PERIMETER APPROXIMATION OF CONVEX DISCS IN THE HYPERBOLIC PLANE AND ON THE SPHERE

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ABSTRACT. Eggleston [5] proved that in the Euclidean plane the best approximating convex n-gon to a convex disc K is always inscribed in K if we measure the distance by perimeter deviation. We prove that the analogue of Eggleston's statement holds in the hyperbolic plane, and we give an example showing that it fails on the sphere.

1. INTRODUCTION AND MAIN RESULTS

We call a compact, convex set $K \subset \mathbb{R}^2$ whose interior is non-empty a *convex disc*. The perimeter of K is denoted by $\mathrm{Per}(K)$. Let K and L be both convex discs. The *perimeter deviation of K and L* is defined as

$$\mathrm{dev}(K, L) = \mathrm{Per}(K \cup L) - \mathrm{Per}(K \cap L).$$

We note that although the perimeter deviation is often used to measure the distance of convex figures, it does not define a proper metric on the set of all convex discs as it does not satisfy the triangle inequality, see Besau, Hoehner, Kur [2, Appendix A]. However, perimeter deviation is an important concept as it is an example of intrinsic volume deviations, which are used to measure distance in approximations of convex bodies by polytopes, see [2, (2) on p. 2]. For another notion of perimeter deviation, which is in fact a metric, see, for example, Florian [9] and the references therein.

Eggleston [5], among other questions, investigated how well a convex disc can be approximated by convex polygons of a given number of vertices in the sense of perimeter deviation. For a positive integer $n \geq 3$, let $\mathcal{P}(n)$ denote the set of convex polygons with at most n vertices. Let

$$\delta_{\mathrm{dev}}(K, n) = \inf\{\mathrm{dev}(K, P) : P \in \mathcal{P}(n)\}.$$

A simple compactness argument shows that for each convex disc K and positive integer $n \geq 3$, there exists a $P \in \mathcal{P}(n)$ which minimizes the perimeter deviation from K, that is, $\mathrm{dev}(K, P) = \delta_{\mathrm{dev}}(K, n)$.

Eggleston proved the following beautiful statement, cf. [5, Lemma 4 on p. 353].

Theorem 1.1 (Eggleston, 1957)**.** *Let K be a convex disc and $n \geq 3$ a positive integer. If $P \in \mathcal{P}(n)$ is such that $\mathrm{dev}(K, P) = \delta_{\mathrm{dev}}(K, n)$, then P is inscribed in K, that is, $P \subset K$ and the vertices of P are on the boundary of K.*

According to a classical result of Dowker [4], the minimum area of convex n-gons containing a given convex disc K is a convex function of n, and the maximum area of convex n-gons contained in K is a concave function of n. This result was later extended for perimeter in place of area by L. Fejes Tóth [7], Eggleston [5], and Molnár [12], independently from each other. Thus, it follows from Theorem 1.1 that for a fixed convex disc K, the minimum perimeter deviation of convex n-gons from K is also a concave function of n.

Let $\mathbb{H}^2$ denote the hyperbolic plane, and for two points $p, q \in \mathbb{H}^2$ let $d_H(p, q)$ denote their hyperbolic distance, and pq the segment with endpoints p and q. Let $\mathcal{P}_H(n)$ be the set of all convex polygons in $\mathbb{H}^2$ with at most n vertices for $n \geq 3$. Similarly to the Euclidean case, we define

$$\delta_{\mathrm{dev}_H}(K, n) = \inf\{\mathrm{dev}(K, P) : P \in \mathcal{P}_H(n)\}.$$

For any fixed K and positive integer $n \geq 3$, there exits a convex polygon $P \in \mathcal{P}_H(n)$ such that $\mathrm{dev}_H(K, P) = \delta_{\mathrm{dev}_H}(K, n)$. We extend Theorem 1.1 to the hyperbolic plane $\mathbb{H}^2$ as follows.

Theorem 1.2. *Let K be a convex disc in $\mathbb{H}^2$ and $n \geq 3$ a positive integer. If $P \in \mathcal{P}_H(n)$ is such that $\mathrm{dev}(K, P) = \delta_{\mathrm{dev}_H}(K, n)$, then P is inscribed in K, that is, $P \subset K$ and the vertices of P are on the boundary of K.*

The analogues of Dowker's theorem both for area and perimeter also hold on the sphere $\mathbb{S}^2$ and the hyperbolic plane $\mathbb{H}^2$. These were proved by Molnár [12] and L. Fejes Tóth [8]. Thus, Theorem 1.2, combined with the hyperbolic version of Dowker's theorem for the maximum perimeter of convex (hyperbolic) n-gons contained in a given convex disc K, implies the following statement.

Corollary 1.3. *The minimum perimeter deviation of convex n-gons from a given convex disc K is a concave function of n in the hyperbolic plane $\mathbb{H}^2$.*

On the unit sphere $\mathbb{S}^2$, the distance of two non-antipodal points p, q is the length of the shorter arc of the unique great circle through p and q. The distance of two antipodal points is π. We call a closed set K on $\mathbb{S}^2$ a (spherically) convex disc if it is contained in an open hemisphere and for any $p, q \in K$, the shorter arc of the unique great circle connecting p and q is also contained in K. One may naturally define the perimeter deviation $\mathrm{dev}_S(K, L)$ of two convex discs K, L on the unit sphere as in the Euclidean plane and hyperbolic plane. Again, for a convex disc K and $n \geq 3$, there exists a convex spherical polygon P with at most n vertices such that $\mathrm{dev}_S(K, P) = \delta_{\mathrm{dev}_S}(K, n)$. However, P may not necessarily be contained in K (or contain K) as shown by the example in Section 3.

We note that approximations of convex bodies in d dimensional Euclidean space with respect to all intrinsic volume deviations, where the relative position of the polytope and body is not restricted, have recently been studied by Besau, Hoehner, Kur [2]. They prove asymptotic estimates for best approximations of the unit ball in these deviations measures. For more detailed information and further references on best and random approximations of convex bodies by polytopes we refer to the surveys by Bárány [1] and Schneider [14] and the books by Gruber [10] and Schneider [13]; the latter two also serve as references on fundamental properties of convex bodies.

2. Proof of Theorem 1.2

In this section we work in the hyperbolic plane $\mathbb{H}^2$, thus all notions, such as distance, convexity, perimeter, perimeter deviation, etc. are always understood in the hyperbolic sense without mentioning this fact explicitly. We think of $\mathbb{H}^2$ as a 2-dimensional Riemannian manifold of constant curvature -1, such as the Beltrami-Klein model, see more on this below. By the *curvature* of a C^2 curve in $\mathbb{H}^2$ we mean its geodesic curvature. A compact set $K \subset \mathbb{H}^2$ is (geodesically) convex, if for any $x, y \in K$, the geodesic segment xy is contained in K.

We follow an argument that is based on ideas of Eggleston in [5] but is somewhat more complicated due to the hyperbolic setting.

First, note that the set of all compact, convex sets forms a complete metric space with respect to the Hausdorff distance in $\mathbb{H}^2$. Furthermore, the perimeter deviation function is continuous on this space. Thus, it is enough to prove the theorem on a suitable dense subspace of convex discs. We select this dense subspace the following way: We assume that the boundary bd K of K is C_+^2 smooth, meaning that it is twice continuously differentiable at every point and the geodesic curvature is strictly positive everywhere.

We start the proof by examining the difference between the length of a chord and the corresponding arc of bd K cut off by the chord.

In the following argument, we will work in the Beltrami-Klein model D of the hyperbolic plane $\mathbb{H}^2$ whose points are the interior points of the unit radius circular disc B^2 centred at the origin, and whose lines (geodesics) are the Euclidean open line segments with endpoints on the boundary S^1 of B^2. We define the distance of two points $p, q \in D$ as

$$d_H(p, q) = \frac{1}{2}|\ln(abpq)|,$$

where a and b are the intersection points of the line pq with S^1 such that a is on the side of p and b is on the side of q. The symbol $(abpq)$ denotes the cross-ratio of the points a, b, p, q in this order. It is well-known that the Gaussian curvature of this model is constant -1, with this particular metric. Now, if $p(x, y)$ is a point of D, where x and y are its Euclidean coordinates in a Cartesian coordinate system centred at the origin, then the hyperbolic coordinates of $p(x_h, y_h)$ are the following

$$x_h = \frac{1}{2} \ln \frac{1+x}{1-x}, \quad y_h = \frac{1}{2} \ln \frac{\sqrt{1-x^2}+y}{\sqrt{1-x^2}-y}.$$

The first fundamental form of D is

$$ds^2 = \frac{(1-y^2)dx^2 + 2xydxdy + (1-x^2)dy^2}{(1-x^2-y^2)^2},$$

see, for example, [3].

Let $K \subset D$ be a (geodesically) convex disc in D whose boundary is C_+^2 smooth. Since geodesic segments in D are exactly the Euclidean segments, the disc K is convex in the hyperbolic sense exactly if it is convex in the Euclidean sense. Assume that $o \in \text{bd}\, K$ and that the x-axis supports K at o. Then, in a suitably small neighbourhood of o, the boundary of K can be represented by a convex function f such that $f(x) = (\kappa/2)x^2 + o(x^2)$ as $x \to 0$, and $\kappa > 0$. A standard calculation shows that the geodesic curvature of bd K at o is κ.

For sufficiently small x, let $s(x)$ denote the arc-length of bd K between o and the point $(x, f(x))$. Then

$$
\begin{aligned}
s(x) &= \int_0^x \frac{((1 - f^2(\tau)) + 2\tau f(\tau) f'(\tau) + (1 - \tau^2)(f'(\tau)^2))^{1/2}}{1 - \tau^2 - f^2(\tau)} d\tau \\
&= \int_0^x \frac{\sqrt{1 + \kappa^2 \tau^2 + o(\tau^2)}}{1 - \tau^2 + o(\tau^2)} d\tau.
\end{aligned} \tag{1}
$$

After substituting the Taylor series of $\sqrt{1 + z}$ around $z = 0$ and that of $(1 - z)^{-1}$ around $z = 0$ in (1), we obtain

$$
\begin{aligned}
s(x) &= \int_0^x \left(1 + \frac{\kappa^2 \tau^2}{2} + o(\tau^2)\right)\left(1 + \tau^2 + o(\tau^2)\right) d\tau \\
&= \int_0^x 1 + \frac{\kappa^2 + 2}{2}\tau^2 + o(\tau^2) d\tau \\
&= \left(\tau + \frac{\kappa^2 + 2}{6}\tau^3 + o(\tau^3)\right]_0^x \\
&= x + \frac{\kappa^2 + 2}{6}x^3 + o(x^3) \text{ as } x \to 0^+.
\end{aligned} \tag{2}
$$

First, let $l = l(\delta)$ be the line with Euclidean equation $y = \delta$. For sufficiently small $\delta > 0$, the line l intersects bd K at $x_+(\delta) > 0$ $(x_-(\delta) < 0)$ such that $f(x_+(\delta)) = \delta$ $(f(x_-(\delta)) = \delta)$. Due to the definition of f, $x_+(\delta) = \sqrt{2/\kappa}\delta^{1/2} + o(\delta^{1/2})$ $(x_-(\delta) = -\sqrt{2/\kappa}\delta^{1/2} + o(\delta^{1/2}))$ as $\delta \to 0^+$.

Thus, by (2), the arc of bd K between o and the positive intersection point of l and bd K has length

$$
s(x_+(\delta)) = x_+(\delta) + \frac{\kappa^2 + 2}{6}x_+^3(\delta) + o(x_+^3(\delta)) \text{ as } \delta \to 0^+. \tag{3}
$$

Clearly, a similar formula holds for the length of the arc of bd K between the intersection point with (negative) x-coordinate $x_-(\delta)$ and o.

The (hyperbolic) length of the segment between the y-axis and the (positive) intersection point with bd K is the following

$$
\begin{aligned}
s_l(\delta) &= \int_0^{x_+(\delta)} \frac{\sqrt{1 - \delta^2}}{1 - x^2 - \delta^2} dx \\
&= \frac{1}{2} \ln \frac{\sqrt{1 - \delta^2} + x_+(\delta)}{\sqrt{1 - \delta^2} - x_+(\delta)} \\
&= x_+(\delta) + \frac{1}{3}x_+^3(\delta) + o(\delta^2 x_+(\delta)) \text{ as } x_+(\delta) \to 0^+,
\end{aligned} \tag{4}
$$

and, again, a similar formula holds for the length of the segment between the negative intersection point of l and bd K and the y-axis.

From (3) and (4), and the expressions of $x_+(\delta)$ and $x_-(\delta)$, we obtain that the difference of the arc of bd K and the chord at (Euclidean) height δ is

$$
\frac{\kappa^2}{3}(x_+^3(\delta) + x_-^3(\delta)) + o(x_+^3(\delta)) + o(x_-^3(\delta)) = O(\delta^{3/2}) \text{ as } \delta \to 0^+. \tag{5}
$$

Since the hyperbolic height of l is $\delta_H = \tanh^{-1}\delta = \delta + O(\delta^3)$ as $\delta \to 0^+$, the conclusion of (4) holds with δ_H in place of δ as well.

Second, we assume that the Euclidean equation of the line l is $y = \tan\theta \cdot x$, meaning that l passes through o and makes an angle θ with the positive part of the x-axis. If $\theta > 0$ is sufficiently small, then for the x-coordinate $x(\theta)$ of the intersection point of l and bd K, different from o, the following holds

$$f(x(\theta)) = \tan\theta \cdot x(\theta),$$

from which we obtain that

$$x(\theta) = 2\tan\theta/\kappa + o(\tan\theta) = 2\theta/\kappa + o(\theta) \text{ as } \theta \to 0^+.$$

Substituting $x(\theta)$ in (2), we get that the arc-length of bd K between o and the other intersection point of l and bd K is

$$s(\theta) = x(\theta) + \frac{\kappa^2 + 2}{6}x^3(\theta) + o(x^3(\theta)) \text{ as } x(\theta) \to 0^+. \tag{6}$$

At the same time, the length of the segment $l \cap K$ is

$$\begin{aligned} s_l(\theta) &= \tanh^{-1}(\sqrt{x^2(\theta) + f^2(x(\theta))}) \\ &= \tanh^{-1}(\sqrt{x^2(\theta) + \tan^2\theta x^2(\theta)}) \\ &= \tanh^{-1}(x(\theta)\sec\theta) \\ &= x(\theta)\sec\theta + \frac{1}{3}x^3(\theta)\sec^3\theta + O(x^3(\theta)\sec^3\theta) \\ &= x(\theta) + \frac{1}{2}x(\theta)\theta^2 + +\frac{1}{3}x^3(\theta)\sec^3\theta + O(x^3(\theta)\sec^3\theta). \end{aligned} \tag{7}$$

Now, by (6) and (7), the difference between the chord of l and the corresponding part of bd K is

$$s(\theta) - s_l(\theta) = \left(\frac{8\kappa^2}{6\kappa^3} - \frac{1}{\kappa}\right)\theta^2 + o(\theta^3) = \frac{1}{3\kappa}\theta^2 + o(\theta^3) = O(\theta^2) \text{ as } \theta \to 0^+. \tag{8}$$

The observations (5) and (8) are elementary and known. We only included their detailed proofs because we could not find an explicit argument in the literature.

Now, we turn to the actual proof of Theorem 1.2. Let $P \in \mathcal{P}_H(n)$ be an n-gon which minimizes the perimeter deviation from K, that is, $\text{dev}(K, P) = \delta_{\text{dev}_H}(K, n)$. We will denote the vertices of P by $x_1, \ldots, x_n$ in a counter-clockwise cyclic order along P. It is clear that each side $x_i x_{i+1}$ has a common point with K, otherwise we could move it inwards and decrease the perimeter deviation using the monotonicity of perimeter in the hyperbolic plane, cf. [11, Proposition 1.3]. The assumption that bd K is C_+^2 yields that K is strictly convex, that is, bd K contains no geodesic segment, and that bd K has a unique supporting line at each point, and therefore it cannot have vertices.

The proof of Theorem 1.2 is indirect: we assume, on the contrary, that P is not inscribed in K and seek a contradiction. It is clear that if $P \subset K$, then the vertices of P must be on bd K, similarly to the Euclidean case, or otherwise we could increase the perimeter of P by moving the vertices out to the boundary of K. Therefore, the indirect assumption yields that P has a side with at least one endpoint outside of K. There are several possibilities how this may happen. We treat each such case and show that they all contradict to the best approximation property of P.

We use the following notation, similar to [5, Section 2]. Let the vertex x_i be outside of K. We denote the internal angle of P at x_i by α_i. Let b_i be the last

common point of the side $x_{i-1}x_i$ and bd K, and let c_i be the first common point of x_ix_{i+1} and bd K in the counter-clockwise direction along P. Let the angle of the tangent of bd K at b_i and $x_{i-1}x_i$ be denoted by β_i, and the angle of the tangent of bd K at c_i and x_ix_{i+1} be γ_i. Then, clearly, $\alpha_i + \beta_i + \gamma_i < \pi$.

Let us first consider the case when P has a side, say x_1x_2, such that both x_1 and x_2 are outside of K. Let $\delta > 0$ be small and let $h = h(\delta)$ be the hypercycle that is the equidistant curve from the line x_1x_2 at distance δ in the half-plane of x_1x_2 containing P. Let x_1' and x_2' be the intersection points of the sides x_nx_1 and x_2x_3 with $h(\delta)$, respectively. Assume that δ is so small that both x_1' and x_2' are outside of K. Then the n-gon P' with vertices $x_1', x_2', x_3 \ldots, x_n$, in this order, is contained in P, and the intersection of the side $x_1'x_2'$ and K is of positive length. Let the feet of the perpendiculars from x_1' and x_2' to x_1x_2 be x_1'' and x_2'', respectively. Then $x_1''x_2''x_2'x_1'$ is a Saccheri quadrilateral. It is known that the line through the midpoints of the segments $x_1'x_2'$, and $x_1''x_2''$ is perpendicular to both lines and thus it cuts $x_1''x_2''x_2'x_1'$ into two congruent Lambert quadrilaterals. Using known trigonometric relations for Lambert quadrilaterals, we obtain that

$$\sinh(d(x_1'', x_2'')/2) = \sinh(d(x_1', x_2')/2)\cosh\delta,$$

from which it follows that

$$d(x_1'x_2') = d(x_1'', x_2'') + O(\delta^2) \text{ as } \delta \to 0^+.$$

By hyperbolic trigonometry, we obtain for $i = 1, 2$ that

$$\sinh d(x_ix_i') = \sinh\delta\csc\alpha_i,$$

thus

$$d(x_ix_i') = \delta\csc\alpha_i + O(\delta^3) \text{ as } \delta \to 0^+,$$

and

$$\sinh d(x_ix_i'') = -\tanh\delta\cot\alpha_i,$$

thus

$$d(x_ix_i'') = -\delta\cot\alpha_i + O(\delta^3) \text{ as } \delta \to 0^+.$$

If x_1x_2 is tangent to K at a relative interior point $x' \in x_1x_2$, then let a and b denote the intersection points of the segment $x_1'x_2'$ with bd K so that a is closer to x_1'. Then $d(x', ab) \leq \delta$. By the positivity of the geodesic curvature of bd K at x' and by (5), it holds that the difference of the arc-length of bd K between a and b and the length of the segment ab is $O(\delta^{3/2})$ as $\delta \to 0$, and thus, by using the estimates obtained above, we get that

$$\mathrm{dev}(K, P') = \mathrm{dev}(K, P) - \delta(\csc\alpha_1 + \csc\alpha_2 + \cot\alpha_1 + \cot\alpha_2) + O(\delta^{3/2}) \text{ as } \delta \to 0^+.$$

Since $\csc\alpha + \cot\alpha \geq 0$ for any $\alpha \in (0, \pi)$, the coefficient of δ is negative in the above expression. This contradicts the minimality of P, and thus P cannot have such a side.

If the side x_1x_2 cuts the boundary in two distinct points that are relatively interior to x_1x_2, then according to the previously introduced notation these intersection points are c_1 and b_2, and the tangents to bd K make an angle γ_1 and β_2 with x_1x_2, respectively. We introduce the following notations, see Figure 1. Let the last intersection point of $x_1'x_2'$ and bd K be b_2'. Let b_2'' be the perpendicular projection of b_2' onto x_1x_2. Let $\bar{b}_2$ be the point on $x_1'x_2'$ whose perpendicular projection onto x_1x_2 is b_2. Let b_2^* be the intersection point of the tangent line of bd K through b_2 and $x_1'x_2'$. Finally, let b_2^{**} be the perpendicular projection of b_2^* onto x_1x_2.

FIGURE 1.

We first note that, using hyperbolic trigonometry, we can conclude that

$$d(b_2, b_2'') = d(\bar{b}_2, b_2') + O(\delta^2) \text{ as } \delta \to 0^+,$$

and similarly,

$$d(b_2'', b_2^{**}) = d(b_2', b_2^*) + O(\delta^2) \text{ as } \delta \to 0^+.$$

Now, let $\delta' = d(b_2^*, b_2^{**})$. Clearly, $\delta' < \delta$. Similar as above, we obtain by hyperbolic trigonometry applied to the triangle $b_2 b_2^* b_2^{**}$ that

$$\sinh d(b_2, b_2^*) = \sinh \delta' \csc \beta_2,$$

from where

$$d(b_2, b_2^*) = \delta' \csc \beta_2 + O(\delta'^3) \text{ as } \delta \to 0^+,$$

moreover,

$$\sinh d(b_2, b_2^{**}) = -\tanh \delta' \cot \beta_2,$$

and

$$d(b_2, b_2^{**}) = -\delta' \cot \beta_2 + O(\delta'^3) \text{ as } \delta \to 0^+.$$

From trigonometric formulas for the corresponding Lambert quadrilateral we get that

$$\delta' = \delta + O(\delta^3) \text{ as } \delta \to 0^+.$$

Also, it is clear from the C_+^2 property of bd K that

$$d(b_2', b_2^*) = O(d(b_2, b_2^*)) = O(\delta^2) \text{ as } \delta \to 0^+,$$

and thus from all of the above,

$$d(b_2, b_2'') = d(b_2, b_2^{**}) + O(\delta^2) \text{ as } \delta \to 0^+,$$

and

$$d(\bar{b}_2, b_2') = d(\bar{b}_2, b_2^*) + O(\delta^2) \text{ as } \delta \to 0^+.$$

Let $l(\delta)$ denote the length of the arc of bd K between b_2 and b_2'. From (2), we obtain that

$$l(\delta) - d(b_2, b_2^*) = O(\delta^2) \text{ as } \delta \to 0^+.$$

Finally, putting everything together, we obtain (similar to (19) in [5]) that

$$\text{dev}_H(K, P') = \text{dev}_H(K, P) - \delta(2 \cot \beta_2 - 2 \csc \beta_2 + \csc \alpha_2 + \cot \alpha_2$$
$$+ 2 \cot \gamma_1 - 2 \csc \gamma_1 + \csc \alpha_1 + \cot \alpha_1) + O(\delta^2) \text{ as } \delta \to 0^+,$$

and thus, by the optimality of P, it must hold that

$$\cot\frac{1}{2}\alpha_1 + \cot\frac{1}{2}\alpha_2 = 2\left(\tan\frac{1}{2}\beta_2 + \tan\frac{1}{2}\gamma_1\right). \tag{9}$$

In the following case we do not give all small details of the calculations as those are very similar to the ones discussed above. We rather just point out the main conclusions of these calculations.

Next, assume that for the side $x_1 x_2$ it holds that $x_1 \in K$ and $x_2 \notin K$. Rotate the line $x_1 x_2$ around x_1 by a sufficiently small positive angle φ such that the intersection point x_2' of the rotated line with the side $x_2 x_3$ is still outside K. Let P' be the polygon with vertices $x_1 x_2' x_3 \ldots x_n$. Clearly, $P' \subset P$. Let b_2 be the last intersection point of the side $x_1 x_2$ with bd K, as before.

If the line $x_1 x_2$ is not a supporting line of K at x_1, then we obtain by hyperbolic trigonometry that

$$\mathrm{dev}(K,P') = \mathrm{dev}(K,P) + \varphi(2\sinh d(x_1,b_2)(\csc\beta_2 - \cot\beta_2)$$
$$- (\csc\alpha_2 + \cot\alpha_2)\sinh d(x_1,x_2)) + O(\varphi^2) \text{ as } \varphi \to 0^+.$$

Due to the optimality of P, it must hold that

$$\frac{\sinh d(x_1,x_2)}{\sinh d(x_1,b_2)}\cot\frac{1}{2}\alpha_2 = 2\tan\frac{1}{2}\beta_2. \tag{10}$$

Note that $d(x_1,x_2) > d(x_1,b_2)$, and thus by the strictly monotonically increasing property of the sinh function it follows that the coefficient of $\cot(\alpha_2/2)$ in (10) is larger than 1.

If $x_1 x_2$ is a supporting line of K, then, using (8), we get that

$$\mathrm{dev}(K,P') = \mathrm{dev}(K,P) - \varphi(\csc\alpha_2 + \cot\alpha_2)\sinh d(x_1,x_2) + O(\varphi^2) \text{ as } \varphi \to 0^+.$$

As the coefficient of φ is negative, this clearly contradicts the minimality of P, so P cannot have such a side.

Now, the proof can be finished as in [5, cf. (24)–(25) on p. 357]: For each $x_i \notin K$, the angle α_i appears in exactly two equations of type (9) or (10), and β_i and γ_i in exactly one such equation. Thus, by adding the two equations in which α_i appears, the coefficient of $\cot(\alpha_i/2)$ will be at least 2. If $2 + \varepsilon_i$ denotes the coefficient of $\cot(\alpha_i/2)$, then summing all equations of type (9) and (10) yields that

$$\sum(2 + \varepsilon_i)\cot\frac{1}{2}\alpha_i = \sum 2\left(\tan\frac{1}{2}\beta_i + \tan\frac{1}{2}\gamma_i\right)$$
$$< \sum 2\tan\frac{1}{2}(\beta_i + \gamma_i)$$
$$\leq \sum 2\tan\left(\frac{\pi}{2} - \alpha_i\right)$$
$$= \sum 2\cot\frac{1}{2}\alpha_i,$$

which is clearly a contradiction as all $\varepsilon_i \geq 0$. This finishes the proof of Theorem 1.2.

3. Counterexample on the sphere

It is known that among spherical triangles contained in a (spherical) circle the inscribed regular triangle has the maximal perimeter, cf. L. Fejes Tóth [6]. Thus, among triangles contained in the circle, the inscribed regular one has the minimum

perimeter deviation from the circle. However, below we show an example of a triangle and circle, where the triangle is neither inscribed nor circumscribed, and approximates the circle better than either the inscribed or the circumscribed regular triangle.

Let $K(r)$ be the spherical circle with centre P and radius r. Consider the regular spherical triangle $T(d) = ABC\triangle$ with centre P and circumradius d. Let $l = l(d)$ denote the side length of $T(d)$ and $m = m(d)$ its inradius, see Figure 2.
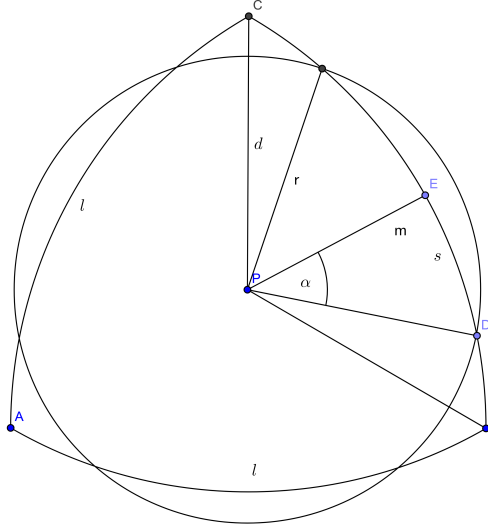
FIGURE 2.

Then
$$\cos l = \cos^2 d + \sin^2 d \cos \frac{2\pi}{3},$$
and
$$\cos m = \frac{\cos d}{\cos \frac{l}{2}}.$$

Let D be the intersection point of the side BC with the circle $K(r)$ that is closer to B. Let E be the intersection of the side BC and the great circle through P perpendicular to BC. Then m is the distance of P and E. Denote by $s = s(d)$ the length of the arc between D and E. Furthermore, let $\alpha = \alpha(d)$ be the central angle $\angle EPD$. Then
$$\cos s = \frac{\cos r}{\cos m},$$
and
$$\cos \alpha = \frac{\cos s - \cos m \cos r}{\sin m \sin r}.$$

Thus
$$f(r, d) = \operatorname{dev}_S(K(r), T(d)) = 6(2\alpha(d) \sin r - 2s(d) + l(d)/2) - 2\pi \sin r.$$

The graph of $f(\frac{\pi}{2} - 0.1, d)$ over the interval $[\frac{\pi}{2} - 0.1, 1.52]$ is shown below, which clearly has its minimum inside the interval. We note that at the left endpoint of the interval the triangle is inscribed and at the right endpoint it is circumscribed.

In fact, the circumscribed regular triangle approximates $K(r)$ better than the inscribed one, and the minimum occurs for a triangle that is neither inscribed nor circumscribed. Since all of these triangles are contained in the open hemisphere centred at P, they are convex in the spherical sense.
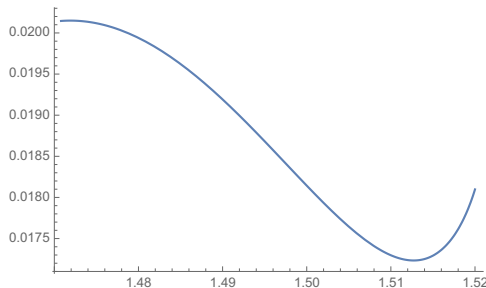

FIGURE 3. The graph of $f(\frac{\pi}{2} - 0.1, d)$ over the interval $[\frac{\pi}{2} - 0.1, 1.52]$

4. ACKNOWLEDGEMENTS

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