}. \tag{2.2}$$

The theoretical and practical parallels to equation (2.1) are abundant. For example, when Q is a single variable (rather than a vector) and $Z(\theta|Q)$ represents the newsvendor payoff function, equation (2.1) reduces to the model proposed by Scarf (1958) in his pioneering article that inspires a large and growing stream of literature on robust inventory management.

The characteristics of the ambiguity set could determine whether the worst-case expected profit is attained or approached. Observe that the ambiguity set Ω in equation (2.2) includes an infinite number of probability distributions (which can be continuous, discrete or mixed) satisfying mean-variance constraints. Many of these distributions may not satisfy log-concave, increasing failure rate, monotonic local likelihood ratio, or convex distribution function conditions, which are common in the economics and supply chain literature. Including some of these conditions could have an undesirable consequence by rendering the worst-case expected profit approached rather than attained. For example, Carroll (2015, page 542) mentions that "the worst-case payoff may be approached, but not actually attained for any technology. This is why we defined it as an infimum and not a minimum." In contrast, we impose only the moment constraints on Ω, enabling us to broaden the application and attain a minimum.

It is well-known that equation (2.1) formulates a zero-sum game between the firm and Adverse Nature. While the firm wishes to maximize her expected profit by choosing a vector $\mathbf{Q}$, Adverse Nature wishes to minimize the firm's expected profit by choosing a distribution from Ω due to the mean-variance constraints. Let $(\mathbf{Q}^*, F^*)$ be the equilibrium of the zero-sum game in equation (2.1). We refer to $\mathbf{Q}^*$ as the firm's equilibrium strategy (or her robust optimal solution), F^* as Adverse Nature's equilibrium strategy, and $Z(\mathbf{Q}^*, F^*) = Z^*$ as the value of the zero-sum game (or the firm's optimized worst-case expected profit). An important feature of the zero-sum game in equation (2.1) is that the firm employs a pure strategy (which occurs in supply chains) but Adverse Nature employs a mixed strategy. Both the firm and Adverse Nature have infinite feasible actions while the finite zero-sum game (pioneered by Nash 1951) restricts the number of feasible actions to be finite.

To cope with the infinite nature of the zero-sum game, the traditional method applies SIP tools to reformulate equation (2.1) as follows:

$$
\begin{aligned}
P = & \max_{\mathbf{Q}\geq\mathbf{0}} \max_{y_0,y_1,y_2} \left\{ y_0 + y_1\mu + y_2\left(\mu^2+\sigma^2\right) \right\} \\
\text{s.t.} \quad & y_0 + y_1\theta + y_2\theta^2 \leq Z\left(\theta|\mathbf{Q}\right), \forall\theta\geq 0.
\end{aligned} \tag{2.3}
$$

For any given $\mathbf{Q}$, the inner maximization problem in equation (2.3) is a *linear* SIP model, in which probabilistic resources are being traded. In the literature of linear programming, the notion of trading resources is commonly used when interpreting a primal-dual relationship. Because our model involves probabilistic concepts (such as mean), we introduce probabilistic resources. The three decision variables y_0, y_1, and y_2 are the shadow prices of the total probability, the mean, and the variance resources, respectively. When Adverse Nature chooses a realization θ, the firm obtains her ex post payoff $Z(\theta|\mathbf{Q})$ and simultaneously purchases 1 unit of the total probability resource, θ units of the mean resource, and θ^2 units of the variance resource from Adverse Nature. By supplying these probabilistic resources, Adverse Nature generates an ex post income that equals $y_0 + y_1\theta + y_2\theta^2$. However, the firm is protected by a limited liability such that the ex post payment cannot exceed the firm's ex post payoff, giving rise the SIP constraints in equation (2.3). Observe that the expected income of Adverse Nature equals $E\left(y_0 + y_1\theta + y_2\theta^2\right) = y_0 + y_1\mu + y_2\left(\mu^2+\sigma^2\right)$, which is the objective function in equation (2.3). In the outer maximization problem, while the firm chooses her action vector $\mathbf{Q}$, Adverse Nature determines the shadow prices to maximize the expected income subject to the limited liability constraints.

We use the subscript wst to indicate the worst case. When the firm plays an arbitrary strategy $\mathbf{Q}$, let $F_{wst}(\mathbf{Q})$ be the best response of Adverse Nature (or the firm's most unfavorable distribution). When the firm plays her equilibrium strategy $\mathbf{Q}^*$, it must hold that $F_{wst}(\mathbf{Q}^*) = F^*$. However, for any $\mathbf{Q} \neq \mathbf{Q}^*$, $F_{wst}(\mathbf{Q})$ may not be identical to F^*. Therefore, $Z_{wst}(\mathbf{Q}) = Z(\mathbf{Q}, F_{wst}(\mathbf{Q}))$ formulates the firm's objective function, in which the firm anticipates that Adverse Nature plays the best response $F_{wst}(\mathbf{Q})$ if she plays $\mathbf{Q}$. The first step of the traditional method is to explicitly derive $Z_{wst}(\cdot)$ and the second step is to optimize $Z_{wst}(\cdot)$. We briefly describe the first step. For any given $\mathbf{Q}$, the inner maximization problem in equation (2.3) is a linear SIP model such that KKT conditions are sufficient and necessary. If the number of binding constraints in equation (2.3) is finite, we can obtain the most unfavorable distribution using complementary slackness and generalized finite sequence. If the number of binding constraints is infinite, we can solve a differential equation to determine the most unfavorable distribution (Carrasco et al. 2018). Thus, we can guarantee that the equilibrium $(\mathbf{Q}^*, F^*)$ is attained rather than approached.

The success of the first step critically depends on the context of the model but inevitably affects the tractability of the second step. Unfortunately, in a range of circumstances, the objective function $Z_{wst}(\cdot)$ could be either analytically unavailable or overly complex. For example, Section 3 studies the $1+t$ model, in which Das et al. (2021) apply the traditional method and find that the explicit form of $Z_{wst}(q)$ is unavailable when q is large. Guo et al. (2022) advance the analysis by deriving a semi-closed form of $Z_{wst}(q)$ for large q. Whenever the analytical form of the objective function $Z_{wst}(q)$ is absent, the traditional method is unable to solve the robust solution q^* in closed form. In contrast, our new method overcomes this technical challenge from a new perspective, enabling us to derive a semi-closed form for q^*. To understand the importance of our approach, in Section 4, we develop a capacity planning model with $n \geq 1$ option contracts. When $n = 1$, the model reduces to Scarf's model where the ex post payoff function has two pieces. When $n = 2$, the ex post payoff function has three pieces. In the context of strangle option contracts, Natarajan and Zhou (2007) apply the traditional method and complete the first step to obtain the objective function $Z_{wst}(q_1, q_2)$ for any given pair of (q_1, q_2). However, Natarajan and Zhou have not optimized (q_1, q_2) because their goal is to evaluate the performance of any given strangle option contract. Figure 2 of Natarajan and Zhou (2007) illustrates the piece-wise objective function $Z_{wst}(q_1, q_2)$, which has 4 cases and the boundary of each case is given by non-trivial quadratic curves. When n increases, the number of cases will grow to 2^n, making the objective function $Z_{wst}(\mathbf{Q})$ too complex to be tractable. Without explicitly deriving $Z_{wst}(\mathbf{Q})$, the traditional method encounter difficulty in analyzing the case with an arbitrary $n \geq 1$. We by-pass this issue completely by focusing on finding the optimal $\mathbf{Q}^*$.

2.2. Adverse Nature's Perspective

The innovative feature of our method is that we bypass the obstacles in deriving $Z_{wst}(\cdot)$ and directly attack the equilibrium $(\mathbf{Q}^*, F^*)$. The crucial difference is that we solve the equilibrium from the perspective of Adverse Nature as follows:

$$\underbrace{\max_{Q\geq 0}\left\{\inf_{F\in\Omega}\int_0^{\infty} Z\left(\theta|Q\right)dF\left(\theta\right)\right\}}_{\text{Traditional Method}} \leq \underbrace{\inf_{F\in\Omega}\left\{\max_{Q\geq 0}\int_0^{\infty} Z\left(\theta|Q\right)dF\left(\theta\right)\right\}}_{\text{New Method}}. \tag{2.4}$$

The right hand side of inequality (2.4) formulates a robust moral hazard model, in which Adverse Nature jointly chooses a distribution and an action vector to minimize the firm's expected profit subject to the firm's IC constraints, which state that the chosen action vector must maximize the firm's expected profit. Under the assumption that $Z(\mathbf{Q}|F)$ is quasi-concave with respect to $\mathbf{Q}$, we can simplify the firm's IC constraint by using her FOCs: $\int_0^{\infty} \frac{\partial Z(\theta|\mathbf{Q})}{\partial Q_i} dF(\theta) = 0$, for $i = 1, 2, ..., n$. This first-order approach is prevalent in economics. We observe that the firm's FOCs become additional moment constraints and equation (2.4) changes to:

$$Z_1 = \inf_{\mathbf{Q}\geq \mathbf{0}, F\in\Omega}\left\{\int_0^{\infty} Z\left(\theta|\mathbf{Q}\right)dF\left(\theta\right)\right\},$$
$$\text{s.t.} \begin{cases} \int_0^{\infty} dF(\theta) = 1, \\ \int_0^{\infty} \theta dF(\theta) = \mu, \\ \int_0^{\infty} \theta^2 dF(\theta) = \mu^2 + \sigma^2, \\ \int_0^{\infty} \frac{\partial Z(\theta|\mathbf{Q})}{\partial Q_i} dF(\theta) = 0, \text{ for } i = 1, 2, ..., n. \end{cases} \tag{2.5}$$

Using duality, we can conveniently reformulate model Z_1 as the following SIP model:

$$P_1 = \max_{y_0, y_1, y_2, \mathbf{Q}, \mathbf{a}} \left\{y_0 + y_1\mu + y_2\left(\mu^2 + \sigma^2\right)\right\}$$
$$\text{s.t.} \quad y_0 + y_1\theta + y_2\theta^2 + \sum_{i=1}^{n} a_i \frac{\partial Z(\theta|\mathbf{Q})}{\partial Q_i} \leq Z(\theta|\mathbf{Q}), \forall\theta\geq 0, \tag{2.6}$$

where $\mathbf{a} = (a_1, a_2, ..., a_n)$ is the vector of Lagrangian multipliers associated with the firm's FOCs. We use $*$ to indicate the optimal objective value or optimal solution.

Contrasting model P in equation (2.3) with model P_1 in equation (2.6), we find that they both have the same objective function $y_0 + y_1\mu + y_2(\mu^2 + \sigma^2)$. Any solution that is feasible in model P is also feasible in model P_1 but the opposite is not true unless all $a_i = 0$. Thus, it must hold that $Z^* \leq Z_1^*$, which is consistent with the well-known min-max inequality (e.g., Sion 1958). Moreover, equality holds if and only if model P_1 encompasses an optimal solution satisfying $a_i^* = 0$ for all $i = 1, 2, ..., n$. Moreover, it is well-known that if $Z^* = Z_1^*$, then the zero-sum game in equation (2.1) has an equilibrium.

Theorem 1 *If the zero-sum game in equation (2.1) has an equilibrium, the following results hold: i)* $Z^* = Z_1^*$ *and ii) the SIP model* P_1 *in equation (2.6) must encompass an optimal solution satisfying the property that* $a_i^* = 0$ *for* $i = 1, 2, ..., n$.

Remark 1 *Model* P_1 *can be seen as an extended reformulation of model* P*, as it has an additional set of variables in* $a_i, i = 1, \ldots, n$. *This lifting into a higher dimensional space is beneficial for finding a closed form solution to the problem. To see this, consider the case* $Z = \min(\theta, Q)$. *In model* P_1*, for any fixed* $Q > 0$*, the constraints can be separated into two sets:*

$$\begin{cases} y_0 + y_1\theta + y_2\theta^2 & \leq \theta, \quad \forall \theta < Q, \\ y_0 + y_1\theta + y_2\theta^2 + a \leq Q, & \forall \theta \geq Q. \end{cases}$$

If $\theta_1 < Q$ *and* $\theta_2 > Q$ *are the points where the two constraints in* P_1 *are binding, together with the tangent conditions:*

$$\begin{cases} y_1 + 2y_2\theta_1 = 1, \\ y_1 + 2y_2\theta_2 = 0. \end{cases}$$

we have a system of four equations with four unknowns (in y_0, y_1, y_2 *and* a*). This allows us to express* a *in terms of* Q *and* θ_1, θ_2. *In equilibrium, we have* $a^* = 0$*, and hence we can compute the optimum* Q^* *as a function of* θ_1^* *and* θ_2^*. *The latter are the points with positive probability mass function (given by the value of the Lagrange multipliers to the constraints in* P_1*) for the worst case distribution, and can be pegged down in most cases using the moment and optimality conditions.*

Theorem 1 describes how to directly attack the equilibrium $(\mathbf{Q}^*, F^*)$ by avoiding the intermediate step of deriving $Z_{wst}(\cdot)$. The first derivative $\frac{\partial Z(\theta|\mathbf{Q})}{\partial Q_i}$ (or the marginal payoff), which appears in equation (2.6), plays a crucial role. This approach can be readily applied to derive closed-form solutions for numerous robust inventory management problems, including the well-known Scarf's model with a single product and information on the first and second moments. In the remaining sections, we focus on hitherto unknown closed-form solutions to new classes of inventory problems.

3. The First and t-th Moment

To assist readers in deepening their understanding of our method and to prepare for the capacity planning model in Section 4, we consider the following newsvendor model with a single capacity. The price of the product is $p > 0$ and the capacity reservation cost is $c \geq 0$ per unit. When the realized demand θ arrives, the firm uses the available capacity to satisfy the demand by incurring a processing cost $r \geq 0$. Any unused capacity expires and the reservation cost is not refunded. The demand in excess of the available capacity is lost. We find that the ex post payoff function equals $Z(\theta|q) = (p - r)\min(\theta, q) - cq$ and the first derivative satisfies $\frac{\partial Z(\theta|q)}{\partial q} = -c$ if $\theta < q$ and $\frac{\partial Z(\theta|q)}{\partial q} =$

$p-r-c$ if $\theta > q$. When $\theta = q$, $Z(\theta|q)$ is not differentiable. By default, we let $\frac{\partial Z(\theta|q)}{\partial q} = (p-r-c)$ if $\theta = q$ so that the first derivative and the cumulative distribution function are both right continuous. We slightly change the ambiguity set in equation (2.2) as follows:

$$\Omega = \left\{ F(\theta) \mid \int_0^\infty dF(\theta) = 1, \int_0^\infty \theta dF(\theta) = \mu, \int_0^\infty \theta^t dF(\theta) = m_t \right\}, \tag{3.1}$$

where m_t is known and represents the t-th moment of demand. In other words, we broaden the analysis to the case in which the mean and the t-th moment of the demand distribution (where $t > 1$ is a real number) are known.

Based on Theorem 1, we formulate the following SIP model:

$$\begin{aligned} & P_1 = \max_{q \geq 0, y_0, y_1, y_2} \{ y_0 + y_1 \mu + y_2 m_t \} \\ \text{s.t.} \quad & y_0 + y_1 \theta + y_2 \theta^t - ac \leq (p-r)\theta - cq, \forall \theta \in [0, q), \\ & y_0 + y_1 \theta + y_2 \theta^t + a(p-r-c) \leq (p-r-c)q, \forall \theta \geq q. \end{aligned} \tag{3.2}$$

Our new method enables us to rapidly solve the robust capacity level q^*. Before characterizing the equilibrium, we first characterize the two points θ_1 and θ_2 that make the constraints in (3.2) binding, using the following nonlinear equation:

$$H(x) = \frac{p-r-c}{p-r} x^t + \frac{c}{p-r} \left[x + \frac{p-r}{c}(\mu - x) \right]^t = m_t. \tag{3.3}$$

With $t > 1$, Jensen's inequality implies that $m_t \geq \mu^t$.

Lemma 1 *a) If*

$$\mu^t \leq m_t \leq \frac{\mu^t}{\left(\frac{c}{p-r}\right)^{t-1}}, \tag{3.4}$$

then equation (3.3) has a root θ_1 satisfying $0 \leq \theta \leq \mu$. Let $\theta_2 = \theta_1 + \left(\frac{p-r}{c}\right)(\mu - \theta_1)$ and define the following two-point distribution:

$$\begin{cases} \Pr\left(\tilde{\theta} = \theta_1\right) = \frac{p-r-c}{p-r}, \\ \Pr\left(\tilde{\theta} = \theta_2\right) = \frac{c}{p-r}. \end{cases} \tag{3.5}$$

The distribution in equation (3.5) satisfies the conditions on the mean and t-th moment.

b) If

$$m_t \geq \frac{\mu^t}{\left(\frac{c}{p-r}\right)^{t-1}} \tag{3.6}$$

then the root of equation (3.3) is negative. We let $\theta_h = \left(\frac{m_t}{\mu}\right)^{\frac{1}{t-1}}$ and define the following two-point distribution:

$$\begin{cases} \Pr\left(\tilde{\theta} = 0\right) = 1 - \frac{\mu}{\theta_h}, \\ \Pr\left(\tilde{\theta} = \theta_h\right) = \frac{\mu}{\theta_h}. \end{cases}, \tag{3.7}$$

which also satisfies the conditions on the mean and t-th moment.

3.1. Robust Inventory Level

The extant literature regards the $1+t$ model as notoriously difficult because the objective function $Z_{wst}(q)$ cannot be explicitly derived by the traditional method. For example, Das et al. (2021, pages 1097-1098) explain the "implausibility" in deriving $Z_{wst}(q)$. Similarly, Guo et al. (2022, page 14) suggest that a semi-closed form for $Z_{wst}(q)$ is "probably the best one can hope for". When the explicit expression of the objective function $Z_{wst}(q)$ remains analytically unavailable, the first derivative $\frac{\partial Z_{wst}(q)}{\partial q}$ is unknown, preventing the extant literature from solving the equilibrium strategy q^* in closed form. In contrast, our new method overcomes the relevant technical challenge.

Proposition 1 *In the $1+t$ model formulated in equation (3.2), the equilibrium is the following:*

a) If condition (3.4) holds, the firm's equilibrium strategy q^* satisfies that

$$q^* = \theta_1 + \frac{\theta_2^{t-1}(\theta_2-\theta_1)}{\theta_2^{t-1}-\theta_1^{t-1}} - \frac{\theta_2^t-\theta_1^t}{t\left(\theta_2^{t-1}-\theta_1^{t-1}\right)} = \frac{t-1}{t}\frac{\theta_2^t-\theta_1^t}{\theta_2^{t-1}-\theta_1^{t-1}}, \tag{3.8}$$

while Adverse Nature's equilibrium strategy satisfies equation (3.5). Thus, the value of the zero-sum game equals $P_1^* = (p-r-c)\,\theta_1$.

b) If condition (3.6) holds, the firm's equilibrium strategy satisfies $q^*=0$ while Adverse Nature's equilibrium strategy satisfies equation (3.7). Thus, the value of the zero-sum game equals $P_1^*=0$.

Proposition 1 highlights the key advantage of our method over the extant literature. We can derive q^* despite that the objective function $Z_{wst}(q)$ could be analytically unavailable. For instance, when $t=3$, we let $d=\frac{c}{(p-r)}$ to simplify equation (3.3) as follows:

$$\begin{aligned} H(x) &= (1-d)\,x^3 + d\left(x+\frac{1}{d}\mu-\frac{1}{d}x\right)^3 \\ &= x^3\left(-\frac{1}{d^2}+\frac{3}{d}-2\right)+3\mu x^2\left(\frac{1}{d^2}-\frac{2}{d}+1\right)+3\mu^2 x\left(-\frac{1}{d^2}+\frac{1}{d}\right)+\frac{\mu^3}{d^2}=m_3. \end{aligned}$$

When $d=0.5$ (implying that $p-r=2c$), the above equation becomes the following quadratic equation: $3\mu x^2-6\mu^2 x+4\mu^3-m_3=0$, yielding $\theta_1=\mu-\frac{\sqrt{3\mu\left(m_3-\mu^3\right)}}{3\mu}$ and $q^*=\frac{2}{3}\frac{m_3+8\mu^3}{6\mu^3}$. When $d\neq 0.5$, we can use software to solve the following equivalent cubic equation:

$$x^3(2d-1)(1-d)+3\mu x^2(1-d)^2-3\mu^2 x(1-d)+\mu^3-m_3 d^2=0$$

to obtain θ_1 and then q^*.

Proposition 1 also enriches our understanding about the effect of heavy-tail (or light-tail) behaviors on inventory planning. For example, because $H(\theta_1)=m_t$ and $H(\cdot)$ is decreasing, we observe that the value of the zero-sum game decreases in m_t. When $t=2$, Proposition 1 becomes Scarf's model. We explain the relevant details in Part A of Online Appendix.

Proposition 1 reveals an important property that the firm's worst-case distribution F^* exhibits. If the ex post profit function is piece-wise linear and has *two* pieces, then F^* must be a *two*-point distribution with realizations θ_1^* and θ_2^* (where $\theta_1^* \leq \theta_2^*$). This two-point distribution not only satisfies the moment constraints but also allocates a mass probability to the low realization θ_1^* according to the firm's newsvendor ratio (i.e., $\Pr\left(\tilde{\theta} = \theta_1^*\right)$ equals her newsvendor ratio). In general, to identify a two-point distribution, we need to determine four parameters (i.e., two probability masses and two realized values). Because the two probability masses are based on the newsvendor ratio, we only need to determine the two realized values by applying the moment conditions. The relevant intuition becomes clearer if we examine the equilibrium from the perspective of Adverse Nature. When the ex post profit function is piece-wise linear and has only two pieces, the marginal ex post payoff (i.e., the first derivative $\frac{\partial Z(\theta|q)}{\partial q}$) takes only two possible values. To randomize the demand, the optimal strategy for Adverse Nature must be choosing between two realized values θ_1 and θ_2 such that θ_1 gives the firm a lower marginal ex post payoff and θ_2 gives the firm a higher marginal ex post payoff. This explains why Proposition 1 converges to a two-point distribution. On the other hand, Adverse Nature anticipates that the firm must play her optimal response, which follows her newsvendor ratio. Therefore, when determining how frequently to choose the low realized value θ_1, Adverse Nature also follows the firm's newsvendor ratio, explaining the equilibrium strategy F^* in equation (3.5).

An early criticism of Scarf's result is that the worst-case distribution is a two-point distribution. However, a two-point distribution is the most natural response of Adverse Nature because the marginal payoff takes only two values in the standard newsvendor model. The relevant intuition can also be generalized to the capacity model in Section 4. Specifically, when the ex post payoff function is concave piece-wise linear and has $(n+1)$ pieces (suggesting that the marginal payoff takes $(n+1)$ values), the equilibrium strategy played by Adverse Nature is characterized by a $(n+1)$-point distribution. The probability mass to be allocated to each point depends on the firm's best response. Certainly, mean and variance alone are insufficient to specify the relevant $(n+1)$-point distribution when $n \geq 2$ and hence, additional steps are required before deriving the closed-form solution.

4. Capacity Planning Model

Sourcing strategy is critical for many firms to excel in the current volatile business environment. Despite low administrative costs, single sourcing can expose firms to significant risks caused by demand or supply uncertainties. To gain a competitive advantage, many firms diversify their supply chains to enhance their flexibility to cope with uncertainties. For example, Hewlett-Packard is a pioneer that uses option contracts to manage the supply of memory devices (Fu et al. 2010). Each option contract specifies the premium (or the reservation fee) and the exercising fee for each memory device. After observing the realized demand, Hewlett-Packard determines which option contracts to exercise. Any unexercised option contract expires but the exercising fee is avoided. If demand exceeds the total quantities that can be satisfied under the purchased option contracts, the excess demand is lost when the required parts cannot be replaced by the standard parts from the spot market.[1] The second example is E.ON (`www.eonenergy.com`), one of the leading energy companies in the United Kingdom. E.ON reserves capacity from fossil fuel generators, wind farms, and nuclear generators. When electricity demand arrives, E.ON uses the reserved capacity to generate electricity. There could be additional costs of generating electricity from the use of reserved generators and the transmission of the generated electricity to external customers. These additional costs could be source dependent and are incurred only in the fulfillment stage. A minor difference is that the excess demand for electricity can be satisfied by the spot market rather than lost. The third example is Xerox Australia, which recycles and remanufactures photocopiers (Kerr and Ryan 2001). When photocopiers are returned to Xerox after the end of lease contracts, Xerox inspects their condition and sorts them into four grades: grade 1 (suitable for refurbishment), grade 2 (suitable for reprocessing), grade 3 (suitable for remanufacturing), and grade 4 (suitable for asset recovery or disposal). After an order for used photocopiers arrives, Xerox implements a priority rule that exhausts an alphabetically lower grade (which has a better condition) before using another grade. Grade 1 and grade 2 are cleaned and repaired. High-frequency-service parts are replaced regardless of their condition or use. Other parts are replaced depending on their condition and expected remaining life. Grade 3 and grade 4 photocopiers must undergo a labor-intensive disassembly process. Good-quality parts are cleaned, tested, and reconditioned. Some photocopiers are then reassembled, while others are sent without reassembly to a disposal facility. Customers who purchase used photocopiers are unable to discern the initial grade of the machines. If the quantity of used photocopiers is insufficient, the demand is lost or the customer is persuaded to buy other machines.

4.1. The Robust Capacity Planning Model

The operations of Hewlett-Packard, E.On, and Xerox Australia share two common characteristics: i) the firm can access $n \geq 1$ supply sources that are substitutes for each other, and ii) the firm makes the procurement and fulfillment decisions in two stages. Accordingly, we develop the following inventory model. The firm faces an uncertain demand and procures from n sources, where each source can represent a different supplier, grade, or option contract. Let $\tilde{\theta}$ be a nonnegative random variable denoting the external demand for the firm's end product. The cumulative distribution function of demand θ is $F(\theta)$. We make a critical departure from the extant literature (e.g., Martínez-de Albéniz and Simchi-Levi 2009, Fu et al. 2010) by assuming that $F(\theta)$ is *unknown*. We assume that the firm knows only the mean ($\mu > 0$) and the variance ($\sigma^2 > 0$) associated with demand $\tilde{\theta}$. Without complete knowledge of the demand distribution, the firm applies distributionally robust approach to creating her capacity plan.

The sourcing decisions proceed in two stages. In Stage 1 (which we refer to as the procurement stage before the selling season), the firm reserves q_i units of capacity from each source i ($i = 1, 2, \cdots, n$) by paying a reservation cost c_i per unit. In Stage 2 (which we refer to as the fulfillment stage), the external demand θ is realized, and the firm uses available capacities to produce the end products. When using the capacity of source i to produce one unit of end product, the firm incurs an additional processing cost $r_i \geq 0$. The end products delivered from each source have the same quality and functionality so that every external customer regards them as indistinguishable. By satisfying one unit of demand, the firm receives the same amount of revenue p from each external customer. If the reserved capacities are insufficient to satisfy all of the external demand, the excess demand is lost.[2] Any unused capacities expire without any salvage value but the processing cost is also avoided. For expositional simplicity, we define source-$(n+1)$ as an artificial source representing lost sales such that $c_{n+1} = 0$ and $r_{n+1} = p$. We use a bold letter to represent a vector. Let $\mathbf{q} = (q_i)$ be the capacity vector (where $i = 1, 2, \cdots, n$) chosen by the firm in the procurement stage.

We refer to all supply sources as a portfolio. Before introducing the assumptions on cost parameters, we define two sequences as follows.

Definition 1 *Let $\alpha_0 = 0$ and $\alpha_{n+1} = 1$. Define*

$$\begin{cases} \alpha_i = 1 - \frac{c_i - c_{i+1}}{r_{i+1} - r_i}, & \text{for } i = 1, 2, \cdots, n, \\ \beta_i = \alpha_i - \alpha_{i-1}, & \text{for } i = 1, 2, \cdots, n+1. \end{cases} \tag{4.1}$$

For ease of exposition and given the need to eliminate unattractive supply sources, we assume that the sequence α_i is strictly increasing in i, which is consistent with the extant literature (e.g., Martínez-de Albéniz and Simchi-Levi 2009, Fu et al. 2010) and ensures that in the robust optimal solution, the order quantity for each supply source is positive. Otherwise, some supply sources will never be used and can be eliminated from the analysis.

Remark 2 *If $\{\alpha_i\}$ is increasing in i, then the following results hold: i) $c_1 + r_1 < c_2 + r_2 < \cdots < c_n + r_n < p$ and ii) $r_1 < r_2 < \cdots < r_n$ and $c_1 > c_2 > \cdots > c_n$.*

We omit the proof of Remark 2 and refer readers to Lemma 1 of Fu et al. (2010). We can visualize the portfolio by using the reservation cost c_i as the vertical coordinate and the processing cost r_i as the horizontal coordinate. Remark 2 implies that the path of the portfolio (which starts from supply source 1 and connects all sources including the artificial source-$(n+1)$ that represents lost sales) is convex decreasing. Remark 2 also implies that if the capacity vector is fixed, the firm will use a priority rule to use the available capacity to satisfy the realized demand. Specifically, source $(i+1)$ will not be used unless the reserved capacity of source i is exhausted.

Let $Q_i = \sum_{j=1}^{i} q_i$ be the total capacity of the first i sources. By default, we let $Q_0 = 0$ such that $Q_i \geq Q_{i-1}$, which we refer to as the monotonicity constraints on Q_i. Because the mapping between vectors $\mathbf{Q} = (Q_i)$ and $\mathbf{q} = (q_i)$ is unique, hereafter, we regard Q_i as the decision variables to facilitate analysis. Let $(\cdot)^+ = \max(0, \cdot)$.

Lemma 2 *For any given capacity vector* $\mathbf{Q}$*, the firm's ex post profit equals*

$$Z(\theta|\mathbf{Q}) = \sum_{i=1}^{n} \left[(p - r_i) \left(\min(Q_i, \theta) - \min(Q_{i-1}, \theta) \right) - c_i (Q_i - Q_{i-1}) \right], \tag{4.2}$$

which is continuous, concave, and increasing in the realized demand θ*.*

The $Z(\theta|\mathbf{Q})$ function in equation (4.2) is piece-wise and has $(n+1)$ different cases, depending on the value of the realized demand θ. Let $\delta_i = \frac{Z(\theta|\mathbf{Q})}{\partial Q_1}$ be the first derivative for the ith such case. The proof of Lemma 2 shows that the sequence $\{\delta_i\}$ satisfies the following recursive equation:

$$\delta_i = r_i - r_1 - c_1, \text{ for } i = 1, 2, \cdots, n+1. \tag{4.3}$$

The two sequences $\{\beta_i\}$ and $\{\delta_i\}$ determine the equilibrium. We refer to δ_i as the marginal impact and provide the managerial interpretation of δ_i after we derive the robust capacity plan.

4.2. Adverse Nature's Model

To apply Theorem 1, we need to simplify the relevant IC constraints. Assume that the demand distribution $F(\theta)$ is known. The firm's expected profit equals $Z(\mathbf{Q}) = \int_0^\infty Z(\theta|\mathbf{Q})\, dF(\theta)$. Let $\tilde{\mathbf{Q}}$ be the firm's distribution-dependent optimal capacity plan.

Lemma 3 *In the benchmark without ambiguity, the firm's expected profit* $Z(\mathbf{Q})$ *is concave in* $\mathbf{Q}$*, and her optimal capacity vector* $\tilde{\mathbf{Q}}$ *satisfies* $F\left(\tilde{Q}_i\right) = \alpha_i$*, implying that the firm's optimal capacity level for source-*i *equals*

$$\tilde{q}_i = F^{-1}(\alpha_i) - F^{-1}(\alpha_{i-1}) \text{ for any } i = 1, 2, \cdots, n, \tag{4.4}$$

where F^{-1} *is the inverse function of* F*.*

Lemma 3 indicates that the optimal capacity plan can be described by a sequence of percentiles, explaining why the cumulative probability α_i must be increasing in i to ensure that $\tilde{q}_i > 0$. In the proof of Lemma 3 (we refer readers to equation (B-7) in the Appendix), we expand the firm's expected profit as the sum of n separate identities. Each of these n identities involves only one Q_i (which is the total quantity of the first i sources) and is concave in Q_i. The definition of Q_i must imply that $Q_i \geq Q_{i-1}$. After relaxing the monotonic constraint on Q_i and solving the FOC, we obtain that $F\left(\tilde{Q}_i\right) = \alpha_i$. Thus, if α_i is monotonic in i, then the candidate solution $\tilde{Q}_i$ is increasing in i, making $\tilde{q}_i = \tilde{Q}_i - \tilde{Q}_{i-1}$ optimal and positive. If $\alpha_i \leq \alpha_{i-1}$, then the monotonicity constraint $Q_i \geq Q_{i-1}$ must be binding, making $q_i = Q_i - Q_{i-1} = 0$ (implying that source i is not used in the optimal solution). We emphasize that the binding status of the monotonicity constraint $Q_i \geq Q_{i-1}$ depends on the monotonicity of α_i rather than on the demand distribution F. Thus, we can apply the process of elimination to reduce the number of supply sources.

The concavity of $Z(\mathbf{Q})$ shown in Lemma 3 indicates that we can replace the IC constraints with the FOCs. However, for ease of analysis (i.e., $\frac{Z(\theta|\mathbf{Q})}{\partial Q_i}$ is rather complex) and the need to involve the mass probabilities $\{\beta_i\}$, we use a different form of FOCs as follows: $F(Q_i) = \alpha_i$, where α_i is uniquely determined by exogenous cost parameters according to Definition 1. The constraints $F(Q_i) = \alpha_i$ are equivalent to

$$\int_{Q_{i-1}}^{Q_i} dF(\theta) = \beta_i, \text{ for } i = 1, 2, \cdots, n, \tag{4.5}$$

which remain to be moment constraints.[3] Thus, the SIP model in equation (2.6) becomes:

$$\begin{aligned} P_1 = & \max_{\substack{y_0, y_1, y_2, \mathbf{Q} \\ a_1, a_2, \cdots, a_n}} \left\{ a_1\beta_1 + a_2\beta_2 + \cdots + a_n\beta_n + y_0 + y_1\mu + y_2\left(\mu^2 + \sigma^2\right) \right\} \\ \text{s.t.} \quad & \begin{cases} a_i + y_0 + y_1\theta + y_2\theta^2 \leq Z(\theta|\mathbf{Q}), \forall\theta \in [Q_{i-1}, Q_i], i = 1, 2, \cdots, n, \\ y_0 + y_1\theta + y_2\theta^2 \leq Z(\theta|\mathbf{Q}), \forall\theta \geq Q_n. \end{cases}, \end{aligned} \tag{4.6}$$

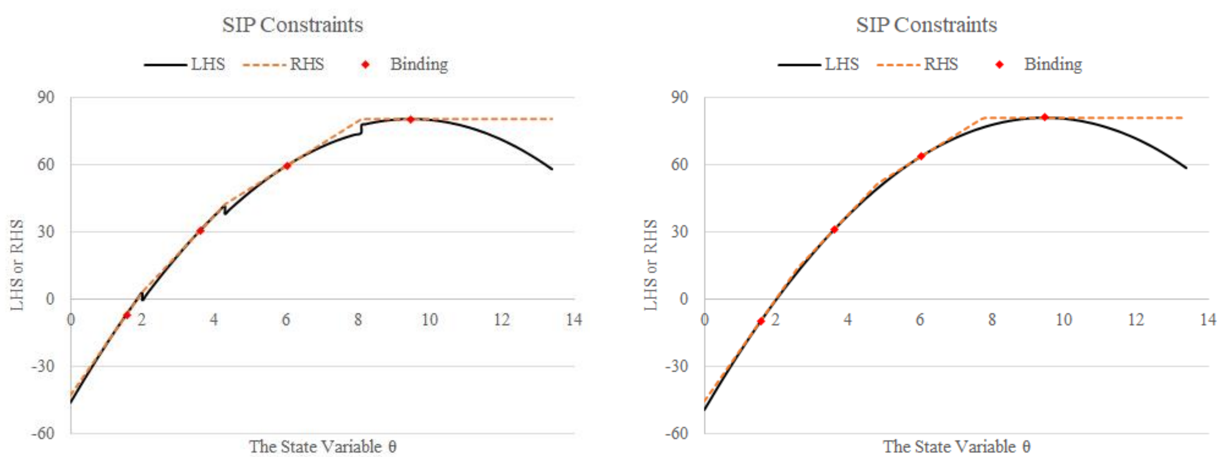
where a_i is the shadow price for the i-th moment constraint in equation (4.5). Contrasting the two SIP models in equations (2.3) and (4.6), we observe that Theorem 1 implies that one of the optimal solutions for model P_1 satisfies $a_i = 0$ and gives the same optimal objective value for both models P and P_1.

4.3. Worst-Case Distribution

To solve the equivalent SIP model in equation (4.6), we first conjecture the binding SIP constraints and then apply a relaxation method. Figure 1a) visualizes the SIP constraints in equation (4.6) by showing that the left-hand side of the constraints is quadratic and piece-wise in θ (specifically, there exist $(n+1)$ pieces with n jumps due to $a_1, a_2, \cdots, a_n$). On the other hand, the right-hand side

(RHS) of the constraints is continuous, concave, and piece-wise linear in θ as Lemma 2 suggested. We conjecture that there exist $(n+1)$ binding constraints at points labeled as θ_i $(i=1,2,\cdots,n+1)$ such that the first n points satisfy $\theta_i \in [Q_{i-1}, Q_i]$ and the last point satisfies $\theta_{n+1} \geq Q_n$. The condition $\theta_i \in [Q_{i-1}, Q_i]$ is consistent with Proposition 3 and incorporates the monotonicity constraints on $\mathbf{Q}$. Based on the conjectured binding constraints, we derive the relaxed solution for equation (4.6). The final step is to verify that the relaxed solution satisfies the omitted constraints and hence is optimal for equation (4.6).

Figure 1 SIP constraints



(a) When a Nonrobust $\mathbf{Q}$ Vector is Used

(b) When the Robust Vector $\mathbf{Q}^*$ is Used

Before presenting the major results, we define an important distribution as follows.

Definition 2 *Let Δ be a positive constant satisfying $\Delta = \sqrt{\sum_{i=1}^{n+1} \beta_i \delta_i^2}$, where $\{\beta_i\}$ is the mass probability sequence given by Definition 1 and $\{\delta_i\}$ is the coefficient sequence given by equation (4.3). Let*

$$\theta_i^* \overset{def}{=} \mu + \delta_i \frac{\sigma}{\Delta}, \text{ for } i = 1, 2, \cdots ., n+1 \tag{4.7}$$

be the i-th possible realization of $\tilde{\theta}$. Using $\{\theta_i^*\}$ and $\{\beta_i\}$, we define the following distribution:

$$\Pr(\theta = \theta_i^*) = \beta_i, \text{ for } i = 1, 2, \cdots, n+1, \tag{4.8}$$

which is a discrete $(n+1)$-point distribution.

We assume $(p - r_1 - c_1)\mu \geq \Delta\sigma$ to avoid the uninteresting case in which the firm ceases operations under mean-variance ambiguity. Because we require demand to be nonnegative, we also assume that $\theta_1^* = \mu - c_1 \frac{\sigma}{\Delta} \geq 0$.

Corollary 1 *i) It holds that* $\sum_{i=1}^{n+1} \beta_i \delta_i = 0$. *ii) The* $(n+1)$*-point discrete distribution in equation (4.8) is one of the feasible distributions in the ambiguity set* Ω.

The next proposition identifies the firm's worst-case demand distribution.

Proposition 2 *The* $(n+1)$*-point distribution in equation (4.8) is the firm's worst-case demand distribution* F^*.

Proposition 2 significantly advances our analysis by paving the way toward determining the firm's worst-case expected profit. Using the firm's worst-case demand distribution F^*, we can compute the firm's expected profit $Z(\mathbf{Q}|F^*)$.

Proposition 3 *There exist an infinite number of* $\mathbf{Q}$ *vectors that maximize the firm's expected profit* $Z(\mathbf{Q}|F^*)$. *However, the firm's optimal expected profit under her worst-case demand distribution* F^* *is unique and equals*

$$Z^* = \max_{\mathbf{Q} \geq 0} Z(\mathbf{Q}|F^*) = (p - r_1 - c_1)\mu - \Delta\sigma. \tag{4.9}$$

Proposition 3 shows that the firm's optimal worst-case expected profit Z^* has a clean and neat form. Observe that $(p - r_1 - c_1)$ is the understock cost of source-1 capacity, which is the firm's most profitable source. Equation (4.9) reveals that Z^* is increasing in the mean of demand and the understock cost of source-1 capacity and is decreasing in the constant Δ and the standard deviation of demand.

4.4. Robust Optimal Capacity Vector

Because optimizing $Z(\mathbf{Q}|F^*)$ does not result in a unique capacity vector, our remaining challenge is to solve the robust optimal capacity vector $\mathbf{Q}^*$ using Theorem 1.

Proposition 4 *The firm's robust optimal capacity vector satisfies* $\mathbf{Q}^* = (Q_i^*) = \left(\frac{\theta_i^* + \theta_{i+1}^*}{2}\right)$ *for* $i = 1, 2, \cdots, n$, *implying that the robust optimal capacity level for source-*i *capacity, for* $i \geq 2$, *equals*

$$q_i^* = \frac{\theta_i^* + \theta_{i+1}^*}{2} - \frac{\theta_{i-1}^* + \theta_i^*}{2} = \frac{\theta_{i+1}^* - \theta_{i-1}^*}{2} = \frac{\delta_{i+1} - \delta_{i-1}}{2} \frac{\sigma}{\sqrt{\sum_{i=1}^{n+1} \beta_i \delta_i^2}}. \tag{4.10}$$

Proposition 4 solves the robust optimal capacities in closed form by demonstrating that in the robust optimal solution, the total capacities of the first i sources equal the midpoint of the closed interval $\left[\theta_i^*, \theta_{i+1}^*\right]$. In particular, in the traditional setting where the fixed price contract has parameters $r_1 = 0, c_1 = c$, and $n = 1$, we obtain

$$q_1^* = \mu + \frac{(p - 2c)\sigma}{2\sqrt{(p - c)c}}.$$

Hence the amount procured from the fixed price contract, under the robust model, exceeds the mean demand only when the spot price p is at least twice the fixed price c.

Corollary 2 *The optimal solution for equation (2.3) satisfies* $y_0^* = -\frac{\Delta}{2\sigma}(\mu^2+\sigma^2)$, $y_1^* = (p-r_1-c_1)+\Delta\frac{\mu}{\sigma}$, $y_2^* = -\frac{\Delta}{2\sigma}$, *and* $Q_i^* = \left(\frac{\theta_i^*+\theta_{i+1}^*}{2}\right)$.

Corollary 2 is a direct result of Propositions 2 to 4. With all $a_i^*=0$, the left-hand side of the SIP constraints in equation (2.3) becomes one smooth piece of a quadratic curve (see Figure 1 b) for an illustration). The discrete distribution in equation (4.8) is critical for unlocking all the results. Because we can ex ante construct the sequences $\{\beta_i\}$ and $\{\delta_i\}$ by using exogenous cost parameters, the robust optimal capacity vector $\mathbf{Q}^*$ is easy to compute.

Recall that 1) the mass probabilities $\{\beta_i\}$ are based on the unambiguous solution and exogenous cost parameters and 2) the marginal impact $\delta_i = \frac{Z(\theta|\mathbf{Q})}{\partial Q_1}$ is the first derivative with respect to Q_1 in the i-th case. The cost coefficients $\{\delta_i\}$ have an interesting managerial interpretation. In our model, the firm can access $n\geq 1$ sources with source 1 being her most preferred source. If demand is deterministic, the firm can use only source 1 in her capacity plan. However, due to demand uncertainty, the firm chooses more sources with lower reservation costs even if the sum of the reservation and processing costs increases. In the i-th case, the realized demand satisfies that $Q_{i-1}\leq\theta<Q_i$ (where by default $Q_0=0$), source-i still has some unused capacity. If the firm increases the capacity of source-1 from q_1 to $q_1+\varepsilon$ and keeps all the other q_i's ($i\geq 2$) unchanged, then the sales quantity of source-1 increases by ε while that of source-i deceases by ε units (where $\varepsilon\in(0,Q_i-\theta)$ is a small positive number). The marginal impact includes two parts: 1) the *change* in the sales revenue due to an additional ε units of source-1 capacity and 2) the reservation cost of an additional ε units of source-1 capacity. The net impact equals

$$(p-r_1)-(p-r_i)-c_1 = r_i - r_1 - c_1 \stackrel{def}{=} \delta_i.$$

In a notable special case where $i=(n+1)$, the firm suffers lost sales when the artificial source $(n+1)$ is used. With $r_{n+1}=p$ and $c_{n+1}=0$, we observe that $\delta_{n+1}=r_{n+1}-r_1-c_1=p-r_1-c_1>0$.

5. Component Commonality

The risk-pooling effect is an important topic in the supply chain literature. For example, Bimpikis and Markakis (2016) study the effect of heavy-tailed demands on risk-pooling while Govindarajan et al. (2021) study a multi-location model with transshipment. The risk-pooling effect involves multi-dimensional random variables rather than the single-dimensional random variable that Sections 3 and 4 consider. To demonstrate that our method is capable of solving multi-dimensional problems, we revisit the inventory model that Baker et al. (1986) study.

The firm manufactures two products: product 1 and product 2. Product 1 requires one unit of product-specific component 1 and one unit of common component 0 while product 2 requires one unit of product-specific component 2 and one unit of common component 0. The production cost of component i is c_i $(i=0,1,2)$ and the selling price of product j $(j=1,2)$ is p_j. By operating an assemble-to-order system, the firm pre-stocks q_i $(i=0,1,2)$ units of component i (where $q_1+q_2 \geq q_0$) and then assembles the final products only after receiving the realized demands (θ_1, θ_2), where θ_j is the realized demand for product j. The available information includes: i) the mean of the demand for product j equals $\mu_j = E\left(\tilde{\theta}_j\right)$, ii) the variance of the demand for product j equals $\sigma_j^2 = Var\left(\tilde{\theta}_j\right)$, and iii) the correlation coefficient of $\tilde{\theta}_1$ and $\tilde{\theta}_2$ equals $\rho \in [-1,1]$.

5.1. Symmetric Case

To facilitate closed-form solution, we consider a special case with the following symmetric parameters: i) the production costs are $c_1 = c_2 = c$ while c_0 may not be identical to c; ii) the prices $p_1 = p_2 = p$, $\mu_1 = \mu_2 = \mu$, and $\sigma_1 = \sigma_2 = \sigma$.

At the assembly stage, the production cost of all the components equals $c_1 q_1 + c_2 q_2 + c_0 q_0$ and is sunk. When the realized demands are (θ_1, θ_2), the firm assembles s_j units of product j to maximize her total sales revenue by solving the following linear programming model:

$$\begin{aligned} Z\left(\theta_1, \theta_2 | q_1, q_2, q_0\right) &= \max_{s_1, s_2 \geq 0} \left\{ps_1 + ps_2\right\} - cq_1 - cq_2 - c_0 q_0 \\ \text{s.t.} \quad & s_j \leq \min\left(\theta_j, q_j\right) \text{ and } s_1 + s_2 \leq q_0. \end{aligned}$$

When the supply of component 0 is inadequate, we give a higher priority to product 1. We obtain the marginal payoffs in the following Table 2 and illustrate the four regions associated with the realized demands (θ_1, θ_2) in Figure 2.

Table 2 The marginal payoff $\frac{\partial Z(\theta_1, \theta_2)}{\partial q_i}$

Circumstance	$\frac{\partial Z}{\partial q_1}$	$\frac{\partial Z}{\partial q_2}$	$\frac{\partial Z}{\partial q_0}$	$Z\left(\theta_1, \theta_2 \| q_1, q_2, q_0\right)$
a) $\theta_1 \leq q_1$, $\theta_2 \leq q_2$, $\theta_1 + \theta_2 \leq q_0$	$-c$	$-c$	$-c_0$	$p\theta_1 + p\theta_2$ $-cq_1 - cq_2 - c_0 q_0$
b) $\theta_1 \leq q_1$, $\theta_2 > q_2$, $\theta_1 + q_2 \leq q_0$	$-c$	$p-c$	$-c_0$	$p\theta_1 + pq_2$ $-cq_1 - cq_2 - c_0 q_0$
c) $\theta_1 > q_1$, $\theta_2 \leq q_2$, $q_1 + \theta_2 \leq q_0$	$p-c$	$-c$	$-c_0$	$pq_1 + p\theta_2$ $-cq_1 - cq_2 - c_0 q_0$
d) $\min(q_1, \theta_1) + \min(q_2, \theta_2) \geq q_0$	$-c$	$-c$	$p - c_0$	$pq_1 + p\left(q_0 - q_1\right)$ $-cq_1 - cq_2 - c_0 q_0$

The dual model of Adverse Nature is the following:

$$D = \inf_F \max_{q_i \geq 0} \left\{ \int_0^\infty \int_0^\infty Z\left(\theta_1, \theta_2 | q_1, q_2, q_0\right) \right\}$$

$$
\begin{aligned}
\text{s.t.} \quad & \int_0^\infty \int_0^\infty dF(\theta_1,\theta_2) = 1 \\
& \int_0^\infty \int_0^\infty \theta_j dF(\theta_1,\theta_2) = \mu \\
& \int_0^\infty \int_0^\infty \theta_j^2 dF(\theta_1,\theta_2) = \mu^2+\sigma^2 \\
& \int_0^\infty \int_0^\infty \theta_1\theta_2 dF(\theta_1,\theta_2) = \rho\sigma^2+\mu^2 \\
& \int_0^\infty \int_0^\infty \frac{\partial Z(\theta_1,\theta_2|q_1,q_2,q_0)}{\partial q_i} dF(\theta_1,\theta_2) = 0,
\end{aligned}
\tag{5.1}
$$

We clarify the two limiting cases with $\rho=-1$ and $\rho=+1$ in Part C of Online Appendix.

Initially, the objective function of the corresponding SIP model is the following:

$$P_1 = \max\left\{y_0 + y_{11}\mu + y_{12}\mu + y_{21}\left(\mu^2+\sigma^2\right) + y_{22}\left(\mu^2+\sigma^2\right) + y_3\left(\rho\sigma^2+\mu^2\right)\right\},$$

where the decision variables include y_0, y_{11}, y_{12}, y_{21}, y_{22}, y_3, a_i, and q_i. Symmetry yields that $q_1=q_2=q$, $y_{11}=y_{12}=y_1$, and $y_{21}=y_{22}=y_2$, where q, y_1 and y_2 are to be determined. Thus, we can simplify the objective function as follows:

$$P_1 = \max_{y_i,q,q_0,a_i}\left\{y_0 + 2y_1\mu + 2y_2\left(\mu^2+\sigma^2\right) + y_3\left(\rho\sigma^2+\mu^2\right)\right\}.$$

Next, we formulate the SIP constraints in four cases by referring to Table 2. When circumstance a) occurs, the SIP constraints are the following:

$$y_0 + y_1\theta_1 + y_1\theta_2 + y_2\theta_1^2 + y_2\theta_2^2 + y_3\theta_1\theta_2 - a_0c_0 - a_1c - a_2c \le p\theta_1 + p\theta_2 - 2cq - c_0q_0.$$

When circumstance b) occurs, it gives rise to the following SIP constraints:

$$y_0 + y_1\theta_1 + y_1\theta_2 + y_2\theta_1^2 + y_2\theta_2^2 + y_3\theta_1\theta_2 - a_0c_0 - a_1c + a_2(p-c) \le p\theta_1 + pq - 2cq - c_0q_0.$$

When circumstance c) occurs, it gives rise to the following SIP constraints:

$$y_0 + y_1\theta_1 + y_1\theta_2 + y_2\theta_1^2 + y_2\theta_2^2 + y_3\theta_1\theta_2 - a_0c_0 + a_1(p-c) - a_2c \le pq + p\theta_2 - 2cq - c_0q_0.$$

When circumstance d) occurs, it gives rise to the following SIP constraints:

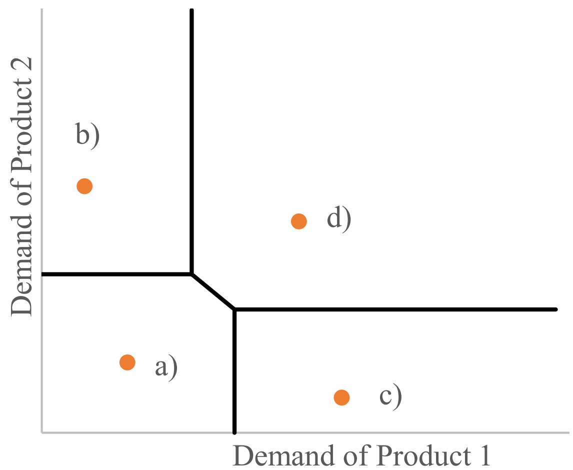
$$y_0 + y_1\theta_1 + y_1\theta_2 + y_2\theta_1^2 + y_2\theta_2^2 + y_3\theta_1\theta_2 + a_0(p-c_0) - a_1c - a_2c \le pq_0 - 2cq - c_0q_0.$$

Because there exist four different vectors of marginal payoffs, the number of binding constraints is four. Due to symmetry, the coordinates of these four binding points are conjectured to be the following: (x_1,x_1), point (x_2,x_3), point (x_3,x_2), and point (x_4,x_4), where $x_1\le q$, $x_2\le q_0-q$, $x_3\ge q$, and $x_4 \geq q$. Let L be the Lagrangian based on four binding constraints. The FOCs with respect to a_i include:

$$\frac{\partial L}{\partial a_0} = -c_0 (\lambda_1 + \lambda_2 + \lambda_3) + (p - c_0) \lambda_4 = 0.$$
$$\frac{\partial L}{\partial a_1} = -c (\lambda_1 + \lambda_3 + \lambda_4) + (p - c) \lambda_2 = 0,$$
$$\frac{\partial L}{\partial a_2} = -c (\lambda_1 + \lambda_2 + \lambda_4) + (p - c) \lambda_3 = 0,$$

Using the total probability ($\frac{\partial L}{\partial y_0} = 1 - \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 = 0$) and the above 3 equations, we can immediately obtain that $\lambda_1 = 1 - \frac{2c+c_0}{p}$, $\lambda_2 = \lambda_3 = \frac{c}{p}$, and $\lambda_4 = \frac{c_0}{p}$. These four Lagrangian multipliers determine the probability masses that Adverse Nature allocates to the four binding points.

Figure 2 Four Circumstances Related to Table 2



We define the following identities:

$$x_1^* = \mu - (c + c_0) \sigma \sqrt{\frac{(1+\rho)}{A}},$$
$$x_2^* = \mu + \left(\frac{p}{2} - c - c_0\right) \sigma \sqrt{\frac{(1+\rho)}{A}} - \frac{\sigma}{2} \sqrt{\frac{(1-\rho)p}{c}},$$
$$x_3^* = \mu + \left(\frac{p}{2} - c - c_0\right) \sigma \sqrt{\frac{(1+\rho)}{A}} + \frac{\sigma}{2} \sqrt{\frac{(1-\rho)p}{c}},$$
$$x_4^* = \mu + (p - c - c_0) \sigma \sqrt{\frac{(1+\rho)}{A}},$$

where $A = p(c + 2c_0) - 2(c + c_0)^2$ must be positive (otherwise, the firm's robust solution is zero). Define $\rho_{nc} = \frac{A - pc}{A + pc}$ as a threshold on the correlation coefficient.

Lemma 4 *With symmetric data and $\rho \le \rho_{nc}$, in the zero-sum game formulated in equation (5.1), the equilibrium strategy of Adverse Nature is the following four-point distribution:*

$$\left(\tilde{\theta}_1, \tilde{\theta}_2\right) = \begin{cases} (x_1^*, x_1^*) \text{ with probability } 1 - \frac{2c+c_0}{p}, \\ (x_2^*, x_3^*) \text{ with probability } \frac{c}{p}, \\ (x_3^*, x_2^*) \text{ with probability } \frac{c}{p}, \\ (x_4^*, x_4^*) \text{ with probability } \frac{c_0}{p}. \end{cases}$$

A relevant benchmark is the case without component commonality. In this benchmark, $2q = q_0$ such that the firm manages the inventory of product j separately. For product j, she pre-stocks $q_j = q$ pairs of product-specific component j and common component 0. Equation (C-2) in Part C of Online Appendix characterizes the equilibrium strategy of Adverse Nature for this benchmark. Because product j uses product-specific component j, this benchmark differs from the case of using a "universal" product, which basically faces an aggregate demand $\theta_1 + \theta_2$. A critical difference is that the joint distribution in equation (C-2) satisfies the additional requirements on the marginal distributions and hence, substantiating Scarf's rule as the firm's equilibrium strategy. In contrast, the distribution in Lemma 4 does not satisfy the requirements on the marginal distributions. Scarf's rule implies that the firm's inventory level of product j equals

$$q_{nc}^* = \mu + \frac{\sigma}{2}\left(\sqrt{\frac{p-c-c_0}{c+c_0}} - \sqrt{\frac{c+c_0}{p-c-c_0}}\right), \tag{5.2}$$

and the value of the zero-sum game without component commonality equals

$$Z_{nc} = 2\mu(p-c-c_0) - 2\sigma\sqrt{(p-c-c_0)(c+c_0)}.$$

When demands are deterministic, the symmetric profit margin of each product equals $(p-c-c_0)$ and the firm's profit equals $2(p-c-c_0)\mu$, which is the first term in Z_{nc}. Next, we derive the firm's equilibrium strategy and compute the value of the zero-sum game with common component.

Proposition 5 *The firm's equilibrium strategy is one of the following two cases:*

a) If $\rho \le \rho_{nc}$, the robust optimal inventory level of the product-specific component is

$$q^* = \frac{x_1^* + x_3^*}{2} = \mu + \sigma\left[\left(\frac{p}{4} - c - c_0\right)\sqrt{\frac{1+\rho}{A}} + \frac{1}{4}\sqrt{\frac{(1-\rho)p}{c}}\right]. \tag{5.3}$$

and that of the common component is

$$q_0^* = x_1^* + x_4^* = 2\mu + \sigma(p-2c-2c_0)\sqrt{\frac{1+\rho}{A}}. \tag{5.4}$$

The value of the zero-sum game equals

$$Z^* = 2(p-c-c_0)\mu - \left[\sqrt{(1+\rho)A} + \sqrt{(1-\rho)pc}\right]\sigma. \tag{5.5}$$

b) If $\rho \geq \rho_{nc}$, the firm's equilibrium strategy is identical to the case without component commonality and the value of the zero-sum game equals Z_{nc}.

Proposition 5 provides several useful insights. First, equation (5.4) indicates that the sign of $(p-2c-2c_0)$ determines whether the inventory level of the common component is increasing or decreasing in the correlation coefficient ρ. Second, equation (5.3) reveals that correlated demands create two opposite effects on the inventory levels of product-specific components. Specifically, the term $\left(\frac{p}{4}-c-c_0\right)\sqrt{\frac{1+\rho}{A}}$ in equation (5.3) captures the increasing (decreasing) effect if $p > 4c+4c_0$ (if $p \leq 4c+4c_0$). The firm also benefits from component commonality and the term $\frac{1}{4}\sqrt{\frac{(1-\rho)p}{c}}$ in equation (5.3) captures the decreasing effect as ρ increases. Which effect dominates depends on the value of $(p-4c-4c_0)$. Third, the two opposite effects on q^* also influence the value of the zero-sum game. In equation (5.5), the term $\sqrt{(1+\rho)A}$ quantifies the impact of higher inventory levels of product-specific components (i.e., the increasing effect); while the term $\sqrt{(1-\rho)pc}$ quantifies the impact of lower inventory level of common component (i.e., the decreasing effect). Because the decreasing effect dominates, Z^* in equation (5.5) is decreasing in ρ such that when the correlation coefficient exceeds the closed-form threshold ρ_{nc}, the firm does not benefit from the risk-pooling effect. We summarize the comparative statics as follows.

Corollary 3 *It holds that $\frac{\partial Z^*}{\partial \rho} < 0$.*

5.2. When Joint Distribution is Known

The comparative statics in Corollary 3 can be generalized to the case when the joint distribution function is known to be $F(\theta_1,\theta_2)$. We let $Z(q,q_0|F)$ be the firm's expected profit. Let $\beta_b = \int_0^{q_0-q}\int_q^{\infty} dF(\theta_1,\theta_2)$ be the probability that circumstance b) in Table 2 occurs. Similarly, let

$$\beta_d = \int_{q_0-q}^{q}\int_{q_0-\theta_1}^{\infty} dF(\theta_1,\theta_2) + \int_q^{\infty}\int_{q_0-q}^{\infty} dF(\theta_1,\theta_2)$$

be the probability that circumstance d) in Table 2 occurs. Because the integral limits are unaffected by the decision variables (q,q_0), Leibniz's rule indicates that the expectation and derivative operators can be interchanged. We obtain the following FOCs:

$$\frac{\partial Z(q,q_0|F)}{\partial q} = p\beta_b - c = 0 \text{ and } \frac{\partial Z(q,q_0|F)}{\partial q_0} = p\beta_d - c_0 = 0.$$

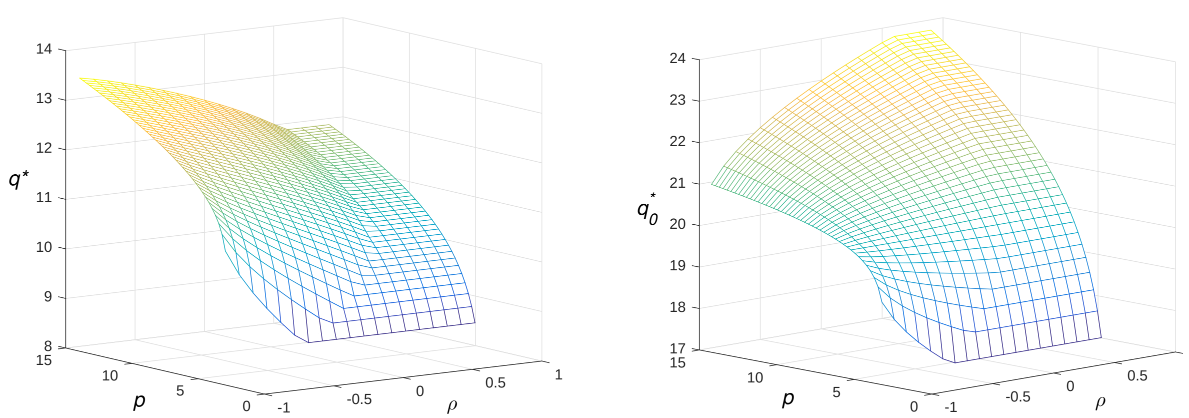
The two probabilities β_b and β_d are non-trivial if demands are dependent. For example, if $F(\theta_1,\theta_2)$ is the standard bi-variate normal distribution, the integrals do not provide tractable functional forms.[4] The tractability issue again underscores the advantage of using the closed-form solution in Proposition 5.

With (q, q_0) being fixed, it is perceivable that when correlation coefficient ρ increases, β_b decreases while β_d increases. To maintain the FOCs intact, the firm must either increase the common component (so that the region in Figure 2 associated with circumstance d) shrinks) or reduce the specific component (so that the region in Figure 2 associated with circumstance b) or c) expands). Either option makes $2q \geq q_0$ binding when ρ exceeds a threshold. Therefore, when correlation increases, the benefit of risk-pooling vanishes. While the break-even ρ can be difficult to determine for many distributions, Proportion 5 shows that when $\rho \geq \rho_{nc} = \frac{A-pc}{A+pc}$, the risk-pooling benefit vanishes under the max-min decision rule. The closed-form ρ_{nc} helps managers to quickly determine whether to take advantage of the risk-pooling effect.

5.3. Numerical Examples

We illustrate comparative statics of the robust order quantities in Figure 3. When the price p increases, order quantities of both product-specific and common components increase. Consistent with the Corollary 3, the order quantity for product-specific component decreases in the demand correlation coefficient ρ. However, the order quantity for common component decreases in ρ when the price is low ($p < 2c + 2c_0$), and increases in ρ otherwise. When $\rho > \rho_{nc}$, the robust order quantities are identical to the case of no component commonality.

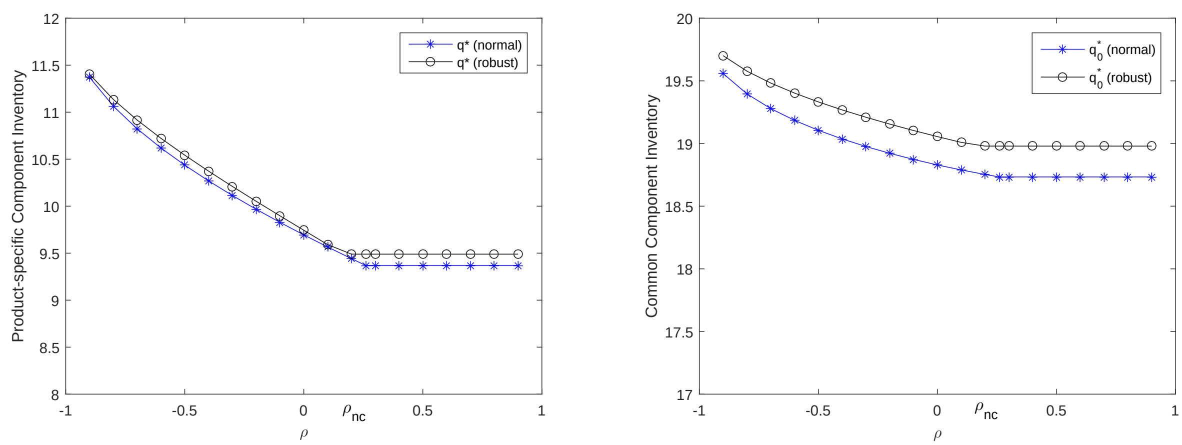
Figure 3 Impact of Price and Demand Correlation on Robust Order Quantities $c = 1$, $c_0 = 2$, $\mu = 10$, $\sigma = 2.5$



(a) Product-specific Component (b) Common Component

We also provide numerical examples to compare the robust solution with the distribution-dependent solution based on the assumption that the demands follow a bivariate normal distribution with symmetric moments. When the correlation coefficient ρ varies, Figure 4 and Figure 5 present the cases with low ($p < 2c + 2c_0$) and high ($p > 2c + 2c_0$) prices, respectively. We observe that the robust order quantities and those obtained under normal distribution exhibit a similar pattern and are quite close to each other. For both solutions, the risk-pooling effect emerges only when the correlation coefficient ρ is less than a certain threshold. When ρ exceeds the threshold, the order quantities are the same as the case of without component commonality. We also find that the threshold on ρ under normal distribution is slightly higher than that of the robust model, ρ_{nc}.

Figure 4 Comparison of Robust Order Quantities with Optimal Order Quantities under Bivariate Normal Distribution ($p < 2c + 2c_0$) with $p = 5$, $c = 1$, $c_0 = 2$, $\mu = 10$, $\sigma = 2.5$



(a) Product-specific Component (b) Common Component

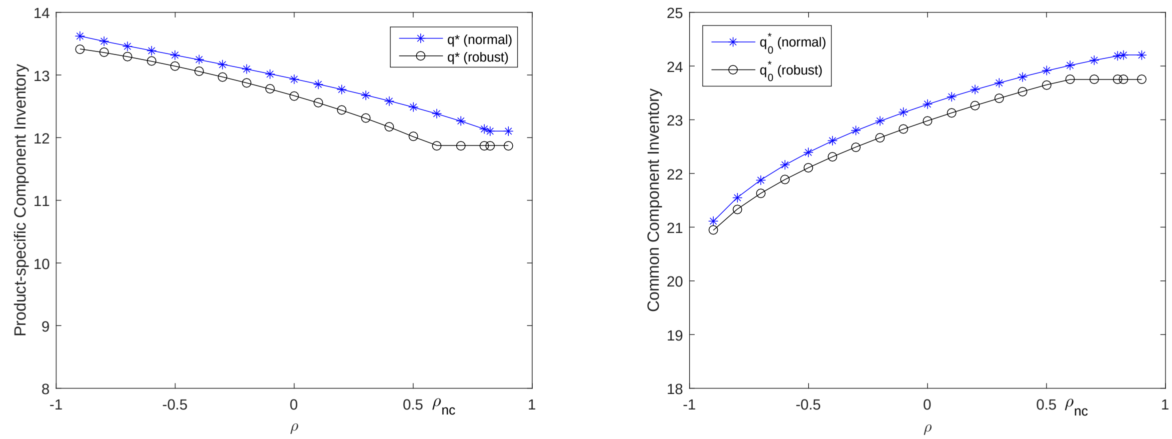
When creating Figures 6 and 7, we increase the cost of the common component from $c_0 = 2$ to $c_0 = 5$ and keep other parameters intact. We observe the same pattern in the order quantities.

6. Discussions

6.1. Existence of an Equilibrium

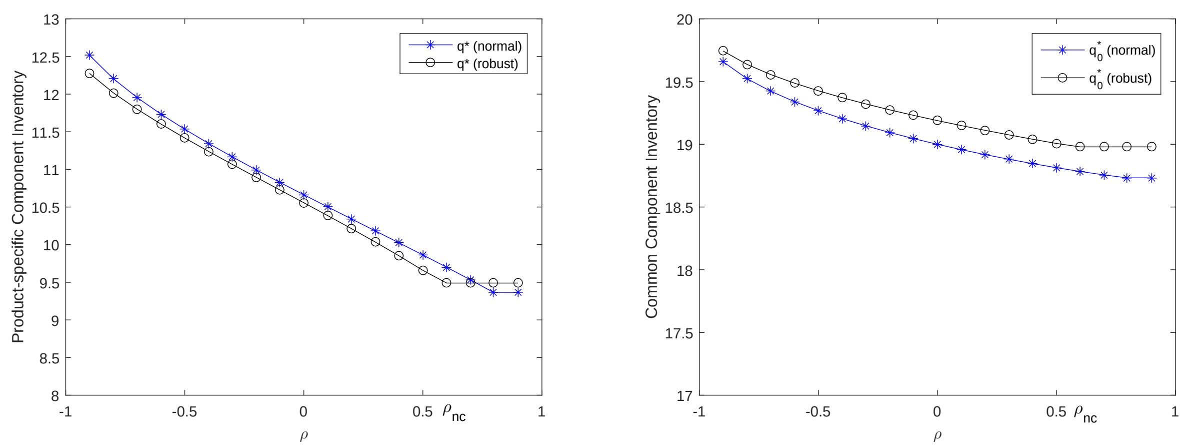
Zero-sum games have been extensively studied in various disciplines (e.g., economics, computer science, and operations research) since John Nash proved that an equilibrium exists in finite games. By definition, a finite game restricts the action space of each player to be finite and discrete. The well-known Debreu-Glicksberg-Fan Theorem extends Nash's result to infinite games, where players can take an infinite number of actions. Specifically, the existence of an equilibrium in an infinite game requires three conditions: i) the action space of each player i is compact and convex; ii) the payoff function of player i is continuous with respect to other players' actions; and iii) the payoff function of player i is continuous and quasi-concave with respect to her own action.

Figure 5 Comparison of Robust Order Quantities with Optimal Order Quantities under Bivariate Normal Distribution ($p > 2c + 2c_0$) with $p = 15$, $c = 1$, $c_0 = 2$, $\mu = 10$, $\sigma = 2.5$



(a) Product-specific Component (b) Common Component

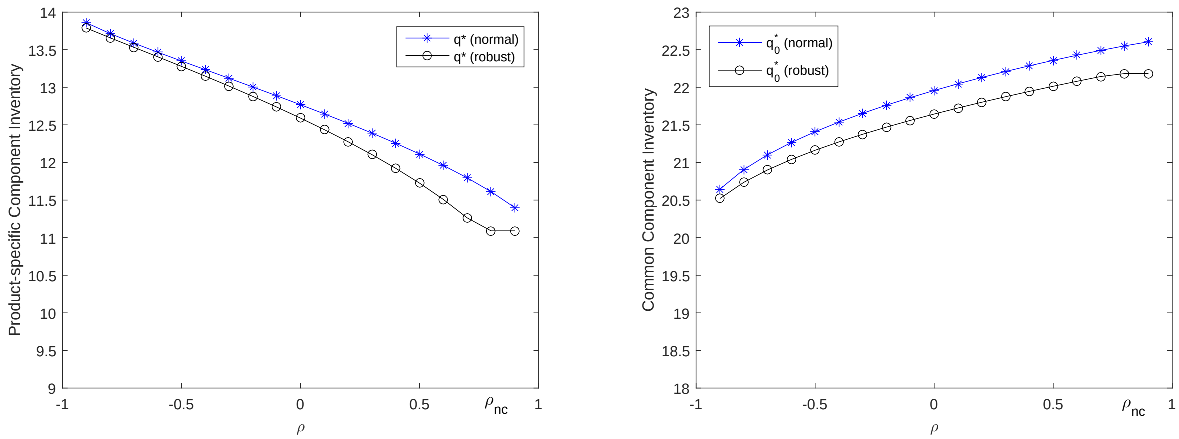
Figure 6 Comparison of Robust Order Quantities with Optimal Order Quantities under Bivariate Normal Distribution ($p < 2c + 2c_0$) with $p = 10$, $c = 1$, $c_0 = 5$, $\mu = 10$, $\sigma = 2.5$



(a) Product-specific Component (b) Common Component

We can extend the analysis to the case with $n \geq 1$ moments. Example 2.6 of Hettich and Kortanek (1993) indicates that we can first use a generalized finite sequence to construct the so-called "moment cones". The convex hull of the moment cones then specifies the corresponding ambiguity set Ω_n (where the subscript n indicates that the first $n \geq 1$ moments are known). It is well-known that the convex hull of the moment cones is convex and compact so that condition i) holds in our examples. In the examples that we consider, the ex post payoff functions are continuous and quasi-concave and hence, conditions ii) and iii) also hold. A minor issue is that we do require the

Figure 7 Comparison of Robust Order Quantities with Optimal Order Quantities under Bivariate Normal Distribution ($p > 2c + 2c_0$) with $p = 20$, $c = 1$, $c_0 = 5$, $\mu = 10$, $\sigma = 2.5$



(a) Product-specific Component (b) Common Component

ex post payoff function $Z(\theta|\mathbf{Q})$ and its first derivative $\frac{\partial Z(\theta|\mathbf{Q})}{\partial Q_i}$ both be finite for any $\theta \geq 0$ so that duality holds. We believe that the assumption of a bounded payoff and a bounded marginal payoff is mild, especially in supply chains.

Discontinuous payoff functions can arise in various circumstances, for example, when the (s, S) inventory policy is used. Reny (1999, page 1033) indicates that many discontinuous games can still have an equilibrium (we refer readers to Reny 2020, for an updated literature review). A promising path for future research is to consider discontinuous payoff functions. Notably, Theorem 1 provides an alternative method to determine whether an equilibrium exists when payoff functions are discontinuous. Specifically, we solve the SIP model of Adverse Nature. If one of the optimal solutions satisfies $a_i = 0$ for all i, then an equilibrium exists; otherwise, the original zero-sum game does not have an equilibrium.

If quasi-concavity is not assumed, we either retrospectively verify the firm's second-order conditions or include them as part of the IC constraints in Adverse Nature's model. To illustrate, we use a single-action model as an example. The FOC includes $\int_0^\infty \frac{\partial Z(\theta|Q)}{\partial Q} dF(\theta) = 0$ and the second-order condition includes $\int_0^\infty \frac{\partial^2 Z(\theta|Q)}{\partial Q^2} dF(\theta) \leq 0$. Consequently, the SIP constraints in equation (2.6) change to the following:

$$y_0 + y_1\theta + y_2\theta^2 + a\frac{\partial Z(\theta|Q)}{\partial Q} + b\frac{\partial^2 Z(\theta|Q)}{\partial Q^2} \leq Z(\theta|Q), \forall \theta \geq 0,$$

where b is nonpositive. If an equilibrium exists, both a^* and b^* are zero.[5] Nonetheless, quasi-concavity not only validates the first-order approach that economists advocate but also ensures the existence of an equilibrium in the zero-sum games that we study.

6.2. Uniqueness

In the two-player zero-game that we study, the firm's payoff equals $Z(q, F)$ and Adverse Nature's payoff equals $-Z(q, F)$. The zero-sum feature makes the existence and uniqueness of an equilibrium the same theoretical issue. The well-known von Neumann's Minmax Theorem indicates that if the zero-sum game has an equilibrium, then the value of the zero-sum game Z^* must be unique. Thus, the equilibrium strategy of the firm who uses a pure strategy is also unique.

However, the equilibrium strategy of Adverse Nature (who employs a mixed strategy) may not be unique. A relevant example is the newsvendor model with asymmetric demand distribution that Natarajan et al. (2018) study. In the second case of their Theorem 2.2 (on page 3152 Natarajan et al. 2018), the equilibrium strategy F^* of Adverse Nature exhibits the following two properties: i) F^* allocates a probability mass $\frac{p-c}{p}$ to a unique binding point $\theta_L = \mu - \sigma\sqrt{\frac{p(1-s)}{2(p-c}}$. ii) F^* allocates the remaining density to all the points above μ subject to the conditions on the mean and variance. There exist multiple mixed distributions satisfying these two properties. In summary, the firm's equilibrium strategy and the value of zero-sum game are unique but Adverse Nature's equilibrium strategy may not be unique.

6.3. Other Ambiguity Sets

The available information affects the ambiguity set and critically determines the mathematical properties of the robust solution. For example, Mulvey et al. (1995) propose a scenario-based model. Ben-Tal and Nemirovski (1999) propose an ellipsoid model to model contaminated data or parameter uncertainty. Bertsimas and Sim (2004) propose a cardinality-constrained model to control the level of conservatism. Esfahani and Kuhn (2018) consider Wasserstein balls. To extend our method to these models, the key step is to develop the IC constraints from the perspective of Adverse Nature. If the relevant IC constraints have a simple form, we believe that our method remains effective.

On the other hand, economists have considered several types of ambiguities, including unknown actions, unknown prior distributions, non-Bayesian beliefs, unknown strategic behaviors, and unknown interactions among agents (we refer readers to a recent survey done by Carroll 2019). In supply chains, an unknown prior distribution is the most relevant type of ambiguity for random demands or random yields. Again, the key step to apply our method is to simplify the IC constraints in the min-max version of the model. By following Perakis and Roels (2008), we can easily incorporate other information such as mode and range into the analysis. In future research, we plan to examine the ambiguities that economists have considered.

7. Concluding Remark

This paper proposes a new and efficient method to solve a large class of zero-sum games under moment conditions. By solving this class of zero-sum games, we solve the corresponding max-min optimization models, which have abundant theoretical and practical applications. Our method is based on the min-max inequality and reformulates the zero-sum game as a robust moral hazard model from the perspective of Adverse Nature. We show that the IC constraints of the moral hazard model become moment constraints. While the marginal payoff function (i.e., the first derivative of the ex post payoff function) appears in the SIP constraints, the number of corner points to be considered declines drastically. The key advantage of our method is that we can solve the robust solution without explicitly deriving the objective function. For example, both the $(1+t)$ model and the n-option model presented in this paper have an overly complex objective function. However, using the property of zero Lagrangian multipliers, we conveniently determine many equilibriums that the traditional method is unable to derive in closed forms.

Endnotes

1. In September 2000, Sony (which also used option contracts) announced that it failed to meet customer demand for the new PlayStation console due to shortages in capacitors, LCDs, and flash memory chips (Fu et al. 2010).

2. Instead of lost sales, future research could consider other scenarios. For instance, the firm could use the spot market or persuade customers to buy alternative products. These extensions will enhance the application of our model.

3. We emphasize that the right hand side (RHS) of the IC constraints can be non-zero. For example, the IC constraint for Scarf's model is $\int_0^q (-c)dF(\theta) + \int_q^\infty (p-c)dF(\theta) = 0$, which is equivalent to $\int_0^q dF(\theta) = \frac{p-c}{p}$. Using either version to formulate the SIP model must produce the same solution. In the n-option model, the second version of the IC constraints to formulate equation (4.6) is more convenient. To ensure that the solution in model P_1 is feasible in model P, the Lagrangian multipliers of the IC constraints must be zero despite the non-zero RHS of the IC constraints.

4. In a notable special case with $2q = q_0$, we observe that $\beta_b = \int_0^q \int_q^\infty dF(\theta_1, \theta_2)$ and $\beta_d = \int_q^\infty \int_q^\infty dF(\theta_1, \theta_2)$ such that

$$\beta_b + \beta_d = \int_0^\infty \int_q^\infty dF(\theta_1, \theta_2) = \Pr(\theta_2 \geq q),$$

implying that q_F satisfies the following FOC: $\Pr(\theta_2 \geq q_F) = \frac{c+c_0}{p}$. We refer to q_F as the inventory level without component commonality.

5. With multiple actions, where $\mathbf{Q} = (Q_1, Q_2, ..., Q_n)$, the second-order conditions are complex and involve bordered Hessian matrix. In the special case where the cross derivative $\frac{\partial^2 Z(\theta|\mathbf{Q})}{\partial Q_i \partial Q_j}$ is zero for any $i \neq j$, the second-order conditions create a summation of $b_i \frac{\partial^2 Z(\theta|\mathbf{Q})}{\partial Q_i^2}$ at the left-hand-side of the SIP constraints.

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Online Appendix: Technical Proofs

Part A: Proofs for Sections 2 and 3

Proof of Theorem 1

The first step is to establish strong duality (i.e., $P = D$ and $P_1 = D_1$). This step is rather routine and we omit the details. The second step has been fully explained in the paragraph preceding Theorem 1. Q.E.D.

Proof of Lemma 1

a) We consider the following nonlinear system:

$$\begin{cases} \frac{p-r-c}{p-r}x_1 + \frac{c}{p-r}x_2 = \mu, \\ \frac{p-r-c}{p-r}x_1^t + \frac{c}{p-r}x_2^t = m_t. \end{cases}, \tag{A-1}$$

which has two unknowns and two equations with the power of t. Using the equation $\frac{c}{p-r}x_2 = \mu - \frac{p-r-c}{p-r}x_1$, we can rewrite equation (A-1) as the following one-variable equation:

$$H(x_1) = \frac{p-r-c}{p-r}x_1^t + \frac{c}{p-r}\left[x_1 + \frac{p-r}{c}(\mu - x_1)\right]^t = m_t.$$

It is easy to verify that for $x_1 \in [0, \mu]$, $H(x_1)$ is decreasing. When $x_1 = 0$, $H(x_1) = \frac{\mu^t}{\left(\frac{c}{p-r}\right)^{t-1}}$ and when $x_1 = \mu$, $H(\mu) = \mu^t$. Hence, when condition (3.4) holds, $H(x_1) = m_t$ has a nonnegative solution θ_1 satisfying $0 \le \theta_1 \le \mu$. We then obtain that

$$\theta_2 = \theta_1 + \left(\frac{p-r}{c}\right)(\mu - \theta_1) \ge \theta_1.$$

Because (θ_1, θ_2) satisfy the nonlinear system in (A-1), the two-point distribution in equation (3.5) satisfies the conditions on the mean and t-th moment.

b) It is readily verified that the distribution in equation (3.7) satisfies the conditions on the mean and t-th moment. Q.E.D.

Proof of Proposition 1

a) We conjecture that binding constraints occur at θ_1 and θ_2, where $\theta_1 \le q \le \theta_2$ such that overstock (lost sales) occurs when the realized demand is θ_1 (θ_2). If the conjectured constraints satisfy $\theta_1 \le \theta_2 \le q$, then the firm can lower the inventory level to avoid overstocking. If the conjectured constraints satisfy $q \le \theta_1 \le \theta_2$, then the firm can increase the inventory level to reduce lost sales. The Lagrangian equals

$$\begin{aligned} L = {} & y_0 + y_1\mu + y_2 m_t - \lambda_1\left[y_0 + y_1\theta_1 + y_2\theta_1^t - ac - (p-r)\theta_1 + cq\right] \\ & -\lambda_2\left[y_0 + y_1\theta_2 + y_2\theta_2^t + a(p-r-c) - (p-r-c)q\right]. \end{aligned}$$

The FOCs with respect to y_0 and a are the following:

$$\frac{\partial L}{\partial y_0} = 1 - \lambda_1 - \lambda_2 = 0 \text{ and } \frac{\partial L}{\partial a} = \lambda_1 c - \lambda_2 (p - r - c) = 0.$$

These two equations yield that $\lambda_1 = \frac{p-r-c}{p-r}$ and $\lambda_2 = \frac{c}{p-r}$. Observe that the multiplier $\lambda_1 = \frac{p-r-c}{p-r}$ corresponds to the newsvendor ratio. The FOCs with respect to y_1 and y_2 are the following:

$$\frac{\partial L}{\partial y_1} = \mu - \lambda_1 \theta_1 - \lambda_2 \theta_2 = 0 \text{ and } \frac{\partial L}{\partial y_2} = m_t - \lambda_1 \theta_1^t - \lambda_2 \theta_2^t = 0.$$

Because $\lambda_1 = \frac{p-r-c}{p-r}$ and $\lambda_2 = \frac{c}{p-r}$ are known, the above two equations yield the following nonlinear system:

$$\begin{cases} \frac{p-r-c}{p-r}\theta_1 + \frac{c}{p-r}\theta_2 = \mu, \\ \frac{p-r-c}{p-r}\theta_1^t + \frac{c}{p-r}\theta_2^t = m_t. \end{cases}, \tag{A-2}$$

which involves two equations and two unknowns. Lemma 1a) solves this nonlinear system and identifies the two-point distribution in equation (3.5). We label this two-point distribution as F^*, which is independent of q and is the strategy that Adverse Nature plays in the equilibrium.

Without the shadow prices, the analysis remains incomplete. To solve for the shadow prices, we use the binding constraints and the tangent conditions as follows:

$$\begin{aligned} & y_0 + y_1\theta_1 + y_2\theta_1^t - ac - (p-r)\theta_1 + cq = 0, \\ & y_0 + y_1\theta_2 + y_2\theta_2^t + a(p-r-c) - (p-r)q + cq = 0, \\ & y_1 + ty_2\theta_1^{t-1} = p - r \text{ and } y_1 + ty_2\theta_2^{t-1} = 0. \end{aligned}$$

We emphasize that in model P_1, the tangent condition satisfies that $y_1 + ty_2\theta^{t-1} + a\frac{\partial^2 Z(\theta|q)}{\partial q \partial \theta} = \frac{\partial Z(\theta|q)}{\partial \theta}$ while in (the traditional) model P, the tangent condition satisfies that $y_1 + ty_2\theta^{t-1} = \frac{\partial Z(\theta|q)}{\partial \theta}$. Due to the characteristics of the newsvendor model (in which the ex post profit function is piece-wise linear), the cross derivative $\frac{\partial^2 Z(\theta|q)}{\partial q \partial \theta}$ is zero.

Solving this system of 4 unknowns and 4 equations, we obtain that

$$a = q - \theta_1 - \frac{\theta_2^{t-1}(\theta_2 - \theta_1)}{\theta_2^{t-1} - \theta_1^{t-1}} + \frac{\theta_2^t - \theta_1^t}{t\left(\theta_2^{t-1} - \theta_1^{t-1}\right)}.$$

According to Theorem 1, we let $a = 0$ to obtain the firm's equilibrium strategy q^*, which is shown in equation (3.8).

The shadow prices y_1 and y_2 are the following:

$$y_1 = \theta_2^{t-1}\left(\frac{p-r}{\theta_2 - \theta_1}\right) \text{ and } y_2 = -\frac{p-r}{t\left(\theta_2^{t-1} - \theta_1^{t-1}\right)},$$

and the fourth shadow price y_0 equals

$$\begin{aligned}
y_0 &= -y_1\theta_1 - y_2\theta_1^t + ac + (p-r)\theta_1 - cq \\
&= -\theta_2^{t-1}\theta_1\left(\frac{p-r}{\theta_2-\theta_1}\right) + \frac{p-r}{t\left(\theta_2^{t-1}-\theta_1^{t-1}\right)}\theta_1^t + c\left(q - \theta_1 - \frac{\theta_2^{t-1}(\theta_2-\theta_1)}{\theta_2^{t-1}-\theta_1^{t-1}} + \frac{\theta_2^t-\theta_1^t}{t\left(\theta_2^{t-1}-\theta_1^{t-1}\right)}\right) \\
&+ (p-r)\theta_1 - cq \\
&= (p-r)\left[\theta_1 + \frac{\theta_1^t}{t\left(\theta_2^{t-1}-\theta_1^{t-1}\right)} - \frac{\theta_2^{t-1}\theta_1}{\theta_2-\theta_1}\right] - c\left(\theta_1 + \frac{\theta_2^{t-1}(\theta_2-\theta_1)}{\theta_2^{t-1}-\theta_1^{t-1}} - \frac{\theta_2^t-\theta_1^t}{t\left(\theta_2^{t-1}-\theta_1^{t-1}\right)}\right).
\end{aligned}$$

There are two alternatives to compute P_1^*, which is the value of the zero-sum game. The first alternative is to compute P_1^* from the firm's perspective. When Adverse Nature plays the two-point distribution in equation (3.5), we can easily verify that for any $q \in [\theta_1, \theta_2]$,

$$\begin{aligned}
Z(q, F^*) &= \Pr(\tilde{\theta}=\theta_1)Z(\theta_1|q) + \Pr(\tilde{\theta}=\theta_2)Z(\theta_2|q) = \frac{p-r-c}{p-r}[(p-r)\theta_1] + \frac{c}{p-r}[(p-r)q] - cq \\
&= (p-r-c)\theta_1 + cq - cq = (p-r-c)\theta_1.
\end{aligned}$$

The second alternative is to substitute the optimal shadow prices that we have developed into the objective function of model P_1. Note that (θ_1, θ_2) satisfy the nonlinear system in equation (A-2). We find that

$$\begin{aligned}
P_1^* &= y_0 + y_1\mu + y_2 m_t = y_0 + y_1\left(\frac{p-r-c}{p-r}\theta_1 + \frac{c}{p-r}\theta_2\right) + y_2\left(\frac{p-r-c}{p-r}\theta_1^t + \frac{c}{p-r}\theta_2^t\right) \\
&= \frac{p-r-c}{p-r}\left(y_0 + y_1\theta_1 + y_2\theta_1^t\right) + \frac{c}{p-r}\left(y_0 + y_1\theta_2 + y_2\theta_2^t\right).
\end{aligned}$$

The binding SIP constraints imply that $y_0 + y_1\theta_1 + y_2\theta_1^t = ac + (p-r)\theta_1 - cq$ and $y_0 + y_1\theta_2 + y_2\theta_2^t = -a(p-r-c) + (p-r)q - cq$. Thus, we obtain that

$$\begin{aligned}
P_1^* &= \frac{p-r-c}{p-r}\left[ac + (p-r)\theta_1 - cq\right] + \frac{c}{p-r}\left[-a(p-r-c) + (p-r)q - cq\right] \\
&= a\left[\frac{(p-r-c)c}{p-r} - \frac{c(p-r-c)}{p-r}\right] + (p-r-c)\theta_1 + cq - cq = (p-r-c)\theta_1.
\end{aligned}$$

Both alternatives lead to the same conclusion. Note that the values of θ_1 and θ_2 satisfy Lemma 1a) and condition (3.4) ensures that θ_1 is positive.

b) Suppose that condition (3.6) holds and Adverse Nature chooses the distribution in equation (3.7) as the equilibrium strategy. We can easily verify that the firm's expected profit is decreasing with respect to q. Hence, the firm's best response is $q^* = 0$, which results in zero profit. Because Adverse Nature's objective is to minimize the value of the zero-sum game, $P_1 \geq 0$ must hold. Because the distribution in equation (3.7) makes $P_1 = 0$, we conclude that $q^* = 0$ is the firm's equilibrium strategy and the distribution in equation (3.7) is Adverse Nature's equilibrium strategy. Q.E.D.

Scarf's Model

In a special case where $t=2$ (such that $m_2=\mu^2+\sigma^2$), the analysis reduces to Scarf's model and condition (3.4) requires that $\mu^2 \geq \frac{c}{p-r}\left(\mu^2+\sigma^2\right)$, which is equivalent to $(p-r-c) \geq c\rho^2$. It is well-known that when $(p-r-c)<c\rho^2$, the firm's robust optimal solution is $q^*=0$. Based on Lemma 1, we re-write the nonlinear equation of (3.3) as follows:

$$H(x)=\frac{p-r-c}{p-r}x^2+\frac{c}{p-r}\left[x+\frac{p-r}{c}(\mu-x)\right]^2=\mu^2+\sigma^2,$$

which readily yields that $\theta_1=\mu-\sigma\sqrt{\frac{c}{p-r-c}}$ and $\theta_2=\mu+\sigma\sqrt{\frac{p-r-c}{c}}$. As a result, we obtain the following distribution:

$$\begin{cases}\Pr\left(\tilde{\theta}=\mu-\sigma\sqrt{\frac{c}{p-r-c}}\overset{def}{=}\theta_1^*\right)=\frac{p-r-c}{p-r}, \\ \Pr\left(\tilde{\theta}=\mu+\sigma\sqrt{\frac{p-r-c}{c}}\overset{def}{=}\theta_2^*\right)=\frac{c}{p-r}.\end{cases} \tag{A-3}$$

Corollary 4 *In Scarf's model, when $(p-r-c)\geq c\rho^2$ holds, the equilibrium strategy F^* chosen by Adverse Nature is the two-point distribution shown in equation (A-3). The SIP model in equation (3.2) encompasses multiple solutions satisfying*

$$\begin{cases}a=q-\frac{\theta_1^*+\theta_2^*}{2}, \\ y_0=-\frac{\Delta}{2\sigma}\left(\mu^2+\sigma^2\right), \\ y_1=(p-r-c)+\Delta\frac{\mu}{\sigma}, \\ y_2=-\frac{\Delta}{2\sigma},\end{cases} \tag{A-4}$$

where $q\in[\theta_1^*,\theta_2^*]$ and $\Delta=\sqrt{(p-r-c)\,c}$ such that the value of the zero-sum game equals

$$P_1^*=(p-r-c)\theta_1^*=(p-r-c)\,\mu-\Delta\sigma.$$

Proof: Let $\Delta=\sqrt{(p-r-c)\,c}$. Based on the proof of Proposition 1, we obtain that

$$a=q-\theta_1-\frac{\theta_2(\theta_2-\theta_1)}{\theta_2-\theta_1}+\frac{\theta_2^2-\theta_1^2}{2(\theta_2-\theta_1)}=q-\frac{\theta_1+\theta_2}{2}.$$

On the other hand, the remaining shadow prices include: $y_0=-\frac{\Delta}{2\sigma}\left(\mu^2+\sigma^2\right)$, $y_1=(p-r-c)+\Delta\frac{\mu}{\sigma}$, $y_2=\frac{-(p-r)}{2(\theta_2-\theta_1)}=-\frac{\Delta}{2\sigma}$, confirming the results in equation (A-4). Notice that when q is fixed, the SIP model P_1 is also linear with respect to decision variables y_i and a, making KKT conditions sufficient and necessary. Using Theorem 1, we let $a^*=0$ to obtain $q^*=\frac{\theta_1+\theta_2}{2}$. Q.E.D.

From the firm's perspective, if the distribution in equation (A-3) arises, any $q\in[\theta_1^*,\theta_2^*]$ gives her the same expected profit, which equals $P_1^*=(p-r-c)\,\theta_1^*$. Although any $q\in[\theta_1^*,\theta_2^*]$ does not affect the optimal value of model P_1, the firm cannot arbitrarily choose any q from the interval $[\theta_1^*,\theta_2^*]$.

To construct a solution that is feasible for both models P and P_1, Theorem 1 suggests that we let $a=0$ such that

$$q^* = \frac{1}{2}\left(\theta_1^* + \theta_2^*\right) = \mu + \frac{1}{2}\sigma\left(\sqrt{\frac{p-r-c}{c}} - \sqrt{\frac{c}{p-r-c}}\right),$$

which is the well-known Scarf's rule and happens to be the midpoint of the closed interval $[\theta_1^*, \theta_2^*]$. A few useful observations can be made.

- When a nonrobust solution is used, the shadow price a associated with the IC constraint is non-zero. The shadow price a could be positive or negative depending on which direction that the nonrobust q deviates from the equilibrium. Whenever $a \neq 0$, the solution (a, y_i, q), which is feasible for model P_1, cannot be implemented in model P, making $P < P_1$ and q a nonrobust solution.

- In terms of computational complexity, we bypass the middle step of solving the inner SIP model in equation (2.3) and directly attack the robust solution. Because the first derivative is a step function, we encounter multiple solutions in model P_1. We then apply Theorem 1 to solve the robust solution.

- If the firm deviates from the equilibrium, it is well-known that Adverse Nature plays the following strategy:

$$\int_0^q dF\left(\theta|q\right) = \frac{1}{2} + \frac{q-\mu}{2\sqrt{\left(q-\mu\right)^2 + \sigma^2}},$$

which represents a credible threat to the firm such that she has no incentive to deviate from her equilibrium strategy. Interestingly, by letting $q = \frac{1}{2}\left(\theta_1^* + \theta_2^*\right)$ in the above equation, we obtain the two-point distribution equation (A-3), which is labeled as F^*. If distribution F^* realizes, the firm's expected profit is monotonically increasing in q if $q < \theta_1^*$ and is monotonically decreasing in q if $q > \theta_2^*$. We can conclude that the firm never plays a q outside of the closed interval $[\theta_1^*, \theta_2^*]$. In other words, to satisfy the IC constraints in Adverse Nature's model, the capacity level must come from the interval $[\theta_1^*, \theta_2^*]$.

- Let $k = \frac{p-r-c}{p-c}$. In a special case with $k = 0.5$, equation (A-3) yields that $\theta_1^* = \mu - \sigma$ and $\theta_2^* = \mu + \sigma$, which are the theoretical bounds on the median. The bounds on the median are well-known in statistics. However, equation (A-3) generalizes the bounds to any $k \times 100\%$ percentile, where $0 < k < 1$. Specifically, the upper bound on the $k \times 100\%$ percentile is $\mu + \sigma\sqrt{\frac{k}{1-k}}$ and the lower bound is $\mu - \sigma\sqrt{\frac{1-k}{k}}$. In relation to supply chains, if the newsvendor ratio is known to be k, the firm should not order less than $\mu - \sigma\sqrt{\frac{1-k}{k}}$ or order more than $\mu + \sigma\sqrt{\frac{k}{1-k}}$; otherwise, she behaves off the equilibrium. Certainly, using Lemma 1, we can generalize the bounds on percentile by using the mean and the t-th moment.

Part B: Proofs for Section 4

Proof of Lemma 2:

In the fulfillment stage, the capacity vector $\mathbf{q}$ is already chosen. After observing the realized demand θ, the firm solves the following linear programming model to maximize her ex post profit:

$$Z(\theta|\mathbf{q}) = -\sum_{i=1}^{n} c_i q_i + \max_{x_i \geq 0} \left\{ \sum_{i=1}^{n} (p - r_i) x_i \right\}, \tag{B-1}$$

subject to the capacity availability constraints:

$$x_i \leq q_i, \ \forall i \in \{1, 2, \cdots, n\}, \tag{B-2}$$

and the total demand constraint:

$$x_1 + x_2 + \cdots + x_n \leq \theta. \tag{B-3}$$

The nonnegative decision variable x_i represents the quantity of the end products delivered by source-i. The capacity availability constraints (B-2) ensure that the quantity of the end products delivered by source-i does not exceed the available capacity q_i, and the total demand constraint (B-3) ensures that the total delivered quantities do not exceed the realized demand θ. In equation (B-1), the total cost $\sum_{i=1}^{n} c_i q_i$ is sunk, and the coefficient $(p - r_i)$ is decreasing in i due to Definition 1. Hence, the firm must prefer source i to source $(i+1)$, implying that the firm's optimal fulfillment plan follows a priority rule such that source $(i+1)$ is not used unless the reserved capacity of source i is exhausted.

In equation (B-1), the coefficient of decision variable x_i equals $(p - r_i)$, which is decreasing in i. The firm must exhaust all of source-i capacities before using any source-$(i+1)$ capacity. Thus, it holds that

$$x_i^* = \min \left(q_i, \left(\theta - \sum_{j=1}^{i-1} q_i \right)^+ \right),$$

which is optimal for equation (B-1). Using the definition of Q_i, we obtain that

$$x_i^* = \min \left(Q_i - Q_{i-1}, (\theta - Q_{i-1})^+ \right) = \min (Q_i, \theta) - \min (Q_{i-1}, \theta),$$

indicating that the sales quantity contributed by source-i capacity equals the sales of the first i sources minus the sales of the first $(i-1)$ sources. Thus, we obtain that

$$\begin{aligned} Z(\theta|\mathbf{q}) &= Z(\theta|\mathbf{Q}) = -\sum_{i=1}^{n} c_i (Q_i - Q_{i-1}) + \sum_{i=1}^{n} (p - r_i) x_i^* \\ &= \sum_{i=1}^{n} \left[(p - r_i) (\min (Q_i, \theta) - \min (Q_{i-1}, \theta)) - c_i (Q_i - Q_{i-1}) \right], \end{aligned}$$

which yields equation (4.2). It is easy to verify that $Z(\theta|\mathbf{Q})$ is continuous in θ.

To understand the concave and increasing properties of $Z(\theta|\mathbf{Q})$, we expand equation (4.2) into $(n+1)$ cases. i) When $\theta \in [0, Q_1]$:

$$Z(\theta|\mathbf{Q}) = (p - r_1)\,\theta - \sum_{i=1}^{n} c_i q_i, \tag{B-4}$$

in which $\frac{\partial Z(\theta|\mathbf{Q})}{\partial \theta} = p - r_1$ and $\frac{\partial Z(\theta|\mathbf{Q})}{\partial q_1} = -c_1$. ii) When $\theta \in [Q_1, Q_2]$:

$$Z(\theta|\mathbf{Q}) = (p - r_2)(\theta - q_1) + (p - r_1)\,q_1 - \sum_{i=1}^{n} c_i q_i = (p - r_2)\,\theta + (r_2 - r_1 - c_1)\,q_1 - \sum_{j=2}^{n} c_j q_j,$$

in which $\frac{\partial Z(\theta|\mathbf{Q})}{\partial \theta} = p - r_2$ and $\frac{\partial Z(\theta|\mathbf{Q})}{\partial q_1} = r_2 - r_1 - c_1$. iii) In general, when $\theta \in [Q_{i-1}, Q_i]$ and $2 \le i \le n$:

$$Z(\theta|\mathbf{Q}) = (p - r_i)\,\theta + \sum_{j=1}^{i-1} (r_i - r_j - c_j)\,q_j - \sum_{j=i}^{n} c_j q_j, \tag{B-5}$$

in which $\frac{\partial Z(\theta|\mathbf{Q})}{\partial \theta} = p - r_i$ and $\frac{\partial Z(\theta|\mathbf{Q})}{\partial q_1} = r_i - r_1 - c_1$. iv) Finally, when $\theta \ge Q_n$:

$$Z(\theta|\mathbf{Q}) = \sum_{i=1}^{n} (p - r_i - c_i)\,q_i, \tag{B-6}$$

in which $\frac{\partial Z(\theta|\mathbf{Q})}{\partial \theta} = 0$ and $\frac{\partial Z(\theta|\mathbf{Q})}{\partial q_1} = p - r_1 - c_1$.

The first derivative $\frac{\partial Z(\theta|\mathbf{Q})}{\partial \theta}$ is nonnegative and decreasing in θ, making $Z(\theta|\mathbf{Q})$ concave and increasing in θ. The first derivative $\frac{\partial Z(\theta|\mathbf{Q})}{\partial q_1} = \delta_i$ represents the marginal impact of source-i. Q.E.D.

Proof of Lemma 3:

Using equation (4.2), we obtain that the firm's expected profit equals:

$$\begin{aligned} Z(\mathbf{Q}) &= \sum_{i=1}^{n} (p - r_i)\left[E\min(Q_i, \theta) - E\min(Q_{i-1}, \theta)\right] - c_i\,(Q_i - Q_{i-1}) \\ &= \sum_{i=1}^{n-1} \left\{\left[(p - r_i) - (p - r_{i+1})\right] E\left[\min(Q_i, \theta)\right] - (c_i - c_{i+1})\,Q_i\right\} \\ &\quad + (p - r_n)\,E\min(Q_n, \theta) - c_n Q_n \\ &= \sum_{i=1}^{n-1} \left\{(r_{i+1} - r_i)\,E\left[\min(Q_i, \theta)\right] - (c_i - c_{i+1})\,Q_i\right\} \\ &\quad + (r_{n+1} - r_n)\,E\min(Q_n, \theta) - c_n Q_n. \end{aligned} \tag{B-7}$$

In equation (B-7), we regard Q_i as the decision variable. Based on the assumptions on the cost parameters, we observe that $(r_{i+1} - r_i) \ge 0$ and $(c_i - c_{i+1}) \ge 0$. The expected sales quantity $E[\min(Q_i,\theta)]$ is continuous and concave with respect to Q_i. Thus, the term $(r_{i+1}-r_i)E[\min(Q_i,\theta)]-(c_i-c_{i+1})Q_i$ is concave in Q_i. The last term of equation (B-7) is also concave because $(r_{n+1}-r_n)>0$. We conclude that equation (B-7) is concave in Q_i.

By relaxing the monotonicity constraints $Q_i \geq Q_{i-1}$, we can separately optimize each term in equation (B-7). Observe that $\frac{\partial}{\partial Q_i}E[\min(Q_i,\theta)]=1-F(Q_i)$. The FOC yields a candidate solution $\tilde{Q}_i$ satisfying

$$F\left(\tilde{Q}_i\right)=1-\frac{c_i-c_{i+1}}{r_{i+1}-r_i}=\alpha_i,$$

for $i=1,2,\cdots,n-1$. For $i=n$, the FOC yields that

$$F\left(\tilde{Q}_n\right)=1-\frac{c_n-c_{n+1}}{r_{n+1}-r_n}=1-\frac{c_n}{p-c_n}=\alpha_n.$$

Because α_i is increasing in i, the candidate solution $\tilde{Q}_i$ satisfies the monotonicity constraints and hence is optimal. We obtain that

$$\tilde{q}_i=F^{-1}\left(\tilde{Q}_i\right)-F^{-1}\left(\tilde{Q}_{i-1}\right)=F^{-1}(\alpha_i)-F^{-1}(\alpha_{i-1})\geq 0,$$

where F^{-1} is the inverse function of F. Q.E.D.

Proof of Corollary 1:

i) Equation (4.3) indicates that $\delta_i=r_i-r_1-c_1$ for $i=1,2,\cdots n+1$.

$$\sum_{i=1}^{n+1}\beta_i\delta_i=\left(\sum_{i=1}^{n+1}\beta_i r_i\right)-\left(\sum_{i=1}^{n+1}\beta_i(r_1+c_1)\right)=\left(\sum_{i=1}^{n+1}\beta_i r_i\right)-c_1-r_1.$$

Observe that

$$\begin{aligned}\sum_{i=1}^{n+1}\beta_i r_i &= \left(1-\frac{c_1-c_2}{r_2-r_1}\right)r_1+\left(\frac{c_1-c_2}{r_2-r_1}-\frac{c_2-c_3}{r_3-r_2}\right)r_2+\cdots+\left(\frac{c_{n-1}-c_n}{r_n-r_{n-1}}-\frac{c_n}{p-r_n}\right)r_n+\frac{pc_n}{p-r_n}\\ &= r_1-\frac{c_1-c_2}{r_2-r_1}r_1+\frac{c_1-c_2}{r_2-r_1}r_2-\frac{c_2-c_3}{r_3-r_2}r_2+\cdots+\frac{c_{n-1}-c_n}{r_n-r_{n-1}}r_n-\frac{c_n r_n}{p-r_n}+\frac{pc_n}{p-r_n}\\ &= r_1+\frac{c_1-c_2}{r_2-r_1}(r_2-r_1)+\cdots+\frac{c_{n-1}-c_n}{r_n-r_{n-1}}(r_n-r_{n-1})-\frac{c_n r_n}{p-r_n}+\frac{pc_n}{p-r_n}\\ &= r_1+c_1-c_2+\cdots+c_{n-1}-c_n-\frac{c_n r_n}{p-r_n}+\frac{pc_n}{p-r_n}\\ &= r_1+c_1-c_2+\cdots+c_{n-1}-c_n+c_n=r_1+c_1.\end{aligned}$$

Thus, we conclude that $\sum_{i=1}^{n+1}\beta_i\delta_i=0$.

ii) We shall verify that the discrete distribution shown in equation (4.8) satisfies the mean and variance constraints. Equation (4.7) indicates that θ_i^* is a linear transformation of δ_i. Thus, we immediately obtain that

$$\sum_{i=1}^{n+1}\beta_i\theta_i^*=\sum_{i=1}^{n+1}\beta_i\left(\mu+\frac{\sigma}{\Delta}\delta_i\right)=\mu+\frac{\sigma}{\Delta}\sum_{i=1}^{n+1}\beta_i\delta_i=\mu$$

and

$$\sum_{i=1}^{n+1}\beta_i\left(\theta_i^*-\mu\right)^2=\sum_{i=1}^{n+1}\beta_i\left(\frac{\sigma}{\Delta}\delta_i\right)^2=\frac{\sigma^2}{\Delta^2}\sum_{i=1}^{n+1}\beta_i\delta_i^2=\frac{\sigma^2}{\Delta^2}\Delta^2=\sigma^2,$$

where the third equality is based on the definition of Δ. We conclude that the discrete distribution shown in equation (4.8) satisfies the mean and variance constraints, becoming an element of the ambiguity set Ω. Q.E.D.

Proof of Proposition 2:

Step 1): Conjecture binding constraints. We conjecture that the SIP model in equation (4.6) has $(n+1)$ binding constraints at points θ_i (where $i=1,2,\cdots,n+1$) such that $\theta_i\in[Q_{i-1},Q_i]$. At the first segment with $\theta\in[0,Q_1]$, using equation (B-4), we find that the SIP constraint is

$$a_1+y_0+y_1\theta+y_2\theta^2\leq(p-r_1)\,\theta-\sum_{i=1}^{n}c_iq_i.$$

By assuming that $\theta=\theta_1$ is the binding constraint at the first segment, we obtain

$$a_1+y_0+y_1\theta_1+y_2\theta_1^2=(p-r_1)\,\theta_1-\sum_{i=1}^{n}c_iq_i=Z\left(\theta_1|\mathbf{Q}\right).$$

The above condition alone is insufficient to ensure that θ_1 is a locally binding constraint for the first segment. Similar to in Scarf (1958), the *tangent* condition $y_1+2y_2\theta_1=p-r_1$ must also hold. By repeating the same procedure on equations (B-5) and (B-6), we obtain all of the binding and tangent conditions. We summarize the $(n+1)$ binding conditions as follows:

$$\begin{cases}a_1+y_0+y_1\theta_1+y_2\theta_1^2=Z\left(\theta_1|\mathbf{Q}\right)\\ \vdots\\ a_i+y_0+y_1\theta_i+y_2\theta_i^2=Z\left(\theta_i|\mathbf{Q}\right)\\ \vdots\\ a_n+y_0+y_1\theta_n+y_2\theta_n^2=Z\left(\theta_n|\mathbf{Q}\right)\\ y_0+y_1\theta_{n+1}+y_2\theta_{n+1}^2=Z\left(\theta_{n+1}|\mathbf{Q}\right)\end{cases}\tag{B-8}$$

and $(n+1)$ tangent conditions as follows:

$$\begin{cases}y_1+2y_2\theta_1=p-r_1\\ \vdots\\ y_1+2y_2\theta_i=p-r_i\\ \vdots\\ y_1+2y_2\theta_n=p-r_n\\ y_1+2y_2\theta_{n+1}=p-r_{n+1}=0\end{cases}\tag{B-9}$$

Step 2: Solve the KKT conditions. The Lagrangian of equation (4.6) equals:

$$\begin{aligned}L=\;&a_1\beta_1+a_2\beta_2+...+a_n\beta_n+y_0+y_1\mu+y_2\left(\mu^2+\sigma^2\right)\\&-\sum_{i=1}^{n}\lambda_i\left[a_i+y_0+y_1\theta_i+y_2\theta_i^2-Z\left(\theta_i|\mathbf{Q}\right)\right]-\lambda_{n+1}\left[y_0+y_1\theta_{n+1}+y_2\theta_{n+1}^2-Z\left(\theta_{n+1}|\mathbf{Q}\right)\right],\end{aligned}$$

where $\lambda_i \geq 0$ is the Lagrangian multiplier.

After solving the FOC with respect to a_i, we obtain that $\frac{\partial L}{\partial a_i} = \beta_i - \lambda_i = 0$, implying that the first n Lagrangian multipliers satisfy that $\lambda_i = \beta_i$ for $i = 1, 2, \cdots, n$. Solving the FOC with respect to y_0, we obtain that $1 = \sum_{i=1}^{n+1} \lambda_i$. Because we have just shown that $\lambda_i = \beta_i$ for $i = 1, 2, \cdots, n$, we obtain that $\lambda_{n+1} = 1 - \sum_{i=1}^{n} \beta_i = \beta_{n+1}$. We conclude that the mass probabilities β_i shown in Definition 1 happen to be the Lagrangian multipliers.

We derive the conjectured binding points θ_i. We solve the FOCs with respect to y_1 and y_2 to obtain

$$\mu = \sum_{i=1}^{n+1} \beta_i \theta_i \text{ and } \mu^2 + \sigma^2 = \sum_{i=1}^{n+1} \beta_i \theta_i^2. \tag{B-10}$$

Equations (B-9) and (B-10) form a system with $(n+3)$ unknown variables (i.e., θ_i, y_1, y_2) and $(n+3)$ equations. To streamline the expressions, let $x_i = \theta_i - \mu$ for $i = 1, 2, \cdots, n+1$. We also define $\{v_i\}$ and $\{h_i\}$ for $i = 1, 2, \cdots, n$ in Table 3 below.

Table 3 The $\{v_i\}$ and $\{h_i\}$ sequences and their relationship to sequences $\{\beta_i\}$ and $\{\alpha_i\}$.

i	v_i	h_i	β_i	α_i
1	$c_1 - c_2$	$r_2 - r_1$	$1 - \frac{v_1}{h_1}$	$1 - \frac{v_1}{h_1}$
2	$c_2 - c_3$	$r_3 - r_2$	$\frac{v_1}{h_1} - \frac{v_2}{h_2}$	$1 - \frac{v_2}{h_2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	$c_{n-1} - c_n$	$r_n - r_{n-1}$	$\frac{v_{i-1}}{h_{i-1}} - \frac{v_i}{h_i}$	$1 - \frac{v_{i-1}}{h_{i-1}}$
n	c_n	$p - r_n$	$\frac{v_{n-1}}{h_{n-1}} - \frac{v_n}{h_n}$	$1 - \frac{c_n}{p - r_n}$
$n+1$	N/A	N/A	$\frac{c_n}{p - r_n}$	1

Using the sequences $\{v_i\}$ and $\{h_i\}$ given in Table 3, we rewrite the mean condition shown in equation (B-10) as:

$$\left(1 - \frac{v_1}{h_1}\right) x_1 + \cdots + \left(\frac{v_{i-1}}{h_{i-1}} - \frac{v_i}{h_2}\right) x_i + \cdots + \left(\frac{v_{n-1}}{h_{n-1}} - \frac{v_n}{h_n}\right) x_n + \frac{v_n}{h_n} x_{n+1} = 0.$$

We also simplify the tangent conditions (B-9) to obtain that $2y_2(x_i - x_{i+1}) = h_i$ for $i = 1, 2, \cdots, n$. After substituting the recursive equation $2y_2(x_i - x_{i+1}) = h_i$ into the mean condition and performing some algebra, we obtain that

$$\begin{cases} x_1 = \frac{\sum_{j=1}^{n} v_j}{2y_2} = \frac{c_1}{2y_2}, \\ x_i = \frac{\sum_{k=1}^{i} v_k - \sum_{k=1}^{i-1} h_k}{2y_2} = \frac{c_1 + r_1 - r_i}{2y_2}, \text{ for } i = 2, ..., n, \\ x_{n+1} = \frac{\sum_{i=1}^{n} v_i - \sum_{i=1}^{n} h_i}{2y_2} = \frac{c_1 + r_1 - p}{2y_2}. \end{cases}$$

We can simplify x_i by using equation (4.3) to obtain $x_i = -\frac{\delta_i}{2y_2}$, where y_2 is the remaining unknown variable. Substituting $x_i = -\frac{\delta_i}{2y_2}$ into the variance condition, we obtain that

$$\sigma^2 = \sum_{i=1}^{n+1} \beta_i x_i^2 = \sum_{i=1}^{n+1} \beta_i \left(-\frac{\delta_i}{2y_2}\right)^2 = \frac{1}{(2y_2)^2} \sum_{i=1}^{n+1} \beta_i \delta_i^2 = \frac{1}{(2y_2)^2} \Delta^2,$$

where the last equality follows the definition of Δ in Definition 2. We obtain that $(y_2)^2 = \left(\frac{\Delta}{2\sigma}\right)^2$. Recall that for $\theta \geq Q_n$, the SIP constraints of equation (4.6) include:

$$y_0 + y_1\theta + y_2\theta^2 \leq \sum_{i=1}^{n} (p - r_i - c_i)\, q_i,$$

where the RHS is a positive constant, indicating that $y_2 \leq 0$ (otherwise, the constraint cannot hold when $\theta \to \infty$). We obtain that $y_2^* = -\frac{\Delta}{2\sigma}$. We find that each conjecture binding point θ_i satisfies

$$\theta_i = x_i + \mu = -\frac{\delta_i}{2y_2^*} + \mu = -\frac{\delta_i}{-\frac{\Delta}{\sigma}} + \mu = \mu + \delta_i \frac{\sigma}{\Delta} = \theta_i^*.$$

Complementary slackness indicates that the discrete distribution satisfying $\Pr(\theta = \theta_i) = \lambda_i$ (for $i = 1, 2, \cdots, n+1$) represents the firm's worst-case distribution. Because we have demonstrated that the Lagrangian multipliers satisfy $\lambda_i = \beta_i$ and the conjectured binding points satisfy $\theta_i = \theta_i^*$, we conclude that the discrete distribution shown in equation (4.8) is the firm's worst-case distribution.

Step 3: We derive the remaining shadow prices. The final tangent condition in equation (B-9) indicates that:

$$y_1^* = -2y_2^*\theta_{n+1}^* = \frac{\Delta}{\sigma}\left[\mu + \frac{\sigma}{\Delta}(p - r_1 - c_1)\right] = (p - r_1 - c_1) + \Delta\frac{\mu}{\sigma}.$$

With y_1^* and y_2^* being known, the number of unknown variables in equation (B-8) decreases from $(n+3)$ to $(n+1)$. By solving the remaining $(n+1)$ equations in (B-8), we obtain

$$\begin{cases} y_0^* = y_2^* \left(\theta_{n+1}^*\right)^2 + Z\left(\theta_{n+1}^* | \mathbf{Q}\right) \\ a_n^* = -y_0^* - y_1^*\theta_n^* - y_2^*\left(\theta_n^*\right)^2 + Z\left(\theta_n^* | \mathbf{Q}\right) \\ \vdots \\ a_2^* = -y_0^* - y_1^*\theta_2^* - y_2^*\left(\theta_2^*\right)^2 + Z\left(\theta_2^* | \mathbf{Q}\right) \\ a_1^* = -y_0^* - y_1^*\theta_1^* - y_2^*\left(\theta_1^*\right)^2 + Z\left(\theta_1^* | \mathbf{Q}\right) \end{cases}. \tag{B-11}$$

Because we shall apply Theorem 1, it is unimportant to simplify equation (B-11) at this stage.

With $y_2^* < 0$, we can verify that the proposed shadow prices satisfy all of the SIP constraints in equation (4.6) because the tangent conditions (B-9) are the FOCs and $y_2 < 0$ is the second-order condition for guaranteeing a locally binding constraint. Q.E.D.

Proof of Proposition 3:

As the proof of Proposition 1 indicated, we can compute the value of the zero-sum game from the perspective of the firm or Adverse Nature. When both players play their equilibrium strategy, either perspective will lead to the same value of the zero-sum game. Because we defer the characterization of the firm's equilibrium strategy in the subsequent Proposition 4, we compute the value of the zero-sum game from the firm's perspective. Suppose that the firm's worst-case distribution F^* given by equation (4.8) is realized. For simplicity of exposition, we suppress the superscript $*$ in θ_i^* (i.e., we write θ_i^* in equation (4.8) as θ_i in this proof) but retain the superscript $*$ in F^*. Let $\mathbf{Q}$ be a capacity vector satisfying $Q_i \in [\theta_i, \theta_{i+1}]$.

Recall that $r_{n+1} = p$ and hence, in Table 3, $h_n = r_{n+1} - r_n = p - r_n$. Using r_i for all $i = 1, 2, \cdots, n+1$, we find that equation (B-5) becomes valid for all i for all $i = 1, 2, \cdots, n+1$. We obtain that when the realized demand is $\theta = \theta_i$, the ex post profit equals

$$Z(\theta_i|\mathbf{Q}) = (p - r_i)\theta_i + \sum_{j=1}^{i-1}(r_i - r_j - c_j)q_j - \sum_{j=i}^{n} c_j q_j = (p - r_i)\theta_i + \sum_{j=1}^{i-1}(r_i - r_j)q_j - \sum_{j=1}^{n} c_j q_j,$$

where $\sum_{j=1}^{n} c_j q_j = C$ is the total cost associated with the given capacity vector $\mathbf{Q}$. Thus, the firm's expected profit equals:

$$\begin{aligned} Z(\mathbf{Q}|F^*) &= \sum_{i=1}^{n+1}\beta_i Z(\theta_i|\mathbf{Q}) = \sum_{i=1}^{n+1}\beta_i(p - r_i)\theta_i + \sum_{i=1}^{n+1}\beta_i\left(\sum_{j=1}^{i-1}(r_i - r_j)q_j\right) - \sum_{i=1}^{n+1}\beta_i C \\ &= \sum_{i=1}^{n+1}\beta_i(p - r_i)\theta_i + \sum_{i=1}^{n+1}\beta_i\left(\sum_{j=1}^{i-1}(r_i - r_j)q_j\right) - C. \end{aligned} \tag{B-12}$$

We simplify equation (B-12) in the next two steps.

First, we find that the first summation in equation (B-12) equals

$$\sum_{i=1}^{n+1}\beta_i(p - r_i)\theta_i = p\sum_{i=1}^{n+1}\beta_i\theta_i - \sum_{i=1}^{n+1}\beta_i\theta_i r_i. \tag{B-13}$$

According to Corollary 1, the first term in equation (B-13) is $p\mu$. Using equation (4.3), we find that for $i = 1, 2, \cdots, n+1$, $\delta_i - r_i = -r_1 - c_1$, implying that

$$\theta_i r_i = \left(\mu + \delta_i\frac{\sigma}{\Delta}\right)(\delta_i + r_1 + c_1) = \mu\delta_i + \mu(r_1 + c_1) + \delta_i^2\frac{\sigma}{\Delta} + (r_1 + c_1)\delta_i\frac{\sigma}{\Delta}.$$

Applying Corollary 1, we simplify the second term in equation (B-13) as:

$$\begin{aligned} \sum_{i=1}^{n+1}\beta_i\theta_i r_i &= \sum_{i=1}^{n+1}\beta_i\left[\mu\delta_i + \mu(r_1 + c_1) + \delta_i^2\frac{\sigma}{\Delta} + (r_1 + c_1)\delta_i\frac{\sigma}{\Delta}\right] \\ &= 0\cdot\mu + \mu(r_1 + c_1) + \Delta^2\frac{\sigma}{\Delta} + (r_1 + c_1)\frac{\sigma}{\Delta}\cdot 0 = \mu(r_1 + c_1) + \Delta\sigma. \end{aligned}$$

Thus, equation (B-13) becomes

$$\sum_{i=1}^{n+1} \beta_i (p - r_i)\theta_i = (p - c_1 - r_1)\,\mu - \Delta\sigma.$$

Next, we find that the second summation in equation (B-12) equals

$$\begin{aligned}
S &= \sum_{i=1}^{n+1} \beta_i \left(\sum_{j=1}^{i-1} (r_i - r_j)\, q_j \right) = \beta_1 \cdot 0 + \beta_2 \left[(r_2 - r_1)\, q_1 \right] + \beta_3 \left[(r_2 - r_1) q_1 + (r_3 - r_2)(q_1 + q_2) \right] \\
&\quad + \beta_4 \left[(r_2 - r_1) q_1 + (r_3 - r_2)(q_1 + q_2) + (r_4 - r_3)(q_1 + q_2 + q_3) \right] \\
&\quad + \cdots + \beta_n \begin{bmatrix} (r_2 - r_1) q_1 + (r_3 - r_2)(q_1 + q_2) \\ + \cdots + (r_n - r_{n-1})(q_1 + q_2 + \cdots + q_{n-1}) \end{bmatrix} \\
&\quad + \beta_{n+1} \begin{bmatrix} (r_2 - r_1) q_1 + (r_3 - r_2)(q_1 + q_2) \\ + \cdots + (r_n - r_{n-1})(q_1 + q_2 + \cdots + q_{n-1}) \\ + (r_{n+1} - r_n) \sum_{i=1}^{n} q_i \end{bmatrix}.
\end{aligned}$$

By reorganizing the terms, we obtain

$$\begin{aligned}
S &= (\beta_2 + \beta_3 + \cdots + \beta_{n+1})\,(r_2 - r_1) q_1 + (\beta_3 + \cdots + \beta_{n+1})(r_3 - r_2)(q_1 + q_2) \\
&\quad + \cdots + (\beta_n + \beta_{n+1})\,(r_n - r_{n-1}) \left(\sum_{i=1}^{n-1} q_i \right) + \beta_{n+1} (r_{n+1} - r_n) \left(\sum_{i=1}^{n} q_i \right) \\
&= (1 - \alpha_1)\,(r_2 - r_1) q_1 + (1 - \alpha_2)\,(r_3 - r_2)(q_1 + q_2) \\
&\quad + \cdots + (1 - \alpha_{n-1})\,(r_n - r_{n-1}) \left(\sum_{i=1}^{n-1} q_i \right) + (1 - \alpha_n)\,(r_{n+1} - r_n) \left(\sum_{i=1}^{n} q_i \right) \\
&= \left(\frac{c_1 - c_2}{r_2 - r_1} \right) (r_2 - r_1) q_1 + \left(\frac{c_2 - c_3}{r_3 - r_2} \right) (r_3 - r_2)(q_1 + q_2) \\
&\quad + \cdots + \left(\frac{c_{n-1} - c_n}{r_n - r_{n-1}} \right) (r_n - r_{n-1}) \left(\sum_{i=1}^{n-1} q_i \right) + \left(\frac{c_n}{r_{n+1} - r_n} \right) (r_{n+1} - r_n) \left(\sum_{i=1}^{n} q_i \right) \\
&= (c_1 - c_2) q_1 + (c_2 - c_3)(q_1 + q_2) + \cdots + (c_{n-1} - c_n)(q_1 + q_2 + \cdots + q_{n-1}) \\
&\quad + c_n (q_1 + q_2 + \cdots + q_n) \\
&= q_1 (c_1 - c_2 + c_2 - c_3 + \cdots + c_{n-1} - c_n + c_n) + q_2\,(c_2 - c_3 + c_3 - c_4 + \cdots + c_{n-1} - c_n + c_n) \\
&\quad + \cdots + q_{n-1}\,(c_{n-1} - c_n + c_n) + c_n q_n = \sum_{i=1}^{n} c_i q_i = C.
\end{aligned}$$

Hence, equation (B-12) becomes $Z(\mathbf{Q}|F^*) = (p - r_1 - c_1)\mu - \Delta\sigma$, which is a constant whenever $\theta_i \in [Q_{i-1}, Q_i]$ holds. We conclude that under the firm's worst-case distribution F^*, the firm is indifferent among an infinite number of capacity vectors, and her optimal expected profit is a constant that equals $(p - r_1 - c_1)\mu - \Delta\sigma$. Q.E.D.

Proof of Proposition 4:

For simplicity of exposition, we suppress the superscript $*$ (i.e., we write θ_i^* and y_i^* as θ_i and y_i in this proof). In the proof of Proposition 2, we find that the shadow prices satisfy $y_2 = -\frac{\Delta}{2\sigma}$,

$y_1 = (p - r_1 - c_1) + \Delta\frac{\mu}{\sigma}$, and the remaining shadow prices y_0 and a_i in equations (B-11). By forcing $a_i = 0$ for all $i = 1, 2, \cdots n$, we can solve the firm's optimal capacity levels. Note that we never force $y_0 = 0$. To illustrate the recursive procedure, we consider $a_n = 0$. Equations (B-11) show that

$$\begin{cases} y_0 = y_2 (\theta_{n+1})^2 + Z(\theta_{n+1}|\mathbf{Q}) \\ a_n = -y_0 - y_1\theta_n - y_2(\theta_n)^2 + Z(\theta_n|\mathbf{Q}) = 0 \end{cases}$$

We obtain that

$$y_2 (\theta_{n+1})^2 + Z(\theta_{n+1}|\mathbf{Q}) = -y_1\theta_n - y_2(\theta_n)^2 + Z(\theta_n|\mathbf{Q}).$$

Using the tangent condition $y_1 = -2y_2\theta_{n+1}$, we can rewrite the above equation as:

$$Z(\theta_{n+1}|\mathbf{Q}) - Z(\theta_n|\mathbf{Q}) = 2y_2\theta_{n+1}\theta_n - y_2(\theta_n)^2 - y_2(\theta_{n+1})^2 = -y_2(\theta_{n+1} - \theta_n)^2. \tag{B-14}$$

Because the total cost is sunk, by using equation (4.2), we find that:

$$Z(\theta_{n+1}|\mathbf{Q}) - Z(\theta_n|\mathbf{Q}) = (p - r_n)(Q_n - \theta_n).$$

Applying the two tangent conditions related to $i = n$ and $i = n+1$, we obtain that $y_1 + 2y_2\theta_n = p - r_n$ and $y_1 = -2y_2\theta_{n+1}$. We find that $2y_2(\theta_{n+1} - \theta_n) = -(p - r_n)$ and equation (B-14) becomes:

$$(p - r_n)(Q_n - \theta_n) = \frac{1}{2}(p - r_n)(\theta_{n+1} - \theta_n),$$

which results in $Q_n = \frac{1}{2}(\theta_{n+1} + \theta_n)$.

Applying the same method, we can generalize equation (B-14) for any $i = 1, 2, \cdots, n-1$ by establishing that:

$$Z(\theta_{i+1}|\mathbf{Q}) - Z(\theta_i|\mathbf{Q}) = (r_{i+1} - r_i)(Q_i - \theta_i) = -y_2(\theta_{i+1} - \theta_i)^2 = \frac{1}{2}(r_{i+1} - r_i)(\theta_{i+1} - \theta_i),$$

resulting in $Q_i = \frac{1}{2}(\theta_{i+1} + \theta_i)$. Q.E.D.

Proof of Corollary 2:

We reintroduce the superscript $*$ in y_i^*. While Proposition 4 solves for the robust optimal capacity vector, the proof of Proposition 2 shows that $y_2^* = -\frac{\Delta}{2\sigma}$ and $y_1^* = (p - r_1 - c_1) + \Delta\frac{\mu}{\sigma}$, leaving y_0^* as the only unknown variable. Using the equivalence $P^* = P_1^* = (p - r_1 - c_1)\mu - \sigma\Delta$, we obtain that

$$y_0^* + y_1^*\mu + y_2^*\left(\mu^2 + \sigma^2\right) = (p - r_1 - c_1)\mu - \sigma\Delta.$$

By reorganizing the terms in the above equation, we obtain

$$y_0^* = (p - r_1 - c_1)\mu - \sigma\Delta - \left[(p - r_1 - c_1) + \Delta\frac{\mu}{\sigma}\right]\mu + \frac{\Delta}{2\sigma}\left(\mu^2 + \sigma^2\right) = -\frac{\Delta}{2\sigma}\left(\mu^2 + \sigma^2\right).$$

The final remark is that equation (B-11) is more complex than the counterpart equation (A-4), prompting us to derive q_i^* by forcing $a_i = 0$. Readers might concern that this step could weakly reduce the value of P_1^*. Because we compute the value of the zero-sum game based on Adverse Nature's equilibrium strategy F^* in the proof of Proposition 3 and find that the solution in Corollary 2 yields the same value of the zero-sum game, we conclude that the optimal solution shown in Corollary solves P_1 and satisfies Theorem 1. Thus, the robust capacity vector in Proposition 4 is the firm's equilibrium strategy. Q.E.D.

Part C: Proofs for Section 5

Proof of Lemma 4

We obtain the following symmetric tangent conditions:

$$\begin{cases} y_1 + 2y_2x_1 + y_3x_1 = p, \\ y_1 + 2y_2x_2 + y_3x_3 = p, \\ y_1 + 2y_2x_3 + y_3x_2 = 0, \\ y_1 + 2y_2x_4 + y_3x_4 = 0. \end{cases}$$

We regard y_1, y_2, and y_3 as input parameters and x_i as unknown variables. We obtain that

$$\begin{aligned} x_1 &= \frac{p - y_1}{2y_2 + y_3}, \ x_2 = \frac{2(y_1 - p)y_2 - y_1y_3}{y_3^2 - 4y_2^2}, \\ x_3 &= \frac{(p - y_1)\, y_3 + 2y_1y_2}{y_3^2 - 4y_2^2}, \text{ and } x_4 = \frac{-y_1}{2y_2 + y_3}. \end{aligned}$$

The above equations imply that

$$x_1 + x_4 = x_2 + x_3 = \frac{p - 2y_1}{2y_2 + y_3},$$

which we refer to as the symmetric property. The effective moment conditions include:

$$\begin{cases} \mu = \lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3 + \lambda_4x_4, \\ \mu^2 + \sigma^2 = \lambda_1 (x_1)^2 + \lambda_2 (x_2)^2 + \lambda_3 (x_3)^2 + \lambda_4 (x_4)^2, \\ \rho\sigma^2 + \mu^2 = \lambda_1 (x_1)^2 + \lambda_2 (x_2x_3) + \lambda_3 (x_3x_2) + \lambda_4 (x_4)^2. \end{cases}$$

Combining these three effective moment conditions with the symmetric property $x_1 + x_4 = x_2 + x_3$, we obtain the four-point distribution shown in Lemma 4. We shall explain the condition on ρ after deriving the optimal inventory levels. Q.E.D.

Remark 3 *If parameters are asymmetric, we apply eight tangent conditions to determine four pairs of coordinates by using the shadow prices as input parameters. Five shadow prices* $(y_{11}, y_{12}, y_{21}, y_{22}, y_3)$ *are involved in this step and five moment conditions remain (because the total probability condition is already used when deriving* λ_i*). We can solve the nonlinear system with five* *variables and five equations to obtain the shadow prices, which in turn determine the coordinates of the four binding points. This step could be too complex to be analytically tractable. However, we can easily obtain the numerical solution using commercial software such as EXCEL. The symmetric special case, however, gives rise to the symmetric property $x_1 + x_4 = x_2 + x_3$, which enables us to conveniently characterize the equilibrium strategy of Adverse Nature in Lemma 4.*

Proof of Proposition 5

For exposition simplicity, we omit the superscript $*$ when involving x_i^*. Using the first and fourth tangent conditions shown in the proof of Lemma 4, we obtain that

$$(y_1 + 2y_2x_1 + y_3x_1)\,x_4 - (y_1 + 2y_2x_4 + y_3x_4)\,x_1 = px_4 - 0 \cdot x_1,$$

which yields that $y_1 = \frac{px_4}{x_4 - x_1}$. Similarly, we obtain that $y_2 = \frac{p}{4}\left(\frac{1}{x_1 - x_4} + \frac{1}{x_2 - x_3}\right)$ and $y_3 = \frac{p}{2}\left(\frac{1}{x_1 - x_4} - \frac{1}{x_2 - x_3}\right)$.

With $a_i = 0$, we simplify the binding constraints as follows:

$$\begin{aligned}
y_0 + y_1x_1 + y_1x_1 + y_2\,(x_1)^2 + y_2\,(x_1)^2 + y_3x_1x_1 &= p\,(x_1 + x_1) - 2cq - c_0q_0\\
y_0 + y_1x_2 + y_1x_3 + y_2\,(x_2)^2 + y_2\,(x_3)^2 + y_3x_2x_3 &= p\,(q + x_2) - 2cq - c_0q_0\\
y_0 + y_1x_3 + y_1x_2 + y_2\,(x_3)^2 + y_2\,(x_2)^2 + y_3x_3x_2 &= p\,(x_2 + q) - 2cq - c_0q_0\\
y_0 + y_1x_4 + y_1x_4 + y_2\,(x_4)^2 + y_2\,(x_4)^2 + y_3x_4x_4 &= pq_0 - 2cq - c_0q_0
\end{aligned}$$

Using the first and fourth binding constraints, we obtain

$$\begin{aligned}
p\,(q_0 - x_1 - x_1) &= 2y_1\,(x_4 - x_1) + 2y_2\left(x_4^2 - x_1^2\right) + y_3\left(x_4^2 - x_1^2\right)\\
&= y_1\,(x_4 - x_1) - px_1 = \frac{px_4}{x_4 - x_1}\,(x_4 - x_1) - px_1 = p\,(x_4 - x_1)\,,
\end{aligned}$$

which yields that $q_0^* = x_1 + x_4$ and proves equation (5.4).

The sum of the first and fourth binding constraints minus that of the second and third binding constraints yields that

$$p\,(q_0 - 2q - 2x_2 + 2x_1) = 2y_2\left(x_1^2 + x_4^2 - x_2^2 - x_3^2\right) + y_3\left(x_1^2 + x_4^2 - 2x_2x_3\right).$$

Using tangent conditions, we obtain

$$\begin{aligned}
y_1x_1 + 2y_2x_1^2 + y_3x_1^2 &= px_1\\
y_1x_2 + 2y_2x_2^2 + y_3x_2x_3 &= px_2\\
y_1x_3 + 2y_2x_3^2 + y_3x_2x_3 &= 0\\
y_1x_4 + 2y_2x_4^2 + y_3x_4^2 &= 0.
\end{aligned}$$

Hence, we find that

$$\begin{aligned} px_1 - px_2 &= y_1 (x_1 + x_4 - x_2 - x_3) + 2y_2 \left(x_1^2 + x_4^2 - x_2^2 - x_3^2\right) + y_3 \left(x_1^2 + x_4^2 - 2x_2 x_3\right) \\ &= 2y_2 \left(x_1^2 + x_4^2 - x_2^2 - x_3^2\right) + y_3 \left(x_1^2 + x_4^2 - 2x_2 x_3\right). \end{aligned}$$

We find that $p(q_0 - 2q - 2x_2 + 2x_1) = p(x_1 - x_2)$, which is equivalent to

$$q^* = \frac{q_0 - x_2 + x_1}{2} = \frac{x_1 + x_4 - x_2 + x_1}{2} = \frac{x_2 + x_3 - x_2 + x_1}{2} = \frac{x_1 + x_3}{2},$$

confirming equation (5.3). The last shadow price y_0 is complex but can be easily derived from any one of the binding constraints.

Under the robust optimal production plan, the total cost equals

$$\begin{aligned} TC &= 2cq^* + c_0 q_0^* = c(x_1 + x_3) + c_0 (x_1 + x_4) \\ &= 2(c + c_0)\mu + \sigma \left[A - \frac{1}{2} p(c + 2c_0)\right] \sqrt{\frac{1+\rho}{A}} + \frac{\sigma}{2} \sqrt{(1-\rho)pc}. \end{aligned}$$

The value of the zero-sum game equals

$$\begin{aligned} Z^* &= \left(1 - \frac{2c + c_0}{p}\right) \cdot 2px_1 + \frac{c}{p} \cdot 2p(x_2 + q) + \frac{c_0}{p} \cdot p(x_1 + x_4) - TC \\ &= \left(1 - \frac{2c + c_0}{p}\right) 2px_1 + c(2x_2 + x_1 + x_3) + c_0 (x_1 + x_4) - c(x_1 + x_3) - c_0 (x_1 + x_4) \\ &= 2\left[(p - 2c - c_0) x_1 + cx_2\right] = 2(p - c - c_0)\mu - \left[\sqrt{(1+\rho)A} + \sqrt{(1-\rho)pc}\right] \sigma. \end{aligned}$$

We obtain that the deference between Z^* with Z_{nc} equals

$$Z^* - Z_{nc} = \sigma \left[2\sqrt{(p - c - c_0)(c + c_0)} - \sqrt{(1+\rho)A} - \sqrt{(1-\rho)pc}\right].$$

Observe that

$$A + pc = p(c + 2c_0) - 2(c + c_0)^2 + pc = 2(p - c - c_0)(c + c_0).$$

We find that

$$Z^* - Z_{nc} = \sigma \left[\sqrt{2A + 2pc} - \sqrt{(1+\rho)A} - \sqrt{(1-\rho)pc}\right].$$

We can verify that

$$\begin{aligned} & 2A + 2pc - \left(\sqrt{(1+\rho)A} + \sqrt{(1-\rho)pc}\right)^2 \\ &= 2A + 2pc - (1+\rho)A - (1-\rho)pc - 2\sqrt{(1+\rho)A(1-\rho)pc} \\ &= (1-\rho)A + (1+\rho)pc - 2\sqrt{(1+\rho)A(1-\rho)pc} \geq 0, \end{aligned}$$

where the last inequality is due to the well-known geometric inequality. Thus, $Z^* \geq Z_{nc}$, where the equal sign holds if $(1-\rho)A = (1+\rho)pc$, which yields that $\rho = \frac{A-pc}{A+pc} = \rho_{nc}$.

Up to this point, we have relaxed the constraints that $2q \geq q_0$. It is readily verified that when $\rho = \rho_{nc} = \frac{A-pc}{A+pc}$, several coincidences occur. First,

$$2q^* = q_0^* = 2\mu + \sigma\left(\sqrt{\frac{p-c-c_0}{c+c_0}} - \sqrt{\frac{c+c_0}{p-c-c_0}}\right) = 2q_{nc}^*,$$

meaning that the robust inventory levels are identical to the case without commonality. Thus, to ensure that the robust inventory levels are non-zero, it must hold that $Z_{nc} > 0$. Second, we have already shown that Z^* in equation (5.5) also equals Z_{nc} when $\rho = \rho_{nc}$.

We now determine the equilibrium strategy of Adverse Nature when $\rho \geq \rho_{nc}$. Based on Scarf's model, the marginal distribution must display the following property:

$$\begin{cases} \Pr\left(\tilde{\theta}_j = \mu - \sigma\sqrt{\frac{c+c_0}{p-c-c_0}} = \theta_l\right) = \frac{p-c-c_0}{p}, \\ \Pr\left(\tilde{\theta}_j = \mu + \sigma\sqrt{\frac{p-c-c_0}{c+c_0}} = \theta_h\right) = \frac{c+c_0}{p}. \end{cases} \tag{C-1}$$

The new issue is whether the covariance constraint can hold. Let $\beta = \frac{p-c-c_0}{p}$ be the newsvendor ratio of product j without commonality. We construct the following four-point distribution:

$$\left(\tilde{\theta}_1, \tilde{\theta}_2\right) = \begin{cases} (\theta_l, \theta_l) & \text{with probability } \beta^2 + \rho(1-\beta)\beta, \\ (\theta_l, \theta_h) & \text{with probability } \beta(1-\beta) - \rho(1-\beta)\beta, \\ (\theta_h, \theta_l) & \text{with probability } \beta(1-\beta) - \rho(1-\beta)\beta, \\ (\theta_h, \theta_h) & \text{with probability } (1-\beta)^2 + \rho(1-\beta)\beta. \end{cases} \tag{C-2}$$

With $\rho \geq \rho_{nc}$, the probabilities in equation (C-2) are positive. The four-point distribution in equation (C-2) satisfies the marginal distribution specified by equation (C-1) and all the moment conditions. The third coincidence is that when $\rho = \rho_{nc}$, the distribution in equation (C-2) is identical to that in Lemma 4.

Because the firm manages the inventory separately (i.e., $2q = q_0$), we can decompose the inventory system as two Scarf's models. When the marginal distribution of θ_j satisfies the properties of Scarf's model, equation (C-2) emerges as Adverse Nature's equilibrium strategy in the benchmark without commonality. Q.E.D.

Proof of Corollary 3

When $\rho > \rho_{nc}$, the firm's equilibrium strategy satisfies $2q = q_0 = 2q_{nc}^*$, giving rise to the benchmark without commonality and $\frac{\partial Z^*}{\partial \rho} = 0$. We focus on the case with $\rho \leq \rho_{nc}$ and show that the comparative static $\frac{\partial Z^*}{\partial \rho}$ is negative. Equation (5.5) indicates that

$$\frac{\partial Z^*}{\partial \rho} = \sigma\left(\frac{\sqrt{pc}}{2\sqrt{1-\rho}} - \frac{\sqrt{A}}{2\sqrt{1+\rho}}\right).$$

The pre-condition $\rho \leq \frac{A-pc}{A+pc} = \rho_{nc}$ implies that $A(1-\rho) \geq pc(1+\rho)$, which is equivalent to

$$\frac{\sqrt{pc}}{2\sqrt{1-\rho}} \leq \frac{\sqrt{A}}{2\sqrt{1+\rho}}.$$

Thus, we conclude that $\frac{\partial Z^*}{\partial \rho} < 0$ when $\rho < \rho_{nc}$. Q.E.D.

Perfect Correlation

When $\rho = 1$, the demand vector changes to one-dimensional with $(\theta_1, \theta_2) = (\theta, \theta)$. We can apply Scarf's model to confirm that $q^* = q^*_{nc}$, which is consistent with Proposition 5 when $\rho \geq \rho_0$.

When $\rho = -1$, the demand vector changes to $(\theta, 2\mu - \theta)$, which also becomes one-dimensional. It is readily verified that $\sigma^2 \leq \mu^2$ otherwise, θ could be negative. The linear programming model changes to the following:

$$\begin{aligned} Z(\theta, 2\mu - \theta | q, q_0) &= \max_{s_1, s_2 \geq 0} \{p(s_1 + s_2)\} - 2cq - c_0 q_0 \\ \text{s.t.} \quad & s_1 \leq \min(\theta, q), \; s_2 \leq \min(2\mu - \theta, q), \text{ and } s_1 + s_2 \leq q_0. \end{aligned}$$

Note that $s_1 + s_2 \leq \theta + 2\mu - \theta = 2\mu$ (i.e., the total assembled quantity cannot exceed the total demand). We find that $q_0 = 2\mu$. Due to the inventory balance constraint $2q \geq q_0$, we find that $q \geq \mu$. The maximum demand for product j is 2μ and hence, $q \leq 2\mu$. Therefore, the ex post payoff equals

$$Z(\theta|q) = p \min\{2\mu, q + \theta, q + 2\mu - \theta\} - 2cq - 2c_0\mu,$$

which is piece-wise linear (and has three pieces). The marginal payoff is the following: i) $\frac{Z(\theta|q)}{\partial q} = p - 2c$ when $\theta \leq 2\mu - q$; ii) $\frac{Z(\theta|q)}{\partial q} = -2c$ when $2\mu - q < \theta < q$; and iii) $\frac{Z(\theta|q)}{\partial q} = p - 2c$ when $\theta \geq q$. We formulate the SIP model for Adverse Nature as follows:

$$\begin{aligned} P = & \max_{y_i, q, a} \left\{ y_0 + y_1 \mu + y_2 \left(\mu^2 + \sigma^2\right) \right\} \\ \text{s.t.} \quad & y_0 + y_1 \theta + y_2 \theta^2 + a(p - 2c) \leq p(q + \theta) - 2cq - 2c_0\mu, \; \forall 0 \leq \theta \leq 2\mu - q; \\ & y_0 + y_1 \theta + y_2 \theta^2 - 2ac \leq 2\mu p - 2cq - 2c_0\mu, \; \forall \theta \in 2\mu - q \leq \theta \leq q; \\ & y_0 + y_1 \theta + y_2 \theta^2 + a(p - 2c) \leq p(q + 2\mu - \theta) - 2cq - 2c_0\mu, \; \forall \theta \in q \leq \theta \leq 2\mu. \end{aligned}$$

The conjectured binding constraints include $\theta = x$, $\theta = \mu$, and $\theta = 2\mu - x$ (where x is to be determined). In other words, the marginal distribution of θ is a three-point distribution.

After solving this SIP model, we obtain that $x^* = \mu - \frac{\sigma}{2}\sqrt{\frac{2p}{c}}$ and $q^* = \mu + \frac{\sigma}{4}\sqrt{\frac{2p}{c}}$, which is identical to equation (5.3). Hence, the three-point marginal distribution is the following: $\Pr\left(\tilde{\theta}_1 = x^*\right) = \frac{c}{p}$,

$\Pr\left(\tilde{\theta}_1 = \mu\right) = 1 - \frac{2c}{p}$, and $\Pr\left(\tilde{\theta}_1 = 2\mu - x^*\right) = \frac{c}{p}$. Using the relationship $\theta_2 = 2\mu - \theta_1$, we convert this marginal distribution into the following joint distribution:

$$\left(\tilde{\theta}_1, \tilde{\theta}_2\right) = \begin{cases} (\mu, \mu) & \text{with probability } 1 - \frac{2c}{p}, \\ (x^*, 2\mu - x^*) & \text{with probability } \frac{c}{p}, \\ (2\mu - x^*, x^*) & \text{with probability } \frac{c}{p}. \end{cases}$$

Interestingly, with $\rho = -1$, Lemma 4 indicates that $x_1^* = x_4^* = \mu$ and thus, the four-point distribution in Lemma 4 is identical to the above three-point distribution. Thus, Lemma 4 holds when $\rho = -1$.