 doi:10.3934/amc.2022016

ON SOME CODES FROM RANK 3 PRIMITIVE ACTIONS OF THE SIMPLE CHEVALLEY GROUP $G_2(q)$

TUNG LE AND BERNARDO G. RODRIGUES*

Department of Mathematics and Applied Mathematics
University of Pretoria
Hatfield 0028, South Africa

Dedicated to the Memory of Kay Magaard

ABSTRACT. Let $G_2(q)$ be a Chevalley group of type G_2 over a finite field $\mathbb{F}_q$. Considering the $G_2(q)$-primitive action of rank 3 on the set of $\frac{q^3(q^3-1)}{2}$ hyperplanes of type $O_6^-(q)$ in the 7-dimensional orthogonal space $\mathrm{PG}(7, q)$, we study the designs, codes, and some related geometric structures. We obtained the main parameters of the codes, the full automorphism groups of these structures, and geometric descriptions of the classes of minimum weight codewords.

1. INTRODUCTION

In recent studies [17, 18], it was shown that codes from the row span over finite fields of incidence matrices of regular graphs have uniform properties that can result in the graphs being retrieved from the code. It was observed in those papers that under certain hypothesis the minimum weight of the code is precisely the valency k of the graph, and the minimum weight codewords are the rows of the incidence matrix of the graph and their scalar multiples. These properties and the gap in the weight enumerator between k and $2(k-1)$ seem to be particularly characteristic to codes of incidence matrices of regular graphs, (see [18] and its references). This is in stark contrast with the codes from adjacency matrices. However, the binary codes defined by the row span of the adjacency matrices of a number of strongly regular graphs enjoy the property that their minimum weight is the valency of the graph and the minimum weight codewords are exactly the rows of the adjacency matrix and their scalar multiples. In addition, the reflexive graphs, i.e. graphs obtained by including a loop at every vertex have been shown to also satisfy these properties.

For reflexive graphs whose codes possess these properties and which are related to strongly regular graphs, see [33] for the binary codes from the strongly regular graph related to the Conway group Co_2, [34] for the binary codes of the strongly regular graph related to the simple group Ru of Rudvalis, and [35] for the ternary codes of the complement of the Higman-Sims graph. Other classes of reflexive graphs also yielded interesting codes, see for example [21, 22] for the study of the binary and ternary codes from reflexive uniform subset graphs on the 3-element subsets of a set of size n.

2020 *Mathematics Subject Classification:* Primary: 05B05, 94B05; Secondary: 20D45.
Key words and phrases: Strongly regular graph, modular representation theory, linear code.
The second author acknowledges research support by the National Research Foundation of South Africa (Grant Number 120846).
* Corresponding author.

Note that the simple group $G = G_2(q)$ acts primitively of rank 3 on the set of hyperplanes of type $O_6^-(q)$ in the 7-dimensional orthogonal space $\mathrm{PG}(7, q)$ only if $q \in \{3, 4\}$. Let $\mathcal{G}$ denote the the strongly regular graph on $\frac{q^3(q^3-1)}{2}$ vertices, $\overline{\mathcal{G}}$ its complementary graph, $\mathcal{G}^R$ and $\overline{\mathcal{G}}^R$ their corresponding reflexive graphs. For every prime divisor p of $|G|$ we examine the codes $C_p(\mathcal{G})$, $C_p(\overline{\mathcal{G}})$, $C_p(\mathcal{G}^R)$ and $C_p(\overline{\mathcal{G}}^R)$, obtained by taking the p-ary row span of the adjacency matrices of the graphs. We show that some codes of these matrices satisfy the property that their minimum weight is the valency of the graph and the minimum weight codewords are the rows of the adjacency matrix of the graph and their scalar multiples. In this way, we extend the class of strongly regular graphs whose codes satisfy these properties.

Codes from the reflexive graphs can be quite different from those form the graph itself, although they are clearly related (for example $C_p(\mathcal{G})^\perp \subseteq C_p(\mathcal{G}^R)$). There are however, some significant differences that make most of these codes worthy examining. In particular, in the paper we present an instance where $C_p(\overline{\mathcal{G}}) = C_p(\mathcal{G}^R)$, whereas some of the codes $C_p(\mathcal{G}^R)$ or $C_p(\overline{\mathcal{G}}^R)$ are the full space $\mathbb{F}_p^{|V|}$. We summarize these findings in Theorem 1.1 and through a series of lemmas, propositions and theorems we prove our results in Sections 5 and 6.

Theorem 1.1. *Let $G = G_2(q)$ be a Chevalley group of type G_2 over a finite field $\mathbb{F}_q$ where $q \in \{3, 4\}$. Let Γ and Λ be the strongly regular graphs defined by the rank 3 action of G of degree $\frac{q^3(q^3-1)}{2}$ on the set of hyperplanes of type $O_6^-(q)$ in the 7-dimensional orthogonal space $\mathrm{PG}(7, q)$, and let $\overline{\Gamma}$ and $\overline{\Lambda}$ be their complements. Let Γ^R and Λ^R be their reflexive associates (including the loops), $\overline{\Gamma}^R$ and $\overline{\Lambda}^R$ be their complementary graphs. For $\mathcal{G} \in \{\Gamma, \Lambda, \overline{\Gamma}, \overline{\Lambda}, \Gamma^R, \Lambda^R, \overline{\Gamma}^R, \overline{\Lambda}^R\}$ and $p||G|$, let $C_p(\mathcal{G})$ denote the code obtained as a p-ary row span of the adjacency matrix of $\mathcal{G}$.*

1. *Assume that $G = G_2(3)$.*
   - (a) *If $\mathcal{G} = \Gamma$ or $\mathcal{G} = \Gamma^R$ then*
     - (i) *$C_p(\Gamma)$ is a code of codimension 1 in $\mathbb{F}_p^{351}$ for $p = 2, 7$, and $C_{13}(\Gamma) = \mathbb{F}_{13}^{351}$.*
     - (ii) *$C_3(\Gamma)$ is a self-orthogonal $[351, 27, 126]_3$ code and an irreducible $G_2(3)$-module, and $\mathrm{Aut}(C_3(\Gamma)) \cong O_7(3){:}2$. Its dual code $C_3(\Gamma)^\perp$ is a $[351, 324, 6]_3$ code. The minimum weight of $C_3(\Gamma)$ is the valency of the graph Γ and the minimum weight codewords are the rows of the adjacency matrix of Γ and their scalar multiples.*
     - (iii) *$C_p(\Gamma^R) = \mathbb{F}_p^{351}$ for $p \neq 2$.*
     - (iv) *$C_2(\Gamma^R) = [351, 79, 48]_2$ and $C_2(\Gamma^R) \cap C_2(\Gamma^R)^\perp = [351, 78, 48]_2$.*
   - (b) *If $\mathcal{G} = \overline{\Gamma}$ or $\mathcal{G} = \overline{\Gamma}^R$ then*
     - (i) *$C_2(\overline{\Gamma}) = C_2(\overline{\Gamma}^R)$ is a self-orthogonal doubly-even $[351, 78, 48]_2$ code. Moreover, $C_2(\overline{\Gamma})$ is a faithful irreducible $G_2(3)$-module and* Aut *$(C_2(\overline{\Gamma})) \cong O_7(3){:}2$.*
     - (ii) *$C_p(\overline{\Gamma}) = \mathbb{F}_p^{351}$ for $p = 3, 13$ and $C_7(\overline{\Gamma})$ is a code of codimension 1 in $\mathbb{F}_7^{351}$.*
     - (iii) *$C_p(\overline{\Gamma}^R) = \mathbb{F}_p^{351}$ for $p \neq 3$.*
     - (iv) *$C_3(\overline{\Gamma}^R) = [351, 28, 108]_3$ and $C_3(\overline{\Gamma}^R) = C_3(\Gamma) + \langle \mathbf{1} \rangle$.*
2. *Assume that $G = G_2(4)$.*
   - (a) *If $\mathcal{G} = \Lambda$ or $\mathcal{G} = \Lambda^R$ then*

(i) $C_p(\Lambda) = \mathbb{F}_p^{2016}$ *for* $p = 2, 7$, *and* $C_{13}(\Lambda)$ *is a code of codimension* 1 *in* $\mathbb{F}_{13}^{2016}$.
(ii) $C_3(\Lambda)$ *is a self-orthogonal* $[2016, 651, d]_3$ *code with* $d \geq 975$.
(iii) $\mathbb{F}_5^{2016} = C_5(\Lambda) + C_5(\Lambda)^{\perp}$ *and* $\dim(C_5(\Lambda)) = 650$.
(iv) $C_2(\Lambda^R)$ *is a self-orthogonal triply even* $[2016, 14, 976]_2$ *code, and its dual code* $C_2(\Lambda^R)^{\perp}$ *is a* $[2016, 2002, 4]_2$ *code. The minimum weight of* $C_2(\Lambda^R)$ *is the valency of the reflexive graph* Λ^R, *and the words of minimum weight are the rows of the adjacency matrix of* Λ^R. *Moreover,* $\mathrm{Aut}(C_2(\Lambda^R)) \cong \mathrm{PSp}_6(4)$.
(v) $C_p(\Lambda^R) = \mathbb{F}_p^{2016}$ *for* $p \neq 2$.

(b) *If* $\mathcal{G} = \overline{\Lambda}$ *or* $\mathcal{G} = \overline{\Lambda}^R$ *then*
(i) $C_2(\overline{\Lambda}) = C_2(\Lambda^R)$.
(ii) $C_p(\overline{\Lambda}) = \mathbb{F}_p^{2016}$ *for* $p = 3, 7$ *and* $C_p(\overline{\Lambda})$ *is a code of codimension* 1 *in* $\mathbb{F}_p^{2016}$ *for* $p = 5, 13$.
(iii) $C_p(\overline{\Lambda}^R) = \mathbb{F}_p^{2016}$ *for* $p = 2, 7, 13$.
(iv) $C_3(\overline{\Lambda}^R)$ *is a self-orthogonal* $[2016, 651, d]_3$ *code with* $d \geq 1041$.
(v) $\mathbb{F}_5^{2016} = C_5(\overline{\Lambda}^R) + C_5(\overline{\Lambda}^R)^{\perp}$ *and* $\dim(C_5(\overline{\Lambda}^R)) = 651$.

Theorem 1.1 2(a) (iv) led us in Proposition 6.4 to examine a 13-dimensional subcode $\mathcal{L}$ of the $[2016, 14, 976]_2$ code, which is invariant under the action of $G_2(4)$ as a permutation group of automorphisms of the code. We noticed that $\mathcal{L}$ is an indecomposable $\mathbb{F}_2$-module of $G_2(4)$ whose radical is of dimension 1, and the full automorphism group of $\mathcal{L}$ explodes in size in comparison to that given in Theorem 1.1 2(a) (iv). In fact the full automorphism group of this 13-dimensional subcode uncovers an embedding of the groups $G_2(4)$ and $\mathrm{PSp}_6(4)$ into the symplectic group $\mathrm{PSp}_{12}(2)$. Note from [24, p. 273] that $G_2(4)$ has an irreducible representation of degree 6 over $\mathbb{F}_4$. This 6-dimensional representation gives a natural embedding of $G_2(4)$ into $\mathrm{PSp}_6(4)$. Now from [16, Lemma 11, Lemma 12] one can see that $\mathrm{PSp}_6(4)$ is embedded into $\mathrm{PSp}_{12}(2)$ as $\mathcal{C}_8$-family in Aschbacher's classification. Thus, $G_2(4)$ is embedded into $\mathrm{PSp}_{12}(2)$, giving an irreducible representation of dimension 12 over $\mathbb{F}_2$.

The paper is organised as follows. In Section 2 we give some basic terminology on graphs, designs and codes. In Section 3 we give the necessary background on $G_2(q)$ and Section 4 gives a brief overview on the interplay between designs, graphs and codes from $G_2(3)$ and $G_2(4)$, respectively. In Sections 5 and 6 we present our results and their proofs. In the last section of the paper, we pose two open questions which could lead to further development of the results presented in this paper. We have placed a sample of the computations carried out using Magma in [28].

2. Terminology

The notation for designs and codes is as in [1]. An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{I}$ is a 2-(v, k, λ) design, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points, and every two distinct points are together incident with precisely λ blocks. The design $\mathcal{D}$ is *symmetric* if it has the same number of points and blocks. A *residual* structure of $\mathcal{D}$ is the design obtained by deleting a block of $\mathcal{D}$ and retaining those points not incident with the block. A residual structure at any block of $\mathcal{D}$ is a 2-$(v-k, k-\lambda, \lambda)$ design. A *derived* structure of $\mathcal{D}$ is the design obtained by deleting a block and retaining those points incident with the block. A derived structure of $\mathcal{D}$ is a 2-$(k, \lambda, \lambda-1)$ design. The numbers that occur as the size of the intersection of two distinct blocks are the *intersection numbers* of the design. A 2-design is *quasi-symmetric* with intersection numbers x, y $(x < y)$ if any two distinct blocks intersect in either x or y points. A 2-(v, k, λ) design is called *self-orthogonal* if the intersection numbers have the same parity as the block size. An automorphism of a design $\mathcal{D}$ is a permutation on $\mathcal{P}$ which sends blocks to blocks. The set of all automorphisms of $\mathcal{D}$ forms its full automorphism group denoted by $\mathrm{Aut}(\mathcal{D})$.

The *code* C_F *of the design* $\mathcal{D}$ over the finite field F is the space spanned by the incidence vectors of the blocks over F. If the point set of $\mathcal{D}$ is denoted by $\mathcal{P}$ and the block set by $\mathcal{B}$, and if $\mathcal{Q}$ is any subset of $\mathcal{P}$, then we will denote the incidence vector of $\mathcal{Q}$ by $v^{\mathcal{Q}}$. Thus $C_F = \left\langle v^B \mid B \in \mathcal{B} \right\rangle$, and is a subspace of $F^{\mathcal{P}}$, the full vector space of functions from $\mathcal{P}$ to F.

Terminology for *graphs* is standard: the *graphs* $\mathcal{G} = (V, E)$ with vertex set V and edge set E, discussed here are undirected with no loops, apart from the case where all loops are included, in which case the graph is called *reflexive* and denoted $\mathcal{G}^R$. If $u, v \in V$ and u and v are adjacent, we write $u \sim v$, and uv or $[u, v]$ for the edge in E that they define. We also consider the *complementary* graph, $\overline{\mathcal{G}} = (V, \overline{E})$ where for $u, v \in V$, $u \neq v$, $u \sim v$ in $\mathcal{G}$ if and only if $u \not\sim v$ in $\overline{\mathcal{G}}$. The set of neighbours of $u \in V$ is denoted by $N(u)$, and the valency of u is $|N(u)|$. A graph is *regular* if all the vertices have the same valency. An *adjacency matrix* A of a graph of order n is an $n \times n$ matrix with entries a_{ij} such that $a_{ij} = 1$ if vertices v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. With a slight abuse of notation, we also write $\mathcal{G} = (n, k, \lambda, \mu)$ to denote a *strongly regular* of type (n, k, λ, μ), i.e. a regular graph that has n vertices, degree k, in which any two adjacent vertices are together adjacent to λ vertices, while any two non-adjacent vertices are together adjacent to μ vertices. The *neighbourhood design* of a regular graph $\mathcal{G}$ is the symmetric 1-$(|V|, k, k)$ design formed by taking the points to be the vertices of the graph and the blocks to be the sets of neighbours of a vertex, for each vertex, i.e. an adjacency matrix is an incidence matrix for the design. If $\mathcal{G} = (V, E)$ is a graph with adjacency matrix A then $A + I_{|V|}$ is an adjacency matrix for the reflexive graph $\mathcal{G}^R$.

A *rank* 3 *graph* is a graph that admits an automorphism group which is transitive on the vertices, edges, and nonedges. Note that any rank 3 graph is a strongly regular graph. The converse is not always true. The complementary graph of a strongly regular graph with parameters (n, k, λ, μ) is a strongly regular graph with parameters $(n, n-k-1, n-2k+\mu-2, n-2k+\lambda)$. A connected strongly regular graph has diameter 2. If v and w are vertices of a connected strongly regular graph $\mathcal{G}$ such that $d(v, w) = i$, $i = 0, 1, 2$, then the number p_{ij} of neighbors of w whose distance from v is j, $j = 0, 1, 2$, are the intersection numbers of $\mathcal{G}$. The 3×3-matrix with entries p_{ij}, $i, j = 0, 1, 2$, is called the *intersection matrix* of $\mathcal{G}$.

The *code* of a graph $\mathcal{G}$ over a finite field F is the row span of an adjacency matrix A over the field F, denoted by $C_F(\mathcal{G})$ or $C_F(A)$. The dimension of the code is the rank of the matrix over F, also written $\mathrm{rk}_p(A)$ if $F = \mathbb{F}_p$, in which case we will speak of the *p-rank* of A or $\mathcal{G}$, and write $C_p(\mathcal{G})$ or $C_p(A)$ (respectively $C_p(\mathcal{G}^R)$ or $C_p(A + I_{|V|})$) for the code. The ambient space of these codes is $\mathbb{F}_p^{|V|}$.

All our codes will be *linear codes*, i.e. subspaces of the ambient vector space. If a code C over a field of order q is of length n, dimension k, and minimum weight d, then we write $[n, k, d]_q$ to summarize this information. A *generator matrix* for the code is a $k \times n$ matrix made up of a basis for C. The *dual* code $C_{\mathcal{G}}^{\perp}$ is the orthogonal complement under the standard inner product $(,)$, i.e. $C_{\mathcal{G}}^{\perp} = \{v \in F^n \,|\, (v,c) = 0 \text{ for all } c \in C_{\mathcal{G}}\}$. A code $C_{\mathcal{G}}$ is *self-orthogonal* if $C_{\mathcal{G}} \subseteq {C_{\mathcal{G}}}^{\perp}$. The *hull* of a code is the intersection of a code and its dual. A linear code C over any field is a *linear code with complementary dual (LCD)* code if $C \cap C^{\perp} = \{0\}$.

The all-one vector will be denoted by $\mathbf{1}$, and is a constant vector of weight the length of the code. A binary code $C_{\mathcal{G}}$ is *doubly-even* if all codewords of $C_{\mathcal{G}}$ have weight divisible by 4. A *triply-even code* is a binary linear code in which the weight of every codeword is divisible by 8. The weight enumerator of $C_{\mathcal{G}}$ is defined as $W_{C_{\mathcal{G}}}(x) = \sum_{i=0}^{n} A_i x^i$, where A_i denotes the number of codewords of weight i in $C_{\mathcal{G}}$.

Two linear codes are *isomorphic* if they can be obtained from one another by permuting the coordinate positions. An *automorphism* of a code is any permutation of the coordinate positions that maps codewords to codewords and will be denoted $\mathrm{Aut}(C)$. Any automorphism clearly preserves each weight class of C. In this note we restrict our attention to permutation automorphisms.

If F is a field and G is a group then FG denotes the group algebra of G over F. If U is an FG-module then U^* denotes the dual FG-module. $U{=}_{FG}X \oplus Y$ denotes a direct decomposition of the FG-module U into FG-submodules X and Y. If G is a finite group acting on a set Ω then the *permutation module* $F\Omega$ is by definition the F-vector space with basis $\overline{\Omega} = \{\overline{\alpha} \,|\, \alpha \in \Omega\}$ where $\overline{\alpha} = (\delta_{\beta,\alpha})_{\beta \in \Omega}$ and δ denotes the Kronecker δ symbol. The action of G on Ω naturally extends by linearity to $F\Omega$ giving the canonical structure of an FG-module. Usually $\overline{\alpha}$ is identified with α and $\overline{\Omega}$ with Ω, but for the sake of clarity it makes sense to keep this distinction in this paper. Note that $\overline{\Omega}$ will be considered as ambient basis for codes which naturally will admit G as an automorphism group acting by permuting the coordinate positions. The *canonical* bilinear form on $F\Omega$ has $\overline{\Omega}$ as orthonormal basis. It turns out that any code over a field F admitting G can be obtained as a submodule of the permutation module $F\Omega$ over F, considering Ω as the ambient basis. The reader is encouraged to consult [30] for details on permutation modules.

For the structure of groups and their maximal subgroups we follow the $\mathbb{ATLAS}$ notation, see [13]. The groups $G.H$, $G{:}H$, and $G{}^{\cdot}H$ denote a general extension, a split extension and a non-split extension respectively. If $N \trianglelefteq G$ is a normal subgroup with quotient $Q = G/N$ then G is an extension of Q by N. When the sequence splits, that is G has a subgroup isomorphic to Q that meets N trivially, G is called a split extension of Q by N, or a semi-direct product of N and Q. Otherwise G is a non-split extension of Q by N. For a prime p, the symbol p^m denotes an elementary abelian p-group of that order. The notation p_+^{1+2n} and p_-^{1+2n} are used for extraspecial groups of order p^{1+2n}. If p is an odd prime, the subscript is $+$ or $-$ according as the group has exponent p or p^2. For $p = 2$ it is $+$ or $-$ according as the central product has an even or odd number of quaternionic factors. Throughout the paper $O_n(q)$ denotes the simple orthogonal group in dimension n over $\mathbb{F}_q$.

3. The group $G_2(q)$

Here we give a brief overview of the simple exceptional group $G_2(q)$ and its primitive permutation representations via the coset action on the set Ω of hyperplanes of type $O_6^-(q)$. For more information on the group we refer the reader to [29, Proposition 1] or [37, Section 4.3]. The Chevalley group $G_2(q)$ of type G_2 is isomorphic to a subgroup of the orthogonal group $O_7(q)$ and acts transitively on the set Ω of hyperplanes of type $O_6^-(q)$ in the 7-dimensional orthogonal geometry V over $\mathbb{F}_q$ related to the group $O_7(q)$. The group $G_2(q)$ has order $q^6(q^6-1)(q^2-1) = q^6(q-1)^2(q+1)^2(q^2+q+1)(q^2-q+1)$ and $|\Omega| = \frac{q^3(q^3-1)}{2}$. In this action the point stabilizer is a subgroup isomorphic to $\mathrm{PSU}_3(q^2){\cdot}2$. The rank of $G_2(q)$ is $\frac{q+1+(2,q-1)}{2}$. Each suborbit is self-paired and the non-trivial subdegrees of $G_2(q)$ are

$$(q^3+1)(q^2-1),\ \frac{q^2(q^3+1)}{2},\ \text{ and } \frac{(q-3)}{2} \times q^2(q^3+1), \text{if } q \text{ is odd,} \tag{1}$$

$$(q^3+1)(q^2-1) \text{ and } \frac{(q-2)}{2} \times q^2(q^3+1), \text{if } q \text{ is even.} \tag{2}$$

Moreover, any suborbit graph arising from this action of $G_2(q)$ when $q > 2$ has diameter 2. It should be obvious from this that the only rank 3 representations of exceptional type G_2 having $\mathrm{PSU}_3(q^2)\cdot 2$ as a stabilizer of a point are those of degree 351 associated with the exceptional group $G_2(3)$ and of degree 2016 associated with the exceptional group $G_2(4)$, respectively, see for example [8, Table 8, p. 19].

4. The graphs, designs and codes

In Section 3 we observed that the graphs derived from the suborbits of the rank 3 action of $G_2(q)$ when $q = 3$ and $q = 4$, respectively, are strongly regular on a set Ω with $|\Omega| = \frac{q^3(q^3-1)}{2}$ vertices. The stabilizer of a vertex $u \in \Omega$ is a maximal subgroup isomorphic to $\mathrm{PSU}_3(q^2){\cdot}2$, producing orbits $\Gamma_0 = \{u\}$, Γ_1, and Γ_2 of lengths 1, 126 and 224 using Equation (1), or orbits $\Lambda_0 = \{u\}$, Λ_1, and Λ_2 of lengths 1, 975 and 1040, when Equation (2) is used. The regular graphs Γ, Γ^R and their complementary graphs $\overline{\Gamma}$, $\overline{\Gamma}^R$, that are examined in Section 5, result from the sets Γ_1, $\Gamma_0 \cup \Gamma_1$, Γ_2, and $\Gamma_0 \cup \Gamma_2$ respectively. If A denotes an adjacency matrix for Γ then $A + I_{|V|}$ is an adjacency matrix of the reflexive graph Γ^R while $\overline{A} = J - I - A$, where J is the all-one and I the identity $|V| \times |V|$ matrix, will be an adjacency matrix for the complementary graph $\overline{\Gamma}$ on the same vertices. Thus, we examine the neighbourhood designs of the graphs described earlier and corresponding codes $C_p(A)$, $C_p(A+I)$, $C_p(\overline{A})$ and $C_p(\overline{A}+I)$ defined by the p-ary row span of A, $A+I$, $\overline{A}$, $\overline{A}+I$. Note that $A+I$ and $\overline{A}+I$ are adjacency matrices for the graphs $\Gamma^R, \overline{\Gamma}^R$ obtained from Γ and $\overline{\Gamma}$, respectively, by including all loops, and thus referred to as reflexive graphs.

A similar discussion is carried out in Section 6 where we examine the p-ary codes $C_p(\Lambda)$, $C_p(\Lambda^R)$, $C_p(\overline{\Lambda})$ and $C_p(\overline{\Lambda}^R)$ constructed from the graphs denoted by Λ, Λ^R, $\overline{\Lambda}$ and $\overline{\Lambda}^R$, respectively. The adjacency matrices of the corresponding graphs are denoted B, $B+I$, $\overline{B}$ and $\overline{B}+I$, respectively. Note that the latter codes are constructed from the orbits $\Lambda_0 \cup \Lambda_1, \Lambda_1, \Lambda_2$ and $\Lambda_0 \cup \Lambda_2$, respectively.

5. Graphs, designs and codes from $G_2(3)$ of degree 351

In this section we discuss the examples arising in Theorem 1.1, considering the group $G_2(3)$ of degree 351. For this, let G be $G_2(3)$. Notice that $|G_2(3)| = 2^6{\cdot}3^6{\cdot}7{\cdot}13$ and G has an involutory outer automorphism, so its automorphism group is a split extension of $G_2(3)$ by $\mathbb{Z}_2$. Notice from the $\mathbb{ATLAS}$ [13, p. 60] that there are two classes of maximal subgroups (i.e. two pairwise inequivalent rank 3 primitive permutation representations) of index 351, each having a representative isomorphic to $\mathrm{PSU}_3(3){:}2$. These representations are interchanged by an outer automorphism of

G. Therefore, it suffices to consider one action for G of degree 351. We consider only those structures constructed from the first representation of this degree, since the graphs, designs and codes obtained from the two representations are isomorphic.

We need some more detailed notation for the action of G on Ω. From Section 4 recall that G has the orbits $\Gamma_0 = \{u\}$, Γ_1, and Γ_2 on Ω^2 where

$$\begin{aligned}\Gamma_0 &= \{(\alpha,\alpha)\,|\,\alpha\in\Omega\} \text{ is the diagonal}\\ \Gamma_1 &= \{(\alpha,\beta)\,|\,\{\alpha,\beta\}\in E\}\\ \Gamma_2 &= \{(\alpha,\beta)\,|\,\alpha,\beta\in\Omega,\alpha\neq\beta,\{\alpha,\beta\}\notin E\}.\end{aligned}$$

We use the notation $\Gamma_i(\alpha) = \{\beta\,|\,(\alpha,\beta)\in\Gamma_i\}$ for the corresponding G_α-orbits. Thus $\Gamma_0 =$ the diagonal graph, $\Gamma_1 = \Gamma$, and $\Gamma_2 = \overline{\Gamma}$ are the suborbit graphs of Γ with $|\Gamma_i(\alpha)| = 1, 126, 224$. The matrix A_i in the centralizer algebra of (G,Ω) is defined by

$$A_i = (f_i(\alpha,\beta))_{(\alpha,\beta)\in\Omega\times\Omega},$$

where $f_i(\alpha,\beta) = 1$, if $(\alpha,\beta)\in\Gamma_i$ and $f_i(\alpha,\beta) = 0$, otherwise, where $0\leq i\leq 2$. Recall that $\mathbb{F}$ is the finite field $\mathbb{F}_q$. Now, set $\mathbb{F}\Omega$ to be the permutation module of (G,Ω) over $\mathbb{F}$ so that to each A_i there is a naturally assigned endomorphism $\mathbf{a}_i$ such that

$$\alpha\mapsto\alpha\mathbf{a}_i = \sum_\beta f_i(\alpha,\beta)\beta.$$

The endomorphism algebra $E(\mathbb{F}\Omega) = \mathrm{End}_{\mathbb{F}G}\mathbb{F}\Omega$ has basis $(\mathbf{a}_0,\mathbf{a}_1,\mathbf{a}_2)$ where $\mathbf{a}_0 = \mathrm{Id}_{\mathbb{F}\Omega}$. This basis is called Schur basis in [30]. According to [30, Theorem 1.2.20] (see also [10, Chapter 3] or [27]),

$$E(\mathbb{F}\Omega)\to\mathbb{F}^{3\times 3},\ \mathbf{a}_i\mapsto A_i = [\mathbf{a}_{ijk}]_{j,k=1,\dots,3}\ \ (0\leq i\leq 2),$$

gives the regular matrix representation of $E(\mathbb{F}\Omega)$ with respect to the Schur basis. The matrices $A_i = ((\mathbf{a}_i)_{jk})$ are called the *intersection matrices* of the orbital graphs (Ω,Γ_i) if $\mathrm{char}(\mathbb{F}) = 0$.

The structure of the graph Γ and of its complement $\overline{\Gamma}$ give the following values:

$$A_0 = \mathbf{I}_3,\quad A_1 = \begin{bmatrix} 0 & 1 & 0\\ 126 & 45 & 45\\ 0 & 80 & 81\end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 & 1\\ 0 & 80 & 81\\ 224 & 144 & 142\end{bmatrix}.$$

Remark 5.1. From results of [11, Chapter 2], for example, we deduce that the eigenvalues and multiplicities for Γ are λ_i for $0\leq i\leq 2$ with multiplicities m_i respectively, those for Γ^R are $\lambda_i^* = \lambda_i + 1$ with multiplicities m_i for $0\leq i\leq 2$, and those for $\overline{\Gamma}$ and $\overline{\Gamma}^R$ are $\overline{\lambda_0} = n-1-k$, $\overline{\lambda_i} = -\lambda_i - 1$, $\overline{\lambda}_i^* = \overline{\lambda_i}+1$ for $i = 1,2$, where

- $\lambda_0 = 126$, $\lambda_0^* = 127$, $\overline{\lambda_0} = 224$, $\overline{\lambda}_0^* = 225, m_0 = 1$;
- $\lambda_1 = 9, \lambda_1^* = 10, \overline{\lambda_1} = -10, \overline{\lambda}_1^* = -9$, $m_1 = 168$;
- $\lambda_2 = -9, \lambda_2^* = -8, \overline{\lambda_2} = 8, \overline{\lambda}_2^* = 9$, $m_2 = 182$.

Note that $\lambda_1 + \lambda_2 = \lambda - \mu$.

- The rows of the adjacency matrix A for Γ give the blocks of the neighbourhood design of Γ which we denote $\mathcal{D}_{126}$. Note that $\mathcal{D}_{126}$ is in fact a self-dual symmetric 2-$(351,126,45)$ design. We write $C_p(\Gamma)$ to denote the p-ary codes spanned by the rows of the incidence matrix of $\mathcal{D}_{126}$.
- From the rows of an adjacency matrix $A+I$ of the reflexive graph Γ^R we obtain the self-dual symmetric 1-$(351,127,127)$ design $\mathcal{D}_{127}$, and the p-ary code $C_p(\Gamma^R)$.

- The rows of an adjacency matrix $\overline{A}$ for $\overline{\Gamma}$ give the self-dual symmetric 1-$(351, 224, 224)$ design $\mathcal{D}_{224}$, and p-ary code $C_p(\overline{\Gamma})$.
- From the rows of an adjacency matrix $\overline{A} + I$ of the reflexive graph $\overline{\Gamma}^R$ we get the self-dual symmetric 2-$(351, 225, 144)$ design $\mathcal{D}_{225}$. We write $C_p(\overline{\Gamma}^R)$ to denote the p-ary codes of $\mathcal{D}_{225}$.

In the ensuing results of this section we examine the p-ary codes $C_p(\Gamma)$, $C_p(\overline{\Gamma})$, $C_p(\Gamma^R)$ and $C_p(\overline{\Gamma}^R)$, where $p||G|$, from the graph Γ, its complementary graph and those of their respective reflexive associates. Notice that the Chevalley group $G_2(3)$ acts on each of these graphs, designs and codes.

5.1. THE 2-$(351, 126, 45)$ DESIGN. Recall that in Section 2 we established that the neighborhood design of a regular graph is a symmetric 1-$(|V|, k, k)$ design. This is the case here, for the 1-$(351, 126, 126)$ design constructed from the regular graph Γ on 351 vertices invariant under $G_2(3)$. The graph Γ is strongly regular with parameters $(351, 126, 45, 45)$. Since $\lambda = \mu = 45$, Γ is called a $(351, 126, 45)$-graph. The 351 vertices serve as both points and blocks of the design and adjacency in the graph is incidence in the design. Conversely, a 2-$(351, 126, 45)$-design having a polarity with no absolute points (meaning that it has a symmetric incidence matrix with zero diagonal), corresponds to a strongly regular $(351, 126, 45, 45)$ graph. In this section, we will make use of the incidence matrix of the design and the adjacency matrix of the graph interchangeably.

Symmetric 2-$(351, 126, 45)$ designs belong to the series with parameters

$$v = \frac{3^l(3^l - 1)}{2}, k = \frac{3^{l-1}(3^l + 1)}{2} \text{ and } \lambda = \frac{3^{l-1}(3^{l-1} + 1)}{2},$$

where $l > 1$. Let (V, f) be a non-degenerate orthogonal space of dimension $2l + 1$ over $\mathbb{F}_3$ with discriminant $(-1)^l$. Then all anisotropic 1-dimensional subspaces $W = \langle w \rangle \leq V$ for which $f(w, w) = 1$ form the set $\mathcal{P}$ of points of this design, while the blocks have the form $B(W) = \{U \in \mathcal{P} : f(W, U) = 0\}$, $W \in \mathcal{P}$, see [3]. For $l = 3$ we obtain a 2-$(351, 126, 45)$ design isomorphic to the neighbourhood design of Γ, namely $\mathcal{D}_{126}$.

Dempwolff in [19], and more recently Braić et al in [3] determined all symmetric designs that admit a group which has a non-abelian socle and is primitive rank 3 on points and on blocks. As a by-product, the existence and uniqueness of the symmetric 2-$(351, 126, 45)$ design having the simple Chevalley group $G_2(3)$ as a non-abelian socle of the automorphism group and acting primitively of rank 3 on points and on blocks of the design was established.

Lemma 5.2. *Let $G = G_2(3)$ and let $\mathcal{D}_{126} = (\Omega, \mathcal{B})$. Then $\mathcal{D}_{126}$ is a self-dual, symmetric and self-orthogonal 2-$(351, 126, 45)$ design with $G \leq \mathrm{Aut}(\mathcal{D}_{126}) \cong O_7(3){:}2$ acting flag-transitively, and point primitively. Moreover, up to isomorphism, $\mathcal{D}_{126}$ is the only flag-transitive, point primitive symmetric 2-$(351, 126, 45)$ design with these parameters admitting G as an automorphism group.*

Proof. From the definition of Ω and $\mathcal{B}$ it is clear that $G \subseteq \mathrm{Aut}(\mathcal{D}_{126})$. It follows from the $\mathbb{ATLAS}$ [13, p. 60] that G acts primitively on both Ω and $\mathcal{B}$ of degree $|\Omega| = |\mathcal{B}| = 351$, and the stabilizer G_W of a point W has exactly three orbits in Ω. From the definition of $B(W)$ we have that G_W fixes setwise each of $\{W\}$, $B(W)$ and $\Omega \setminus (B(W) \cup \{W\})$ and these are all G_W-orbits. This shows that $\mathcal{D}_{126}$ is a flag-transitive, point primitive, symmetric 1-design. Using an argument similar to that of [19, Lemma 3.3] we deduce that $\mathcal{D}_{126}$ is a 2-design, and self-duality of $\mathcal{D}_{126}$ follows readily from [26, Proposition 1]. Since for an isometry $h \in I(V, f)$ we have $hB(W) = B(hW)$, it follows that $O_7(3){:}2$ is an automorphism group of $\mathcal{D}_{126}$. Moreover, since $O_7(3){:}2$ is a maximal subgroup of the symmetric group S_{351}, it is the full automorphism group of $\mathcal{D}_{126}$. Again, from [19, Lemma 3.3] we have $\mathcal{D}_{126}$ is a rank 3 design, and $G_2(3)$ is a primitive rank 3 group on $\mathcal{D}_{126}$. Now from the embedding $G_2(3) \leq O_7(3)$ and $|\mathrm{Aut}(\mathcal{D}_{126})| = |O_7(3){:}2|$ we deduce that $\mathrm{Aut}(\mathcal{D}_{126}) \cong O_7(3){:}2$. Finally if $i \neq j$, consider two distinct blocks B_i and B_j in $\mathcal{D}_{126}$. Since $|B_i \cap B_j| \equiv k \equiv 0 \pmod 3$, (where $i, j \in \{1, \ldots, b\}$, and b and k are respectively the number of blocks and the block size) we have $\mathcal{D}_{126}$ is self-orthogonal. The proof of last statement of the theorem follows from [3, Section 3, Case (3^0)] or [19, Lemma 3.3]. □

5.2. THE CODES OF THE GRAPHS Γ AND $\overline{\Gamma}$ AND THOSE OF THEIR REFLEXIVE ASSOCIATES. Recall that if A is an incidence matrix of a 2-(v, k, λ) design and $\mathrm{rk}_p(A) < v - 1$, then it is well-known (see [36, Theorem 1.86]) that this code is interesting only when p divides $r - \lambda$, the order of the design. Notice that for the particular case of a design with parameters those of $\mathcal{D}_{126}$ and $\mathcal{D}_{225}$, the order $r - \lambda = 126 - 45 = 225 - 144 = 81 = 3^4$, and so only the ternary codes of such designs will be of interest for characterization purposes. Thus, in Theorem 5.3 and in Theorem 5.4 we deal with the ternary codes of these designs and examine their combinatorial properties. In particular, we show that the code $C_3(\Gamma)$ is spanned by the minimum weight codewords, which are the rows of the adjacency matrix and their scalar multiples, and the minimum weight of the code is the valency of the graph.

Theorem 5.3. *Let $C_p(\Gamma)$ and $C_p(\Gamma^R)$ denote the p-ary codes of Γ and of Γ^R, respectively. Then*
(i) *$C_3(\Gamma)$ is a self-orthogonal $[351, 27, 126]_3$ code and $C_3(\Gamma)^\perp$ is a $[351, 324, 6]_3$ code with 458640 words of weight 6. The minimum weight of $C_3(\Gamma)$ is the valency of Γ, and the minimum weight codewords are the rows of the adjacency matrix A of Γ and their scalar multiples; $C_3(\Gamma)$ is spanned by its minimum weight codewords.*
(ii) *$\mathbf{1} \in C_3(\Gamma)$ and $\mathrm{Aut}(C_3(\Gamma)) \cong O_7(3){:}2$.*
(iii) *$C_p(\Gamma)$ is a code of codimension 1 in $\mathbb{F}_p^{351}$ for $p = 2, 7$.*
(iv) *$C_{13}(\Gamma) = \mathbb{F}_{13}^{351}$.*
(v) *$C_p(\Gamma^R) = \mathbb{F}_p^{351}$ for $p \neq 2$.*
(vi) *$C_2(\Gamma^R) = [351, 79, 48]_2$ and $C_2(\Gamma^R) \cap C_2(\Gamma^R)^\perp = [351, 78, 48]_2 = C_2(\overline{\Gamma})$.*

Proof. (i) We start by determining the 3-rank of A, i.e. the dimension of $C_3(\Gamma)$. By Remark 5.1, the eigenvalues of the adjacency matrix A of Γ are $\lambda_0 = 126$, $\lambda_1 = 9$, and $\lambda_2 = -9$ and their corresponding multiplicities are $m_0 = 1$, $m_1 = 168$ and $m_2 = 182$. Since $p \mid (\lambda_1 - \lambda_2)$ i.e., $3 \mid 18$, we have from [6, Section 3] that $\mathrm{rk}_3(A) \leq \min(m_1 + 1, m_2 + 1) = 169$. However, the 3-rank of A is much smaller and equals 27. To show this we rely on the $\mathbb{ATLAS}$ [13] and the Atlas of Brauer characters [24]. According to [24, p. 141], the irreducible 3-modular characters of $G_2(3)$ have degree 1, 7, 27, 49, 189 and 729. Let χ_π denote the ordinary permutation character π of $G_2(3)$ of degree 351, and φ_π denote its 3-Brauer character. From the $\mathbb{ATLAS}$ [13, p. 60] we have that the permutation character of $G_2(3)$ of degree 351 is the sum of three irreducibles, i.e., $\chi_\pi = \chi_1 + \chi_{168a} + \chi_{182b}$, where the subscript numbers denote the degree, and the subscript letters indicate the sequence as in

TABLE 1. The weight distribution of $C_3(\Gamma)$

i	A_i	i	A_i
0	1	243	1899969548750
126	702	252	376258697100
144	132678	261	22893588900
162	264810	270	1272627720
180	15877134	279	107557632
189	125095689	288	3027024
198	2147437656	306	88452
207	26912233530	315	88452
216	395941284648	324	21840
225	1844882687232	351	756
234	3055067224272		

the $\mathbb{ATLAS}$. Restricting these to the elements of order prime to 3 and decomposing into 3-modular irreducibles we obtain from [24, p. 141] (see also [4]) that $\varphi_{168a} = 2\varphi_1+\varphi_{7_a}+\varphi_{7_b}+\varphi_{27_a}+\varphi_{27_b}+2\varphi_{49}$ and $\varphi_{182b} = 2\varphi_1+\varphi_{7_a}+2\varphi_{7_b}+\varphi_{27_a}+\varphi_{27_b}+2\varphi_{49}$. It follows from this that $\varphi_\pi = 5\varphi_1+3\varphi_{7a}+3\varphi_{7b}+2\varphi_{27a}+2\varphi_{27b}+4\varphi_{49}$. This shows that the dimension of the smallest non-trivial irreducible submodule (code) $C \leq \mathbb{F}_3^{351}$ is at least 7. We argue using the weight distribution calculated through computations with Magma [2] and given in Table 1, where i represents the weight of a codeword w_i in $C_3(\Gamma)$ and A_i denotes the number of codewords of weight i. We can easily see that $C_3(\Gamma)$ does not contain an invariant subspace of dimension 1, and also has no invariant subspace of dimension 7. Moreover, we establish that $G_2(3)$ of degree 351 has no irreducible modules over $\mathbb{F}_3$ with dimensions between 7 and 26. Hence $C_3(\Gamma)$ is a 27-dimensional $\mathbb{F}_3$-module on which $G_2(3)$ and $\mathrm{Aut}(C_3(\Gamma))$ act absolutely irreducibly. Now, from [24] (see also [38]) we deduce that the 27-dimensional module is not unique.

Note that the code $C_3(\Gamma)$ is the code spanned by the adjacency matrix A of Γ. Moreover, self-orthogonality of $C_3(\Gamma)$ follows readily by noticing that in the second row of A_1 all values are divisible by 3. Alternatively, recall that Γ and $\mathcal{D}_{126}$ are being used interchangeably when necessary. So, for example we could use the fact that $\mathcal{D}_{126}$ is self-orthogonal, and deduce the self-orthogonality of $C_3(\Gamma)$ since the block-point incidence matrix of $\mathcal{D}_{126}$ spans a self-orthogonal code of length 351. Since the block size of $\mathcal{D}_{126}$ is divisible by 3, we have that $\mathbf{1} \in C_3(\Gamma)^\perp$.

The minimum distance 126 can be deduced from the weight distribution of $C_3(\Gamma)$ which is given in Table 1.

Denote by $d^\perp$ the minimum weight of $C_3(\Gamma)^\perp$. From [1, Lemma 2.4.2] we have that $d^\perp \geq \frac{r}{\lambda} + 1 = \frac{126}{45} + 1 > 3$. We will show next that for $d^\perp \geq 6$ and we do so by showing that if $d^\perp \in \{4, 5\}$ we get a contradiction. Since the argument goes through smoothly for either choices of $d^\perp$ we consider $d^\perp = 5$. Let p be a fixed point in the support S of a non-zero codeword $u \in C_3(\Gamma)^\perp$ of weight $s = d^\perp$ and p_i be the number of blocks of the design $\mathcal{D}_{126}$ (recall here that Γ is a (v, k, λ)-graph which is identified with $\mathcal{D}_{126}$) passing through p and meeting S in i points. A counting argument gives

$$\sum_{i=1}^{k} p_i = r, \ \sum_{i=2}^{k}(i-1)p_i = (s-1)\lambda. \tag{3}$$

From Equation (3) we obtain

$$\sum_{i=3}^{k}(i-2)p_i = (s-1)\lambda - r, \tag{4}$$

and Equations (3) and (4) imply that $p_2 = r - \sum_{i=3}^{k} p_i \geq r - \sum_{i=3}^{k}(i-2)p_i = r - [(s-1)\lambda - r] = 2r - (s-1)\lambda$. Hence we have $p_2 \geq 252 - 180 = 72$ for any point of S. Now examine the entries of u. Denote $S = \{q_i \mid 1 \leq i \leq 5\}$, since $\mathbf{1} \in {C_\Gamma}^\perp$ we must have entries $+1$ at four points, say q_i, for $i = 1$ to 4 and -1 at q_5. However, every block meeting S in two points and passing through q_1 must pass through q_5, but there are only four points remaining once q_1 is chosen; thus not all 72 blocks which meet S in two points can pass through q_5; thus we have a contradiction. Now, direct calculations show that the weights of the rows of the generator matrix for $C_3(\Gamma)^\perp$ equals 6, so the minimum weight $d^\perp \geq 6$, and the assertion follows.

(ii) That $\mathbf{1} \in C_3(\Gamma)$ follows since the sum (modulo 3) of all rows of a generator matrix of $C_3(\Gamma)$ is the all-one vector.

Notice from Table 1 that there are 702 codewords of minimum weight 126 in $C_3(\Gamma)$. Thus the words of weight 126 in $C_3(\Gamma)$ are the incidence vectors of the blocks of $\mathcal{D}_{126}$ and their scalar multiples. Moreover, these codewords form a spanning set for $C_3(\Gamma)$. Since by Lemma 5.2 we have $\mathrm{Aut}(\mathcal{D}_{126}) = O_7(3){:}2 \subseteq \mathrm{Aut}(C_3(\Gamma))$, and since $|O_7(3){:}2| = |\mathrm{Aut}(C_3(\Gamma))|$ it follows that $\mathrm{Aut}(C_3(\Gamma)) \cong O_7(3){:}2$.

(iii) Since λ_0 is the only eigenvalue which vanishes modulo p, we obtain $\mathrm{rk}_p(A) = n - m_0 = 350$.

(iv) and (v) follow from [5, Proposition 13.7.1(iv)], since none of the λ_i vanishes modulo p, for p as given in the proposition. So, $\mathrm{rk}_p(A) = 351$.

(vi) See proof of Proposition 5.4(i) below. $\square$

We now examine the codes of the complementary graph $\overline{\Gamma}$ and those of its reflexive graph $\overline{\Gamma}^R$.

Theorem 5.4. *Let $C_p(\overline{\Gamma})$ and $C_p(\overline{\Gamma}^R)$ denote the p-ary codes of $\overline{\Gamma}$ and of $\overline{\Gamma}^R$, respectively. Then*

(i) *$C_2(\overline{\Gamma})$ is a self-orthogonal doubly-even $[351, 78, 48]_2$ code with 9828 codewords of weight 48. Moreover, $C_2(\overline{\Gamma})$ is an irreducible and faithful $\mathbb{F}_2$-module invariant under $G_2(3)$ and $\mathrm{Aut}(C_2(\overline{\Gamma})) \cong O_7(3){:}2$.*

(ii) *$C_p(\overline{\Gamma}) = \mathbb{F}_p^{351}$ for $p = 3, 13$.*

(iii) *$C_7(\overline{\Gamma})$ is a code of codimension 1 in $\mathbb{F}_7^{351}$.*

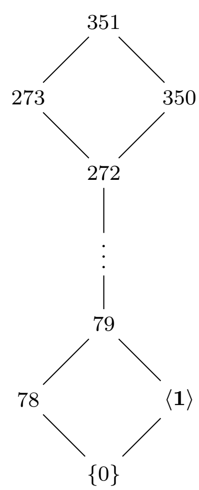
(iv) *$C_p(\overline{\Gamma}^R) = \mathbb{F}_p^{351}$ for $p \neq 3$.*

(v) *$C_3(\overline{\Gamma}^R) = [351, 28, 108]_3$ and $C_3(\overline{\Gamma}^R) = C_3(\Gamma) + \langle \mathbf{1} \rangle$.*

Proof. (i) Since $\overline{\lambda_1} - \overline{\lambda_2} = -18$, and divisible by 2 we have from [6, Section 3] that $\mathrm{rk}_2(\overline{A}) \leq \min(m_1 + 1, m_2 + 1) = 169$. In what follows we sketch an argument that shows that the 2-rank of $\overline{A}$ is 78. From [24, p. 140] it follows that the irreducible 2-modular characters of $G_2(3)$ have degrees 1, 14, 64, 78, 90, 378, 448 and 832, respectively. Recall that the permutation character of $G_2(3)$ of degree 351 is the sum of three irreducibles: $\chi_\pi = \chi_1 + \chi_{168a} + \chi_{182b}$. Restricting these to the elements of order prime to 2 and decomposing into 2-modular irreducibles we obtain from [24, p. 140] (see also [4]) that $\varphi_{168a} = \varphi_{78} + \varphi_{90_a}$ and $\varphi_{182b} = \varphi_{14} + \varphi_{78} + \varphi_{90_c}$. It follows from this that $\varphi_\pi = \varphi_1 + \varphi_{14} + 2\varphi_{78} + 2\varphi_{90}$. Now $\mathrm{rk}_2(J - I - A) = 78$, and $C_2(\overline{\Gamma}) = \langle J - I - A\rangle_2$ is the unique irreducible (absolutely irreducible) $\mathbb{F}_2$-module of dimension 78 invariant under $G_2(3)$.

For the reader's convenience, in Table 2 we give the 2-module structure of the permutation module $\mathbb{F}_2\Omega = \mathbb{F}_2^{351}$ of dimension 351 computed using Magma [2]. Since the table is symmetric about the diagonal we omit the lower half for clarity. In addition, we place a 1 or . in the table according to whether or not a submodule is contained in a given module (sometimes itself). The socle of $\mathbb{F}_2^{351}$, denoted here by $\mathrm{Soc}(\mathbb{F}_2^{351})$ has dimension $\dim(\mathrm{Soc}(\mathbb{F}_2^{351})) = 79 = 78 + 1$. From this we deduce that $\mathrm{Soc}(\mathbb{F}_2^{351}) = C_2(\Gamma^R) = C_2(\overline{\Gamma}) + \langle \mathbf{1}\rangle$. See also, the partial lattice of submodules for $\mathbb{F}_3\Omega$ given in Figure 1 which depicts the code inclusions.

FIGURE 1. Partial submodule lattice for the 351-dimensional representation over $\mathbb{F}_2$



The code $C_2(\overline{\Gamma})$ is the code spanned by the matrix $\overline{A}$ over $\mathbb{F}_2$. The self-orthogonality of $C_2(\overline{\Gamma})$ follows by noticing the divisibility of the parameters of $\overline{\Gamma}$ by 2. This can be read off from the third row of A_2 found above Remark 5.1.

For items (ii), (iii) and (iv), respectively, the proof is virtually the same as that given for Proposition 5.3 with Γ and Γ^R replaced by $\overline{\Gamma}$ and $\overline{\Gamma}^R$, respectively. Thus we omit it.

(v) The code $C_3(\overline{\Gamma}^R)$ is the code of the complementary 2-$(351, 225, 144)$ design, and so it has parameters $[351, 28, 108]_3$ and thus contains $C_3(\Gamma)$. Moreover, $\mathbf{1} \in C_3(\overline{\Gamma}^R)$. □

5.3. CODES OF RESIDUAL AND DERIVED DESIGNS. In this section we examine the ternary codes of the residual and derived designs of $\mathcal{D}_{126}$ and describe some of their properties. The residual design of the 2-$(351, 126, 45)$ design $\mathcal{D}_{126}$ is a 2-$(225, 81, 45)$ design, denoted here $\mathcal{D}_{81}$ and the derived design is a 2-$(126, 45, 44)$ design denoted $\mathcal{D}_{45}$.

TABLE 2. Incidence matrix of the poset of submodules of ${\mathbb{F}_2}^{351\times 1}$

dim	0	1	78	79	92	93	168	168	169	169	182	182	183	183	258	259	272	273	350	351
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	.	1	.	1	.	1	.	.	1	1	.	.	1	1	.	1	.	1	.	1
78	.	.	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
79	.	.	.	1	.	1	.	.	1	1	.	.	1	1	.	1	.	1	.	1
92	.	.	.	.	1	1	.	.	.	.	1	1	1	1	.	.	1	1	1	1
93	.	.	.	.	.	1	.	.	.	.	.	.	1	1	.	.	.	1	.	1
168	.	.	.	.	.	.	1	.	1	.	1	.	1	.	1	1	1	1	1	1
168	.	.	.	.	.	.	.	1	.	1	.	1	.	1	1	1	1	1	1	1
169	.	.	.	.	.	.	.	.	1	.	.	.	1	.	.	1	.	1	.	1
169	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	1	.	1	.	1
182	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	.	1	1	1	1
182	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	.	1	1	1	1
183	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	1	.	1
183	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	.	1
258	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1	1	1
259	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1	.	1
272	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	1
273	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	1
350	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1
351	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1

Proposition 5.5. (i) *Let* C_{225} *denote the ternary code of the residual* 2-$(225, 81, 45)$ *design* $\mathcal{D}_{81}$*. Then* C_{225} *is a self-orthogonal* $[225, 26, 81]_3$ *code, with* 700 *codewords of weight* 81*. The minimum weight codewords of* C_{225} *are the blocks of* $\mathcal{D}_{81}$ *and their scalar multiples. The dual code* $\mathrm{C}_{225}^{\perp}$ *of* C_{225} *is a* $[225, 199, 6]_3$ *code. Further,* $\mathrm{Aut}(\mathcal{D}_{81}) \cong 2{\cdot}\mathrm{PSU}_4(3){\cdot}(2^2)_{122} \cong \mathrm{Aut}(\mathrm{C}_{225})$.
(ii) *Let* C_{126} *denote the ternary code of the derived* 2-$(126, 45, 44)$ *design* $\mathcal{D}_{45}$*. Then* C_{126} *is a self-orthogonal* $[126, 21, 36]_3$ *code, with* 252 *codewords of weight* 36*. The minimum weight codewords of* C_{126} *span the code, and its dual code* $\mathrm{C}_{126}^{\perp}$ *is a* $[126, 105, 6]_3$ *code with* 23250 *words of weight* 6.

Remark 5.6. (a) Notice that $\mathrm{Aut}(\mathcal{D}_{81}) = \mathrm{Stab}_{\mathrm{Aut}(\mathcal{D}_{126})}(B) \cong 2{\cdot}\mathrm{PSU}_4(3){\cdot}(2^2)_{122}$ where $B \in \mathcal{B}$ is the block used to construct $\mathcal{D}_{81}$. For the notation $(2^2)_{122}$, consult the $\mathbb{ATLAS}$ [13, p. 109].
(b) The central involution of $\mathrm{Aut}(\mathcal{D}_{81})$ acts trivially on C_{225}. As stated in Proposition 5.5(i) the central involution of $\mathrm{Aut}(\mathrm{C}_{225})$ sends v to $-v$ for every codeword $v \in \mathrm{C}_{225}$.
(c) Observe that the minimum weight of C_{225} is 81 and this is the block size of $\mathcal{D}_{81}$.
(d) The code C_{126} is isomorphic to the code discussed in [15, Section 5.2]. Observe that the code examined in [15, Section 5.2] has been obtained as the code of a strongly regular $(126, 45, 12, 18)$ graph. The latter graph denoted $\mathcal{G}_{126}$ is one of two strongly regular locally $GQ(4, 2)$ graphs, see [31]. $\mathcal{G}_{126}$ is isomorphic to $NO_6^-(3)$, the graph on one class of nonisotropic points of $\mathrm{PG}(5, 3)$ equipped with a nondegenerate quadratic form, where two points are joined when they are orthogonal (i.e. when the connecting line is elliptic).

6. GRAPHS, DESIGNS AND CODES FROM $G_2(4)$ OF DEGREE 2016

The action of the group $\mathrm{PGO}_{2m+1}(2^t)$ on an orbit of non-singular hyperplanes of $\mathrm{PG}(2m, 2^t)$ is isomorphic to the action of $\mathrm{PSp}_{2m}(2^t)$ on $\mathcal{Q}$, the set of quadratic forms of V, polarizing into the given symplectic form f, where V is a $2m$-dimensional vector space over $\mathbb{F}_{2^t}$. The quadratic form could be hyperbolic or elliptic. It was proved in [20, Lemma 1] that if $Q(x) \in \mathcal{Q}$, then the members of $\mathcal{Q}$ are the various

$Q(x) + (f(x,q))^2$ for $q \in V$. In addition, $Q(x) + (f(x,q))^2$ are in the same orbit under $\mathrm{Sp}_{2m}(2^t)$ if and only if $Q(q) = s^2 + s$ for some $s \in \mathbb{F}_{2^t}$. Hence $\mathrm{Sp}_{2m}(2^t)$ is of rank 3 when $t = 2$, i.e., $Q(q) \in \{0, 1\}$. It was proven in [14, Section 5] that $G_2(q)$ is a maximal subgroup of $\mathrm{PSp}_6(q)$ for $q > 2$, even. In particular, here we consider $G = G_2(4) < \mathrm{PSp}_6(4)$. So, let Q be an elliptic form. As stated earlier, the two orbits of G_Q correspond to the quadratic form $Q(x)+(f(x,q))^2$ for $Q(q) = 0$ or 1, but in this case the stabilizer G_Q is isomorphic to $\mathrm{PSU}_3(4){:}2$ acting naturally on $V = \mathbb{F}_4^6$. One can deduce from [14, Section 6] that the diagonal of a hermitian form on $\mathbb{F}_{q^2}^3$ is an elliptic quadratic form on $\mathbb{F}_q^6$, so $\mathrm{PSU}_3(4)$ is a subgroup isomorphic to $O_6^-(4)$, which is in fact the stabilizer of an elliptic form in $\mathrm{PSp}_6(4)$. Notice that $|G_2(4)| = 2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$ and $G_2(4)$ has an involutory outer automorphism, so its automorphism group is a split extension of $G_2(4)$ by $\mathbb{Z}_2$. Recall from Section 3 that $G_2(4)$ of degree 2016 has a rank 3 action on the set $\mathcal{Q}$ of elliptic forms. Here, we consider the designs, graphs and codes defined by this rank 3 action. By Equation (2), the orbits of G_Q are of lengths 1, 975, and 1040, respectively. We denote by Λ the strongly regular $(2016, 975, 462, 480)$ graph constructed from the orbit of length 975, and denote $\overline{\Lambda}$ its complement which has parameters $(2016, 1040, 544, 528)$. The structure of Λ and $\overline{\Lambda}$ give the following values:

$$B_0 = \mathbf{I}_3, \quad B_1 = \begin{bmatrix} 0 & 1 & 0 \\ 975 & 462 & 480 \\ 0 & 512 & 495 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 512 & 495 \\ 1040 & 528 & 544 \end{bmatrix}.$$

The common eigenspaces $\xi_i(b_j)$ of the intersection matrices over a field of characteristic zero are displayed in the "character table" $[\xi_i(b_j)]_{1 \le i,j \le 3}$ given below

$$[\xi_i(b_j)] = \begin{bmatrix} 1 & 975 & 1040 \\ 1 & 15 & -16 \\ 1 & -33 & 32 \end{bmatrix}.$$

- The rows of the adjacency matrix B for Λ give the blocks of the neighbourhood design of Λ which we denote $\mathcal{D}_{975}$. Notice that $\mathcal{D}_{975}$ is a self-dual symmetric 1-$(2016, 975, 975)$ design. We write $C_p(\Lambda)$ to denote the p-ary codes spanned by the rows of an incidence matrix of $\mathcal{D}_{975}$.
- From the rows of an adjacency matrix $B + I$ of the reflexive graph Λ^R we obtain the self-dual symmetric 1-$(2016, 976, 976)$ design $\mathcal{D}_{976}$, and the p-ary code $C_p(\Lambda^R)$.
- The rows of an adjacency matrix $\overline{B}$ for $\overline{\Lambda}$ give the self-dual symmetric 1-$(2016, 1040, 1040)$ design $\mathcal{D}_{1040}$, and p-ary code $C_p(\overline{\Lambda})$.
- From the rows of an adjacency matrix $\overline{B} + I$ of the reflexive graph $\overline{\Lambda}^R$ we get the self-dual symmetric 1-$(2016, 1041, 1041)$ design $\mathcal{D}_{1041}$. We write $C_p(\overline{\Lambda}^R)$ to denote the p-ary codes of $\mathcal{D}_{1041}$.

Remark 6.1. It follows from [11, Chapter 2] and the $\mathbb{ATLAS}$ [13, p. 97] that the eigenvalues and multiplicities for Λ are α_i for $0 \le i \le 2$ with multiplicities m_i respectively, those for Λ^R are $\alpha_i^* = \alpha_i + 1$ with multiplicities m_i for $0 \le i \le 2$, and those for $\overline{\Lambda}$ and $\overline{\Lambda}^R$ are $\overline{\alpha_0} = n - 1 - k$, $\overline{\alpha_i} = -\alpha_i - 1$, $\overline{\alpha}_i^* = \overline{\alpha_i} + 1$ for $i = 1, 2$, where

- $\alpha_0 = 975$, $\alpha_0^* = 976$, $\overline{\alpha_0} = 1040, \overline{\alpha}_0^* = 1041, m_0 = 1$;
- $\alpha_1 = 15, \alpha_1^* = 16, \overline{\alpha_1} = 32, \overline{\alpha}_1^* = 33$, $m_1 = 1365$;
- $\alpha_2 = -33, \alpha_2^* = -32, \overline{\alpha_2} = -16, \overline{\alpha}_2^* = -15$, $m_2 = 650$.

Let B denote the adjacency matrix of the graph Λ and $\overline{B}$ the adjacency matrix of its complementary graph $\overline{\Lambda}$. In this section we examine the p-ary codes $C_p(B)$, $C_p(B+I)$, $C_p(\overline{B})$ and $C_p(\overline{B}+I)$ from the matrices B, $\overline{B}$, $B+I$ and $\overline{B}+I$. We denote these codes according to the notation of their graphs, i.e. $C_p(\Lambda)$, $C_p(\Lambda^R)$, $C_p(\overline{\Lambda})$ and $C_p(\overline{\Lambda}^R)$, respectively. We start by examining the codes $C_p(\Lambda)$ and $C_p(\Lambda^R)$.

Theorem 6.2. *Let $C_p(\Lambda)$ denote the p-ary code of Λ and $C_p(\Lambda^R)$ be the p-ary code of the reflexive graph Λ^R of Λ. Then*
(i) $C_p(\Lambda) = \mathbb{F}_p^{2016}$ *for* $p = 2, 7$.
(ii) $C_3(\Lambda)$ *is a self-orthogonal* $[2016, 651, d]_3$ *code with* $d \geq 975$.
(iii) $\mathbb{F}_5^{2016} = C_5(\Lambda) + C_5(\Lambda)^\perp$ *and* $\dim(C_5(\Lambda)) = 650$. *Moreover,* $C_5(\overline{\Lambda}^R)$ *and* $C_5(\overline{\Lambda}^R)^\perp$ *are LCD codes.*
(iv) $C_{13}(\Lambda)$ *is a code of codimension* 1 *in* $\mathbb{F}_{13}^{2016}$.
(v) $C_2(\Lambda^R)$ *is a self-orthogonal triply-even* $[2016, 14, 976]_2$ *code, and* $C_2(\Lambda^R)^\perp$ *is a* $[2016, 2002, 4]_2$ *code with* 83701800 *codewords of weight* 4. *The minimum weight of* $C_2(\Lambda^R)$ *is the valency of* Λ^R *and the words of minimum weight are the rows of the adjacency matrix of* Λ^R. *Moreover,* $C_2(\Lambda^R)$ *is spanned by its minimum weight codewords.*
(vi) $\mathbf{1} \in C_2(\Lambda^R)$ *and* $\mathrm{Aut}(C_2(\Lambda^R)) \cong \mathrm{PSp}_6(4)$.
(vii) $C_p(\Lambda^R) = \mathbb{F}_p^{2016}$ *for* $p \neq 2$.

Proof. (i) By Remark 6.1 we have that the eigenvalues of B are $\alpha_0 = 975, \alpha_1 = 15$, and $\alpha_2 = -33$ with multiplicities $m_0 = 1, m_1 = 1365$ and $m_2 = 650$. Since none of the α_i vanishes mod p, for $p = 2, 7$ it follows from [5, Proposition 13.7.1(iv)] that $\mathrm{rk}_p(B) = 2016$.

(ii) From [5, Proposition 13.7.1] we obtain an upper bound on the 3-rank of B, i.e., $\mathrm{rk}_3(B) \leq \min(m_1 + 1, m_2 + 1) = 651$. Since by computations with Magma [2] we have $\dim(C_3(\Lambda))$ equals 651, the result follows. Furthermore, self-orthogonality of $C_3(\Lambda)$ follows since in Λ all parameters are divisible by 3. Now, $d \geq 975$ follows since the valency of Λ is 975.

(iii) $\mathrm{rk}_5(B) = 650$, since $\alpha_0 \equiv \alpha_1 \pmod 5$, $\alpha_2 \not\equiv 0 \pmod 5$ and $5 \mid \mu$, by [5, Proposition 13.3.2 (iii)]. Now, $C_5(\Lambda) \cap C_5(\Lambda)^\perp = \{0\}$ implies that $\mathbb{F}_5^{2016} = C_5(\Lambda) + C_5(\Lambda)^\perp$ and the result follows. Observe that this shows that $C_5(\Lambda)$ and $C_5(\Lambda)^\perp$ are LCD codes.

(iv) Finally, since α_0 is the only eigenvalue which vanishes modulo 13, we obtain $\mathrm{rk}_{13}(\overline{B}) = n - m_0 = 2015$.

(v) That the dimension of $C_2(\Lambda^R)$ is as stated in the proposition follows by using the Atlas of Brauer characters [24, p. 273]. Moreover, $C_2(\Lambda^R) \cap C_2(\Lambda^R)^\perp = C_2(\Lambda^R)$ and thus $C_2(\Lambda^R)$ is self-orthogonal. That the minimum weight equals the valency of Λ^R follows at once. By computations with Magma [2] we obtain the weight distribution of $C_2(\Lambda^R)$ which is given as follows

$$A_0 = A_{2016} = 1,\ A_{976} = A_{1040} = 2016,\ A_{1008} = 4160,\ A_{992} = A_{1024} = 4095. \tag{5}$$

Now, notice from Equation (5) that there are exactly 2016 codewords of minimum weight 976 in $C_2(\Lambda^R)$. Thus the adjacency matrix of Λ^R is determined up to a column permutation by the set of all minimum weight codewords, and these correspond to the rows of Λ^R. Consequently, these are spanning vectors of $C_2(\Lambda^R)$, and since the spanning words of $C_2(\Lambda^R)$ have weight divisible by eight, it follows that $C_2(\Lambda^R)$ is triply-even.

(vi) That $\mathbf{1} \in C_2(\Lambda^R)$ can be deduced from Equation (5). From the fact that an automorphism of the code must preserve weight classes and thus the minimum words which correspond to the rows of Λ^R we deduce that $\mathrm{Aut}(\Lambda^R) \subseteq \mathrm{Aut}(C_2(\Lambda^R))$. Order considerations shows that $\mathrm{Aut}(\Lambda^R) = \mathrm{Aut}(C_2(\Lambda^R)) \cong \mathrm{PSp}_6(4)$.

(vii) Follows as in (i). □

In Theorem 6.3 we list the properties of the p-ary codes from the graphs $\overline{\Lambda}$ and $\overline{\Lambda}^R$. The proof follows by arguing similarly as in Theorem 6.2.

Theorem 6.3. *Let $C_p(\overline{\Lambda})$ denote the p-ary code of $\overline{\Lambda}$ and $C_p(\overline{\Lambda}^R)$ be the p-ary code of $\overline{\Lambda}^R$ the reflexive graph of $\overline{\Lambda}$. Then*
(i) $C_2(\overline{\Lambda}) = C_2(\Lambda^R)$.
(ii) $C_p(\overline{B}) = \mathbb{F}_p^{2016}$ *for* $p = 3, 7$.
(iii) $C_p(\overline{\Lambda})$ *is a code of codimension* 1 *in* $\mathbb{F}_p^{2016}$ *for* $p = 5, 13$.
(iv) $C_p(\overline{\Lambda}^R) = \mathbb{F}_p^{2016}$ *for* $p = 2, 7, 13$.
(v) $C_3(\overline{\Lambda}^R)$ *is a self-orthogonal* $[2016, 651, d]_3$ *code with* $d \leq 1041$.
(vi) $\mathbb{F}_5^{2016} = C_5(\overline{\Lambda}^R) + C_5(\overline{\Lambda}^R)^\perp$ *and* $\dim(C_5(\overline{\Lambda}^R)) = 651$. *Moreover,* $C_5(\overline{\Lambda}^R)$ *and* $C_5(\overline{\Lambda}^R)^\perp$ *are LCD codes.*

The preceding theorems, lemmas and propositions give the proof of Theorem 1.1 stated in the introduction.

Codes with few weights have gained recent interest. In particular, three-weight codes have been studied in [9]. We show in the next result that a subcode of codimension one of $C_2(\Lambda^R)$ is a projective triply-even three-weight code.

Proposition 6.4. *The codewords of weight* 992 *in* $C_2(\Lambda^R)$ *span an indecomposable code* $\mathcal{L}$ *of co-dimension* 1. *The code* $\mathcal{L}$ *is a projective three-weight* $[2016, 13, 992]_2$ *self-orthogonal and triply-even code and its dual* $\mathcal{L}^\perp$ *is a* $[2016, 2003, 4]_2$ *code with* 167567400 *codewords of weight* 4. *Furthermore,* $\mathbf{1} \in \mathcal{L}$, $\mathrm{Aut}(\mathcal{L}) \cong \mathrm{PSp}_{12}(2)$ *and* $\mathcal{L}$ *meets the Grey-Rankin bound with equality.*

Proof. Observe first that $C_2(\Lambda^R) = \mathcal{L} \oplus \langle \mathbf{1} \rangle$. The weight distribution

$$A_0 = A_{2016} = 1, \ A_{992} = A_{1024} = 4095 \tag{6}$$

of $\mathcal{L}$ can be deduced from Equation (5).

Now, since $\mathcal{L} \subset C_2(\Lambda^R)$ it follows that $\mathcal{L}$ is self-orthogonal and triply-even, since subcodes of triply-even codes must be triply-even. It is clear from Theorem 6.2 (vi) that $\mathbf{1} \in \mathcal{L}$, see also [25, Lemma 2.2(iv)].

Taking the support of the codewords of weight 992 in $\mathcal{L}$ we observe that these span a 1-$(2016, 992, 2015)$ design $\mathcal{D}$ with 4095 blocks. This is in fact a quasi-symmetric 2-$(2016, 992, 991)$ design $\mathcal{D}$ with 4095 blocks, i.e. totally isotropic 1-spaces. By computations with Magma we observe that $\mathcal{D}$ admits a 2-transitive automorphism group of order $2^{36} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 31$ with point stabilizer isomorphic to $\mathrm{PGO}_{12}^-(2)$ and with block stabiliser isomorphic to $2^{11}{:}\mathrm{PSp}_{10}(2)$. Now by the classification of primitive groups of degree 2016 we have $\mathrm{Aut}(\mathcal{D}) \cong \mathrm{PSp}_{12}(2)$. Since $\mathrm{Aut}(\mathcal{D}) \subseteq \mathrm{Aut}(\mathcal{L})$ we deduce by order considerations that $\mathrm{Aut}(\mathcal{L}) \cong \mathrm{PSp}_{12}(2)$.

The fact that $\mathcal{L}$ meets the Grey-Rankin bound with equality follows from [25, Theorem 2.8]. □

As a direct application of [12, Lemma 2.4], we deduce the following:

Proposition 6.5. *The code generated by the incidence matrix of the design $\mathcal{D}$ is contained in $\mathcal{L}$. There are no other self-orthogonal 2-$(2016, 2k, \lambda)$ designs invariant under* $\mathrm{PSp}_{12}(2)$.

Remark 6.6. The design $\mathcal{D}$ of Proposition 6.4 forms part of a family of non-symmetric 2-(v, k, λ) designs with parameters

$$v = 2^{2m-1} - 2^{m-1},\ k = 2^{2m-2} - 2^{m-1},\ \lambda = 2^{2m-2} - 2^{m-1} - 1 \tag{7}$$

known to have the symmetric difference property (SDP), i.e. designs for which the symmetric difference of any two blocks is either a block or the complement of a block. According to [25, Lemma 2.2(ii)] designs of the form (7) are derived designs of symmetric designs with the symmetric difference property with parameters

$$v = 2^{2m},\ k = 2^{2m-1} - 2^{m-1},\ \lambda = 2^{2m-2} - 2^{m-1}. \tag{8}$$

The number of non-isomorphic quasi-symmetric SDP designs grows exponentially and the exact number of non-isomorphic symmetric SDP designs with parameters 2-$(2^{2m}, 2^{2m-1}-2^{m-1}, 2^{2m-2}-2^{m-1})$ depends on the number of inequivalent univariate bent functions over $\mathbb{F}_{2^m}$. According [7, Section 7.6] this number equals 896 for $m = 6$. In Corollary 6.4 the constructed 2-$(2016, 992, 991)$ design $\mathcal{D}$ admits a 2-transitive automorphism group. To the benefit of the reader, we remark that our construction differs from that presented in [7, Proposition 7.6.2] and [25].

By enumerating all $G_2(4)$-submodules of $G_2(4)$ of degree 2016 over $\mathbb{F}_2$, we determine the number of distinct $G_2(4)$-invariant codes of length 2016. We use this fact to give a non-existence result on self-dual codes of length 2016 over $\mathbb{F}_2$ invariant under $G_2(4)$.

Proposition 6.7. *Let $\mathbb{F}$ be an algebraically closed field of characteristic* 2. *Then there is no $G_2(4)$-invariant self-dual code of length* 2016.

Proof. Consider $G = G_2(4) \leq \mathrm{PSp}_6(4)$. Recall that in this case we are dealing with the rank 3 representation of degree 2016. For this let φ_π denote the 2-Brauer character of the permutation character of $G_2(4)$ of degree 2016, and χ_π denote its ordinary permutation character. By the $\mathbb{ATLAS}$ [13, p. 97] we have $\chi_\pi = \chi_1 + \chi_{650} + \chi_{1365}$, and from [4] φ_π decomposes into

$$\varphi_\pi = 28\varphi_1 + 19\varphi_{6_a} + 19\varphi_{6_b} + 10\varphi_{14_a} + 10\varphi_{14_b}$$
$$+8\varphi_{36} + \varphi_{64_a} + \varphi_{64_b} + 4\varphi_{84_a} + 4\varphi_{84_b} + 2\varphi_{196}.$$

Since all simple submodules of φ_π appear with even multiplicity, then [23, Theorem 2.1] would imply existence of a self-dual code of length 2016 invariant under $G_2(4)$. However, examining the Atlas of Brauer characters [24, p. 273] (see also [38]) we see that the representations of degrees 6, 14, 64 and 84 are not realizable over $\mathbb{F}_2$. The smallest field of realization of these representations is $\mathbb{F}_4$. Hence, there is no self-dual code of length 2016 invariant under G.

In the case of arbitrary fields $\mathbb{F} \supseteq \mathbb{F}_2$ we have essentially the same situation, since $\mathbb{F}\Omega \cong_{\mathbb{F}\otimes\mathbb{F}_2} \mathbb{F}_2\Omega$ and almost all completely reducible factors are multiplicity-free. □

7. Concluding remarks

We note here that the p-ary codes with parameters those listed in Theorem 1.1 are codes with moderately large length and small dimension and as such are not listed in any known database of linear codes. The interest in studying them was mainly prompted by their geometric connections to groups of exceptional type, irreducibility properties, by the large size of their automorphism groups which in the binary case might make them useful to permutation decoding. An additional motivation for their study is the fact that they satisfy the property that their minimum weight is the valency of the graph and the minimum weight codewords are exactly the rows of the adjacency matrix and their scalar multiples. For the reader's convenience in Table 3 we list the main parameters of the codes obtained in the paper and provide a structure description of their automorphism groups.

Table 3. p-ary codes from the rank 3 actions of $G_2(3)$ and $G_2(4)$

p	$G_2(3)$	code	Aut	$G_2(4)$	code	Aut
2		$[351, 78, 48]$	$O_7(3){:}2$		$[2016, 14, 976]$	$\mathrm{PSp}_6(4)$
2		$[351, 79, 48]$	$O_7(3){:}2$		$[2016, 2002, 4]$	$\mathrm{PSp}_6(4)$
2					$[2016, 13, 992]$	$\mathrm{PSp}_{12}(2)$
2					$[2016, 2003, 4]$	$\mathrm{PSp}_{12}(2)$
3		$[351, 27, 126]$	$O_7(3){:}2$		$[2016, 651, d]$, $d \geq 975$	$-$
3		$[351, 324, 6]$	$O_7(3){:}2$		$[2016, 651, d]$, $d \geq 1041$	$-$
3		$[351, 28, 108]$	$O_7(3){:}2$			
5					$[2016, 650, d]$	$-$
5					$[2016, 651, d]$	$-$

We note that the designs that we examine from the given strongly regular graphs are in accordance with the designs constructed from finite primitive groups by using a method of construction outlined by Key and Moori in [32]. For $q \geq 5$, the action of the group $G_2(q)$ on $\frac{q^3(q^3-1)}{2}$ points is no longer of rank 3. However, the current work together with some additional computer experimentations (up to $q = 9$) led us to the following questions which are of theoretical significance.

Question 7.1. *For odd $q \geq 5$, let $G = G_2(q)$ act primitively on the set Ω of degree $\frac{q^3(q^3-1)}{2}$ and $\alpha \in \Omega$. Let Δ be the orbit of size $q^2(q^3+1)$ of the stabilizer $\mathrm{Stab}_G(\alpha)$. Let $\mathcal{D} = (\Omega, \mathcal{B})$ be the design constructed from G with $\mathcal{B} = \{\Delta^g : g \in G\}$. Then is the core of the full automorphism group $\mathrm{Aut}(\mathcal{D})$ of $\mathcal{D}$ isomorphic to $O_7(q)$?*

A computer search performed in Magma with limited computing power, has not yielded another example of subcodes whose properties are similar to those of the code studied in Proposition 6.4 for $G_2(4)$. Thus, we pose the following

Question 7.2. *Is there any generalization for the construction of this $G_2(2^m)$-subcode, i.e., does there exist a $G_2(2^m)$-code $\mathcal{C}$ over $\mathbb{F}_2$ having a nontrivial proper subcode $\mathcal{L}$ such that*

- *$\mathcal{L}$ is a $G_2(2^m)$-subcode of $\mathcal{C}$ and*
- *The full automorphism group of $\mathcal{L}$ explodes in size, to possibly a group isomorphic to $\mathrm{Sp}_{6m}(2)$?*

Acknowledgments

The authors thank the anonymous referees whose comments and suggestions led to a better presentation of the paper.

References

[1] E. F. Assmus, Jr. and J. D. Key, *Designs and Their Codes*, Cambridge Tracts in Mathematics, Vol. 103. Cambridge: Cambridge University Press, 1992.

[2] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system I: The user language, *J. Symbolic Comput.*, **24** (1997), 235–265.

[3] S. Braić, A. Golemac, J. Mandić and T. Vučičić, Primitive symmetric designs with up to 2500 points, *J. Combin Designs*, **19** (2011), 463–474.

[4] T. Breuer, *Decomposition Matrices*, Available at: The Modular Atlas homepage. http://www.math.rwth-aachen.de/~MOC/decomposition. 1999.

[5] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*, Universitext, Springer, New York, 2012.

[6] A. E. Brouwer and C. A. van Eijl, On the p-rank of the adjacency matrices of strongly regular graphs, *J. Algebraic Combin.*, **1** (1992), 329–346.

[7] A. E. Brouwer and H. van Maldeghem, *Strongly Regular Graphs*, Encyclopedia of Mathematics and its Applications, 182. Cambridge University Press, Cambridge, 2022. https://homepages.cwi.nl/~aeb/math/srg/rk3/srgw.pdf.

[8] F. Buekenhout and H. Van Maldeghen, A characterization of some rank 2 incidence geometries by their automorphism group, *Mitt. Math. Sem. Giessen*, **218** (1994), 1–70.

[9] A. R. Calderbank and J. -M. Goethals, Three weight codes and association schemes, *Philips J. Res.*, **39** (1984), 143–152.

[10] P. J. Cameron, *Permutation Groups*, London Mathematical Society Student Texts, 45, Cambridge: Cambridge University Press, 1999.

[11] P. J. Cameron and J. H. van Lint, *Designs, Graphs, Codes and Their Links*, London Mathematical Society Student Texts, 22, Cambridge: Cambridge University Press, 1991.

[12] N. Chigira, M. Harada and M. Kitazume, Permutation groups and binary self-orthogonal codes, *J. Algebra*, **309** (2007), 610–621.

[13] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *An Atlas of Finite Groups*, Oxford: Oxford University Press, 1985.

[14] A. Cossidente and O. H. King, On the geometry of the exceptional group $G_2(q)$, q even, *Des. Codes Cryptogr.*, **47** (2008), 145–157.

[15] D. Crnković, V. Mikulić and B. G. Rodrigues, Some strongly regular graphs and self-orthogonal codes from the unitary group $U_4(3)$, *Glas. Mat. Ser. III,* **45** (2010), 307–323.

[16] H. J. Coutts, M. Quick and C. M. Roney-Dougal, The primitive permutation groups of degree less than 4096, *Comm. Algebra*, **39** (2011), 3526–3546.

[17] P. Dankelmann, J. D. Key and B. G. Rodrigues, A characterization of graphs by codes from their incidence matrices, *Electron. J. of Combin.*, **20** (2013), Paper 18, 22 pp.

[18] P. Dankelmann, J. D. Key and B. G. Rodrigues, Codes from incidence matrices of graphs, *Des. Codes Cryptogr.*, **68** (2013), 373–393.

[19] U. Dempwolff, Primitive rank-3 groups on symmetric designs, *Des. Codes Crypt.*, **22** (2001), 191–207.

[20] R. H. Dye, Interrelations of symplectic and orthogonal groups in characteristic two, *J. Algebra*, **59** (1979), 202–221.

[21] W. Fish, J. D. Key and E. Mwambene, Binary codes from reexive uniform subset graphs on 3-sets, *Adv. Math. Commun.*, **9** (2015), 211–232.

[22] W. Fish, J. D. Key and E. Mwambene, Ternary codes from some reflexive uniform subset graphs, *Appl. Algebra Engrg. Comm. Comput.*, **25** (2014), 363–382.

[23] A. Günther and G. Nebe, Automorphisms of doubly even self-dual codes, *Bull. London Math. Soc.*, **41** (2009), 769–778.

[24] C. Jansen, K. Lux, R. Parker and R. Wilson, *An Atlas of Brauer Characters*, LMS Monographs New Series 11. Oxford: Oxford Science Publications, Clarendon Press, 1995.

[25] D. Jungnickel and V. D. Tonchev, Exponential number of quasi-symmetric SDP designs and codes meeting the Grey-Rankin bound, *Des. Codes and Cryptogr.*, **1** (1991), 247–253.

[26] J. D. Key and J. Moori, Codes, designs and graphs from the Janko groups J_1 and J_2, *J. Combin. Math and Combin. Comput.*, **40** (2002), 143–159.

[27] W. Knapp and H.-J. Schaeffer, On the codes related to the Higman-Sims graph, *Electron. J. of Combin.*, **22** (2015), Paper 1.19, 58 pp.

[28] T. Le and B. G. Rodrigues, Magma computations for $G_2(q)$ of rank 3, Available at: https://bgrodrigues.weebly.com/uploads/1/2/8/4/12846738/g2qrank3.txt

[29] M. W. Liebeck, C. E. Praeger and J. Saxl, On the 2-closures of finite permutation groups, *J. Lond. Math. Soc.*, **37** (1987), 241–252.

[30] K. Lux and H. Pahlings, *Representations of Groups: A Computational Approach*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2010.

[31] A. A. Makhnev, $GQ(4,2)$-extensions, the strongly regular case, *Math. Notes*, **68**, 97–102.

[32] J. Moori, Finite groups, designs and codes, Information security, coding theory and related combinatorics, 202–230, NATO Sci. Peace Secur. Ser. D Inf. Commun. Secur., 29, IOS, Amsterdam, 2011.

[33] J. Moori and B. G. Rodrigues, Some designs and codes invariant under the simple group Co_2, *J. of Algebra*, **316** (2007), 649–661.

[34] J. Moori and B. G. Rodrigues, Some designs and binary codes preserved by the simple group Ru of Rudvalis, *J. Algebra*, **372** (2012), 702–710.

[35] B. G. Rodrigues, Linear codes with complementary duals related to the complement of the Higman-Sims graph, *Bull. Iranian Math. Soc.*, **43** (2017), 2183–2204.

[36] V. D. Tonchev, Codes, in: Handbook of Combinatorial Designs, 2^{nd} ed. (C. J. Colbourn and J. H. Dinitz, Eds.), Chapman and Hall/CRC, Boca Raton, 2007, 667–702.

[37] R. A. Wilson, *The Finite Simple Groups*, London: Springer-Verlag London Ltd., 2009. Graduate Texts in Mathematics, Vol. 251.

[38] R. A. Wilson, P. Walsh, J. Tripp, I. Suleiman, R. A. Parker, S. P. Norton, S. Nickerson, S. Linton, J. Bray and R. Abbott, *Atlas of Finite Group Representations* - version 3, http://brauer.maths.qmul.ac.uk/Atlas/v3/exc/.

Received August 2021; revised February 2022; early access March 2022.

E-mail address: lttung96@yahoo.com
E-mail address: bernardo.rodrigues@up.ac.za