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Approximating Least Fixpoints

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Everything should be made as simple as possible, but not simpler.
– Attributed to Albert Einstein

Abstract I try to come up with general techniques for approximating least fixpoints from below and greatest fixpoints from above. In this, I try to place as few restrictions as possible on the underlying partial orders; in particular, I avoid the use of linear orders. The approach is intended to abstract and thus generalise approximation techniques used in [10,1]. I hope that I am not falling into Einstein's trap with this note.

1 Basics

A *poset* is a set P with a partial order $\leq$ on it. A *chain* of P is a non-empty subset of P on which $\leq$ is linear.

A function $f : P \to P$ is *isotone* or *monotonically increasing* if $x \leq y \Rightarrow f(x) \leq f(y)$, and *antitone* or *monotonically decreasing* if $x \leq \Rightarrow f(y) \leq f(x)$ (cf. [2], §2).

Following [14], a *subset system* is a function Z which assigns to each poset P a set $Z[P]$ of subsets of P such that

1. there exists a poset P such that $Z[P]$ contains some non-empty set;
2. If $f : P \to P'$ for posets P, P' is isotone and $S \in Z[P]$ then $f(S) \in Z[P']$.

The elements of $Z[P]$ are called the *Z-sets of P*.

We will use the following versions of Z-sets.

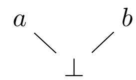
1. $\mathcal{P}[P]$: the set of *all* subsets of P.
2. $\Delta[P]$: the set of all directed subsets of P.
3. $\aleph_0[P]$: the set of all non-empty countable subsets of P.
4. $\Gamma[P]$: the set of all countable chains of P.
5. $\Gamma_{\aleph_0}[P]$: the set of all countable chains of P.

Given a subset system Z, we say that a poset P is *Z-complete* iff it has a least element $\bot$ and every Z-set S of P has a supremum $\bigsqcup S \in P$. A function $f : P \to P'$ is *Z-continuous* if for every Z-set S in P such that $\bigsqcup S$ exists, we have $f(\bigsqcup S) = \bigsqcup f(S)$.

A poset is *countably complete (CC)* if it is $\aleph_0$-complete. By elementary order theory, in an upper semilattice this is equivalent to $\Gamma_{\aleph_0}$-completeness.

A poset is a *complete lattice* if it is $\mathcal{P}$-complete.

Every complete lattice is Δ-complete, $\aleph_0$-complete and Γ-complete, every Δ-complete poset is Γ-complete, but not necessarily $\aleph_0$-complete. A counterexample is the poset



In a complete lattice every subset S also has an infimum $\sqcap S$.

For a function $f : P \to P$ and $x \in P$ we define, as in Kleene's iteration [5], the set $\hat{f}(x) =_{df} \{f^i(x) \mid i \in \mathbb{N}\}$, where

$$f^0(x) =_{df} x \qquad f^{i+1}(x) = f(f^i(x))$$

$\hat{f}(x)$ is non-empty, since $x = f^0(x) \in \hat{f}(x)$. Moreover, by construction $\hat{f}(x)$ is countable.

Lemma 1.1 *Assume that $f : P \to P$ is isotone.*

1. *$\hat{f}(x)$ is a chain iff x is contracted or expanded by f.*
2. *In particular, if P has a least element $\bot$ then $\hat{f}(\bot)$ is a chain.*

Proof.

1. ($\Rightarrow$) By definition $x = f^0(x)$ and $f(x) = f^1(x)$ are in $\hat{f}(x)$ and hence $f(x) \leq x$ or $x \leq f(x)$.
   ($\Leftarrow$) Assume $x \leq f(x)$. A straightforward induction using isotony of f shows $f^i(x) \leq f^{i+1}(x)$ for all $i \in \mathbb{N}$, which entails $f^j(x) \leq f^k(x)$ for all j, k with $j \leq k$.
   The proof in case $f(x) \leq x$ is symmetric.
2. This follows from Part 1, since $\bot$ as the least element is trivially expanded by f. □

We use the well known fixpoint theorems. The set of fixpoints of a function $f : P \to P$ is denoted by *fix*(f).

First we deal with least fixpoints.

Theorem 1.2 *Let P be a Δ-complete poset and $f : P \to P$ an isotone function.*

1. *[3,8] f has a least fixpoint μf which is the infimum of the contracted elements or pre-fixpoints.*
2. *This entails the least fixpoint induction rule*
$$\frac{f(x) \leq x}{\mu f \leq x}$$
3. *[6] If P has a least element $\bot$ then $\sqcup \hat{f}(\bot) \leq \mu f$. If f is $\Gamma_{\aleph_0}$-continuous then this strengthens to an equality.*

The case of greatest fixpoints is similar. However, one usually does not work with the dual of Δ-completeness but rather with complete lattices.

Theorem 1.3 *Let P be a complete lattice and $f : P \to P$ an isotone function.*

1. *[13] f has a greatest fixpoint νf which is the supremum of the expanded elements or post-fixpoints.*

2. *This entails the greatest fixpoint co-induction rule*

$$\frac{x \leq f(x)}{x \leq \nu f}$$

3. *[5] If P has a greatest element $\top$ then $\nu f \leq \bigsqcap \hat{f}(\top)$. If f is $\Gamma_{\aleph_0}$-co-continuous, i.e., preserves infima of non-empty countable chains, then this strengthens to an equality.*

In the remainder of the paper we mostly deal with least fixpoints; the treatment of greatest fixpoints is symmetric.

2 Some Closure Properties

For $X \subseteq P$, by $\downarrow X =_{df} \{y \mid \exists x \in X : y \leq x$ we denote the *downward closure* or *downset* of X. For $x \in P$ we abbreviate $\downarrow\{x\}$ to $\downarrow x$. The *upward closure* or *upset* $\uparrow X$ is defined symmetrically.

Lemma 2.1 *Consider an arbitrary $x \in P$, an isotone function $f : P \to P$ and $y, z \in P$ such that y is contracted and z is expanded by f.*

1. *$\downarrow x$ is closed under arbitrary existing suprema and under arbitrary existing suprema of non-empty sets.*
2. *$\downarrow y$ and $\uparrow z$ are closed under f.*
3. *If $u \in \downarrow y$ then $\hat{f}(u)$ is a countable set with $\hat{f}(u) \subseteq \downarrow y$. Hence, if P is CC then $\bigsqcup \hat{f}(u) \in \downarrow y$ as well.*
4. *$\hat{f}(z)$ is a countable chain.*

Proof.

1. Assume $X \subseteq \downarrow x$. Then x is an upper bound of X. Hence, if X has a supremum $y = \bigsqcup X$ then $y \leq x$ and hence $y \in \downarrow x$.
   Assume $Y \subseteq \uparrow x$ with $Y \neq \emptyset$ and $v = \bigsqcup Y$. Choose a $u \in Y$. By definition $x \leq u$ and $u \leq v$. Hence also $v \in \uparrow x$ by transitivity of $\leq$.
2. Assume $u \in \downarrow y$, i.e., $u \leq y$. Therefore isotony of f implies $f(u) \leq f(y) \leq y$ and hence $u \in \downarrow y$ as well.
   The second claim is shown symmetrically.
3. This is immediate from Parts 1 and 2.
4. This follows by an easy induction. □

3 Relativised Fixpoints

We want to find "relativised" fixpoints of f above some element $x \in P$. For this we define the set $\mathit{ufix}(f, x) =_{df} \mathit{fix}(f) \cap \uparrow x$ of fixpoints of f above x and denote, when existing, the least element of $\mathit{ufix}(f, x)$ by $\mathit{lfp}(f, x)$.

To apply our earlier results we form the up-set $P' =_{df} \uparrow x$ and restrict f to P'. If P is Δ-complete then so is P' with least element x, and hence in P' we have $\bigsqcup \emptyset = x$.

Theorem 3.1

1. *$x \in \mathit{fix}(f) \Rightarrow x = \mathit{lfp}(f, x)$.*
2. *The restriction $f|P'$ is an endofunction on P' iff x is expanded by f.*
3. *In this case $\mathit{lfp}(f, x) = \mu(f|P')$. In particular, $\mu f = \mathit{lfp}(f, \bot)$. In P' we have the rule*
$$\frac{f(y) \le y}{\mathit{lfp}(f, x) \le y}$$
4. *Let $E_f =_{df} \{x \in P \mid x \le f(x)$ be the set of elements expanded by f. Then E_f is closed under f and $\mathit{lfp}(f, \_)$ is a closure operator on E_f. In particular, $x \le \mathit{lfp}(f, x)$.*
5. *For $x, y \in E_f$, if $x \le y \le \mathit{lfp}(f, x)$ then $\mathit{lfp}(f, y) = \mathit{lfp}(f, x)$.*
6. *For $x, y \in E_f$, if $x \le y \le \mathit{lfp}(f, x)$ and $y \in \mathit{fix}(f)$ then $y = \mathit{lfp}(f, x)$.*

Proof.

1. By assumption $z \in \mathit{ufix}(f, z)$. Consider an arbitrary $y \in \mathit{ufix}(f, x) = \mathit{fix}(f) \cap \uparrow x$. Then by definition $x \le y$. Thus x is the least element of $\mathit{ufix}(f, x)$ and hence $x = \mathit{lfp}(f, x)$.
2. The implication ($\Rightarrow$) is immediate from the definition of $\uparrow x$, while ($\Leftarrow$) follows from Lm. 2.1.2.
3. This follows by Th. 1.2.1 and 1.2.2 with Lm. 1.1.2.
4. First, by definition, isotony of f and definition again,
$$x \in E_f \Leftrightarrow x \le f(x) \Rightarrow f(x) \le f(f(x)) \Leftrightarrow f(x) \in E_f$$
Next, by Part 3 $\mathit{lfp}(f, x)$ exists for all $x \in E_f$. Now we show the properties of a closure operator.
   - Extensivity: By definition $\mathit{lfp}(f, x) \in \uparrow x$, i.e., $x \le \mathit{lfp}(f, x)$.
   - Isotony: We have $x \le y$ iff $\uparrow y \subseteq \uparrow x$. Hence $x \le y$ implies $\mathit{ufix}(f, y) \subseteq \mathit{ufix}(f, x)$, and hence the least element $\mathit{lfp}(f, x)$ of $\mathit{ufix}(f, x)$ is below all elements of $\mathit{ufix}(f, y)$, particular below $\mathit{lfp}(f, y)$.
   - Idempotence is immediate from Part 1.
5. Assume $x \le y \le \mathit{lfp}(f, x)$. From the first inequation and Part 4 we infer $\mathit{lfp}(f, x) \le \mathit{lfp}(f, y)$. From the second inequation and again Part 4 we infer
$$\mathit{lfp}(f, y) \le \mathit{lfp}(f, \mathit{lfp}(f, x)) = \mathit{lfp}(f, x)$$
6. Immediate from Parts 5 and 1. □

Parts 5 and 6 generalise to arbitrary closure operators.

Theorem 3.2 *Consider a Δ-complete countable upper semilattice P which additionally is* Noetherian*, i.e., has no infinite properly ascending chains and let $f : P \to P$ be isotone.*

1. *For every $x \in P$ we have $\mathit{lfp}(f, x) = \bigsqcup \hat{f}(x)$, even if x is not expanded by f.*
2. *$\mathit{lfp}(f, x)$ is computed by the following basic iterative algorithm:*

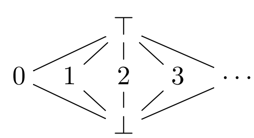
$y := x\ ; z := x$
$\{\,\mathsf{inv}\ y \le \mathit{lfp}(f, x)\}$
$\mathsf{while}\ (f(y) \ne y)$

do $y := y \sqcup z \;\; ; z := f(z)$
od $\{y = \mathit{lfp}(f, x)\}$

Proof.

1. We define $y_i =_{df} \bigsqcup_{j \leq i} f^j(x)$. The y_i are well defined, since P is an upper semilattice. Clearly $y_i \leq y_{i+1}$ and hence the y_i form a countable ascending chain. Set $u =_{df} \bigsqcup_{i \in \mathbb{N}} y_i$. By Noetherity f is $\Gamma_{\aleph_0}$-continuous. Therefore, as in Kleene's theorem, $u \in \mathit{fix}(f)$. By construction, $x \leq u$ and thus $u \in \mathit{ufix}(f, x)$. Consider an arbitrary $v \in \mathit{ufix}(f, x)$. We show by induction on i that v is an upper bound of the $f^i(x)$ and the y_i.
   <u>$i = 0$.</u> By $v \in \mathit{ufix}(f, x)$ and the definitions,
   $$y_0 = f^0(x) = x \leq v$$
   <u>$i \to i + 1$.</u> First, by the first part of the induction hypothesis $f^i(x) \leq v$. Hence by the definitions, isotony of f and $v \in \mathit{fix}(f)$,
   $$f^{i+1} = f(f^i(x)) \leq f(v) = v$$
   Second, by the definitions, the second part of the induction hypothesis and the just shown inequation,
   $$y_{i+1} = y_i \sqcup f^{i+1}(x) \leq v \sqcup v = v$$
   Therefore $u \leq v$, as claimed.
2. Termination of the algorithm follows by the assumed Noetherity. The initialisation establishes the invariant. The variable z successively contains the $f^i(x)$ and, as shown in the proof of Part 1, therefore the loop body preserves the invariant. By standard Hoare logic, after the loop also the negated loop condition holds, i.e., $y \leq \mathit{lfp}(f, x) \wedge y = f(y)$. By the initialisation $x \leq y$ and hence $y \in \mathit{ufix}(f, x)$. Since $\mathit{lfp}(f, x)$ is least in $\mathit{ufix}(f, x)$, we conclude $y = \mathit{lfp}(f, x)$. □

Example 3.3 An example where the starting element is not expanded is the complete lattice



with the isotone function $f(\bot) = \bot, f(\top) = \top$ and $f(i) = i + 1$. None of the $i \in \mathbb{N}$ is expanded by f and yet $\mathit{lfp}(f, i) = \top = \bigsqcup \hat{f}(i) = \{j \mid j > i\}$.

Here the loop for $x = i$ stops after the second pass, since then $y = \top = \mathit{lfp}(f, x)$. □

4 Under-Approximating Least Fixpoints

Th. 3.1.4 provides a first rule for showing that an element is below a least fixpoint, namely $x \leq \mathit{lfp}(f, x)$. However, we want to find other ones in which the element and the fixpoint are less tightly coupled, i.e., where the fixpoint is given “independently” of the element.

Let P be Z-complete and $X, Y \in P[Z]$. Then Y is a *majorant* of X if for every $x \in X$ there is a $y \in Y$ with $x \le y$. This implies $\bigsqcup X \le \bigsqcup Y$.

Theorem 4.1 *Assume an isotone function $f : P \to P$, an element x expanded by f and and a sequence $(y_i)_{i \in \mathbb{N}}$ of $y_i \in P$ such that $y_0 = x$ and $y_{i+1} \le f(y_i)$ for all $i \in \mathbb{N}$.*
1. *$\hat{f}(x)$ is a majorant of the set $Q =_{df} \{y_i \mid i \in \mathbb{N}\}$.*
2. *$\bigsqcup Q \le lfp(f, x)$.*

Proof.
1. We show $y_i \le f^i(x)$ by induction on i.
   $\underline{i = 0}$: by reflexivity and the definitions,
   $$y_0 = x \le x = f^0(x)$$
   $\underline{i \to i+1}$: by the assumption, the induction hypothesis with isotony of f and the definitions,
   $$y_{i+1} \le f(y_i) \le f(f^i(x)) = f^{i+1}(x)$$
2. Immediate from Th. 1.2.3 and Part 1 with order theory. □

This results abstracts and generalises the result in Lemma 3.4 of the paper [10] that the inference rule
$$\frac{\forall\, n \in \mathbb{N} : [\, p_n \,]\ a\ [\, p_{n+1} \,]}{[\, p_0 \,]\ a^*\ [\, \bigvee_{n \in \mathbb{N}} p_n \,]} \qquad (\mathit{Iteration})$$
of Incorrectness Logic is sound.

5 Star-Like Recursions

We now study recursions in the form of Kleene’s classical definition [6] of the star operator. However, we abstract from semirings or quantales as discussed in [4,7,12].

Assume a CC upper semilattice P as well as an isotone “step function” $g : P \to P$. We define the isotone function $f : P \to P$ by
$$f(x) =_{df} x \sqcup g(x) \qquad (1)$$

Example 5.1
1. In a quantale S the recursion for a^* has the pattern $a^* = f(a^*)$, where $P = S, \sqcup = +$ and $g(x) = a \cdot x$.
2. In a modal Kleene algebra S the recursion for $q =_{df} \langle a^* | p$ has the pattern $q = f(q)$, where $P = \mathsf{test}(S), \sqcup = +$ and $g(x) = \langle a | x$. □

Theorem 5.2 *Let f be given as in* (1) *and assume a sequence $(z_i)_{i \in \mathbb{N}}$ of $z_i \in P$ such that $z_0 = x$ and $z_{i+1} \le g(z_i)$ for all $i \in \mathbb{N}$.*
1. *$\bigsqcup \hat{g}(x) \le lfp(f, x)$.*

2. *If g is $\Gamma_{\aleph_0}$-continuous then so is f and the above inequation strengthens to an equality.*

Proof.
1. Setting $y_i =_{df} x \sqcup z_i$ establishes the assumptions of Th. 4.1. Hence the claim follows by Th. 4.1.1.
2. Immediate from Th. 1.3. □

6 Approximation by Measure

In this section we try to generalise an approach by Baldan et al. [1]. There some kind of measure function is employed. It tells "how far" elements are from the closest fixpoint and allows showing that elements are below or above extremal fixpoints without using standard fixpoint iteration, as in the previous sections.

Since the main theorem centrally uses greatest fixpoints, in this section we first work with complete lattices rather than cpos.

We present the abstraction right away, because is is fairly simple. After that we show how it mirrors the approach of [1].

6.1 Basic results

Let P be a complete lattice. The aim is to study isotone endofunctions $f : P \to P$. In particular, we are interested whether $\nu f \leq x$ for some element $x \in P$. For this, we proceed as follows, leaning notationally on [1].

- Find a complete lattice Q of "measures"
- and a function $\gamma : P \times P \times P \to Q$ such that $\gamma(x, y, \delta)$ yields the "distance" between two P-elements relative to a threshold δ.
- γ is required to be to be antitone in its first argument. The motivation for this is given in the proof of Th. 6.1 below.
- Moreover, γ needs to be "sharp" in that
$$(\forall\, \delta : \delta \neq \bot \Rightarrow \gamma(x, y, \delta) = \bot) \Rightarrow x \leq y$$
- Next, find a *simulation operator* $\# : (P \to P) \times P \to (Q \to Q)$ which "mimics" the action of an endofunction f on P in the measure lattice Q. For abbreviation we denote $\#(f, x)$ by $f^\#_x$ and call a function f *#-well-behaved* when for all $\delta \neq \bot$ we have
$$\gamma(f(x), f(y), \delta) \leq f^\#_x(\gamma(x, y, \delta))$$
and $f^\#_x$ is isotone. An example is provided by the non-expansive functions of [1].

This admits the following result.

Theorem 6.1 *Consider some $\delta \neq \bot$, an element $x \in P$ and a #-well-behaved function $f : P \to P$ such that $\nu f^\#_x = \bot$.*
1. *If x is contracted by f, i.e., satisfies $f(x) \leq x$, then $\nu f \leq x$.*
2. *If x is even a fixpoint of f then $x = \nu f$.*

Proof.

1. We first show that $\gamma(x, \nu f, \delta)$ is expanded by $f_x^{\#}$:

$$
\begin{array}{ll}
 & f_x^{\#}(\gamma(x, \nu f, \delta)) \\
\geq & \quad \{\!\![\ f \text{ is } \#\text{-well-behaved }]\!\!\} \\
 & \gamma(f(x), f(\nu f)) \\
= & \quad \{\!\![\ \nu f \text{ is a fixpoint of } f\ ]\!\!\} \\
 & \gamma(f(x), \nu f, \delta) \\
\geq & \quad \{\!\![\ x \text{ is contracted by } f \text{ and } \gamma \text{ is antitone in its first argument }]\!\!\} \\
 & \gamma(x, \nu f, \delta)
\end{array}
$$

   Hence the Knaster-Tarski theorem entails $\gamma(x, \nu f, \delta) \leq \nu f_x^{\#} = \bot$. Since δ is arbitrary, the claim now follows by sharpness of γ.

2. Since νf is the greatest fixpoint of f, we know $x \leq \nu f$. Moreover, since every fixpoint is contracted by f, Part 1 shows the reverse inequation. □

Note that nowhere a (component) order is assumed to be linear.

In using this result it has to be efficiently checkable whether $\nu f_x^{\#} = \bot$. This can hopefully be achieved if a finite set Q is used.

The result only shows "soundness", i.e., $f(x) \leq x \wedge \nu f_x^{\#} = \bot \Rightarrow \nu f \leq x$. For "completeness" $f(x) \leq x \wedge \nu f \leq x \Rightarrow \nu f_x^{\#} = \bot$ one would need stronger assumptions on $f_x^{\#}$.

6.2 A Concrete Instance of the Results

As announced, we now present the example of [1]. The essential operators there are addition $\oplus$ and subtraction $\ominus$ of measures, axiomatised by MV-algebras (e.g. [11]). We use a simplified axiomatisation, not using a complement operator, that is sufficient for our purposes.

A *difference algebra* is a commutative monoid $(M, \oplus, 0)$ with a partial order $\leq$ in which 0 is the least element and there is another binary operator $\ominus : M \times M \to M$ satisfying the Galois connection

$$x \ominus y \leq z \Leftrightarrow x \leq y \oplus z$$

In this, the function $f_y(x) =_{df} x - y$ is the lower and $g_y(z) =_{df} y + z$ the upper adjoint. Therefore, by standard Galois theory, existence of $\ominus$ can be guaranteed when M is a completely distributive complete lattice und $\leq$, since then g_y is universally conjunctive, and hence determines $\ominus$ uniquely. This holds, in particular, when M is a complete Boolean algebra or a complete chain.

The above axioms entail most of the negation-free properties mentioned in [1]:

$$
\begin{array}{ll}
x \leq x \oplus y & x \oplus y \leq 0 \Rightarrow x \leq 0 \\
y \leq z \Rightarrow x \oplus y \leq x \oplus z & x \leq y \Rightarrow x \oplus z \leq y \oplus z \\
y \leq z \Rightarrow x \ominus z \leq x \ominus y & x \leq y \Leftrightarrow x \ominus y \leq 0 \\
x \ominus 0 = x & y \ominus x \leq y \\
x \ominus x = 0 & 0 \ominus y = 0 \\
x \ominus (y \oplus z) = (x \ominus y) \ominus z & (x \oplus y) \ominus y \leq x \\
x \ominus (x \ominus y) \leq x & y = x \oplus (y \ominus x) \Rightarrow x \leq y
\end{array}
$$

$$\exists\, z(y = x \oplus z) \Rightarrow x \leq y$$

They are readily automatically proved by Prover9 [9] in no time.

The approach of [1] can be represented with difference algebras as follows. One uses a finite set Y and a linearly ordered difference algebra M having a greatest element 1. Then $P =_{df} M^Y$; this is the set of all Y-tuples over M. The lattice Q is simply the power set of Y. For $x \in M^Y$ one defines its norm as $||a|| = max\{a(u) \mid u \in Y\}$. Linearity of the order $\leq$ has been assumed to make this well defined. If linearity is undesired then one can axiomatise a supremum operator $\sqcup$ by

$$x \sqcup y = x + (y \ominus x) \qquad y \sqcup x = x \sqcup y$$

This is the negation-free analogue of the corresponding axioms in [11]. Then one can set $||a|| = \sqcup_{u \in Y} a(u)$.

The operators $\oplus, \ominus$ are extended pointwise to tuples. Then we can adapt the definition of γ from [1]:

$$\gamma(x, y, \delta) =_{df} \{u \in Y \mid x(u) \neq 1 \wedge y(u) \ominus x(u) \geq \delta\}$$

For antitony of γ we have, assuming $x \leq x'$,

$u \in \gamma(x, y, \delta)$
$\Leftrightarrow$ {[definition]}
$x(u) \neq 1 \wedge y(u) \ominus x(u) \geq \delta$
$\Leftarrow$ {[$x \leq x'$ and antitony of $\ominus$ in its right argument]}
$x'(u) \neq 1 \wedge y(u) \ominus x'(u) \geq \delta$
$\Leftrightarrow$ {[definition]}
$u \in \gamma(x', y, \delta)$

To see sharpness of γ, we use the contrapositive $x \not\leq y \Rightarrow \exists\, \delta : \delta \neq \bot \wedge \gamma(x, y, \delta) \neq \bot = \emptyset$. By the Galois connection the premise is equivalent to $x \ominus y \not\leq 0$ and hence $\delta =_{df} ||x \ominus y|| \not\leq 0$. In this case there must be a $u \in Y$ with $(x \ominus y)(u) = \delta$, so that $\gamma(x, y, \delta) \neq \emptyset$.

Next, Th. 10.a of [1] shows that the non-expanding functions are #-well-behaved.

Finally, an assumption of $\nu f \not\leq x$ is by the Galois connection equivalent to $\delta =_{df} \nu f \ominus x \not\leq 0$ and hence $\delta \neq 0$, so that Th. 6.1 yields $\nu f \leq x$, a contradiction.

6.3 Dualising to Least Fixpoints

To deal with least fixpoints one essentially employs the order in a "mirrored way". Only this time we do not require P to be a complete lattice but are satisfied with a cpo. However, for Q we still use a complete lattice because we are still working with greatest fixpoints of endofunctions on measures. This leads to the following requirements and definitions.

- Find a function $\gamma' : P \times P \times P \to Q$ such that $\gamma(x, y, \delta)$ yields the "distance" between two P-elements relative to a threshold δ.
- γ' is now required to be to be *iso*tone in its first argument and sharpness now means

$$\delta \neq \bot \wedge \gamma'(x, y, \delta) = \bot \Rightarrow y \leq x$$

In the concrete instance of Sect. 6.2 this can be achieved by swapping the roles of the first and second arguments of γ, i.e., by setting

$$\gamma'(x, y, \delta) =_{df} \{u \in Y \mid x(u) \neq 1 \wedge x(u) \ominus y(u) \geq \delta\}$$

– Next, find again a *simulation operator* $\# : (P \to P) \times P \to (Q \to Q)$. For abbreviation we denote now $\#(f, x)$ by $f^x_\#$ and call a function f *#-co-well-behaved* when for all $\delta \not\leq \bot$ we have

$$\gamma'(f(x), f(y), \delta) \leq f^x_\#(\gamma'(x, y, \delta))$$

and $f^x_\#$ is isotone.

This admits the following result.

Theorem 6.2 *Assume some $\delta \neq \bot$ and a #-co-well-behaved function $f : P \to P$ such that $\nu f^\#_x = \bot$.*

1. *If $x \in P$ is expanded by f, i.e., satisfies $x \leq f(x)$, then $x \leq \mu f$.*
2. *If x is even a fixpoint of f then $x = \mu f$.*

Proof.

1. We first show that $\gamma'(x, \mu f, \delta)$ is expanded by $f^x_\#$:

$$\begin{array}{ll} & \gamma'(x, \mu f, \delta) \\ \leq & \quad \{\![\ x \text{ is expanded by } f \text{ and } \gamma' \text{ is isotone in its first argument }]\!\} \\ & \gamma'(f(x), \mu f, \delta) \\ = & \quad \{\![\ \nu f \text{ is a fixpoint of } f\ ]\!\} \\ & \gamma'(f(x), f(\mu f)) \\ \leq & \quad \{\![\ f \text{ is \#-co-well-behaved }]\!\} \\ & f^x_\#(\gamma'(x, \mu f, \delta)) \end{array}$$

   Hence the Knaster-Tarski theorem entails $\gamma'(x, \mu f, \delta) \leq \nu f^\#_x = \bot$ and the claim follows by sharpness of γ'.
2. Since μf is the least fixpoint of f, we know $\mu f \leq x$. Moreover, since every fixpoint is expanded by f, Part 1 shows the reverse inequation. □

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